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Bachelor Thesis

Constrained-degree percolation on d-ary trees

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Abstract

We investigate the constrained-degree percolation model on d-ary trees, $\mathbb{T}_d = (\mathbb{V}_d, \mathbb{E}_d)$. It is defined based on the continuous-time percolation model, where aside from the standard sequence of uniform random variables $(U_e)_{e \in \mathbb{E}_d}$ on $[0, 1]$, a constraint $k \in \mathbb{N}$ is given. Each edge $e \in \mathbb{E}_d$ opens at time U_e, unless one of its end-vertices is a neighbour to k already open edges at this time. The main result of this thesis is establishing the upper and lower bounds on the critical time of the constrained-degree percolation model on d-ary trees. Using these bounds we conduct initial research on the monotonicity and asymptotic behaviour of the critical time.

Contents

1 Introduction

Ever since the first mention of the percolation model in [1], mathematicians have been experimenting with its different variations seeking for interesting results. The percolation model is defined on an infinite connected graph $G = (\mathbb{V}, \mathbb{E})$, with vertex set $\mathbb{V}$ and set of edges $\mathbb{E}$, such that each $v \in \mathbb{V}$ is of a finite degree. We fix a number $p \in [0, 1]$ and close each edge in $\mathbb{E}$ independently with probability $(1 - p)$. In such a reduced graph, we are interested in the existence of an infinite open path, in case such exists we say that *percolation* occurs. The answer to the questions of existence and number of disjoint infinite connected components depend on the structure of the underlying graph and value of p. It is well known that this model undergoes a phase transition at a certain value of $p = p_c(G)$, called the *critical probability*, which is defined as $p_c(G) := \inf\{p \in [0, 1] : \text{percolation occurs in } G\}$. The most commonly used graphs are d-dimensional lattices, relevant results for these graphs are discussed in the following paragraphs.

The importance of this class of models stems from the fact that their macroscopic characteristics follow deterministically from the parameters used to define them. This in turn allows for its applications to real life problems which involve graph-like structures with random edges, and predict a phase transition regarding the existence of a path between distant vertices. A notable example of such is the Ising Model [2], a model in statistical mechanics which explains the phenomenon of ferromagnetism. The effort by Hugo Duminil-Copin to prove that this model undergoes a phase-transition in dimensions three and four has been awarded a Fields Medal, exemplifying the relevance of percolation theory in modern mathematics [3].

A variation of the percolation model that is examined in this thesis is the constrained-percolation model (we shall abbreviate it as CDP and refer to the regular model as unconstrained percolation), first defined in [4] and further researched in [5]–[7]. In CDP, each vertex of the graph is assigned a constraint, which determines the maximum number of open edges which can start at that vertex. This model is defined based on an alternative representation of the unconstrained percolation. Once again, consider an infinite connected graph $G = (\mathbb{V}, \mathbb{E})$ with $\deg(v) < \infty$ for all $v \in \mathbb{V}$. Let $(k_v)_{v\in\mathbb{V}}$ be a sequence of degree constraints, that is, integers such that $k_v \leq \deg(v)$ for all vertices in $\mathbb{V}$. To each bond $e \in \mathbb{E}$ assign an independent uniformly distributed random variable $U_e \in [0, 1]$. This allows to define *continuous-time percolation model* with all bonds closed at $t = 0$ and each bond $e = \langle v_1, v_2\rangle$ opening at time U_e, provided that the current number of open bonds connected to v_1 or v_2 does not exceed k_{v_1} or k_{v_2} respectively. This assures that at every vertex v there will be at most k_v open bonds, since once vertex v has k_v neighbouring open edges, no more neighbouring edges can open. In such a case we say that v is *saturated*. A more in-depth definition of this model is presented in [5]. Note that in the absence of constraints, such defined continuous-time percolation model at time t corresponds to the unconstrained percolation model in which each edge is open with probability t.

In CDP, a similar phase transition could occur as in the unconstrained model. Due to the opening times characterisation, instead of the name *critical probability* we shall call this value *critical time*. For each pair of a graph G and a set of constraints $(k_v)_{v\in\mathbb{V}}$, we define it equivalently as

$$t_c(G,(k_v)_{v\in\mathbb{V}}) := \inf\{t\in[0,1];\ \text{percolation occurs in } G \text{ with constraints } (k_v)\}\,, \tag{1}$$

where we set $t_c(G,(k_v)_{v\in\mathbb{V}})=\infty$ if the model does not undergo a phase transition. We note that in this paper we will deal with constant sets of constraints, that is $k_v=k$ for all $v\in\mathbb{V}$ and some constant k. Therefore, we simplify the notation of the critical time to $t_c(G,k)$. It is worth pointing out that there have been other CDP models researched which utilise different types of constraints. For example, the models in [8], [9] research the cases when only specific configurations of the open edges are allowed.

One of the first relevant breakthroughs in the area of percolation theory was the proof of $p_c=\frac{1}{2}$ on a square lattice $\mathbb{Z}^2$. Conjectured already in the 1950's, it had to wait nearly 25 years for a complete argument. The first part of the proof, namely $p_c\geq\frac{1}{2}$ has been proved by Harris in 1960 [10]. It took however twenty more years before Kesten has shown that the inverted inequality also holds, and hence validating the conjecture [11]. Since then, more questions regarding important quantities associated to this model have been asked, such as the value of the critical probability for higher dimensional lattices or the number of infinite clusters for $p>p_c$ and $p=p_c$. In [12], Kesten and Hara show that $p_c(\mathbb{Z}^d)$ is of order $1/d$ for large d, more precisely:

$$p_c(\mathbb{Z}^d)=(1+o_d(1))\frac{1}{2d}$$

where the notation $f=o_d(g)$ means that the expression f/g tends to 0 as $d\to\infty$. Regarding the number of infinite clusters, it is known that for $p>p_c(\mathbb{Z}^d)$ there is exactly one infinite cluster for all $d\geq 2$, a result shown in [13]. However, the case when $p=p_c(\mathbb{Z}^d)$ has proved to be more difficult. For $d=2$ there is no infinite cluster, a result first proved by Harris with the Russo–Seymour–Welsh theorem, which then later contributed to the lower bound $t_c(\mathbb{Z}^d)\geq\frac{1}{2}$ [10]. However, the methods used to prove the planar case are not generalisable to higher dimensions and different approaches were needed to tackle this problem. The best effort up to date is the proof that there is no infinite cluster at critical probability for all $d\geq 11$ [14]. Hence, no infinite clusters at $p=p_c(\mathbb{Z}^d)$ is still a conjecture for $3\leq d\leq 10$.

Since CDP has not been researched nearly as much as the unconstrained percolation model, there is less results that were shown. Most importantly, for $d=2$

we know that $t_c(\mathbb{Z}^2, 2) = 0$ and $\frac{1}{2} < t_c(\mathbb{Z}^2, 3) < 1$ [5]. The same paper also provides an argument that for $t > t_c(\mathbb{Z}^2, 3)$ there is a unique infinite cluster. Regarding higher dimensions, in [6] the authors establish that $t_c(\mathbb{Z}^d, k)$ is of order $1/d$ for a substantial number of pairs (d, k). They also show that:

$$\lim_{n\to\infty} d_n t_c(\mathbb{Z}^d, (k_n)_n) = \frac{1}{2}$$

for all sequences $(d_n)_n$ and $(k_n)_n$ (where $d_n \geq k_n$ for all $n \in \mathbb{N}$) which diverge as $n \to \infty$.

These results and the techniques which authors use to prove them are an inspiration to research for similar outcomes for different graph structures. The natural structure to extend this research to, also mentioned and initially studied in [5], are trees.

In this thesis we restrict ourselves to researching CDP on d-ary trees. Denoted as $\mathbb{T}_d$, these are the trees in which each vertex (apart from the root) has exactly one *parent* vertex and d *offspring* vertices, in total $d + 1$ neighboring vertices. The constraints we consider are $k \in \{1, ..., d\}$; we don't consider the case $k = d + 1$ since it is equivalent to the unconstrained percolation. We will make use of the notations from [5]. Fix $d \geq 2$ and denote $\mathbb{T}_d = (\mathbb{V}_d, \mathbb{E}_d)$, with vertex set $\mathbb{V}_d$ and set of edges $\mathbb{E}_d$. Furthermore, denote $[d] := \{1, 2, ..., d\}$ for all $d \in \mathbb{N}$, and define $[d]^n$ as the set of n-dimensional vectors whose entries take values in $[d]$. This allows to identify the set of vertices $\mathbb{V}_d$ as $[d]_\star$, defined as:

$$[d]_\star = \bigcup_{n\in\mathbb{N}\cup\{0\}} [d]^n$$

Here, $[d]^0$ is considered to be the root vertex o, and each $[d]^n$ represents the set of vertices in the n-th generation, which we denote as $x = (x_1, ..., x_n)$ (see Figure 1). For each $x = (x_1, ..., x_n) \in [d]^n$ and $a \in [d]$ define their concatenation: $x \cdot a := (x_1, ..., x_n, a) \in [d]^{n+1}$. The set of edges can be hence defined equivalently as $\mathbb{E}_d := \{\langle x, x \cdot a\rangle; x \in \mathbb{V}_d,\ a \in [d]\}$.

Given some $t \in [0, 1]$ and uniformly distributed independent random variables U_e for all edges $e \in \mathbb{E}_d$, we shall denote by ω_t the configuration of all open and closed edges at time t in the CDP model. That is, $\omega_t \in \{0, 1\}^{\mathbb{E}_d}$, with $\omega_{t,e} = \mathbb{1}(e \text{ is open at time } t)$ for all $e \in \mathbb{E}_d$, which is a random variable equal to 1 when edge e is open at time t, and 0 otherwise. In a similar fashion define the configuration $\tilde{\omega}_t$ and indicator function of open edges $\tilde{\omega}_{t,e}$ for the unconstrained percolation model, where no restriction on the degree of the vertex is assumed. Observe that this construction implies $\tilde{\omega}_{t,e} \geq \omega_{t,e}$ for all $t \in [0, 1]$ and $e \in \mathbb{E}_d$, since, by definition, in order for an edge to be open in CDP, it clearly has to be open in unconstrained model as well. This in turn provides a trivial lower bound $t_c(\mathbb{T}_d) \leq t_c(\mathbb{T}_d, k)$.

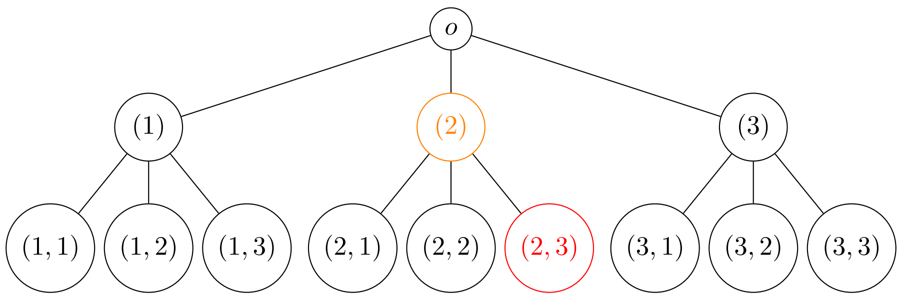

Figure 1: Example of a 3-ary tree $\mathbb{T}_3$. Considering the introduced notation, the red vertex is denoted as a two dimensional vector $(2, 3)$. It can be written alternatively using concatenation of the orange vertex and the number 3: $(2) \cdot 3 = (2, 3)$

As is usually the case in the area of percolation theory, we are interested in evaluating the critical times for all permissible values of parameters d, k in CDP. What makes it difficult for this model is the fact that its properties disallow for standard coupling arguments, which are commonly used to find bounds for t_c in other percolation models. Such arguments often use the fact that a model is m-dependent, which means that an object can influence only the objects which are within distance m from it (in the graph theoretical sense). However, in CDP configurations exist where the events of opening of arbitrarily distant edges depend on each other. This is elaborated on in the further part of this section.

It is also not possible to trivially couple models on the d-regular tree with different constraints. For an example, see Figure 2.

The standard coupling arguments assume that the set of open edges of one model is a subset of the set of open edges of the other model. However, the example above shows that in CDP an edge can be open for the model with constraint $k - 1$ when it is closed for the one with constraint k. This means that not always existence of an infinite path in one model will trivially imply its existence in the other.

This is as an example of how an edge connecting generations 2 and 3 influences could connect an edge connecting generations 0 and 1. In a similar way, the state of an edge can depend on the state of another edge m distance away for an arbitrary $m \in \mathbb{N}$. Opening on an edge from m-th generation can allow for opening of the $m - 2$ edges above it, and that in turn could disallow for an opening of the edge connecting generations 0 and 1.

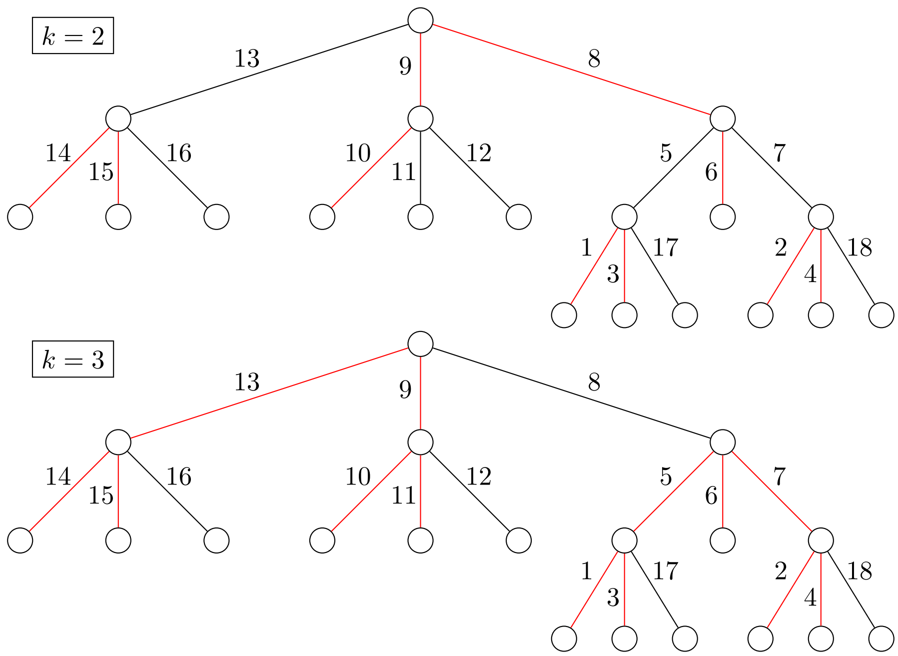

Figure 2: A realisation of ω_t at $t = 1$ for CDP with constraints $k = 2, 3$. Open edges are coloured red and the numbers above each edge denotes the order in which they try to open in a given realization of the opening times U_e. Moreover, edges connecting generations 2 and 3, and 3 and 4, are considered to have larger opening times than edges presented on the graphs. In such a case, the open part of ω_1 shown on the graphs is independent of the remaining values of U. Note that the edge $\langle o, (3)\rangle$ (with number 8) is open for $k = 2$, but closed for $k = 3$.

Regarding the unconstrained percolation model on d-ary trees, it is well known that $t_c(\mathbb{T}_d) = 1/d$ [15]. The proof of this statement is based on the fact that at time t, the cluster at the origin has the same distribution as the family tree of a branching process with offspring distribution $\mathrm{Bin}(d, t)$. Since in the CDP model the edges open at a slower rate than the ones in the unconstrained percolation model, it simply follows that $t_c(\mathbb{T}_d) \leq t_c(\mathbb{T}_d, k)$ for all $k \leq d$. We will in fact prove something slightly stronger.

Theorem 1. For all $d \geq 3$ and $k \leq d$ we have

$$\frac{1}{d} = t_c(\mathbb{T}_d) < t_c(\mathbb{T}_d, k).$$

It is also important to learn whether this model undergoes a phase transition at all (that is, whether $t_c < \infty$, where by $t_c = \infty$ we mean that the model does not percolate). Already in [5], the authors establish $t_c(\mathbb{T}_d, 2) = \infty$ and $t_c(\mathbb{T}_d, 3) < 1$

for arbitrary $d \geq 3$. We will provide a generalisation of the argument used to prove the aforementioned result to extend it to an arbitrary constraint.

Theorem 2. For all $d \geq 3$ we have

$$\begin{cases} t_c(\mathbb{T}_d, k) = \infty & \text{for } k = 1, 2, \\ t_c(\mathbb{T}_d, k) < 1 & \text{for } k \geq 3. \end{cases}$$

Nevertheless, the upper bound of 1 is not particularly interesting to us since it fails to establish the order of magnitude of $t_c(\mathbb{T}_d, k)$. The main contribution of this thesis is obtaining of the upper bound of the order $1/d$.

Theorem 3. For all $d \geq 4$ and $4 \leq k \leq d$, the critical time satisfies

$$t_c(\mathbb{T}^d, k) < \frac{1.2}{d} \,.$$

Unfortunately, the reasoning used in this proof fails to account for the case $d = 3$.

Establishing a closed form expression of the upper bound for $t_c(\mathbb{T}_d, k)$ which would depend on k is still to be achieved. Instead, we focused on the asymptotic behaviour of the critical time to obtain the following result.

Theorem 4. For all increasing sequences $(d_n)_n, (k_n)_n \in \mathbb{N}$ where $d_n \geq k_n$ for all $n \in \mathbb{N}$, the critical time satisfies

$$\lim_{n \to \infty} d_n \cdot t_c(\mathbb{T}^d, k_n) = 1 \,.$$

Lastly, we investigate the monotonicity of the critical time in k and d for small values of these parameters. Since as these parameters grow more edges should be able to open, we expect this variable to be non-increasing in both k and d. Once again, this result is far from straightforward due to the difficulty in posing standard coupling arguments in the CDP model. To show monotonicity, we will try make the upper bound of the model with parameters d, k smaller than the lower bound for the model with either $k' = k - 1$ or $d' = d - 1$.

Theorem 5. For all $3 \leq d \leq 100$ and $3 \leq k \leq d$, the critical time $t_c(\mathbb{T}^d, k)$ is monotonously decreasing in k and d.

The paper is structured as follows. In Section 2 we present the necessary theoretical background for the further parts of the thesis. Section 3 introduces two relevant branching processes which are used in Section 4, where the proofs of the theorems outlined above are conducted. In Section 5 we conclude on the results and state conjectures to be researched further.

2 Preliminary results

In this Section we present results which are necessary for the proofs conducted in Section 3. Firstly, we define branching processes and present some of their useful characteristics. We then proceed to present preliminary ideas regarding CDP on d-ary trees, and how they can be understood in the language of branching processes.

2.1 Branching processes

To formulate the concept of branching processes, we use the notations from [16]. Let X be a random variable which takes values in $\mathbb{N}$. A *branching process* $(Z_n)_{n\in\mathbb{N}}$ is defined as follows. We let $Z_0 = 1$. Then, for all $n \geq 1$, Z_n is defined inductively as a sum of Z_{n-1} independent realisations of X

$$Z_n := \sum_{i=1}^{Z_{n-1}} X_i, \quad \text{where } \forall_i \, X_i \sim X. \tag{2}$$

One can understand the sequence of random variables $(Z_n)_n$ as a number of individuals in consecutive generations during a reproduction process. $Z_0 = 1$ denotes the first individual in generation 0. Then, generation 1 consists of Z_1 offspring of the first individual, whose number follows the offspring distribution X. Next, each of the individual i from the 1-st generation has $X_i \sim X$ children, and so the total number of offspring in the 2-nd generation is distributed as

$$\sum_{i=1}^{Z_1} X_i \overset{\text{def}}{\sim} Z_2 \,.$$

Continuing this procedure leads to creation of the *family tree*, also known as the *Galton–Watson tree*. The vertices of this graph are identified with individuals and each individual is connected through an edge with their parent and all their offspring. In such a way, for all $n \in \mathbb{N}$ the n-th generation consists of Z_n individuals connected to their predecessors and offspring. Observe that if $Z_i = 0$ for some $i \in \mathbb{N}$, then, by definition, $Z_j = 0$ for all $j > i$. If there exists such i, we say that the process *goes extinct*. Choosing minimal i with this property, the family tree of this branching process consists of $i - 1$ generations. However, if such i doesn't exist, the process is said to *survive* and it is continued indefinitely.

The following lemma presents a useful characteristic regarding the expected value of the offspring distribution.

Lemma 1. Consider a branching process $(Z_n)_n$ with offspring distribution X. We have

$$\mathbb{P}((Z_n)_n \text{ goes extinct}) = 1 \iff \mu \leq 1 \,.$$

What this implies is that $\mathbb{P}(\text{survival}) > 0$ whenever the branching process is *supercritical*, what means that $\mu > 1$. In case that $\mu < 1$, we say the branching process is *subcritical*, and from Lemma 1 there is $\mathbb{P}(\text{survival}) = 0$.

The notion of branching processes can be generalised to models where each individual has its own offspring distribution, and not necessarily independent from the others. We will refer to such as *generalised branching processes*. For these processes, the problem of establishing the probability of survival becomes much more challenging. The following lemma achieves this for a very simple generalised branching process.

Lemma 2. Let $X, Y \in \mathbb{N}$ be finite random variables with expectations μ, θ respectively. Consider the branching process $(Z_n^*)_n$ where the root has offspring distribution Y and the remaining vertices have offspring distribution X. If $\theta > 0$, then the probability of its survival is nonzero if and only if $\mu > 1$.

Proof. Suppose that $\mu \leq 1$ and let m be the largest possible value that Y can realise. Observe that a process starting any vertex from the 1-st generation is a branching process with offspring distribution X. Then

$$\mathbb{P}((Z_n^*)_n \text{ survives}) \leq m \cdot P(\{(Z_{n+1}^*)_n \text{ survives}\}) = m \cdot 0 = 0\,,$$

where the first equality follows from Lemma 1 and the fact that $\mu \leq 1$. Now let $\mu > 1$ and let l be the smallest non-zero value that Y can realise. In a similar fashion we obtain the following inequality

$$\mathbb{P}((Z_n^*)_n \text{ survives}) \geq \mathbb{P}(Y \geq 0) P(\{(Z_{n+1}^*)_n \text{ survives}\}) > 0\,,$$

as the probability of survival is at least the probability of survival of the first generation and the survival of the remainder of the process. The fact that $\mathbb{P}(Y \geq 0) > 0$ follows from $\theta > 0$ and $P(\{(Z_{n+1}^*)_n \text{ survives}\}) > 0$ is the result of Lemma 1 and $\mu > 1$. □

In further sections we are going to use the relation between the event of percolation in CDP and survivability of some branching process. To this end we introduce the notation $BP(\mathbb{T}_d, k, t)$, which signifies a generalised branching process, whose offspring distribution matches the distribution of the open cluster containing the origin of the CDP on a d-ary tree at time t with constraint k. Note that unlike in the unconstrained percolation model, the offspring distributions of $BP(\mathbb{T}_d, k, t)$ are non-trivial and might vary between the generations. We also let $BP(X)$ denote the branching process with offspring distribution X.

2.2 Percolation on d-ary trees

We prove a well known result for the critical time of the unconstrained percolation model on trees, which is a necessary tool for the proof of the lower bound of CDP's.

Lemma 3. The critical time of the unconstrained percolation model on a d-ary tree satisfies

$$t_c(\mathbb{T}_d) = \frac{1}{d}\,.$$

Proof. Fix an arbitrary $t \in [0,1]$. Since in the unconstrained model the event of an edge opening is independent of the opening of any other edge, clearly all vertices will have the same distributions of open edges. Fix an arbitrary vertex $v \in \mathbb{V}_d$. Each of the edges $e_a = \langle v, v \cdot a\rangle, a \in [d]$ is assigned an independent uniform random variable $U_{e_a} \in [0,1]$. Observe that the probability that exactly i edges e_a are open at v at time t is equal to

$$\mathbb{P}(\#\{a \in [d];\ U_{e_a} < t\} = i).$$

It is left to observe that this event is equivalent to obtaining i successes in d trials, where each trial is sampling a uniform random variable $U \in [0,1]$, with success defined as $\{U < t\}$, hence

$$\mathbb{P}(\{\text{exactly } i \text{ edges } e_a \text{ neighboring } v \text{ are open at time } t\}) = \mathbb{P}(\mathrm{Bin}(d,t) = i).$$

Since these distributions are the same, the event of percolation in the unconstrained percolation model is equivalent with there being a non-zero probability of survival of the branching process. Since the expected value of the binomial distribution with parameters d, t is equal to dt, by Lemma 1, we see that this branching process will have a nonzero probability of survival exactly when $dt > 1$, hence if and only if $t > \frac{1}{d}$. Therefore

$$\begin{aligned} t_c(\mathbb{T}_d) &= \inf\{t \in [0,1] :\ \text{percolation occurs in } \mathbb{T}_d\} \\ &= \inf\left\{t \in [0,1] :\ t > \frac{1}{d}\right\} = \frac{1}{d}\,. \end{aligned}$$

□

This lemma in combination with a result from introduction allow to establish a natural lower bound for CDP on d-ary trees.

Corollary 1. The critical time for CDP on a d-ary tree with constraint k satisfies

$$\frac{1}{d} = t_c(\mathbb{T}_d) \le t_c(\mathbb{T}_d, k)\,.$$

3 Comparison processes

In this section we introduce two branching processes which will be used to bound the critical times of CDP's for various values of parameters d and k. We also provide a tool which will allow us to compare two branching processes with respect to their supercriticality.

3.1 Lower bound process

We first define a random variable which is the offspring distribution of the branching process which we will later use to bound the critical time of CDP from below.

Definition 1. Given parameters $d, k \in \mathbb{N}$ with $d \geq 3$ and $d \geq k$ and $t \in [0, 1]$, the random variable $X^-(d, k, t)$ is defined as

$$\mathbb{P}(X^-(d,k,t) = l) = \begin{cases} \mathbb{P}(\mathrm{Bin}(d,t) = l) & \text{if } l \in \{0, ..., k-2\}\,, \\ \mathbb{P}(\mathrm{Bin}(d,t) \geq k-1) & \text{if } l = k-1\,. \end{cases}$$

This yields a random variable that realises an outcome of $\mathrm{Bin}(d, t)$ when it is less than $k - 1$, and takes value $k - 1$ when the outcome is greater than $k - 2$. Using this notation, we can then define the *lower bound process* $(Z^-(d, k, t)_n)_n$ as the branching process with the offspring distribution $X^-(d, k + 1, t)$ at the root and $X^-(d, k, t)$ for all remaining vertices. For simplicity, we denote it as $Z^-(d, k, t) := Z^-(d, k, t)_n)_n$. While considering constant d, k, we will use the shortened notations $Z^-(t) := Z^-(d, k, t)$ and $X^-(t) := X^-(d, k, t)$.

From Lemma 3, it follows that this process is supercritical whenever the branching process with offspring distribution $X^-(d, k, t)$ is supercritical. Since for constant d, k this random variable depends only on the time parameter, we introduce the notation of the *critical time of the lower bound process*

$$t_c(Z^-(t)) := \inf\{t \in [0, 1] :\ \mathbb{E}[X^-(t)] > 1\}\,,$$

and set $t_c(Z^-(t)) = \infty$ if the condition fails to be satisfied for all $t \in [0, 1]$. Observe that if $t_c(Z^-(t)) < \infty$, then, using Lemma 1, the critical time of the lower bound process is the smallest value of the parameter t, such that for all $t' > t$ the process $Z^-(t')$ is supercritical. Lastly, the expected value of $X^-(d, k, t)$ satisfies

$$\mathbb{E}[X^-(d,k,t)] = \sum_{l=1}^{k-1} l \binom{d}{l} t^l (1-t)^{d-l} + (k-1) \sum_{l=k}^{d} \binom{d}{l} t^l (1-t)^{d-l}\,.$$

3.2 Upper bound process

In the similar manner as above, we define the following random variable.

Definition 2. Given parameters $d, k \in \mathbb{N}$ with $d \geq 3$ and $d \geq k$ and $t \in [0,1]$, the random variable $X^+(d,k,t)$ is defined as

$$\mathbb{P}(X^+(d,k,t) = l) = \begin{cases} \mathbb{P}(\mathrm{Bin}(d,t) = l) & \text{if } l \in \{1, ..., k-1\}\,, \\ (1-t)^d + \mathbb{P}(\mathrm{Bin}(d,t) \geq k) & \text{if } l = 0\,. \end{cases}$$

Intuitively, one can understand this as a random variable $\mathrm{Bin}(d,t)$ taking values $1, ..., k-1$ if they are its realisation, but having value 0 if the outcome is 0 or greater than $k-1$. Using this notation, we can then define the *upper bound process* $(Z^+(d,k,t)_n)_n$ as the branching process with the offspring distribution $X^+(d,k+1,t)$ at the root and $X^+(d,k,t)$ for all remaining vertices. Again, simplicity, we denote it as $Z^+(d,k,t) := Z^+(d,k,t)_n)_n$, and for constant parameters d,k we will use the shortened notations $Z^+(t) := Z^+(d,k,t)$ and $X^+(t) := X^+(d,k,t)$.

The process $Z^+(d,k,t)$ is supposed to behave closely to CDP in order to be useful in finding an upper bound on the critical time of the CDP. Observe that, by definition, the degree of each vertex never exceeds k. Each individual has offspring only if the realisation of $\mathrm{Bin}(d,t)$ is less than k, meaning that in case when more than $k-1$ edges could open, none of them do open instead. This strict rule will make it possible to show that when the upper bound process is supercritical, then necessarily there is a non-zero probability of an infinite open path from the origin in the CDP with parameters d,k,t.

Similarly to the previous section, Lemma 3, yields that upper bound process is supercritical whenever the branching process $BP(X^+(d,k,t))$ is supercritical. For constant d,k, we introduce the notation of the *critical time of the lower bound process*

$$t_c(Z^+(t)) := \inf\{t \in [0,1] :\ \mathbb{E}[X^+(t)] > 1\}\,,$$

and set $t_c(Z^+(t)) = \infty$ if the condition fails to be satisfied for all $t \in [0,1]$. Analogously to above, $t_c(Z^+(t)) < \infty$ implies that the critical time of the lower bound process is the smallest value of the parameter t, such that for all $t' > t$ the process $Z^+(t')$ is supercritical. We also find the expected value of $X^+(d,k,t)$ to be

$$\mathbb{E}[X^+(d,k,t)] = \sum_{l=1}^{k-1} l \binom{d}{l} t^l (1-t)^{d-l}\,.$$

In order to prove Theorem 3, we will need the following result regarding the expected value of the offspring distribution $X^+(d,k,t)$.

Lemma 4. The time parameter $t_0 = 1.2/d$ is such that $\mathbb{E}(X^+(d,k,t_0)) > 1$ for all $k \geq 4$ and $d \geq k$.

Proof. Consider the function:

$$h(d) = \mathbb{P}(X^+(d,k,t_0) = 1) + 2\mathbb{P}(X^+(d,k,t_0) = 2) + 3\mathbb{P}(X^+(d,k,t_0) = 3)$$

Observe that h does not depend on k, since $k \geq 4$. Clearly $h(d) < g(d)$ for all regarded d, k. One can show that:

$$\begin{aligned} h'(d) =& (((-1.2 + d)/d)^d(-6.2208 + d(16.5888 + d(-14.3424 + 4.2048d))+ \\ & d(-6.2208 + d(15.552 + d(-12.8448 + 3.504d)))\cdot \\ & \log((-1.2 + d)/d)))/(-1.2 + d)^4 \end{aligned}$$

It can be checked numerically that for all $d \leq 100$ there is $h(d) > 1.05$. We use the well known Maclaurin series for the function $\log(1 + x)$ to evaluate:

$$\log(1 - 1.2/d) = \sum_{i=1}^{\infty}(-1)^{i+1}\frac{(-1.2/d)^i}{i} = -1.2/d - \sum_{i=2}^{\infty}\frac{(1.2/d)^i}{i}$$

We can use this formula to simplify the form of the derivative above:

$$\begin{aligned} h'(d) = ... = & \frac{(d-1.2)^{d-4}}{d^d}((1.24416 - 2.0736d + 1.07136d^2) - \\ & d\left(\sum_{i=2}^{\infty}\frac{(1.2/d)^i}{i}\right)(-6.2208 + 15.552d - 12.8448d^2 + 3.504d^3)) \end{aligned}$$

We would like to bound the absolute value of $h'(d)$ and use this bound to say something about the behaviour of the function $h(d)$. To do this, we first bound the infinite sum in the expression:

$$\left|-\sum_{i=2}^{\infty}\frac{(1.2/d)^i}{i}\right| < \frac{0.72}{d^2}\sum_{i=0}^{\infty}(1.2/d)^i = \frac{0.72}{d^2}\frac{d}{d-1.2} = \frac{0.72}{d(d-2)}$$

Moreover, observe that the first polynomial (in the expression for the derivative) is positive for all d and the second polynomial is positive for all $d > 2$. Lastly, note that $(d - 1.2)^{d-4} < d^{d-4}$ for the d's that are considered. Therefore:

$$\begin{aligned} |h'(d)| = ... < & \frac{1}{d^4}((1.24416 - 2.0736d + 1.07136d^2) + \\ & \frac{0.72}{d-2}(-6.2208 + 15.552d - 12.8448d^2 + 3.504d^3)) =: \theta(d) \end{aligned}$$

Since $h(d) = h(a) + \int_a^d h'(x)dx$, we can substitute $a = 100$ to obtain for all $d > 100$:

$$|h(d) - h(100)| = \left| \int_{100}^{d} h'(x)dx \right| \leq \int_{100}^{d} |h'(x)|dx < \int_{100}^{d} \theta(x)dx \leq \int_{100}^{\infty} \theta(x)dx$$

It can be computed that:

$$\int_{100}^{\infty} \theta(x)dx = \frac{16875000 \log(100) - 16875000 \log(98) + 8357796}{244140625} < 0.036$$

In combination with previous results, this implies:

$$h(d) > h(100) - 0.036 > 1.05 - 0.036 > 1$$

for all $d > 100$, what with the numerically found $h(d) > 100$ for $d \leq 100$ gives us the wanted result. □

Remark. From Poisson Limit Theorem we have that for $t = c/d$:

$$\lim_{d \to \infty} h(d) = c(1 + 2c + 2c^2)e^{-c}$$

The value $c = 1.2$ is one of the possible values that yield $\lim_{d \to \infty} h(d) > 1$. If $h(d)$ were to be monotonously decreasing, there would have to be $g(d) > h(d) > 1$ for all d. Unfortunately, due to the log function approaching 0 and the non-vanishing largest power of the polynomials (which on its own would make the function diverge), for large d there is a lot of fluctuations around 0, some of which take positive value. Therefore another method had to be used to bound the function.

3.3 Comparison Lemma

In order to use the branching process $Z^-(d, k, t)$ to reach conclusions about the critical times of the CDP's, we need a tool to compare these two models. The following lemma will serve as such.

Lemma 5 (**Comparison Lemma**)**.** If the branching process $Z^-(d, k, t)$ is subcritical, then also the cluster of the origin of CDP on a d-ary tree with constraint k is subcritical.

Proof. Fix the parameters d, k, t and let $Z^- := Z^-(d, k, t)$ as well as $X^- := X^-(d, k, t)$. We can identify the cluster of the origin of CDP as some branching process $(\tilde{Z}_n)_n$, where each individual has a different offspring distribution. Specifically, let $\tilde{Z}_0 := \{o\}$ and for each m let $\tilde{Z}_m := \{(x_1, \ldots, x_m) \in \mathbb{T}_d : o \to (x_1, \ldots, x_m)\}$, the set of vertices in the cluster of the root at height m. Denote the number of offspring of the i-th vertex of generation m in this tree by $\tilde{X}_{m,i}$ for $i \in [\tilde{Z}_m]$. We will first show that for all $m \in \mathbb{Z}$, the random variables

$\tilde{X}_{m,1},...,\tilde{X}_{m,\tilde{Z}_{m-1}}$ conditioned on the previous generations are stochastically dominated by X^-. More precisely, we want for each individual $i \in [\tilde{Z}_m]$ that

$$\mathbb{P}(\tilde{X}_{m,i} \leq x \,|\, \tilde{Z}_0,...,\tilde{Z}_{m-1}) \geq \mathbb{P}(X^- \leq x) \ \ \forall_{x\in\mathbb{R}}. \tag{3}$$

Using the notation from introduction, let this individual be denoted as $v \in [d]^m$ in the CDP model. Since we know that both $\tilde{X}_{m,i}$ and X^- take values in $\{0,1,...,k-1\}$, it is sufficient that we show inequality (3) holds for x taking these values. Since there clearly is $\mathbb{P}(X^- \leq k) = (\tilde{X}_{m,i} \leq k \,|\, \tilde{Z}_0,...,\tilde{Z}_{m-1}) = 1$, the inequality holds for $x = k$. Fix an arbitrary $x \in \{0,1,...,k-1\}$. Observe that the value of $\tilde{X}_{m,i}$ conditioned on previous generations depends only on the event that the edge $e = \langle u,v\rangle$ connecting $(m-1)$-st and (m)-th generations at $i-th$ individual is open at time t. Therefore we can bound this conditioned random variable by

$$\begin{aligned}&\mathbb{P}(\tilde{X}_{m,i} \leq x \,|\, \tilde{Z}_0,...,\tilde{Z}_{m-1}) \geq \\ &\mathbb{P}(\#\{a \in [d] : U_{\langle v,v\cdot a\rangle} \ \leq t\} \ \leq x \,|\, \text{at time } U_e, u \text{ and } v \text{ are not saturated})\,,\end{aligned}$$

where the condition is equivalent with the fact that at time U_e, the edge e can open. This inequality holds since the event that at most x edges having an opening time less than t is a sub-event of at most x edges opening at time t. If we can now show that

$$\begin{aligned}&\mathbb{P}(\#\{a \in [d] : U_{\langle v,v\cdot a\rangle} \ \leq t\} \leq x \,|\, \text{at time } U_e, u \text{ and } v \text{ are not saturated}) \\ &\geq \mathbb{P}(X^- \leq x)\,,\end{aligned}$$

then we will have proved the assertion. To this end, observe that for $x \leq k-1$, the edge e can always be open from the bottom, and hence

$$\begin{aligned}&\mathbb{P}(\#\{a \in [d] : U_{\langle v,v\cdot a\rangle} \ \leq t\} \leq x \,|\, \text{at time } U_e, u \text{ and } v \text{ are not saturated}) = \\ &\mathbb{P}(\#\{a \in [d] : U_{\langle v,v\cdot a\rangle} \ \leq t\} \leq x)/\mathbb{P}(\text{at time } U_e, u \text{ and } v \text{ are not saturated}) \geq \\ &\mathbb{P}(\#\{a \in [d] : U_{\langle v,v\cdot a\rangle} \ \leq t\} \leq x) = \mathbb{P}(X^- \leq x)\,.\end{aligned}$$

We have learned that X^- stochastically dominates the offspring distribution of the branching process which matches the distribution of the cluster at the origin of the CDP at all vertices. Therefore, in the sense of (reference with the theorem coupling stoch. dom. random variables), we can couple this CDP cluster with the family tree of $(Z_n^-)_n$ in a way that for all generations $m \in \mathbb{Z}$ there is $\tilde{Z}_m \leq Z_m^-$. This in turn implies that if $Z_l^- = 0$ for some $l \in \mathbb{Z}$, then necessarily $\tilde{Z}_m = 0$, and hence the lemma follows. □

4 Results

The introduction of the preliminary concepts and lemmas allows to prove further results for CDP on d-ary trees, expanding the current knowledge established in [5].

4.1 Phase transition occurs for all constraints $3 \leq k \leq d$

In the paper [5], it has been shown that $t_c(\mathbb{T}^d, 3) < 1$ for arbitrary $d \geq 3$. We generalise the reasoning from this proof to show a stronger result.

Theorem 2. For all $d \geq 3$ we have:

$$\begin{cases} t_c(\mathbb{T}_d, k) = \infty & \text{for } k = 1, 2 \\ t_c(\mathbb{T}_d, k) < 1 & \text{for } k \geq 3 \end{cases}$$

Proof. If $k = 1$, then existence of an infinite path is trivially impossible. Observe that each vertex belonging to a path, which is neither its beginning or end, needs to have a degree equal to 2. The case $k = 2$ has been resolved in Proposition 6 from [5].

For the remainder of the proof assume $k \geq 3$ Given a sequence of uniformly distributed random times $U \in [0,1]^{\mathbb{E}}$. We define a random forest $\mathfrak{F}(U) = (\mathbb{V}, \mathbb{E}(\mathfrak{F}))$, where

$$\mathbb{E}(\mathfrak{F}) = \{\langle x, x \cdot a\rangle \in \mathbb{E}_d; \#\{b \in [d]\backslash\{a\}; U(e) > U(\langle x, x \cdot a\rangle)\} \leq k - 2\}$$

This set contains all edges of $\mathbb{T}_d$ which have opening times greater than at most $k-1$ of its siblings. As this definition implies that $\mathfrak{F}$ is a collection of $k-1$-ary trees, let $\mathcal{T}$ be the one which contains the origin. Given any $e = \langle x, x \cdot a\rangle \in \mathcal{T}$, we call this bond *blue* if:

$$\#\{b \in [d]; U(e) > U(\langle x \cdot a, x \cdot a \cdot b\rangle)\} \leq k - 1$$

That means that blue edges belong to $\mathcal{T}$ and have opening times greater than at most $k-1$ of their offspring. Clearly, if e is blue, it will open at time U_e. Therefore it is sufficient to prove that there exists a percolation of blue bonds in $\mathcal{T}$.

Define the $k-1$ edges of n-th generation of $\mathcal{T}$ as $e_n^{(i)} = \langle x, x \cdot a_i\rangle$ for $i \in [k-1]$, $x \in [d]^{n-1}$ and $a_i \in [d]$, so that $U(e_n^{(1)}) < ... < U(e_n^{(k-1)})$. Given any $e \in \mathbb{E}(\mathcal{T})$, it holds that

$$\begin{aligned} \mathbb{P}(e \text{ is blue}) &= \mathbb{P}(e \text{ is blue}, e = e_n^{(1)}) + ... + \mathbb{P}(e \text{ is blue}, e = e_n^{(k-1)}) \\ &= \frac{1}{k-1}\mathbb{P}(e \text{ is blue} \,|\, e = e_n^{(1)}) + ... + \frac{1}{k-1}\mathbb{P}(e \text{ is blue} \,|\, e = e_n^{(k-1)}) \end{aligned}$$

Pick any $l \in [k-1]$ and consider the expression $\mathbb{P}(e \text{ is blue} \,|\, e = e_n^{(l)})$. This is an event that the chosen edge is blue, given that it has the l-th smallest opening time from the d edges from the same node. This event depends only on $2d$ edges: sons and siblings of e and e itself. For e to be blue, U_e can be larger than at most $k-1$ of the opening times of its sons. Assume that U_e is greater than exactly $0 \leq m \leq k-1$ of its sons. Then the number of possible combinations of these $2d$ edges given the restrictions above is equal to

$$(d)_l (d)_m \binom{m+l-1}{l-1} (2d-m-l)!$$

where we introduce the notation $(x)_n = x!/(x-n)!$. This holds due to the fact that we can choose with order $l-1$ siblings and one edge e in $(d)_l$ ways, choose m sons with order in $(d)_m$ ways, order the preceding edge e edges in $\binom{m+l-1}{l-1}$ ways (since they are already ordered in their respective groups, we order them as if they were indistinguishable) and lastly order the remaining $(2d-m-l)$ elements in $(2d-m-l)!$ ways. Since this holds for all $m \in \{0, ..., k-1\}$ and there are a total of $(2d)!$ permutations of a $2d$-element set, we conclude that

$$\mathbb{P}(e \text{ is blue} \,|\, e = e_n^{(l)}) = \sum_{m=0}^{k-1} \frac{(d)_l (d)_m \binom{m+l-1}{l-1} (2d-m-l)!}{(2d)!}$$

Combining it with the expression for the probability that the edge e is blue, we obtain:

$$\mathbb{P}(e \text{ is blue}) = \frac{1}{k-1} \sum_{l=1}^{k-1} \sum_{m=0}^{k-1} \frac{(d)_l (d)_m \binom{m+l-1}{l-1} (2d-m-l)!}{(2d)!}$$

One can check that this is consistent with the paper [5] in case $k = 3$. For the blue edges to have the distribution of a supercritical branching process, we need that $\mathbb{E}[X] > 1$, where X is a binomial distribution with parameters $p = \mathbb{P}(e \text{ is blue})$ and $s = k-1$. If this will turn out to be the case, then there must exist a percolation of blue edges and we are done. Since $X \sim \text{Bin}(s, p)$, we know that $\mathbb{E}[X] = sp = (k-1)\mathbb{P}(e \text{ is blue})$, hence the process is supercritical if $\mathbb{P}(e \text{ is blue}) > \frac{1}{k-1}$. To this end, observe:

$$\begin{aligned}
\mathbb{P}(e \text{ is blue}) &= \frac{1}{k-1} \sum_{l=1}^{k-1} \sum_{m=0}^{k-1} \frac{(d)_l (d)_m \binom{m+l-1}{l-1} (2d-m-l)!}{(2d)!} \\
&> \frac{1}{k-1} \sum_{l=1}^{2} \sum_{m=0}^{2} \frac{(d)_l (d)_m \binom{m+l-1}{l-1} (2d-m-l)!}{(2d)!} \\
&> \frac{1}{k-1} \cdot 1 > \frac{1}{k-1}
\end{aligned}$$

where the second inequality follows from simple evaluation of the double sum in the second line. □

4.2 Strict lower bound for the critical time $t_c(\mathbb{T}_d, k)$

We already know that for $k = 1, 2$ the critical time is infinite. The following result provides a trivial lower bound for CDP on d-ary trees.

Theorem 1. For all $d \geq 3$ and $3 \leq k \leq d$ we have:

$$\frac{1}{d} = t_c(\mathbb{T}_d) < t_c(\mathbb{T}_d, k)$$

Proof. The first equation is the statement of Lemma 2. To show that the inequality holds, consider the processes $Z^-(d, d+1, t)$ and $Z^-(d, k, t)$ for an arbitrary $t \in [0, 1]$. Observe that:

$$\mathbb{E}[X^-(d, d+1, t)] = \sum_{l=1}^{d} l \binom{d}{l} t^l (1-t)^{d-l} > \sum_{l=1}^{k-1} l \binom{d}{l} t^l (1-t)^{d-l} + (k-1) \sum_{l=k}^{d} \binom{d}{l} t^l (1-t)^{d-l} = \mathbb{E}[X^-(d, k, t)]$$

Therefore $\frac{1}{d} = t_c(Z^-(d, d+1, t)) < t_c(Z^-(d, k, t))$. Now, consider the CDP on $\mathbb{T}_d$ with constraint k at time t, and the branching process $BP(\mathbb{T}_d, k, t)$ corresponding to it. From Lemma 5 it follows that if the branching process $BP(\mathbb{T}_d, k, t)$ is supercritical, then so is $Z^-(d, k, t)$. Moreover, whenever $BP(\mathbb{T}_d, k, t)$ is supercritical, there is a non-zero probability of existence of an infinite open cluster containing the origin in the CDP with parameters d, k, t. Therefore $t_c(Z^-(d, k, t)) \leq t_c(\mathbb{T}_d, k)$ and hence

$$\frac{1}{d} = t_c(Z^-(d, d+1, t)) < t_c(Z^-(d, k, t)) \leq t_c(\mathbb{T}_d, k)\,.$$

□

4.3 Upper bound for $t_c(\mathbb{T}_d, k)$

So far we know that $\frac{1}{d} < t_c(\mathbb{T}^d, k) < 1$ for all choices of natural $k \geq 3$ and $d > k$. We wish to obtain a better upper bound for the critical times, ideally one of the order $o(1/d)$. To obtain that, we will make use of the properties of the upper bound process.

Theorem 3. For all $d \geq 4$ and $4 \leq k \leq d$, the critical time satisfies

$$t_c(\mathbb{T}^d, k) < \frac{1.2}{d}\,.$$

Proof. We want to show that $t_c(\mathbb{T}^d, k) < t_0 = \frac{1.2}{d}$ for all $k \geq 4$ and $d > k$. This relies on the coupling of CDP to the percolation model, in which the cluster of the origin has the same distribution as the family tree of $Z^+(d,k,t_0)$.

Consider the tree $\mathbb{T}_d$ and any realisation of uniform random variables $U \in [0,1]^{\mathbb{E}(\mathbb{T}_d)}$. Let $x \in [d]^n$ be an arbitrary vertex and $e = \langle x, x \cdot a\rangle$ an edge for some $a \in [d]$. We write $v(e) = x$ for the end vertex of e which is in the earlier generation. Define the sets $E_x := \{\langle x, x \cdot b\rangle;\ b \in [d]\}$ and $D_x := \{U_{e^*};\ e^* \in E_x\}$, which is a collection of opening times of the edges stemming from x. Lastly, we denote the random variable $F_{t,x} := \#\{u \in D_x;\ u \leq t\}$. We define the model $\breve{\omega}$ as

$$\breve{\omega}_{t,e} = \begin{cases} \tilde{\omega}_{t,e} \cdot \mathbb{1}(F_{t,v(e)} \leq k) & e = \langle o, a\rangle\ ,\ a \in [d]\,, \\ \tilde{\omega}_{t,e} \cdot \mathbb{1}(F_{t,v(e)} \leq k-1) & \text{otherwise}\,, \end{cases}$$

where $\tilde{\omega}_t$ is the unconstrained percolation model as defined in the introduction. Therefore, an edge $e = \langle x, x \cdot a\rangle$ is open in $\breve{\omega}_t$ if and only if at time t at vertex x there are at most $k-1$ edges $\langle x, x \cdot b\rangle$, $b \in [d]$, with opening times lower than t (and k edges in case they are neighbouring the root).

Similarly to reasoning from Section 3.2, this construction yields a model where the root has distribution $X^+(d,k+1,t)$, and all remaining vertices has the distribution $X^+(d,k,t)$. Observe that this implies that for all considered pairs of (d,k), the model $\breve{\omega}_t$ at time $t = t_0$ has a non-zero probability of containing an infinite open cluster containing the origin. This follows from Lemma 4, and that $Z^+(d,k,t)$ is supercritical for $t = t_0$. Therefore $t_0 > t_c(Z^+(d,k,t))$. If we now show that

$$\{\text{Cluster of origin is infinite in } \breve{\omega}_{t,e}\} \implies \{\text{Cluster of origin is infinite in } \omega_{t,e}\}$$

for all fixed t and all e, then trivially $t_c(\mathbb{T}_d, k) \leq t_c(Z^+(d,k,t)) < t_0$, what in turn yields the inequality from Theorem 3. To this end, suppose an infinite path starting at the origin exists in $\breve{\omega}_{t_0}$ and denote the subsequent vertices it visits as $x_0, x_1, ...$ where $x_0 = o$. For an arbitrary $n \in \mathbb{N}$, consider the edge $e = \langle x_n, x_{n+1}\rangle$. Since it is open in $\breve{\omega}_{t_0}$, it must be the case that $U_e < t_0$ and at the vertices x_n, x_{n+1} there is at most $k-1$ edges open (k if $n = 0$). This description clearly implies that edge e will open be open at time t_0 in the configuration ω_{t_0}. Since this holds for all $n \in \mathbb{N}$, we obtain that the infinite path $x_0, x_1, ...$ is also open in ω_{t_0}, what concludes the proof. □

4.4 Limiting behaviour of the expression $d \cdot t_c(\mathbb{T}_d, k)$

Theorem 4. For all increasing sequences $(d_n)_n, (k_n)_n \in \mathbb{N}$ where $d_n \geq k_n$ for all $n \in \mathbb{N}$, the critical time satisfies

$$\lim_{n\to\infty} d_n \cdot t_c(\mathbb{T}^d, k_n) = 1\,.$$

Proof. First observe that for all such sequences $(d_n)_n, (k_n)_n$ the result from Theorem 3 implies

$$1 < d_n \cdot t_c(\mathbb{T}^d, k_n) < d_n \cdot t_c(Z^+(d_n, k_n, t))\,.$$

Hence if we prove that the last expression goes to 1 as $n \to \infty$, by the squeeze theorem it will imply Theorem 4. To show this, it is sufficient to prove that for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n > N$ there is $t_c(Z^+(d_n, k_n, t)) \leq (1+\epsilon)/d_n$. Remembering the definition of the critical time of the upper bound process (1), the statement above is equivalent with proving that such $N \in \mathbb{N}$ exists which for all $n > N$ satisfies $\mathbb{E}[X^+(d_n, k_n, t)] > 1$. Hence we need

$$\sum_{l=1}^{k_n-1} l \binom{d_n}{l} \left(\frac{1+\epsilon}{d_n}\right)^l \left(\frac{d_n - 1 - \epsilon}{d_n}\right)^{d_n - l} > 1 \ \forall_{n>N}$$

Before working on this expression further, consider the Maclaurin series of $e^{-(1+\epsilon)}$. Since it converges to the value $e^{-(1+\epsilon)}$, there exists $N_0 \in \mathbb{N}$ such that for all $n > N_0$

$$e^{1+\epsilon} - \sum_{l=0}^{n-2} \frac{(1+\epsilon)^l}{l!} < \alpha \,,$$

where $\alpha := (\epsilon e^{-(1+\epsilon)})/(2+2\epsilon) > 0$. Let $k_{N_0} = \kappa$. We also know that as $n \to \infty$, by Poisson Limit theorem the Binomial distribution with parameters $d_n, (1+\epsilon)/d_n$ converges to the Poisson distribution with parameter $(1+\epsilon)$. Then, considering $n = N_0$, for all $\kappa - 1$ summands of the expected value sum, there exists $N_l \in \mathbb{N}$ (where $l \in \{1, ..., \kappa - 1\}$) such that for all $n > N_l$:

$$\frac{(1+\epsilon)^l}{l!} e^{-(1+\epsilon)} - \binom{d_n}{l} \left(\frac{1+\epsilon}{d_n}\right)^l \left(\frac{d_n - 1 - \epsilon}{d_n}\right)^{d_n - l} < \frac{\epsilon}{\kappa(\kappa - 1)}$$

Pick $N := \max\{N_0, ..., N_{\kappa-1}\}$. Then for all $n > N$:

$$\sum_{l=1}^{k_N-1} l \binom{d_N}{l} \left(\frac{1+\epsilon}{d_N}\right)^l \left(\frac{d_N - 1 - \epsilon}{d_N}\right)^{d_N - l} > \sum_{l=1}^{k_N-1} l \left(\frac{(1+\epsilon)^l}{l!} e^{-(1+\epsilon)} - \frac{\epsilon}{\kappa(\kappa-1)}\right)$$
$$= (1+\epsilon)e^{-(1+\epsilon)} \sum_{l=0}^{k_N-2} \left(\frac{(1+\epsilon)^l}{l!}\right) - \frac{\epsilon}{2} > (1+\epsilon)e^{-(1+\epsilon)}(e^{1+\epsilon} - \alpha) - \frac{\epsilon}{2}$$
$$= 1 + \frac{\epsilon}{2} - \frac{\epsilon}{2} = 1$$

what ends the proof. □

4.5 Monotonicity of the critical times in k and d

Observe that the previous sections have proved the following relation for the critical time $t_c(\mathbb{T}_d, k)$:

$$t_c(Z^-(d,k,t)) \leq t_c(\mathbb{T}_d, k) \leq t_c(Z^+(d,k,t))$$

To prove the existence of the upper bound of order $o(1/d)$ we have used an upper bound on the rightmost expression. In order to say something about the monotonicity of the critical time we will need a tighter bound; one that depends on k, what is not provided by Theorem 3. Observe that monotonicity in d, k can be shown if we prove the following assertions for all $k \geq 3$ and $d \geq k$:

$$t_c(\mathbb{T}_d, k) \leq t_c(Z^+(d,k,t)) < t_c(Z^-(d,k-1,t)) \leq t_c(\mathbb{T}_d, k-1)$$
$$t_c(\mathbb{T}_d, k) \leq t_c(Z^+(d,k,t)) < t_c(Z^-(d-1,k,t)) \leq t_c(\mathbb{T}_{d-1}, k)$$

Although we weren't able to show this explicitly, numerical calculations performed to evaluate these inequalities lead to the following conclusions:

Theorem 5. For all $3 \leq d \leq 100$ and $3 \leq k \leq d$, the critical time $t_c(\mathbb{T}^d, k)$ is monotonously decreasing in k and d.

Proof. We only need to prove the second inequality of both cases since the remaining ones follow from previous results. Since we know that $\frac{1}{d} < t_c(\mathbb{T}_d, k) < \frac{1.2}{d}$ for all considered d, k, the inequality would follow from showing that for all $t \in [\frac{1}{d}, \frac{1.2}{d}]$ there is

$$\mathbb{E}[X^+(d,k,t)] - \mathbb{E}[X^-(d,k-1,t)] > 0\,,$$
$$\mathbb{E}[X^+(d,k,t)] - \mathbb{E}[X^-(d-1,k,t)] > 0\,.$$

This can be done numerically by calculating the difference between the expected values and checking whether it is always greater than 0 for all t on the considered interval. The code in the appendix strongly suggests the correctness of this hypothesis for all $d < 100$ and $3 \leq k \leq d$. However, it cannot be treated as an explicit proof, since not all values of t are considered. We would nevertheless expect a certain smoothness from these functions on the considered interval and hence it is very unlikely that significant fluctuations would appear at the unchecked arguments. □

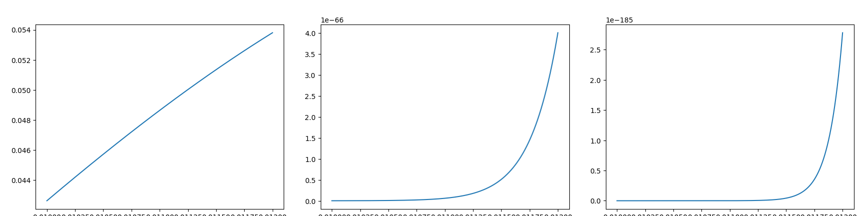

Figure 3: Graphs of the difference of expectations for $d = 100$ and $k = 4, 50, 99$ respectively on the interval $[0.01, 0.012]$. Importantly, the functions take only non-negative values

5 Conclusion

An insight which allowed us to hypothesise the order of the critical time of the CDP's to be $o(1/d)$ is that the constraint doesn't play a major role in cases where t is small. More precisely, for small values of parameter t there is a very small probability that more than k edges neighbouring a single vertex have opening times smaller than t. If this intuition were to be correct, CDP should behave closely to the unconstrained percolation model when $t \approx 1/d$, and hence the motivation to search for bounds of such order. The methods used to prove these bounds have also adopted this observation, since we were able to compare the CDP to other models whose behaviour was partially mimicking the behaviour of the unconstrained model. Further research is needed to extend the upper bound to the case $d = 3$ and to optimise its value of $1.2/d$.

The thesis investigates the asymptotic behavior of the critical time in CDP's, deriving the asymptotic formula for the critical time in the case when both parameters d, k go to infinity. Unfortunately the method used to prove Theorem 4 is not sufficient to show similar result when the constraint k is allowed to be constant. The critical time $t_c(Z^+(d_n, k, t))$ does not converge to $1/d$ as $n \to \infty$, but this does not imply that the same holds for $\lim_{d\to\infty} t_c(\mathbb{T}_d, k)$. Intuition suggests that as $d \to \infty$, the critical time $1/d$ of the unconstrained percolation model approaches 0, and hence the constraint k should have less and less impact on the critical time of the CDP. This supports the following conjecture.

Conjecture 1. For all increasing sequences $(d_n)_n \in \mathbb{N}$ where $d_0 \geq k$, the critical time satisfies

$$\lim_{n\to\infty} d_n \cdot t_c(\mathbb{T}^d, k) = 1 \,.$$

As we discussed before, it is to be expected that as d, k grow, more and more edges are able to open and hence the critical time of the CDP should decrease. The way such results are usually proved in the percolation theory is by making use of standard coupling arguments, where one modelh is shown to produce configurations which are subsets of configurations produced by the second model at all times. However, as was shown in the introduction, these are not applicable for the CDP's. Hence another method needs to be found in order to prove this result rigorously. The computer assisted solution outlined in Theorem 5 was merely an attempt at showing this result for a limited number of parameters d, k. Both the intuitive insight and the numerical results from the thesis allow to formulate the following conjecture.

Conjecture 2. For all $3 \leq d$ and $3 \leq k \leq d$, the critical time $t_c(\mathbb{T}^d, k)$ is monotonously decreasing in k and d.

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Appendix: Python code

```
import math
import numpy as np

def binomial(n,k):
  expression = 1
  for i in range(n-k+1, n+1):
    expression *= i
  expression = expression / math.factorial(k)
  return expression

def Difference(k, d, p):
  array_probabilities = np.zeros(d-k+1)
  expected_value = 0
  sum = 0

  for i in range(k,d+1):
    array_probabilities[i-k]=binomial(d, i)*(p ** i)*(1 - p)**(d - i)
    expected_value+=array_probabilities[i-k]*i
    sum += array_probabilities[i-k]

  expected_value *= (k-2)
  expected_value = binomial(d, k-1)*(p**(k-1))*(1-p)**(d-k+1)-sum
  return array_probabilities, expected_value

d_max = 100
N = 10000
Fault_Counter = 0

for d in range(3, d_max+1):
  probabilities = np.linspace(1/d, 1.2/d, N)
  print(d)

  for k in range(3, d+1):
    for l in range(N):
      if Difference(k, d, probabilities[l])[1] < 0:
        Fault_Counter += 1
        print('Fault at', d, k, 'at time', probabilities[l])

if Fault_Counter == 0:
  print("Success")

else:
  print("Failure")
```