Harmonic Oscillators in Spaces of Constant Curvature

Bachelor's Project Mathematics

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Abstract

In this thesis the N-dimensional simple harmonic oscillator is considered in three spaces of constant curvature, that is, in Euclidean, spherical and hyperbolic geometry. The orbits are derived in all three spaces: they are flat, spherical, and hyperbolic ellipses, respectively. For both classical and quantum mechanics, the tensor constants of motion are derived using Poisson bracket and commutator algebra, respectively. With these constants of motion, the Hamiltonian of the three dynamical systems are expressed as quadratic Casimir functions or operators of its symmetry group SU(N). Furthermore, the energy eigenvalues and eigenfunctions are derived by solving the Schrödinger equation in the three N-dimensional spaces.

Contents

Introduction

The motion of a particle that moves along a single straight line with acceleration directed towards a fixed point at all times and whose magnitude is proportional to its distance from this fixed point is known as simple harmonic motion [12]. Physically, this type of motion can be modeled by a spring-mass system, called the simple harmonic oscillator.

If the dynamical system is left at rest at the equilibrium position of the spring, no force is acting on the particle. If the particle is displaced from this point, a non-zero force is produced to restore the system to equilibrium. When the system does not experience any loss of energy, the particle continues to oscillate and its motion is periodic. Characteristic of the harmonic oscillator is its potential energy function, which says that the particle's potential energy is directly proportional to the distance from its equilibrium point squared.

The simple harmonic oscillator occurs frequently in physics, both in classical and quantum mechanics. For an potential energy function with non-vanishing second derivative, its Taylor expansion around an energy minimum can be approximated by a quadratic function, that is, by a harmonic oscillator. Then, the solution to the simple harmonic oscillator provides an approximate solution in the vicinity of an equilibrium point.

The harmonic oscillator can be generalised in many ways. For example, one could account for the loss of energy by considering an additional frictional force that damps the particle's motion. This dynamical system is known as the damped harmonic oscillator [12]. Another generalisation can be the dimension of the dynamical system. Physically, this can be modeled by attaching additional springs to the particle in different directions.

The latter generalisation can be further divided in two separated cases. If we use different springs, each with their own unique stiffness, the restoring force acting on the particle depends on the direction of displacement. This dynamical model is known as the non-isotropic oscillator [12]. By assuming that the restoring force is the same in all directions, we consider the motion of the isotropic oscillator. In this thesis, we study the N-dimensional isotropic simple harmonic oscillator.

In addition to our description of the simple harmonic oscillator from a classical mechanics standpoint, an analog formulation exists for quantum mechanics. The quantum harmonic oscillator is one of the few mechanical systems for which an analytic solution is known and can be derived [15]. Just as its classical counterpart, the quantum harmonic oscillator is an approximate model in the vicinity of an equilibrium for more general models in quantum mechanics.

A further generalisation of both the classical and quantum harmonic oscillator is concerned with the geometry of the dynamical system. Until now, our description of the simple harmonic oscillator takes place in the general framework of Euclidean geometry, which has zero curvature everywhere. We can make the extension to spaces of constant, non-zero curvature and formulate the harmonic oscillator in these geometries.

Both the classical and quantum harmonic oscillator in a spherical geometry, which has constant positive curvature, have been studied by Higgs [18] and Leemon [21]. They showed that the isotropic oscillator in N-dimensional spherical geometry is dynamically symmetric both in classical and quantum mechanics, similarly as in N-dimensional Euclidean space. The harmonic oscillator in non-Euclidean N-dimensional hyperbolic geometry, which has constant negative curvature, has not yet been studied extensively.

In this thesis, we study both the classical and the quantum dynamical systems that correspond to replacing N-dimensional Euclidean geometry by N-dimensional hyperbolic geometry. To understand the formulation of the quantum harmonic oscillator, we provide in Chapter 1 the mathematical preliminaries that are necessary for N-dimensional quantum mechanics. Subsequently in Chapter 2, we first study the N-dimensional simple harmonic oscillator in the framework of Euclidean geometry. Here, we first concentrate on the classical formulation of the dynamical system, after which we aim our efforts to the quantum mechanical version.

In the second half of this thesis we show how the N-dimensional simple harmonic oscillator can be generalised for positive and negative curved geometries. In Chapter 3, we first consider the spherical geometry as discussed by Higgs [18] and Leemon [21]. We formulate a model representing the N-sphere to analyse both the classical and quantum harmonic oscillator, for which we derive the particle's orbit and solve the radial wave function, respectively. Thereafter, we follow a similar method to extend this generalisation to the hyperbolic geometry. In Chapter 4, we construct a model for hyperbolic N-space and discuss the classical and quantum harmonic oscillator, for which we again derive the orbits and solution for the radial wave function, respectively.

1 Mathematical Preliminaries on Quantum Mechanics

In advance of our exploration of the harmonic oscillator, we establish the mathematical foundations of quantum mechanics [15] we shall be working with in future chapters. This general framework of quantum mechanics can be summarised by a set of so-called postulates [3], each with their own consequences and mathematics. These fundamental rules of quantum mechanics form the main structure of this Chapter and cover the necessary mathematical preliminaries.

In Section 1.1, we discuss the quantum state together with the space it belongs to, and we introduce Dirac notation. Thereafter, in Section 1.2, we focus on quantum observables, linear operators and the connection between them. Subsequently, we study Born's statistical interpretation of the wave function and its consequences in Section 1.3. Thereupon, in Section 1.4, we develop the notion of commutation and the uncertainty present in quantum mechanics. Finally, in Section 1.5, we introduce the time evolution of the quantum state and work out the general solution of Schrödinger's equation.

1.1 Hilbert Spaces

Consider a particle of mass m moving through N-dimensional space. In quantum mechanics, the state of this particle is described by a wave function $\Psi(\mathbf{x}, t)$. The quantum state of the particle at time $t \in \mathbb{R}$, denoted by $|\Psi(t)\rangle$, is a mathematical object that describes the quantum system at a fixed moment in time. All quantum states belong to a separable Hilbert space over the complex numbers [27].

Definition 1.1. A complex vector space $\mathcal{H}$ is called an inner product space if there is a complex-valued function $\langle\cdot|\cdot\rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$, called the inner product, that satisfies for all $x, y, z \in \mathcal{H}$ and all $\alpha \in \mathbb{C}$ the following conditions:

(i) $\langle x|y\rangle = \langle y|x\rangle^*$, where $\langle y|x\rangle^*$ denotes the complex conjugate of $\langle y|x\rangle$;

(ii) $\langle x|y + z\rangle = \langle x|y\rangle + \langle x|z\rangle$;

(iii) $\langle x|\alpha y\rangle = \alpha \langle x|y\rangle$;

(iv) $\langle x|x\rangle \geq 0$ and $\langle x|x\rangle = 0$ if and only if $x = 0$.

A complete[1] inner product space $\mathcal{H}$ is called a Hilbert space.

Before we proceed with establishing the mathematical foundations of quantum mechanics, we define the Dirac notation, a useful formulation for linear algebra on complex vector spaces. The inner product $\langle x|y\rangle$ is divided into two pieces, namely a bra $\langle x|$ and a ket $|y\rangle$. As we have seen with the quantum state $|\Psi(t)\rangle$ before, a ket $|y\rangle \in \mathcal{H}$ represents a vector. A bra $\langle x| \in \mathcal{H}^*$, where $\mathcal{H}^*$ denotes the dual of $\mathcal{H}$, is a linear form $x : \mathcal{H} \to \mathbb{C}$, that is, it maps a ket in $\mathcal{H}$ to a complex number. Thus, combining a bra $\langle x|$ and a ket $|y\rangle$ yields precisely the inner product,

$$\langle x|y\rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}. \tag{1.1}$$

There is an immediate relation between bras and kets, which is described by the Hermitian conjugate [16], namely $|x\rangle = (\langle x|)^\dagger$ and $\langle x| = (|x\rangle)^\dagger$.

Throughout this thesis, instead of working with a general Hilbert space $\mathcal{H}$, we work with a specific example [24], namely the space of square-integrable functions $L^2(\mathbb{R}^N)$.

Definition 1.2. A complex-valued function $f : \mathbb{R}^N \to \mathbb{C}$ is called square-integrable in $\mathbb{R}^N$ if it has finite L^2 norm, that is,

$$||f||^2 = \int_{\mathbb{R}^N} |f(\mathbf{x})|^2 d^N\mathbf{x} < \infty. \tag{1.2}$$

The Hilbert space $\mathcal{H} = L^2(\mathbb{R}^N)$ is the vector space consisting of all complex-valued square-integrable functions.

Consider the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^N)$ and let $|\psi\rangle, |\varphi\rangle \in \mathcal{H}$. The inner product is defined by

$$\langle\psi|\varphi\rangle = \int_{\mathbb{R}^N} \psi^*(\mathbf{x})\varphi(\mathbf{x}) d^N\mathbf{x}, \tag{1.3}$$

[1] Here, completeness means that every Cauchy sequence in $\mathcal{H}$ also converges in $\mathcal{H}$.

where $\psi^*(\mathbf{x})$ denotes the complex conjugate of $\psi(\mathbf{x})$. The inner product is guaranteed to exist, since the integral converges to a finite value. Together with the notation $|\psi(\mathbf{x})|^2 = \psi^*(\mathbf{x})\psi(\mathbf{x})$, the inner product of $|\psi\rangle \in \mathcal{H}$ with itself is given by

$$\langle\psi|\psi\rangle = \int_{\mathbb{R}^N} |\psi(\mathbf{x})|^2 d^N\mathbf{x}, \tag{1.4}$$

which is both real and non-negative.

A ket $|\psi\rangle \in \mathcal{H}$ is said to be normalised if its inner product with itself is 1, and two kets $|\psi\rangle, |\varphi\rangle \in \mathcal{H}$ are orthogonal if their inner product is 0. Moreover, a discrete set of kets $\{|u_i\rangle\}$ is said to be orthonormal if they are normalised and mutually orthogonal, that is,

$$\langle u_i|u_j\rangle = \delta_{ij}, \tag{1.5}$$

where δ_{ij} denotes the Kronecker delta[2] and $i, j = 1, \ldots, N$.

1.2 Quantum Observables

In physics, any dynamical variable or physical quantity that can be measured is called an observable. In the field of quantum mechanics, they are represented by linear operators [3].

Definition 1.3. Let V be a complex vector space. A linear operator on V is a map $\hat{O} : V \to V$ such that

$$\hat{O}(\alpha|\psi\rangle + \beta|\varphi\rangle) = \alpha\hat{O}|\psi\rangle + \beta\hat{O}|\varphi\rangle, \tag{1.6}$$

for all vectors $|\psi\rangle, |\varphi\rangle \in V$ and all scalars $\alpha, \beta \in \mathbb{C}$.

With each observable quantity O, we associate the linear operator $\hat{O}$. Since Hilbert spaces are complex vector spaces, the operator $\hat{O}$ relates with every vector $|\psi\rangle \in \mathcal{H}$ another vector $|\varphi\rangle \in \mathcal{H}$, that is,

$$\hat{O}|\psi\rangle = |\varphi\rangle.$$

Trivial examples of operators are the identity operator $\hat{I}$ and the null operator $\hat{0}$, which satisfy $\hat{I}|\psi\rangle = |\psi\rangle$ and $\hat{0}|\psi\rangle = |0\rangle$, respectively. While $\langle\varphi|\psi\rangle$ represents the inner product in Dirac notation, the expression $|\psi\rangle\langle\varphi|$ represents an operator [7].

Sometimes, the linear operator $\hat{O}$ associates with the non-zero vector $|\psi\rangle$ another vector that is its scalar multiple, that is,

$$\hat{O}|\psi\rangle = \lambda|\psi\rangle \tag{1.7}$$

for some $\lambda \in \mathbb{C}$. This equation is known as the eigenvalue equation for the operator $\hat{O}$. Furthermore, we call the scalar λ and the vector $|\psi\rangle$ that satisfy (1.7) the eigenvalue and eigenvector,[3] respectively, of the operator $\hat{O}$.

Given any observable O, there are many mathematical expressions we can choose to represent the operator $\hat{O}$. Because these operators reflect the mathematical structure, different choices of the operators lead to different representations of quantum mechanics [3]. Choosing a representation is equivalent to choosing a orthonormal basis [7] for the Hilbert space $\mathcal{H}$.

A discrete set $\{|u_i\rangle\}$ forms a basis for the Hilbert space $\mathcal{H}$ if it is complete,[4] that is, any $|\psi\rangle \in \mathcal{H}$ can be written as a linear combination,

$$|\psi\rangle = \sum_j c_i|u_i\rangle. \tag{1.8}$$

The coefficients c_i in this equation can be calculated using the expression

$$c_i = \langle u_i|\psi\rangle, \tag{1.9}$$

given that the discrete set $\{|u_i\rangle\}$ is orthonormal. Combining equations (1.8) and (1.9), we find

$$|\psi\rangle = \sum_i c_i|u_i\rangle = \sum_i \langle u_i|\psi\rangle|u_i\rangle = \sum_i |u_i\rangle\langle u_i|\psi\rangle = \left(\sum_i |u_i\rangle\langle u_i|\right)|\psi\rangle. \tag{1.10}$$

[2] The Kronecker delta δ_{ij} is defined by

$$\delta_{ij} = \begin{cases} 0, & \text{if } i \neq j, \\ 1, & \text{if } i = j. \end{cases}$$

[3] If the Hilbert space $\mathcal{H}$ is a function space, the eigenvector of an operator is also known as an eigenfunction.

[4] The completeness of a set we find here is unrelated to the completeness of a space we encountered before.

This result implies that the sum in the parentheses represents the identity operator, that is,

$$\sum_i |u_i\rangle \langle u_i| = \hat{I}. \tag{1.11}$$

Equation (1.11) is called the closure relation [7], which is a mathematical statement that a basis exists and is complete. Therefore, the only formulas we require for calculations in the $\{|u_i\rangle\}$ representation of quantum mechanics are given by the equations (1.5) and (1.11), that is, the orthonormality and closure relation of the discrete set $\{|u_i\rangle\}$, respectively.

Definition 1.4. For the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^N)$, the basis $\{|\mathbf{x}\rangle\}$ constitutes the so-called position representation [3], for which the orthonormality and closure relation, respectively, are given by

$$\langle \mathbf{x}|\mathbf{x}'\rangle = \delta(\mathbf{x}-\mathbf{x}') \qquad \text{and} \qquad \int_{\mathbb{R}^N} |\mathbf{x}\rangle \langle \mathbf{x}|\, d^N\mathbf{x} = \hat{I}, \tag{1.12}$$

where $\delta(\mathbf{x}-\mathbf{x}')$ is the Dirac delta function.[5]

Using the position representation to express linear operators, the position operator $\hat{\mathbf{x}}$ is represented by multiplication by $\mathbf{x}$ and the momentum operator $\hat{\mathbf{p}}$ is represented by differentiation with respect to position and multiplication by $-i\hbar$, where $\hbar$ is the reduced Planck constant.[6] Stated explicitly,

$$\hat{x}_j f = x_j f \qquad \text{and} \qquad \hat{p}_j f = -i\hbar \frac{\partial f}{\partial x_j}, \tag{1.13}$$

for any function f and $j = 1, \ldots, N$.

In physics, all classical dynamical variables, and therefore all observables, can be expressed in terms of position and momentum [15]. Hence, let us denote these observables by $O(\mathbf{x}, \mathbf{p})$. To find the operator $\hat{O}$ associated with these observables, we replace the position $\mathbf{x}$ and momentum $\mathbf{p} = m\dot{\mathbf{x}}$ in the classical dynamical variables by the position operator $\hat{\mathbf{x}}$ and momentum operator $\hat{\mathbf{p}}$, respectively.

To illustrate this procedure, let us consider a simple example. The kinetic energy K of a particle is a classical dynamical variable, whose value is given by the formula

$$K = \frac{1}{2} m |\dot{\mathbf{x}}|^2 = \frac{|\mathbf{p}|^2}{2m}.$$

Then, using the momentum operator $\hat{\mathbf{p}}$ as defined in equation (1.13), the kinetic energy operator $\hat{K}$ can be written as

$$\hat{K} f = \frac{(\hat{\mathbf{p}} \cdot \hat{\mathbf{p}})}{2m} f = -\frac{\hbar^2}{2m} \nabla^2 f, \tag{1.14}$$

where the operator ∇^2 is the Laplace operator and f is an arbitrary function. Similarly, the potential energy operator $\hat{U}$ is represented by multiplication by $U(\mathbf{x})$, which is the potential energy function,[7]

$$\hat{U} f = U(\mathbf{x}) f. \tag{1.15}$$

The total energy of a system in classical mechanics, called the Hamiltonian and denoted by H, is given by the sum of kinetic energy K and potential energy U. Therefore, using (1.14) and (1.15), the Hamiltonian operator $\hat{H}$ is given by

$$\hat{H} f = (\hat{K} + \hat{U}) f = -\left(\frac{\hbar^2}{2m} \nabla^2 + U(\mathbf{x}) \right) f. \tag{1.16}$$

[5] The 1-dimensional Dirac delta function [15], denoted by $\delta(x)$, is defined by

$$\delta(x) = \begin{cases} 0 & \text{if } x \neq 0, \\ \infty & \text{if } x = 0, \end{cases} \qquad \text{with} \qquad \int_{-\infty}^{\infty} \delta(x) dx = 1.$$

The N-dimensional Dirac delta function $\delta(\mathbf{x})$, with $\mathbf{x} = (x_1, \ldots, x_N)$, is defined as

$$\delta(\mathbf{x}) = \delta(x_1) \cdots \delta(x_N) \qquad \text{with} \qquad \int_{\mathbb{R}^N} \delta(\mathbf{x}) d^N\mathbf{x} = 1.$$

[6] The reduced Planck constant $\hbar$ is a fundamental physical constant in quantum mechanics. Its value is given by

$$\hbar = \frac{h}{2\pi} = 1.054571871 \cdot 10^{-34} \text{ J} \cdot \text{s},$$

where $h = 6.62607015 \cdot 10^{-34}$ J $\cdot$ s is the original Planck constant.

[7] In general, the potential energy is a function of both position $\mathbf{x}$ and time t, that is, $U(\mathbf{x}, t)$. However, we make the simplification that U is independent of time, and a function of position only.

Consider the quantum state $|\Psi(t)\rangle \in \mathcal{H}$. Using the closure relation for the position representation given by (1.12), we can write

$$|\Psi(t)\rangle = \int_{\mathbb{R}^N} |\mathbf{x}\rangle \langle \mathbf{x}|\Psi(t)\rangle \, d^N\mathbf{x}. \tag{1.17}$$

In particular, for the coefficient $\langle \mathbf{x}|\Psi(t)\rangle$ we have

$$\langle \mathbf{x}|\Psi(t)\rangle = \Psi(\mathbf{x}, t), \tag{1.18}$$

that is, the wave function $\Psi(\mathbf{x}, t)$ is the component of the quantum state $|\Psi(t)\rangle$ on the basis vector $\{|\mathbf{x}\rangle\}$ of the position representation [7].

1.3 The Born Interpretation

In the previous section, we concluded with the relation between the quantum state $|\Psi(t)\rangle$ and the wave function $\Psi(\mathbf{x}, t)$. We can interpret the quantum state as a vector, since it is an element of a vector space, namely the Hilbert space. However, we do not yet have an interpretation for the wave function.

A widely accepted interpretation of the wave function is known as Born's statistical interpretation, which states that the probability of finding a particle in the region R at the position $\mathbf{x}$ is proportional to $|\Psi(\mathbf{x}, t)|^2 d^N\mathbf{x}$, where $d^N\mathbf{x}$ is a volume element [3]. Formulated in the language of probability theory, the statistical interpretation reads

$$\mathbb{P}(\text{particle is in region } R) = \int_R |\Psi(\mathbf{x}, t)|^2 d^N\mathbf{x}. \tag{1.19}$$

Born's interpretation of the wave function implies that $|\Psi(\mathbf{x}, t)|^2$ is a probability density, and it introduces the concept of indeterminancy into quantum mechanics.

One of the most important conditions in the statistical interpretation is the normalisation condition of the wave function,

$$\mathbb{P}(\text{particle exists in } \mathbb{R}^N) = \int_{\mathbb{R}^N} |\Psi(\mathbf{x}, t)|^2 d^N\mathbf{x} = 1. \tag{1.20}$$

Physically, this condition states that the particle has to be somewhere in N-dimensional space. It is for this reason that we require wave functions to be normalised, that is, $\langle \Psi(t)|\Psi(t)\rangle = 1$ for all t.

Suppose we measure the observable $O(\mathbf{x}, \mathbf{p})$ in the quantum state $|\Psi(t)\rangle$ on an ensemble of identically prepared systems. As a result of the indeterminancy of quantum mechanics, these measurements do not have the same result each time. The act of measurement is probabilistic by nature [22]. If we treat the measured value of the observable as a random variable, we can compute its statistical properties. The expectation value of the quantum observable $O(\mathbf{x}, \mathbf{p})$, which we denote by $\langle O(\mathbf{x}, \mathbf{p})\rangle$, is the average of measurements on an assemble of identically-prepared systems [15], and can be computed by

$$\langle O(\mathbf{x}, \mathbf{p})\rangle = \langle \Psi| \hat{O} |\Psi\rangle = \int_{\mathbb{R}^N} \Psi^* \left(O \left(\mathbf{x}, -i\hbar \frac{d}{d\mathbf{x}} \right) \right) \Psi d^N\mathbf{x}. \tag{1.21}$$

Physically, the outcome of a measurement can only be a real value, and consequently, the average of many measurements is real as well. Therefore, for quantum observables, we have the restriction

$$\langle O(\mathbf{x}, \mathbf{p})\rangle = \langle \Psi| O |\Psi\rangle = \langle \Psi| O |\Psi\rangle^* = \langle O(\mathbf{x}, \mathbf{p})\rangle^* . \tag{1.22}$$

Linear operators that satisfy this condition are called Hermitian operators [3]. A Hermitian operator has the property that it is equal to its Hermitian conjugate, that is,

$$\hat{O} = \hat{O}^\dagger.$$

In quantum mechanics, all quantum observables $O(\mathbf{x}, \mathbf{p})$ are represented by Hermitian operators $\hat{O}$. The normalised eigenfunctions of a Hermitian operator have two important mathematical properties [15].

Theorem 1.5. *The eigenvalues of a Hermitian operator $\hat{O}$ are real.*

Proof. Consider the eigenvalue equation

$$\hat{O} |\psi\rangle = \lambda |\psi\rangle ,$$

where the eigenfunction $|\psi\rangle$ is normalised. Multiplication from the left by $\langle \psi|$ and using the normalisation condition gives

$$\langle \psi| \hat{O} |\psi\rangle = \langle \psi| \lambda |\psi\rangle = \lambda \langle \psi|\psi\rangle = \lambda.$$

Now, taking the complex conjugate of both sides of this equation gives us

$$\langle\psi|\,\hat{O}\,|\psi\rangle^* = \lambda^*.$$

Because $\hat{O}$ is a Hermitian operator, we have $\langle\psi|\,\hat{O}\,|\psi\rangle = \langle\psi|\,\hat{O}\,|\psi\rangle^*$, and therefore, we find

$$\lambda = \langle\psi|\,\hat{O}\,|\psi\rangle = \langle\psi|\,\hat{O}\,|\psi\rangle^* = \lambda^*,$$

which proves that the eigenvalue λ is real. □

Theorem 1.6. *Eigenfunctions corresponding to different eigenvalues of an Hermitian operator $\hat{Q}$ are orthogonal.*

Proof. Consider the two eigenvalue equations

$$\hat{O}\,|\psi\rangle = \lambda_1\,|\psi\rangle \qquad \text{and} \qquad \hat{O}\,|\varphi\rangle = \lambda_2\,|\varphi\rangle\,,$$

where the eigenfunctions $|\psi\rangle$ and $|\varphi\rangle$ are normalised and $\lambda_1 \neq \lambda_2$. Multiplying these two eigenvalue equations by $\langle\varphi|$ and $\langle\psi|$, respectively, we find

$$\langle\varphi|\,\hat{O}\,|\psi\rangle = \langle\varphi|\,\lambda_1\,|\psi\rangle = \lambda_1\,\langle\varphi|\psi\rangle\,, \qquad \text{and} \qquad \langle\psi|\,\hat{O}\,|\varphi\rangle = \langle\psi|\,\lambda_2\,|\varphi\rangle = \lambda_2\,\langle\psi|\varphi\rangle\,.$$

Taking the complex conjugate of the second equation and subtracting it from the first gives

$$\langle\varphi|\,\hat{O}\,|\psi\rangle - \langle\psi|\,\hat{O}\,|\varphi\rangle^* = \lambda_1\,\langle\varphi|\psi\rangle - \lambda_2^*\,\langle\psi|\varphi\rangle^*\,.$$

Since $\hat{O}$ is a Hermitian operator, we have $\langle\varphi|\,\hat{O}\,|\psi\rangle = \langle\psi|\,\hat{O}\,|\varphi\rangle^*$ and $\lambda_2^* = \lambda_2$ by Theorem 1.5. Moreover, using that $\langle\psi|\varphi\rangle^* = \langle\varphi|\psi\rangle$, we find

$$0 = \lambda_1\,\langle\varphi|\psi\rangle - \lambda_2\,\langle\psi|\varphi\rangle^* = (\lambda_1 - \lambda_2)\,\langle\varphi|\psi\rangle\,.$$

Because $\lambda_1 \neq \lambda_2$, this equation can only hold when $\langle\varphi|\psi\rangle = 0$, which is to say that the eigenfunctions $|\psi\rangle$ and $|\varphi\rangle$ corresponding to the distinct eigenvalues λ_1 and λ_2, respectively, are orthogonal. □

Let $\hat{O}$ be the Hermitian operator corresponding to the particle's observable $O(\mathbf{x},\mathbf{p})$. Then, if $|\Psi\rangle$ is an eigenfunction of the operator $\hat{O}$ with corresponding eigenvalue λ, we have the eigenvalue equation

$$\hat{O}\,|\Psi\rangle = \lambda\,|\Psi\rangle\,.$$

Multiplying both sides of this eigenvalue equation by $\langle\Psi|$ yields

$$\langle\Psi|\,\hat{O}\,|\Psi\rangle = \langle\Psi|\,\lambda\,|\Psi\rangle = \lambda\,\langle\Psi|\Psi\rangle = \lambda. \tag{1.23}$$

Note that equation (1.23) is precisely the expectation value of the quantum observable $O(\mathbf{x},\mathbf{p})$,

$$\langle O(\mathbf{x},\mathbf{p})\rangle = \langle\Psi|\,\hat{O}\,|\Psi\rangle = \lambda. \tag{1.24}$$

Therefore, if $|\Psi(t)\rangle$ is an eigenfunction of the operator $\hat{O}$, the determination of the observable $O(\mathbf{x},\mathbf{p})$ yields a single result, namely its eigenvalue [3]. We call the eigenfunctions of $\hat{O}$ the determinate states of $O(\mathbf{x},\mathbf{p})$. The measurement of the observable on such a state is certain to yield the eigenvalue λ as an outcome [15].

As we mentioned before, the outcome of a measurement can only be a real value to make physically sense. Fortunately, since obervables $O(\mathbf{x},\mathbf{p})$ are represented by Hermitian operators, Theorem 1.5 ensures that the eigenvalue λ is real, and therefore that the outcome of a measurement on an observable is always a real value.

1.4 The Uncertainty Principle

Let $\hat{A}$ and $\hat{B}$ be two linear operators and let $|\psi\rangle \in \mathcal{H}$. The product of two operators $\hat{A}$ and $\hat{B}$, written as $\hat{A}\hat{B}$, is defined by

$$(\hat{A}\hat{B})\,|\psi\rangle = \hat{A}(\hat{B}\,|\psi\rangle). \tag{1.25}$$

This is to say that $\hat{B}$ acts first on $|\psi\rangle$, after which $\hat{A}$ acts on $\hat{B}\,|\psi\rangle$. In general, $\hat{A}\hat{B} \neq \hat{B}\hat{A}$, which implies that the order in which the operators are carried out is of great importance.

Definition 1.7. Let $\hat{A}$ and $\hat{B}$ be two linear operators. The commutator of $\hat{A}$ and $\hat{B}$ is given by

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}. \tag{1.26}$$

We say that $\hat{A}$ and $\hat{B}$ commute if $\hat{A}\hat{B} = \hat{B}\hat{A}$, that is, if $[\hat{A}, \hat{B}] = \hat{0}$.

For example, consider the position and momentum operators $\hat{\mathbf{x}}$ and $\hat{\mathbf{p}}$ in the position representation given by equation (1.13). With these operators, we associate the operators $\hat{x}_1$ and $\hat{p}_1$ given by

$$\hat{x}_1 = x_1 \qquad \text{and} \qquad \hat{p}_1 = -i\hbar\frac{\partial}{\partial x_1}, \tag{1.27}$$

Using an arbitrary wave function $\Psi(\mathbf{x}, t)$, we find

$$\begin{aligned}
[\hat{x}_1, \hat{p}_1]\Psi(\mathbf{x}, t) &= (\hat{x}_1\hat{p}_1 - \hat{p}_1\hat{x}_1)\Psi(\mathbf{x}, t) \\
&= \hat{x}_1\hat{p}_1\Psi(\mathbf{x}, t) - \hat{p}_1\hat{x}_1\Psi(\mathbf{x}, t) \\
&= x_1\left(-i\hbar\frac{\partial}{\partial x_1}\Psi(\mathbf{x}, t)\right) + i\hbar\frac{\partial}{\partial x_1}(x_1\Psi(\mathbf{x}, t)) \\
&= i\hbar\left(\Psi(\mathbf{x}, t) + x_1\frac{\partial\Psi(\mathbf{x}, t)}{\partial x_1} - x_1\frac{\partial\Psi(\mathbf{x}, t)}{\partial x_1}\right) \\
&= i\hbar\Psi(\mathbf{x}, t).
\end{aligned}$$

Since the wave function $\Psi(\mathbf{x}, t)$ was arbitrary, we find $[\hat{x}_1, \hat{p}_1] = i\hbar\hat{I}$, which shows that $\hat{x}_1$ and $\hat{p}_1$ do not commute with eachother.

Theorem 1.8. *Let $\hat{x}_i$ and $\hat{p}_i$ be the components of the operators $\hat{\mathbf{x}}$ and $\hat{\mathbf{p}}$ as defined by equation* (1.13). *Then, we have the fundamental commutation relations*

$$[\hat{x}_i, \hat{p}_i] = \hat{0}, \qquad [\hat{p}_i, \hat{p}_i] = \hat{0}, \qquad \text{and} \qquad [\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}\hat{I}, \tag{1.28}$$

where $i, j = 1, \dots, N$.

Proof. The fundamental commutation relations follow from straightforward calculations [7], similarly as we have preformed for $\hat{x}_1$ and $\hat{p}_1$. □

To compute the commutator of two arbitrary linear operators, the following theorem provides the necessary properties which make calculations much easier [17].

Lemma 1.9. *Given the three linear operators $\hat{A}$, $\hat{B}$ and $\hat{C}$ and the constants $\alpha, \beta \in \mathbb{C}$, the commutator satisfies:*

(i) $[\hat{A}, \hat{A}] = \hat{0}$*;*

(ii) $[\hat{A}, \hat{B}] = -[\hat{B}, \hat{A}]$ *(anticommutativity);*

(iii) $[\alpha\hat{A} + \beta\hat{B}, \hat{C}] = \alpha[\hat{A}, \hat{C}] + \beta[\hat{B}, \hat{C}]$ *and* $[\hat{C}, \alpha\hat{A} + \beta\hat{B}] = \alpha[\hat{C}, \hat{A}] + \beta[\hat{C}, \hat{B}]$ *(bilinearity);*

(iv) $[\hat{A}\hat{B}, \hat{C}] = [\hat{A}, \hat{C}]\hat{B} + \hat{A}[\hat{B}, \hat{C}]$ *(derivation);*

(v) $[\hat{A}, [\hat{B}, \hat{C}]] + [\hat{B}, [\hat{C}, \hat{A}]] + [\hat{C}, [\hat{A}, \hat{B}]] = \hat{0}$ *(Jacobi identity).*

Let A and B be two observables, with which we associate the Hermitian operators $\hat{A}$ and $\hat{B}$. For any quantum state $|\Psi\rangle$, the variance of A is given by

$$\sigma_A^2 = \langle (A - \langle A \rangle)^2 \rangle = \langle (\hat{A} - \langle A \rangle)\Psi | (\hat{A} - \langle A \rangle)\Psi \rangle, \tag{1.29}$$

and similarly for B. The standard deviation σ_A is the positive square root of the variance σ_A^2, which is by nature always positive. The commutator $[\hat{A}, \hat{B}]$ of the operators $\hat{A}$ and $\hat{B}$ plays an important role in the uncertainty of quantum mechanics [15].

Theorem 1.10. *Given two observables A and B, the generalised uncertainty principle is given by*

$$\sigma_A^2\sigma_B^2 \geq \left(\frac{1}{2i}\langle [\hat{A}, \hat{B}] \rangle\right)^2, \tag{1.30}$$

where $[\hat{A}, \hat{B}]$ is the commutator of the operators $\hat{A}$ and $\hat{B}$.

For example, consider the operators $\hat{x}_1$ and $\hat{p}_1$ given by equation (1.27). For the commutator, we found $[\hat{x}_1, \hat{p}_1] = i\hbar\hat{I}$, hence we have the uncertainty relation

$$\sigma_{x_1}^2 \sigma_{p_1}^2 \geq \left(\frac{1}{2i} i\hbar\right)^2 = \left(\frac{\hbar}{2}\right)^2.$$

Using the standard deviation, we find Heisenberg's uncertainty principle of position and momentum,

$$\sigma_{x_1}\sigma_{p_1} \geq \frac{\hbar}{2}. \tag{1.31}$$

For every pair of quantum observables whose operators do not commute, there exists an uncertainty principle [15]. Both the position and momentum of a particle as well as its energy and time constitute important uncertainty relations in quantum mechanics. Two observables whose associated operators do not commute are known as imcompatible observables.

1.5 The Schrödinger Equation

We return to the quantum state $|\Psi(t)\rangle \in \mathcal{H}$. A physical system is dynamic by nature, which means that its state changes over time. In quantum mechanics, the time evolution [22] of a quantum state $|\Psi(t)\rangle$ is governed by the Schrödinger equation,

$$i\hbar \frac{\partial}{\partial t} |\Psi(t)\rangle = \hat{H} |\Psi(t)\rangle, \tag{1.32}$$

where $\hat{H}$ is the Hamiltonian operator. Using equation (1.16), we obtain the time-dependent Schrödinger equation in N dimensions,

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi + U(\mathbf{x})\Psi. \tag{1.33}$$

To obtain the wave function $\Psi(\mathbf{x}, t)$, and hence to derive information about the quantum state $|\Psi(t)\rangle$ of the physical system, we need to solve this partial differential equation.

Because the potential $U(\mathbf{x})$ is independent of time, we are able to solve equation (1.33) using the method of separation of variables [24]. This method seeks solutions that can be written as the product

$$\Psi(\mathbf{x}, t) = \psi(\mathbf{x})\varphi(t), \tag{1.34}$$

where ψ is a function of position $\mathbf{x} \in \mathbb{R}^N$ only and φ is a function of time t only. Then, (1.33) can be written as

$$i\hbar \frac{1}{\varphi}\frac{d\varphi}{dt} = -\frac{\hbar^2}{2m}\frac{1}{\psi}\nabla^2\psi + U(\mathbf{x}). \tag{1.35}$$

Note that the left-hand side of this equation depends solely on t, while the right-hand side depends only on $\mathbf{x}$. Therefore, both sides of equation (1.35) must be constant, which we shall call E due to its physical relation with energy. Then, we obtain a pair of differential equations,

$$i\hbar \frac{1}{\varphi}\frac{d\varphi}{dt} = E, \qquad \text{and} \qquad -\frac{\hbar^2}{2m}\frac{1}{\psi}\nabla^2\psi + U(\mathbf{x}) = E.$$

Rewriting the first differential equation yields

$$\frac{d\varphi}{dt} = -\frac{iE}{\hbar}\varphi, \tag{1.36}$$

for which $\varphi(t) = e^{-iEt/\hbar}$ is the solution.[8] The second differential equation can be written as

$$-\frac{\hbar^2}{2m}\nabla^2\psi + U(\mathbf{x})\psi = E\psi, \tag{1.37}$$

and is called the time-independent Schrödinger equation. Until the potential energy function $U(\mathbf{x})$ is specified, we cannot solve it.

[8] The general solution of equation (1.36) is $\varphi(t) = Ce^{-iEt/\hbar}$ for some constant C. However, because we try to find the product $\Psi(\mathbf{x}, t) = \psi(\mathbf{x})\varphi(t)$, we can ignore the constant C and absorb it into $\psi(\mathbf{x})$.

The time-independent Schrödinger equation (1.37), which we can write as the eigenvalue equation $\hat{H}\psi = E\psi$, has an infinite collection of solutions $\psi_0(\mathbf{x}), \psi_1(\mathbf{x}), \psi_2(\mathbf{x}), \ldots$, each with its own separation constant $E_0, E_1, E_2, \ldots$, known as energy levels. For each allowed energy level, there is different a wave function,

$$\Psi_0(\mathbf{x}, t) = \psi_0(\mathbf{x})e^{-iE_0 t/\hbar}, \qquad \Psi_1(\mathbf{x}, t) = \psi_1(\mathbf{x})e^{-iE_1 t/\hbar}, \qquad \Psi_2(\mathbf{x}, t) = \psi_2(\mathbf{x})e^{-iE_2 t/\hbar}, \ldots.$$

These separable solutions $\Psi_n(\mathbf{x}, t) = \psi_n(\mathbf{x})e^{-iE_n t/\hbar}$ are called stationary states, where the non-negative integer n denotes the principal quantum number [15]. As time progresses, the quantum system remains in the same state. Consequently, it follows that the stationary states are the determinate states of the Hamiltonian operator [15]. If we measure the energy of a particle in the stationary state Ψ_n, the outcome always is the energy level E_n.

The Schrödinger equation (1.33), being a linear partial differential equation, has the property that any linear combination of solutions is itself a solution [15]. Using the separable solutions, the general solution is of the form

$$\Psi(\mathbf{x}, t) = \sum_{n=1}^{\infty} c_n \psi_n(\mathbf{x}) e^{-iE_n t/\hbar}, \tag{1.38}$$

for some coefficients c_n. They are determined by the initial condition $\Psi(\mathbf{x}, 0)$ and are of great importance to the generalised statistical interpretation [15] of quantum mechanics.[9]

[9] Explained in a nutshell, for an observable O and a particular eigenvalue λ_n of the Hermitian operator $\hat{O}$, the probability of measuring $O = \lambda_n$ in the quantum state $|\Psi(t)\rangle$ is given by $|c_n|^2$.

2 Harmonic Oscillators in Euclidean N-Spaces

Prior to concentrating on curved spaces, we begin with the simple harmonic oscillator in flat Euclidean geometry. For both classical and quantum mechanics, we generalise our framework to the N-dimensional harmonic oscillator in Euclidean space.

In Section 2.1, we formulate the classical dynamics for the N-dimensional simple harmonic oscillator, both using Newton's laws of motion and Hamiltonian mechanics. Thereafter, we derive the particle's orbits and discuss the constants of motion for the harmonic oscillator in Section 2.2. Subsequently in Section 2.3, the quadratic Casimir function is constructed from the constants of motion. In Section 2.4, we start discussing the dynamics of the N-dimensional quantum harmonic oscillator, after which we solve the 1-dimensional Schrödinger equation in Section 2.5. Finally in Section 2.6, the solutions and energy levels for the N-dimensional quantum harmonic oscillator are constructed.

2.1 Classical Dynamics in the Euclidean N-Space

Consider a particle of mass m moving in the N-dimensional Euclidean space $\mathbb{E}^N$, an affine space [5] with which we can associate the Euclidean vector space $\mathbb{R}^N$. Using cartesian coordinates, the position of the particle at the point $X \in \mathbb{E}^N$ is represented by the position vector $\mathbf{x} = (x_1, \dots, x_N) \in \mathbb{R}^N$. To compute the distance between two positions in the Euclidean N-space, we apply the metric tensor [20].

Definition 2.1. For two positions $\mathbf{x}, \mathbf{x}+d\mathbf{x} \in \mathbb{R}^N$, let $d\mathbf{x} = (dx_1, \dots, dx_N) \in \mathbb{R}^N$ denote the infinitesmial displacement vector. The distance ds between the positions is called the line element, whose square is given by the formula

$$ds^2 = |d\mathbf{x}|^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} g_{ij} dx_i dx_j, \tag{2.1}$$

where the constants g_{ij} are the matrix elements representing the metric tensor g.

In the Euclidean N-space, the metric tensor g is defined by $g_{ij} = \delta_{ij}$. Then, the line element ds that corresponds to the Euclidean metric reads

$$ds^2 = dx_1^2 + \cdots + dx_N^2, \tag{2.2}$$

which we shall use to compute the distance between two positions in the Euclidean N-space.

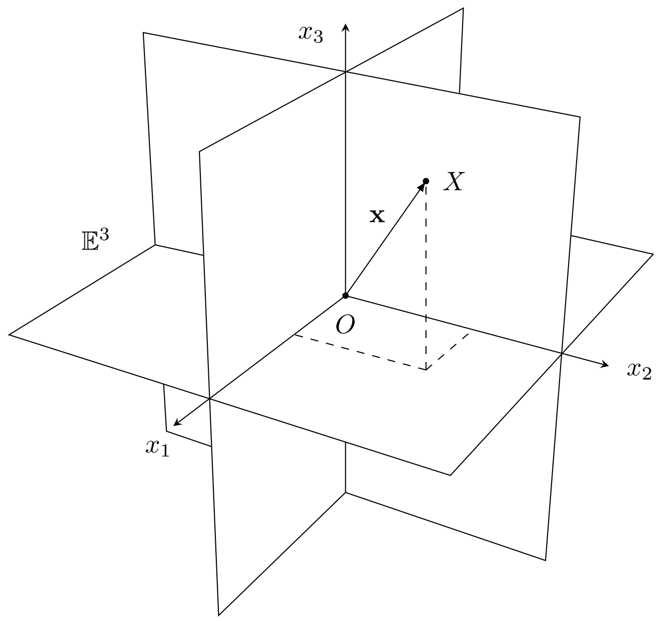

Figure 2.1 A point X in the Euclidean 3-space $\mathbb{E}^3$, represented using the Euclidean vector space $\mathbb{R}^3$.

Suppose that the origin O of the Euclidean N-space is the sole stable mechanical equilibrium point of the particle. Then, the position vector $\mathbf{x}$ represents the displacement of the particle in $\mathbb{E}^N$ away from its equilibrium point.

Additionally, suppose that the particle experiences a linear restoring force [12], a force proportional to the displacement of the particle. That is, its force vector $\mathbf{F}$ is described by Hooke's law,

$$\mathbf{F} = -k\mathbf{x}, \tag{2.3}$$

where $k > 0$ is constant. If the particle is displaced from the origin, the restoring force continuously acts to return the particle to its equilibrium point. Thus, the motion of the particle in the Euclidean N-space is dictated by this restoring force.

In classical mechanics, the state of the particle is described by its position $\mathbf{x}(t)$ and its velocity $\dot{\mathbf{x}}(t)$. We can use Newton's laws of motion, together with the initial state, to derive the state of the particle at any time $t \in \mathbb{R}$. If the restoring force (2.3) is the only force acting on the particle, Newton's second law reads

$$m\ddot{\mathbf{x}} = -k\mathbf{x}. \tag{2.4}$$

The general solution to this differential equation is given by

$$\mathbf{x}(t) = \mathbf{x}(0)\cos(\omega t) + \frac{1}{\omega}\dot{\mathbf{x}}(0)\sin(\omega t), \tag{2.5}$$

where $\mathbf{x}(0)$ and $\dot{\mathbf{x}}(0)$ are the initial position and velocity of the particle, respectively, and the constant

$$\omega = \sqrt{\frac{k}{m}} \tag{2.6}$$

is known as the angular frequency of oscillation. The particle's motion is a sinusoidal oscillation of its displacement $\mathbf{x}$. It is for this reason that equation (2.4) is called the differential equation of the simple harmonic oscillator [12].

To bridge the gap between classical and quantum mechanics, we derive the equations of motion using Hamiltonian mechanics, an alternative formulation of classical mechanics that is equivalent to Newton's laws of motion. The kinetic energy K of the particle is

$$K = \frac{1}{2}m|\dot{\mathbf{x}}|^2 = \frac{1}{2}m\left(\dot{x}_1^2 + \cdots + \dot{x}_N^2\right), \tag{2.7}$$

and its linear momentum $\mathbf{p} = (p_1, \ldots, p_N) \in \mathbb{R}^N$ is given by

$$\mathbf{p} = \frac{\partial K}{\partial \dot{\mathbf{x}}} = m\dot{\mathbf{x}}. \tag{2.8}$$

Using this formula for the linear momentum of the particle, its kinetic energy can also be written as

$$K = \frac{|\mathbf{p}|^2}{2m} = \frac{1}{2m}\left(p_1^2 + \cdots + p_N^2\right). \tag{2.9}$$

For the linear restoring force $\mathbf{F}$, there exists a scalar-valued function $U(\mathbf{x})$, which is called the potential energy of the particle [12], such that we have the differential equation

$$\mathbf{F} = -\frac{dU}{d\mathbf{x}}. \tag{2.10}$$

Using equation (2.3), the potential U for the classical harmonic oscillator is given by

$$U(\mathbf{x}) = \frac{1}{2}k|\mathbf{x}|^2 + C = \frac{1}{2}k\left(x_1^2 + \cdots + x_N^2\right) + C, \tag{2.11}$$

for some constant of integration C. By choosing the origin as the reference point of the potential, that is, the sole point in Euclidean N-space where $U(\mathbf{x}) = 0$, we set $C = 0$. Moreover, let r denote the length of the position vector $\mathbf{x}$, that is,

$$|\mathbf{x}|^2 = r^2 = x_1^2 + \cdots + x_N^2. \tag{2.12}$$

Then, using equation (2.6) to rewrite the constant k, the potential U can alternatively be written as

$$U(\mathbf{x}) = \frac{1}{2}m\omega^2 r^2. \tag{2.13}$$

Using the equations (2.9) and (2.13) for the kinetic and potential energy, respectively, the total energy of the particle reads

$$H = K + U = \frac{|\mathbf{p}|^2}{2m} + \frac{1}{2}m\omega^2 r^2, \tag{2.14}$$

and is called the Hamiltonian of the particle. In Hamiltonian mechanics, the motion of the particle can be described by a system of $2N$ ordinary differential equations,

$$\begin{cases} \dot{x}_i = \dfrac{\partial H}{\partial p_i}, \\ \dot{p}_i = -\dfrac{\partial H}{\partial x_i}, \end{cases} \tag{2.15}$$

for $i = 1, \ldots, N$. These equations are known as Hamilton's equations [12], which describe the motion of the particle in terms of the $2N$ state space coordinates $x_i(t)$ and $p_i(t)$.

Before we proceed, let us verify the equivalence between the Hamiltonian formulation of classical mechanics and Newton's laws of motion. Using the Hamiltonian of the harmonic oscillator described by equation (2.14), Hamilton's equations (2.15) read

$$\dot{x}_i = \frac{\partial}{\partial p_i}\left[\frac{|\mathbf{p}|^2}{2m} + \frac{1}{2}m\omega^2 r^2\right] = \frac{p_i}{m} \qquad \text{and} \qquad \dot{p}_i = -\frac{\partial}{\partial x_i}\left[\frac{|\mathbf{p}|^2}{2m} + \frac{1}{2}m\omega^2 r^2\right] = -m\omega^2 x_i = -kx_i,$$

where we used equation (2.6) to recover the constant k. Then, upon combining both sets of equations gives us N differential equations, namely

$$m\ddot{x}_i = p_i = -kx_i.$$

Collecting these N equations in a vector yields precisely equation (2.4) which we derived using Newton's second law. Hence, we conclude that both formulations of classical mechanics give the same result and are therefore equivalent.

Throughout this thesis, we shall use the formulation of Hamiltonian mechanics instead of Newton's laws of motion, due to its close link to quantum mechanics as we discussed in Chapter 1.

2.2 Constants of Motion

In classical mechanics, an observable is described by a real-valued function $F(\mathbf{x}, \mathbf{p}, t)$, that is, a function of the state space coordinates and time. The total time derivative of F accounts for all possible variations this observable can have and reads

$$\dot{F} = \frac{\partial F}{\partial x_1}\dot{x}_1 + \cdots + \frac{\partial F}{\partial x_N}\dot{x}_N + \frac{\partial F}{\partial p_1}\dot{p}_1 + \cdots + \frac{\partial F}{\partial p_N}\dot{p}_N + \frac{\partial F}{\partial t} = \sum_{k=1}^{N}\left(\frac{\partial F}{\partial x_k}\dot{x}_k + \frac{\partial F}{\partial p_k}\dot{p}_k\right) + \frac{\partial F}{\partial t}. \tag{2.16}$$

Using Hamilton's equation (2.15), we can relate the observable quantity F and the Hamiltonian H,

$$\dot{F} = \sum_{k=1}^{N}\left(\frac{\partial F}{\partial x_k}\frac{\partial H}{\partial p_k} - \frac{\partial F}{\partial p_k}\frac{\partial H}{\partial x_k}\right) + \frac{\partial F}{\partial t}. \tag{2.17}$$

Therefore, the total time derivative of an observable $F(\mathbf{x}, \mathbf{p}, t)$ can be written in terms of a summation and a partial time derivative. The summation in equation (2.17) is of particular interest in the Hamiltonian formulation of classical mechanics [13].

Definition 2.2. For two real-valued functions $A(\mathbf{x}, \mathbf{p}, t)$ and $B(\mathbf{x}, \mathbf{p}, t)$, the Poisson bracket of A and B with respect to the state space coordinates, denoted by $\{A, B\}$, is defined as

$$\{A, B\} = \sum_{k=1}^{N}\left(\frac{\partial A}{\partial x_k}\frac{\partial B}{\partial p_k} - \frac{\partial A}{\partial p_k}\frac{\partial B}{\partial x_k}\right). \tag{2.18}$$

The Poisson bracket A, B of two real-valued functions $A(\mathbf{x}, \mathbf{p}, t)$ and $B(\mathbf{x}, \mathbf{p}, t)$ is also a real-valued function of the state space coordinates and time. Furthermore, the Poisson bracket satisfies the following properties [13], which are helpful for computations.

Lemma 2.3. *Given the three real-valued functions $A(\mathbf{x},\mathbf{p},t)$, $B(\mathbf{x},\mathbf{p},t)$ and $C(\mathbf{x},\mathbf{p},t)$ and the constants $\alpha,\beta\in\mathbb{R}$, the Poisson bracket satisfies:*

(i) $\{A,A\}=0$*;*

(ii) $\{A,B\}=-\{B,A\}$ *(anticommutativity);*

(iii) $\{\alpha A+\beta B,C\}=\alpha\{A,C\}+\beta\{B,C\}$ *and* $\{C,\alpha A+\beta B\}=\alpha\{C,A\}+\beta\{C,B\}$ *(bilinearity);*

(iv) $\{AB,C\}=\{A,C\}B+A\{B,C\}$ *(derivation);*

(v) $\{A,\{B,C\}\}+\{B,\{C,A\}\}+\{C,\{A,B\}\}=0$ *(Jacobi identity).*

The Poisson brackets of the state space coordinates $\mathbf{x}$ and $\mathbf{p}$ themselves are called the canonical Poisson brackets [13]. One can use Lemma 2.3 to show that, for $i,j=1,\dots,N$,

$$\{x_i,x_j\}=0,\qquad \{p_i,p_j\}=0\qquad\text{and}\qquad \{x_i,p_j\}=\delta_{ij}. \tag{2.19}$$

The canonical Poisson brackets are useful when computing the Poisson bracket of an observable $O(\mathbf{x},\mathbf{p},t)$ and the Hamiltonian H. Furthermore, Poisson brackets are used frequently to describe observables of a particle that are conserved throughout its motion [13].

Definition 2.4. A constant of motion is an observable $O(\mathbf{x},\mathbf{p},t)$ that is conserved throughout the motion, that is, its total time derivative (2.17) is zero.

If the observable $O(\mathbf{x},\mathbf{p},t)$ associated with our particle is a constant of motion, equation (2.17) reads

$$0=\{O,H\}+\frac{\partial O}{\partial t}.$$

Moreover, if the constant of motion O does not depend explicitly on time, then

$$\{O,H\}=0. \tag{2.20}$$

Stated conversely, if the observable $O(\mathbf{x},\mathbf{p})$ does not depend explicitly on time and its Poisson bracket with the Hamiltonian $\{O,H\}$ is zero, then its total time derivative is zero as well and O is a constant of motion. Using this method, it is straightforward to show that energy, which is an observable for our system, is conserved.

Theorem 2.5. *The Hamiltonian H of the harmonic oscillator is a constant of motion.*

Proof. Equation (2.14) shows that the Hamiltonian H does not depend explicitly on time. Furthermore, Lemma 2.3, (i), shows that $\{H,H\}=0$, and therefore the Hamiltonian is a constant of motion. □

Before we derive other constants of motion of the harmonic oscillator, we first analyse the motion of the particle. Let us recall the general solution (2.5) of the differential equation of the simple harmonic oscillator,

$$\mathbf{x}(t)=\mathbf{x}(0)\cos(\omega t)+\frac{1}{\omega}\dot{\mathbf{x}}(0)\sin(\omega t).$$

The vectors $\mathbf{x}(0)$ and $\dot{\mathbf{x}}(0)$ span a plane, in which the motion of the particle occurs [12]. To describe the path of motion in this plane, we confine ourselves to the two-dimensional harmonic oscillator.

For simplicity, let us assume that the vectors $\mathbf{x}(0)$ and $\dot{\mathbf{x}}(0)$ form the same plane that is spanned by the coordinates x_1 and x_2. Then, the solution can alternatively be written as

$$x_1(t)=A_1\cos(\omega t-\varphi_1)\qquad\text{and}\qquad x_2(t)=A_2\cos(\omega t-\varphi_2), \tag{2.21}$$

where for $i=1,2$, we have

$$A_i=\sqrt{x_i^2(0)+\frac{\dot{x}_i^2(0)}{\omega^2}},\qquad \cos(\varphi_i)=\frac{x_i(0)}{A_i}\qquad\text{and}\qquad \sin(\varphi_i)=\frac{\dot{y}_i(0)}{\omega A_i}.$$

Let us introduce the phase difference $\Delta=\varphi_2-\varphi_1$ such that we can write

$$x_2(t)=A_2\cos(\omega t-\varphi_1-\Delta)=A_2[\cos(\omega t-\varphi_1)\cos(\Delta)+\sin(\omega t-\varphi_1)\sin(\Delta)]. \tag{2.22}$$

Then, by using the first equation of (2.21), it follows that we can write

$$\cos(\omega t - \varphi_1) = \frac{x_1(t)}{A_1} \qquad \text{and} \qquad \sin(\omega t - \varphi_1) = \sqrt{1 - \frac{x_1^2(t)}{A_1^2}},$$

and we find that the motion of the particle described by $x_1(t)$ and $x_2(t)$ must satisfy the equation

$$\frac{x_2}{A_2} = \frac{x_1}{A_1}\cos(\Delta) + \sqrt{1 - \frac{x_1^2}{A_1^2}}\sin(\Delta).$$

By squaring and rewriting the result, the motion in the plane satisfies

$$\frac{x_1^2}{A_1^2} - x_1 x_2 \frac{2\cos(\Delta)}{A_1 A_2} + \frac{x_2^2}{A_2^2} = \sin^2(\Delta). \tag{2.23}$$

We recognise equation (2.23) as the formula for an ellipse [12], where the constants A_i correspond to the stretch of the ellipse in the x_i direction. Therefore, for the N-dimensional simple harmonic oscillator, the motion of the particle is given by an elliptical orbit that is confined to the plane spanned by the vectors $\mathbf{x}(0)$ and $\dot{\mathbf{x}}(0)$.

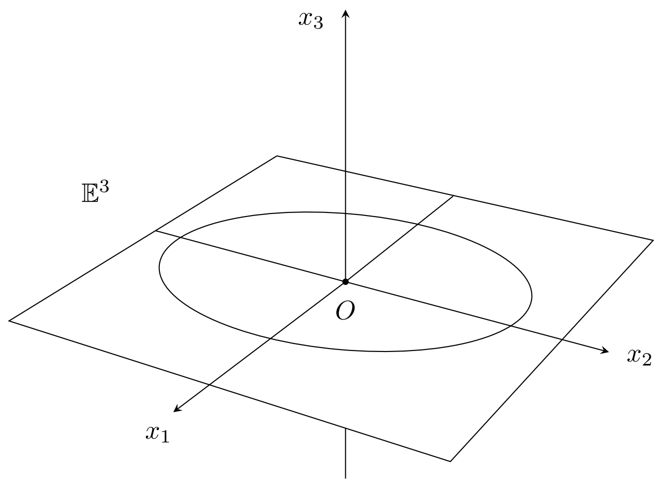

Figure 2.2 Elliptical path of motion in the Euclidean 3-space $\mathbb{E}^3$.

Remark 2.6. While the particle's orbit generally describes an ellipse, if the phase difference Δ is precisely 0 or π, the path of motion reduces to a straight line,

$$x_2 = \pm\frac{A_2}{A_1}x_1.$$

Commonly, the positive sign is assigned to $\Delta = 0$, while the negative sign is taken if $\Delta = \pi$.

In addition to linear momentum, the particle also has orbital angular momentum about the center of its elliptical orbit. The angular momentum tensor L can be represented by a matrix, whose elements L_{ij} are given by

$$L_{ij} = x_i p_j - x_j p_i, \tag{2.24}$$

for $i, j = 1, \ldots, N$. It is clear from equation (2.24) that the matrix elements L_{ij} satisfy the relations $L_{ij} = -L_{ji}$ and $L_{ii} = 0$. Therefore, the matrix L representing the angular momentum tensor is skew-symmetric. The angular momentum of a particle is a well-known example of a constant of motion for the harmonic oscillator.

Theorem 2.7. *The angular momentum tensor L of the harmonic oscillator is a constant of motion.*

Proof. Equation (2.24) shows that the matrix elements L_{ij} do not depend explicitly on time. Hence, it suffices to show that $\{L_{ij}, H\} = 0$. To that end, let us derive the partial derivatives of the Hamiltonian and the angular momentum tensor. For $k = 1, \ldots, N$, we have

$$\frac{\partial H}{\partial p_k} = \frac{\partial}{\partial p_k}\left[\frac{|\mathbf{p}|^2}{2m} + \frac{1}{2}m\omega^2 r^2\right] = \frac{p_k}{m} \qquad \text{and} \qquad \frac{\partial H}{\partial x_k} = \frac{\partial}{\partial x_k}\left[\frac{|\mathbf{p}|^2}{2m} + \frac{1}{2}m\omega^2 r^2\right] = m\omega^2 x_k. \tag{2.25}$$

Furthermore, for $k \neq i, j$, the partial derivatives of the angular momentum tensor are

$$\frac{\partial L_{ij}}{\partial x_k} = \frac{\partial}{\partial x_k}[x_i p_j - x_j p_i] = 0 \qquad \text{and} \qquad \frac{\partial L_{ij}}{\partial p_k} = \frac{\partial}{\partial p_k}[x_i p_j - x_j p_i] = 0,$$

while for $k = i$ and $k = j$, respectively, we find

$$\frac{\partial L_{ij}}{\partial x_i} = \frac{\partial}{\partial x_i}[x_i p_j - x_j p_i] = p_j \qquad \text{and} \qquad \frac{\partial L_{ij}}{\partial p_i} = \frac{\partial}{\partial p_i}[x_i p_j - x_j p_i] = -x_j,$$
$$\frac{\partial L_{ij}}{\partial x_j} = \frac{\partial}{\partial x_j}[x_i p_j - x_j p_i] = -p_i \qquad \text{and} \qquad \frac{\partial L_{ij}}{\partial p_j} = \frac{\partial}{\partial p_j}[x_i p_j - x_j p_i] = x_i.$$

Then, for the Poisson bracket of the angular momentum tensor L_{ij} and the Hamiltonian H, we find

$$\begin{aligned}\{L_{ij}, H\} &= \sum_{k=1}^{N} \left(\frac{\partial L_{ij}}{\partial x_i}\frac{\partial H}{\partial p_i} - \frac{\partial L_{ij}}{\partial p_i}\frac{\partial H}{\partial x_i} \right) = \frac{\partial L_{ij}}{\partial x_i}\frac{\partial H}{\partial p_i} - \frac{\partial L_{ij}}{\partial p_i}\frac{\partial H}{\partial x_i} + \frac{\partial L_{ij}}{\partial x_j}\frac{\partial H}{\partial p_j} - \frac{\partial L_{ij}}{\partial p_j}\frac{\partial H}{\partial x_j} \\ &= p_j \frac{p_i}{m} - x_j m\omega^2 x_i - p_i \frac{p_j}{m} + x_i m\omega^2 x_j = 0,\end{aligned}$$

which concludes the proof. □

Geometrically, the angular momentum tensor L describes the orientation of the plane of motion in Euclidean N-space. For the particle's position $\mathbf{x} \in \mathbb{R}^N$ and momentum $\mathbf{p} \in \mathbb{R}^N$, the angular momentum tensor L can be written as

$$L = \mathbf{x}\mathbf{p}^T - \mathbf{p}\mathbf{x}^T, \tag{2.26}$$

where $\mathbf{x}^T$ and $\mathbf{p}^T$ denote the transpose of the column vectors $\mathbf{x}$ and $\mathbf{p}$, respectively. Since the angular momentum is a constant of motion, we can also write

$$L = \mathbf{x}(0)\mathbf{p}^T(0) - \mathbf{p}(0)\mathbf{x}^T(0),$$

where $\mathbf{x}(0)$ and $\mathbf{p}(0)$ are the initial position and momentum, respectively. Then, for any nonzero vector $\mathbf{v} \in \mathbb{R}^N$ that is perpendicular to the plane of motion, we have

$$L\mathbf{v} = \mathbf{x}(0)(\mathbf{p}(0) \cdot \mathbf{v}) - \mathbf{p}(0)(\mathbf{x}(0) \cdot \mathbf{v}). \tag{2.27}$$

However, since both $\mathbf{x}(0)$ and $\mathbf{p}(0)$ lie in the plane of motion, $\mathbf{v}$ is orthogonal to both $\mathbf{x}(0)$ and $\mathbf{p}(0)$, thus $(\mathbf{x}(0) \cdot \mathbf{v}) = 0$ and $(\mathbf{p}(0) \cdot \mathbf{v}) = 0$. Hence, for any nonzero vector $\mathbf{v} \in \mathbb{R}^N$ orthogonal to the span of the vectors $\mathbf{x}(0)$ and $\mathbf{p}(0)$, we have that

$$\sum_{j=1}^{N} L_{ij} v_j = 0, \tag{2.28}$$

for $i = 1, \ldots, N$. The result above specifies the particle's plane of motion by using the angular momentum tensor L.

Besides the angular momentum tensor L, we can also consider a symmetric tensor S whose matrix elements S_{ij} are given by

$$S_{ij} = \frac{p_i p_j}{2m} + \frac{1}{2} m\omega^2 x_i x_j, \tag{2.29}$$

for $i, j = 1, \ldots, N$. Clearly, we notice that S is symmetric, and the diagonal elements are precisely the terms occurring in the Hamiltonian (2.14), that is,

$$S_{ii} = \frac{p_i^2}{2m} + \frac{1}{2} m\omega^2 x_i^2,$$

and thus, we have

$$\operatorname{tr}(S) = \sum_{i=1}^{N} S_{ii} = \frac{1}{2m}\left(p_1^2 + \cdots + p_N^2\right) + \frac{1}{2} m\omega^2 \left(x_1^2 + \cdots + x_N^2\right) = \frac{|\mathbf{p}|^2}{2m} + \frac{1}{2} m\omega^2 r^2 = H. \tag{2.30}$$

Because the Hamiltonian is a constant of motion, the diagonal components S_{ii} of the symmetric tensor also have vanishing Poisson bracket with the Hamiltonian H. This occurrence leads us to believe that the symmetric tensor is a constant of motion for the harmonic oscillator as well [13].

Theorem 2.8. *The symmetric tensor S of the harmonic oscillator is a constant of motion.*

Proof. Equation (2.29) shows that the matrix elements S_{ij} do not depend explicitly on time. Therefore, it suffices to show that $\{S_{ij}, H\} = 0$. For $k = 1, \dots, N$, the partial derivatives of H are given by equation (2.25). Then, for $k \neq i, j$, the partial derivatives of the symmetric tensor are

$$\frac{\partial S_{ij}}{\partial x_k} = \frac{\partial}{\partial x_k}\left[\frac{p_i p_j}{2m} + \frac{1}{2}m\omega^2 x_i x_j\right] = 0 \qquad \text{and} \qquad \frac{\partial S_{ij}}{\partial p_k} = \frac{\partial}{\partial p_k}\left[\frac{p_i p_j}{2m} + \frac{1}{2}m\omega^2 x_i x_j\right] = 0,$$

while for $k = i$ and $k = j$, respectively, we find

$$\frac{\partial S_{ij}}{\partial x_i} = \frac{\partial}{\partial x_i}\left[\frac{p_i p_j}{2m} + \frac{1}{2}m\omega^2 x_i x_j\right] = \frac{1}{2}m\omega^2 x_j \qquad \text{and} \qquad \frac{\partial S_{ij}}{\partial p_i} = \frac{\partial}{\partial p_i}\left[\frac{p_i p_j}{2m} + \frac{1}{2}m\omega^2 x_i x_j\right] = \frac{p_j}{2m},$$
$$\frac{\partial S_{ij}}{\partial x_j} = \frac{\partial}{\partial x_j}\left[\frac{p_i p_j}{2m} + \frac{1}{2}m\omega^2 x_i x_j\right] = \frac{1}{2}m\omega^2 x_i \qquad \text{and} \qquad \frac{\partial S_{ij}}{\partial p_j} = \frac{\partial}{\partial p_j}\left[\frac{p_i p_j}{2m} + \frac{1}{2}m\omega^2 x_i x_j\right] = \frac{p_i}{2m}.$$

Then, for the Poisson bracket of the symmetric tensor S_{ij} and the Hamiltonian H, we find

$$\begin{aligned}\{S_{ij}, H\} &= \sum_{k=1}^{N}\left(\frac{\partial S_{ij}}{\partial x_i}\frac{\partial H}{\partial p_i} - \frac{\partial S_{ij}}{\partial p_i}\frac{\partial H}{\partial x_i}\right) = \frac{\partial S_{ij}}{\partial x_i}\frac{\partial H}{\partial p_i} - \frac{\partial S_{ij}}{\partial p_i}\frac{\partial H}{\partial x_i} + \frac{\partial S_{ij}}{\partial x_j}\frac{\partial H}{\partial p_j} - \frac{\partial S_{ij}}{\partial p_j}\frac{\partial H}{\partial x_j} \\ &= \frac{1}{2}m\omega^2 x_j \frac{p_i}{m} - \frac{p_j}{2m}m\omega^2 x_i + \frac{1}{2}m\omega^2 x_i \frac{p_j}{m} - \frac{p_i}{2m}m\omega^2 x_j = 0,\end{aligned}$$

which concludes the proof. □

Geometrically, the symmetric tensor S describes the orientation of the elliptical orbit in the plane of motion. To illustrate, let us write equation (2.29) as a linear combination of vector projections [25].

Definition 2.9. The projection of a vector $\mathbf{v}$ onto a nonzero vector $\mathbf{u}$, denoted by $\mathrm{Proj}_{\mathbf{u}}(\mathbf{v})$, is given by

$$\mathrm{Proj}_{\mathbf{u}}(\mathbf{v}) = (\mathbf{u}\cdot\mathbf{v})\mathbf{u}. \tag{2.31}$$

Now, let $\mathbf{v} \in \mathbb{R}^N$ be an arbitrary column vector. We note that using equation (2.31), the projection can also be written as

$$\mathrm{Proj}_{\mathbf{u}}(\mathbf{v}) = |\mathbf{u}|^2(\hat{\mathbf{u}}\cdot\mathbf{v})\hat{\mathbf{u}} = |\mathbf{u}|^2\,\mathrm{Proj}_{\hat{\mathbf{u}}}(\mathbf{v}),$$

where $\hat{\mathbf{u}}$ is the unit vector of $\mathbf{u}$ in our notation.[1] Then, the product $S\mathbf{v}$ of the matrix S representing the symmetric tensor and $\mathbf{v}$ is

$$S\mathbf{v} = \frac{1}{2m}\mathbf{p}\mathbf{p}^T\mathbf{v} + \frac{1}{2}m\omega^2\mathbf{x}\mathbf{x}^T\mathbf{v} = \frac{1}{2m}(\mathbf{p}\cdot\mathbf{v})\mathbf{p} + \frac{1}{2}m\omega^2(\mathbf{x}\cdot\mathbf{v})\mathbf{x}.$$

Using the vector projection, we can write

$$S\mathbf{v} = \frac{|\mathbf{p}|^2}{2m}\,\mathrm{Proj}_{\hat{\mathbf{p}}}(\mathbf{v}) + \frac{1}{2}m\omega^2|\mathbf{x}|^2\,\mathrm{Proj}_{\hat{\mathbf{x}}}(\mathbf{v}), \tag{2.32}$$

where $\hat{\mathbf{x}}$ and $\hat{\mathbf{p}}$ denote the unit position and momentum vectors, respectively.[2] By choosing our time variable appropriately, suppose the particle is positioned at the right vertex of the elliptical orbit at $t = 0$, such that the initial position vector $\mathbf{x}(0)$ and initial momentum vector $\mathbf{p}(0)$ represent the semi-major and semi-minor axes of the elliptical orbit, respectively, and $\mathbf{x}(0)\cdot\mathbf{p}(0) = 0$. Similarly then for the unit vectors $\hat{\mathbf{x}}(0)$ and $\hat{\mathbf{p}}(0)$, we have $\hat{\mathbf{x}}(0)\cdot\hat{\mathbf{p}}(0) = 0$.

Using the equations (2.31) and (2.32), we derive the eigenvalue equations

$$S\mathbf{x}(0) = \frac{1}{2}m\omega^2|\mathbf{x}(0)|^2\mathbf{x}(0), \qquad S\mathbf{p}(0) = \frac{|\mathbf{p}(0)|^2}{2m}\mathbf{p}(0) \qquad \text{and} \qquad S\mathbf{v} = \mathbf{0} = 0\cdot\mathbf{v}, \tag{2.33}$$

for any nonzero vector $\mathbf{v} \in \mathbb{R}^N$ that is perpendicular to the plane of motion. Hence, equation (2.33) shows that the symmetric tensor S has two nonzero eigenvalues with corresponding eigenvectors $\mathbf{x}(0)$ and $\mathbf{p}(0)$, which determine the orientation of the axes of the elliptical orbit.

[1] The notation of a unit vector $\hat{\mathbf{u}}$ we use is not related to the notation of the quantum operator $\hat{O}$ from Chapter 1.
[2] Again, this notation is not related to the position and momentum operators of quantum mechanics.

With the geometric interpretations of the angular momentum tensor L and the symmetric tensor S established, we can now summarise how the particle's motion is dictated by its constants of motion. Let $\mathbf{v} \in \mathbb{R}^N$ by an arbitrary vector that is perpendicular to the particle's motion. Then, the elliptical orbit of the particle is confined to the plane that satisfies the component relation

$$\sum_{j=1}^{N} L_{ij} v_j = 0 \tag{2.34}$$

for $i = 1, \ldots, N$ which involves the angular momentum tensor L, and the orientation and eccentricity of the elliptical orbit is described by the eigenvectors of the symmetric tensor S that lie in the plane of motion.

2.3 Lie Algebras and Symmetry Groups

The conservation of the Hamiltonian, the angular momentum tensor and the symmetric tensor can be used to understand the symmetry of the harmonic oscillator in Euclidean N-space. To illustrate this, we first provide the following algebraic connection between the angular momentum tensor and the Poisson bracket of the symmetric tensor with itself.

Theorem 2.10. *The Poisson bracket of the symmetric tensor S with itself is given by*

$$\{S_{ij}, S_{kl}\} = \frac{\omega^2}{4}(L_{ik}\delta_{jl} + L_{il}\delta_{jk} + L_{jk}\delta_{il} + L_{jl}\delta_{ik}), \tag{2.35}$$

for $i, j, k, l = 1, \ldots, N$.

Proof. To confine our notation, let us introduce the abbreviated notation

$$\mathcal{L} = L_{ik}\delta_{jl} + L_{il}\delta_{jk} + L_{jk}\delta_{il} + L_{jl}\delta_{ik}.$$

With these shortened forms, equation (2.35) can simply be written as

$$\{S_{ij}, S_{kl}\} = \frac{\omega^2 \mathcal{L}}{4}. \tag{2.36}$$

To show that this holds, we consider some cases. First of all, suppose that all indices are distinct. Using equation (2.18), the left-hand side of (2.36) reads

$$\{S_{ij}, S_{kl}\} = \sum_{n=1}^{N} \left(\frac{\partial S_{ij}}{\partial x_n} \frac{\partial S_{kl}}{\partial p_n} - \frac{\partial S_{ij}}{\partial p_n} \frac{\partial S_{kl}}{\partial x_n} \right) = 0,$$

while for the right-hand side, we have $\mathcal{L} = 0$ due to the Kronecker deltas. Thus, it is clear that equation (2.36) holds. Second, suppose that one pair of indices are equal and that the remaining two are distinct. Without loss of generality, we assume that $i = k$ and $j \neq l$. Again with equation (2.18), the left-hand side of equation (2.36) is given by

$$\{S_{ij}, S_{kl}\} = \{S_{ij}, S_{il}\} = \sum_{n=1}^{N} \left(\frac{\partial S_{ij}}{\partial x_n} \frac{\partial S_{il}}{\partial p_n} - \frac{\partial S_{ij}}{\partial p_n} \frac{\partial S_{il}}{\partial x_n} \right) = \frac{\partial S_{ij}}{\partial x_i} \frac{\partial S_{il}}{\partial p_i} - \frac{\partial S_{ij}}{\partial p_i} \frac{\partial S_{il}}{\partial x_i}.$$

Furthermore, evaluating the right-hand side yields

$$\frac{\omega^2 \mathcal{L}}{4} = \frac{\omega^2 L_{jl}}{4} = \frac{\omega^2}{4}(x_j p_l - x_l p_j) = \frac{1}{2} m\omega^2 x_j \frac{p_l}{2m} - \frac{p_j}{2m} \frac{1}{2} m\omega^2 x_l = \frac{\partial S_{ij}}{\partial x_i} \frac{\partial S_{il}}{\partial p_i} - \frac{\partial S_{ij}}{\partial p_i} \frac{\partial S_{il}}{\partial x_i}.$$

Therefore, equation (2.36) holds. Finally, suppose that two pair of indices are equal. Without loss of generality, we assume that $i = k$ and $j = l$. By Lemma 2.3, (i) the left-hand side of (2.36) reads

$$\{S_{ij}, S_{kl}\} = \{S_{ij}, S_{ij}\} = 0.$$

For the right-hand side, using the Kronecker deltas and the skew-symmetry of the angular momentum tensor L, we have

$$\mathcal{L} = L_{ii} \cdot 1 + L_{ij} \cdot 0 + L_{ji} \cdot 0 + L_{jj} \cdot 1 = 0.$$

Therefore, it follows that equation (2.36) holds. This concludes the proof. $\square$

The continuous symmetric behaviour of a dynamical system can be represented by a Lie group [6], which is closely related to a Lie algebra [10]. Before we analyse the symmetry of the harmonic oscillator, let us introduce these two related concepts.

Definition 2.11. A real Lie group is a set G with a group structure and a real manifold structure such that the mapping

$$G \times G \to G, \qquad (x, y) \mapsto xy^{-1}$$

is a smooth mapping from the product manifold to G.

Definition 2.12. A complex Lie algebra is a complex vector space $\mathfrak{g}$ together with a mapping

$$\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, \qquad (x, y) \mapsto [x, y],$$

called the Lie bracket, that satisfies, for all $x, y, z \in \mathfrak{g}$ and all $\alpha, \beta \in \mathbb{C}$:

(i) $[x, x] = 0$;

(ii) $[\alpha x + \beta y, z] = \alpha[x, z] + \beta[y, z]$ and $[x, \alpha y + \beta z] = \alpha[x, y] + \beta[x, z]$ (bilinearity);

(iii) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ (Jacobi identity).

We note that the mapping associated with the Lie algebra $\mathfrak{g}$ has similar characteristics as the Poisson bracket, as we discussed in Lemma 2.3 before.[3] Furthermore, we can associate a set of generators with the Lie group G. These generators specify a special class of Lie groups [13].

Definition 2.13. The Lie group G whose generators can be identified with the constants of motion is called the dynamical group of the system.

For the simple harmonic oscillator, the dynamical group associated with the Hamiltonian, the angular momentum tensor and the symmetric tensor is the special unitary group SU(N) of degree N,

$$\mathrm{SU}(N) = \{M \in \mathrm{GL}(N, \mathbb{C}) \mid M^\dagger M = I,\ \det(M) = 1\}. \tag{2.37}$$

The generators [18] of the dynamical group SU(N) are the angular momentum tensor L and a traceless symmetric tensor T that satisfy

$$\{T_{ij}, H\} = 0 \qquad \text{and} \qquad \{T_{ij}, T_{kl}\} = L_{ij}\delta_{jl} + L_{il}\delta_{jk} + L_{jk}\delta_{il} + L_{jl}\delta_{ik}. \tag{2.38}$$

To derive this tensor, we note the resemblance between the second part of equation (2.38) and (2.35).

Proposition 2.14. *The traceless symmetric tensor T that satisfies equation* (2.38) *is given by*

$$T_{ij} = \frac{2S_{ij}}{\omega} - \frac{2H}{N\omega}\delta_{ij}, \tag{2.39}$$

for $i, j = 1, \ldots, N$.

Proof. To show that the first half of equation (2.38) holds for the traceless symmetric tensor given by equation (2.39), we use the properties of the Poisson bracket we listed in Lemma 2.3. Since the symmetric tensor is a constant of motion, we find

$$\{T_{ij}, H\} = \left\{\frac{2S_{ij}}{\omega} - \frac{2H}{N\omega}\delta_{ij}, H\right\} = \frac{2}{\omega}\{S_{ij}, H\} - \frac{2\delta_{ij}}{N\omega}\{H, H\} = 0.$$

We use similar argument to show that the second half of equation (2.38) holds for the traceless symmetric tensor (2.39). Computing the Poisson bracket of T with itself yields

$$\begin{aligned}\{T_{ij}, T_{kl}\} &= \left\{\frac{2S_{ij}}{\omega} - \frac{2H}{N\omega}\delta_{ij}, \frac{2S_{kl}}{\omega} - \frac{2H}{N\omega}\delta_{kl}\right\} \\ &= \frac{4}{\omega^2}\{S_{ij}, S_{kl}\} + \frac{4\delta_{ij}}{N\omega^2}\{S_{kl}, H\} - \frac{4\delta_{kl}}{N\omega^2}\{S_{ij}, H\} + \frac{4\delta_{ij}\delta_{kl}}{N^2\omega^2}\{H, H\} \\ &= L_{ik}\delta_{jl} + L_{il}\delta_{jk} + L_{kl}\delta_{il} + L_{jl}\delta_{ik},\end{aligned}$$

where we used the algebra given by equation (2.35) in the last line. ☐

[3] Likewise, the Lie algebra $\mathfrak{g}$ has similar characteristics as the commutator, as we discussed in Lemma 1.9.

From the matrix elements S_{ij} of the symmetric tensor, we can construct two scalars [18], namely

$$I_1 = \operatorname{tr}(S) = H \qquad \text{and} \qquad I_2 = \operatorname{tr}(S^2) - \operatorname{tr}(S)^2 = -2\omega^2|L|^2. \tag{2.40}$$

Since the trace of a matrix equals the sum of its eigenvalues, it follows that the scalars I_1 and I_2 are independent. Furthermore, we can derive another scalar I_3 with these two scalars,

$$I_3 = \sum_{i=1}^{N}\sum_{j=1}^{N} T_{ij}T_{ji} = \frac{4(N-1)H^2}{N\omega^2} - 2|L|^2, \tag{2.41}$$

which is the only independent scalar we can construct from the matrix elements T_{ij} of the traceless symmetric tensor T. We can use the scalar given by equation (2.41) to express the Hamiltonian H as a function of the Casimir function C associated with the dynamical group $\mathrm{SU}(N)$, that is,

$$H = \omega\sqrt{C}, \tag{2.42}$$

where the quadratic Casimir function C satisfies the relation

$$\frac{4(N-1)}{N}C = \sum_{i=1}^{N}\sum_{j=1}^{N}(T_{ij}T_{ij} + L_{ij}L_{ij}). \tag{2.43}$$

In the upcoming chapters, we use the relation between the Hamiltonian H and the quadratic Casimir function C to understand the rotational symmetry of the harmonic oscillator.

2.4 Quantum Dynamics in the Euclidean N-space

In quantum mechanics, the state of the particle is described by the quantum state $|\Psi(t)\rangle$, an element of the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^N)$. The quantum state is the solution of the Schrödinger equation,

$$i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle = \hat{H}|\Psi(t)\rangle, \tag{2.44}$$

where $\hat{H}$ is the Hamiltonian operator. For the quantum harmonic oscillator, the Hamiltonian operator is given by

$$\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + \frac{1}{2}m\omega^2 r^2. \tag{2.45}$$

With the quantum state $|\Psi(t)\rangle$, we associate a wave function $\Psi(\mathbf{x}, t)$ such that equation (2.44) reads

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi + \frac{1}{2}m\omega^2 r^2\Psi. \tag{2.46}$$

The general solution to the Schrödinger equation (2.46), and thus a representation for the quantum state of the particle,[4] is given by

$$\Psi(\mathbf{x}, t) = \sum_{n=1}^{\infty} c_n\psi_n(\mathbf{x})\exp\left(-\frac{iE_n t}{\hbar}\right). \tag{2.47}$$

The constants c_n can be determined using the initial state $\Psi(\mathbf{x}, 0)$, and the functions $\psi_n(\mathbf{x})$ with their associated energy E_n are solutions to the time-independent Schrödinger equation,

$$-\frac{\hbar^2}{2m}\nabla^2\psi + \frac{1}{2}m\omega^2 r^2\psi = E\psi. \tag{2.48}$$

Once we have found the solution set $\{\psi_n\}$ of the time-independent Schrodinger equation together with the associated energies $\{E_n\}$, we can determine the quantum state $|\Psi(t)\rangle$ by constructing the associated wave function $\Psi(\mathbf{x}, t)$ given by equation (2.47).

To solve the time-independent Schrödinger equation for the quantum harmonic oscillator, we use the method of seperation of variables [24], that is, we seek a solution of the particular form

$$\psi(\mathbf{x}) = X_{x_1}(x_1)\cdots X_{x_N}(x_N). \tag{2.49}$$

[4] For a more-detailed derivation of these results, we refer to Section 1.5.

Substituting this in equation (2.49) gives

$$-\frac{\hbar^2}{2m}\left(X_{x_2}\cdots X_{x_N}\frac{\partial^2 X_{x_1}}{\partial x_1^2}+\cdots+X_{x_1}\cdots X_{x_{N-1}}\frac{\partial^2 X_{x_N}}{\partial x_N^2}\right)+\frac{1}{2}m\omega^2 r^2 X_{x_1}\cdots X_{x_N}=EX_{x_1}\cdots X_{x_N}.$$

Dividing by $X_{x_1}\cdots X_{x_N}$ and using equation (2.12) yields

$$-\frac{\hbar^2}{2m}\left(\frac{1}{X_{x_1}}\frac{\partial^2 X_{x_1}}{\partial x_1^2}+\cdots+\frac{1}{X_{x_N}}\frac{\partial^2 X_{x_N}}{\partial X_N^2}\right)+\frac{1}{2}m\omega^2\left(x_1^2+\cdots+x_N^2\right)=E.$$

Now, if we collect terms with common variables, the equation reads

$$\left(-\frac{\hbar^2}{2m}\frac{1}{X_{x_1}}\frac{\partial^2 X_{x_1}}{\partial x_1^2}+\frac{1}{2}m\omega^2 x_1^2\right)+\cdots+\left(-\frac{\hbar^2}{2m}\frac{1}{X_{x_N}}\frac{\partial^2 X_{x_N}}{\partial x_N^2}+\frac{1}{2}m\omega^2 x_N^2\right)=E. \tag{2.50}$$

Because the energy E is constant, all terms on the left-hand side of equation (2.50) are constant as well. Consequently, let us write

$$E=E_{x_1}+\cdots+E_{x_N}, \tag{2.51}$$

such that we have

$$-\frac{\hbar^2}{2m}\frac{1}{X_{x_i}}\frac{\partial^2 X_{x_i}}{\partial x_i^2}+\frac{1}{2}m\omega^2 x_i^2=E_{x_i}, \tag{2.52}$$

for $i=1,\ldots,N$. Multiplying both sides of equation (2.52) by X_{x_i} gives a familiar equation, namely

$$-\frac{\hbar^2}{2m}\frac{\partial^2 X_{x_i}}{\partial x_i^2}+\frac{1}{2}m\omega^2 x_i^2 X_{x_i}=E_{x_i}X_{x_i}. \tag{2.53}$$

This is precisely the one-dimensional time-independent Schrödinger equation. Therefore, we have found that solving the N-dimensional time-independent Schrödinger equation (2.48) is equivalent to solving equation (2.53) for $i=1,\ldots,N$.

2.5 The 1-Dimensional Quantum Harmonic Oscillator

Let $x\in\mathbb{R}$ denote the position of the particle. For the one-dimensional quantum harmonic oscillator, our goal is to solve the Schrödinger equation,

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}+\frac{1}{2}m\omega^2x^2\psi=E\psi. \tag{2.54}$$

We solve it directly by the power series method [28]. To clean up the differential equation, we introduce a new dimensionless variable,

$$\xi=\sqrt{\frac{m\omega}{\hbar}}x=\alpha x,$$

such that equation (2.54) reads

$$\frac{d^2\psi}{d\xi^2}=(\xi^2-\mathcal{E})\psi, \tag{2.55}$$

where $\mathcal{E}$ is the dimensionless version of the energy E,

$$\mathcal{E}=\frac{2E}{\hbar\omega}. \tag{2.56}$$

Remark 2.15. Note that as ξ grows without bound, the term ξ^2 dominates equation (2.55) completely. Then, we have the approximation

$$\frac{d^2\psi}{d\xi^2}\approx\xi^2\psi,$$

which yields an approximate solution

$$\psi(\xi)\approx Ae^{-\xi^2/2}+Be^{\xi^2/2}.$$

Normalisation requires $B=0$, which implies that the wave function must approximate the form

$$\psi(\xi)\approx e^{-\xi^2/2}$$

for sufficiently large ξ.

Using this observation, our change of variables suggests that the exact solution of the differential equation (2.55) is of the form

$$\psi(\xi) = H(\xi)e^{-\xi^2/2}, \tag{2.57}$$

where $H(\xi)$ is a polynomial.[5] Substituting the derivatives of (2.57) in equation (2.55) yields

$$\frac{d^2H}{d\xi^2} - 2\xi\frac{dH}{d\xi} + (\mathcal{E} - 1)H = 0. \tag{2.58}$$

To solve this differential equation, we use the power series representation [24] of the polynomial H,

$$H(\xi) = a_0 + a_1\xi + a_2\xi^2 + \cdots = \sum_{j=0}^{\infty} a_j\xi^j. \tag{2.59}$$

Differentiating the series term by term and substituting the results in equation (2.58) gives

$$\sum_{j=0}^{\infty} \big((j+1)(j+2)a_{j+2} - (2j+1-\mathcal{E})a_j\big)\xi^j = 0. \tag{2.60}$$

By the uniqueness of the power series expansions [2], it follows that the coefficient of each power ξ must vanish,

$$(j+1)(j+2)a_{j+2} - (2j+1-\mathcal{E})a_j = 0,$$

or equivalently,

$$a_{j+2} = \frac{2j+1-\mathcal{E}}{(j+1)(j+2)}a_j. \tag{2.61}$$

This recursion formula gives the coefficients of the power series representation of our polynomial H that satisfies the differential equation (2.58).

Remark 2.16. Let us investigate the asymptotic behaviour of the polynomial $H(\xi)$. Using (2.61), we find the limit

$$\lim_{j\to\infty} \frac{j}{2}\frac{a_{j+2}}{a_j} = \lim_{j\to\infty} \frac{2j^2 + j(1-\mathcal{E})}{2(j+1)(j+2)} = \lim_{j\to\infty} \frac{2 + (1-\mathcal{E})/j}{2(1+1/j)(1+2/j)} = 1.$$

Hence, a_{j+2}/a_j behaves like $2/j$ as j grows without bound. Now, consider the function $f(\xi) = e^{\xi^2}$, for which we have the series representation

$$f(\xi) = e^{\xi^2} = \sum_{j=0}^{\infty} \frac{\xi^{2j}}{j!} = 1 + \xi^2 + \frac{\xi^4}{2} + \cdots + \frac{\xi^n}{(n/2)!} + \frac{\xi^{n+2}}{(n/2+1)!} + \cdots,$$

where n is even. Using a similar approach for the asymptotic behaviour of f, we find the limit

$$\lim_{k\to\infty} \frac{k}{2}\frac{a_{k+2}}{a_k} = \lim_{k\to\infty} \frac{k}{2}\frac{(k/2)!}{(k/2+1)!} = \lim_{k\to\infty} \frac{k}{k+2} = \lim_{k\to\infty} \frac{1}{1+2/k} = 1,$$

thus a_{k+2}/a_k behaves like $2/k$ as k grows without bound. Hence, we conclude that the polynomial $H(\xi)$ and f have similar asymptotic behaviour [11]. Consequently, equation (2.57) gives

$$\psi(\xi) \approx e^{\xi^2/2},$$

which is not normalisable. Therefore the power series of the polynomial $H(\xi)$ must terminate.

The power series representation of the polynomial $H(\xi)$ can only terminate if there exists some $j = n$ such that the recursion formula (2.61) returns $a_{n+2} = 0$. This can only happen whenever

$$\mathcal{E} = 2n + 1, \tag{2.62}$$

for some non-negative integer n called the principal quantum number [15]. Returning to (2.56), we find that the energy E is quantised in energy levels,

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right). \tag{2.63}$$

[5] The notation for our polynomial $H(\xi)$ is not related to the Hamiltonian H previously used.

Furthermore, the condition provided by equation (2.62) quantifies the differential equation (2.58),

$$\frac{d^2 H_n}{d\xi^2} - 2\xi \frac{dH_n}{d\xi} + 2nH_n = 0. \tag{2.64}$$

The solution $H_n(\xi)$ is called a Hermite polynomial [2]. This is a polynomial of degree n in ξ, involving even powers only if n is even, and odd powers only if n is odd. Furthermore, the coefficients of $H_n(\xi)$ alternate and the leading coefficient has the form 2^n. The first few Hermite polynomials $H_n(\xi)$ are listed in Table 2.1.

$H_0(\xi) = 1$
$H_1(\xi) = 2\xi$
$H_2(\xi) = 4\xi^2 - 2$
$H_3(\xi) = 8\xi^3 - 12\xi$
$H_4(\xi) = 16\xi^4 - 48\xi^2 + 12$
$H_5(\xi) = 32\xi^5 - 160\xi^3 + 120\xi$
$H_6(\xi) = 64\xi^6 - 480\xi^4 + 720\xi^2 - 120$

Table 2.1 The first few Hermite polynomials, $H_n(\xi)$.

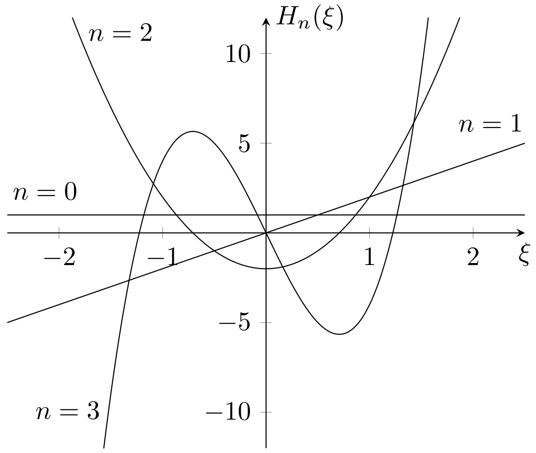

Figure 2.3 The first few Hermite polynomials, $H_n(\xi)$, graphed.

There exist many different methods to compute the nth Hermite polynomial [2], one of which is given by Rodrigues' formula,

$$H_n(\xi) = (-1)^n e^{\xi^2} \frac{d^n}{d\xi^n} \left[e^{-\xi^2} \right]. \tag{2.65}$$

Now that we have solved the Hermite differential equation (2.64), we also have a solution to the time-independent Schrödinger equation for the one-dimensional quantum harmonic oscillator. Expressed in our original variable x rather than ξ, we have

$$\psi(x) = N_n H_n \left(\sqrt{\frac{m\omega}{\hbar}} x \right) \exp\left(-\frac{m\omega}{2\hbar} x^2 \right), \tag{2.66}$$

where N_n is the normalisation constant. Before computing N_n, we show that the Hermite polynomials are orthogonal polynomials [19].

Proposition 2.17. *The Hermite polynomials $H_m(\xi)$ and $H_n(\xi)$ are orthogonal with respect to the weight function $w(\xi) = e^{-\xi^2}$ over $\mathbb{R}$, that is,*

$$\int_{-\infty}^{\infty} H_m(\xi) H_n(\xi) w(\xi) d\xi = \begin{cases} 2^n n! \sqrt{\pi} & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}$$

Proof. Without loss of generality, let $m \le n$. We have the following recursive formula [2] for $H_n(\xi)$,

$$H_n'(\xi) = 2nH_{n-1}(\xi). \tag{2.67}$$

Let $w^{(n)}(\xi)$ denote the nth derivative of the weight function. Then,

$$\int_{-\infty}^{\infty} H_m(\xi)H_n(\xi)w(\xi)d\xi = \int_{-\infty}^{\infty} H_m(\xi)(-1)^n e^{\xi^2}\frac{d^n}{d\xi^n}\left[e^{-\xi^2}\right]e^{-\xi^2}d\xi = (-1)^n\int_{-\infty}^{\infty} H_m(\xi)w^{(n)}(\xi)d\xi.$$

To evaluate the latter integral, we use integration by parts. Note that

$$\begin{aligned}\int_{-\infty}^{\infty} H_m(\xi)w^{(n)}(\xi)d\xi &= H_m(\xi)w^{(n-1)}(\xi)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} 2mH_{m-1}(\xi)w^{(n-1)}(\xi)d\xi\\ &= -2m\int_{-\infty}^{\infty} H_{m-1}(\xi)w^{(n-1)}(\xi)d\xi.\end{aligned}$$

Hence, after integrating by parts m times, we obtain

$$\int_{-\infty}^{\infty} H_m(\xi)w^{(n)}(\xi)d\xi = (-1)^m 2^m m!\int_{-\infty}^{\infty} H_0(\xi)w^{(n-m)}(\xi)d\xi = (-1)^m 2^m m!\int_{-\infty}^{\infty} w^{(n-m)}(\xi)d\xi,$$

since $H_0(\xi) = 1$. Now, if $m = n$, we find

$$\int_{-\infty}^{\infty} H_n^2(\xi)w(\xi)d\xi = (-1)^{2n}2^n n!\int_{-\infty}^{\infty} w(\xi)d\xi = 2^n n!\int_{-\infty}^{\infty} e^{-x^2}d\xi = 2^n n!\sqrt{\pi},$$

whereas if $m < n$, after integrating once more, we find

$$\int_{-\infty}^{\infty} H_m(\xi)H_n(\xi)w(\xi)d\xi = (-1)^{n+m}2^m m!\int_{-\infty}^{\infty} w^{(n-m)}(\xi)d\xi = (-1)^{n+m}2^m m!\left[w^{(n-m-1)}(\xi)\Big|_{-\infty}^{\infty}\right] = 0,$$

since $w(\xi)$ and its derivatives go to zero as $\xi \to \pm\infty$. □

Using the orthogonality of Hermite polynomials, we can compute the normalisation constant N_n,

$$\int_{-\infty}^{\infty} |\psi_n(x)|^2 dx = |N_n|^2\int_{-\infty}^{\infty} H_n^2(\alpha x)e^{-\alpha^2x^2}dx = \frac{|N_n|^2}{\alpha}\int_{-\infty}^{\infty} H_n^2(\xi)e^{-\xi^2}d\xi = \frac{|N_n|^2}{\alpha}2^n n!\sqrt{\pi} = 1,$$

and thus

$$N_n = \sqrt{\frac{\alpha}{\sqrt{\pi}}\frac{1}{2^n n!}} = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}. \tag{2.68}$$

Finally, the normalised stationary states $\psi_n(x)$ for the quantum harmonic oscillator are given by

$$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right)\exp\left(-\frac{m\omega}{2\hbar}x^2\right). \tag{2.69}$$

The first few stationary states $\psi_n(x)$, for which we have used the convention $m = \omega = \hbar = 1$, together with their corresponding probability densities $\psi_n^2(x)$ are shown in Figure 2.4.

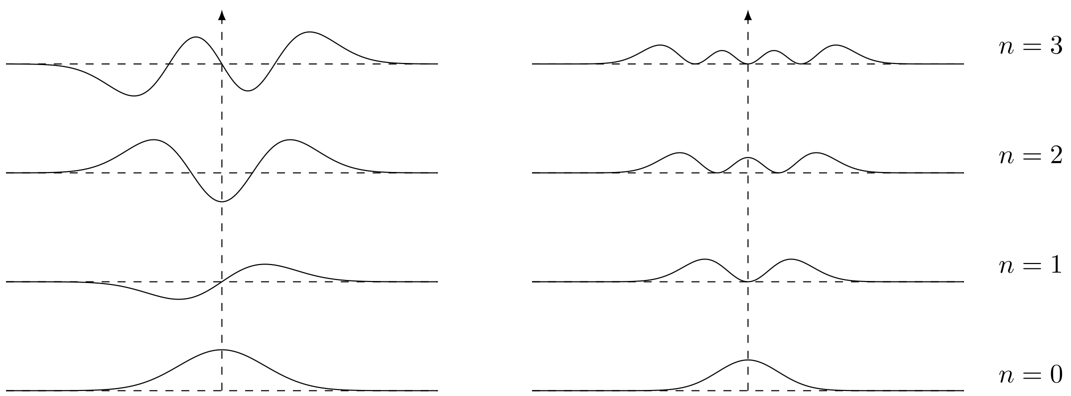

Figure 2.4 The first few stationary states $\psi_n(x)$ and corresponding probability densities $\psi_n^2(x)$ graphed.

To construct the stationary states for the N-dimensional quantum harmonic oscillator, we combine our results from section 2.4 and the stationary states we found for the 1-dimensional quantum harmonic oscillator.

2.6 The *N*-Dimensional Quantum Harmonic Oscillator

As we discussed in section 2.4, the eigenstates of the time-independent Schrödinger equation (2.48) are the product of N eigenstates for the 1-dimensional quantum harmonic oscillator (2.49), and the eigenvalues are the sum of N eigenvalues (2.51). To start with the latter, our result from section 2.5 shows that we have

$$E = E_{n_1} + \cdots + E_{n_N} = \hbar\omega\left(n_1 + \frac{1}{2}\right) + \cdots + \hbar\omega\left(n_N + \frac{1}{2}\right), \tag{2.70}$$

where $n_1, \ldots, n_N$ denote the principal quantum numbers corresponding to the components of Euclidean N-space. To summarise all this information, let us define the principal quantum number n by

$$n = n_1 + \cdots + n_N. \tag{2.71}$$

Then, the energy levels E_n of the N-dimensional quantum harmonic oscillator can compactly be written as

$$E_n = \hbar\omega\left(n + \frac{N}{2}\right). \tag{2.72}$$

Using the same convention for the principal quantum number n, the eigenfunctions $\psi_n(\mathbf{x})$ of the time-independent Schrödinger equation are given by

$$\psi_n(\mathbf{x}) = \left(\frac{m\omega}{\pi\hbar}\right)^{N/4} \prod_{i=1}^{N} \frac{1}{\sqrt{2^{n_i} n_i!}} H_{n_i}\left(\sqrt{\frac{m\omega}{\hbar}} x_i\right) \exp\left(-\frac{m\omega}{2\hbar} x_i^2\right). \tag{2.73}$$

Using the properties of exponentials and square roots together with $r^2 = x_1^2 + \cdots + x_N^2$ as is given by equation (2.12), this can also be written as

$$\psi_n(\mathbf{x}) = \left(\frac{m\omega}{\pi\hbar}\right)^{N/4} \frac{1}{\sqrt{2^n n_1! \cdots n_N!}} H_{n_1}\left(\sqrt{\frac{m\omega}{\hbar}} x_1\right) \cdots H_{n_N}\left(\sqrt{\frac{m\omega}{\hbar}} x_N\right) \exp\left(-\frac{m\omega}{2\hbar} r^2\right). \tag{2.74}$$

With these solutions, we are able to construct the general solution given by (2.47) of the time-dependent Schrödinger equation (2.46).

3 Harmonic Oscillators in N-Spheres

Now that we have established both the classical and quantum harmonic oscillator in Euclidean N-space in Chapter 2, we proceed with our first generalisation of space curvature, that is, the simple harmonic oscillator in N-dimensional positively curved spherical space.

In Section 3.1, we use gnomonic projection to formulate the spherical dynamics as Euclidean dynamics and derive the corresponding Hamiltonian. Then, we describe the particle's motion in the N-sphere in Section 3.2 and we discuss its constants of motion and their relation to the quadratic Casimir function. Subsequently in Section 3.3, the classical dynamics are formulated according to quantum mechanics, upon which we solve the Schrödinger equation in Section 3.4.

3.1 Classical Dynamics in the N-Sphere

Consider a particle of mass m moving in the N-dimensional sphere[1] $\mathbb{S}^N$ of radius $R > 0$ embedded in the $(N+1)$-dimensional Euclidean space $\mathbb{E}^{N+1}$. Using cartesian coordinates, the position of the particle at the point $Q \in \mathbb{S}^N$ is represented by the position vector $\mathbf{q} = (q_0, q_1, \ldots, q_N) \in \mathbb{R}^{N+1}$. To compute distances in the N-sphere, we can use the Euclidean metric tensor $g_{ij} = \delta_{ij}$, because $\mathbb{S}^N$ is embedded in $\mathbb{E}^{N+1}$. Hence, we use the spherical metric[2]

$$ds^2 = dq_0^2 + dq_1^2 + \cdots + dq_N^2. \tag{3.1}$$

Instead of positioning the center C of the N-sphere at the origin O, we let its south pole coincide with the origin, such that

$$(q_0 - R)^2 + q_1^2 + \cdots + q_N^2 = R^2. \tag{3.2}$$

The vector $\mathbf{c} = (R, 0, \ldots, 0) \in \mathbb{R}^{N+1}$ represents the position of the center C. Then, the tangent space of the N-sphere at the origin spans the Euclidean space $\mathbb{E}^N$,

$$T_{\mathbf{0}}\mathbb{S}^N = \left\{(0, x_1, \ldots, x_N) \in \mathbb{R}^{N+1} \mid x_1, \ldots, x_N \in \mathbb{R}\right\}. \tag{3.3}$$

Let us restrict the movement of the particle to the southern hemisphere of the N-sphere,

$$\mathbb{S}^N_- = \left\{\mathbf{q} \in \mathbb{R}^{N+1} \mid (q_0 - R)^2 + q_1^2 + \cdots + q_N^2 = R^2,\ q_0 - R < 0\right\}, \tag{3.4}$$

for which the tangent space at the origin $T_{\mathbf{0}}\mathbb{S}^N$ remains unchanged. This restriction allows us to connect motion in the N-sphere with motion in Euclidean N-space.

With every point $Q \in \mathbb{S}^N_-$ we can associate a point $X \in T_{\mathbf{0}}\mathbb{S}^N$ by using gnomonic projection [8]. We draw radially a straight line from the center C to the point Q, whose extension intersects the tangent space at the point X. Similarly, the position of the particle at the point X is represented by the position vector $\mathbf{x} = (0, x_1, \ldots, x_N) \in \mathbb{R}^{N+1}$.

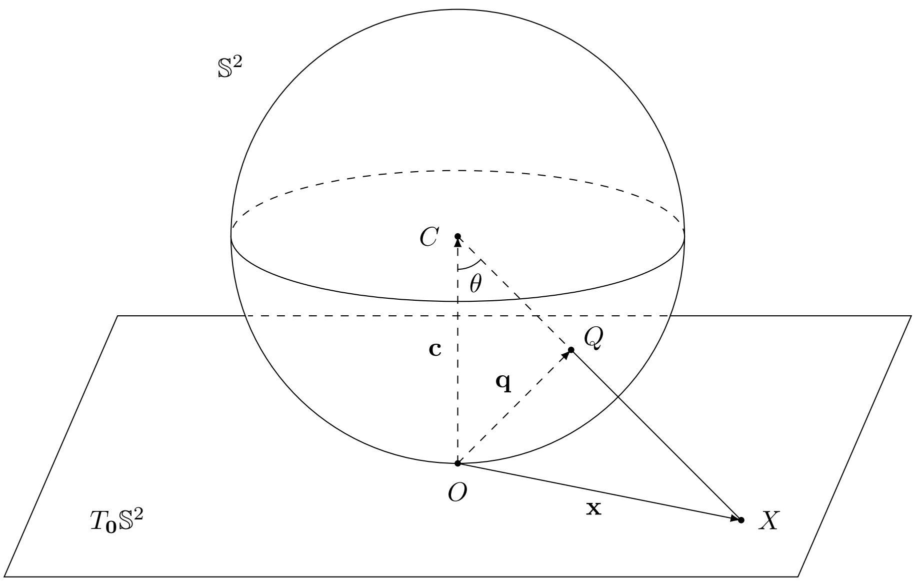

Figure 3.1 Gnomonic projection of Q in the southern hemisphere of $\mathbb{S}^2$ to X in the tangent space $T_{\mathbf{0}}\mathbb{S}^2$.

[1] To confine notation, we let $\mathbb{S}^N$ denote the N-sphere for any radius $R > 0$, rather than the unit N-sphere specifically.
[2] That is, the metric restricted to curves in the N-sphere.

We can use the construction given by gnomonic projection to derive a relation between the points Q and X. The line between Q and the center C can be written as a vector equation, which intersects the tangent space at X for some $\lambda \in \mathbb{R}$, that is,

$$\mathbf{x} = \mathbf{c} + \lambda(\mathbf{q} - \mathbf{c}).$$

Using our expressions for the position vectors $\mathbf{c}$, $\mathbf{q}$ and $\mathbf{x}$, this vector equation can be written as a system of $N+1$ equations, that is,

$$\begin{cases} 0 = R + \lambda(q_0 - R), \\ x_i = \lambda q_i, \end{cases} \tag{3.5}$$

for $i = 1, \ldots, N$. The first equation of this system can be used to derive an expression for λ,

$$\lambda = -\frac{R}{q_0 - R} = \frac{R}{R - q_0} > 0,$$

which is positive due to the restriction $q_0 - R < 0$. Substituting this result in the other N equations of the system, we find the relations

$$x_i = \frac{R}{R - q_0} q_i \qquad \text{and} \qquad q_i = \frac{R - q_0}{R} x_i. \tag{3.6}$$

Now, combining equation (3.2) and the second part of equation (3.6) yields

$$(q_0 - R)^2 + \frac{(R - q_0)^2}{R^2}\left(x_1^2 + \cdots + x_N^2\right) = R^2,$$

from which we then derive

$$R - q_0 = \frac{R^2}{\sqrt{R^2 + x_1^2 + \cdots + x_N^2}}.$$

Writing $r^2 = x_1^2 + \cdots + x_N^2$ as in equation (2.12), the coordinates q_0 and q_i of the spherical point Q in relation to the projection point X are given by

$$q_0 = R - \frac{R^2}{\sqrt{R^2 + r^2}} \qquad \text{and} \qquad q_i = \frac{R x_i}{\sqrt{R^2 + r^2}}. \tag{3.7}$$

To formulate the equations of motion of the particle in the N-sphere, we use Hamiltonian mechanics rather than Newton's laws of motion. Moreover, we use the relations between Q and X given by equation (3.7) to formulate the Hamiltonian in terms of $\mathbf{x}$ rather than in terms of $\mathbf{q}$. To begin with, the kinetic energy of the particle in terms of $\mathbf{q}$ is

$$K = \frac{1}{2} m |\dot{\mathbf{q}}|^2 = \frac{1}{2} m \left(\dot{q}_0^2 + \dot{q}_1^2 + \cdots + \dot{q}_N^2\right).$$

Substituting the relations from equation (3.7), we obtain the kinetic energy in terms of $\mathbf{x}$,

$$K = \frac{1}{2} m \left(\frac{R^2 |\dot{\mathbf{x}}|^2}{R^2 + r^2} - \frac{R^2 (\mathbf{x} \cdot \dot{\mathbf{x}})^2}{(R^2 + r^2)^2} \right). \tag{3.8}$$

Then, the momentum $\mathbf{p} = (p_1, \ldots, p_N) \in \mathbb{R}^N$ of the particle conjugate to $\mathbf{x}$ is given by

$$\mathbf{p} = \frac{\partial K}{\partial \dot{\mathbf{x}}} = m \left(\frac{R^2 \dot{\mathbf{x}}}{R^2 + r^2} - \frac{R^2 \mathbf{x} (\mathbf{x} \cdot \dot{\mathbf{x}})}{(R^2 + r^2)^2} \right). \tag{3.9}$$

Remark 3.1. As the radius R increases, the surrounding region of the origin of the N-sphere flattens out towards the tangent space, and as the radius R grows without bound, $\mathbb{S}^N_-$ will coincide with the tangent space $T_{\mathbf{0}}\mathbb{S}^N \simeq \mathbb{E}^N$. Because we have the limits

$$\lim_{R\to\infty} \frac{R^2}{R^2 + r^2} = \lim_{R\to\infty} \frac{1}{1 + r^2/R^2} = 1 \qquad \text{and} \qquad \lim_{R\to\infty} \frac{R^2}{(R^2 + r^2)^2} = \lim_{R\to\infty} \frac{1}{R^2 + 2r^2 + r^4/R^2} = 0,$$

the formulas for the kinetic energy (3.8) and conjugate momentum (3.9) of the particle in $\mathbb{S}^N_-$ become

$$\lim_{R\to\infty} K = \frac{1}{2} m |\dot{\mathbf{x}}|^2 \qquad \text{and} \qquad \lim_{R\to\infty} \mathbf{p} = m \dot{\mathbf{x}},$$

which are the formulas for the kinetic energy (2.7) and linear momentum (2.8) of the particle in $\mathbb{E}^N$.

An alternative formulation for the relation between the points Q and X is provided by trigonometric functions. Let $\theta \in [0, \pi/2)$ denote the polar angle between the radial line and the vertical direction, as depicted in Figure 3.1. With $|\mathbf{c}| = R$ and $|\mathbf{x}| = r$, we have the trigonometric relations

$$\sin(\theta) = \frac{r}{\sqrt{R^2 + r^2}}, \qquad \cos(\theta) = \frac{R}{\sqrt{R^2 + r^2}} \qquad \text{and} \qquad \tan(\theta) = \frac{r}{R}.$$

Consequently, the relations from equation (3.7) can alternatively be written as

$$q_0 = R(1 - \cos(\theta)) \qquad \text{and} \qquad q_i = \cos(\theta) x_i. \tag{3.10}$$

Moreover, this trigonometric approach can be used to derive the linear momentum $\boldsymbol{\pi}$ of the particle. For a free particle moving in $\mathbb{S}^N$, its motion is confined to a great circle, that is, the circular intersection of the N-sphere and a plane through the center C. Using the gnomonic projection and the trigonometric relations, the linear momentum [18] of the particle is given by

$$\boldsymbol{\pi} = \mathbf{p} + \frac{\mathbf{x}(\mathbf{x} \cdot \mathbf{p})}{R^2} = \frac{mR^2 \dot{\mathbf{x}}}{R^2 + r^2}, \tag{3.11}$$

for which we have

$$|\boldsymbol{\pi}|^2 = \frac{m^2 R^4 |\dot{\mathbf{x}}|^2}{(R^2 + r^2)^2}. \tag{3.12}$$

In addition to linear momentum, the particle further has orbital angular momentum about the center of the N-sphere. The elements L_{ij} of the angular momentum tensor L are given by

$$L_{ij} = x_i p_j - x_j p_i = \frac{mR^2}{R^2 + r^2} \left(x_i \dot{x}_j - x_j \dot{x}_i \right), \tag{3.13}$$

for $i, j = 1, \ldots, N$. Applying our expressions (3.11) and (3.13) for the linear and angular momentum of the particle, respectively, the total kinetic energy of the particle can alternatively be written as

$$K = \frac{1}{2m} \left(|\boldsymbol{\pi}|^2 + \frac{|L|^2}{R^2} \right), \tag{3.14}$$

where we have

$$|L|^2 = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} L_{ij} L_{ij} = |\mathbf{x}|^2 |\mathbf{p}|^2 - (\mathbf{x} \cdot \mathbf{p})^2. \tag{3.15}$$

Using this formulation of the kinetic energy rather than equation (3.8), the Hamiltonian of the harmonic oscillator in the N-sphere is given by

$$H = \frac{1}{2m} \left(|\boldsymbol{\pi}|^2 + \frac{|L|^2}{R^2} \right) + \frac{1}{2} m \omega^2 r^2. \tag{3.16}$$

Then, using the equations (3.12) and (3.15), we can express the Hamiltonian in terms of the conjugate momentum $\mathbf{p}$ of the particle,

$$H = \frac{1}{2m} \left(\frac{(R^2 + r^2)|\mathbf{p}|^2}{R^2} + \frac{(R^2 + r^2)(\mathbf{x} \cdot \mathbf{p})^2}{R^4} \right) + \frac{1}{2} m \omega^2 r^2. \tag{3.17}$$

Remark 3.2. As we discussed in Remark 3.1, suppose the radius R grows without bound. Since we have the limits

$$\lim_{R \to \infty} \frac{R^2 + r^2}{R^2} = \lim_{R \to \infty} \left(1 + \frac{r^2}{R^2} \right) = 1 \qquad \text{and} \qquad \lim_{R \to \infty} \frac{R^2 + r^2}{R^4} = \lim_{R \to \infty} \left(\frac{1}{R^2} + \frac{r^2}{R^4} \right) = 0,$$

the formula for the Hamiltonian (3.17) of the particle in $\mathbb{S}^N_-$ becomes in the limit

$$\lim_{R \to \infty} H = \frac{|\mathbf{p}|^2}{2m} + \frac{1}{2} m \omega^2 r^2,$$

which is precisely the formula for the Hamiltonian (2.14) of the particle in Euclidean N-space.

Using the trigonometric relations of the N-sphere once more, we note that the potential U of the harmonic oscillator has the particular form

$$U = \frac{1}{2} m\omega^2 r^2 = \frac{1}{2} m\omega^2 R^2 \tan^2(\theta). \tag{3.18}$$

Clearly then, the potential of the particle is symmetric between the northern and southern hemisphere of the N-sphere, and U is not defined on the equator $\theta = \pi/2$. Consequently then, the possible orbits of the particle are confined to one of the two hemispheres, in which the path of motion takes the same form [18].

3.2 Constants of Motion

Before we derive the particle's path of motion in the N-sphere, let us discuss the constants of motion of the particle. We use equation (3.17) for the Hamiltonian of the harmonic oscillator in the tangent space $T_{\mathbf{0}}\mathbb{S}^N$ to show that an observable is a constant of motion in the N-sphere.

Theorem 3.3. *The Hamiltonian H of the spherical harmonic oscillator is a constant of motion.*

Proof. Equation (3.17) shows that the Hamiltonian H does not depend explicitly on time. Furthermore, Lemma 2.3, (i), shows that $\{H, H\} = 0$, and therefore the Hamiltonian is a constant of motion. □

Furthermore, notice that the angular momentum tensor L given by (2.24) and (3.13) in the Euclidean N-space and in the N-sphere, respectively, are defined similarly. This resemblance suggests that the angular momentum of a particle moving in the N-sphere is a constant of motion as well.

Theorem 3.4. *The angular momentum tensor L of the spherical harmonic oscillator is a constant of motion.*

Proof. Equation (3.13) shows that the matrix elements L_{ij} do not depend explicitly on time, hence it suffices to show that $\{L_{ij}, H\} = 0$. For $k = 1, \ldots, N$, the partial derivatives of the Hamiltonian are

$$\frac{\partial H}{\partial x_k} = \frac{x_k}{m}\left(\frac{(\mathbf{x}\cdot\mathbf{p})^2}{R^4} + \frac{|\mathbf{p}|^2}{R^2}\right) + \frac{(R^2+r^2)(\mathbf{x}\cdot\mathbf{p})p_k}{mR^4} + m\omega^2 x_k, \tag{3.19}$$

$$\frac{\partial H}{\partial p_k} = \frac{R^2+r^2}{m}\left(\frac{p_k}{R^2} + \frac{(\mathbf{x}\cdot\mathbf{p})x_k}{R^4}\right). \tag{3.20}$$

For $k \neq i, j$, the partial derivatives of the angular momentum tensor are

$$\frac{\partial L_{ij}}{\partial x_k} = \frac{\partial}{\partial x_k}[x_i p_j - x_j p_i] = 0 \qquad \text{and} \qquad \frac{\partial L_{ij}}{\partial p_k} = \frac{\partial}{\partial x_k}[x_i p_j - x_j p_i] = 0,$$

while for $k = i$ and $k = j$, respectively, we find

$$\frac{\partial L_{ij}}{\partial x_i} = \frac{\partial}{\partial x_i}[x_i p_j - x_j p_i] = p_j \qquad \text{and} \qquad \frac{\partial L_{ij}}{\partial p_i} = \frac{\partial}{\partial p_i}[x_i p_j - x_j p_i] = -x_j,$$
$$\frac{\partial L_{ij}}{\partial x_j} = \frac{\partial}{\partial x_j}[x_i p_j - x_j p_i] = -p_i \qquad \text{and} \qquad \frac{\partial L_{ij}}{\partial p_j} = \frac{\partial}{\partial p_j}[x_i p_j - x_j p_i] = x_i.$$

Then, for the Poisson bracket of the angular momentum tensor L_{ij} and the Hamiltonian H, we find

$$\begin{aligned}
\{L_{ij}, H\} &= \sum_{k=1}^{N}\left(\frac{\partial L_{ij}}{\partial x_k}\frac{\partial H}{\partial p_k} - \frac{\partial L_{ij}}{\partial p_k}\frac{\partial H}{\partial x_k}\right) = \frac{\partial L_{ij}}{\partial x_i}\frac{\partial H}{\partial p_i} - \frac{\partial L_{ij}}{\partial p_i}\frac{\partial H}{\partial x_i} + \frac{\partial L_{ij}}{\partial x_j}\frac{\partial H}{\partial p_j} - \frac{\partial L_{ij}}{\partial p_j}\frac{\partial H}{\partial x_j} \\
&= p_j \frac{R^2+r^2}{m}\left(\frac{p_i}{R^2} + \frac{(\mathbf{x}\cdot\mathbf{p})x_i}{R^4}\right) + x_j\left(\frac{x_i}{m}\left(\frac{(\mathbf{x}\cdot\mathbf{p})^2}{R^4} + \frac{|\mathbf{p}|^2}{R^2}\right) + \frac{(R^2+r^2)(\mathbf{x}\cdot\mathbf{p})p_i}{mR^4} + m\omega^2 x_i\right) \\
&\quad - p_i \frac{R^2+r^2}{m}\left(\frac{p_j}{R^2} + \frac{(\mathbf{x}\cdot\mathbf{p})x_j}{R^4}\right) - x_i\left(\frac{x_j}{m}\left(\frac{(\mathbf{x}\cdot\mathbf{p})^2}{R^4} + \frac{|\mathbf{p}|^2}{R^2}\right) + \frac{(R^2+r^2)(\mathbf{x}\cdot\mathbf{p})p_j}{mR^4} + m\omega^2 x_j\right) \\
&= \frac{(R^2+r^2)(\mathbf{x}\cdot\mathbf{p})x_i p_j}{mR^4} + \frac{(R^2+r^2)(\mathbf{x}\cdot\mathbf{p})x_j p_i}{mR^4} - \frac{(R^2+r^2)(\mathbf{x}\cdot\mathbf{p})x_j p_i}{mR^4} - \frac{(R^2+r^2)(\mathbf{x}\cdot\mathbf{p})x_i p_j}{mR^4} \\
&= 0,
\end{aligned}$$

which concludes the proof. □

Because the angular momentum of the particle is a constant of motion, every projected orbit lies in the plane that corresponds to the angular momentum tensor [18]. For the Euclidean N-space, we derived that the particle's path of motion is an ellipse whose center coincides with the origin. As it turns out, a similar result holds for a particle moving in the N-sphere.

Since every projected orbit is confined to a plane, let us use the polar coordinates (r, φ) to denote the particle's position. In this coordinate system, the conservation of the particle's angular momentum and energy (3.16) read

$$\frac{mR^2r^2}{R^2+r^2}\dot{\varphi} = |L| \qquad \text{and} \qquad \frac{1}{2m}\left(\frac{m^2R^4}{(R^2+r^2)^2}\dot{r}^2 + \frac{m^2R^2r^2}{R^2+r^2}\dot{\varphi}^2\right) + \frac{1}{2}m\omega^2 r^2 = E. \tag{3.21}$$

Combining both equations yields a single differential equation that describes the particle's orbit,

$$\frac{|L|^2}{2m}\left(\frac{1}{r^4}\left(\frac{dr}{d\varphi}\right)^2 + \frac{1}{r^2}\right) + \frac{1}{2}m\omega^2 r^2 = E - \frac{|L|^2}{2mR^2}. \tag{3.22}$$

The general solution $r(\varphi)$ of equation (3.22) takes the form

$$\frac{1}{r^2} = \left(\frac{mE}{|L|^2} - \frac{1}{2R^2}\right)\left(1 + \sqrt{1 - \frac{4m^2\omega^2|L|^2R^4}{(2mER^2-|L|^2)^2}}\cos(2(\varphi-\varphi_0))\right), \tag{3.23}$$

where φ_0 is just a constant of integration [23]. For simplicity, let us assume that the plane of motion is spanned by the position vectors x_1 and x_2, and furthermore let $\varphi_0 = 0$. By introducing the constants

$$\mathcal{E} = E - \frac{|L|^2}{2mR^2} \qquad \text{and} \qquad \varepsilon = \sqrt{1 - \frac{\omega^2|L|^2}{\mathcal{E}^2}},$$

we can write equation (3.23) in a much more compact form, namely,

$$\frac{1}{r^2} = \frac{m\mathcal{E}}{|L|^2}(1 + \varepsilon\cos(2\varphi)). \tag{3.24}$$

Furthermore, since the plane of motion is spanned by x_1 and x_2, we have $r^2 = x_1^2 + x_2^2$. Then, equation (3.24) can be rewritten as

$$\frac{|L|^2}{m\mathcal{E}} = x_1^2 + x_2^2 + \varepsilon(x_1^2 - x_2^2),$$

or equivalently,

$$\frac{m\mathcal{E}(1+\varepsilon)}{|L|^2}x_1^2 + \frac{m\mathcal{E}(1-\varepsilon)}{|L|^2}x_2^2 = 1. \tag{3.25}$$

Equation (3.25) describes an ellipse whose center is positioned at the origin of the plane of motion. To relate this projected path of motion and the particle's path of motion in the N-sphere, we use gnomonic projection.

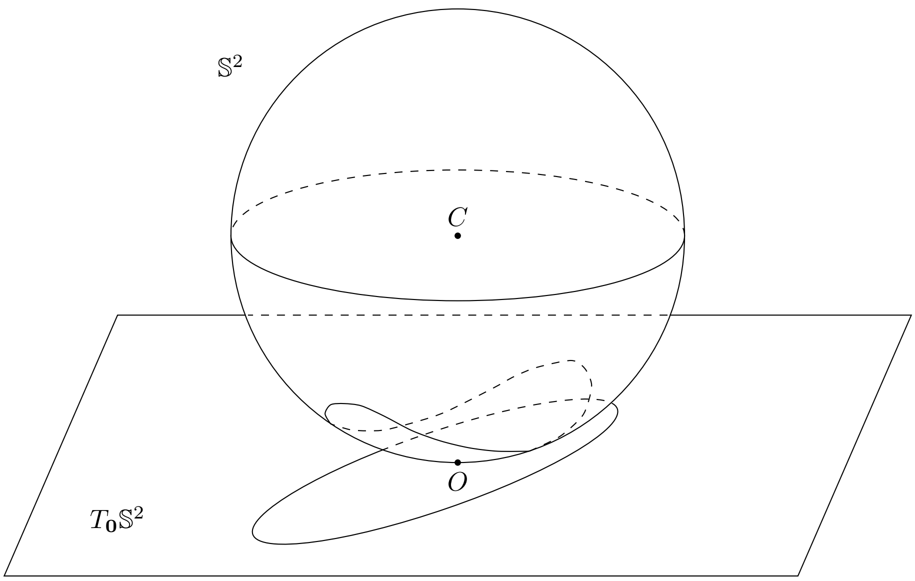

Figure 3.2 Elliptical path of motion in the southern hemisphere of $\mathbb{S}^2$ and in the tangent space $T_{\mathbf{0}}\mathbb{S}^2$.

The projected path of motion is restricted to a plane through the origin, which is a 2-dimensional subspace of the Euclidean N-space. Thus, the motion of the particle prior to the gnomonic projection was confined to the southern hemisphere of a 2-sphere. Using our relations (3.6) between the N-sphere and and the Euclidean N-space, we can project the elliptical orbit (3.25) to the southern hemisphere of the N-sphere.

To summarise the particle's motion for the spherical harmonic oscillator, the path of motion of a particle moving in the N-sphere is an spherical ellipse [29] that is confined to the southern hemisphere of a 2-sphere.

Let us now return to the constants of motion. In addition to the Hamiltonian H and the angular momentum tensor L, we consider a symmetric tensor S whose matrix elements S_{ij} are given by

$$S_{ij} = \frac{\pi_i \pi_j}{m} + m\omega^2 x_i x_j. \tag{3.26}$$

This tensor is very similar to the symmetric tensor discussed in Euclidean N-space, and it has similar properties for the N-sphere.

Theorem 3.5. *The symmetric tensor S of the spherical harmonic oscillator is a constant of motion.*

Proof. Instead of using the Poisson bracket technique to show that the symmetric tensor S_{ij} is a constant of motion, we show that its total time derivative is zero. Differentiating equation (3.26) with respect to time, we obtain

$$\dot{S}_{ij} = \frac{\dot{\pi}_i \pi_j + \pi_i \dot{\pi}_j}{m} + m\omega^2 (\dot{x}_i x_j + x_i \dot{x}_j). \tag{3.27}$$

To evaluate this equation, we first derive expressions for $\dot{\pi}_i, \dot{\pi}_j$ and $\dot{x}_i, \dot{x}_j$. Using Hamilton's equations (2.15) and the partial derivatives (3.20) of the Hamiltonian, we find

$$\dot{x}_i = \frac{\partial H}{\partial p_i} = \frac{R^2 + r^2}{m} \left(\frac{p_i}{R^2} + \frac{(\mathbf{x} \cdot \mathbf{p}) x_i}{R^4} \right) \qquad \text{and} \qquad \dot{x}_j = \frac{\partial H}{\partial p_j} = \frac{R^2 + r^2}{m} \left(\frac{p_j}{R^2} + \frac{(\mathbf{x} \cdot \mathbf{p}) x_j}{R^4} \right).$$

Differentiating the components of equation (3.11) with respect to time yields

$$\dot{\pi}_i = \dot{p}_i + \frac{(\mathbf{x} \cdot \mathbf{p}) \dot{x}_i}{R^2} + \frac{(\dot{\mathbf{x}} \cdot \mathbf{p} + \mathbf{x} \cdot \dot{\mathbf{p}}) x_i}{R^2} \qquad \text{and} \qquad \dot{\pi}_j = \dot{p}_j + \frac{(\mathbf{x} \cdot \mathbf{p}) \dot{x}_j}{R^2} + \frac{(\dot{\mathbf{x}} \cdot \mathbf{p} + \mathbf{x} \cdot \dot{\mathbf{p}}) x_j}{R^2}.$$

Again, using Hamilton's equations (2.15) and the partial derivatives (3.19) of the Hamiltonian, we find

$$\begin{aligned}
\dot{p}_i &= -\frac{\partial H}{\partial x_i} = -\frac{x_i}{m} \left(\frac{(\mathbf{x} \cdot \mathbf{p})^2}{R^4} + \frac{|\mathbf{p}|^2}{R^2} \right) - \frac{(R^2 + r^2)(\mathbf{x} \cdot \mathbf{p}) p_i}{mR^4} - m\omega^2 x_i, \\
\dot{p}_j &= -\frac{\partial H}{\partial x_j} = -\frac{x_j}{m} \left(\frac{(\mathbf{x} \cdot \mathbf{p})^2}{R^4} + \frac{|\mathbf{p}|^2}{R^2} \right) - \frac{(R^2 + r^2)(\mathbf{x} \cdot \mathbf{p}) p_j}{mR^4} - m\omega^2 x_j.
\end{aligned}$$

Using the expressions for $\dot{x}_i, \dot{x}_j$ and $\dot{p}_i, \dot{p}_j$, we can construct the vector equations

$$\begin{aligned}
\dot{\mathbf{x}} &= \frac{R^2 + r^2}{m} \left(\frac{\mathbf{p}}{R^2} + \frac{\mathbf{x}(\mathbf{x} \cdot \mathbf{p})}{R^4} \right), \\
\dot{\mathbf{p}} &= -\frac{\mathbf{x}}{m} \left(\frac{(\mathbf{x} \cdot \mathbf{p})^2}{R^4} + \frac{|\mathbf{p}|^2}{R^2} \right) - \frac{(R^2 + r^2)\mathbf{p}(\mathbf{x} \cdot \mathbf{p})}{mR^4} - m\omega^2 \mathbf{x}.
\end{aligned}$$

Substituting the expressions for $\dot{\mathbf{x}}, \dot{\mathbf{p}}$ and $\dot{p}_i, \dot{p}_j$ to derive expressions for $\dot{\pi}_i, \dot{\pi}_j$ eventually gives

$$\dot{\pi}_i = -\frac{m\omega^2 (R^2 + r^2) x_i}{R^2} \qquad \text{and} \qquad \dot{\pi}_j = -\frac{m\omega^2 (R^2 + r^2) x_j}{R^2}.$$

Then, using our expressions for $\dot{\pi}_i, \dot{\pi}_j$ and $\dot{x}_i, \dot{x}_j$ in equation (3.27) for the total time derivative $\dot{S}_{ij}$ of the symmetric tensor, we obtain

$$\begin{aligned}
\dot{S}_{ij} &= \frac{1}{m} \left(-\frac{m\omega^2 (R^2 + r^2) x_i}{R^2} \left(p_j + \frac{(\mathbf{x} \cdot \mathbf{p}) x_j}{R^2} \right) - \left(p_i + \frac{(\mathbf{x} \cdot \mathbf{p}) x_i}{R^2} \right) \frac{m\omega^2 (R^2 + r^2) x_j}{R^2} \right) \\
&\quad + m\omega^2 \left(\frac{R^2 + r^2}{m} \left(\frac{p_i}{R^2} + \frac{(\mathbf{x} \cdot \mathbf{p}) x_i}{R^4} \right) x_j + x_i \frac{R^2 + r^2}{m} \left(\frac{p_j}{R^2} + \frac{(\mathbf{x} \cdot \mathbf{p}) x_j}{R^4} \right) \right) \\
&= -\omega^2 (R^2 + r^2) \left(\frac{x_i p_j + x_j p_i}{R^2} + \frac{2(\mathbf{x} \cdot \mathbf{p}) x_i x_j}{R^4} \right) + \omega^2 (R^2 + r^2) \left(\frac{x_i p_j + x_j p_i}{R^2} + \frac{2(\mathbf{x} \cdot \mathbf{p}) x_i x_j}{R^4} \right) \\
&= 0,
\end{aligned}$$

which concludes the proof. □

Not only the Poisson bracket of the symmetric tensor S with the Hamiltonian yields an interesting result. As we discussed in the Euclidean N-space, the Poisson bracket of the symmetric tensor with itself yields a useful algebra in the N-sphere.

Theorem 3.6. *The Poisson bracket of the symmetric tensor S with itself is given by*

$$\{S_{ij}, S_{kl}\} = \omega^2(L_{ik}\delta_{jl} + L_{jk}\delta_{il} + L_{jl}\delta_{ik}) + \frac{1}{R^2}(L_{ik}S_{jl} + L_{il}S_{jk} + L_{jk}S_{il} + L_{jl}S_{ik}), \tag{3.28}$$

for $i, j, k, l = 1, \dots, N$.

Proof. To confine our notation, let us introduce the abbreviated notation

$$\mathcal{L} = L_{ik}\delta_{jl} + L_{il}\delta_{jk} + L_{jk}\delta_{il} + L_{jl}\delta_{ik} \qquad \text{and} \qquad \mathcal{S}_{\mathcal{L}} = L_{ik}S_{jl} + L_{il}S_{jk} + L_{jk}S_{il} + L_{jl}S_{ik}.$$

With these shortened forms, equation (3.28) can simply be written as

$$\{S_{ij}, S_{kl}\} = \omega^2\mathcal{L} + \frac{\mathcal{S}_{\mathcal{L}}}{mR^2}. \tag{3.29}$$

To show that (3.29) holds, we consider some individual cases. First of all, suppose that all indices are distinct. Using equation (2.18), the left-hand side of equation (3.29) reads

$$\{S_{ij}, S_{kl}\} = \sum_{n=1}^{N}\left(\frac{\partial S_{ij}}{\partial x_n}\frac{\partial S_{kl}}{\partial p_n} - \frac{\partial S_{ij}}{\partial p_n}\frac{\partial S_{kl}}{\partial x_n}\right) = D_i + D_j + D_k + D_l,$$

where we used the notation

$$D_i = \frac{\partial S_{ij}}{\partial x_i}\frac{\partial S_{kl}}{\partial p_i} - \frac{\partial S_{ij}}{\partial p_i}\frac{\partial S_{kl}}{\partial x_i} \qquad \text{and} \qquad D_j = \frac{\partial S_{ij}}{\partial x_j}\frac{\partial S_{kl}}{\partial p_j} - \frac{\partial S_{ij}}{\partial p_j}\frac{\partial S_{kl}}{\partial x_j},$$
$$D_k = \frac{\partial S_{ij}}{\partial x_k}\frac{\partial S_{kl}}{\partial p_k} - \frac{\partial S_{ij}}{\partial p_k}\frac{\partial S_{kl}}{\partial x_k} \qquad \text{and} \qquad D_l = \frac{\partial S_{ij}}{\partial x_l}\frac{\partial S_{kl}}{\partial p_l} - \frac{\partial S_{ij}}{\partial p_l}\frac{\partial S_{kl}}{\partial x_l}.$$

Evaluating these linear combinations of partial derivatives yields

$$\begin{aligned}
D_i = &-\frac{(x_kp_l + x_lp_k)p_ip_j}{m^2R^2} - \frac{2(\mathbf{x}\cdot\mathbf{p})x_kx_lp_ip_j}{m^2R^4} + \frac{(\mathbf{x}\cdot\mathbf{p})^2(x_kp_l + x_lp_k)x_ix_j}{m^2R^6} + \frac{\omega^2(x_kp_l + x_lp_k)x_ix_j}{R^2} \\
&+ \frac{(\mathbf{x}\cdot\mathbf{p})(x_ip_j - x_jp_i)}{m^2R^6}\left(2(\mathbf{x}\cdot\mathbf{p})x_kx_l + R^2(x_kp_l + x_lp_k)\right) + \frac{2(\mathbf{x}\cdot\mathbf{p})x_ix_jx_kx_l}{m^2R^8}\left((\mathbf{x}\cdot\mathbf{p})^2 + m^2\omega^2R^4\right), \\
D_j = &-\frac{(x_kp_l + x_lp_k)p_ip_j}{m^2R^2} - \frac{2(\mathbf{x}\cdot\mathbf{p})x_kx_lp_ip_j}{m^2R^4} + \frac{(\mathbf{x}\cdot\mathbf{p})^2(x_kp_l + x_lp_k)x_ix_j}{m^2R^6} + \frac{\omega^2(x_kp_l + x_lp_k)x_ix_j}{R^2} \\
&- \frac{(\mathbf{x}\cdot\mathbf{p})(x_ip_j - x_jp_i)}{m^2R^6}\left(2(\mathbf{x}\cdot\mathbf{p})x_kx_l + R^2(x_kp_l + x_lp_k)\right) + \frac{2(\mathbf{x}\cdot\mathbf{p})x_ix_jx_kx_l}{m^2R^8}\left((\mathbf{x}\cdot\mathbf{p})^2 + m^2\omega^2R^4\right), \\
D_k = &\frac{(x_ip_j + x_jp_i)p_kp_l}{m^2R^2} + \frac{2(\mathbf{x}\cdot\mathbf{p})x_ix_jp_kp_l}{m^2R^4} - \frac{(\mathbf{x}\cdot\mathbf{p})^2(x_ip_j + x_jp_i)x_kx_l}{m^2R^6} - \frac{\omega^2(x_ip_j + x_jp_i)x_kx_l}{R^2} \\
&- \frac{(\mathbf{x}\cdot\mathbf{p})(x_kp_l - x_lp_k)}{m^2R^6}\left(2(\mathbf{x}\cdot\mathbf{p})x_ix_j + R^2(x_ip_j + x_lp_i)\right) - \frac{2(\mathbf{x}\cdot\mathbf{p})x_ix_jx_kx_l}{m^2R^8}\left((\mathbf{x}\cdot\mathbf{p})^2 + m^2\omega^2R^4\right), \\
D_l = &\frac{(x_ip_j + x_jp_i)p_kp_l}{m^2R^2} + \frac{2(\mathbf{x}\cdot\mathbf{p})x_ix_jp_kp_l}{m^2R^4} - \frac{(\mathbf{x}\cdot\mathbf{p})^2(x_ip_j + x_jp_i)x_kx_l}{m^2R^6} - \frac{\omega^2(x_ip_j + x_jp_i)x_kx_l}{R^2} \\
&+ \frac{(\mathbf{x}\cdot\mathbf{p})(x_kp_l - x_lp_k)}{m^2R^6}\left(2(\mathbf{x}\cdot\mathbf{p})x_ix_j + R^2(x_ip_j + x_lp_i)\right) - \frac{2(\mathbf{x}\cdot\mathbf{p})x_ix_jx_kx_l}{m^2R^8}\left((\mathbf{x}\cdot\mathbf{p})^2 + m^2\omega^2R^4\right).
\end{aligned}$$

For the right-hand side, it is clear that $\mathcal{L} = 0$ as a result of the Kronecker deltas, and using equations (3.13) and (3.26) for the angular momentum tensor and symmetric tensor, respectively, one derives

$$\begin{aligned}
\mathcal{S}_{\mathcal{L}} = &\frac{2(x_ip_j + x_jp_i)p_kp_l}{m} + \frac{4(\mathbf{x}\cdot\mathbf{p})x_ix_jp_kp_l}{mR^2} + \frac{2(\mathbf{x}\cdot\mathbf{p})^2(x_kp_l + x_lp_k)x_ix_j}{mR^4} + 2m\omega^2(x_kp_l + x_lp_k)x_ix_j \\
&- \frac{2(x_kp_l + x_lp_k)p_ip_j}{m} - \frac{4(\mathbf{x}\cdot\mathbf{p})x_kx_lp_ip_j}{mR^2} - \frac{2(\mathbf{x}\cdot\mathbf{p})^2(x_ip_j + x_jp_i)x_kx_l}{mR^4} - 2m\omega^2(x_ip_j + x_jp_i)x_kx_l.
\end{aligned}$$

By comparison of these results, it is clear that we have

$$\{S_{ij}, S_{kl}\} = D_i + D_j + D_k + D_l = \omega^2\mathcal{L} + \frac{\mathcal{S}_{\mathcal{L}}}{mR^2}.$$

Second, suppose that one pair of indices are equal and that the remaining two indices are distinct. Without loss of generality, assume $i = k$ and $j \neq l$. Then, the left-hand side of equation (3.29) reads

$$\{S_{ij}, S_{kl}\} = \{S_{ij}, S_{il}\} = \sum_{n=1}^{N} \left(\frac{\partial S_{ij}}{\partial x_n} \frac{\partial S_{il}}{\partial p_n} - \frac{\partial S_{ij}}{\partial p_n} \frac{\partial S_{il}}{\partial x_n} \right) = D_i + D_j + D_l.$$

Again, evaluating the linear combinations of partial derivatives gives

$$\begin{aligned}
D_i &= (x_i p_l - x_l p_i) \left(\frac{p_j p_i}{m^2 R^2} + \frac{(\mathbf{x}\cdot\mathbf{p})^2 x_j x_i}{m^2 R^6} + \frac{\omega^2 x_j x_i}{R^2} \right) + \frac{2(\mathbf{x}\cdot\mathbf{p})^2 x_i^2 x_l p_j}{m^2 R^6} + \omega^2 (x_j p_l - x_l p_j) \\
&\quad + (x_j p_i - x_i p_j) \left(\frac{p_i p_l}{m^2 R^2} + \frac{(\mathbf{x}\cdot\mathbf{p})^2 x_i x_l}{m^2 R^6} + \frac{\omega^2 x_i x_l}{R^2} \right) - \frac{2(\mathbf{x}\cdot\mathbf{p})^2 x_i^2 x_j p_l}{m^2 R^6}, \\
D_j &= \frac{(x_i p_j + x_j p_i) p_i p_l}{m^2 R^2} + \frac{2(\mathbf{x}\cdot\mathbf{p}) x_i x_j p_i p_l}{m^2 R^4} + \frac{(\mathbf{x}\cdot\mathbf{p})^2 (x_i p_l + x_l p_i) x_i x_j}{m^2 R^6} + \frac{\omega^2 (x_i p_l + x_l p_i) x_i x_j}{R^2} \\
&\quad + (x_j p_i - x_i p_j) \frac{(\mathbf{x}\cdot\mathbf{p})(x_i p_l + x_l p_i)}{m^2 R^4} + \frac{2(\mathbf{x}\cdot\mathbf{p}) x_i x_j x_l}{m^2 R^8} \left((\mathbf{x}\cdot\mathbf{p})^2 x_i + R^2 (\mathbf{x}\cdot\mathbf{p}) p_i + m^2 \omega^2 R^4 x_i \right) \\
&\quad - \frac{2(\mathbf{x}\cdot\mathbf{p})^2 x_i^2 x_l p_j}{m^2 R^6}, \\
D_l &= -\frac{(x_i p_l + x_l p_i) p_i p_j}{m^2 R^2} - \frac{2(\mathbf{x}\cdot\mathbf{p}) x_i x_l p_i p_j}{m^2 R^4} - \frac{(\mathbf{x}\cdot\mathbf{p})^2 (x_i p_j + x_j p_i) x_i x_l}{m^2 R^6} - \frac{\omega^2 (x_i p_j + x_j p_i) x_i x_l}{R^2} \\
&\quad + (x_i p_l - x_l p_i) \frac{(\mathbf{x}\cdot\mathbf{p})(x_j p_i + x_i p_j)}{m^2 R^4} - \frac{2(\mathbf{x}\cdot\mathbf{p}) x_i x_j x_l}{m^2 R^8} \left((\mathbf{x}\cdot\mathbf{p})^2 x_i + R^2 (\mathbf{x}\cdot\mathbf{p}) p_i + m^2 \omega^2 R^4 x_i \right) \\
&\quad + \frac{2(\mathbf{x}\cdot\mathbf{p})^2 x_i^2 x_j p_l}{m^2 R^6}.
\end{aligned}$$

For the right-hand side, we find $\mathcal{L} = L_{jl} = x_j p_l - x_l p_j$ due to the Kronecker deltas, and again using equations (3.13) and (3.26) for the angular momentum tensor and symmetric tensor, respectively, one finds

$$\begin{aligned}
\mathcal{S}_{\mathcal{L}} &= (x_i p_l - x_l p_i) \left(\frac{p_j p_i}{m} + \frac{(\mathbf{x}\cdot\mathbf{p})(x_j p_i + x_i p_j)}{m R^2} + \frac{(\mathbf{x}\cdot\mathbf{p})^2 x_j x_i}{m R^4} + m\omega^2 x_j x_i \right) \\
&\quad + (x_j p_i - x_i p_j) \left(\frac{p_i p_l}{m} + \frac{(\mathbf{x}\cdot\mathbf{p})(x_i p_l + x_l p_i)}{m R^2} + \frac{(\mathbf{x}\cdot\mathbf{p})^2 x_i x_l}{m R^4} + m\omega^2 x_i x_l \right) \\
&\quad + (x_j p_l - x_l p_j) \left(\frac{p_i^2}{m} + \frac{2(\mathbf{x}\cdot\mathbf{p}) x_i p_i}{m R^2} + \frac{(\mathbf{x}\cdot\mathbf{p})^2 x_i^2}{m R^4} + m\omega^2 x_i^2 \right).
\end{aligned}$$

Then, by comparing the sums on the left-hand and the right-hand side of (3.29), one yields the result

$$\{S_{ij}, S_{il}\} = D_i + D_j + D_l = \omega^2 \mathcal{L} + \frac{\mathcal{S}_{\mathcal{L}}}{m R^2}.$$

Finally, suppose that two pairs of indices are equal. Without loss of generality, assume that $i = k$ and $j = l$. By Lemma 2.3, (i) the left-hand side of (3.29) reads

$$\{S_{ij}, S_{kl}\} = \{S_{ij}, S_{ij}\} = 0.$$

For the right-hand side, using the Kronecker deltas and the skew-symmetry of the angular momentum tensor L, we have

$$\mathcal{L} = L_{ii} \cdot 1 + L_{ij} \cdot 0 + L_{ji} \cdot 0 + L_{jj} \cdot 1 = 0,$$

and additionally by the symmetry of the symmetric tensor S, we find

$$\mathcal{S} = L_{ii} S_{jj} + L_{ij} S_{ji} + L_{ji} S_{ij} + L_{jj} S_{ii} = L_{ij} S_{ij} - L_{ij} S_{ij} = 0.$$

Hence, we find that equation (3.29) holds. This concludes the proof. □

As we discussed for the Euclidean N-space, we can construct two scalars I_1 and I_2 from the matrix elements S_{ij} of the symmetric tensor [18]. They are

$$I_1 = \operatorname{tr}(S) = 2H - \frac{|L|^2}{R^2} \qquad \text{and} \qquad I_2 = \operatorname{tr}(S^2) - \operatorname{tr}(S)^2 = -2\omega^2 |L|^2, \tag{3.30}$$

which are again independent from each other. The generators of the dynamical group SU(N) are again the angular momentum tensor L and a traceless symmetric tensor T that satisfy the Poisson bracket relations

$$\{T_{ij}, H\} = 0 \qquad \text{and} \qquad \{T_{ij}, T_{kl}\} = L_{ik}\delta_{jl} + L_{il}\delta_{jk} + L_{jk}\delta_{il} + L_{jl}\delta_{ik}. \tag{3.31}$$

However, the construction of this tensor is more difficult as a result of the non-linearity of the Poisson brackets (3.28). For two arbitrary functions f and g, the most general traceless symmetric tensor $\mathcal{T}$ we can consider is

$$\mathcal{T}_{ij} = fS_{ij} + \omega^2 g L_{ik} L_{kj},$$

for $i, j, k = 1, \ldots, N$. The corresponding two scalars J_1 and J_2 associated with this tensor are

$$J_1 = \mathrm{tr}(\mathcal{T}) = fI_1 + gI_2 \qquad \text{and} \qquad J_2 = \mathrm{tr}(\mathcal{T}^2) - \mathrm{tr}(\mathcal{T})^2 = \left(f^2 - fgI_1 - \frac{1}{2}g^2 I_2\right) I_2. \tag{3.32}$$

The Poisson brackets of such a general symmetric tensor with itself is given by

$$\{\mathcal{T}_{ij}, \mathcal{T}_{kl}\} = L_{ik}U_{jl} + L_{il}U_{jk} + L_{jk}L_{il} + L_{jl}U_{ik},$$

for $i, j, k, l = 1, \ldots, N$, where we have

$$U_{ij} = \frac{\omega^2 J_2 \delta_{ij}}{I_2} - \mathcal{T}_{ij}\frac{\partial}{\partial |L|^2}[J_1] - L_{ik}L_{kj}\frac{\partial}{\partial |L|^2}\left[\frac{\omega^2 J_2}{I_2}\right]. \tag{3.33}$$

Then similar to the tensor (2.39) we derived in the Euclidean N-space, the Poisson brackets (3.31) are obtained for

$$T_{ij} = \mathcal{T}_{ij} - \frac{\mathrm{tr}(\mathcal{T})}{N}\delta_{ij}, \tag{3.34}$$

on the condition that the functions f and g are chosen such that

$$J_1 = A(H) \qquad \text{and} \qquad J_2 = -2|L|^2, \tag{3.35}$$

where $A(H)$ is an arbitrary function of the Hamiltonian H. As in equation (2.43), the quadratic Casimir function C of the special unitary group SU(N) satisfies the relation

$$\sum_{i=1}^{N}\sum_{j=1}^{N}(T_{ij}T_{ij} + L_{ij}L_{ij}) = \frac{4(N-1)}{N}C(H), \tag{3.36}$$

where

$$C(H) = \frac{1}{4}A(H)^2. \tag{3.37}$$

To determine this arbitrary function $A(H)$ of the Hamiltonian, we consider the circular orbits of the particle [18]. For the harmonic oscillator, the orbits are characterised by the two non-zero eigenvalues of the symmetric tensor S, which occurs whenever we have

$$I_1^2 + 2I_2 = 0.$$

This condition can be written in a more explicit form, which reads

$$E = \omega|L| - \frac{|L|^2}{2R^2}. \tag{3.38}$$

We have the same condition for the traceless symmetric tensor T, that is,

$$J_1^2 + 2J_2 = 0,$$

which can be written in the explicit form

$$\frac{1}{4}A(E)^2 = |L|^2. \tag{3.39}$$

The two requirements that are given by equations (3.38) and (3.39) can be used to determine the function $A(E)$. Consequently then, the Hamiltonian can be expressed as a function of the quadratic Casimir function C, that is,

$$H = \omega\sqrt{C} + \frac{C}{2R^2}. \tag{3.40}$$

Remark 3.7. As the radius R grows without bound, the Hamiltonian H as a function of the quadratic Casimir function C becomes in the limit

$$\lim_{R\to\infty} H = \omega\sqrt{C},$$

which is exactly the Hamiltonian as a function of the quadratic Casimir function C (2.42) in the Euclidean N-space.

3.3 Quantum Dynamics in the N-Sphere

As we discussed in previous chapters, observables in quantum mechanics are required to be Hermitian operators. For this reason the classical linear momentum (3.11) is replaced by its Hermitian counterpart, which reads

$$\hat{\boldsymbol{\pi}} = \hat{\mathbf{p}} + \frac{\hat{\mathbf{x}}(\hat{\mathbf{x}}\cdot\hat{\mathbf{p}}) + (\hat{\mathbf{p}}\cdot\hat{\mathbf{x}})\hat{\mathbf{x}}}{2R^2}. \tag{3.41}$$

The Hamiltonian operator of the quantum harmonic oscillator is

$$\hat{H} = \frac{1}{2m}\left((\hat{\boldsymbol{\pi}}\cdot\hat{\boldsymbol{\pi}}) + \frac{|\hat{L}|^2}{R^2}\right) + \frac{1}{2}m\omega^2 r^2, \tag{3.42}$$

where $\hat{L}_{ij} = \hat{x}_i\hat{p}_j - \hat{x}_j\hat{p}_i$. Furthermore, the classical symmetric tensor S as given by equation (3.26) is replaced by a Hermitian symmetric operator $\hat{S}$, whose matrix elements $\hat{S}_{ij}$ read

$$\hat{S}_{ij} = \frac{\hat{\pi}_i\hat{\pi}_j + \hat{\pi}_j\hat{\pi}_i}{2m} + m\omega^2\hat{x}_i\hat{x}_j. \tag{3.43}$$

In quantum mechanics, the role of the Poisson bracket is replaced by the commutator. If $[\hat{O},\hat{H}] = 0$, then the expectation value of the quantum observable O is independent of time [17]. In addition to the fundamental commutation relations given by (1.28), we have the following identities [15].

Theorem 3.8. *For $i,j,k = 1,\dots,N$ and $k \neq i,j$, we have the commutation relations*

$$[\hat{L}_{ij},\hat{x}_i] = i\hbar\hat{x}_j, \qquad [\hat{L}_{ij},\hat{x}_j] = -i\hbar\hat{x}_i \qquad \textit{and} \qquad [\hat{L}_{ij},\hat{x}_k] = 0, \tag{3.44}$$

and

$$[\hat{L}_{ij},\hat{p}_i] = i\hbar\hat{p}_j, \qquad [\hat{L}_{ij},\hat{p}_j] = -i\hbar\hat{p}_i \qquad \textit{and} \qquad [\hat{L}_{ij},\hat{p}_k] = 0. \tag{3.45}$$

Proof. Using the commutation properties established in Theorem 1.8 and Lemma 1.9, we find that

$$\begin{aligned}
[\hat{L}_{ij},\hat{x}_i] &= [\hat{x}_i\hat{p}_j - \hat{x}_j\hat{p}_i,\hat{x}_i] = [\hat{x}_i\hat{p}_j,\hat{x}_i] - [\hat{x}_j\hat{p}_i,\hat{x}_i] = \hat{x}_j[\hat{x}_i,\hat{p}_i] = i\hbar\hat{x}_j,\\
[\hat{L}_{ij},\hat{x}_j] &= [\hat{x}_i\hat{p}_j - \hat{x}_j\hat{p}_i,\hat{x}_j] = [\hat{x}_i\hat{p}_j,\hat{x}_j] - [\hat{x}_j\hat{p}_i,\hat{x}_j] = -\hat{x}_i[\hat{x}_j,\hat{p}_j] = -i\hbar\hat{x}_i,\\
[\hat{L}_{ij},\hat{x}_k] &= [\hat{x}_i\hat{p}_j - \hat{x}_j\hat{p}_i,\hat{x}_k] = [\hat{x}_i\hat{p}_j,\hat{x}_k] - [\hat{x}_j\hat{p}_i,\hat{x}_k] = 0,
\end{aligned}$$

and

$$\begin{aligned}
[\hat{L}_{ij},\hat{p}_i] &= [\hat{x}_i\hat{p}_j - \hat{x}_j\hat{p}_i,\hat{p}_i] = [\hat{x}_i\hat{p}_j,\hat{p}_i] - [\hat{x}_j\hat{p}_i,\hat{p}_i] = \hat{p}_j[\hat{x}_i,\hat{p}_i] = i\hbar\hat{p}_j,\\
[\hat{L}_{ij},\hat{p}_j] &= [\hat{x}_i\hat{p}_j - \hat{x}_j\hat{p}_i,\hat{p}_j] = [\hat{x}_i\hat{p}_j,\hat{p}_j] - [\hat{x}_j\hat{p}_i,\hat{p}_j] = -\hat{p}_i[\hat{x}_j,\hat{p}_j] = -i\hbar\hat{p}_i,\\
[\hat{L}_{ij},\hat{p}_k] &= [\hat{x}_i\hat{p}_j - \hat{x}_j\hat{p}_i,\hat{p}_k] = [\hat{x}_i\hat{p}_j,\hat{p}_k] - [\hat{x}_j\hat{p}_i,\hat{p}_k] = 0,
\end{aligned}$$

for $i,j,k = 1,\dots,N$ and $i,j \neq k$. □

Together with the fundamental commutation relations (1.28) and the commutation properties that we established in Lemma 1.9 and Theorem 3.8, we can show that

$$[\hat{H},\hat{H}] = \hat{0}, \qquad [\hat{L}_{ij},\hat{H}] = \hat{0} \qquad \text{and} \qquad [\hat{S}_{ij},\hat{H}] = \hat{0}. \tag{3.46}$$

Hence, the Hamiltonian operator $\hat{H}$, the angular momentum tensor $\hat{L}$ and the symmetric tensor $\hat{S}$ are constants of motion in the N-sphere. Additionally [18], the the Poisson bracket relation (3.28) are replaced by the commutation relation

$$\begin{aligned}
[\hat{S}_{ij},\hat{S}_{kl}] = {}& i\hbar\omega^2(\hat{L}_{ik}\delta_{jl} + \hat{L}_{il}\delta_{jk} + \hat{L}_{jk}\delta_{il} + \hat{L}_{jl}\delta_{ik}) - \frac{i\hbar(\hat{L}_{ik}\delta_{jl} + \hat{L}_{il}\delta_{jk} + \hat{L}_{jk}\delta_{il} + \hat{L}_{jl}\delta_{ik})}{4R^4}\\
&+ \frac{i\hbar(\hat{L}_{ik}\hat{S}_{jl} + \hat{L}_{il}\hat{S}_{jk} + \hat{L}_{jk}\hat{S}_{il} + \hat{L}_{jl}\hat{S}_{ik}) + i\hbar(\hat{S}_{ik}\hat{L}_{jl} + \hat{S}_{il}\hat{L}_{jk} + \hat{S}_{jk}\hat{L}_{il} + \hat{S}_{jl}\hat{L}_{ik})}{2R^2}.
\end{aligned} \tag{3.47}$$

Similar to the independent scalars (3.30) we derived from the classical symmetric tensor, the quantum symmetric tensor has the independent commuting operators

$$\hat{I}_1 = \mathrm{tr}(\hat{S}) = 2\hat{H} - \frac{|\hat{L}|^2}{R^2}, \tag{3.48}$$

and

$$\hat{I}_2 = \mathrm{tr}(\hat{S}^2) - \mathrm{tr}(\hat{S})^2 = -\omega^2(2|\hat{L}|^2 + N(N-1)\hat{I}) - \frac{1}{R^2}\left(2(N-1)\hat{H} - \frac{N|\hat{L}|^2}{R^2} - \frac{|\hat{L}|^2}{2R^2}\right). \tag{3.49}$$

Just as the equations (3.28) and (3.30) provided us with a method to formulate the Hamiltonian as a function of the quadratic Casimir function C of the dynamical group SU(N), the equations (3.47), (3.48) and (3.49) give

$$\hat{H} = \sqrt{\omega^2 + \frac{1}{4R^4}}\sqrt{\hat{C} + \frac{N^2\hat{I}}{4}} + \frac{\hat{C}}{2R^2} + \frac{N\hat{I}}{4R^2}. \tag{3.50}$$

Therefore, the spherical quantum harmonic oscillator is symmetric upon rotation of the N-sphere [21] and $\hat{C}$ is the quadratic Casimir operator.

3.4 Energy Levels and Eigenfunctions in the N-Sphere

To solve the eigenvalue problem and find the energy levels of the particle, we follow a method that applies the Schrödinger equation [21]. Since the harmonic oscillator is symmetric upon rotation of the N-sphere, let us use the hyperspherical coordinates (r, Φ) to denote the particle's position, where Φ represents the $N-1$ angular variables $\varphi_1, \dots, \varphi_{N-1}$. To solve the N-dimensional time-independent Schrödinger equation

$$\hat{H}\psi_{n,\ell,m_\ell}(r,\Phi) = E_n\psi_{n,\ell,m_\ell}(r,\Phi), \tag{3.51}$$

where E_n denotes the nth energy level en the integers n, ℓ and m_ℓ represent the principal, azimuthal and magnetic quantum numbers [15], respectively, we use the method of separation of variables,

$$\psi_{n,\ell,m_\ell}(r,\Phi) = X_{n,\ell}(r)Y_\ell^{m_\ell}(\Phi). \tag{3.52}$$

This method provides us with two separate equations, which are the hyperangular and radial equation, respectively. This first equation reads

$$\Lambda_{N-1}^2 Y_\ell^{m_\ell}(\Phi) = \ell(\ell + N - 2)Y_\ell^{m_\ell}(\Phi), \tag{3.53}$$

where $\ell(\ell + N - 2)$ are the eigenvalues [9] of the hyperangular momentum operator Λ_{N-1}^2 and the functions $Y_\ell^{m_\ell}(\Phi)$ are known as the hyperspherical harmonics. Using our geometric relations established previously and the convention $m = \hbar = 1$, the radial equation is given by

$$\left(\frac{d^2}{d\theta^2} + (N-1)\cot(\theta)\frac{d}{d\theta} - \ell(\ell+N-2)\csc^2(\theta) - \omega^2R^4\tan^2(\theta) + 2R^2E_n\right)X_{n,\ell} = 0. \tag{3.54}$$

To rewrite the radial equation (3.54), let us define the differential operators

$$\hat{O}_+ = (2\ell+N)\cot(\theta)\frac{d}{d\theta} - \ell(2\ell+N)\cot^2(\theta) - \ell(\ell+N-1) + 2R^2E_n \tag{3.55}$$

and

$$\hat{O}_- = -(2\ell+N)\cot(\theta)\frac{d}{d\theta} - (\ell+N)(2\ell+N)\cot^2(\theta) - (\ell+1)(\ell+N) + 2R^2E_n. \tag{3.56}$$

Then, one finds that we can write

$$\begin{aligned}\hat{O}_+\hat{O}_-X_{n,\ell+2} &- \left(\left[2E_nR^2 - (\ell+1)(\ell+N)\right]\left[2E_nR^2 - \ell(\ell+N-1)\right] - (2\ell+N)^2\omega^2R^4\right)X_{n,\ell+2} \\ &= 0,\end{aligned} \tag{3.57}$$

and

$$\begin{aligned}\hat{O}_+\hat{O}_-X_{n,\ell} &- \left(\left[2E_nR^2 - (\ell+1)(\ell+N)\right]\left[2E_nR^2 - \ell(\ell+N-1)\right] - (2\ell+N)^2\omega^2R^4\right)X_{n,\ell} \\ &= 0.\end{aligned} \tag{3.58}$$

We notice that $\hat{O}_- X_{n,\ell+2}$ is a solution of the second equation while $\hat{O}_+ X_{n,\ell}$ is a solution of the first. Hence, it follows that the operators $\hat{O}_+$ and $\hat{O}_-$ are raising and lowering operators, respectively.

If the eigenfunctions $\psi_{n,\ell,m_\ell}(r,\Phi)$ are normalised according to the normalisation condition and the hyperspherical harmonics $Y_\ell^{m_\ell}(\Phi)$ are normalised to unity, then the normalisation condition for the radial function $X_{n,\ell}$ reads

$$R^N \int_0^{\pi/2} \sin^{N-1}(\theta) X^*_{n',\ell} X_{n,\ell} d\theta = \delta_{n'n}. \tag{3.59}$$

By combining the equations (3.54) and (3.59), it follows that the radial function satisfies the condition

$$\int_0^{\pi/2} \sin^{N-1}(\theta) X^*_{n,\ell'} X_{n,\ell} d\theta = 0 \tag{3.60}$$

for $\ell' \neq \ell$. This latter condition implies that the raising and lowering operators $\hat{O}_+$ and $\hat{O}_-$ are Hermitian operators [21]. Consequently, the constant term occurring in both (3.57) and (3.58) is non-negative, and there exists some integer n such that this constant term is zero for $\ell = n$. With this condition, we find that the energy levels of the spherical quantum harmonic oscillator are given by

$$E_n = \left(n + \frac{N}{2}\right)\sqrt{\omega^2 + \frac{1}{4R^4}} + \frac{1}{2R^2}\left(n^2 + Nn + \frac{N}{2}\right). \tag{3.61}$$

Remark 3.8. As we discussed in previous remarks, as the radius R grows without bound, the energy levels E_n of the quantum harmonic oscillator in the N-sphere become

$$\lim_{R\to\infty} E_n = \omega\left(n + \frac{N}{2}\right),$$

which are precisely the energy levels (2.72) of the quantum harmonic oscillator in the Euclidean N-space for $\hbar = 1$.

We can use the raising and lowering operators $\hat{O}_+$ and $\hat{O}_-$ to compute the eigenfunctions for the spherical quantum harmonic oscillator [21]. The action of the raising and lowering operators, respectively, is given by

$$\hat{O}_+ X_{n,\ell} = -(2\ell+N)|C_{n,\ell}|R^2 X_{n,\ell+2} \qquad \text{and} \qquad \hat{O}_- X_{n,\ell+2} = -(2\ell+N)|C_{n,\ell}|R^2 X_{n,\ell}, \tag{3.62}$$

where $|C_{n,\ell}|$ satisfies the relation

$$\left(\ell + \frac{N}{2}\right)^2 |C_{n,\ell}|^2 = \frac{(n-\ell)(n+\ell+N)}{R^4}\left(\frac{n-\ell}{2} + R^2\varpi\right)\left(\frac{n+\ell+N}{2} + R^2\varpi\right), \tag{3.63}$$

and we have used the abbreviation

$$\varpi = \sqrt{\omega^2 + \frac{1}{4R^4}}. \tag{3.64}$$

To simplify the recurrence relations (3.62), let us introduce the definitions

$$X_{n,\ell} = A_{n,\ell}(\sin(\theta))^{-(\ell+N-2)}(\cos(\theta))^{\frac{1}{2}+R^2\varpi} Z_{n,\ell} \qquad \text{and} \qquad A_{n,\ell} = \frac{A_{n,\ell+2}}{(2\ell+N-4)|C_{n,\ell}|R^2}. \tag{3.65}$$

With these definitions, the second recurrence relation of equation (3.63) can be written as

$$Z_{n,\ell} = \csc^2(\theta)\left((2\ell+N-4)\cot(\theta)\frac{d}{d\theta} - (n+\ell+N-2)(n-\ell+2+2R^2\varpi)\right) Z_{n,\ell+2}, \tag{3.66}$$

while the first recurrence relation for $\ell = n$ becomes

$$\left(\frac{d}{d\theta} - (2n+N-2)\cot(\theta)\right) Z_{n,n}(\theta) = 0. \tag{3.67}$$

This differential equation has a simple solution, which reads

$$Z_{n,n}(\theta) = (\sin(\theta))^{2n+N-2}. \tag{3.68}$$

It remains now to compute the general solutions $Z_{n,\ell}(\theta)$ from the specific solution $Z_{n,n}(\theta)$. By using induction, we find that

$$Z_{n,n-2D}(\theta) = \sum_{i=1}^{D}(-1)^i B_{D,i}^{n,N}(\sin(\theta))^{2n+N-4D+2i-2}(\cos(\theta))^{2D-2i}, \tag{3.69}$$

where the constant $B_{D,i}^{n,N}$ is defined as

$$B_{D,i}^{n,N} = \frac{2^{2i-2D}D!\Gamma(D+1+R^2\varpi)}{(D-i)!i!\Gamma(D+1+R^2\varpi-i)}\frac{\Gamma(2n+N-1)\Gamma(\frac{1}{2}(2n+N-4D+2i-1))}{\Gamma(\frac{1}{2}(2n+N-1)\Gamma(2n+N-4D+2i-1)}$$

and Γ denotes the Gamma function [1].

Definition 3.9. For $z \in \mathbb{C}$ such that $\mathrm{Re}(z) > 0$, the gamma function $\Gamma(z)$ is defined by the definite integral

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt. \tag{3.70}$$

Additionally, the gamma function works as an extension of the factorial function, because for any positive integer n, the gamma function satisfies

$$\Gamma(n) = (n-1)!. \tag{3.71}$$

To reduce the amount of Gamma functions, let us introduce the constant

$$\gamma = \frac{\Gamma(2n+N-2D-1)\Gamma(\frac{1}{2}(2n+N-1))}{\Gamma(\frac{1}{2}(2n+N-2D-1))\Gamma(2N+N-1)}.$$

With a change of variables, we find the result

$$\left(\frac{d}{d(\sin^2(\theta))}\right)^D\left[(\sin^2(\theta))^{\frac{1}{2}(2n+N-2D-2)}(\cos^2(\theta))^{D+R^2\varpi}\right] = \gamma(\cos^2(\theta))^{R^2\varpi}Z_{n,n-2D}(\sin^2(\theta)). \tag{3.72}$$

Upon normalisation, the solution $X_{n,\ell}(\sin^2(\theta))$ of the radial equation (3.54) is given by

$$X_{n,\ell}(\sin^2(\theta)) = \sqrt{\frac{2(n+\frac{1}{2}N+R^2\varpi)\Gamma(\frac{1}{2}(n+\ell+N)+R^2\varpi)}{R^N(\frac{1}{2}(n-\ell))!\Gamma(\frac{1}{2}(n+\ell+N))\Gamma(\frac{1}{2}(n-\ell)+1+R^2\varpi)}}(\sin^2(\theta))^{-\frac{1}{2}(\ell+N-2)}$$
$$(\cos^2(\theta))^{\frac{1}{4}-\frac{R^2}{2}\varpi}\left(\frac{d}{d(\sin^2(\theta))}\right)^{\frac{1}{2}(n-\ell)}\left[(\sin^2(\theta))^{\frac{1}{2}(n+\ell+N-2)}(\cos^2(\theta))^{\frac{1}{2}(n-\ell)+R^2\varpi}\right]. \tag{3.73}$$

To write this result more compactly, we introduce the notion of hypergeometric functions [4].

Definition 3.10. For $z \in \mathbb{C}$ with $|z| < 1$, the hypergeometric function ${}_2F_1(a,b,c,z)$ is a power series of parameters a, b, c and variable z given by

$${}_2F_1(a,b,c,z) = \sum_{i=0}^{\infty}\frac{(a)_i(b)_i}{(c)_i}\frac{z^i}{i!} = 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \cdots, \tag{3.74}$$

where for $q = a, b, c$, we have

$$(q)_i = \begin{cases} 1, & \text{if } i = 0, \\ q(q+1)\cdots(q+i-1), & \text{if } i > 0, \end{cases} \tag{3.75}$$

which is called the Pochhammer symbol. The series terminates if either a or b is a non-positive integer, and is undefined if c is a non-positive integer.

The hypergeometric function ${}_2F_1(a,b,c,z)$ appears as the solution of a second-order linear ordinary differential equation. Furthermore, every polynomial is a specific example of the hypergeometric function, depending on the choice of parameters a, b, c.

Finally, using the hypergeometric function (3.74), the solution $X_{n,\ell}(\sin^2(\theta))$ of the radial function can be written as

$$X_{n,\ell}(\sin^2(\theta)) = \text{constant} \cdot \frac{\Gamma(\frac{1}{2}(n+\ell+N))}{\Gamma(\frac{1}{2}(2\ell+N))}(\sin^2(\theta))^{\frac{1}{2}\ell}(\cos^2(\theta))^{\frac{1}{4}+\frac{R^2}{2}\varpi}$$
$$ {}_2F_1\left(\frac{1}{2}(n+\ell+N)+R^2\varpi, -\frac{(n-\ell)}{2}, \ell+\frac{N}{2}, \sin^2(\theta)\right). \quad (3.76)$$

One finds the solution $X_{n,\ell}(\theta)$ by undoing a final change of variable as we defined previously [21], and we derive the radial wave function $X_{n,\ell}(r)$ by working out the trigonometric relation in the N-sphere.

4 Harmonic Oscillators in Hyperbolic N-Spaces

After working out the flat Euclidean N-space and the N-dimensional positively curved spherical space, we proceed with our final generalisation, that is, the simple harmonic oscillator in N-dimensional negatively curved hyperbolic space.

The structure of this Chapter is equivalent to Chapter 3 to highlight the similarities between spherical and hyperbolic space. In Section 4.1, we derive the classical dynamics and the Hamiltonian with the gnomonic projection. Subsequently in Section 4.2, the particle's orbits and its constants of motion are discussed and used to construct the associated quadratic Casimir function. Then, the classical dynamics are translated to quantum mechanics in Section 4.3, which we use to find the particle's energy levels and solve the radial wave function in Section 4.4.

4.1 Classical Dynamics in the Hyperbolic N-Space

To represent the N-dimensional hyperbolic N-space $\mathbb{H}^N$, we use the hyperboloid model [26]. However, we first introduce the notion of an N-dimensional pseudo-Euclidean space [14].

Definition 4.1. A pseudo-Euclidean space $\mathbb{E}^{k,N-k}$ is an affine Euclidean N-space together with a non-degenerate quadratic form $\mathcal{Q}(\mathbf{x})$ such that

$$\mathcal{Q}(\mathbf{x}) = x_1^2 + \cdots + x_k^2 - \left(x_{k+1}^2 + \cdots + x_N^2\right),$$

for $\mathbf{x} = (x_1, \ldots, x_N) \in \mathbb{R}^N$. If $k = N$, we end up with the Euclidean N-space, that is, $\mathbb{E}^{N,0} = \mathbb{E}^N$.

In the hyperboloid model, the hyperbolic N-space is represented as a subset of the $(N+1)$-dimensional pseudo-Euclidean space $\mathbb{E}^{N,1}$, that is, the set of all vectors $\mathbf{x} = (x_0, x_1, \ldots, x_N) \in \mathbb{R}^{N+1}$ such that

$$\mathcal{Q}(\mathbf{x}) = x_1^2 + \cdots + x_N^2 - x_0^2 = -R^2 \qquad \text{and} \qquad x_0 < 0, \tag{4.1}$$

for some positive constant R. The geometric object described by equation (4.1) is the lower sheet of an N-dimensional two-sheeted hyperboloid, for which $R > 0$ is the distance between its center of symmetry and its sheets. We use this model to study the hyperbolic N-space.

Now, consider a particle of mass m moving in the N-dimensional hyperboloid $\mathbb{H}^N$ that is embedded in the $(N+1)$-dimensional pseudo-Euclidean space $\mathbb{E}^{N,1}$. Using cartesian coordinates, the position of the particle at the point $Q \in \mathbb{H}^N$ is represented by the position vector $\mathbf{q} = (q_0, q_1, \ldots, q_N) \in \mathbb{R}^{N+1}$. In the hyperbolic N-space, the pseudo-metric tensor [26] is given by

$$g_{ij} = \begin{cases} -1, & \text{if } i = j = 1, \\ 1, & \text{if } i = j \neq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, to compute the distance between positions in $\mathbb{H}^N$, we use the hyperbolic pseudo-metric[1]

$$ds^2 = -dq_0^2 + dq_1^2 + \cdots + dq_N^2. \tag{4.2}$$

In a similar fashion, rather than positioning the center of symmetry C of the N-hyperboloid at the origin O, we let the highest point of the lower sheet coincide with the origin, such that

$$-(q_0 - R)^2 + q_1^2 + \cdots + q_N^2 = -R^2. \tag{4.3}$$

The position vector $\mathbf{c} = (R, 0, \ldots, 0) \in \mathbb{R}^{N+1}$ represents the position of the center of symmetry C. Again, the tangent space of the N-hyperboloid at the origin spans the Euclidean space $\mathbb{E}^N$,

$$T_{\mathbf{0}}\mathbb{H}^N = \left\{(0, x_1, \ldots, x_N) \in \mathbb{R}^{N+1} \mid x_1, \ldots, x_N \in \mathbb{R}\right\}. \tag{4.4}$$

In line with the hyperboloid model, we restrict the movement of the particle to the lower sheet of the N-hyperboloid,

$$\mathbb{H}^N_- = \left\{\mathbf{q} \in \mathbb{R}^{N+1} \mid -(q_0 - R)^2 + q_1^2 + \cdots + q_N^2 = -R^2,\ q_0 - R < 0\right\}, \tag{4.5}$$

for which the tangent space at the origin $T_{\mathbf{0}}\mathbb{H}^N$ remains unchanged. Analogously, this restriction leads to a connection between motion in the N-hyperboloid and the motion in the Euclidean N-space.

[1] That is, the pseudo-metric restricted to curves in the hyperbolic N-space.

With every point $Q \in \mathbb{H}^N_-$ we can associate a point $X \in T_{\mathbf{0}}\mathbb{H}^N$ by using gnomonic projection [26]. We draw a straight line from the center of symmetry C to the point Q, which intersects the tangent space at the point X. Similarly then, the position of the particle at the point X is represented by the position vector $\mathbf{x} = (0, x_1, \ldots, x_N) \in \mathbb{R}^{N+1}$.

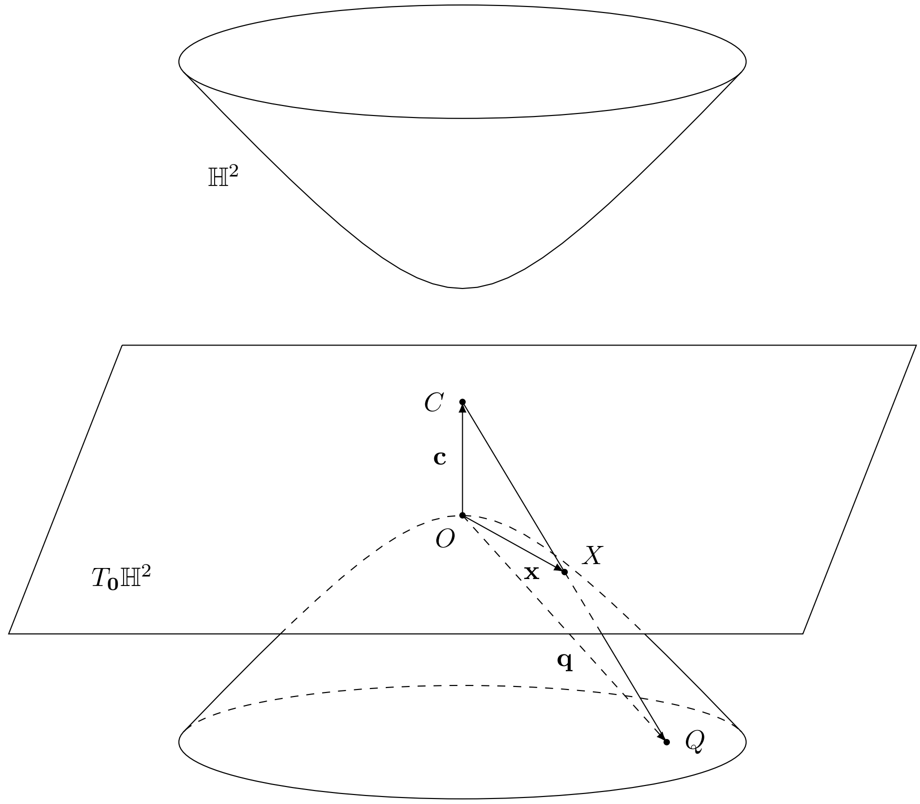

Figure 4.1 Gnomonic projection of Q in the upper sheet of $\mathbb{H}^2$ to X in the tangent space $T_{\mathbf{0}}\mathbb{H}^2$.

The construction given by the gnomonic projection does not differ from the method we described in section 3.1. Hence, similarly to equation (3.6), we have the relations

$$x_i = \frac{R}{R - q_0} q_i \qquad \text{and} \qquad q_i = \frac{R - q_0}{R} x_i \tag{4.6}$$

between the points Q in the lower sheet of the N-hyperboloid and X in the tangent space. Combining equation (4.3) and the second part of equation (4.6) gives

$$-(q_0 - R)^2 + \frac{(R - q_0)^2}{R^2}\left(x_1^2 + \cdots + x_N^2\right) = -R^2,$$

from which we derive

$$R - q_0 = \frac{R^2}{\sqrt{R^2 - x_1^2 - \cdots - x_N^2}}.$$

Similarly to equation (3.7) then, by writing $r^2 = x_1^2 + \cdots + x_N^2$ as in equation (2.12), the coordinates q_0 and q_i of he hyperbolic point Q in relation to the projection point X are given by

$$q_0 = R - \frac{R^2}{\sqrt{R^2 - r^2}} \qquad \text{and} \qquad q_i = \frac{R x_i}{\sqrt{R^2 - r^2}}. \tag{4.7}$$

To formulate the equations of motion of the particle in the N-hyperboloid, we again use Hamiltonian mechanics and the relations between Q and X given by equation (4.7) to write H in terms of $\mathbf{x}$ rather than $\mathbf{q}$. The kinetic energy of the particle in terms of $\mathbf{q}$ is

$$K = \frac{1}{2} m |\dot{\mathbf{q}}|^2 = \frac{1}{2} m \left(-\dot{q}_0^2 + \dot{q}_1^2 + \cdots + \dot{q}_N^2\right).$$

Substituting the relations from equation (4.7), we obtain the kinetic energy in terms of $\mathbf{x}$,

$$K = \frac{1}{2}m\left(\frac{R^2|\dot{\mathbf{x}}|^2}{R^2 - r^2} + \frac{R^2(\mathbf{x}\cdot\dot{\mathbf{x}})^2}{(R^2 - r^2)^2}\right). \tag{4.8}$$

The momentum $\mathbf{p} = (p_1, \ldots, p_N) \in \mathbb{R}^N$ of the particle conjugate to $\mathbf{x}$ is given by

$$\mathbf{p} = \frac{\partial K}{\partial \dot{\mathbf{x}}} = m\left(\frac{R^2\dot{\mathbf{x}}}{R^2 - r^2} + \frac{R^2\mathbf{x}(\mathbf{x}\cdot\dot{\mathbf{x}})}{(R^2 - r^2)^2}\right). \tag{4.9}$$

Remark 4.1. As the distance R between the center of symmetry and the sheets increases, the surrounding region of the origin of the N-hyperboloid flattens out towards the tangent space, and as R grows without bound, $\mathbb{H}^N_-$ will coincide with the tangent space $T_{\mathbf{0}}\mathbb{H}^N \simeq \mathbb{E}^N$. Because we have the limits

$$\lim_{R\to\infty} \frac{R^2}{R^2 - r^2} = \lim_{R\to\infty} \frac{1}{1 - r^2/R^2} = 1 \qquad \text{and} \qquad \lim_{R\to\infty} \frac{R^2}{(R^2 - r^2)^2} = \lim_{R\to\infty} \frac{1}{R^2 - 2r^2 + r^4/R^2} = 0,$$

the formulas for the kinetic energy (4.8) and conjugate momentum (4.9) of the particle in $\mathbb{H}^N_-$ become

$$\lim_{R\to\infty} K = \frac{1}{2}m|\dot{\mathbf{x}}|^2 \qquad \text{and} \qquad \lim_{R\to\infty} \mathbf{p} = m\dot{\mathbf{x}},$$

which are the formulas for the kinetic energy (2.7) and linear momentum (2.8) of the particle in $\mathbb{E}^N$.

Similar to the trigonometric relation between the points Q and X in the N-sphere, there exists an hyperbolic relation between Q and X in the hyperboloid model. Let $u \in [0, \infty)$ denote twice the area between $\mathbf{c}$, the radial line between C and Q and the lower sheet of the N-hyperboloid. Then, we have the hyperbolic relations

$$\sinh(u) = \frac{r}{\sqrt{R^2 - r^2}}, \qquad \cosh(u) = \frac{R}{\sqrt{R^2 - r^2}} \qquad \text{and} \qquad \tanh(u) = \frac{r}{R}.$$

Consequently, the relations from equation (4.7) can alternatively be written as

$$q_0 = R(1 - \cosh(u)) \qquad \text{and} \qquad q_i = \cosh(u)x_i. \tag{4.10}$$

Furthermore, this hyperbolic approach can be used to derive the linear momentum $\boldsymbol{\pi}$ of the particle. For a free particle moving in $\mathbb{H}^N$, its motion is confined to a great hyperbola, that is, the hyperbolic intersection of the N-sphere and a plane through the center of symmetry C. Again, using the gnomonic projection and the hyperbolic relations, the linear momentum of the particle is

$$\boldsymbol{\pi} = \mathbf{p} - \frac{\mathbf{x}(\mathbf{x}\cdot\mathbf{p})}{R^2} = \frac{mR^2\dot{\mathbf{x}}}{R^2 - r^2}, \tag{4.11}$$

for which we have

$$|\boldsymbol{\pi}|^2 = \frac{m^2R^4|\dot{\mathbf{x}}|^2}{(R^2 - r^2)^2}. \tag{4.12}$$

Besides the linear momentum, the particle also has orbital angular momentum about the center of symmetry of the N-hyperboloid. Similarly as before, the elements L_{ij} of the angular momentum tensor L are given by

$$L_{ij} = x_ip_j - x_jp_i = \frac{mR^2}{R^2 - r^2}\left(x_i\dot{x}_j - x_j\dot{x}_i\right), \tag{4.13}$$

for $i, j = 1, \ldots, N$. Then, by using our expressions (4.11) and (4.13) for the linear and angular momentum of the particle in the N-hyperboloid, respectively, the total kinetic energy of the particle can alternatively be written as the form

$$K = \frac{1}{2}m\left(|\boldsymbol{\pi}|^2 - \frac{|L|^2}{R^2}\right), \tag{4.14}$$

where, just as in equation (3.15), we have

$$|L|^2 = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} L_{ij}L_{ij} = |\mathbf{x}|^2|\mathbf{p}|^2 - (\mathbf{x}\cdot\mathbf{p})^2. \tag{4.15}$$

Using this formulation of the kinetic energy rather than equation (4.8), the Hamiltonian of the harmonic oscillator in the N-hyperboloid is given by

$$H = \frac{1}{2m}\left(|\boldsymbol{\pi}|^2 - \frac{|L|^2}{R^2}\right) + \frac{1}{2}m\omega^2 r^2. \tag{4.16}$$

Then, using the equations (4.12) and (4.15), we can express the Hamiltonian in terms of the conjugate momentum $\mathbf{p}$ of the particle,

$$H = \frac{1}{2m}\left(\frac{(R^2 - r^2)|\mathbf{p}|^2}{R^2} - \frac{(R^2 - r^2)(\mathbf{x}\cdot\mathbf{p})^2}{R^4}\right) + \frac{1}{2}m\omega^2 r^2. \tag{4.17}$$

Remark 4.2. Again, suppose the distance R between the center of symmetry and the sheets grows without bound. Because of the limits

$$\lim_{R\to\infty} \frac{R^2 - r^2}{R^2} = \lim_{R\to\infty}\left(1 - \frac{r^2}{R^2}\right) = 1 \qquad \text{and} \qquad \lim_{R\to\infty} \frac{R^2 - r^2}{R^4} = \lim_{R\to\infty}\left(\frac{1}{R^2} - \frac{r^2}{R^4}\right) = 0,$$

the formula for the Hamiltonian (4.17) of the particle in $\mathbb{H}_-^N$ becomes

$$\lim_{R\to\infty} H = \frac{|\mathbf{p}|^2}{2m} + \frac{1}{2}m\omega^2 r^2,$$

which is the familiar result for the Hamiltonian (2.14) of the particle in Euclidean N-space.

Finally, applying the hyperbolic relations of the N-hyperboloid again, we note that the potential U of the harmonic oscillator has the particular form

$$U = \frac{1}{2}m\omega^2 r^2 = \frac{1}{2}m\omega^2 R^2 \tanh^2(u). \tag{4.18}$$

Again, the potential of the particle is symmetric between the upper and lower sheet of the N-hyperboloid, and U is not defined for $u = \infty$. Once more, the possible orbits of the particle are confined to one of the two sheets, in which the path of motion takes the same form.

4.2 Constants of motion

Prior to deriving the particle's path of motion in the N-hyperboloid, we analyse the constants of motion of the particle. Similarly to the N-sphere, we use equation (4.17) for the Hamiltonian of the harmonic oscillator to show that an observable is conserved in the N-hyperboloid.

Theorem 4.3. *The Hamiltonian H of the hyperbolic harmonic oscillator is a constant of motion.*

Proof. Equation (4.17) shows that the Hamiltonian H does not depend explicitly on time. Furthermore, Lemma 2.3, (i), shows that $\{H, H\} = 0$, and therefore the Hamiltonian is a constant of motion. □

Once more, we notice that the angular momentum tensor L given by (2.24) and (4.13) in Euclidean N-space and in the N-hyperboloid, respectively, are defined in similar fashion. Again, this similarity suggests that the angular momentum in the N-hyperboloid is also a constant of motion.

Theorem 4.4. *The angular momentum tensor L of the hyperbolic harmonic oscillator is a constant of motion.*

Proof. Equation (4.13) shows that the matrix elements L_{ij} do not depend explicitly on time, hence it suffices to show that $\{L_{ij}, H\} = 0$. For $k = 1, \ldots, N$, the partial derivatives of the Hamiltonian are

$$\frac{\partial H}{\partial x_k} = \frac{x_k}{m}\left(\frac{(\mathbf{x}\cdot\mathbf{p})^2}{R^4} - \frac{|\mathbf{p}|^2}{R^2}\right) - \frac{(R^2 - r^2)(\mathbf{x}\cdot\mathbf{p})p_k}{mR^4} + m\omega^2 x_k, \tag{4.19}$$

$$\frac{\partial H}{\partial p_k} = \frac{R^2 - r^2}{m}\left(\frac{p_k}{R^2} - \frac{(\mathbf{x}\cdot\mathbf{p})x_k}{R^4}\right). \tag{4.20}$$

For $k \neq i, j$, the partial derivatives of the angular momentum tensor are given by

$$\frac{\partial L_{ij}}{\partial x_k} = \frac{\partial}{\partial x_k}[x_i p_j - x_j p_i] = 0 \qquad \text{and} \qquad \frac{\partial L_{ij}}{\partial p_k} = \frac{\partial}{\partial x_k}[x_i p_j - x_j p_i] = 0,$$

while for $k = i$ and $k = j$, respectively, we find the partial derivatives

$$\frac{\partial L_{ij}}{\partial x_i} = \frac{\partial}{\partial x_i}[x_i p_j - x_j p_i] = p_j \qquad \text{and} \qquad \frac{\partial L_{ij}}{\partial p_i} = \frac{\partial}{\partial p_i}[x_i p_j - x_j p_i] = -x_j,$$
$$\frac{\partial L_{ij}}{\partial x_j} = \frac{\partial}{\partial x_j}[x_i p_j - x_j p_i] = -p_i \qquad \text{and} \qquad \frac{\partial L_{ij}}{\partial p_j} = \frac{\partial}{\partial p_j}[x_i p_j - x_j p_i] = x_i.$$

Then, the Poisson bracket of the angular momentum tensor L_{ij} and the Hamiltonian H, we find

$$\begin{aligned}
\{L_{ij}, H\} &= \sum_{k=1}^{N}\left(\frac{\partial L_{ij}}{\partial x_k}\frac{\partial H}{\partial p_k} - \frac{\partial L_{ij}}{\partial p_k}\frac{\partial H}{\partial x_k}\right) = \frac{\partial L_{ij}}{\partial x_i}\frac{\partial H}{\partial p_i} - \frac{\partial L_{ij}}{\partial p_i}\frac{\partial H}{\partial x_i} + \frac{\partial L_{ij}}{\partial x_j}\frac{\partial H}{\partial p_j} - \frac{\partial L_{ij}}{\partial p_j}\frac{\partial H}{\partial x_j} \\
&= p_j\frac{R^2 - r^2}{m}\left(\frac{p_i}{R^2} - \frac{(\mathbf{x}\cdot\mathbf{p})x_i}{R^4}\right) + x_j\left(\frac{x_i}{m}\left(\frac{(\mathbf{x}\cdot\mathbf{p})^2}{R^4} - \frac{|\mathbf{p}|^2}{R^2}\right) - \frac{(R^2 - r^2)(\mathbf{x}\cdot\mathbf{p})p_i}{mR^4} + m\omega^2 x_i\right) \\
&\quad - p_i\frac{R^2 - r^2}{m}\left(\frac{p_j}{R^2} - \frac{(\mathbf{x}\cdot\mathbf{p})x_j}{R^4}\right) - x_i\left(\frac{x_j}{m}\left(\frac{(\mathbf{x}\cdot\mathbf{p})^2}{R^4} - \frac{|\mathbf{p}|^2}{R^2}\right) - \frac{(R^2 - r^2)(\mathbf{x}\cdot\mathbf{p})p_j}{mR^4} + m\omega^2 x_j\right) \\
&= \frac{(R^2 - r^2)(\mathbf{x}\cdot\mathbf{p})x_i p_j}{mR^4} + \frac{(R^2 - r^2)(\mathbf{x}\cdot\mathbf{p})x_j p_i}{mR^4} - \frac{(R^2 - r^2)(\mathbf{x}\cdot\mathbf{p})x_j p_i}{mR^4} - \frac{(R^2 - r^2)(\mathbf{x}\cdot\mathbf{p})x_i p_j}{mR^4} \\
&= 0,
\end{aligned}$$

which concludes the proof. □

As we discussed for the N-sphere, since the angular momentum of the particle is conserved throughout the motion, every projected orbit lies in the plane that is normal to the plane that corresponds to the angular momentum tensor. Geometrically, the same result holds for the orbit of a particle moving in hyperbolic N-space.

Let us use the polar coordinates (r, φ) to denote the particle's position in the plane of motion. Then, the conservation of the particle's angular momentum and energy (4.16) read

$$\frac{mR^2 r^2}{R^2 - r^2}\dot{\varphi} = |L| \qquad \text{and} \qquad \frac{1}{2m}\left(\frac{m^2 R^4}{(R^2 - r^2)^2}\dot{r}^2 + \frac{m^2 R^2 r^2}{R^2 - r^2}\dot{\varphi}^2\right) + \frac{1}{2}m\omega^2 r^2 = E. \tag{4.21}$$

Combining both equations yields a single differential equation that describes the particle's orbit,

$$\frac{|L|^2}{2m}\left(\frac{1}{r^4}\left(\frac{dr}{d\varphi}\right)^2 + \frac{1}{r^2}\right) + \frac{1}{2}m\omega^2 r^2 = E + \frac{|L|^2}{2mR^2}. \tag{4.22}$$

This result is similar to equation (3.22). The general solution $r(\varphi)$ of equation (4.22) takes the form

$$\frac{1}{r^2} = \left(\frac{mE}{|L|^2} + \frac{1}{2R^2}\right)\left(1 + \sqrt{1 - \frac{4m^2\omega^2|L|^2 R^4}{(2mR^2 E + |L|^2)^2}}\cos(2(\varphi - \varphi_0))\right), \tag{4.23}$$

where φ_0 is simply a constant of integration. For simplicity, let us assume that the plane of motion is spanned precisely by the vectors x_1 and x_2 and let $\varphi_0 = 0$. By introducing the constants

$$\mathcal{E} = E + \frac{|L|^2}{2mR^2} \qquad \text{and} \qquad \varepsilon = \sqrt{1 - \frac{\omega^2|L|^2}{\mathcal{E}^2}},$$

we are able to formulate equation (4.23) equivalently as

$$\frac{1}{r^2} = \frac{m\mathcal{E}}{|L|^2}(1 + \varepsilon\cos(2\varphi)), \tag{4.24}$$

which is precisely the same result as equation (3.24) with other constants $\mathcal{E}$ and ε. Consequently, equation (4.24) can be written as

$$\frac{m\mathcal{E}(\varepsilon + 1)}{|L|^2}x_1^2 + \frac{m\mathcal{E}(\varepsilon - 1)}{|L|^2}x_2^2 = 1, \tag{4.25}$$

which describes an ellipse whose center is positioned at the origin of the plane of motion. To relate this projected path of motion and the particle's path of motion in the hyperbolic N-space, we use gnomonic projection.

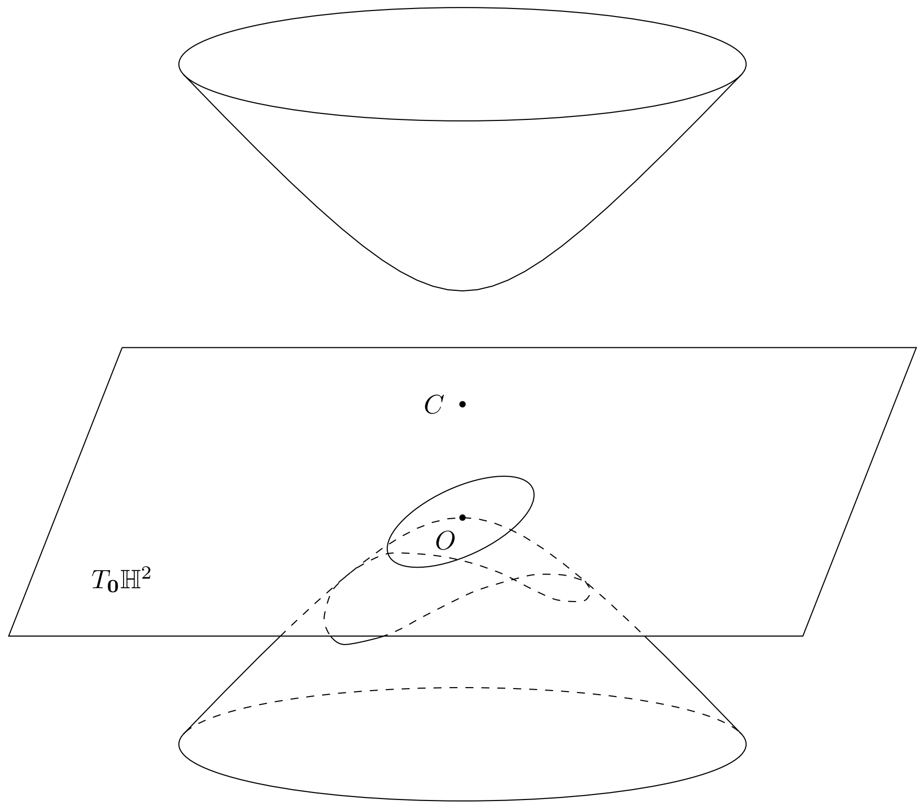

Figure 4.2 Elliptical path of motion in the lower sheet of $\mathbb{H}^2$ and in the tangent space $T_{\mathbf{0}}\mathbb{H}^2$.

The projected path of motion is restricted to a plane through the origin, which is a 2-dimensional subspace of the Euclidean N-space. Hence, the motion of the particle prior to the gnomonic projection was confined to the lower sheet of a 2-hyperboloid. Using our relation (4.6) between the N-hyperboloid and the Euclidean N-space, we are able to project the elliptical orbit (4.25) to the lower sheet of the N-hyperboloid.

Now, to summarise the particle's motion for the hyperbolic harmonic oscillator, the path of motion of a particle moving in the N-hyperboloid is a hyperbolic ellipse that is confined to the lower sheet of a 2-hyperboloid.

Let us return to our discussion on the constants of motion. In addition to the Hamiltonian H and the angular momentum tensor L, we consider a symmetric tensor S whose matrix elements S_{ij} are given by

$$S_{ij} = \frac{\pi_i \pi_j}{m} + m\omega^2 x_i x_j. \tag{4.26}$$

This tensor is equivalent to the symmetric tensor discussed in the N-sphere, and has similar properties for the hyperbolic N-space.

Theorem 4.5. *The symmetric tensor S of the hyperbolic harmonic oscillator is a constant of motion.*

Proof. Instead of using the Poisson bracket technique to show that the symmetric tensor S_{ij} is a constant of motion, we show that its total time derivative is zero. Differentiating equation (4.26) with respect to time, we obtain

$$\dot{S}_{ij} = \frac{\dot{\pi}_i \pi_j + \pi_i \dot{\pi}_j}{m} + m\omega^2(\dot{x}_i x_j + x_i \dot{x}_j). \tag{4.27}$$

To evaluate this equation, we first derive expressions for $\dot{\pi}_i, \dot{\pi}_j$ and $\dot{x}_i, \dot{x}_j$. Using Hamilton's equations (2.15) and the partial derivatives (4.20) of the Hamiltonian, we find

$$\dot{x}_i = \frac{\partial H}{\partial p_i} = \frac{R^2 - r^2}{m}\left(\frac{p_i}{R^2} - \frac{(\mathbf{x}\cdot\mathbf{p})x_i}{R^4}\right) \qquad \text{and} \qquad \dot{x}_j = \frac{\partial H}{\partial p_j} = \frac{R^2 - r^2}{m}\left(\frac{p_j}{R^2} - \frac{(\mathbf{x}\cdot\mathbf{p})x_j}{R^4}\right).$$

Differentiating the components of equation (4.11) with respect to time yields

$$\dot{\pi}_i = \dot{p}_i - \frac{(\mathbf{x}\cdot\mathbf{p})\dot{x}_i}{R^2} - \frac{(\dot{\mathbf{x}}\cdot\mathbf{p} + \mathbf{x}\cdot\dot{\mathbf{p}})x_i}{R^2} \qquad \text{and} \qquad \dot{\pi}_j = \dot{p}_j - \frac{(\mathbf{x}\cdot\mathbf{p})\dot{x}_j}{R^2} - \frac{(\dot{\mathbf{x}}\cdot\mathbf{p} + \mathbf{x}\cdot\dot{\mathbf{p}})x_j}{R^2}.$$

Again, using Hamilton's equations (2.15) and the partial derivatives (4.19) of the Hamiltonian, we find

$$\dot{p}_i = -\frac{\partial H}{\partial x_i} = -\frac{x_i}{m}\left(\frac{(\mathbf{x}\cdot\mathbf{p})^2}{R^4} - \frac{|\mathbf{p}|^2}{R^2}\right) + \frac{(R^2-r^2)(\mathbf{x}\cdot\mathbf{p})p_i}{mR^4} - m\omega^2 x_i,$$
$$\dot{p}_j = -\frac{\partial H}{\partial x_j} = -\frac{x_j}{m}\left(\frac{(\mathbf{x}\cdot\mathbf{p})^2}{R^4} - \frac{|\mathbf{p}|^2}{R^2}\right) + \frac{(R^2-r^2)(\mathbf{x}\cdot\mathbf{p})p_j}{mR^4} - m\omega^2 x_j.$$

Using the expressions for $\dot{x}_i, \dot{x}_j$ and $\dot{p}_i, \dot{p}_j$, we can construct the vector equations

$$\dot{\mathbf{x}} = \frac{R^2-r^2}{m}\left(\frac{\mathbf{p}}{R^2} - \frac{\mathbf{x}(\mathbf{x}\cdot\mathbf{p})}{R^4}\right),$$
$$\dot{\mathbf{p}} = -\frac{\mathbf{x}}{m}\left(\frac{(\mathbf{x}\cdot\mathbf{p})^2}{R^4} - \frac{|\mathbf{p}|^2}{R^2}\right) + \frac{(R^2-r^2)\mathbf{p}(\mathbf{x}\cdot\mathbf{p})}{mR^4} - m\omega^2\mathbf{x}.$$

Substituting the expressions for $\dot{\mathbf{x}}, \dot{\mathbf{p}}$ and $\dot{p}_i, \dot{p}_j$ to derive expressions for $\dot{\pi}_i, \dot{\pi}_j$ eventually gives

$$\dot{\pi}_i = -\frac{m\omega^2(R^2-r^2)x_i}{R^2} \qquad \text{and} \qquad \dot{\pi}_j = -\frac{m\omega^2(R^2-r^2)x_j}{R^2}.$$

Then, using our expressions for $\dot{\pi}_i, \dot{\pi}_j$ and $\dot{x}_i, \dot{x}_j$ in equation (4.27) for the total time derivative $\dot{S}_{ij}$ of the symmetric tensor, we obtain

$$\begin{aligned}
\dot{S}_{ij} &= \frac{1}{m}\left(-\frac{m\omega^2(R^2-r^2)x_i}{R^2}\left(p_j - \frac{(\mathbf{x}\cdot\mathbf{p})x_j}{R^2}\right) - \left(p_i - \frac{(\mathbf{x}\cdot\mathbf{p})x_i}{R^2}\right)\frac{m\omega^2(R^2-r^2)x_j}{R^2}\right) \\
&\quad + m\omega^2\left(\frac{R^2-r^2}{m}\left(\frac{p_i}{R^2} - \frac{(\mathbf{x}\cdot\mathbf{p})x_i}{R^4}\right)x_j + x_i\frac{R^2-r^2}{m}\left(\frac{p_j}{R^2} - \frac{(\mathbf{x}\cdot\mathbf{p})x_j}{R^4}\right)\right) \\
&= -\omega^2(R^2-r^2)\left(\frac{x_ip_j + x_jp_i}{R^2} - \frac{2(\mathbf{x}\cdot\mathbf{p})x_ix_j}{R^4}\right) + \omega^2(R^2-r^2)\left(\frac{x_ip_j + x_jp_i}{R^2} - \frac{2(\mathbf{x}\cdot\mathbf{p})x_ix_j}{R^4}\right) \\
&= 0,
\end{aligned}$$

which concludes the proof. □

Similar to the Euclidean N-space and the N-sphere, not only the Poisson bracket of the symmetric tensor S with the Hamiltonian yields an useful relation. Just as we discussed before, the Poisson bracket of the symmetric tensor with itself gives the following algebra on the N-hyperboloid.

Theorem 4.6. *The Poisson bracket of the symmetric tensor S with itself is given by*

$$\{S_{ij}, S_{kl}\} = \omega^2(L_{ik}\delta_{jl} + L_{jk}\delta_{il} + L_{jl}\delta_{ik}) - \frac{1}{R^2}(L_{ik}S_{jl} + L_{il}S_{jk} + L_{jk}S_{il} + L_{jl}S_{ik}), \tag{4.28}$$

for $i, j, k, l = 1, \ldots, N$.

Proof. To confine our notation, let us introduce the abbreviated notation

$$\mathcal{L} = L_{ik}\delta_{jl} + L_{il}\delta_{jk} + L_{jk}\delta_{il} + L_{jl}\delta_{ik} \qquad \text{and} \qquad \mathcal{S}_{\mathcal{L}} = L_{ik}S_{jl} + L_{il}S_{jk} + L_{jk}S_{il} + L_{jl}S_{ik}.$$

With these shortened forms, equation (4.28) can simply be written as

$$\{S_{ij}, S_{kl}\} = \omega^2\mathcal{L} - \frac{\mathcal{S}_{\mathcal{L}}}{mR^2}. \tag{4.29}$$

To show that (4.29) holds, we consider some individual cases. First of all, suppose that all indices are distinct. Using equation (2.18), the left-hand side of equation (4.29) reads

$$\{S_{ij}, S_{kl}\} = \sum_{n=1}^{N}\left(\frac{\partial S_{ij}}{\partial x_n}\frac{\partial S_{kl}}{\partial p_n} - \frac{\partial S_{ij}}{\partial p_n}\frac{\partial S_{kl}}{\partial x_n}\right) = D_i + D_j + D_k + D_l,$$

where we used the notation

$$D_i = \frac{\partial S_{ij}}{\partial x_i}\frac{\partial S_{kl}}{\partial p_i} - \frac{\partial S_{ij}}{\partial p_i}\frac{\partial S_{kl}}{\partial x_i} \qquad \text{and} \qquad D_j = \frac{\partial S_{ij}}{\partial x_j}\frac{\partial S_{kl}}{\partial p_j} - \frac{\partial S_{ij}}{\partial p_j}\frac{\partial S_{kl}}{\partial x_j},$$
$$D_k = \frac{\partial S_{ij}}{\partial x_k}\frac{\partial S_{kl}}{\partial p_k} - \frac{\partial S_{ij}}{\partial p_k}\frac{\partial S_{kl}}{\partial x_k} \qquad \text{and} \qquad D_l = \frac{\partial S_{ij}}{\partial x_l}\frac{\partial S_{kl}}{\partial p_l} - \frac{\partial S_{ij}}{\partial p_l}\frac{\partial S_{kl}}{\partial x_l}.$$

Evaluating these linear combinations of partial derivatives yields

$$
\begin{aligned}
D_i =& \frac{(x_k p_l + x_l p_k) p_i p_j}{m^2 R^2} - \frac{2(\mathbf{x}\cdot\mathbf{p}) x_k x_l p_i p_j}{m^2 R^4} - \frac{(\mathbf{x}\cdot\mathbf{p})^2 (x_k p_l + x_l p_k) x_i x_j}{m^2 R^6} - \frac{\omega^2 (x_k p_l + x_l p_k) x_i x_j}{R^2} \\
&- \frac{(\mathbf{x}\cdot\mathbf{p})(x_i p_j - x_j p_i)}{m^2 R^6} \left(2(\mathbf{x}\cdot\mathbf{p}) x_k x_l - R^2 (x_k p_l + x_l p_k)\right) + \frac{2(\mathbf{x}\cdot\mathbf{p}) x_i x_j x_k x_l}{m^2 R^8} \left((\mathbf{x}\cdot\mathbf{p})^2 + m^2\omega^2 R^4\right), \\
D_j =& \frac{(x_k p_l + x_l p_k) p_i p_j}{m^2 R^2} - \frac{2(\mathbf{x}\cdot\mathbf{p}) x_k x_l p_i p_j}{m^2 R^4} - \frac{(\mathbf{x}\cdot\mathbf{p})^2 (x_k p_l + x_l p_k) x_i x_j}{m^2 R^6} - \frac{\omega^2 (x_k p_l + x_l p_k) x_i x_j}{R^2} \\
&+ \frac{(\mathbf{x}\cdot\mathbf{p})(x_i p_j - x_j p_i)}{m^2 R^6} \left(2(\mathbf{x}\cdot\mathbf{p}) x_k x_l - R^2 (x_k p_l + x_l p_k)\right) + \frac{2(\mathbf{x}\cdot\mathbf{p}) x_i x_j x_k x_l}{m^2 R^8} \left((\mathbf{x}\cdot\mathbf{p})^2 + m^2\omega^2 R^4\right), \\
D_k =& -\frac{(x_i p_j + x_j p_i) p_k p_l}{m^2 R^2} + \frac{2(\mathbf{x}\cdot\mathbf{p}) x_i x_j p_k p_l}{m^2 R^4} + \frac{(\mathbf{x}\cdot\mathbf{p})^2 (x_i p_j + x_j p_i) x_k x_l}{m^2 R^6} + \frac{\omega^2 (x_i p_j + x_j p_i) x_k x_l}{R^2} \\
&+ \frac{(\mathbf{x}\cdot\mathbf{p})(x_i p_j - x_j p_i)}{m^2 R^6} \left((\mathbf{x}\cdot\mathbf{p}) x_k x_l - R^2 (x_k p_l + x_l p_k)\right) - \frac{2(\mathbf{x}\cdot\mathbf{p}) x_i x_j x_k x_l}{m^2 R^8} \left((\mathbf{x}\cdot\mathbf{p})^2 + m^2\omega^2 R^4\right), \\
D_l =& -\frac{(x_i p_j + x_j p_i) p_k p_l}{m^2 R^2} + \frac{2(\mathbf{x}\cdot\mathbf{p}) x_i x_j p_k p_l}{m^2 R^4} + \frac{(\mathbf{x}\cdot\mathbf{p})^2 (x_i p_j + x_j p_i) x_k x_l}{m^2 R^6} + \frac{\omega^2 (x_i p_j + x_j p_i) x_k x_l}{R^2} \\
&- \frac{(\mathbf{x}\cdot\mathbf{p})(x_i p_j - x_j p_i)}{m^2 R^6} \left(2(\mathbf{x}\cdot\mathbf{p}) x_k x_l - R^2 (x_k p_l + x_l p_k)\right) - \frac{2(\mathbf{x}\cdot\mathbf{p}) x_i x_j x_k x_l}{m^2 R^8} \left((\mathbf{x}\cdot\mathbf{p})^2 + m^2\omega^2 R^4\right).
\end{aligned}
$$

For the right-hand side, it is clear that $\mathcal{L} = 0$ as a result of the Kronecker deltas, and using equations (4.13) and (4.26) for the angular momentum tensor and symmetric tensor, respectively, one derives

$$
\begin{aligned}
\mathcal{S}_{\mathcal{L}} =& \frac{2(x_i p_j + x_j p_i) p_k p_l}{m} - \frac{4(\mathbf{x}\cdot\mathbf{p}) x_i x_j p_k p_l}{m R^2} + \frac{2(\mathbf{x}\cdot\mathbf{p})^2 (x_k p_l + x_l p_k) x_i x_j}{m R^4} + 2m\omega^2 (x_k p_l + x_l p_k) x_i x_j \\
&- \frac{2(x_k p_l + x_l p_k) p_i p_j}{m} + \frac{4(\mathbf{x}\cdot\mathbf{p}) x_k x_l p_i p_j}{m R^2} - \frac{2(\mathbf{x}\cdot\mathbf{p})^2 (x_i p_j + x_j p_i) x_k x_l}{m R^4} - 2m\omega^2 (x_i p_j + x_j p_i) x_k x_l.
\end{aligned}
$$

By comparison of these results, it is clear that we have

$$
\{S_{ij}, S_{kl}\} = D_i + D_j + D_k + D_l = \omega^2 \mathcal{L} - \frac{\mathcal{S}_{\mathcal{L}}}{m R^2}.
$$

Second, suppose that one pair of indices are equal and that the remaining two indices are distinct. Without loss of generality, assume that $i = k$ and $j \neq l$. With equation (2.18) and similar notation as before, the left-hand side of (4.29) reads

$$
\{S_{ij}, S_{kl}\} = \sum_{n=1}^{N} \left(\frac{\partial S_{ij}}{\partial x_n} \frac{\partial S_{il}}{\partial p_n} - \frac{\partial S_{ij}}{\partial p_n} \frac{\partial S_{il}}{\partial x_n} \right) = D_i + D_j + D_l.
$$

Again, evaluating the linear combinations of partial derivatives gives

$$
\begin{aligned}
D_i =& -(x_i p_l - x_l p_i) \left(\frac{p_j p_i}{m^2 R^2} + \frac{(\mathbf{x}\cdot\mathbf{p})^2 x_j x_i}{m^2 R^6} + \frac{\omega^2 x_j x_i}{R^2} \right) + \frac{2(\mathbf{x}\cdot\mathbf{p})^2 x_i^2 x_j p_l}{m^2 R^6} + \omega^2 (x_j p_l - x_l p_j) \\
&- (x_j p_i - x_i p_j) \left(\frac{p_i p_l}{m^2 R^2} + \frac{(\mathbf{x}\cdot\mathbf{p})^2 x_i x_l}{m^2 R^6} + \frac{\omega^2 x_i x_l}{R^2} \right) - \frac{2(\mathbf{x}\cdot\mathbf{p})^2 x_i^2 x_l p_j}{m^2 R^6}, \\
D_j =& \frac{(x_i p_l + x_l p_i) p_i p_j}{m^2 R^2} - \frac{2(\mathbf{x}\cdot\mathbf{p}) x_i x_l p_i p_j}{m^2 R^4} - \frac{(\mathbf{x}\cdot\mathbf{p})^2 (x_i p_l + x_l p_i) x_i x_j}{m^2 R^6} - \frac{\omega^2 (x_i p_l + x_l p_i) x_i x_j}{R^2} \\
&+ (x_j p_i - x_i p_j) \frac{(\mathbf{x}\cdot\mathbf{p})(x_i p_l + x_l p_i)}{m^2 R^4} + \frac{2(\mathbf{x}\cdot\mathbf{p}) x_i x_j x_l}{m^2 R^8} \left((\mathbf{x}\cdot\mathbf{p})^2 x_i - R^2 (\mathbf{x}\cdot\mathbf{p}) p_i + m^2\omega^2 R^4 x_i\right) \\
&+ \frac{2(\mathbf{x}\cdot\mathbf{p})^2 x_i^2 x_l p_j}{m^2 R^6}, \\
D_l =& -\frac{(x_i p_j + x_j p_i) p_i p_l}{m^2 R^2} + \frac{2(\mathbf{x}\cdot\mathbf{p}) x_i x_j p_i p_l}{m^2 R^4} + \frac{(\mathbf{x}\cdot\mathbf{p})^2 (x_i p_j + x_j p_i) x_i x_l}{m^2 R^6} + \frac{\omega^2 (x_i p_j + x_j p_i) x_i x_l}{R^2} \\
&+ (x_i p_l - x_l p_i) \frac{(\mathbf{x}\cdot\mathbf{p})(x_j p_i + x_i p_j)}{m^2 R^4} - \frac{2(\mathbf{x}\cdot\mathbf{p}) x_i x_j x_l}{m^2 R^8} \left((\mathbf{x}\cdot\mathbf{p})^2 x_i - R^2 (\mathbf{x}\cdot\mathbf{p}) p_i + m^2\omega^2 R^4 x_i\right) \\
&- \frac{2(\mathbf{x}\cdot\mathbf{p})^2 x_i^2 x_j p_l}{m^2 R^6}.
\end{aligned}
$$

For the right-hand side, we find $\mathcal{L} = L_{jl} = x_j p_l - x_l p_j$ due to the Kronecker deltas, and again using equations (4.13) and (4.26) for the angular momentum tensor and symmetric tensor, respectively, one finds

$$\begin{aligned}\mathcal{S}_{\mathcal{L}} = (x_i p_l - x_l p_i)&\left(\frac{p_j p_i}{m} - \frac{(\mathbf{x}\cdot\mathbf{p})(x_j p_i + x_i p_j)}{mR^2} + \frac{(\mathbf{x}\cdot\mathbf{p})^2 x_j x_i}{mR^4} + m\omega^2 x_j x_i\right)\\ + (x_j p_i - x_i p_j)&\left(\frac{p_i p_l}{m} - \frac{(\mathbf{x}\cdot\mathbf{p})(x_i p_l + x_l p_i)}{mR^2} + \frac{(\mathbf{x}\cdot\mathbf{p})^2 x_i x_l}{mR^4} + m\omega^2 x_i x_l\right)\\ + (x_j p_l - x_l p_j)&\left(\frac{p_i^2}{m} - \frac{2(\mathbf{x}\cdot\mathbf{p})x_i p_i}{mR^2} + \frac{(\mathbf{x}\cdot\mathbf{p})^2 x_i^2}{mR^4} + m\omega^2 x_i^2\right).\end{aligned}$$

Then, by comparing the sums on the both the left-hand and right-hand side of (4.29), one yields the result

$$\{S_{ij}, S_{kl}\} = D_i + D_j + D_l = \omega^2\mathcal{L} - \frac{\mathcal{S}_{\mathcal{L}}}{mR^2}.$$

Finally, suppose that two pairs of indices are equal. Without loss of generality, assume that $i = k$ and $j = l$. By Lemma 2.3, (i) the left-hand side of (4.29) reads

$$\{S_{ij}, S_{kl}\} = \{S_{ij}, S_{ij}\} = 0.$$

For the right-hand side, using the Kronecker deltas and the skew-symmetry of the angular momentum tensor L, we have

$$\mathcal{L} = L_{ii}\cdot 1 + L_{ij}\cdot 0 + L_{ji}\cdot 0 + L_{jj}\cdot 1 = 0,$$

and additionally by the symmetry of the symmetric tensor S, we find

$$\mathcal{S}_{\mathcal{L}} = L_{ii}S_{jj} + L_{ij}S_{ji} + L_{ji}S_{ij} + L_{jj}S_{ii} = L_{ij}S_{ij} - L_{ij}S_{ij} = 0.$$

Hence, we conclude that equation (4.29) holds. This concludes the proof. □

Similar to our discussion for the N-sphere, we can construct two scalars I_1 and I_2 from the matrix elements S_{ij} of the symmetric tensor. They are

$$I_1 = \mathrm{tr}(S) = 2H + \frac{|L|^2}{R^2} \qquad \text{and} \qquad I_2 = \mathrm{tr}(S^2) - \mathrm{tr}(S)^2 = -2\omega^2|L|^2, \tag{4.30}$$

which are independent from each other. The generators of the dynamical group SU(N) are once more the angular momentum tensor L and a traceless symmetric tensor T that satisfy the Poisson bracket relations

$$\{T_{ij}, H\} = 0 \qquad \text{and} \qquad \{T_{ij}, T_{kl}\} = L_{ijk}\delta_{jl} + L_{il}\delta_{jk} + L_{jk}\delta_{il} + L_{jl}\delta_{ik}. \tag{4.31}$$

To construct such a tensor, we can use the same approach as for the N-sphere. For two arbitrary functions f and g, the most general traceless symmetric tensor $\mathcal{T}$ we can consider is

$$\mathcal{T}_{ij} = fS_{ij} + \omega^2 g L_{ik}L_{kj},$$

for $i, j, k = 1, \ldots, N$. The corresponding two scalars J_1 and J_2 associated with this tensor are

$$J_1 = \mathrm{tr}(\mathcal{T}) = fI_1 + gI_2 \qquad \text{and} \qquad J_2 = \mathrm{tr}(\mathcal{T}^2) - \mathrm{tr}(\mathcal{T})^2 = \left(f^2 - fgI_1 - \frac{1}{2}g^2 I_2\right)I_2, \tag{4.32}$$

and the Poisson bracket of such a general symmetric tensor with itself is given by

$$\{\mathcal{T}_{ij}, \mathcal{T}_{kl}\} = L_{ik}U_{jl} + L_{il}U_{jk} + L_{jk}L_{il} + L_{jl}U_{ik},$$

for $i, j, k, l = 1, \ldots, N$, where we have

$$U_{ij} = \frac{\omega^2 J_2 \delta_{ij}}{I_2} - \mathcal{T}_{ij}\frac{\partial}{\partial|L|^2}[J_1] - L_{ik}L_{kj}\frac{\partial}{\partial|L|^2}\left[\frac{\omega^2 J_2}{I_2}\right]. \tag{4.33}$$

As we derived for the N-sphere, the Poisson brackets (4.31) are obtained for

$$T_{ij} = \mathcal{T}_{ij} - \frac{\mathrm{tr}(\mathcal{T})\delta_{ij}}{N}, \tag{4.34}$$

on the condition that the functions f and g are chosen such that

$$J_1 = A(H) \qquad \text{and} \qquad J_2 = -2|L|^2, \tag{4.35}$$

where $A(H)$ is an arbitrary function of the Hamiltonian H. Just as in the previous chapters, the quadratic Casimir function C of the special unitary group SU(N) satisfies the relation

$$\sum_{i=1}^{N}\sum_{j=1}^{N}(T_{ij}T_{ij} + L_{ij}L_{ij}) = \frac{4(N-1)}{N}C(H), \tag{4.36}$$

where

$$C(H) = \frac{1}{4}A(H)^2. \tag{4.37}$$

To determine this arbitrary function $A(H)$ of the Hamiltonian in the hyperbolic N-space, again we consider the circular orbits of the particle. For the harmonic oscillator, these orbits are characterised by the two non-zero eigenvalues of the symmetric tensor S, which occurs whenever we have

$$I_1^2 + 2I_2 = 0.$$

This condition can be written in a more explicit form, which then reads

$$E = \omega|L| + \frac{|L|^2}{2R^2}. \tag{4.38}$$

Again, we have the same condition for the traceless symmetric tensor T, that is,

$$J_1^2 + 2J_2 = 0,$$

which can be written in the explicit form

$$\frac{1}{4}A(E)^2 = |L|^2. \tag{4.39}$$

The two requirements that are given by equations (4.38) and (4.39) can now be used to determine this function $A(E)$. Consequently, the Hamiltonian can again be expressed as a function of the quadratic Casimir function C, that is,

$$H = \omega\sqrt{C} - \frac{C}{2R^2}. \tag{4.40}$$

Remark 4.7. As the radius R grows without bound, the Hamiltonian H as a function of the quadratic Casimir function C becomes in the limit

$$\lim_{R\to\infty} H = \omega\sqrt{C},$$

which is exactly the Hamiltonian as a function of the quadratic Casimir function C (2.42) in the Euclidean N-space.

4.3 Quantum Dynamics in the Hyperbolic N-Space

As we discussed in the previous chapters, observables in quantum mechanics are required to be Hermitian operators. Hence, the classical linear momentum (4.11) is replaced by its Hermitian counterpart, which reads

$$\hat{\boldsymbol{\pi}} = \hat{\mathbf{p}} - \frac{\hat{\mathbf{x}}(\hat{\mathbf{x}}\cdot\hat{\mathbf{p}}) + (\hat{\mathbf{x}}\cdot\hat{\mathbf{p}})\hat{\mathbf{x}}}{2R^2}. \tag{4.41}$$

Then, the Hamiltonian of the quantum harmonic oscillator is given by

$$\hat{H} = \frac{1}{2m}\left((\hat{\boldsymbol{\pi}}\cdot\hat{\boldsymbol{\pi}}) - \frac{|\hat{L}|^2}{R^2}\right) + \frac{1}{2}m\omega^2 r^2, \tag{4.42}$$

where we have $\hat{L}_{ij} = \hat{x}_i\hat{p}_j - \hat{x}_j\hat{p}_i$. Moreover, the classical symmetric tensor S as given by equation (4.26) is replaced by a Hermitian symmetric operator $\hat{S}$, whose matrix elements $\hat{S}_{ij}$ are

$$\hat{S}_{ij} = \frac{\hat{\pi}_i\hat{\pi}_j + \hat{\pi}_j\hat{\pi}_i}{2m} + m\omega^2\hat{x}_i\hat{x}_j. \tag{4.43}$$

Using the fundamental commutation relations (1.26) and the commutation properties established in Lemma 1.9 and Theorem 3.8, we have the same result in hyperbolic N-space as in the N-sphere, namely

$$[\hat{H}, \hat{H}] = \hat{0}, \qquad [\hat{L}_{ij}, \hat{H}] = \hat{0} \qquad \text{and} \qquad [\hat{S}_{ij}, \hat{H}] = \hat{0}. \tag{4.44}$$

Thus, the Hamiltonian operator $\hat{H}$, the angular momentum tensor $\hat{L}$ and the symmetric tensor $\hat{S}$ are also constants of motion in hyperbolic N-space. Similarly, the Poisson bracket relation (4.28) are by the commutation relation

$$\begin{aligned}[\hat{S}_{ij}, \hat{S}_{kl}] = i\hbar\omega^2(\hat{L}_{ik}\delta_{jl} + \hat{L}_{il}\delta_{jk} + \hat{L}_{jk}\delta_{il} + \hat{L}_{jl}\delta_{ik}) - \frac{i\hbar(\hat{L}_{ik}\delta_{jl} + \hat{L}_{il}\delta_{jk} + \hat{L}_{jk}\delta_{il} + \hat{L}_{jl}\delta_{ik})}{4R^4} \\ - \frac{i\hbar(\hat{L}_{ik}\hat{S}_{jl} + \hat{L}_{il}\hat{S}_{jk} + \hat{L}_{jk}\hat{S}_{il} + \hat{L}_{jl}\hat{S}_{ik}) + i\hbar(\hat{S}_{ik}\hat{L}_{jl} + \hat{S}_{il}\hat{L}_{jk} + \hat{S}_{jk}\hat{L}_{il} + \hat{S}_{jl}\hat{L}_{ik})}{2R^2}.\end{aligned} \tag{4.45}$$

Similar to the independent scalars (4.30) we derived from the classical symmetric tensor, the quantum symmetric tensor has the independent commuting operators

$$\hat{I}_1 = \mathrm{tr}(\hat{S}) = 2\hat{H} + \frac{|\hat{L}|^2}{R^2}, \tag{4.46}$$

and

$$\hat{I}_2 = \mathrm{tr}(\hat{S}^2) - \mathrm{tr}(\hat{S})^2 = -\omega^2(2|\hat{L}|^2 + N(N-1)\hat{I}) + \frac{1}{R^2}\left(2(N-1)\hat{H} + \frac{N|\hat{L}|^2}{R^2} + \frac{|\hat{L}|^2}{2R^2}\right). \tag{4.47}$$

In a similar way as equations (4.28) and (4.30) yielded us with a method to formulate the Hamiltonian as a function of the quadratic Casimir function C of the dynamical group SU(N), equations (4.45), (4.46) and (4.47) lead to the formulation

$$\hat{H} = \sqrt{\omega^2 + \frac{1}{4R^4}}\sqrt{\hat{C} + \frac{N^2\hat{I}}{4}} - \frac{\hat{C}}{2R^2} - \frac{N\hat{I}}{4R^2}. \tag{4.48}$$

Hence, we conclude that the hyperbolic quantum harmonic oscillator is also symmetric upon rotation of the N-hyperboloid and $\hat{C}$ is the quadratic Casimir operator.

4.4 Energy Levels and Eigenfunctions in the Hyperbolic N-Space

To solve the eigenvalue problem and find the energy levels of the particle, we follow the same procedure as we previously did for the N-sphere in section 3.4. Because the harmonic oscillator is symmetric upon rotation of the N-hyperboloid, we can use the hyperspherical coordinates (r, Φ) to denote the particle's position, where Φ represents the $N-1$ angular variables $\varphi_1, \dots, \varphi_{N-1}$. To solve the N-dimensional time-independent Schrödinger equation for the quantum harmonic oscillator, that is,

$$\hat{H}\psi_{n,\ell,m_\ell}(r, \Phi) = E_n\psi(r, \Phi), \tag{4.49}$$

where E_n denotes the nth energy level and the integers n, ℓ and m_ℓ represent the principal, azimuthal and magnetic quantum numbers, respectively, we use once more the method of separation of variables,

$$\psi_{n,\ell,m_\ell}(r, \Phi) = X_{n,\ell}(r)Y_\ell^{m_\ell}(\Phi). \tag{4.50}$$

This method yields two separate equations, namely the hyperangular and radial equation, respectively. The first equation reads

$$\Lambda^2_{N-1}Y_\ell^{m_\ell}(\Phi) = \ell(\ell + N - 2)Y_\ell^{m_\ell}(\Phi), \tag{4.51}$$

where $\ell(\ell+N-2)$ are the eigenvalues [9] of the hyperangular momentum operator Λ^2_{N-1} with $N > 1$ and $Y_\ell^{m_\ell}(\Phi)$ are known as the hyperspherical harmonics. Using the hyperbolic relations we have established previously and the additional convention $m = \hbar = 1$, the radial equation is

$$\left(\frac{d^2}{du^2} + (N-1)\coth(u)\frac{d}{du} - \ell(\ell + N - 2)\,\mathrm{csch}^2(u) - \omega^2R^4\tanh^2(u) - 2R^2E_n\right)X_{n,\ell} = 0. \tag{4.52}$$

To rewrite the radial equation (4.52), let us define the differential operators

$$\hat{O}_+ = (2\ell + N)\coth(u)\frac{d}{du} - \ell(2\ell + N)\coth^2(u) - \ell(\ell + N - 1) - 2R^2 E_n \tag{4.53}$$

and

$$\hat{O}_- = -(2\ell + N)\coth(u)\frac{d}{du} - (\ell + N)(2\ell + N)\coth^2(u) - (\ell + 1)(\ell + N) - 2R^2 E_n. \tag{4.54}$$

Consequently, one finds that we can write

$$\begin{aligned}\hat{O}_+\hat{O}_- X_{n,\ell+2} - \left(\left[2E_nR^2 + (\ell+1)(\ell+N)\right]\left[2E_nR^2 + \ell(\ell+N-1)\right] - (2\ell+N)^2\omega^2R^4\right) X_{n,\ell+2} \\ = 0,\end{aligned} \tag{4.55}$$

and

$$\begin{aligned}\hat{O}_-\hat{O}_+ X_{n,\ell} - \left(\left[2E_nR^2 + (\ell+1)(\ell+N)\right]\left[2E_nR^2 + \ell(\ell+N-1)\right] - (2\ell+N)^2\omega^2R^4\right) X_{n,\ell} \\ = 0.\end{aligned} \tag{4.56}$$

We notice that $\hat{O}_- X_{n,\ell+2}$ is a solution of the second equation while $\hat{O}_+ X_{n,\ell}$ is a solution of the first. Thus, it follows that the operators $\hat{O}_+$ and $\hat{O}_-$ are raising and lowering operators, respectively.

If the eigenfunctions $\psi_{n,\ell,m_\ell}(r,\Phi)$ are normalised according to the normalisation condition and the hyperspherical harmonics $Y_\ell^{m_\ell}(\Phi)$ are normalised to unity, then the normalisation condition for the radial function $X_{n,\ell}$ reads

$$R^N \int_0^\infty \sinh^{N-1}(u) X^*_{n',\ell} X_{n,\ell}\, du = \delta_{n'n}. \tag{4.57}$$

By combining the equations (4.55) and (4.57), it follows that the radial function satisfies the condition, namely

$$\int_0^\infty \sinh^{N-1}(u)\operatorname{csch}^2(u) X^*_{n,\ell'} X_{n,\ell}\, du = 0 \tag{4.58}$$

for $\ell' \neq \ell$. This latter condition implies that the raising and lowering operators $\hat{O}_+$ and $\hat{O}_-$ are Hermitian operators. Thus, the constant term occurring in both (4.55) and (4.56) is non-negative, and there is some integer n such that this constant term is zero for $\ell = n$. With this condition, we find that the energy levels of the hyperbolic quantum harmonic oscillator are given by

$$E_n = \left(n + \frac{N}{2}\right)\sqrt{\omega^2 + \frac{1}{4R^4}} - \frac{1}{2R^2}\left(n^2 + Nn + \frac{N}{2}\right). \tag{4.59}$$

Remark 4.8. Just as we discussed in previous remarks, as the distance R grows without bound, the energy levels E_n of the quantum harmonic oscillator in the N-hyperboloid become

$$\lim_{R\to\infty} E_n = \omega\left(n + \frac{N}{2}\right),$$

which are exactly the energy levels (2.72) of the quantum harmonic oscillator in the Euclidean N-space for $\hbar = 1$.

Similarly for the N-sphere, We can use the raising and lowering operators $\hat{O}_+$ and $\hat{O}_-$ to compute the eigenfunctions for the hyperbolic quantum harmonic oscillator. The action of the raising and lowering operators, respectively, is given by

$$\hat{O}_+ X_{n,\ell} = (2\ell + N)|C_{n,\ell}|R^2 X_{n,\ell+2} \qquad \text{and} \qquad \hat{O}_- X_{n,\ell+2} = (2\ell + N)|C_{n,\ell}|R^2 X_{n,\ell}, \tag{4.60}$$

where $|C_{n,\ell}|$ satisfies the relation

$$\left(\ell + \frac{N}{2}\right)^2 |C_{n,\ell}|^2 = \frac{(n-\ell)(n+\ell+N)}{R^4}\left(\frac{n-\ell}{2} - R^2\varpi\right)\left(\frac{n+\ell+N}{2} - R^2\varpi\right), \tag{4.61}$$

where we have used the abbreviation (3.64) once again. To simplify the recurrence relations (4.60), let us introduce the definitions

$$X_{n,\ell} = A_{n,\ell}(\sinh(u))^{-(\ell+N-2)}(\cosh(u))^{\frac{1}{2}-R^2\varpi} Z_{n,\ell} \qquad \text{and} \qquad A_{n,\ell} = -\frac{A_{n,\ell+2}}{(2\ell+N-4)|C_{n,\ell}|R^2}. \tag{4.62}$$

Then, the second recurrence relation of equation (4.60) can be written as

$$Z_{n,\ell} = \operatorname{csch}^2(u)\left((2\ell+N-4)\coth(u)\frac{d}{du} - (n+\ell+N-2)(n-\ell+2-2R^2\varpi)\right)Z_{n,\ell+2}, \qquad (4.63)$$

while the first recurrence relation for $\ell = n$ becomes

$$\left(\frac{d}{du} - (2n+N-2)\coth(u)\right)Z_{n,n}(u) = 0. \qquad (4.64)$$

This differential equation is similar to equation (3.67), and it has a simple and similar solution, namely

$$Z_{n,n}(u) = (\sinh(u))^{2n+N-2}. \qquad (4.65)$$

It finally remains to compute the general solutions $Z_{n,\ell}(u)$ from the specific solution $Z_{n,n}(u)$. With induction, we find that

$$Z_{n,n-2D}(u) = \sum_{i=1}^{D}(-1)^i B_{D,i}^{n,N}(\sinh(u))^{2n+N-4D+2i-2}(\cosh(u))^{2D-2i}, \qquad (4.66)$$

where the constant $B_{D,i}^{n,N}$ is defined as

$$B_{D,i}^{n,N} = \frac{2^{2i-2D}D!\Gamma(D+1-R^2\varpi)}{(D-i)!i!\Gamma(D+1-R^2\varpi-i)}\frac{\Gamma(2n+N-1)\Gamma(\frac{1}{2}(2n+N-4D+2i-1))}{\Gamma(\frac{1}{2}(2n+N-1)\Gamma(2n+N-4D+2i-1)}$$

and Γ denotes the Gamma function. Again, by writing

$$\gamma = \frac{\Gamma(2n+N-2D-1)\Gamma(\frac{1}{2}(2n+N-1))}{\Gamma(\frac{1}{2}(2n+N-2D-1))\Gamma(2N+N-1)}$$

and making a change of variables, we find that

$$\left(\frac{d}{d(\sinh^2(u))}\right)^D\left[(\sinh^2(u))^{\frac{1}{2}(2n+N-2D-2)}(\cosh^2(u))^{D-R^2\varpi}\right] = \gamma(\cosh^2(u))^{-R^2\varpi}$$
$$Z_{n,n-2D}(\sinh^2(u)). \qquad (4.67)$$

Upon normalisation, the solution $X_{n,\ell}(\sinh^2(u))$ of the radial equation (4.54) is given by

$$X_{n,\ell}(\sinh^2(u)) = \sqrt{\frac{2(n+\frac{1}{2}N-R^2\varpi)\Gamma(\frac{1}{2}(n+\ell+N)-R^2\varpi)}{R^N(\frac{1}{2}(n-\ell))!\Gamma(\frac{1}{2}(n+\ell+N))\Gamma(\frac{1}{2}(n-\ell)-R^2\varpi+1)}}(\sinh^2(u))^{-\frac{1}{2}(\ell+N-2)}$$
$$(\cosh^2(u))^{\frac{1}{4}+\frac{R^2}{2}\varpi}\left(\frac{d}{d(\sinh^2(u))}\right)^{\frac{1}{2}(n-\ell)}\left[(\sinh^2(u))^{\frac{1}{2}n+\ell+N-2}(\cosh^2(u))^{\frac{1}{2}(n-\ell)-R^2\varpi}\right]. \qquad (4.68)$$

To write this result more compactly, we can use the hypergeometric function (3.74). Then, the solution $X_{n,\ell}(\sinh^2(u))$ of the radial function can be written as

$$X_{n,\ell}(\sinh^2(u)) = \text{constant} \cdot \frac{\Gamma(\frac{1}{2}(n+\ell+N))}{\Gamma(\frac{1}{2}(2\ell+N))}(\sinh^2(u))^{\frac{1}{2}\ell}(\cosh^2(u))^{\frac{1}{4}-\frac{R^2}{2}\varpi}$$
$${}_2F_1\left(\frac{1}{2}(n+\ell+N)-R^2\varpi, -\frac{(n-\ell)}{2}, \ell+\frac{N}{2}, (\sinh^2(u))\right). \qquad (4.69)$$

To conclude, one finds the solution $X_{n,\ell}(u)$ by undoing our last change of variables, and we find the radial wave function $X_{n,\ell}(r)$ by using the hyperbolic relation in the N-hyperboloid.

Conclusion

We have shown that the simple harmonic oscillator, both for classical and quantum mechanics, can be generalised to N-dimensional Euclidean, spherical and hyperbolic geometry. In Euclidean N-space, we have described the particle's orbit and derived its constants of motion, which we used to find the symmetry group in classical mechanics. As for the quantum harmonic oscillator, we have solved the 1-dimensional Schrödinger equation, whose solutions we have been used to solve the N-dimensional generalisation.

Moreover, we have formulated the relation between motion in the N-sphere and motion in Euclidean N-space by means of gnomonic projection. Both for classical and quantum mechanics, we have derived the constants of motion and constructed the symmetry group. Furthermore, we have demonstrated that the N-dimensional Schrödinger equation in a spherical symmetric geometry can be solved by using hyperspherical coordinates and separation of variables, and we have formulated the exact solution of the radial Schrödinger equation.

To conclude, we have formulated the relation between motion in hyperbolic N-space and motion in Euclidean N-space by using the gnomonic projection. Similarly, we have used the constants of motion to construct the symmetry group in classical and quantum mechanics. Moreover, we have solved the radial Schrödinger equation for negatively curved hyperbolic geometry, which can be used to construct the general solution for the Schrödinger equation in hyperbolic N-space.

As we discussed previously, the harmonic oscillator has many generalisations. In this thesis, we only concentrated on the dimensional and geometric aspects. It is worth to analyse other generalisations of the harmonic oscillator, like the damped or non-isotropic oscillator, in curved geometries. Additionally, one could research the possibilities to generalise our results to spaces of non-constant curvature, such as Riemannian manifolds.

Symbol Index

The following list consists of all the symbols that were commonly used throughout this thesis. In addition, this index can be used swiftly to navigate one's way through this thesis and find the occurrences of these symbols.

σ_O	standard deviation of observable O	10
ψ	time-independent wave function	11, 22, 38, 52
Γ	Gamma function	40, 54
Ψ	time-dependent wave function	5, 22
$\vert\Psi(t)\rangle$	quantum state	5, 22
ω	angular frequency	14
∇^2	Laplace operator	7, 22
$\hat{0}$	null operator	6
${}_2F_1$	hypergeometric function	40, 54

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