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Beyond the Standard Model: Leptoquark Explanations for $R(D)$ and $R(D^*)$ Flavour Anomalies

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Abstract

The Standard Model of particle physics remains an extensive yet incomplete description of the physics behind fundamental particles and their interactions. Among the issues not addressed by the model are the lepton flavour anomalies, suggesting that the third generation of leptons requires an alternative description. The $R(D)$ and $R(D^*)$ anomalies concerning the semileptonic τ decay mode of a B meson constitute a combined 3.3σ deviation from the Standard Model prediction. Potential explanations of these phenomena include novel bosonic particles – leptoquarks, allowing for a direct transition mechanism between quarks and leptons. They appear naturally in generalisations of the Standard Model gauge group to Grand Unified Theories. In this work, three leptoquark models were presented as manifestations of the UV-complete $SU(5)$, $SO(10)$, and the Flavoured Gauge models. The scalar leptoquarks were determined to exist with a mass of an order of TeV within the unified $SU(5)$ and $SO(10)$ models, whereas the last vector leptoquark was identified to emerge from the non-universal gauge model, with a mass of approximately 1.6 TeV. The inclusion of any of the leptoquark particles was proven to resolve the $R(D^{(*)})$ anomalies while simultaneously accounting for the agreement between certain other observables. A potential observation of one of these bosons would constitute evidence for a larger symmetry structure in the Universe.

1 Introduction

The Standard Model of particle physics is widely regarded as one of the most successful theories, describing the realm of fundamental particles and their interactions with remarkable accuracy. Albeit an outstanding model, it is troubled by a wide range of uncertainties and unexplained issues. These include, for instance, the number of free parameters to be determined experimentally or the apparent symmetry between different generations of quarks and leptons [1, p. 223]. Moreover, another crucial problem emerges when considering the universality of lepton flavours, stating that the relative strength of the electroweak interaction involving different generations of leptons is precisely equivalent. It is only broken by the Yukawa interaction coupling the leptons to the Higgs field [2]. In essence, it implies that the decays containing various lepton flavours depend only on the available phase space and helicity suppression effects [3]. This fact has been thoroughly tested through semileptonic decay modes of heavy mesons, such as the measurements of the $R(D)$ and $R(D^*)$ observables, involving the branching fraction of beauty-charm quark transitions in B-meson decays

$$R(D^{(*)}) = \frac{\mathcal{B}(B \to D^{(*)}\tau\bar{\nu}_\tau)}{\mathcal{B}(B \to D^{(*)}\ell\bar{\nu}_\ell)}.$$

The decays involving the third generation of leptons were observed to occur much more frequently than predicted, within three standard deviations from the Standard Model result, constituting a violation of lepton flavour universality [4, p. 21].

This violation of the established expectations naturally hints at possible extensions of the theory, including new physics beyond the Standard Model. Of particular theoretical interest are the elusive Grand Unified Theories (GUTs), where larger symmetry groups containing the Standard Model as a subgroup are employed in hopes of rectifying the anomalies and providing a mathematically concise way to describe elementary particles by utilising representations of a single gauge group. One of the fundamental features of such new physics models is the existence of leptoquarks — bosons coupling simultaneously to quarks and leptons, providing a direct mechanism to transform between the individual particles [5]. In the context of solutions to the $R(D^{(*)})$ anomalies, three leptoquarks often appear: the scalar doublet R_2, scalar singlet S_1, and the vector singlet U_1, all equipped with different transformation properties [6]. Definite experimental observation of one of these particles may provide evidence for the applicability of the Grand Unified Theories, elucidating the larger symmetry structure within the Universe.

This thesis aims to provide a bottom-up approach to the explanations of lepton flavour anomalies in measurements by determining whether the proposed leptoquark models are directly manifested within given Grand Unified Theories and other mathematical structures beyond the Standard Model. The relevant symmetry groups on which the analysis will focus include flavour non-universal gauge groups with a $SU(4) \times SU(3) \times SU(2) \times U(1)$ subgroup. Moreover, more traditional extensions, such as $SO(10)$ and the UV-complete version of $SU(5)$, will also be discussed.

For these purposes, first, a formal overview of the Standard Model will be presented to establish a solid foundation allowing for comprehending the various symmetries of the theory, the convention of labelling particles depending on their transformation characteristics, as well as the concept of spontaneous symmetry breaking. Consequently, the current state of the $R(D)$ and $R(D^*)$ observable measurements will be presented to identify the properties of the predicted particles. Finally, the relevant GUT frameworks will be discussed to determine which ones provide suitable representations of the necessary leptoquarks, constituting possible Standard Model extensions based on the current experimental data.

2 The Standard Model

Modern particle physics is written in the language of gauge theory, in which the Lagrangian of a given system determining its dynamics may be modified with a choice of a *gauge* – a set of additional constraints that reduces the number of degrees of freedom. The symmetry group of the system's action, referred to as the *gauge group*, is comprised of all possibly spacetime-dependent transformations that result in an equivalent description, utilising the existing redundancy in the parameters [7]. For the Standard Model, the gauge group G_{SM} is a direct product of three Lie groups

$$G_{\mathrm{SM}} \equiv SU(3)_C \times SU(2)_L \times U(1)_Y,$$

constituting the colour $SU(3)_C$, isospin $SU(2)_L$, and hypercharge $U(1)_Y$ internal symmetry groups, corresponding to the strong and electroweak interactions, respectively. These are identified with the aforementioned relevant quantum numbers, determining the transformation properties of particles.

Depending on the energy scale, the gauge group describing the symmetry of the theory may degenerate to a different group in a process known as *spontaneous symmetry-breaking*. Within the Standard Model, one such transition is the electroweak symmetry-breaking, where the electroweak sector $SU(2)_L \times U(1)_Y$ breaks into the electromagnetic symmetry group $U(1)_Q$, identified with the electric charge

$$SU(3)_C \times SU(2)_L \times U(1)_Y \rightarrow SU(3)_C \times U(1)_Q.$$

This mechanism is crucial to many extensions of the Standard Model, with Grand Unified Theories being the prime example.

The aim of this section is to establish a suitable mathematical framework for describing the particles and their properties under G_{SM}. Identifying these characteristics will allow for generalising the treatment to certain Grand Unified Theories and recognising the footprint of new physics particles, such as the leptoquarks responsible for lepton flavour anomalies. Moreover, the mechanism of spontaneous symmetry-breaking will also be discussed to provide a solid foundation for the phenomenology of the various symmetry-breaking chains in Standard Model extensions. The arguments presented in this section are largely based on "The Algebra of Grand Unified Theories" by John Baez and John Huerta [8, pp. 487–511].

2.1 Particle representations

In many applications, it is convenient to represent the action of a group using the familiarity of linear algebra. Representations of groups provide a formal way of expressing the elements of a group as invertible matrices acting on a given vector space. In terms of particle physics, the concerned vector space is often a finite-dimensional Hilbert space $\mathcal{H}$ in which particle states exist. Depending on the dimension of the representation n, this vector space may be simply represented through the set of complex numbers $\mathbb{C}^n$.

If G is an arbitrary matrix Lie group, the representation Π of the group is a homomorphism mapping an element g of G to an $n \times n$ invertible matrix $M_{n\times n}$, an element of the general linear group GL, defined on the appropriate n-dimensional Hilbert space $\mathcal{H}$, mirroring the set of transformations imposed by the group in question

$$\begin{aligned} \Pi: \quad G &\to \mathrm{GL}(n; \mathcal{H}) \\ g &\mapsto M_{n\times n}(g). \end{aligned}$$

If under the action of the group, there exist smaller invariant subspaces of the Hilbert space that remain unchanged by the transformations

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \ldots,$$

the representation is deemed *reducible* and may consequently be decomposed into individual *irreducible* representations

$$\Pi = \Pi_1 \oplus \Pi_2 \ldots .$$

Fundamental particles of the Standard Model are modelled as basis vectors of the individual invariant subspaces of the general Hilbert space. In physics literature, it is common to refer to these vector spaces themselves as representations, which is the convention that will be used in this work when referring to particles as representations.

The representations of the groups in G_{SM} provide a means of transforming between particles through the exchange of *bosons*. These particles emerge when considering the Lie algebra $\mathfrak{g}$ of a matrix Lie group G – the set of all matrices X, such that $e^{\lambda X} \in G$, for all real parameters λ [9, pp. 36–37]. In essence, the Lie algebra facilitates approximating the group structure for elements close to the identity, used when applying infinitesimal transformations.

Consider the *adjoint map* Ψ_A obtained by conjugating an element $X \in \mathfrak{g}$ by some $A \in G$ [9, p. 43]

$$\begin{aligned} \Psi_A: \quad G &\to \mathrm{GL}(\mathfrak{g}) \\ A &\mapsto AXA^{-1}. \end{aligned}$$

The mapping induces a linear transformation on the Lie algebra, resulting in a group homomorphism, since if $A \in G$ and $X, Y \in \mathfrak{g}$:

$$\Psi_A(XY) = AXYA^{-1} = AX\overbrace{A^{-1}A}^{\mathbb{I}}YA^{-1} = \Psi_A(X)\Psi_A(Y).$$

By extension, the mapping constitutes a real *adjoint representation* of G, acting on $\mathfrak{g}$ [9, p. 70]. Each Lie group homomorphism induces a corresponding Lie algebra homomorphism, so the adjoint representation extends to a representation of the Lie algebra [9, p. 41]. Furthermore, $\mathfrak{g}$ may be complexified by forming a linear combination of its elements through the following form

$$g_1 + ig_2, \quad g_1, g_2 \in \mathfrak{g}.$$

This results in a complex algebra, denoted as $\mathfrak{g}_{\mathbb{C}}$ [9, pp. 48–49]. Depending on the dimension, there is an appropriate number of generators that span the space. These generators of the complexified adjoint representation of the group are precisely associated with bosons, acting on the corresponding particle space $\mathcal{H}$. The mechanism of nucleon-nucleon interactions serves as a suitable illustration of this principle.

Example (Nucleon-nucleon interactions)
In 1935, Hideki Yukawa conjectured the existence of a hypothetical U-field, mediating the interactions between nucleons. He predicted that the carriers of this force, named *pions*, would be approximately 200 times more massive than the electron [10]. In 1947, Cecil Powell and Giuseppe Occhialini confirmed the existence of two electrically charged states π^+ and π^- in a cosmic ray experiment, with the neutral state π^0 being discovered consequently in 1949 in a particle accelerator experiment at Berkeley [11], [12].

Let protons and neutrons be the basis vectors of the complex space $\mathbb{C}^2$

$$p = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad n = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Guided by the similarity of interactions based on whether the concerned particle is a neutron or a proton, in 1936, Benedict Cassen and Edward Condon suggested that the force is invariant under the symmetry group $SU(2)$ [13]. Even though this proved to be a significant simplification, it is still considered appropriate to a large degree.

The Lie algebra $\mathfrak{su}(2)$, after complexification, becomes a three-dimensional algebra $\mathfrak{sl}(2;\mathbb{C})$, with a basis corresponding to the different pion states [14]

$$\pi^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \pi^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \pi^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

In an example of a nucleon-nucleon interaction, a proton may absorb a negatively charged pion, transforming into a neutron. In terms of representation theory, this is described by the action of one of the basis states of $\mathfrak{sl}(2;\mathbb{C})$ on the Hilbert space of the nucleon $\mathbb{C}^2$, signifying the approximate mechanism of the strong interaction.

$$\pi^- + p \to n$$

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

2.2 Standard Model symmetries

2.2.1 Hypercharge $U(1)$

The unitary group of the first degree, $U(1)$, in its defining representation, consists of all complex phases which may be interpreted geometrically as the rotational symmetry group of a circle

$$U(1) = \{e^{i\varphi} \mid \varphi \in [0, 2\pi)\}.$$

Since this group consists of one-dimensional scalars equipped with regular multiplication, it is an abelian group. Moreover, the hermitian conjugate of any element $U \in U(1)$ is associated with the inverse of the element

$$U^\dagger U = 1 \implies U^\dagger = U^{-1}.$$

The action of the $U(1)$ group on an arbitrary particle state $\psi \in \mathcal{H}$ is represented via the *hypercharge* quantum number Y

$$U \cdot \psi = e^{3iY\varphi}\psi,$$

where the factor of three in the exponential is present purely due to convention. Hence, particles with a nonzero value of hypercharge interact with each other through the action of $U(1)$. Consequently, the vector space in which these particles exist may be labelled by $\mathbb{C}_Y$, since all irreducible representations of the group are one-dimensional. Moreover, the complexified adjoint representation of the Lie algebra is isomorphic to $\mathbb{C}$ – hence, the interactions are mediated by a one-dimensional element, referred to in the literature as the B-boson.

2.2.2 Isospin $SU(2)$

The special unitary group of the second degree encapsulates all 2×2 unitary matrices with complex entries, equipped with unit determinant

$$SU(2) = \{U \in \mathbb{C}^{2\times 2} \,|\, U^\dagger U = \mathbb{I}_{2\times 2} \text{ and } \det(U) = 1\}.$$

An arbitrary element $U \in SU(2)$, may be expressed in terms of three generators σ_i, such that

$$U = e^{-i\varphi\sigma_i/2},$$

where $\sigma_1, \sigma_2, \sigma_3$ correspond to the renowned Pauli matrices [15, p. 39]

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In analogy with the quantum-mechanical spin of particles, the concept of *isospin* I_3 may be introduced, as an internal symmetry of a given system under the weak interaction. The isospin-$n/2$ representation is the unique $(n+1)$-dimensional irreducible representation of $SU(2)$, given by symmetric tensors of rank n. In this basis, the isospin of a given particle ranges from $-n/2$ to $n/2$ in integer steps

$$I_3 \in \left[-\frac{n}{2}, -\frac{n}{2}+1, \dots, \frac{n}{2}-1, \frac{n}{2}\right].$$

Since the weak force interacts with left-handed particles and right-handed antiparticles, the left-handed fermions form *doublets*

$$\begin{pmatrix} \ell \\ \nu_\ell \end{pmatrix}_L, \begin{pmatrix} u \\ d \end{pmatrix}_L \in \mathbb{C}^2,$$

where u and d represent the up and down-type quarks of any flavour, but within the same generation. On the other hand, the right-handed particles transform under the trivial representation of $SU(2)$, forming *singlets*

$$\ell_R, \nu_{\ell_R}, u_R, d_R \in \mathbb{C}.$$

The force is mediated by three W bosons, being the basis of $\mathfrak{sl}(2;\mathbb{C})$

$$W^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad W^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad W^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The quantum numbers identified via $U(1)$ and $SU(2)$ symmetries are related to each other through the electric charge based on the *Gell-Mann-Nishijima formula*, introduced independently by Tadao Nakano and Kazuhiko Nishijima in 1953, as well as Murray Gell-Mann in 1956 [16], [17]

$$Q = I_3 + \frac{Y}{2}, \tag{2.1}$$

where Q is the electric charge of the particle, I_3 is its isospin component, and Y is the hypercharge. Initially, the symmetry was identified with the action of the strong interaction, however, in 1961, it was generalised to the weak force by Sheldon Glashow [18]. This formula allows for a consistent way of categorising the particles based on their quantum numbers, representing the transformation properties under the aforementioned groups.

2.2.3 Colour $SU(3)$

The special unitary group of the third degree is represented by 3×3 unitary complex matrices with unit determinant

$$SU(3) = \left\{ U \in \mathbb{C}^{3\times 3} \,|\, U^\dagger U = \mathbb{I}_{3\times 3} \text{ and } \det(U) = 1 \right\}.$$

Any given element $U \in SU(3)$ may be written as follows

$$U = e^{i\varphi\lambda_i/2},$$

where λ_i belongs to the set of eight traceless hermitian 3×3 matrices, known as the *Gell-Mann* matrices, generating the $\mathfrak{su}(3)$ Lie algebra of the group [15, p. 43], [19, p. 59], [20],

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & -\frac{2}{\sqrt{3}} \end{pmatrix}.$$

After the complexification of the aforementioned basis, the generators of the resulting Lie algebra $\mathfrak{sl}(3;\mathbb{C})$ are associated with gluons, the mediators of the strong interaction. The relevant Hilbert space is identified with the complex space $\mathbb{C}^3$ with the basis vectors corresponding to the three quark colours. Each quark flavour spans $\mathbb{C}^3$, whereas leptons are not affected by the action of the strong force - they are considered "white", spanning the space $\mathbb{C}$, transforming under the trivial representation of the group.

2.3 Labelling particles

In general, any given particle P within the Standard Model representation may be labelled based on its transformation properties under the individual gauge groups

$$P \sim (\mathbf{m}, \mathbf{n}, Y/2),$$

where **m** and **n** are the dimensions of the representations of the $SU(3)$ and $SU(2)$ groups respectively, and $Y/2$ is the rescaled hypercharge quantum number. The reason for the adjustment of the hypercharge is largely conventional since the factor of $1/2$ is also present in the Gell-Mann-Nishijima formula (2.1), and hence allows for relating the isospin and the electric charge straightforwardly.

Example (Left-handed leptons)
Any left-handed lepton ℓ_L is invariant under the strong force, transforming as a singlet in $\mathbb{C}$, being a one-dimensional vector space on which the representation acts. Furthermore, it forms a doublet in $\mathbb{C}^2$ alongside same-generation quarks, implying the $SU(2)$ representation must be two-dimensional. Lastly, the hypercharge associated with it is -1. Therefore, the lepton may be labelled within the Standard Model as follows

$$\ell_L \sim (\mathbf{1}, \mathbf{2}, -1/2).$$

The leptons exist within a isospin-1/2 representation, and hence, $I_3 \in \{-\frac{1}{2}, \frac{1}{2}\}$, where $I_3 = -\frac{1}{2}$ is associated with the electron e and $I_3 = \frac{1}{2}$ refers to the electron neutrino ν_e, analogously to any other generation of leptons. Using equation (2.1), the electric charges of the particles may be found directly

$$Q(e) = -\frac{1}{2} - \frac{1}{2} = -1,$$

$$Q(\nu_e) = \frac{1}{2} - \frac{1}{2} = 0.$$

2.4 Spontaneous symmetry-breaking

Modern theories in the field of particle physics often rely on studying the symmetries of the Lagrangian and the various paths in which they may be broken. Including additional terms in the Lagrangian that violate the preexisting symmetry is known as *explicit symmetry breaking*. Arguably more interesting scenario emerges when the system itself disrupts the given symmetry in a process referred to as *spontaneous symmetry breaking*.

Following the treatment of Zee [21, pp. 223–229, 263–264], consider a generic scalar field Lagrangian as a function of N fields $\vec{\varphi} = (\varphi_1, \varphi_2, \dots \varphi_N)$, including a potential function $V(\varphi)$. A particular example of such a function is the Ginzburg-Landau potential[1]

$$V(\varphi) = -\frac{1}{2}\mu^2\vec{\varphi}^2 + \frac{\lambda}{4}(\vec{\varphi}^2)^2, \tag{2.2}$$

such that the full Lagrangian is given as

$$\mathcal{L} = \frac{1}{2}\partial_\mu\vec{\varphi}\partial^\mu\vec{\varphi} + \frac{1}{2}\mu^2\vec{\varphi}^2 - \frac{\lambda}{4}(\vec{\varphi}^2)^2. \tag{2.3}$$

[1]Named in honour of Vitaly Ginzburg and Lev Landau who utilised this model in their study of energy density in superconducting materials [22, p. 251]. It is also informally referred to as the Mexican hat or wine bottle potential.

Comparing this form to the Klein-Gordon Lagrangian for a real scalar field ϕ [23, p. 8]

$$\mathcal{L}_{\text{KG}} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2, \tag{2.4}$$

it may be concluded that the field given by equation (2.3) produces a particle with $m = i\mu$. However, this situation changes drastically when the symmetry is broken.

For the one-dimensional case $N = 1$, such that $\vec{\varphi} \equiv \varphi$, the graph of the potential is illustrated in figure 1 below.

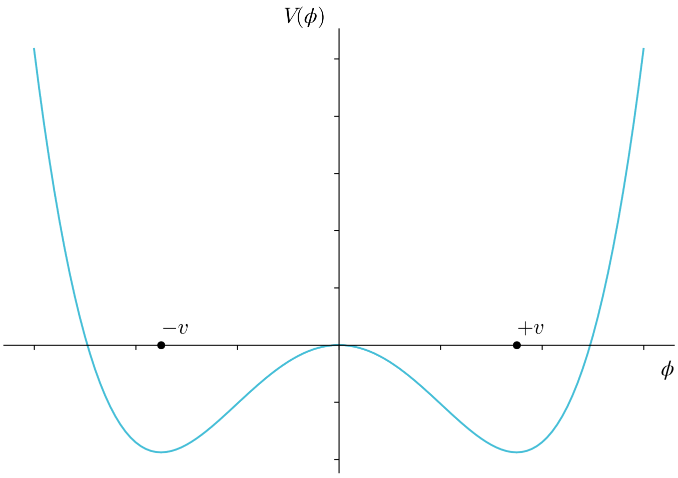

Figure 1: Plot of the Ginzburg-Landau potential form for $N = 1$ with a constant $\lambda > 0$.

The minima v of the potential function may be explicitly calculated.

$$\frac{dV}{d\varphi} = -\mu^2\varphi + \lambda\varphi^3 = 0$$

$$\implies -\mu^2 + \lambda\varphi^2 = 0, \quad (\varphi \neq 0)$$

$$\varphi \equiv v = \pm\sqrt{\frac{\mu^2}{\lambda}}$$

In the framework of Quantum Field Theory, the probability of tunneling between the minima is suppressed, since the width of the barrier is infinite. Therefore, the ground state of the system is concentrated around one of the minima. However, it is arbitrary which minimum it is, as they both represent the state of lowest energy for the system. This fact implies that the reflection symmetry of $\varphi \mapsto -\varphi$ is spontaneously broken.

Choosing the ground state to be at $+v$ and substituting $\varphi = v + \varphi'$ in the original Lagrangian given by equation (2.3)

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}\partial_\mu \varphi' \partial^\mu \varphi' + \frac{\mu^2}{2}(v+\varphi')^2 - \frac{\lambda}{4}(v+\varphi')^4 \\
&= \frac{1}{2}\partial_\mu \varphi' \partial^\mu \varphi' + \frac{\mu^2}{2}(v^2 + 2v\varphi' + \varphi'^2) - \frac{\lambda}{4}(v^4 + 4v^3\varphi' + 6v^2\varphi'^2 + 4v\varphi'^3 + \varphi'^4) \\
&= \frac{1}{2}\partial_\mu \varphi' \partial^\mu \varphi' + \cancel{(\mu^2 v - \lambda v^3)}\varphi' + \left(\frac{\mu^2}{2} - \frac{3\lambda}{2}v^2\right)\varphi'^2 + \frac{\mu^2}{2}v^2 - \frac{\lambda}{4}v^4 + \mathcal{O}(\varphi'^3) \\
&= \frac{1}{2}\partial_\mu \varphi' \partial^\mu \varphi' - \mu^2\varphi'^2 + \mathcal{O}(\varphi'^3),
\end{aligned}$$

where the constant terms have been omitted, as they produce no physical effect in the Lagrangian. Moreover, the terms of order φ'^3 have been neglected, since they constitute higher-order corrections. This form of the Lagrangian is identified with a particle with a real mass parameter $m = \sqrt{2}\mu$. The process of spontaneous symmetry-breaking forces the field to assume one of the two minima and acquire a nonzero *vacuum expectation value*, $\langle\varphi\rangle$.

For higher-dimensional situations, e.g. with $N = 2$, the potential function is very similar, as shown in figure 2.

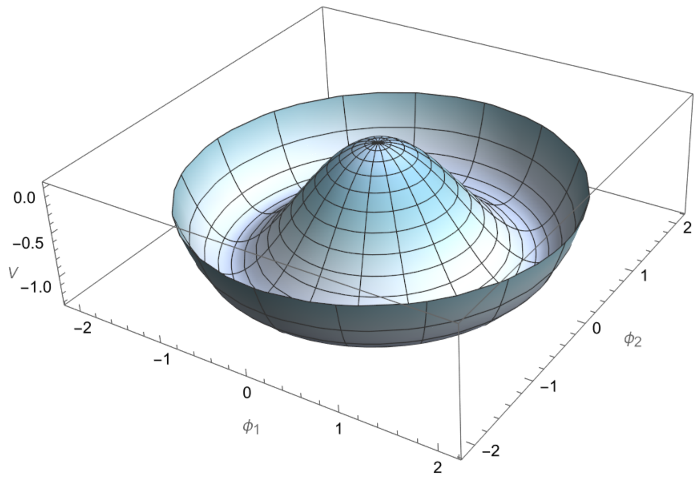

Figure 2: Three-dimensional plot of the Ginzburg-Landau potential form for $N = 2$ with a constant $\lambda > 0$. The fields φ_1 and φ_2 may be identified with the x and y-axes and $V(\varphi_1, \varphi_2)$ with the z-axis of the graph.

This potential has a continuous rotational symmetry, with an infinite spectrum of minima at $\vec{\varphi} = \pm\sqrt{\mu^2/\lambda}$. Since all of the configurations are physically equivalent, a particular choice for the field minima may be made

$$\varphi_1 \equiv v = \sqrt{\frac{\mu^2}{\lambda}}, \quad \varphi_2 = 0.$$

As previously, performing the substitution $\varphi_1 = v + \varphi_1'$ and $\varphi_2 = \varphi_2'$, the relevant Lagrangian after rearranging is given as

$$\mathcal{L} = \frac{1}{2}\partial_\mu \varphi_1' \partial^\mu \varphi_1' + \frac{1}{2}\partial_\mu \varphi_2' \partial^\mu \varphi_2' - \mu^2\varphi_1'^2 + \mathcal{O}(\varphi_{1,2}'^3).$$

The field φ_1 has an associated mass $m = \sqrt{2}\mu$ whereas, since there is no contribution from φ_2^2, the field φ_2 has $m = 0$.

Example (Nambu-Goldstone theorem)
The aforementioned phenomenon entails that with each instance of breaking a continuous symmetry of a Lagrangian, there is a corresponding massless field known as the *Nambu-Goldstone boson*. This realisation is attributed to Yoichiro Nambu, who in 1960 elaborated on the "hidden" nature of spontaneous symmetry breaking in superconductivity, and Jeffrey Goldstone, who a year later illustrated the quantum theory behind the transition from a symmetric to an antisymmetric state, providing foundations for Nambu's idea [22, pp. 56, 61].

To showcase the proof of this statement, known as the Nambu-Goldstone theorem, consider the existence of a conserved charge Q, responsible for a certain continuous symmetry. The statement of conservation is equivalent to noting that Q commutes with the Hamiltonian

$$[H, Q] = 0.$$

Given the vacuum state $|0\rangle$, it is defined as the state of zero energy, such that $H|0\rangle = 0$. Moreover, it possesses an invariance under the transformation associated with the symmetry, i.e. $Q|0\rangle = 0$. If the symmetry is spontaneously broken this requirement no longer applies

$$Q|0\rangle \neq 0.$$

Applying the Hamiltonian to determine the energy of this specific state

$$HQ|0\rangle = HQ|0\rangle - \underbrace{QH|0\rangle}_{0} = (HQ - QH)|0\rangle = [H, Q]|0\rangle = 0.$$

Hence, since the energy is zero, but the state does not correspond to the vacuum, it necessitates the existence of a massless particle.

Goldstone's theorem is crucial in the construction of the *Higgs mechanism*, where the Higgs boson absorbs such a Nambu-Goldstone boson. It acquires mass through the introduction of additional degrees of freedom in gauge theories, such as those based on $U(1)$ symmetry.

2.4.1 Electroweak symmetry-breaking

An example of the mechanism of spontaneous symmetry-breaking within the Standard Model framework is manifested by the electroweak interaction modelled by the $SU(2)_L \times U(1)_Y$ gauge group. Following the outline of [24], the gauge bosons associated with the electroweak interaction are the three W bosons and one B boson, constituting the gauge fields of the theory. A complex scalar $SU(2)$ doublet $\varphi = (\phi^+, \phi^0)^T$ is coupled to the fields producing the following potential function

$$V(\varphi) = \mu^2|\varphi^\dagger\varphi| + \lambda|\varphi^\dagger\varphi|^2.$$

As described previously, there is a freedom of choice for the minimum of the potential, implying a nonzero vacuum expectation value of the field

$$\langle \varphi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} .$$

Due to this choice, the doublet is assigned a hypercharge $Y = 1$, such that the electric charge $Q\langle \varphi \rangle = 0$. This fact implies that even though the electroweak symmetry has been spontaneously broken, the vacuum expectation value is not changed under the action of Q, meaning it is still a symmetry of the system. The effective symmetry of the theory has undergone *electroweak symmetry-breaking*.

3 Lepton Flavour Universality

The three generations of left-handed leptons, i.e. the electron e, the muon μ, and the tau τ, are characterised by the same value of isospin I_3 and electric charge Q. The same holds for their respective neutrinos, where the quantum numbers are also irrespective of the family.

Particle	Isospin I_3	Hypercharge Y	Electric charge Q
$e/\mu/\tau$	$-1/2$	-1	-1
$\nu_e/\nu_\mu/\nu_\tau$	$1/2$	-1	0

Table 1: Summary of all relevant left-handed lepton quantum numbers for the electroweak interaction [8, p. 501].

Due to these characteristics, in the Standard Model framework, the three generations of leptons interact in precisely the same way via the weak force, leading to the same strength of the interaction between families, constituting the concept of *Lepton Flavour Universality* (LFU). The only distinguishing factor among the different flavours is their coupling to the Higgs field through the Yukawa interaction, which results in varying masses of the particles, being significantly higher for the third generation [2, p. 85], [25]

$$m_e = 0.51099 \text{ MeV}/c^2,$$

$$m_\mu = 105.66 \text{ MeV}/c^2,$$

$$m_\tau = 1776.9 \text{ MeV}/c^2.$$

In terms of experimental evidence for that phenomenon, one common approach is to determine the branching fractions of decays including various lepton generations. In the current theory, these should only be affected by the available phase space for the decay and helicity suppression effects [3]. However, data from experiments such as Large Hadron Collider beauty (LHCb) at CERN, Belle at KEK, or BaBar at Stanford have produced tensions with the Standard Model expectations of the decays involving the third generation of leptons, seemingly occurring much more likely than predicted, hinting at violations of flavour universality in phenomena deemed *flavour anomalies*.

Of specific experimental interest are the B-mesons, containing the beauty quark b, being the second most massive quark right after the top t. Semileptonic decays of these hadrons provide an accessible window into the flavour anomalies, as because of their large mass, all three generations of leptons may be produced in the decay [3]. The LFU violations in B-mesons decays are referred to as B flavour anomalies, existing in two specific types. Firstly, the beauty-strange transition

$$b \to s\ell^+\ell^-,$$

producing a lepton-antilepton pair involves an exchange of a neutral boson, resulting in *neutral-current anomalies*. On the other hand, the beauty-charm transition

$$b \to c\ell^-\bar{\nu}_\ell$$

results in a production of a lepton and a corresponding antineutrino, requiring a charged W^- boson to mediate the interaction, constituting *charged-current anomalies* [4, p. 1].

3.1 $R(D)$ and $R(D^*)$ observables

An example of a charged-current semileptonic decay is the decay of a neutral $\bar{B}^0$ to a possibly excited state of a D-meson $D^{(*)^+}$, where the parentheses encapsulate both D^+ and D^{*^+} possibilities. The decay is represented in the Feynman diagram in figure 3 below.

$$\bar{B}^0 \to D^{(*)^+} \tau \bar{\nu}_\tau \tag{3.1}$$

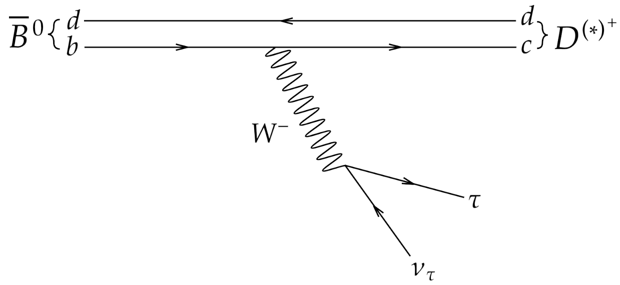

Figure 3: Feynman diagram of the semileptonic $\bar{B}^0$ decay to $D^{(*)^+}\tau\bar{\nu}_\tau$.

Since this decay involves the production of the tau τ, being the heaviest third-generation lepton, of particular interest are the observables $R(D^{(*)})$, being the ratio of the branching fraction of this decay, named the signal mode, to one containing a lepton of the lower generation $\ell = e, \mu$, referred to as the normalisation mode

$$R(D^{(*)}) = \frac{\mathcal{B}(B \to D^{(*)}\tau\bar{\nu}_\tau)}{\mathcal{B}(B \to D^{(*)}\ell\bar{\nu}_\ell)}. \tag{3.2}$$

Based on Lattice QCD calculations, the Standard Model expectations for $R(D)$ and $R(D^*)$ are estimated to be [26], [27]

$$R(D)_{\text{SM}} = 0.299 \pm 0.011, \tag{3.3}$$

$$R(D^*)_{\text{SM}} = 0.252 \pm 0.003. \tag{3.4}$$

From an experimental standpoint, based on the unified results from LHCb, Belle, and Babar, the Heavy Flavour Averaging group (HFLAV) compiled averages of the measurements [28]

$$R(D)_{\text{AVG}} = 0.342 \pm 0.026, \tag{3.5}$$

$$R(D^*)_{\text{AVG}} = 0.287 \pm 0.012. \tag{3.6}$$

constituting a significant discrepancy between the theoretically established values, resulting in 1.6σ deviation in $R(D)$ and 2.5σ deviation in $R(D^*)$ [28]. Furthermore, the anomaly is even more pronounced when simultaneously considering the results from $R(D)$ and $R(D^*)$, which yield a combined 3.3σ disagreement [28].

To compactify the extent of the measurement discrepancy with the current theory, the quantities $\Delta R(D)$ and $\Delta R(D^*)$ may be defined as follows

$$\Delta R(D) \equiv \frac{R(D)_{\text{AVG}}}{R(D)_{\text{SM}}} - 1, \tag{3.7}$$

$$\Delta R(D^*) \equiv \frac{R(D^*)_{\text{AVG}}}{R(D^*)_{\text{SM}}} - 1, \tag{3.8}$$

utilising the comparison between the averaged experimental value and the Standard Model calculation with perfect agreement corresponding to $\Delta R(D^*) = 0$. Using the data from equations (3.3), (3.5), and (3.4), (3.6), the current state of anomaly may be calculated

$$\Delta R(D) \approx 0.14 \pm 0.09, \tag{3.9}$$

$$\Delta R(D^*) \approx 0.14 \pm 0.05. \tag{3.10}$$

Note that due to the nature of the measurement, the uncertainty in the $R(D)$ observable is significantly higher, yielding less reliable predictions. Therefore, most analyses focus only on the $R(D^*)$ ratio. Moreover, it has been observed that in the decays where the involved leptons are of the first or second generations, the observable ratio is approximately 1.04 ± 0.05, yielding a virtually zero $\Delta R(D^*)$, within proper uncertainty bounds [4, p. 21]. This fact provides substantial evidence for treating the new physics effects as affecting only the third generation of leptons since, in that sector, the anomalies are the most apparent.

3.2 Leptoquark hypothesis

In discussions concerning the experimental flavour anomalies, the mention of *leptoquarks* often appears as a potential explanation for the peculiar phenomena. These particles are bosons that simultaneously couple to quarks and leptons, providing a direct transformation mechanism between them. Since in the current state of the Standard Model, the direct coupling of quarks and leptons is not allowed, leptoquarks emerge naturally in the extensions of the theory to *Grand Unified Theories*, aiming to unify matter and fundamental forces.

The distinguishing characteristic of the leptoquarks is the introduction of a novel quantum number, the *fermion number* F, being a combination of the baryon B and lepton L numbers defined for many elementary particles [5, p. 3]

$$F = 3B + L. \tag{3.11}$$

Based on this quantity, in accordance with the convention adopted by Buchmüller, Rückl, and Wyler in 1987, the particles may be divided into categories of $F = -2$ and $F = 0$, respectively [29]. The spin-zero bosons, known as *scalar leptoquarks*, include the S and R states, whereas the spin-one particles - *vector leptoquarks* - consist of V and U states. These particles exist in various multiplets, differing in their transformation properties under the representations of $G_{\text{SM}} = SU(3) \times SU(2) \times U(1)$. The leptoquarks are assigned a subscript denoting the dimension of the $SU(2)$ representation for the particular particle. The summary of all relevant scalar and vector leptoquarks may be found in the tables 2 and 3 below.

Type	$SU(3) \times SU(2) \times U(1)$	F
S_1	$(\bar{\mathbf{3}}, \mathbf{1}, 1/3)$	-2
$\tilde{S}_1$	$(\bar{\mathbf{3}}, \mathbf{1}, 4/3)$	-2
S_3	$(\bar{\mathbf{3}}, \mathbf{3}, 1/3)$	-2
R_2	$(\bar{\mathbf{3}}, \mathbf{2}, 7/6)$	0
$\tilde{R}_2$	$(\bar{\mathbf{3}}, \mathbf{2}, 1/6)$	0

Table 2: Summary of scalar (spin 0) leptoquarks [5, p. 4].

Type	$SU(3) \times SU(2) \times U(1)$	F
V_2	$(\bar{\mathbf{3}}, \mathbf{2}, 5/6)$	-2
$\tilde{V}_2$	$(\bar{\mathbf{3}}, \mathbf{2}, -1/6)$	-2
U_1	$(\bar{\mathbf{3}}, \mathbf{1}, 2/3)$	0
$\tilde{U}_1$	$(\mathbf{3}, \mathbf{1}, 5/3)$	0
U_3	$(\mathbf{3}, \mathbf{3}, 2/3)$	0

Table 3: Summary of vector (spin 1) leptoquarks [5, p. 4].

Alongside the subscript determining the transformation properties of the leptoquarks under the $SU(2)$ group, it is also customary to identify the electric charges of the particles with a superscript. These may be calculated using the Gell-Mann-Nishijima formula (2.1) and the isospin representation.

Example (Electric charge of U_3)
The vector triplet leptoquark

$$U_3 \sim (\mathbf{3}, \mathbf{3}, 2/3),$$

transforms under a three-dimensional representation $\mathbf{3}$ of $SU(2)$. It also corresponds to the isospin-1 representation, furnished with the following values of I_3

$$I_3 \in \{-1, 0, 1\}\,.$$

Moreover, its hypercharge quantum number, $Y = \frac{4}{3}$. Consequently, the electric charges of the individual states contained in U_3 may be calculated using equation (2.1)

$$Q = I_3 + \frac{Y}{2} \in \left\{-\frac{1}{3}, \frac{2}{3}, \frac{5}{3}\right\}.$$

One of the defining characteristics of leptoquarks is their fractional charge assignment. These bosons may couple the up-type quarks ($Q = \pm 2/3$) or down-type quarks ($Q = \pm 1/3$) to the charged or uncharged leptons ($Q = \pm 1$ or $Q = 0$). Hence, all possible electric charge states of leptoquarks include [5, p. 4]

$$Q \in \left\{\pm\frac{5}{3}, \pm\frac{4}{3}, \pm\frac{2}{3}, \pm\frac{1}{3}\right\}.$$

The viable leptoquark models proposed by Sakaki *et al.* as potential explanations of the $R(D^{(*)})$ anomalies in accordance with experimental data accommodate the vector singlet $U_1^{2/3}$, scalar doublet $R_2^{2/3}$, as well as the scalar singlet $S_1^{1/3}$ [6]. The Feynman diagram displaying the $b \to c\tau\bar{\nu}_\tau$ decay is presented in figure 4 below.

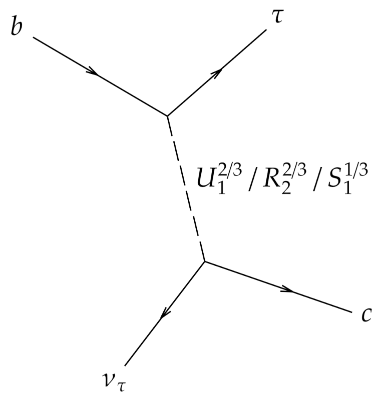

Figure 4: Feynman diagram of the leptoquark-mediated $b \to c\tau\bar{\nu}$ decay.

The analysis was performed under the assumption that all leptoquarks lie within the same mass of the order of 1 TeV. The interaction Lagrangian producing contributions to the $b \to c\ell\bar{\nu}_\ell$ process was introduced, and based on it, the Wilson coefficients, specifying the coupling strength of various terms, were calculated at the aforementioned mass scale. Finally, the synthesised leptoquarks were constrained by the experimental data gathered through $\bar{B} \to X_s\nu\bar{\nu}$ and $\bar{B} \to D^{(*)}\tau\bar{\nu}_\tau$ decays [6]. Based on this procedure and the induced limits, the U_1, R_2, and S_1 particles are the only viable leptoquark models used in hypotheses of $R(D^{(*)})$ observable explanations.

The study of Grand Unified Theory manifestations in the framework of flavour anomalies in the following sections will focus on these three individual leptoquark states. It is important to note that the aforementioned leptoquark study contains several possible limitations, for example, through the choice of experimental constraints imposed on the theoretical predictions. Due to the vast amount of experimental data available within the field of flavour physics, it is not feasible to include all possible restrictions. Therefore, even though the findings are widely accepted in the field, the analysis might not account for certain tensions with other observables.

4 Grand Unified Theories

The search for the origin of the symmetry between various generations of quarks and leptons guided the development of various theoretical frameworks - *Grand Unified Theories* (GUTs), which might conceivably provide a common mathematical structure to these fundamental particles. A defining characteristic of GUTs is the combination of the strong, electromagnetic, and weak interactions into a single force, described by an individual gauge group at extreme temperatures. A comparison of the GUT energy scale to other magnitudes is included in figure 5 below.

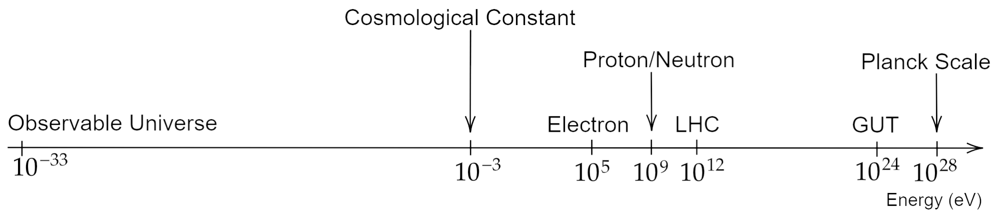

Figure 5: Physical energy scales, including the average GUT unification energy [1, p. 205], [23].

The unified structure experiences a series of spontaneous symmetry-breaking events, causing the group to effectively reduce to a particular subgroup, depending on the energy [1, p. 204]. However, for the new theory to be consistent with the established framework of particle physics, the last stage of the symmetry breaking must include the Standard Model gauge group G_{SM}.

Leptoquarks appear naturally in such theories, due to the necessity of unification of matter imposed by the single gauge group. A symmetry group of a larger dimension requires more generator elements to fully describe it, each corresponding to a different boson, as established previously. Moreover, the dimensions of the Hilbert space on which a given representation of the group acts would also be larger, allowing for more possible particle states. Historically, the discussion of leptoquarks began with Jogesh Pati and Abdus Salam's 1974 paper introducing the symmetry structure known in the literature as the Pati-Salam Model (PSM) [30], [31]

$$SU(4) \times SU(2)_L \times SU(2)_R.$$

The $SU(4)$ gauge group is employed as an attempt to unify matter particles, treating leptons as the fourth colour of quarks. Alongside the familiar $SU(2)_L$, an additional $SU(2)_R$ factor is applied, acting on the right-handed particles, constituting the left-right symmetry of the theory. The relevant particles are treated as doublets or singlets, depending on the transformation properties under the combined group $SU(2)_L \times SU(2)_R$ [8, p. 530]

$$SU(2)_L : \mathbb{C}^2 \otimes \mathbb{C},$$

$$SU(2)_R : \mathbb{C} \otimes \mathbb{C}^2.$$

PSM remains an important group in discussions of various GUTs, where it is employed as an intermediate stage in the symmetry-breaking chain, allowing for the unification of the gauge couplings at high energies.

In the remainder of this section, three different GUT-like models will be introduced, with each one describing one of the leptoquarks shown to be hypothetically responsible for alleviating the $R(D^{(*)})$ anomalies.

4.1 UV-complete $SU(5)$

4.1.1 Georgi-Glashow Model

The original mention of a $SU(5)$-based Grand Unified Theory followed from Howard Georgi and Sheldon Glashow's 1974 paper, describing the precise mechanism in which all elementary forces may be combined [32]. It was proposed as the minimal extension of the Standard Model group, where the symmetry breaking proceeds in a single step at the energy scale $\Lambda_U > 10^{15}$ GeV, estimated based on the proton lifetime $\tau_p > 10^{34}$ years [33]. The degeneration of the symmetry is illustrated in figure 6.

$$\begin{array}{c} SU(5) \\ \Big\downarrow \Lambda_U \\ G_{\mathrm{SM}} \equiv SU(3)_C \times SU(2)_L \times U(1)_Y \end{array}$$

Figure 6: Group diagram of $SU(5)$ including breaking into the Standard Model.

The main idea behind this form of unification is combining the notion of isospin and colour, creating five basis vectors denoted as $u, d, r, g, b \in \mathbb{C}^5$, corresponding to values of isospin up and down, as well as the three quark colours, respectively [8, p. 513]. In essence, within this theory, leptons and quarks are compacted into a single fermion representation. Due to this construction, the direct processes between the two aforementioned types of particles emerge naturally, constituting hints for the physical realisability of leptoquarks.

Since in the confines of the Standard Model, the notions of isospin and colour are distinct, they are treated separately, and the full five-dimensional space is split into

$$\mathbb{C}^5 = \mathbb{C}^2 \oplus \mathbb{C}^3,$$

governing the individual components. The action of the group on this space must proceed through a subgroup of $SU(5)$, consisting of block diagonal matrices of unit determinant of 2×2 and 3×3 unitary block components. This subgroup may be denoted as $S(U(2) \times U(3))$ [8, p. 513]. This subgroup is isomorphic to the Standard Model group modulo $\mathbb{Z}_6$, accounting for the kernel ensuring that the map is injective [8, p. 514]

$$S(U(2) \times U(3)) \cong G_{\mathrm{SM}}/\mathbb{Z}_6.$$

Proof (Group isomorphism by explicit construction)
Following the setup introduced in [8, p. 514], consider the map

$$\begin{aligned} \psi : SU(3) \times SU(2) \times U(1) &\to S(U(2) \times U(3)) < SU(5) \\ (\alpha, \beta, \gamma) &\mapsto \begin{pmatrix} \gamma^3\beta & 0 \\ 0 & \gamma^{-2}\alpha \end{pmatrix}. \end{aligned}$$

This map fulfills the homomorphism property, since for $(\alpha_1, \beta_1, \gamma_1), (\alpha_2, \beta_2, \gamma_2) \in G_{\text{SM}}$:

$$\begin{aligned} \psi((\alpha_1\alpha_2, \beta_1\beta_2, \gamma_1\gamma_2)) &= \begin{pmatrix} (\gamma_1\gamma_2)^3\beta_1\beta_2 & 0 \\ 0 & (\gamma_1\gamma_2)^{-2}\alpha_1\alpha_2 \end{pmatrix} = \begin{pmatrix} \gamma_1^3\beta_1\gamma_2^3\beta_2 & 0 \\ 0 & \gamma_1^{-2}\alpha_1\gamma_2^{-2}\alpha_2 \end{pmatrix} \\ &= \begin{pmatrix} \gamma_1^3\beta_1 & 0 \\ 0 & \gamma_1^{-2}\alpha_1 \end{pmatrix} \begin{pmatrix} \gamma_2^3\beta_2 & 0 \\ 0 & \gamma_2^{-2}\alpha_2 \end{pmatrix} = \psi(\alpha_1, \beta_1, \gamma_1)\psi(\alpha_2, \beta_2, \gamma_2), \end{aligned}$$

where the fact that $U(1)$ is an abelian group was used to commute γ with $\beta \in SU(2)$ and $\alpha \in SU(3)$. Moreover, this map is surjective by construction since all elements of $S(U(2) \times U(3))$ may be formed through appropriate combinations of α, β, and γ. However, the kernel of ψ

$$\ker\psi \equiv \left\{ (\alpha, \beta, \gamma) \in G_{\text{SM}} \,|\, \psi(\alpha, \beta, \gamma) = \begin{pmatrix} \mathbb{I}_{2\times2} & 0 \\ 0 & \mathbb{I}_{3\times3} \end{pmatrix} \right\},$$

consists of all elements of the form $(\gamma^2\mathbb{I}_{3x3}, \gamma^{-3}\mathbb{I}_{2\times2}, \gamma)$. For $\gamma^2\mathbb{I}_{3x3} \in SU(3)$ and for $\gamma^{-3}\mathbb{I}_{2\times2} \in SU(2)$, γ must be the sixth root of unity: $\gamma^6 = 1$, which may be confirmed by calculation of the determinant of the individual matrices.

$$\implies \ker\psi \cong \mathbb{Z}_6$$

Since the kernel is naturally nontrivial, the previously established group homomorphism is not injective, and what follows is not an isomorphism. However, the *first isomorphism theorem* may be utilised, stating that if an arbitrary map $\varphi : G \to H$ is a homomorphism of groups, then

$$G/\ker\varphi \cong \operatorname{Im}\varphi < H.$$

For the surjective map ψ in question, its image is the entire group $S(U(2) \times U(3))$, establishing the required isomorphism and concluding the proof

$$G_{\text{SM}}/\mathbb{Z}_6 \cong S(U(2) \times U(3)).$$

□

The kernel of the map acts trivially on all fermions. As an example, since any left-handed lepton transforms as $\ell_L \sim (\mathbf{1}, \mathbf{2}, -1/2)$, the action of the kernel on the lepton is as follows

$$(\gamma^2 \mathbb{I}_{3\times 3}, \gamma^{-3}\mathbb{I}_{2\times 2}, \gamma) \cdot \ell_L = \gamma^{-3}\gamma^{3\cdot Y}\ell_L = \gamma^{-3}\gamma^{-3}\ell_L = \ell_L,$$

where the the hypercharge $Y = -1$ and the identity $\gamma^6 = 1$ were used. This fact implies that an alternative description for the Standard Model gauge group, resulting in the same physical description, is $G_{\mathrm{SM}}/\mathbb{Z}_6$ [8, p. 516]. With this conclusion, the applicability of the $SU(5)$ model is retained since the regular symmetry group is contained as its subgroup.

The Standard Model fermions are placed in two separate **5**- and **10**-dimensional representations of $SU(5)$, decomposing in the following way into the Standard Model representation [34, pp. 27, 29]

$$\mathbf{5} = (\mathbf{3}, \mathbf{1}, -1/3) \oplus (\mathbf{1}, \mathbf{2}, 1/2),$$
$$\mathbf{10} = (\bar{\mathbf{3}}, 1, -2/3) \oplus (\mathbf{3}, \mathbf{2}, 1/6) \oplus (\mathbf{1}, \mathbf{1}, 1).$$

This structure is mirrored for each lepton generation, implying that to describe the entire fermionic content of the theory, three copies of the aforementioned representations are needed [34, p. 29].

Even though this prescription of the Georgi-Glashow Model is an appealing and a seemingly natural extension of the Standard Model, it has been rejected as a realistic physical theory. From a phenomenological standpoint, the model contains a variety of inconsistent predictions, such as the proton lifetime being shorter than the current lowest bound, fermion masses conflicting with experimental results, as well as the gauge coupling unification being virtually unachievable [8, p. 512], [35, p. 3]. However, these problems may be circumvented by manually introducing additional representations of the group not present in the usual $SU(5)$ structure. In the following section, one such model, referred to as the UV-complete $SU(5)$, will be investigated in hopes of identifying manifestations of scalar leptoquarks at the TeV scale.

4.1.2 R_2 scalar leptoquark

The minimal realistic $SU(5)$ model consists of the previously established Georgi-Glashow model equipped with an additional **45**- and **15**-dimensional representation. Incorporating it ensures the possibility of gauge coupling unification and adjusts the fermion Yukawa couplings, necessitating the agreement of their masses with experimental data [35, p. 3]. Moreover, the appearance of leptoquarks within the theory is simultaneous with the inclusion of a **50**-dimensional representation.

Following the treatment of Bečirević *et al.* [36], the UV-complete structure contains two scalar leptoquarks

$$R_2 \sim \left(\mathbf{3}, \mathbf{2}, \frac{7}{6}\right),$$
$$S_3 \sim \left(\mathbf{3}, \mathbf{3}, \frac{1}{3}\right),$$

manifested within **45** and **50** representations, providing a heavy state of R_2 and light states of R_2 and S_3. As outlined in the previous sections, the R_2 scalar is a candidate for a potential explanation of the $R(D^{(*)})$ observable anomalies.

In the publication by Popov *et al.* [37], the influence of the R_2 leptoquark on $R(D^{(*)})$ is studied, in parallel with an additional experimental constraint from $R(K^{(*)})$ observables, defined as

$$R(K^{(*)}) = \frac{\mathcal{B}(\bar{B} \to \bar{K}^{(*)}\mu^+\mu^-)}{\mathcal{B}(\bar{B} \to \bar{K}^{(*)}e^+e^-)}. \tag{4.1}$$

Through fixing the mass of R_2 to 1 TeV, the analysis of the Yukawa couplings of the leptoquark is performed, imposing constraints from a variety of decays, including $\tau \to e\gamma$ $Z \to \tau\tau$, and $B^+ \to K^+\tau^+e^-$, among others. This numerical analysis yields the Wilson coefficients determining the strength of the couplings responsible for various decays and hence provides a method of calculating the expected branching fractions of decays. The calculation allows for the conclusion that the R_2 leptoquark is a viable explanation for the aforementioned flavour anomalies in $R(D^{(*)})$ and $R(K^{(*)})$ observables.

4.2 $SO(10)$

Soon after his 1974 paper with Sheldon Glashow, in 1975, Howard Georgi published his findings on the $SO(10)$ theory[2], accentuating the mathematical convenience and aesthetic value of the $SU(5)$ symmetry while attempting to accommodate it within a larger structure [38]. Even though the actual symmetry group responsible for the unification is the double cover of $SO(10)$, the Spin(10) group, it is common within physics literature to refer to it as the former [8, p. 522].

4.2.1 Spinor representation

Instead of the regular real representations of the special orthogonal groups, in order to extend the Standard Model, the spinor representation of the group is necessary. These representations are complex for even degrees $2n$ and, therefore, may provide a suitable description for fermions [39]. The key to constructing such a representation are the *gamma matrices* γ_i. Based on the treatment of [21, pp. 421–423], for the case of $SO(2n)$ symmetry groups, it is possible to define a set of $2n$ hermitian matrices of size $2^n \times 2^n$, satisfying the Clifford algebra

$$\{\gamma_i, \gamma_j\} \equiv \gamma_i\gamma_j + \gamma_j\gamma_i = 2\delta_{ij}. \tag{4.2}$$

Moreover, using this definition, another $n(2n-1)$ hermitian matrices may be constructed via the commutator of γ_i and γ_j

$$\sigma_{ij} \equiv \frac{i}{2}[\gamma_i, \gamma_j]. \tag{4.3}$$

Since these matrices are also $2^n \times 2^n$, they act on a *spinor* ψ, a 2^n-dimensional object, transforming it in the following unitary fashion

$$\psi \to e^{i\omega_{ij}\sigma_{ij}}\psi, \quad \psi^\dagger \to \psi^\dagger e^{-i\omega_{ij}\sigma_{ij}},$$

[2] In private communication with Anthony Zee, Georgi claimed that he, in fact, discovered the $SO(10)$ GUT before $SU(5)$ [21].

where ω_{ij} represents an infinitesimal, antisymmetric tensor. Consider the object defined as $v_k \equiv \psi^\dagger \gamma_k \psi$:

$$\begin{aligned}\psi^\dagger \gamma_k \psi \to \psi^\dagger e^{-i\omega_{ij}\sigma_{ij}} \gamma_k e^{i\omega_{ij}\sigma_{ij}} \psi &\approx \psi^\dagger (1 - i\omega_{ij}\sigma_{ij}) \gamma_k (1 + i\omega_{ij}\sigma_{ij}) \psi \\ &= \psi^\dagger \gamma_k \psi - \psi^\dagger i\omega_{ij}\sigma_{ij}\gamma_k \psi + \psi^\dagger \gamma_k i\omega_{ij}\sigma_{ij}\psi + \mathcal{O}(\omega_{ij}^2) \\ &= \psi^\dagger \gamma_k \psi - i\omega_{ij}\psi^\dagger (\sigma_{ij}\gamma_k - \gamma_k\sigma_{ij})\psi \\ &= \psi^\dagger \gamma_k \psi - i\omega_{ij}\psi^\dagger [\sigma_{ij}, \gamma_k]\psi,\end{aligned}$$

where the Taylor series expansion of the exponential and the fact that ω_{ij} is an infinitesimal have been used to omit terms of $\mathcal{O}(\omega_{ij}^2)$. The commutator may also be evaluated using the definitions given by equations (4.2) and (4.3)

$$\begin{aligned}[\sigma_{ij}, \gamma_k] = \frac{i}{2}[[\gamma_i, \gamma_j], \gamma_k] &= \frac{i}{2}\left(\gamma_i\gamma_j\gamma_k - \gamma_j\gamma_i\gamma_k - \gamma_k\gamma_i\gamma_j + \gamma_k\gamma_j\gamma_i\right) \\ &= \frac{i}{2}\left(\gamma_i\gamma_j\gamma_k - \gamma_j(2\delta_{ik} - \gamma_k\gamma_i) - \gamma_k\gamma_i\gamma_j + (2\delta_{jk} - \gamma_j\gamma_k)\gamma_i\right) \\ &= \frac{i}{2}(\gamma_i\gamma_j\gamma_k - 2\delta_{ik}\gamma_j + \cancel{\gamma_j\gamma_k\gamma_i} - \gamma_k\gamma_i\gamma_j + 2\delta_{jk}\gamma_i - \cancel{\gamma_j\gamma_k\gamma_i}) \\ &= \frac{i}{2}(\gamma_i(2\delta_{jk} - \gamma_k\gamma_j) - 2\delta_{ik}\gamma_j - \gamma_k\gamma_i\gamma_j + 2\delta_{jk}\gamma_i) \\ &= \frac{i}{2}(4\delta_{jk}\gamma_i - (2\delta_{ik} - \gamma_k\gamma_i)\gamma_j - 2\delta_{ik}\gamma_j - \gamma_k\gamma_i\gamma_j) \\ &= \frac{i}{2}(4\delta_{jk}\gamma_i - 4\delta_{ik}\gamma_j + \cancel{\gamma_k\gamma_i\gamma_j} - \cancel{\gamma_k\gamma_i\gamma_j}) = -2i(\delta_{ik}\gamma_j - \delta_{jk}\gamma_i).\end{aligned}$$

Substituting this expression in the aforementioned transformation

$$\begin{aligned}\psi^\dagger \gamma_k \psi &\to \psi^\dagger \gamma_k \psi - i\omega_{ij}\psi^\dagger(-2i(\delta_{ik}\gamma_j - \delta_{jk}\gamma_i))\psi \\ &= \psi^\dagger \gamma_k \psi - 2\omega_{kj}\psi^\dagger\gamma_j\psi + 2\omega_{ik}\psi^\dagger\gamma_i\psi\end{aligned}$$

$$\implies v_k \to v_k - 2(\omega_{kj}v_j - \omega_{ik}v_i) = v_k - 2\omega_{kj}v_j - 2\omega_{ki}v_i = v_k - 4\omega_{kj}v_j,$$

where the antisymmetric property $\omega_{ik} = -\omega_{ki}$ was used, alongside relabeling $i \mapsto j$ in the second term. This expression shows that the set of $2n$ objects v_k transforms like a vector in $2n$-dimensional space, where $4\omega_{kj}$ represents the infinitesimal rotation angle. Given

$$\gamma^5 \equiv (-i)^n \gamma_1\gamma_2 \ldots \gamma_{2n}, \tag{4.4}$$

it is possible to define a left-handed spinor ψ_L and right-handed spinor ψ_R

$$\psi_L \equiv \frac{1}{2}(1 - \gamma^5)\psi, \tag{4.5}$$

$$\psi_R \equiv \frac{1}{2}(1 + \gamma^5)\psi. \tag{4.6}$$

These constitute two alternative irreducible spinor representations of the $SO(2n)$ group of dimension 2^{n-1}. Specifically the representation associated with $SO(10)$ is **16**-dimensional. It incorporates all Standard Model fermions and a right-handed neutrino, in left-handed and right-handed representations $\mathbf{16}_L$ and $\mathbf{16}_R$ respectively, following the general decomposition below [39, p. 37], [34, p. 45]

$$\mathbf{16} = (\mathbf{3}, \mathbf{2}, 1/6) \oplus (\mathbf{1}, \mathbf{2}, -1/2) \oplus (\bar{\mathbf{3}}, \mathbf{1}, 1/3) \oplus (\bar{\mathbf{3}}, \mathbf{1}, -2/3) \oplus (\mathbf{1}, \mathbf{1}, 1) \oplus (\mathbf{1}, \mathbf{1}, 0),$$

where the individual components represent the transformation properties under G_{SM}. When put together, these constitute the full **32**-dimensional fermion representation.

4.2.2 Symmetry-breaking

The $SO(10)$ gauge group follows a symmetry-breaking structure which degenerates to the Standard Model group through a single intermediate subgroup - the Pati-Salam group $SU(4) \times SU(2)_L \times SU(2)_R$. This subgroup may also contain an additional factor of $\mathbb{Z}_2$, corresponding to the statement of left-right symmetry of the theory, sometimes referred to as D-parity invariance, responsible for the complete equivalence of left and right-handed particle sectors under the action of $SU(2)$ [40]. The symmetry-breaking diagram for the group is presented in figure 7 below.

$$\begin{tikzcd}
SO(10) \arrow[d, "\Lambda_U"] \\
SU(4) \times SU(2)_L \times SU(2)_R \ (\times \mathbb{Z}_2) \arrow[d, "\Lambda_C"] \\
G_{\mathrm{SM}} \equiv SU(3)_C \times SU(2)_L \times U(1)_Y
\end{tikzcd}$$

Figure 7: Group diagram of $SU(10)$ including breaking into the Pati-Salam group and, consequently, the Standard Model.

The reduction to various subgroups of $SO(10)$ proceeds at two distinct energy scales. The first stage corresponds to the unification scale $\Lambda_U \approx 10^{16}$ GeV, below which the Pati-Salam group is an effective description of the physics. Furthermore, around the energy $\Lambda_C \approx 10^{11}$ GeV, the Pati-Salam group degenerates into the Standard Model description [40].

4.2.3 S_1 scalar leptoquark

Following the treatment of Aydemir *et al.* [40], the S_1 scalar leptoquark is identified via a possible representation of the $SO(10)$ grand unification mechanism

$$S_1 \sim \left(\mathbf{3}, \mathbf{1}, -\frac{1}{3}\right).$$

The principal motivation behind investigating this specific symmetry group as a viable extension of the current physics is the manifestations of a minimal number of leptoquarks at low energies. Considering the fact that up to this date, no leptoquark sightings have been reported at the LHC, it is likely that if such a discovery is reported, only a few or even a single particle will be observed.

The $SO(10)$ model allows for the occurrence of S_1 in a **10**-dimensional representation, decomposed in terms of the Standard Model representation

$$\mathbf{10} = \underbrace{\left(\mathbf{1}, \mathbf{2}, \frac{1}{2}\right)}_{H} \oplus \underbrace{\left(\mathbf{1}, \mathbf{2}, -\frac{1}{2}\right)}_{\bar{H}} \oplus \underbrace{\left(\mathbf{3}, \mathbf{1}, -\frac{1}{3}\right)}_{S_1} \oplus \underbrace{\left(\bar{\mathbf{3}}, \mathbf{1}, \frac{1}{3}\right)}_{\bar{S_1}}.$$

The last two representations correspond to the scalar leptoquark S_1 and its antiparticle $\bar{S_1}$, whereas the rest of the factors represent the Standard Model Higgs boson and its antiparticle. Therefore, through this specific $SO(10)$ representation, it may be concluded that the leptoquark is the only new physics scalar at low energies. Due to this fact, if a sighting of the particle is reported without any other being observed, it would constitute solid evidence for the applicability of $SO(10)$ grand unification.

The study of the leptoquarks in this model is similar to the one described in the $SU(5)$ framework. For a set of benchmark masses of S_1, being 1 TeV and 2 TeV, the range of possible Yukawa couplings was investigated, including constraints from a range of decays, such as $b \to s\bar{\nu}\nu$ and $Z \to \tau\tau$, as well as the pair-production studies at the LHC. The results were also compared to $pp \to \tau\tau$ data from ATLAS. It was shown that a substantial portion of the parameter space was still available after imposing the constraints, implying that the S_1 leptoquark is a viable new physics particle consistent with crucial experimental data. The relevant Wilson coefficients of the leptoquark were then extracted, allowing for calculations of predicted ratios $R(D)$ and $R(D^*)$, showing that it alleviates the flavour anomaly.

5 Flavoured Gauge Model

In juxtaposition with the attempts at resolving the Standard Model inconsistencies through employing Grand Unified Theories, an alternative approach presents itself when assuming that the gauge group is not fundamentally universal at the unification scale. Whereas GUTs provide a unified description for each fermion generation, the *Flavoured Gauge Model*, presented by Davighi, Isidori, and Pesut in 2023 [41], states that the full description relies on splitting the symmetry group into separate structures acting on the first and second, and the third generations, respectively

$$G = G_{12} \times G_3,$$

$$G_{12} = SU(4)_{1+2} \times Sp(4)_L \times Sp(4)_R, \tag{5.1}$$

$$G_3 = SU(4)_3 \times SU(2)_{L,3} \times SU(2)_{R,3}, \tag{5.2}$$

where $Sp(4)$ is a $n = 2$ case of the symplectic group $Sp(2n)$, defined for even dimensions [41, p. 4].

The fermion representations Ψ under G are compactified into **16**-dimensional representations of the group, depending on the generation and handedness of the particles, where the light fermions transform trivially under G_3 and heavy fermions under G_{12}

$$\Psi_L^{1,2} \sim (\mathbf{4},\mathbf{4},\mathbf{1}) \otimes (\mathbf{1},\mathbf{1},\mathbf{1}), \quad \Psi_L^3 \sim (\mathbf{1},\mathbf{1},\mathbf{1}) \otimes (\mathbf{4},\mathbf{2},\mathbf{1}),$$

$$\Psi_R^{1,2} \sim (\mathbf{4},\mathbf{1},\mathbf{4}) \otimes (\mathbf{1},\mathbf{1},\mathbf{1}), \quad \Psi_R^3 \sim (\mathbf{1},\mathbf{1},\mathbf{1}) \otimes (\mathbf{4},\mathbf{1},\mathbf{2}).$$

Even though a separate mechanism is introduced for the light and heavy fermions, each group allows for the unification of quarks and leptons through a $SU(4)$ factor, in analogy with the Pati-Salam Model [41, p. 3]. This approach is much in the same spirit as the unification mechanisms presented before, where larger symmetry groups containing G_{SM} are employed. However, it does not provide the same mathematical conciseness as the aforementioned theories.

Accommodating a separate gauge group acting on the third generation of quarks and leptons is a natural extension of the current theory, based on the experimental flavour anomalies, considering the little difference in the relative behaviour of the first and second generations. Consequently, a key characteristic of this theory is the prediction of the existence of vector leptoquarks at the TeV scale [41, p. 2]. This fact might provide additional evidence for the physical applicability of the theory, given that the masses of vector leptoquarks present in various GUTs tend to lie in the neighbourhood of the unification scale, such as in the $SU(5)$ structure [34, p. 40].

5.1 Symplectic group $Sp(2n)$

Before investigating the implications of the Flavoured Gauge Model, it is beneficial to understand the mathematical foundation of the groups constituting the light fermion sector. As stated in equation (5.1), the first and second generations of fermions are governed by a $SU(4)$ group and a direct product of two *symplectic groups* $Sp(4)$.

In general, the symplectic group $Sp(2n)$ is the set of all special unitary matrices M that preserve the bilinear form B on $\mathbb{C}^n$, defined as

$$B(u,v) = \sum_{i=1}^{n} u_i v_{n+i} - u_{n+i} v_i, \tag{5.3}$$

for all $u, v \in \mathbb{C}^n$, such that $B(Mu, Mv) = B(u,v)$ [9, pp. 12–13].

Proof (Bilinearity of $B(u,v)$)
For two arbitrary vector spaces V, W and a field of scalars $\mathbb{F}$, the function $f : V \times W \to \mathbb{F}$ is referred to as a *bilinear form* if it satisfies the following conditions [42]:

1. For $v_1, v_2 \in V$, $\alpha, \beta \in \mathbb{F}$, and $w \in W$
$$f(\alpha v_1 + \beta v_2, w) = \alpha f(v_1, w) + \beta f(v_2, w).$$

2. For $v \in V$, $\alpha, \beta \in \mathbb{F}$, and $w_1, w_2 \in W$
$$f(v, \alpha w_1 + \beta w_2) = \alpha f(v, w_1) + \beta f(v, w_2).$$

In essence, bilinearity implies the map being linear in each of its arguments.

To check whether the defining form given in equation (5.3) is indeed bilinear, it suffices to show that it fulfils these requirements. Let $u_1, u_2 \in \mathbb{C}^n$, $\alpha, \beta \in \mathbb{C}$, and $v \in \mathbb{C}^n$:

$$\begin{aligned}
B(\alpha u_1 + \beta u_2, v) &= \sum_{i=1}^{n} (\alpha u_{1,i} + \beta u_{2,i}) v_{n+i} - (\alpha u_{1,n+i} + \beta u_{2,n+i}) v_i \\
&= \sum_{i=1}^{n} \alpha u_{1,i} v_{n+i} + \beta u_{2,i} v_{n+i} - \alpha u_{1,n+i} v_i - \beta u_{2,n+i} v_i \\
&= \alpha \left(\sum_{i=1}^{n} u_{1,i} v_{n+i} - u_{1,n+i} v_i \right) + \beta \left(\sum_{i=1}^{n} u_{2,i} v_{n+i} - u_{2,n+i} v_i \right) \\
&= \alpha B(u_1, v) + \beta B(u_2, v).
\end{aligned}$$

Moreover, if $u \in \mathbb{C}^n$, $\alpha, \beta \in \mathbb{C}$, and $v_1, v_2 \in \mathbb{C}^n$:

$$\begin{aligned}
B(u, \alpha v_1 + \beta v_2) &= \sum_{i=1}^{n} u_i (\alpha v_{1,n+i} + \beta v_{2,n+i}) - u_{n+i} (\alpha v_{1,i} + \beta v_{2,i}) \\
&= \sum_{i=1}^{n} \alpha u_i v_{1,n+i} + \beta u_i v_{2,n+i} - \alpha u_{n+i} v_{1,i} - \beta u_{n+i} v_{2,i} \\
&= \alpha \left(\sum_{i=1}^{n} u_i v_{1,n+i} - u_{n+i} v_{1,i} \right) + \beta \left(\sum_{i=1}^{n} u_i v_{2,n+i} - u_{n+i} v_{2,i} \right) \\
&= \alpha B(u, v_1) + \beta B(u, v_2).
\end{aligned}$$

□

Moreover, given a matrix

$$\Omega = \begin{pmatrix} 0 & \mathbb{I}_{n\times n} \\ -\mathbb{I}_{n\times n} & 0 \end{pmatrix},$$

the matrices M satisfy the following identity [41, p. 4]

$$M^T \Omega M = \Omega.$$

This relation introduces an additional requirement for the determinant of the matrices

$$\det(M^T J M) = \det J$$

$$\det J \cdot \det(M)^2 = \det J$$

$$\implies \det M = \pm 1,$$

where the multiplicative property of the determinant map was used. Specifically, $\det M = 1$ for all $M \in Sp(2n)$ [9, p. 13].

The introduction of the two symplectic groups $Sp(4)_L$ and $Sp(4)_R$ in equation (5.1) serves as a means of unifying flavour and electroweak symmetry for the light fermion families. The two individual gauge groups are also responsible for the left-right symmetry of the theory, acting on left- and right-handed particles, respectively. The choice of the symplectic group for the model is also motivated by the lack of various gauge anomalies associated with the group representations [43].

5.2 Symmetry-breaking

The model follows an extensive sequence of symmetry-breaking mechanisms at various energy scales, illustrated in figure 8.

$$\begin{tikzcd}
\overset{G_{12}}{SU(4)_{1+2} \times Sp(4)_L \times Sp(4)_R} \arrow[d, "\Lambda_{12}"'] & \times & \overset{G_3}{SU(4)_3 \times SU(2)_{L,3} \times SU(2)_{R,3}} \arrow[dd] \\
SU(3)_{1+2} \times SU(2)_{L,1} \times SU(2)_{L,2} \times SU(2)_{R,1} \times U(1)''_R \arrow[d, "\epsilon\Lambda_{12}"'] & & \\
SU(3)_{1+2} \times SU(2)_{L,1+2} \times U(1)_{Y,1+2} & \times \arrow[d, "\Lambda_\Sigma"'] & SU(4)_3 \times SU(2)_{L,3} \times SU(2)_{R,3} \\
& SU(4)_3 \times SU(3)_{1+2} \times SU(2)_L \times U(1)'_R \arrow[d, "\Lambda_{4321}"'] & \\
& G_{\mathrm{SM}} \equiv SU(3)_C \times SU(2)_L \times U(1)_Y &
\end{tikzcd}$$

Figure 8: Group diagram of the Flavoured Gauge Model, including the symmetry-breaking structure.

The relevant symmetry-breaking energies are approximately separated by a single order of magnitude, such that [41, p. 22]

$$\Lambda_{12} \approx 1000\,\text{TeV}, \quad \epsilon\Lambda_{12} \approx 100\,\text{TeV}, \quad \Lambda_{\Sigma} \approx 10\,\text{TeV}, \quad \Lambda_{4321} \approx 1\,\text{TeV}.$$

These energies have been estimated by imposing a series of constraints, such as from the experimental bounds on the gauge bosons W' and Z' generated by the transition

$$SU(2)_{L,1+2} \times SU(2)_{L,3} \to SU(2)_L,$$

especially based on the most strict requirements from the $Z' \to \ell\bar{\ell}$ decay. Moreover, these constraints have also been combined with limits from flavour-changing processes in the first and second generations of quarks [41, p. 22].

5.3 U_1 vector leptoquark

The last symmetry breakdown at the scale Λ_{4321} represents the degeneration of the so-called 4321-model

$$SU(4)_3 \times SU(3)_{1+2} \times SU(2)_L \times U(1)'_R$$

into the Standard Model group G_{SM}. This proceeds through a scalar particle ω, transforming under the specific representation of $G_{12} \times G_3$

$$\omega \sim (\mathbf{4}, \mathbf{1}, \mathbf{4}) \otimes (\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2}).$$

Due to this symmetry-breaking step, the vector leptoquark U_1 is generated with a mass of the order of 1 TeV with the following transformation properties under G_{SM}

$$U_1 \sim \left(\mathbf{3}, \mathbf{1}, \frac{5}{3}\right).$$

Similarly to the study of the $SU(5)$ and $SO(10)$ models, the Flavoured Gauge Model is also able to provide a prediction concerning the $R(D^{(*)})$ observables when including leptoquark effects. To obtain numerical values for the predicted branching fractions, the Wilson coefficients must be extracted from the Lagrangian governing $b \to c\ell\nu_\ell$ transitions. The two relevant variables are

$$\mathcal{C}_{LL} = \frac{1}{2}\left(1 + \beta_L^{s\tau}\frac{V_{cs}}{V_{cb}}\right)\left(\frac{\Lambda_{\text{SM}}}{\Lambda_{U_1}}\right)^2, \quad \mathcal{C}_{LR} = \beta_R^*\mathcal{C}_{LL}, \tag{5.4}$$

where $\Lambda_{\text{SM}}, \Lambda_{U_1}$ represent the energy scales associated with the Standard Model and the U_1 leptoquark, respectively, β are the effective couplings between quarks and leptons, and V_{cs}, V_{cb} are the individual Cabibbo-Kobayashi-Maskawa (CKM) matrix elements [41, p. 23]. Using these quantities allows for the calculation of the metrics introduced in equations (3.7) and (3.8)

$$\Delta R(D) \equiv \frac{R(D)}{R(D)_{\text{SM}}} - 1 = \text{Re}(2\mathcal{C}_{LL} - 3.00\mathcal{C}_{LR}^*), \tag{5.5}$$

$$\Delta R(D^*) \equiv \frac{R(D^*)}{R(D^*)_{\text{SM}}} - 1 = \text{Re}(2\mathcal{C}_{LL} - 0.24\mathcal{C}_{LR}^*). \tag{5.6}$$

Following the data analysis from $R(D^{(*)})$ measurements, an adequate depiction of the experimental results is obtained for $\beta_R = -1$. Setting $\Lambda_{\text{SM}} = 0.1\,\text{TeV}, \Lambda_{U_1} = 1.6\,\text{TeV}$, and $\beta_L^{s\tau} = 0$, the numerical values of the ratios may be calculated [41, p. 24]

$$\Delta R(D) \approx 0.06, \tag{5.7}$$

$$\Delta R(D^*) \approx 0.027. \tag{5.8}$$

Comparing equations (5.7), (5.8) to the current state of the $R(D^{(*)})$ observables given by equations (3.9), (3.10), it may be concluded that employing this leptoquark model constitutes an improvement over the current Standard Model predictions. Even though the values do not precisely match the established values of the observables, the model alleviates the significant tension between theory and measurement [41, p. 28].

6 Conclusions

Throughout this thesis, the correspondence between leptoquarks manifested in theories beyond the Standard Model and lepton flavour anomalies was investigated. For that purpose, the necessary background and the current experimental status of lepton flavour anomalies were outlined. The analysis focused on the specific case of $R(D)$ and $R(D^*)$ observables, showing a significant 3.3σ deviation from the Standard Model prediction. This and many other anomalies indicate that the current theory, even though widely successful in many aspects, is still an incomplete description of the underlying physics. In the context of the aforementioned observables, three leptoquark models have been investigated as a means of rectifying the complications in the current models by conjecturing the applicability of larger symmetry groups, forming Grand Unified Theories. These contain three multiplets from three distinct models, including R_2 and S_1 scalars and a U_1 vector leptoquarks, each constituting a possible explanation for the $R(D^{(*)})$ anomalies.

Even though the Georgi-Glashow $SU(5)$ model was shown to be physically not realisable due to countless factors, such as the limits on proton lifetime, its modification as the UV-complete model is still relevant in literature. It remains the minimal realistic extension of the Standard Model, undergoing the symmetry breaking in a single step. Within its **45** and **50**-dimensional representations, it contains states of R_2 and S_3 leptoquarks with a mass of the order of 1 TeV. Incorporating these new physics particles allows for alleviating the $R(D^{(*)})$ anomalies.

The extension of the $SU(5)$ theory is the model based on the $SO(10)$ symmetry group, which resolves many problematic aspects of the previously described theory by introducing an intermediate Pati-Salam subgroup. It utilises two **16**-dimensional spinor representations accommodating fermions. The **10**-dimensional representation contains the S_1 scalar leptoquark alongside the Standard Model Higgs boson, making the $SO(10)$ a theory with a single manifestation of a scalar particle at the TeV scale.

Lastly, an alternative extension of the current physics is obtained by assuming that the gauge group of the model is fundamentally non-universal at considerable energies. An example of such an approach is the Flavoured Gauge Model, assigning a separate symmetry group acting on the first and second, as well as the third generation of fermions. This is a natural generalisation of the Standard Model, considering the experimental data in flavour physics, suggesting that the third generation of fermions displays a range of phenomena diverging from the theoretical expectations. Incorporating such a structure yields an instance of a U_1 vector leptoquark with a mass of approximately 1.6 TeV, allowing for alleviating the observable tensions in question.

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