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What are Crowds?

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Contents

Introduction

What are Crowds? To give a proper answer to this question, a little history is needed. In the mid-1950s, Jacques Tits released a paper [1], which introduced the concept of $\mathbb{F}_1$, the field with one element, and an idea of its associated geometry. Recalling the definition of a field, it is clear such a field cannot exist. Despite the non-existence of such an object, it does not necessarily imply that geometry over $\mathbb{F}_1$ cannot be studied.

Since Tits' paper, there have been various advances regarding algebraic geometry over $\mathbb{F}_1$, including approaches to the ABC conjecture [2] and even the Riemann hypothesis [3]. The papers [4] and [5] give a more in-depth introduction to what has been done in this area.

Regarding algebraic groups over $\mathbb{F}_1$, a few methods have been tried [6]. However, in a recent paper [7], Oliver Lorscheid and Koen Thas proposed a new approach that may shed light on algebraic groups in matroid theory and tropical geometry. This is achieved by modifying the definition of a group, thereby introducing the crowd structure.

In the first section, we cover the definition of a crowd, a set G, a unit element $1 \in G$ and the crowd law: a set $R \subset G^3$ such that $(1,1,1) \in R$, $(a,1,1) \in R$ implies $a = 1$, $(a,b,1) \in R$ implies $(b,a,1) \in R$ and finally $(a,b,c) \in R$ implies $(b,c,a) \in R$. We then cover basic properties of a crowd, crowd morphisms and subcrowds and a few examples, including but not limited to trivial crowds, maximal crowds and groups. In the second section, we cover the definition of bands and band morphisms before giving a few examples, including the Krasner hyperfield, which becomes important in the third section. The Krasner hyperfield can be viewed as a set $\mathbb{K} = \{0,1\}$, with the normal multiplication and hyperaddition $\boxplus$ such that $1 \boxplus 1 = \{0,1\}$ and $0 \boxplus 1 = \{1\}$.

In the third section, we cover the definition of special linear groups over bands, before taking a close look at $\mathrm{SL}_n(\mathbb{K})$, the special linear group over the Krasner hyperfield. The special linear group is a matrix group where the coefficients are in $\mathbb{K}$ and where the determinant equals one. We prove that it is a saturated crowd, and that every element has an inverse for all $n \in \mathbb{N}$. Then we describe properties that hold when $n = 2$ and show that they no longer hold for a larger n. In particular, the crowd based product and the naive "normal product" agree when $n = 2$ but do not agree for $n = 3$. Many of these last properties were found with the aid of a computer, so the code to simulate an element of the Krasner hyperfield is included in the appendix.

As background, we assume basic knowledge of group theory, permutations and sets. If in doubt, then the following lecture notes by Jaap Top can be accessed [8].

1 Crowds

1.1 Definition & Basic Properties

Definition 1.1. A crowd is a triple $(G, 1, R)$ such that G is a set, $1 \in G$ is a unit element, and $R \subset G^3$ with the following properties:

1. $(1,1,1) \in R$.
2. $(a,1,1) \in R$ implies $a = 1$.
3. $(a,b,1) \in R$ implies $(b,a,1) \in R$.
4. $(a,b,c) \in R$ implies $(c,a,b) \in R$.

Definition 1.2 (Inverse)**.** The **inverse** of $a \in G$ is the set $a^{-1} = \{b \in G \mid (a,b,1) \in R\}$.

Definition 1.3 (Products)**.** The **product** of two elements $a, b \in G$ is the set:

$$a \cdot b = \left\{c \in G \mid c \in d^{-1} \text{ and } (a,b,d) \in R\right\}.$$

Note: It follows that without inverses there are no products.

Example 1.4 (Groups)**.** Any group G gives a crowd with unit element 1 and crowd law $R = \{(a,b,c) \in G^3 \mid abc = 1\}$. This is shown as we have $(1,1,1) \in R$, $(a,1,1) \in R$ implies $a = 1$, $(a,b,1) \in R$ implies $ab = 1 = ba$ and thus $(b,a,1) \in R$. And finally $(a,b,c) \in R$ implies $abc = 1$ thus $ab = c^{-1}$, $cab = 1$ and $(c,a,b) \in R$.

Remark. Every element has an inverse that is for all $a \in G$ there is a b such that $ab1 = 1$ and thus $a^{-1} \neq \emptyset$.

Also $\#a^{-1} = 1$. To prove this assume $b, c \in a^{-1}$ then $ab1 = 1 = ac1$, and thus $b = bab1 = bac1 = c$.

Example 1.5 (Trivial crowds)**.** For any non-empty set G with unit element $1 \in G$, the *trivial crowd* is given by $(G, 1, R_{\text{triv}})$ where $R_{\text{triv}} = \{(1,1,1)\}$.

Example 1.6 (Maximal crowds)**.** For any non-empty set G with unit element $1 \in G$ the *maximal crowd* is given by $(G, 1, R_{\max})$ where

$$R_{\max} \;=\; G^3 \;-\; \{(a,1,1),\ (1,a,1),\ (1,1,a) \mid a \in G, a \neq 1\}.$$

Remark. As can be seen, although all groups can be considered crowds, not every crowd is a group. For example, the trivial crowd where $\#G > 1$ has an element which has no inverse. This shows how crowds can be seen as a generalisation of groups.

Table 1: Inverses and products for trivial crowds, maximal crowds and groups.

$a, b \neq 1$	Trivial crowds	Maximal crowds	Groups
a^{-1}	$\emptyset$	$G - \{1\}$	$\{i(a)\}$
$a \cdot b$	$\emptyset$	G	$\{ab\}$

All three examples above behave in a fairly predictable way. However, this is not always the case as we see with the crowd M below.

Example 1.7. The set $M = \{a, b, c, e\}$ with unit element e and crowd law

$$R_M = \{(e,e,e), (a,b,e), (b,a,e), (e,a,b), (b,e,a), (e,b,a), (a,e,b), (c,c,c)\}$$

is a crowd. Checking the axioms one by one shows this is indeed the case. However, unlike the crowds above, a and b have an inverse set while c does not. Similarly, $a \cdot b = \{e\}$ while $c \cdot c$ does not exist.

Other examples, specifically $\mathrm{SL}_n(\mathbb{K})$, are covered in more detail later on in section 3.

Some optional axioms that crowds can satisfy are:

- (E1) For all $a \in G$, $a^{-1} \neq \emptyset$
- (E2) If $(a,b,c) \in R$, then $(b,a,c) \in R$

corresponding to the existence of inverses and the abelian property respectively.

Remark. It is clear to see that the trivial crowd and the maximal crowd satisfy (E2) and that if a group is abelian, then the crowd from that group satisfies (E2) as well. On the other hand, crowds from non-abelian groups, the crowd M given above, and $\mathrm{SL}_n(\mathbb{K})$ do not satisfy (E2).

All crowds from groups, $\mathrm{SL}_n(\mathbb{K})$ and the maximal crowd satisfy $E1$, while the trivial crowd where $\#G \neq 1$ and M do not.

Proposition 1.8 (Basic properties)**.** *The following properties hold for all crowds:*

1. $1^{-1} = \{1\} = 1 \cdot 1$.
2. $a \in b^{-1} \iff 1 \in a \cdot b \iff 1 \in b \cdot a \iff b \in a^{-1}$.
3. $1 \cdot a \;=\; a \cdot 1 \;=\; (a^{-1})^{-1}$.
4. $1 \in a^{-1} \iff a = 1 \iff 1 \in a \cdot 1 = 1 \cdot a$.
5. $1 \cdot a = a \cdot 1 \neq \emptyset \implies a \in 1 \cdot a = a \cdot 1$.

Proof. Proof of 1:

$$\begin{aligned}\{1\} &= \{a \in G \mid (a,1,1) \in R\} = \{a \in G \mid (1,a,1) \in R\} = 1^{-1} \\ &= \{d \in G \mid d \in 1^{-1}\} = \{d \in G \mid d \in 1^{-1} \text{ and } (1,1,1) \in R\} \\ &= \{d \in G \mid d \in c^{-1} \text{ and } (1,1,c) \in R\} = 1 \cdot 1.\end{aligned}$$

Proof of 2:

$$\begin{array}{ccccc} a \in b^{-1} & \iff & (a,b,1) \in R & \iff & 1 \in a \cdot b \\ & & \Updownarrow & & \\ b \in a^{-1} & \iff & (b,a,1) \in R & \iff & 1 \in b \cdot a. \end{array}$$

Proof of 3:

$$\begin{array}{rlrl} 1 \cdot a = & \left\{c \in G \,\middle|\, c \in b^{-1} \text{ and } (1,a,b) \in R\right\} & = & \left\{c \in G \,\middle|\, c \in b^{-1} \text{ and } (a,b,1) \in R\right\} \\ & & & \qquad\qquad \| \\ (a^{-1})^{-1} = & \left\{c \in G \,\middle|\, c \in b^{-1} \text{ and } b \in a^{-1}\right\} & = & \left\{c \in G \,\middle|\, c \in b^{-1} \text{ and } (a,b,1) \in R\right\} \\ & & & \qquad\qquad \| \\ a \cdot 1 = & \left\{c \in G \,\middle|\, c \in b^{-1} \text{ and } (a,1,b) \in R\right\} & = & \left\{c \in G \,\middle|\, c \in b^{-1} \text{ and } (b,a,1) \in R\right\}. \end{array}$$

Proof of 4:

$$1 \in a^{-1} \iff a \in 1^{-1} \iff a = 1.$$
$$1 \in a \cdot 1 \iff 1 \in (a^{-1})^{-1} \iff \exists c \in a^{-1} \text{ such that } 1 \in c^{-1} \iff 1 \in a^{-1} \iff a = 1.$$

Proof of 5:

$$1 \cdot a = a \cdot 1 \neq \emptyset \implies \left\{c \in G \,\middle|\, c \in b^{-1} \text{ and } (1,a,b) \in R\right\} \neq \emptyset \implies a \in b^{-1} \subset 1 \cdot a \implies a \in 1 \cdot a.$$

□

Proposition 1.9. *A crowd $(G, 1, R)$ is also a group $(G, 1, *)$ where $1 = 1$ and $*$ is a group map induced by R if it satisfies the following properties.*

1. *$ae = dc$ if $d \in ab$ and $e \in bc$ for all $a, b, c, d, e \in G$.*
2. *a^{-1} and ab are singletons for all $a, b \in G$.*

Proof. To show this we define $*$ by $a * b = c$ where $c \in a \cdot b$. This is well defined as $\#a \cdot b = \#\left\{d \in c^{-1} \,\middle|\, (a,b,c) \in R\right\} = \#c^{-1} = 1$ by the second property.

For all $a \in G$, $b \in a^{-1}$ implies $a \in b^{-1}$, and thus, $a * b = 1 = b * a$.

Let $a, b, c, d, f \in G$, where $a * b = d$ and $b * c = f$, then by the first property of our proposition we have

$$(a * b) * c = d * c = a * f = a * (b * c)$$

Since all products are singletons, then we have $a * e = a = e * a$ by proposition 1.8 property 5. This shows that the group axioms are satisfied. □

1.2 Crowd morphisms

Definition 1.10 (Crowd morphisms)**.** A *crowd morphism* from $(G, 1_G, R_G)$ to $(H, 1_H, R_H)$ is a map $\phi : G \to H$ such that $\phi(1_G) = 1_H$ and $\big(\phi(a), \phi(b), \phi(c)\big) \in R_H$ for $(a, b, c) \in R_G$.

Definition 1.11 (Crowd isomorphism)**.** A *crowd isomorphism* from $(G, 1_G, R_G)$ to $(H, 1_H, R_H)$ is a morphism $\phi : G \to H$ such that there exists a morphism $\psi : H \to G$ and $\phi \circ \psi$ is the identity.

Example 1.12. Any group morphism is a crowd morphism.

Example 1.13. For any crowd $(G, 1, R)$, the identity map gives a crowd morphism:

$$(G, 1, R_{\text{triv}}) \to (G, 1, R) \to (G, 1, R_{\max}).$$

Remark. Although this is a bijective map with regard to G, it is not an isomorphism except for the case when $R_{\text{triv}} = R_{\max}$.

1.3 Saturated Crowds

A question arises when we look at what crowds can be generated by their own inverses and products.

Definition 1.14. A crowd is *saturated* if for all $a, b, c \in G$,

$$\left\{a^{-1} \subset b \cdot c \quad \text{and} \quad b^{-1} \subset c \cdot a \quad \text{and} \quad c^{-1} \subset a \cdot b\right\} \quad \text{implies} \quad (a, b, c) \in R.$$

Definition 1.15. The *saturation* of $(G, 1, R)$ is $(G, 1, \widehat{R})$ where

$$\widehat{R} = \left\{(a, b, c) \in G^3 \;\middle|\; c^{-1} \subset a \cdot b, b^{-1} \subset c \cdot a \text{ and } a^{-1} \subset b \cdot c\right\}.$$

Proposition 1.16. *The saturation of a crowd is a crowd.*

Proof. Firstly, if $(a, b, c) \in R$, then $(a, b, c) \in \widehat{R}$. This is because if $(a, b, c) \in R$, then $c^{-1} \subset a \cdot b$, $b^{-1} \subset c \cdot a$ and $a^{-1} \subset b \cdot c$.

Axiom 1: $(1, 1, 1) \in R \subset \widehat{R}$.
Axiom 2: If $(a, 1, 1) \in \widehat{R}$, then $1 \in 1^{-1} \subset a \cdot 1$ which implies that $a = 1$.
Axiom 3: If $(a, b, 1) \in \widehat{R}$, then we have $1^{-1} \subset a \cdot b$ and therefore $(a, b, 1) \in R$ and so $(b, a, 1) \in R \subset \widehat{R}$.
Axiom 4: If $(a, b, c) \in \widehat{R}$, then $c^{-1} \subset a \cdot b$, $b^{-1} \subset c \cdot a$, $a^{-1} \subset b \cdot c$ and thus $(c, a, b) \in \widehat{R}$. As $(G, 1, \widehat{R})$ satisfies the four axioms, it is a crowd. □

Proposition 1.17. *The operator $\mathcal{W}$: Crowds $\to$ Crowds that sends a crowd to its saturation is well defined.*

Proof. This follows from 1.16 □

Proposition 1.18. *The saturation of a crowd is a saturated crowd.*

Proof. By definition □

Proposition 1.19. *Properties of saturated morphisms:*

1. $\text{id} : G \to \mathcal{W}(G)$ *is a morphism.*
2. $\mathcal{W}(G) = \mathcal{W}(\mathcal{W}(G))$.

Proof. Proof of 1: Follows from $R \subset \widehat{R}$.

Proof of 2: First, we note that $\mathcal{W}$ is the identity on G and acts on R by generating $\widehat{R}$ from the sets $a \cdot b$ and a^{-1} for all $a, b \in G$.

Therefore, we only need to show that the sets $a \cdot b_R = a \cdot b_{\widehat{R}}$ and $a_R^{-1} = a_{\widehat{R}}^{-1}$.

First we show $a_R^{-1} = a_{\widehat{R}}^{-1}$:

$$b \in a_{\widehat{R}}^{-1} \implies (a, b, 1) \in \widehat{R} \implies 1 \in 1^{-1} \subset a \cdot b_R \implies (a, b, 1) \in R \implies b \in a_R^{-1}.$$

Likewise:

$$b \in a_R^{-1} \implies (a, b, 1) \in R \implies (a, b, 1) \in \widehat{R} \implies b \in a_{\widehat{R}}^{-1}.$$

Now we show, $a \cdot b_R = a \cdot b_{\widehat{R}}$:

$$a \cdot b_{\widehat{R}} = \bigcup_{d \in G | (a,b,d) \in \widehat{R}} d^{-1} = \bigcup_{d \in G | (a,b,d) \in R} d^{-1} = a \cdot b_R.$$

The $\subset$ direction of the middle equality follows from the fact that for $(a, b, d) \in \widehat{R}$ such that $(a, b, d) \notin R$ we have that $d^{-1} \subset a \cdot b_R$, and so we don't need to count it for the union. The $\supset$ direction follows because if $(a, b, d) \in R$, then also $(a, b, d) \in \widehat{R}$.

Since $a \cdot b_R = a \cdot b_{\widehat{R}}$ and $a_R^{-1} = a_{\widehat{R}}^{-1}$ we have that the proposition holds.

□

Remark. A crowd morphism $\phi : G_1 \to G_2$ is not necessarily a crowd morphism $\phi : \mathcal{W}(G_1) \to \mathcal{W}(G_2)$. Below is a simple counterexample. Let $G_1 = \{1, a, b, c, d, e, w\}$, and $G_2 = \{1, a, b, c, d, e, w, f\}$. Let $\phi : G_1 \to G_2$ be the inclusion, and let the crowd laws be generated by the sets:

$$\overbrace{\{(1,1,1),(b,c,d),(a,d,1),(e,d,1),(a,w,1)\}}^{R_1} \text{ and } \overbrace{\{(1,1,1),(b,c,d),(a,d,1),(e,d,1),(a,w,1),(w,f,1)\}}^{R_2}.$$

Then it is clear that $\phi : G_1 \to G_2$ is a crowd morphism. However, as $w_{R_1}^{-1} = \{a\}$, $b_{R_1}^{-1} = c_{R_1}^{-1} = \emptyset$ and $b \cdot c = \{a, e\}$, we have $(b, c, w) \in \widehat{R_1}$. Similarly $w_{R_2}^{-1} = \{a, f\}$, $b \cdot c_{R_2} = \{a, e\}$ which implies $w_{R_2}^{-1} \not\subset b \cdot c_{R_2}$ and thus $(b, c, w) \notin \widehat{R_2}$. This shows that $\phi : \mathcal{W}(G_1) \to \mathcal{W}(G_2)$ is not a crowd morphism.

1.4 Subcrowds

Definition 1.20. Let $(G, 1, R)$ be a crowd, a subcrowd is given by $(H, 1, S)$, where $1 \in H \subset G$ and $S \subset R \cap H^3$.

Remark. The inclusion $H \to G$ is a crowd morphism.

Proof. See the definition of a crowd morphism. □

Definition 1.21. A full subcrowd of G is a subcrowd $(H, 1, S)$ such that $S = R \cap H^3$.

Remark. A full subcrowd is defined by H and the crowd G.

Example 1.22. A subgroup is a full subcrowd.

Proof. Let H be a subgroup of G, then it is obvious that H is a subcrowd. To show that it is a full subcrowd, assume $(a, b, c) \in R$ with $a, b, c \in H$. Then, $abc = 1$ and therefore, (a, b, c) in R_H, and thus, $(H, 1, R_H)$ is a full subcrowd. □

Example 1.23. Let $(G, 1, R)$ be a crowd, then the trivial crowd $(G, 1, R_{triv})$ is a subcrowd, but in most circumstances not a full subcrowd.

Example 1.24. Let $G = \mathrm{SL}_n(\mathbb{K})$, then the permutation matrices form a subcrowd.

2 Bands

To introduce our main example, $\mathrm{SL}_n(\mathbb{K})$, we need to define $\mathbb{K}$ (the Krasner hyperfield), and therefore, we introduce the concept of bands.

Definition 2.1. A band is a quadruple $(B, \mathbf{1}, \mathbf{0}, \cdot)$ along with a *nullset* N_B, where B is a set, $\mathbf{1}, \mathbf{0} \in B$ unit and absorbing elements respectively, $\cdot : B \times B \to B$ commutative and associative, N_B a subset of $B^+ = \{\sum n_a a \mid n_a \in \mathbb{N}, a \in B / \{\mathbf{0}\}$ and $n_a = 0$ for all but finitely many a $\}$, such that the following axioms hold.

1. $\mathbf{0} \cdot a = \mathbf{0}$ for all $a \in B$.
2. $\mathbf{1} \cdot a = a$ for all $a \in B$.
3. $\mathbf{0} \in N_B$ and for all $a \in B$, there exists a unique $b \in B$, such that $a + b \in N_B$.
4. $x + y \in N_B$ for all $x, y \in N_B$, and $a \cdot x \in N_B$ for all $a \in B$ and $x \in N_B$.

Example 2.2. A ring $(R, \mathbf{1}, \mathbf{0}, \cdot)$ is a band if we take the nullset to be $N_R = \{\sum n_a \cdot a = 0 \mid a \in R, n_a \in \mathbb{N}\}$.

Proof. The axioms follow directly from the axioms for a ring. □

Remark. All fields are rings and thus are bands as well.

Definition 2.3. A *band morphism* is a multiplicative map $\phi : G_1 \to G_2$ such that $\phi(0) = 0$, $\phi(1) = 1$ and $\sum n_a f(a) \in N_{G_2}$ for every $\sum n_a a \in N_{G_1}$.

Proposition 2.4. *A ring morphism is a band morphism.*

Proof. Let R_1, R_2 be rings with $\phi : R_1 \to R_2$ a ring morphism, then $\phi(1) = 1$, $\phi(0) = 0$ and

$$\sum n_a a \in N_{R_1} \implies \sum n_a \phi(a) = \sum \phi(n_a \cdot a) = \phi\Big(\sum n_a a\Big) \in N_{R_2}.$$

Therefore, every ring morphism is a band morphism. □

The following examples should be considered with the expected multiplication and addition unless stated otherwise.

Example 2.5. The Krasner hyperfield $\mathbb{K}$ is a band with $\mathbb{K} = \{0, 1\}$ and nullset

$$N_{\mathbb{K}} = \{n \cdot 1 \mid n \in \mathbb{N}, n \neq 1\}.$$

Example 2.6. The zero band is a band with $B = \{0\}$, $0 = 1$ and $N_B = \{0\}$.

Example 2.7. The quadratic field extension of $\mathbb{F}_1$, $\mathbb{F}_1^{\pm} = \{0, 1, -1\}$ is a band with

$$N_{\mathbb{F}_1^{\pm}} = \{n \cdot 1 + n \cdot (-1) \mid n \geq 0\}.$$

Example 2.8. The tropical hyperfield $\mathbb{T} = \mathbb{R}_{\geq 0}$ is a band with nullset

$$N_{\mathbb{T}} = \Big\{\sum n_a a \;\Big|\; n_a = 0 \text{ for all } a \in R_{>0} \text{ or } n_b \geq 2 \text{ for } b = \max\{a \mid n_a \neq 0\}\Big\}.$$

Proposition 2.9. *For all bands B, there is a band morphism ϕ from $\mathbb{F}_1^{\pm}$ to B.*

Proof. Take the morphism that sends $0 \to 0$, $1 \to 1$ and -1 to the unique element $b \in B$ such that $1 + b \in N_B$. □

Proposition 2.10. *For all bands B, there is a band morphism ψ from B to the zero band.*

Proof. The morphism is given by $\psi : g \mapsto 0$ for all $g \in B$. □

3 Special Linear Group

As the previous section has defined bands, we introduce the special linear group over bands in general before going on to show that $\mathrm{SL}_n(\mathbb{K})$ is in fact a crowd.

3.1 General Definition

Definition 3.1. Let B be a band, then matrices over B are defined as the set $Mat_{n\times n}(B) = \{(a_{ij}) \in B^{n\times n}\}$.

Definition 3.2. The determinant of $A \in Mat_{n\times n}(B)$ is given by $\sum_{\sigma\in S_n} \mathrm{sign}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}$.

Definition 3.3. The special linear group $\mathrm{SL}_n(B)$

$$\mathrm{SL}_n(B) = \left\{ A \in Mat_{n\times n}(B) \,\middle|\, \det(A) - 1 \in N_B \right\}$$

with identity element $1 = (\delta_{ij})$ and crowd law

$$R \;=\; \left\{ \left(A^{(1)}, A^{(2)}, A^{(3)}\right) \,\middle|\, \begin{array}{c} \forall \sigma \in A_3 \quad \forall i,j = 1,\dots,n \\ \sum_{k,l=1,\dots n} a_{i,k}^{\sigma(1)} a_{k,l}^{\sigma(2)} a_{l,j}^{\sigma(3)} - \delta_{i,j} \in N_B \end{array} \right\}$$

is a crowd.

3.2 The Krasner hyperfield $\mathbb{K}$

Theorem 3.4. $(\mathrm{SL}_n(\mathbb{K}), 1, R)$ *is indeed a crowd, with* 1 *being the identity matrix, and crowd law*

$$R \;=\; \left\{ \left(A^{(1)}, A^{(2)}, A^{(3)}\right) \,\middle|\, \begin{array}{c} \forall \sigma \in A_3 \quad \forall i,j = 1,\dots,n \\ \delta_{i,j} \in \sum^{\boxplus}_{k,l=1,\dots n} a_{i,k}^{\sigma(1)} a_{k,l}^{\sigma(2)} a_{l,j}^{\sigma(3)} \end{array} \right\}.$$

Proof. We note that $(1,1,1) \in R$ as we have $\delta_{k,l} \in \{\delta_{k,l}\} = \sum_i^{\boxplus} \delta_{k,i}\delta_{i,l} = \sum_{i,j}^{\boxplus} \delta_{k,i}\delta_{i,j}\delta_{j,l}$, and permutations don't matter since the elements are all the same.

If $(a,1,1) \in R$, then we have:

$$\delta_{k,l} \in \{a_{k,l}\} = \sum_{i,j}^{\boxplus} a_{k,i}\delta_{i,j}\delta_{j,l}.$$

This implies that $a_{k,l} = \delta_{k,l}$, and therefore, $a = 1$.

If $(a,b,1) \in R$, then we have:

$$\delta_{k,l} \in \sum_{i}^{\boxplus} a_{k,i}b_{i,l} = \sum_{i,j}^{\boxplus} a_{k,i}b_{i,j}\delta_{j,l} \quad \text{and} \quad \delta_{k,l} \in \sum_{j}^{\boxplus} b_{k,j}a_{j,l} = \sum_{i,j}^{\boxplus} b_{k,i}\delta_{i,j}a_{j,l}.$$

These imply in turn that:

$$\delta_{k,l} \in \sum_{i,j}^{\boxplus} a_{k,i}\delta_{i,j}b_{j,l}, \quad \delta_{k,l} \in \sum_{i,j}^{\boxplus} b_{k,i}a_{i,j}\delta_{j,l} \quad \text{and} \quad \delta_{k,l} \in \sum_{i,j}^{\boxplus} \delta_{k,j}b_{i,j}a_{j,l}.$$

Which by symmetry implies that $(b, a, 1) \in R$.

If $(a, b, c) \in R$, then $(c, b, a) \in R$, as it is a cyclic permutation and the conditions are also cyclic. □

Proposition 3.5. *Let $a, b, \in \mathrm{SL}_n(\mathbb{K})$, then the following are equivalent.*

1. $b \in a^{-1}$.
2. $a \in b^{-1}$.
3. $\delta_{k,l} \in \sum_i^{\boxplus} a_{k,i}b_{i,l}$ *and* $\delta_{k,l} \in \sum_i^{\boxplus} b_{k,i}a_{i,l}$.

Proof. To prove this, we only show the equivalence of 1 and 3, as their equivalence to 2 follows by symmetry. An equivalent statement to $b \in a^{-1}$ is $(a, b, 1) \in R$, taking this, we get:

$$\begin{aligned}(a, b, 1) \in R \iff& \delta_{k,l} \in \sum_{i,j}^{\boxplus} a_{k,i}b_{i,j}\delta_{j,l} \text{ and } \delta_{k,l} \in \sum_{i,j}^{\boxplus} b_{k,i}\delta_{i,j}a_{j,l} \text{ and } \delta_{k,l} \in \sum_{i,j}^{\boxplus} \delta_{k,i}a_{i,j}b_{j,l}\\ \iff& \delta_{k,l} \in \sum_{i}^{\boxplus} a_{k,i}b_{i,l} \text{ and } \delta_{k,l} \in \sum_{i}^{\boxplus} b_{k,i}a_{i,l} \text{ and } \delta_{k,l} \in \sum_{j}^{\boxplus} a_{k,j}b_{j,l}\\ \iff& \delta_{k,l} \in \sum_{i}^{\boxplus} a_{k,i}b_{i,l} \text{ and } \delta_{k,l} \in \sum_{i}^{\boxplus} b_{k,i}a_{i,l}.\end{aligned}$$

□

Definition 3.6. The adjoint of $a \in \mathrm{SL}_n(\mathbb{K})$ is defined pointwise as

$$a^{\#}_{ij} = \begin{cases} 1 & \text{if } 1 \in d_{ji} \\ 0 & \text{otherwise} \end{cases},$$

where

$$d_{ji} = \sum_{\sigma \in S_n | \sigma(j)=i}^{\boxplus} \prod_{k=1,\ldots,\hat{i},\ldots,n} a_{k,\sigma(k)}.$$

Proposition 3.7. *For all $a \in \mathrm{SL}_n(\mathbb{K})$, we have $a^{\#} \in a^{-1}$.*

Proof. For this to be the case, we need the following condition to hold:

$$\sum_{i=1}^{n} a_{k,i}a^{\#}_{i,l} + \delta_{k,l} \in N_{\mathbb{K}} \text{ and } \sum_{i=1}^{n} a^{\#}_{k,i}a_{i,l} + \delta_{k,l} \in N_{\mathbb{K}} \text{ for all } k, l \in \underline{n}.$$

.

First, we show that $\sum_{i=1}^{n} a_{k,i}a^{\#}_{i,l} + \delta_{k,l} \in N_{\mathbb{K}}$ is the case for all $k, l \in \underline{n}$. As $a \in \mathrm{SL}_n(\mathbb{K})$, we know that $det(a) + 1 \in N_{\mathbb{K}}$, and so for some $\sigma \in S_n$, $\prod_{k=1}^{n} a_{k,\sigma(k)} = 1$.

If $k = l$, then $a_{k,\sigma(k)} = 1$ and $a^{\#}_{\sigma(k),k} = 1$. This implies that $a_{k,\sigma(k)}a^{\#}_{\sigma(k),k} = 1$ is an element of the sum, and as $\delta_{k,k} = 1$ this is sufficient.

If $k \neq l$, then we have two options, either $a_{k,i}a^{\#}_{i,l} = 0$ for all $i \in \underline{n}$, or for at least one $i \in \underline{n}$, $a_{k,i}a^{\#}_{i,l} = 1$. If all are 0, then we are done as $\delta_{k,l} = 0$.

If at least one element of the sum is equal to 1, then take $i \in \underline{n}$ such that $a_{k,i}a^{\#}_{i,l} = 1$.

As $a^{\#}_{i,l} = 1$, there exists a $\pi \in S_n$ such that $\pi(l) = i$ and $\prod_{m \in \underline{n-l}} a_{m,\pi(m)} = 1$. Let $j = \pi(k)$, and let $\tau = \pi \circ (kl)$, where (kl) is the permutation that takes k to l and l to k. Then, $\tau(k) = i$ and $\tau(l) = j$ and $\tau(w) = \pi(w)$ for w not equal to k or l.

This implies that $a_{k,j} = 1$ and $a^{\#}_{j,l} = 1$ as τ is such that $\tau(l) = j$ and $\prod_{m \in \underline{n-l}} a_{m,\tau(m)} = 1$.

Since at least one element of the sum equals 1 implies another also equals 1, the relationship is satisfied, and $\sum_i a_{k,i}a^{\#}_{i,l} + \delta_{k,l} \in N_{\mathbb{K}}$ holds for all $k, l \in \underline{n}$.

Similarly, for

$$\sum_i a^{\#}_{k,i}a_{i,l} + \delta_{k,l} \in N_{\mathbb{K}} :$$

If $k = l$, then $a_{\sigma^{-1}(k),k} = 1$ and $a^{\#}_{k,\sigma^{-1}(k)} = 1$. This implies that $a^{\#}_{k,\sigma^{-1}(k)}a_{\sigma^{-1}(k),k} = 1$ is an element of the sum, and as $\delta_{k,k} = 1$, this is sufficient.

If $k \neq l$, then we have two options, either $a^{\#}_{k,i}a_{i,l} = 0$ for all $i \in \underline{n}$, or at least one is equal to 1. If all are 0 then we are done as $\delta_{k,l} = 0$.

If at least one element is equal to 1, then let us call it $a^{\#}_{k,i}a_{i,l}$ for a specific $i \in \underline{n}$.

Then, there exists a $\pi \in S_n$ such that $i = \pi^{-1}(k)$ and $\prod_{m \in \underline{n-l}} a_{m,\pi(m)} = 1$. Let $j = \pi^{-1}(l)$, and let $\tau = \pi \circ (kl)$, where (kl) is the permutation that takes k to l and l to k. Then, $\tau^{-1}(k) = j$ and $\tau^{-1}(l) = i$ and $\tau(w) = \pi(w)$ for w not equal to k or l.

This implies that $a_{j,l} = 1$, as $\pi(j) = l$; and $a^{\#}_{k,j} = 1$, as τ is such that $\tau(j) = k$ and $\prod_{m \in \underline{n-l}} a_{m,\tau(m)} = 1$.

Since at least one element of the sum equals 1 implies another also equals 1, the relationship is satisfied and $\sum_{i=1}^{n} a_{k,i}a^{\#}_{i,l} + \delta_{k,l} \in N_{\mathbb{K}}$ holds for all $k, l \in \underline{n}$.

Therefore, since both sums hold, $a^{\#} \in a^{-1}$. □

Theorem 3.8. $\mathrm{SL}_n(\mathbb{K})$ *is a saturated crowd.*

Proof. To prove this we take $a, b, c \in \mathrm{SL}_n(\mathbb{K})$ such that $a^{-1} \subset b \cdot c$, $b^{-1} \subset c \cdot a$ and $c^{-1} \subset a \cdot b$, and show this implies $(a, b, c) \in R$.

If we expand our assumption, we get:

$$\begin{aligned}
&\forall w \in G \text{ s.t. } (w, b, 1) \in R \quad \exists d \in G \text{ s.t. } (w, d, 1) \in R \text{ and } (c, a, d) \in R.\\
&\forall w \in G \text{ s.t. } (w, c, 1) \in R \quad \exists d \in G \text{ s.t. } (w, d, 1) \in R \text{ and } (a, b, d) \in R.\\
&\forall w \in G \text{ s.t. } (w, a, 1) \in R \quad \exists d \in G \text{ s.t. } (w, d, 1) \in R \text{ and } (b, c, d) \in R.
\end{aligned}$$

Taking the first condition, we have that for $w \in G$ such that $(w, b, 1) \in R$ there exists a $d \in G$ such that $(w, d, 1) \in R$. By Prop 3.7, w exists therefore, d also exists and this implies $\delta_{k,l} \in \sum_i^{\boxplus} b_{k,i}w_{i,l}$ and $\delta_{k,l} \in \sum_i^{\boxplus} w_{k,i}d_{i,l}$. Using this for substitution purposes we have:

$$\begin{aligned}
\delta_{k,l} \in \sum_{i,j}^{\boxplus} c_{k,i} a_{i,j} d_{j,l} \quad &\iff \delta_{k,l} \in \sum_{i,j}^{\boxplus} c_{k,i} a_{i,j} \sum_{t}^{\boxplus} \delta_{j,t} d_{t,l} &&\iff \delta_{k,l} \in \sum_{i,j,t}^{\boxplus} c_{k,i} a_{i,j} \delta_{j,t} d_{t,l} \\
&\iff \delta_{k,l} \in \sum_{i,j,t}^{\boxplus} c_{k,i} a_{i,j} \left(\sum_{r}^{\boxplus} b_{j,r} w_{r,t} \right) d_{t,l} &&\iff \delta_{k,l} \in \sum_{i,j,t,r}^{\boxplus} c_{k,i} a_{i,j} b_{j,r} w_{r,t} d_{t,l} \\
&\iff \delta_{k,l} \in \sum_{i,j,r}^{\boxplus} c_{k,i} a_{i,j} b_{j,r} \sum_{t}^{\boxplus} w_{r,t} d_{t,l} &&\iff \delta_{k,l} \in \sum_{i,j,r}^{\boxplus} c_{k,i} a_{i,j} b_{j,r} \delta_{r,l} \\
&\iff \delta_{k,l} \in \sum_{i,j}^{\boxplus} c_{k,i} a_{i,j} b_{j,l}.
\end{aligned}$$

Note: The first implication is valid as $\sum_t^{\boxplus} \delta_{j,t} = 1$ for fixed j. The third implication is valid as $\delta_{j,t} \in \sum_r^{\boxplus} b_{j,r} w_{r,t}$. The sixth implication is valid as $\delta_{r,l} \in \sum_t^{\boxplus} w_{r,t} d_{t,l}$.

By symmetry, we have:

$$\delta_{k,l} \in \sum_{i,j}^{\boxplus} a_{k,i} b_{i,j} c_{j,l} \quad \text{and} \quad \delta_{k,l} \in \sum_{i,j}^{\boxplus} c_{k,i} b_{i,j} a_{j,l},$$

which implies that $(a, b, c) \in R$. □

Directly from 3.7, we have that $\mathrm{SL}_n(\mathbb{K})$ has the property $E1$.

3.3 $\mathrm{SL}_2(\mathbb{K})$ and $\mathrm{SL}_3(\mathbb{K})$

In the last subsection, we covered the general case for all n. Now we cover $\mathrm{SL}_2(\mathbb{K})$ in detail, describing its elements, products and inverses. It can be noted that when $n = 2$, $\mathrm{SL}_n(\mathbb{K})$ behaves fairly close to $\mathrm{SL}_n(\mathbb{F})$, when $\mathbb{F}$ is a field. However, this behaviour does not last, as can be seen from the counterexamples given for when $n = 3$. Although it is not difficult to compute the information for $\mathrm{SL}_2(\mathbb{K})$ by hand, by the time we are searching for counterexamples in $\mathrm{SL}_3(\mathbb{K})$ and higher, the aid of a computer is beneficial. Indeed, all counterexamples were found with computer aid.

A complete description of $\mathrm{SL}_2(\mathbb{K})$ is given as follows. The elements are:

$$\mathrm{SL}_2(\mathbb{K}) = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}.$$

Inverses are unique, and given by:

$$a^{-1} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \left\{ \begin{bmatrix} a_{22} & a_{12} \\ a_{21} & a_{11} \end{bmatrix} \right\}.$$

The adjoint of an adjoint of an element is the element again.

$$\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{\#} \right)^{\#} = \begin{bmatrix} a_{22} & a_{12} \\ a_{21} & a_{11} \end{bmatrix}^{\#} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

Definition 3.9. The **naive product**, $\cdot^{\boxplus}$, of $a, b \in \mathrm{SL}_n(\mathbb{K})$ is a set $a \cdot^{\boxplus} b \subset \mathrm{SL}_n(\mathbb{K})$ where every element can be derived from matrix multiplication by taking advantage of Krasner hyperaddition combinations.

Example 3.10. As an example, we show the naive multiplication of $\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}$ and $\begin{bmatrix}0 & 1\\ 1 & 1\end{bmatrix}$.

$$\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix} \cdot^{\boxplus} \begin{bmatrix}0 & 1\\ 1 & 1\end{bmatrix} = \begin{bmatrix}1\cdot 0 \boxplus 0\cdot 1 & 1\cdot 1 \boxplus 0\cdot 1\\ 1\cdot 0 \boxplus 1\cdot 1 & 1\cdot 1 \boxplus 1\cdot 1\end{bmatrix} = \begin{bmatrix}\{0\} & \{1\}\\ \{1\} & \{0,1\}\end{bmatrix} = \left\{ \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\ 1 & 1\end{bmatrix} \right\}$$

Proposition 3.11. *The crowd product of a, b is the same as the naive product of a, b, in formula:*

$$\begin{aligned} ab &= \left\{ c \in \mathrm{SL}_2(\mathbb{K}) \;\middle|\; \exists d \in \mathrm{SL}_2(\mathbb{K}),\ c \in d^{-1} \text{ and } (a,b,d) \in R \right\} \\ &= \left\{ c \in \mathrm{SL}_2(\mathbb{K}) \;\middle|\; c_{k,l} + \sum_i^{\boxplus} a_{k,i} b_{i,l} \in N_{\mathbb{K}} \right\} = a \cdot^{\boxplus} b. \end{aligned}$$

Proof. See [7]. □

The following is a case where $ab \neq a \cdot^{\boxplus} b$.

$$a = \begin{bmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{bmatrix},\ b = \begin{bmatrix}0 & 0 & 1\\ 0 & 1 & 1\\ 1 & 1 & 0\end{bmatrix},\ ab = \left\{ \begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix}, \quad \begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix} \right\} \text{ and } a \cdot^{\boxplus} b = \begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix}.$$

The following is a case with non-unique inverses.

$$a = \begin{bmatrix}0 & 0 & 1\\ 0 & 1 & 1\\ 1 & 1 & 1\end{bmatrix}, \quad a^{-1} = \left\{ \begin{bmatrix}0 & 1 & 1\\ 1 & 1 & 0\\ 1 & 0 & 0\end{bmatrix}, \begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 0\\ 1 & 0 & 0\end{bmatrix} \right\}.$$

The adjoint of an adjoint is not the matrix itself.

$$a = \begin{bmatrix}0 & 0 & 1\\ 0 & 1 & 1\\ 1 & 1 & 0\end{bmatrix}, a^{\#} = \begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 0\\ 1 & 0 & 0\end{bmatrix}, (a^{\#})^{\#} = \begin{bmatrix}0 & 0 & 1\\ 0 & 1 & 1\\ 1 & 1 & 1\end{bmatrix}, ((a^{\#})^{\#})^{\#} = \begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 0\\ 1 & 0 & 0\end{bmatrix}.$$

Remark. The crowd $\mathrm{SL}_2(\mathbb{K})$ has 7 elements, $\mathrm{SL}_3(\mathbb{K})$ has 247, $\mathrm{SL}_4(\mathbb{K})$ has 37823 and $\mathrm{SL}_5(\mathbb{K})$ has 23191071 elements. These numbers and the examples above were found using computer aid.

4 Conclusion

In the introduction, we gave a brief history of $\mathbb{F}_1$ and its geometry as background and motivation for the creation of crowds. In the first section, we defined crowds, crowd morphisms and subcrowds and gave examples of each type of structure. In the second section, we defined bands, band morphisms and the Krasner hyperfield ($\mathbb{K}$) as an example of a band.

In the third section, we defined special linear groups over bands. We proved that $\mathrm{SL}_n(\mathbb{K})$ is a crowd, that it is saturated, and that every element in $\mathrm{SL}_n(\mathbb{K})$ has an inverse (given by its adjoint). Then, we gave a complete description of $\mathrm{SL}_2(\mathbb{K})$ showing some properties that only hold when $n = 2$, and then examples where those properties fail when $n = 3$. Finally, in the appendix, we included a Python class that simulates an element in the Krasner hyperfield and gave a short description of how to compute questions for $\mathrm{SL}_n(\mathbb{K})$.

Further ideas could include studying $\mathrm{SL}_n(B)$ for general bands as well as $\mathrm{GL}_n(\mathbb{K})$, $\mathrm{O}_n(\mathbb{K})$ and $\mathrm{SO}_n(\mathbb{K})$. Another possibility is constructing and categorising finite crowds of low cardinality.

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A Appendix

The code below simulates an element of the Krasner hyperfield. Using this class and the Python library numpy to generate matrices, it is fairly easy to compute solutions to problems in $\mathrm{SL}_n(\mathbb{K})$ for small n. It should be noted however that as n increases it starts to take an unreasonable amount of time.

```
class K:

    def __init__(self, n):
        if n == 0 or n == [0]:
            self.zero = True
            self.one = False
        elif n == 1 or n == [1]:
            self.one = True
            self.zero = False
        elif n == [0, 1]:
            self.one = True
            self.zero = True
        else:
            raise TypeError
    def __bool__(self):
        return self.one
    def __str__(self):
        if self.one and self.zero:
            msg = '[0, 1]'
        elif self.one:
            msg = '[1]'
        elif self.zero:
            msg = '[0]'
        return msg

    def __repr__(self):
        return self.__str__()
    def __add__(self, other):
        if isinstance(other, K):
            if self.one and other.one:
                return K([0, 1]) # At least two ones
            if not self.one:
                return other
            if not other.one:
                return self
        elif isinstance(other, int):
            if other == 1:
                return self.__add__(K(1))
            elif other == 0:
                return self.__add__(K(0))
        return NotImplemented
    def __eq__(self, other):
        if isinstance(other, K):
            if self.zero and other.zero and not self.one and not other.one:
                return True
            if self.zero and other.zero and self.one and other.one:
```

```
                return True
            if not self.zero and not other.zero and self.one and other.one:
                return True
            return False
        if isinstance(other, int):
            if other == 1 and self.one and not self.zero:
                return True
            if other == 0 and self.zero and not self.one:
                return True
            return NotImplemented
    def __req__(self, other):
        return self.__eq__(other)
    def __neg__(self, other):
        return self
    def __sub__ (self, other): # the addition of negation
        return self.__add__(other)
    def __mul__(self, other): # Only defined when only one of
        if self.zero and self.one:
            return NotImplemented
        if isinstance(other, K):
            if other.zero and other.one:
                return NotImplemented
            if self.zero:
                return self
            if self.one:
                return other
        elif isinstance(other, int):
            if other == 0:
                return K(0)
            elif other == 1:
                return self
            else:
                return NotImplemented
        elif isinstance(other, W):
            if self.one:
                return 1
            if self.zero:
                return 0
        else:
            return NotImplemented
    def __rmul__(self, other):
        return self.__mul__(other)
    def __radd__(self, other):
        return self.__add__(other)
    def __rsub__(self, other):
        return self.__sub__(other)
    def in_nullset(self):
        return self.zero
class W:
    def __int__(self):
        pass
    def __add__(self, other):
```

```
        if isinstance(other, K):
            if other.one:
                return 1
            if other.zero:
                return 0
            return 0
    def __radd__(self, other):
        return self.__add__(other)
```