INTEGRABLE OPEN QUANTUM SYSTEMS

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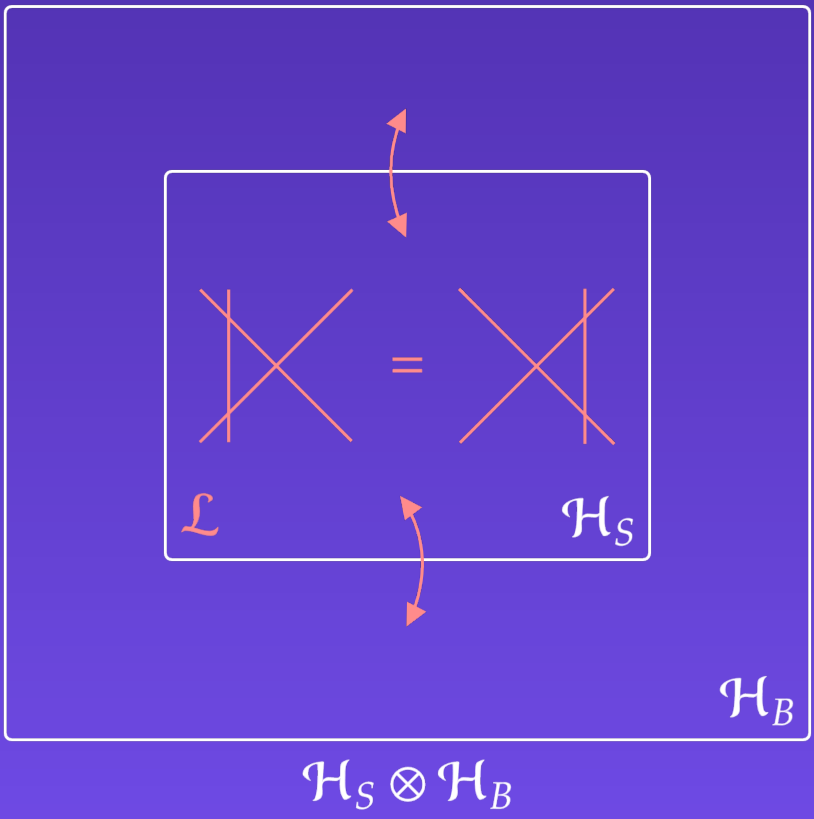

Rijksuniversiteit Groningen

Integrable Open Quantum Systems

Author:
Tunay Oruç
(s4706684)

Supervisor:
dr. Tamás Görbe
Second examiner:
prof. Anupam Mazumdar

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Contents

Abstract

Over the past century, Open Quantum Systems —quantum systems that interact with their environment-— and Quantum Integrable Systems —quantum systems that are exactly solvable-— were viewed as separate fields. However, in recent years, various Integrable Open Quantum Systems have been classified and gained considerable popularity [1, 2]. The approach in classifying such systems is the Boost Automorphism Method – a bottom-up method in which the corresponding R-matrix is constructed from an Hamiltonian ansatz through a boost operator. The resultant R-matrix solves the Yang-Baxter equation, ensuring the integrability of the model. This thesis aims to review the classification of Integrable Open Quantum Systems and examine a concrete example – referred to as Model A1. To this end, this thesis first presents the mathematical formulation of Open Quantum Systems, followed by its dynamics – the derivation of the Lindblad equation. Subsequently, integrability is introduced through the representative example of the Heisenberg Spin Chain, along with the R-matrix and the Yang-Baxter Equation. Moreover, the Boost Automorphism Method is explained in detail with a concrete example. Ultimately, all notions are connected and the classification of Integrable Open Quantum Systems is concluded.

Acknowledgments

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1 Introduction

Between 1609 and 1619, Johannes Kepler formulated his three laws of planetary motion [3] by studying the astronomical data of the solar system recorded by Tycho Brache [4]. Later in 1687, Isaac Newton showed that Kepler's laws of motion would apply in the solar system as a consequence of his own law of universal gravitation [5]. Also in the same year, Newton posed and solved the two-body problem in his work *Principia* [5]; a problem consisting of determining the motion of two massive bodies interacting with a central force with given masses, positions and velocities. Whereas Newton used geometric arguments in solving the problem, other physicists such as the famed Richard Feynman have since proposed alternate solutions [6]. Such problems in classical dynamics date the birth of *Integrable Systems* – models in which the equations of motion can be reduced to a finite number of algebraic operations and integrals, allowing them to have exact solutions [7].

As opposed to classical mechanics, there also exist integrable systems with quantum properties – quantum many-body systems equipped with numerous conserved quantities, allowing them to be exactly solvable. The history of quantum integrable systems dates back to 1931, when Hans Bethe published his article on the Heisenberg spin chain [8]; a toy model of spin-$1/2$ electrons aligned on a one-dimensional lattice with only nearest-neighbour spin interactions. In his article, Bethe constructed an ansatz for the eigenstates of the Hamiltonian of the spin chain, and subsequently utilised it to solve the eigenvalue problem. His method is nowadays known as the *Coordinate Bethe Ansatz*. At the time, Bethe's result was deemed remarkable, but it was mainly considered as "academic exercise". However, ever since then, many other quantum integrable models have been discovered and solved with the Coordinate Bethe Ansatz almost without exception. Consequently, Bethe's work was considered the foundation of quantum integrable systems. As further testament to Bethe's influence, Feynman had "to learn: Bethe Ansatz problems" written on his blackboard at the time of his passing [9]. Following Bethe, between 1979 and 1982, the Leningrad School published a series of papers on the *Inverse Quantum Scattering Method* – an alternate method in studying quantum integrable systems [10, 11, 12]. One such tool in this method was the *Algebraic Bethe Ansatz*, which involved finding an associated R-matrix satisfying the Yang-Baxter equation[1] [13]. Systems possessing this attribute are referred to as *Yang-Baxter Integrable*. Even though there still does not exist a clear-cut definition of quantum integrability, the characteristic of being Yang-Baxter Integrable together with the existence of numerous conserved charges is most commonly accepted as *Quantum Integrability* [1].

Quantum integrability is a fragile property: adding interaction terms on top of a integrable Hamiltonian, in general, breaks its integrable nature. In spite of this delicacy, there also exist quantum integrable models in interacting theories. In recent years, one such field, namely *Integrable Open Quantum Systems*, has gained considerable popularity [1, 2]. In contrast to quantum systems in isolation, *Open Quantum Systems* treats quantum systems which interact with their environments. Due to this key distinction, open quantum systems are occupied by mixed states, which in turn requires an alternate treatment – the density matrix formulation of quantum mechanics. Despite the dynamics of such systems being affected by the interaction with their environments, some still preserve their integrability.

Integrable systems have been known for many years, although the methods used in classifying

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[1] The R-matrix and Yang-Baxter equation are to be defined.

them are relatively new. A prime example is the *Boost-Automorphism Method*, realised by Marius de Leeuw and Chiara Paletta [14, 15, 16, 17]. An identical procedure has also been applied to integrable open quantum systems by the same authors, and their works [1, 2] detail **the first systematic classification** of integrable open quantum systems.

1.1 Thesis Goal and Outline

The goal of this thesis is to review the classification of integrable open quantum systems presented in Paletta's PhD thesis [1] and review letter [2]. Moreover, we aim to reproduce the classification of a concrete example – referred to as Model A1. The thesis is dedicated towards undergraduate students interested in the field of quantum integrable systems, and our ambition is aspiring the next generation of physicists and mathematicians. To this end, we have worked towards writing the thesis in a digestible fashion and have provided examples along with detailed steps at necessary points. The outline of the thesis is as follows:

- **Section 2 - Open Quantum Systems** introduces the reader to the mathematical formulation of Open Quantum Systems. Emphasis is on put on how and why this differs from "standard" quantum mechanics. To achieve this, the reader is first introduced to *mixed states*, followed by the *density matrix*. The properties of the density matrix are discussed in detail. Moreover, the concepts of the *global Hilbert space* and the *partial trace* are also treated in full.

- **Section 3 - The Lindblad Equation** presents a full derivation of the Lindblad equation — the generator of the dynamics, i.e. the time evolution, of Open Quantum Systems. Most importantly, the section introduces the concepts of *Liouville space, quantum dynamical semigroups and the Lindblad superoperator.*

- **Section 4 - Quantum Integrable Spin Chains** introduces the reader to Quantum Integrability through the representative example of the Heisenberg Spin Chain. Even though the eigenvalue problem of the Heisenberg spin chain is not solved explicitly, the reader is introduced to the *Algebraic Bethe Ansatz*. From which, the definitions of the R-matrix and the Yang-Baxter equation emerge. The *boost automorphism method* is also introduced, and afterwards is used in classifying an integrable system – which we present as a detailed example.

- **Section 5 - Integrable Open Quantum Systems** serves to connect all of the preceding sections to provide a framework in classifying of integrable open quantum systems. To achieve this, the adaptation of the Lindblad equation (Section 3) to spin chains is explained. At the end, the boost automorphism method (Section 4) is used in classifying an integrable model – referred to as Model A1.

2 Open Quantum Systems

In an introductory course to Quantum Mechanics, many quantum systems that students encounter are exactly solvable systems such as the infinite square well and the harmonic oscillator. The systems are considered to always be isolated, in the sense that there is no interaction of the system with another quantum system. Any state $|\Psi\rangle$ of such systems lives in an Hilbert space $\mathcal{H}$ and evolves according to the Schrodinger equation

$$i\hbar\frac{d}{dt}\left|\Psi\right\rangle = H\left|\Psi\right\rangle . \tag{2.1}$$

However, most systems in nature are not isolated, they in fact possess interactions with their environments. This poses an interesting discussion; *How do we describe such systems within the aforementioned framework of Quantum Mechanics?* To answer this question, we turn ourselves to *Open Quantum Systems.* Where quantum systems that interact with larger quantum systems – referred to as the environment or bath – are studied. In the following parts of this chapter, we shall introduce the tools required to adequately describe open quantum systems. The arguments presented in this chapter are largely based on the works [18, 19].

2.1 Mixed States

In Quantum Mechanics, one usually considers a system that has exactly known states. For such a system, it is sufficient to find a complete set of commuting observables (C.S.C.O.) in order to determine the state of a system at a given instant in time. However, the state of a system is often not exactly known, rather the states are statistical mixtures. We say that the state of a system is a mixture of pure states $|\psi_k\rangle$ (to be defined), each with an associated probability p_k. Naturally, we require that the probabilities satisfy

$$\begin{cases} 0 \leq p_1, p_2, \ldots, p_k, \ldots \leq 1 \\ \sum_k p_k = 1 \end{cases} \tag{2.2}$$

Looking at the above description of a mixed state, one might be inclined to represent the state of a system, denoted $|\psi\rangle$, as a superposition of the pure states $|\psi_k\rangle$ in the following way:

$$\left|\psi\right\rangle = \sum_k c_k \left|\psi_k\right\rangle , \tag{2.3}$$

with complex coefficients c_k. It is often stated that "$|c_k|^2$ represents the probability of finding the system in the state $|\psi_k\rangle$". However, this is not correct; in fact, the interpretation of a mixed state and a superposition of pure states are completely separate, as we shall illustrate in the following example.

Example 2.1 (First look at a mixed state)

Consider the entangled state

$$|\chi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle),$$

where the vectors $|\uparrow\rangle$ and $|\downarrow\rangle$ are the spin up and down vectors in the usual orthonormal spin basis, respectively. In such a state, measuring the second spin as spin up collapses the first one into spin down and vice versa. Let us consider only the measurements conducted on the first particle. We can say that upon conducting a measurement, the first particle has spin up half the time and spin down half the time. To illustrate this, we construct an ensemble containing the measured state of the first particle and the corresponding probabilities:

$$\begin{aligned} \text{Spin state:} &\quad \{|\uparrow\rangle, |\downarrow\rangle\} \\ \text{Probability:} &\quad \{0.5, 0.5\} \end{aligned}$$

Looking at this ensemble, one might be inclined to describe the state of the first particle by the superposition

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle).$$

In order to see why this is an incorrect description of our system, let us act on both the ensemble and the state $|\psi\rangle$ with the operator S_x. In matrix notation, we have that:

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad S_x = \frac{\hbar}{2}\sigma_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then,

$$\begin{aligned} S_x |\uparrow\rangle &= \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{\hbar}{2}|\downarrow\rangle, \\ S_x |\downarrow\rangle &= \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2}|\uparrow\rangle. \end{aligned}$$

Following this, we calculate the expectation value for the ensemble. We do this in the following way; we measure the observable S_x on all occurring states separately and sum the results with their weighted probabilities. Thus, we get:

$$\begin{aligned} \langle S_x \rangle &= 0.5 \cdot \langle\uparrow| S_x |\uparrow\rangle + 0.5 \cdot \langle\downarrow| S_x |\downarrow\rangle \\ &= \frac{\hbar}{4}\langle\uparrow \mid \downarrow\rangle + \frac{\hbar}{4}\langle\downarrow \mid \uparrow\rangle \\ &= 0. \end{aligned}$$

In other words, using the ensemble, we get that the expectation value of the spin along the x direction is zero. For comparison, we now compute the expectation value of S_x using the superposition state. First, we note that:

$$S_x |\psi\rangle = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{\hbar}{2}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{\hbar}{2}|\psi\rangle.$$

It follows that,

$$\langle S_x \rangle = \langle\psi| S_x |\psi\rangle = \frac{\hbar}{2}.$$

We see that upon measuring the spin along the x direction, each interpretation yields a different result. Therefore, they can not be equivalent descriptions of the same system.

Before moving onto how to represent mixed states, we shall return to the statement made concerning the expression (2.3). We had stated that "the numbers $|c_k|^2$ represent the probability of finding the system in $|\psi_k\rangle$". Precisely, this is what it means: If we have an operator A with $|\psi_k\rangle$ as an eigenstate with eigenvalue a_k, the value of $|c_k|^2$ the probability of finding a_k upon measuring A in the system $|\psi\rangle$. This is completely different than stating $|c_k|^2$ is the weighted probability of the state ψ_k. Therefore, we see that the superposition description is inadequate, and that it is impossible to represent a mixed state as a superposition of pure states. We shall refer to the above example at various points within this chapter to further exemplify future notions.

2.2 The Density Operator

We have now seen that mixed states differ from a superposition of states, and that they can not be expressed directly using pure state vectors. Therefore, it is natural to pose the question: *How do we express mixed states in a way that is completely equivalent to "standard" Quantum Mechanics?* To answer this question, we introduce the density operator – sometimes referred to as the density matrix.

2.2.1 Density Operator for a Pure State

Let us first start with a pure state, a state that is exactly known – not a mixture. Formally, a pure state is simply defined as a state where all probabilities p_k in expression (2.2) except one are zero.

Let the Hilbert space $\mathcal{H}$ have an orthonormal basis $\{|u_n\rangle\}$[2], where $n \in \mathbb{Z}_{>0}$. Then, a pure state that lives in $\mathcal{H}$ is given by:

$$|\psi\rangle = \sum_n c_n |u_n\rangle \,,$$

where c_n are complex constants that satisfy $\sum_n |c_n|^2 = 1$ due to normalisation. The state $|\psi\rangle$ evolves according to the Schrödinger equation (2.1). Now, consider a hermitian operator (i.e. observable) A that acts on the Hilbert space. The matrix elements of A are given by:

$$A_{mn} = \langle u_m| A |u_n\rangle \,.$$

The expectation value of A is expressed as:

$$\begin{aligned} \langle A\rangle_\psi = \langle\psi| A |\psi\rangle &= \left(\sum_m c_m^* \langle u_m|\right) A \left(\sum_n c_n |u_n\rangle\right) \\ &= \sum_{m,n} c_m^* c_n \langle u_m| A |u_n\rangle \\ &= \sum_{m,n} c_m^* c_n A_{mn}. \end{aligned} \tag{2.4}$$

We notice that the terms $c_m^* c_n$ are equivalent to

$$\langle u_n|\psi\rangle \langle\psi|u_m\rangle = \langle u_n| \left(\sum_{n'} c_{n'} |u_{n'}\rangle\right) \left(\sum_{m'} c_{m'}^* \langle u_{m'}|\right) |u_m\rangle$$

---

[2] It is always possible to find an orthonormal basis for a Hilbert space, but here the choice of an orthonormal basis is not necessary. This choice simply serves to make our calculations easier.

$$\begin{aligned}
&= \left(\sum_{n'} c_{n'} \langle u_n | u_{n'} \rangle\right) \left(\sum_{m'} c^*_{m'} \langle u_{m'} | u_m \rangle\right) \\
&= \left(\sum_{n'} c_{n'} \delta_{nn'}\right) \left(\sum_{m'} c^*_{m'} \delta_{m'm}\right) \\
&= c_n c^*_m.
\end{aligned}$$

where in the second equality we used orthonormality. In other words, the terms $c^*_m c_n$ are precisely the matrix elements of the projection $|\psi\rangle \langle\psi|$. Thus, it is natural to introduce this projection as the density operator (or density matrix) ρ:

$$\rho \coloneqq |\psi\rangle \langle\psi| \,. \tag{2.5}$$

Upon the introduction of the density operator, we immediately see that:

$$\rho_{nm} = \langle u_n | \rho | u_m \rangle = c_n c^*_m.$$

Substituting this back into (2.4) results in:

$$\begin{aligned}
\langle A \rangle_\psi &= \sum_{m,n} c^*_m c_n A_{mn} = \sum_{m,n} \rho_{nm} A_{mn} = \sum_n (\rho A)_{nn} \\
&= \mathrm{Tr}(\rho A). \qquad (2.6)
\end{aligned}$$

So, to find the expectation value of any (matrix) operator, we multiply that operator by the density matrix and simply compute the trace. Moreover, since $\rho_{nn} = |c_n|^2$, we see that the density operator has unit trace:

$$\mathrm{Tr}\,\rho = \sum_n \rho_{nn} = 1.$$

Finally, we look at the time evolution of the density operator:

$$\frac{d}{dt}\rho = \frac{d}{dt}(|\psi\rangle \langle\psi|) = \left(\frac{d}{dt}|\psi\rangle\right)\langle\psi| + |\psi\rangle\left(\frac{d}{dt}\langle\psi|\right).$$

Now, we use the Schrodinger equation and its hermitian conjugate:

$$\frac{d}{dt}|\psi\rangle = -\frac{i}{\hbar} H |\psi\rangle \quad \text{and} \quad \frac{d}{dt}\langle\psi| = \frac{i}{\hbar}\langle\psi| H,$$

noting that $H = H^\dagger$. It simply follows that

$$\begin{aligned}
\frac{d}{dt}\rho &= -\frac{i}{\hbar} H \underbrace{|\psi\rangle \langle\psi|}_{=\rho} + \frac{i}{\hbar} \underbrace{|\psi\rangle \langle\psi|}_{=\rho} H \\
&= -\frac{i}{\hbar}(H\rho - \rho H) \\
&= -\frac{i}{\hbar}[H, \rho]. \qquad (2.7)
\end{aligned}$$

This is known as the **von Neumann equation** and is reminiscent of the time evolution of operators in the Heisenberg picture, but with a crucial minus sign difference. Thus, we can summarize three key properties of the (pure) density operator[3];

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[3] We denote these properties with a * as later we will introduce the properties of the general density matrix.

D1*. The density operator has unit trace: $\mathrm{Tr}\,\rho = 1$.

D2*. The expectation value of an observable A is calculated by: $\langle A \rangle = \mathrm{Tr}(\rho A)$.

D3*. The density operator evolves in time according to the von Neumann equation:

$$\dot{\rho} = -\frac{i}{\hbar}[H, \rho].$$

For completeness, let us indicate the probability $\mathcal{P}(a_n)$ of getting an eigenvalue a_n upon measuring A in terms of the density matrix. We have that

$$A \left| u_n^i \right\rangle = a_n \left| u_n^i \right\rangle \Rightarrow \mathcal{P}(a_n) = \langle \psi | P_n | \psi \rangle \,, \tag{2.8}$$

where $|u_n^i\rangle$ are all eigenvectors[4] that correspond to a_n, and P_n is the projector onto the eigensubspace of eigenvectors $|u_n^i\rangle$

$$P_n = \sum_i \left| u_n^i \right\rangle \left\langle u_n^i \right| . \tag{2.9}$$

Note that the eigenvalue a_n does not need to be degenerate. In the case that a_n is not degenerate, there is only one eigenvector $|u_n\rangle$, implying that the sum in equation (2.9) reduces to a single term. Then, we recover $\mathcal{P}(a_n) = |\langle u_n|\psi\rangle|^2 = |c_n|^2$, which is indeed the probability of measuring a_n. Looking at (2.6) and (2.8), we can write

$$\mathcal{P}(a_n) = \langle P_n \rangle = \mathrm{Tr}(\rho P_n).$$

In addition to the properties D1*-D3*, we have the following properties of the (pure) density operator:

D4*. The density operator is hermitian:

$$\rho^\dagger = (|\psi\rangle \langle\psi|)^\dagger = |\psi\rangle \langle\psi| = \rho.$$

D5*. The density operator is idempotent:

$$\rho^2 = (|\psi\rangle \langle\psi|)(|\psi\rangle \langle\psi|) = |\psi\rangle \underbrace{\langle\psi|\psi\rangle}_{=1} \langle\psi| = |\psi\rangle \langle\psi| = \rho.$$

D6*. Since the density operator is idempotent, we have that: $\mathrm{Tr}(\rho^2) = 1$.

With the density operator for a pure state, we move onto the general density operator for a mixed state.

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[4] The set of all eigenvectors $\{|u_n\rangle\}$ of A constitute an orthonormal basis in $\mathcal{H}$, as A is hermitian. This set can be taken as the basis mentioned at the beginning of the section. Here, we use the superscript i to distinguish the degenerate eigenvectors of A, corresponding to a_n, from the basis vectors $|u_n\rangle$. However, these degenerate vectors are still elements of the set of basis vectors $|u_n\rangle$, the superscript is simply a matter of notation.

2.2.2 Density Operator for a Mixed State

We now consider again an ensemble of pure states $|\psi_k\rangle$, each with an associated probability p_k. Then, the density operator is defined as:

$$\rho := \sum_k p_k |\psi_k\rangle \langle\psi_k| \,. \tag{2.10}$$

To see how this operator acts on observables, let us again consider the probability $\mathcal{P}(a_n)$ of finding an eigenvalue a_n after measuring a hermitian operator A. First, we define

$$\mathcal{P}_k(a_n) := \langle\psi_k| P_n |\psi_k\rangle$$

to be the probability of finding a_n if the state vector is $|\psi_k\rangle$, and P_n is the same sum of projections as in equation (2.9). Then, we have that the probability $\mathcal{P}(a_n)$ is given by:

$$\mathcal{P}(a_n) = \sum_k p_k \mathcal{P}_k(a_n).$$

We also define the density matrix constructed from the pure state $|\psi_k\rangle$ as

$$\rho_k := |\psi_k\rangle \langle\psi_k| \Rightarrow \rho = \sum_k p_k \rho_k.$$

Since we have already worked with a density matrix for a pure state, we see that:

$$\mathcal{P}_k(a_n) = \langle P_n \rangle_{\psi_k} = \mathrm{Tr}(\rho_k P_n),$$

and therefore,

$$\begin{aligned}
\mathcal{P}(a_n) &= \sum_k p_k \mathcal{P}_k(a_n) = \sum_k p_k \,\mathrm{Tr}(\rho_k P_n) \\
&= \sum_k \mathrm{Tr}(p_k \rho_k P_n) = \mathrm{Tr}\left(\sum_k p_k \rho_k P_n\right) \\
&= \mathrm{Tr}(\rho P_n),
\end{aligned} \tag{2.11}$$

where in the final equality we have used the definition of ρ. Having defined the density matrix for a mixed state, we move onto its general properties.

2.3 Properties of the Density Operator

Following the definition of the density matrix in equation (2.10), we have the following properties;

D1. The density operator has unit trace:

$$\mathrm{Tr}\,\rho = \mathrm{Tr}\left(\sum_k p_k \rho_k\right) = \sum_k p_k \,\mathrm{Tr}\,\rho_k = \sum_k p_k = 1,$$

where in the third equality we have used that ρ_k has unit trace, as it is a pure density matrix.

D2. The expectation value of an observable A is given by:

$$\begin{aligned}\mathrm{Tr}(\rho A) &= \mathrm{Tr}\left(\sum_k p_k \left|\psi_k\right\rangle \left\langle\psi_k\right| A\right)\\ &= \sum_k p_k \,\mathrm{Tr}(\left|\psi_k\right\rangle \left\langle\psi_k\right| A)\\ &= \sum_k p_k \,\mathrm{Tr}(\left\langle\psi_k\right| A \left|\psi_k\right\rangle)\\ &= \sum_k p_k \left\langle\psi_k\right| A \left|\psi_k\right\rangle\\ &= \left\langle A\right\rangle, \end{aligned}\tag{2.12}$$

where in the third equality we have used that the trace is cyclic, and in the fourth equality that the expression $\left\langle\psi_k\right| A \left|\psi_k\right\rangle$ is simply a scalar.

D3. The time evolution of ρ is given by:

$$\frac{d}{dt}\rho = \frac{d}{dt}\left(\sum_k p_k\rho_k\right) = \sum_k p_k \frac{d}{dt}\rho_k.$$

Now, we can use the time evolution of a pure density operator, which is given by equation (2.7). Therefore, the time evolution of ρ is

$$\begin{aligned}\frac{d}{dt}\rho &= -\frac{i}{\hbar}\sum_k p_k[H,\rho_k] = -\frac{i}{\hbar}\sum_k [H, p_k\rho_k]\\ &= -\frac{i}{\hbar}[H,\rho],\end{aligned}$$

which is once again the von Neumann equation.

D4. The density operator is hermitian $\rho^\dagger = \rho$.

D5. Since ρ is a sum of pure density operators, we have that in general:

$$\rho^2 \neq \rho \Rightarrow \mathrm{Tr}\,\rho^2 \leq 1.$$

As a consequence, if either one of $\rho = \rho^2$ or $\mathrm{Tr}\,\rho^2 = 1$ is satisfied, then ρ is a pure density operator.

D6. For any state vector $\left|\varphi\right\rangle$, we have that:

$$\begin{aligned}\left\langle\varphi\right| \rho \left|\varphi\right\rangle &= \sum_k p_k \left\langle\varphi|\psi_k\right\rangle \left\langle\psi_k|\varphi\right\rangle\\ &= \sum_k \underbrace{p_k}_{\geq 0} \underbrace{|\left\langle\varphi|\psi_k\right\rangle|^2}_{\geq 0} \geq 0,\end{aligned}$$

and therefore ρ is a positive operator, implying that the eigenvalues of ρ are non-negative.

Before we move onto the Hilbert space of an open quantum system, let us discuss an example of the density operator.

Example 2.2 (Example of a density operator)
Consider the entangled state given in example 2.1,

$$|\chi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle).$$

For the mixed state of the first particle, we had constructed the ensemble

$$\begin{array}{rl} \text{Spin state:} & \{|\uparrow\rangle, |\downarrow\rangle\} \\ \text{Probability:} & \{0.5, 0.5\} \end{array}$$

Using this, we can now construct the density operator corresponding to the first particle:

$$\rho = \frac{1}{2}(|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where we have used the same vector representation for the spins as in example 2.1. Now, we can compute the expectation value of the operator S_x through equation (2.12):

$$\rho S_x = \frac{\hbar}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\hbar}{4}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\Rightarrow \langle S_x\rangle = \mathrm{Tr}(\rho S_x) = 0$$

which is indeed what we had computed. However, if we were to instead attempt to describe the first particle by the pure superposition state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

then, the first particle would have a pure density matrix given by:

$$\rho = |\psi\rangle\langle\psi| = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Following this, the expectation value of S_x is:

$$\rho S_x = \frac{\hbar}{4}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\hbar}{4}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\Rightarrow \langle S_x\rangle = \mathrm{Tr}(\rho S_x) = \frac{\hbar}{2}$$

which is indeed different from the mixed state case. In fact, this is precisely the result we had obtained in example 2.1.

2.4 The Hilbert Space of an Open Quantum System

Now, let us consider the Hilbert space of a mixed system – namely a global system consisting of a smaller system and the bath. The global Hilbert space $\mathcal{H}$ is given by the tensor product,

$$\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_B,$$

where $\mathcal{H}_S$ and $\mathcal{H}_B$ are the system's and bath's Hilbert spaces respectively. Let us denote the orthonormal basis of $\mathcal{H}_S$ by $\{|u_n^{(S)}\rangle\}$, and the orthonormal basis of $\mathcal{H}_B$ by $\{|u_k^{(B)}\rangle\}$, where $n, k \in \mathbb{Z}_{>0}$. Then, the set of vectors $\{|u_n^{(S)}\rangle \otimes |u_k^{(B)}\rangle\}$ constitutes an orthonormal basis in $\mathcal{H}$. From which, it follows that:

$$\dim\mathcal{H} = \dim(\mathcal{H}_S \otimes \mathcal{H}_B) = \dim\mathcal{H}_S \cdot \dim\mathcal{H}_B.$$

For legibility, we drop the tensor product and simply write $\{|u_n^{(S)}\rangle\,|u_k^{(B)}\rangle\}$ for the orthonormal basis of $\mathcal{H}$. Similar to the basis, if we have two operators, Q_S and Q_B, which individually act on the system and the bath, respectively, they can be tensored to act on the global system:

$$Q = Q_S \otimes Q_B.$$

Moreover, if we wish to extend an operator that only acts on the system (or bath) onto the global system, we construct the operator

$$\tilde{Q}_S = Q_S \otimes \mathbb{1}_B \quad \text{or} \quad \tilde{Q}_B = \mathbb{1}_S \otimes Q_B,$$

where the identities $\mathbb{1}_B$ only acts on the bath, and $\mathbb{1}_S$ acts only on the system. If the Hilbert spaces of the system and bath are given by an explicit vector basis, the tensor products above are replaced by the Kronecker product.

Definition 2.1. (Kronecker Product) Let A be an $m \times n$ matrix, and B be a $k \times l$ matrix. Then the Kronecker product of A and B is an $mk \times nl$ matrix given by:

$$A \otimes B := \begin{pmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \dots & a_{mn}B \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} & \dots & a_{11}b_{1l} & \dots & a_{1n}b_{11} & \dots & a_{1n}b_{1l} \\ \vdots & \ddots & \vdots & & \vdots & \ddots & \vdots \\ a_{11}b_{k1} & \dots & a_{11}b_{kl} & \dots & a_{1n}b_{k1} & \dots & a_{1n}b_{kl} \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ a_{m1}b_{11} & \dots & a_{m1}b_{1l} & \dots & a_{mn}b_{11} & \dots & a_{mn}b_{1l} \\ \vdots & \ddots & \vdots & & \vdots & \ddots & \vdots \\ a_{m1}b_{k1} & \dots & a_{m1}b_{kl} & \dots & a_{mn}b_{k1} & \dots & a_{mn}b_{kl} \end{pmatrix}$$

In index notation, we write $(A \otimes B)_{ir,js} = A_{ij}B_{rs}$, which should be read as "the element in the rsth entry within the block matrix at the ijth position".

One can see that unlike matrix multiplication, the Kronecker product does not have any requirements on the sizes of the matrices. It is completely allowed to write down a product of, say, a matrix and a vector. We will now illustrate some examples of the Kronecker product.

Example 2.3 (Some simple examples of the Kronecker product)
Consider the vectors representing a spin-up and spin-down particle. Then, we can couple the two spins to construct the state

$$|\uparrow\downarrow\rangle = |\uparrow\rangle \otimes |\downarrow\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\cdot\begin{pmatrix}0\\1\end{pmatrix}\\ 0\cdot\begin{pmatrix}1\\0\end{pmatrix}\end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

Another possible example is extending the Pauli spin matrix σ_z by the use of the 3 × 3 identity matrix:

$$\sigma_z \otimes \mathbb{1}_3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \otimes \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 0 & 0 & -1\end{pmatrix}$$

2.4.1 The Partial Trace

We have seen in the previous section that the Hilbert space of the global system is given by $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_B$. However, it is not true in general that the density operator of the global system can be separated in the same way, i.e. $\rho \neq \rho_S \otimes \rho_B$. In fact, if the density matrix is separable, we say the system and the bath are uncorrelated. Still, we desire a tool to extract the density operator of the system from the global density matrix. To this end, we introduce the partial trace. In rough terms, taking the partial trace over, say, the Hilbert space $\mathcal{H}_B$ means that the vectors of $\mathcal{H}_S$ are left invariant, and we take the trace over $\mathcal{H}_B$. We take "leaving the vectors of $\mathcal{H}_S$ invariant" to be equivalent to acting with the identity. With this, we can state the definition of the partial trace [19].

Definition 2.2. (Partial Trace) Let $\mathcal{H}^{(1)}$ and $\mathcal{H}^{(2)}$ be Hilbert spaces with orthonormal bases $\{|u_n^{(1)}\rangle\}$ and $\{|u_k^{(2)}\rangle\}$ (with n and k being positive integers), respectively. Then consider the global Hilbert space given by $\mathcal{H} = \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}$, with the orthonormal basis $\{|u_n^{(1)}\rangle\,|u_k^{(2)}\rangle\}$. The partial trace over the Hilbert space $\mathcal{H}^{(2)}$ is defined as

$$\mathrm{Tr}_2\,\rho := \sum_k \left(\mathbb{1}^{(1)} \otimes \langle u_k^{(2)}|\right) \rho \left(\mathbb{1}^{(1)} \otimes |u_k^{(2)}\rangle\right)$$

It is customary to denote $\mathrm{Tr}_2\,\rho \equiv \rho^{(1)}$. Similarly, the partial trace over the Hilbert space $\mathcal{H}^{(1)}$ is defined by

$$\mathrm{Tr}_1\,\rho := \sum_n \left(\langle u_n^{(1)}| \otimes \mathbb{1}^{(2)}\right) \rho \left(|u_n^{(1)}\rangle \otimes \mathbb{1}^{(2)}\right)$$

and it is once again customary to denote $\mathrm{Tr}_1\,\rho \equiv \rho^{(2)}$.

The density matrices $\rho^{(1)}$ and $\rho^{(2)}$ satisfy all of the properties D1.-D6., which follows from their definitions. An immediate consequence of the partial trace is

$$\begin{aligned}
\operatorname{Tr}(\operatorname{Tr}_1 \rho) &= \operatorname{Tr}\left(\sum_n \left(\langle u_n^{(1)}| \otimes \mathbb{1}^{(2)}\right) \rho \left(|u_n^{(1)}\rangle \otimes \mathbb{1}^{(2)}\right)\right) \\
&= \sum_k \sum_n \langle u_k^{(2)}| \left(\langle u_n^{(1)}| \otimes \mathbb{1}^{(2)}\right) \rho \left(|u_n^{(1)}\rangle \otimes \mathbb{1}^{(2)}\right) |u_k^{(2)}\rangle \\
&= \sum_k \sum_n \langle u_n^{(1)}| \langle u_k^{(2)}| \, \rho \, |u_n^{(1)}\rangle \, |u_k^{(2)}\rangle = \operatorname{Tr} \rho .
\end{aligned}$$

It is also not difficult to show that $\operatorname{Tr}(\operatorname{Tr}_2 \rho) = \operatorname{Tr} \rho$. Before moving onto how the partial trace appears, let us first discuss an example of the partial trace.

Example 2.4 (The partial trace)

Again, let us consider the entangled state mentioned in examples 2.1 and 2.2,

$$|\chi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle).$$

The state $|\chi\rangle$ lives in the Hilbert space $\mathcal{H} = \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}$, where the Hilbert spaces $\mathcal{H}^{(1)}$ and $\mathcal{H}^{(2)}$ represent the first and second particles, respectively, and are identical copies of each other with the orthonormal spin basis $\{|\uparrow\rangle, |\downarrow\rangle\}$. Then, the density operator ρ of the global system is given by:

$$\rho = |\chi\rangle\langle\chi| .$$

For clarity, let us move to the usual vector representation. First, we express $|\chi\rangle$ as:

$$|\uparrow\downarrow\rangle = |\uparrow\rangle \otimes |\downarrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad |\uparrow\downarrow\rangle = |\downarrow\rangle \otimes |\uparrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow |\chi\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

Following this, the density operator is simply given by:

$$\rho = |\chi\rangle\langle\chi| = \frac{1}{2} \begin{pmatrix} 0 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Now, to retrieve the density operator describing the first particle, we take the partial trace over $\mathcal{H}^{(2)}$:

$$\begin{aligned}
\rho^{(1)} &= \mathrm{Tr}_2\,\rho = \left(\mathbb{1}^{(1)} \otimes \langle\uparrow|\right)\rho\left(\mathbb{1}^{(1)} \otimes |\uparrow\rangle\right) + \left(\mathbb{1}^{(1)} \otimes \langle\downarrow|\right)\rho\left(\mathbb{1}^{(1)} \otimes |\downarrow\rangle\right) \\
&= \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \frac{1}{2}(|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|)
\end{aligned}$$

noting that $\mathbb{1}^{(1)}$ is the 2×2 identity matrix, as it acts on the 2 dimensional Hilbert space $\mathcal{H}^{(1)}$. This resultant density operator is precisely the density operator describing the first particle in example 2.2.

In the discussion that follows, we shall refer to the first Hilbert space $\mathcal{H}^{(1)}$ as the system $\mathcal{H}_S$, and the second Hilbert space $\mathcal{H}^{(2)}$ as the bath $\mathcal{H}_B$. To see how the partial trace appears, let us consider an operator A_S that acts only on $\mathcal{H}_S$, and its extension $\tilde{A}_S = A_S \otimes \mathbb{1}_B$ in $\mathcal{H}$. From equation (2.12), we have that

$$\begin{aligned}
\langle\tilde{A}_S\rangle &= \mathrm{Tr}(\rho\tilde{A}_S) \\
&= \sum_{n,k} \langle u_n^{(S)}| \langle u_k^{(B)}| \rho\tilde{A}_S |u_n^{(S)}\rangle |u_k^{(B)}\rangle \, .
\end{aligned}$$

Now, we introduce the identity $\mathbb{1} = \sum_{n',k'} |u_{n'}^{(S)}\rangle |u_{k'}^{(B)}\rangle \langle u_{n'}^{(S)}| \langle u_{k'}^{(B)}|$. Then,

$$\begin{aligned}
\langle\tilde{A}_S\rangle &= \sum_{n,k}\sum_{n',k'} \langle u_n^{(S)}| \langle u_k^{(B)}| \rho |u_{n'}^{(S)}\rangle |u_{k'}^{(B)}\rangle \cdot \langle u_{n'}^{(S)}| \langle u_{k'}^{(B)}| A_S \otimes \mathbb{1}_B |u_n^{(S)}\rangle |u_k^{(B)}\rangle \\
&= \sum_{n,k}\sum_{n',k'} \langle u_n^{(S)}| \langle u_k^{(B)}| \rho |u_{n'}^{(S)}\rangle |u_{k'}^{(B)}\rangle \cdot \langle u_{n'}^{(S)}| A_S |u_n^{(S)}\rangle \cdot \underbrace{\langle u_{k'}^{(B)}|u_k^{(B)}\rangle}_{=\delta_{k'k}} \\
&= \sum_{n,n'}\sum_{k} \langle u_n^{(S)}| \langle u_k^{(B)}| \rho |u_{n'}^{(S)}\rangle |u_k^{(B)}\rangle \cdot \langle u_{n'}^{(S)}| A_S |u_n^{(S)}\rangle \, .
\end{aligned}$$

From the definition of the partial trace, we see that:

$$\begin{aligned}
\sum_k \langle u_n^{(S)}| \langle u_k^{(B)}| \rho |u_{n'}^{(S)}\rangle |u_k^{(B)}\rangle &= \langle u_n^{(S)}| \left(\sum_k \left(\mathbb{1}^{(S)} \otimes \langle u_k^{(B)}|\right)\rho\left(\mathbb{1}^{(S)} \otimes |u_k^{(B)}\rangle\right)\right) |u_{n'}^{(S)}\rangle \\
&= \langle u_n^{(S)}| \mathrm{Tr}_B\,\rho |u_{n'}^{(S)}\rangle \\
&= \langle u_n^{(S)}| \rho_S |u_{n'}^{(S)}\rangle \, .
\end{aligned}$$

Replacing this result in the above expression gives

$$\langle\tilde{A}_S\rangle = \sum_{n,n'} \langle u_n^{(S)}| \rho_S |u_{n'}^{(S)}\rangle \cdot \langle u_{n'}^{(S)}| A_S |u_n^{(S)}\rangle = \sum_{n,n'} \rho_{S_{nn'}} A_{S_{n'n}}$$

$$= \sum_n (\rho_S A_S)_{nn} = \mathrm{Tr}(\rho_S A_S),$$

where we have switched to index notation in the second equality for clarity. Comparing this to equation (2.12), we see that the partial trace allows us to compute the expectation values of an extended observable that only acts on the system, as if the system were isolated and had ρ_S for a density operator. Moreover, using a reasoning similar to equation (2.11), we see that the partial trace also allows us to obtain the probabilities of all results of measurements done on the system alone.

3 The Lindblad Equation

In the previous section, we introduced Open Quantum Systems as quantum systems that interact with their environment. Moreover, to adequately describe such systems we introduced the density matrix that acts within the global Hilbert space. An illustration visualising an open quantum system along with the Hilbert space and density matrices can be seen below.

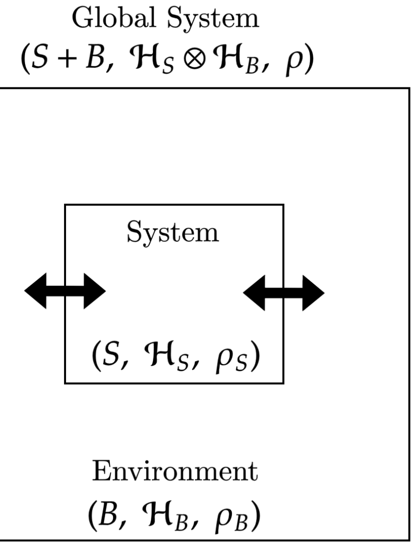

Figure 1: *A diagram visualising an open quantum system, adapted from [20].*

We also saw that the dynamics of the global system, in terms of the density matrix ρ that acts on the global hilbert space $\mathcal{H}$, was expressed by the von Neumann equation

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H,\rho]. \tag{3.1}$$

The exact solution to the von Neumann equation completely describes how the states of our global system evolve in time. However, in general we only wish to understand how the interaction with the environment affects the dynamics of ρ_S. In other words, we only want to know how the density matrix ρ_S evolves in time due to the interaction while excluding the information from the environment. To this end, we had defined the partial trace,

$$\rho_S = \mathrm{Tr}_B\,\rho,$$

which traced over the degrees of freedom of the environment, leaving us with the information of the system alone. One might expect that we can simply express the dynamics of ρ_S by taking the partial trace of both sides in equation (3.1):

$$\frac{d\rho_S}{dt} = -\frac{i}{\hbar}\,\mathrm{Tr}_B[H,\rho]. \tag{3.2}$$

This approach is not incorrect, but it turns out that acquiring the exact solution of equation (3.2) is quite involved. Therefore, in order to describe the dynamics of our system in a less complicated manner, we make the assumption that the time intervals for which the environment retains information are much smaller compared to the system's timescale. As a result of this, we can not unravel the memory effects of the environment, and therefore the system is said to be memoryless. Formally, this simply means that the value of the density matrix at a time $t + dt$ only depends on its value at time t, and no other previous instance in time. This approximation is commonly referred to as the *Markovian approximation*. We will see that this assumption ultimately leads to the Gorini-Kossakowski-Sudarshan-Lindblad master equation[5] in diagonal form:

$$\mathcal{L}\rho_S = \frac{d\rho_S}{dt} = -\frac{i}{\hbar}[H, \rho_S] + \sum_k \gamma_k \left(L_k \rho_S L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho_S\} \right), \tag{3.3}$$

where the terms involved in the equation will be defined in the following sub-sections. For brevity, we will refer to equation (3.3) simply as the Lindblad equation, but it is important to state that it was also derived independently by multiple physicists. Throughout this section, our aim will be to provide the reader with a digestible, although rather long, derivation of the Lindblad equation. The discussion and derivation presented in this section is largely derived from the works [20, 21].

3.1 Liouville Space and Dynamical Maps

In order to derive the Lindblad equation, we ask ourselves the following question: *What is the most general Markovian way of mapping the density matrix onto another density matrix?* However, we do not yet have the framework to think about such a mapping yet. Due to this, we need to introduce some new notions before we move onto mapping density matrices.

Let us first observe how a unitary time evolution is expressed in terms of density matrices. Suppose that the pure states $|\psi_k\rangle \in \mathcal{H}$ evolve in time according to:

$$|\psi_k(t)\rangle = U(t,0)\,|\psi_k(0)\rangle\,,$$

where $U(t,0)$ is unitary the time evolution operator, and $|\psi_k(0)\rangle$ is the pure state at time $t = 0$. Then, the total density matrix at a time t is given by:

$$\begin{aligned}
\rho(t) &= \sum_k p_k\,|\psi_k(t)\rangle\,\langle\psi_k(t)| = \sum_k p_k U(t,0)\,|\psi_k(0)\rangle\,\langle\psi_k(0)|\,U^\dagger(t,0) \\
&= U(t,0)\left(\sum_k p_k\,|\psi_k(0)\rangle\,\langle\psi_k(0)|\right)U^\dagger(t,0) \\
&= U(t,0)\rho(0)U^\dagger(t,0).
\end{aligned}$$

From which we can write:

$$\rho_S(t) = \mathrm{Tr}_B\,\rho(t) = \mathrm{Tr}_B\{U(t,0)\rho(0)U^\dagger(t,0)\}. \tag{3.4}$$

We now see how the density matrix ρ_S evolves unitarily in time[6], but we still have not considered what type of space the density matrix lives in. For this reason, we define the *Liouville space*.

---

[5] Master equations are simply a set of first order differential equations describing the time evolution of a system that is a probabilistic mixture. The von Neumann equation (3.1) can be considered the simplest unitary master equation.

[6] In principle, the time evolution of our density matrix ρ_S is completely governed by equations (3.2) and (3.4). However, as we mentioned before, the exact solutions to these equations are quite involved.

Definition 3.1. (Liouville Space) Given the Hilbert space $\mathcal{H}_S$, the corresponding Liouville space, denoted $\mathcal{B}(\mathcal{H}_S)$ is the space of operators A that act on $\mathcal{H}_S$ for which $\mathrm{Tr}(A^\dagger A)$ is finite. Equipped with the scalar product

$$\langle\!\langle A|B\rangle\!\rangle := \mathrm{Tr}(A^\dagger B), \tag{3.5}$$

for any $A, B \in \mathcal{B}(\mathcal{H}_S)$, the Liouville space becomes a Hilbert space. Sometimes the Liouville space $\mathcal{B}(\mathcal{H}_S)$ is referred to as the space of bounded operators acting on $\mathcal{H}_S$.

Due to the properties D4. and D5. of the density matrix, it is easy to see that $\rho \in \mathcal{B}(\mathcal{H}_S)$. With this definition, we can finally think of how to map a density matrix at time $t = 0$ to a density matrix at time t in the most general Markovian way. For this, we will introduce the concept of a *dynamical map*. Let us first suppose that the system and the environment are uncorrelated at $t = 0$, allowing us to write $\rho(0) = \rho_S(0) \otimes \rho_B$, where $\rho_S(0)$ is the initial density matrix of the system. The map sending $\rho_S(0)$ to a density matrix at a later time $\rho_S(t)$, with $t \in \mathbb{R}_{\geq 0}$, may be written as

$$\begin{aligned} \varphi(t) : \mathcal{B}(\mathcal{H}_S) &\to \mathcal{B}(\mathcal{H}_S) \\ \rho_S(0) &\mapsto \rho_S(t) = \varphi(t)\rho_S(0) := \mathrm{Tr}_B\{U(t,0)[\rho_S(0) \otimes \rho_B]U^\dagger(t,0)\} \end{aligned}$$

The map $\varphi(t)$ which describes the change in states of the open system over a time t is referred to as a *dynamical map*. We can visually illustrate the action of $\varphi(t)$ as in figure 2.

$$\begin{tikzcd}
\rho(0) = \rho_S(0) \otimes \rho_B \arrow[r, "\text{unitary evolution}"] \arrow[d, "\mathrm{Tr}_B"] & \rho(t) = U(t,0)[\rho_S(0) \otimes \rho_B]U^\dagger(t,0) \arrow[d, "\mathrm{Tr}_B"] \\
\rho_S(0) \arrow[r, "\text{dynamical map}"] & \rho_S(t) = \varphi(t)\rho_S(0)
\end{tikzcd}$$

Figure 2: *A diagram visualising the action of a dynamical map, adapted from [20].*

Since the density matrix represents a physical state, it must adhere to its properties of having unit trace and being positive. Therefore, the dynamical map $\varphi(t)$ must preserve the trace of all density matrices ρ_S, and ensure that the density matrix has non-negative eigenvalues at any instance in time. In fact, for the latter it is not sufficient for the map to be simply positive, as we require the preservation of positivity even when the map is acting on a larger system. This more general property is referred to as "complete positivity". Due to the trace preserving and complete positive properties of a dynamical map, it is sometimes referred to as a *completely positive trace preserving (CPTP)* map.

We emphasize that after the unitary transformation of the seperable density matrix $\rho(0)$, or after the dynamical mapping with $\varphi(t)$, the density matrix at a later time t is not seperable. This represents the interaction of the system with it's environment, as it "mixes" both of the density matrices.

3.1.1 Quantum Dynamical Semigroups

In the above subsections, we have introduced the notion of a dynamical map $\varphi(t)$ which describes the change of the density matrix ρ_S over a fixed time t. We can also consider the set of all dynamical maps $\{\varphi(t)|t \in \mathbb{R}_{\geq 0}\}$, with $\varphi(0)$ as it's identity element. Such a family of dynamical maps describes the complete future evolution of our system, and in general is quite complex. Moreover, the elements of the set satisfy property

$$\varphi(t_1)\varphi(t_2) = \varphi(t_1 + t_2), \quad \forall\, t_1, t_2 \in \mathbb{R}_{\geq 0}, \tag{3.6}$$

which is evident as evolving the states by t_2 and then by t_1 is completely equivalent to evolving our states by $t_1 + t_2$. The property (3.6) is often called the "semigroup property", and the one-parameter family of dynamical maps $\{\varphi(t)|t \in \mathbb{R}_{\geq 0}\}$ is called a "quantum dynamical semigroup". In rough terms, a quantum dynamical semigroup is a family of one-parameter dynamical maps satisfying the semigroup property (3.6).

3.2 Derivation of the Lindblad Equation

Upon seeing that the most general Markovian way of mapping a density matrix onto another density matrix is through a CPTP map, and we seek a representation for such a map to derive the Lindblad equation. For this, we turn ourselves to Choi-Kraus' Theorem.

Choi-Kraus' Theorem. Any linear map $\varphi : \mathcal{B}(\mathcal{H}_S) \to \mathcal{B}(\mathcal{H}_S)$ is completely positive and trace preserving if and only if it can be expressed as

$$\varphi\rho_S = \sum_k V_k \rho_S V_k^\dagger, \tag{3.7}$$

where $V_k \in \mathcal{B}(\mathcal{H}_S)$ are called the Kraus operators and they satisfy

$$\sum_k V_k V_k^\dagger = \mathbb{1}_S. \tag{3.8}$$

The sum in both equations runs from 0 to Q, where $Q \leq N^2$, with N being the dimension of the Hilbert space $\mathcal{H}_S$.

The proof of this theorem requires extensive algebra and as a result we refer to the works [21, 22] instead. By Choi-Kraus' theorem, it simply follows from (3.7) that our CPTP map $\varphi(t)$ can be represented as

$$\varphi(t)\rho_S = \sum_k V_k(t)\rho_S V_k^\dagger(t), \tag{3.9}$$

where $V_k(t) \in \mathcal{B}(\mathcal{H}_S)$ satisfy the completeness relation (3.8). Thus, we now know the form of our CPTP map, but we still do not have a master equation that is a set of first order differential equations that describes the time evolution of our system. To this end, we desire a linear, time independent map $\mathcal{L}$ such that

$$\frac{d\rho_S}{dt} = \mathcal{L}\rho_S,$$

which immediately allows us to express our CPTP map in the form $\varphi(t) = e^{\mathcal{L}t}$. The map $\mathcal{L}$ is the so-called *Lindblad superoperator* or *the generator of the quantum dynamical semigroup*. In what follows, we will derive the most general form of $\mathcal{L}$ in order to reach the Lindblad equation.

Let us consider the simple case where the Hilbert space $\mathcal{H}_S$ is finite dimensional, with $\dim \mathcal{H}_S = N$. The corresponding Liouville space $\mathcal{B}(\mathcal{H}_S)$ is N^2-dimensional, and we choose an orthonormal basis of operators $\{F_i\}$, with $i = 1, 2, \ldots, N^2$. Following equation (3.5), orthonormality in this case is expressed as:

$$\langle\langle F_i | F_j \rangle\rangle = \mathrm{Tr}(F_i^\dagger F_j) = \delta_{ij}.$$

Without loss of generality, we choose one of the basis operators to be proportional to the identity $F_{N^2} = 1/\sqrt{N} \mathbb{1}_S$. It is trivial to see that F_{N^2} has norm one. Additionally, F_{N^2} being proportional to the identity implies that the other basis operators have zero trace:

$$\begin{aligned} &\mathrm{Tr}(F_{N^2}^\dagger F_j) = 0 && \forall\, j = 1, 2, \ldots, N^2 - 1 \\ &\Rightarrow \frac{1}{\sqrt{N}} \mathrm{Tr}(F_j) = 0 && \forall\, j = 1, 2, \ldots, N^2 - 1 \\ &\Rightarrow \mathrm{Tr}(F_j) = 0 && \forall\, j = 1, 2, \ldots, N^2 - 1 \end{aligned}$$

Since $\{F_i\}$ constitutes a complete basis in $\mathcal{B}(\mathcal{H}_S)$, we can write the operator $V_k(t) \in \mathcal{B}(\mathcal{H}_S)$ in (3.9) as a linear superposition of the basis operators:

$$V_k(t) = \sum_{i=1}^{N^2} \alpha_{ik} F_i,$$

where α_{ik} are complex constants. Taking the scalar product with F_i yields:

$$\langle\langle F_i | V_k(t) \rangle\rangle = \sum_{j=1}^{N^2} \alpha_{jk} \underbrace{\langle\langle F_i | F_j \rangle\rangle}_{=\delta_{ij}} = \alpha_{ik}.$$

Therefore, the operators $V_k(t)$ can be expressed in the following way:

$$V_k(t) = \sum_{i=1}^{N^2} \langle\langle F_i | V_k(t) \rangle\rangle F_i.$$

This further allows us to write equation (3.9) as

$$\begin{aligned} \varphi(t)\rho_S &= \sum_k V_k(t) \rho_S V_k^\dagger(t) \\ &= \sum_k \left(\sum_{i=1}^{N^2} \langle\langle F_i | V_k(t) \rangle\rangle F_i \right) \rho_S \left(\sum_{j=1}^{N^2} F_j^\dagger \langle\langle V_k(t) | F_j \rangle\rangle \right) \\ &= \sum_k \sum_{i,j=1}^{N^2} \langle\langle F_i | V_k(t) \rangle\rangle \langle\langle V_k(t) | F_j \rangle\rangle F_i \rho_S F_j^\dagger \\ &= \sum_{i,j=1}^{N^2} c_{ij}(t) F_i \rho_S F_j^\dagger, \end{aligned} \tag{3.10}$$

where we have introduced the complex coefficients

$$c_{ij}(t) := \sum_k \langle\langle F_i | V_k(t) \rangle\rangle \langle\langle V_k(t) | F_j \rangle\rangle.$$

Claim: The coefficient matrix $c = (c_{ij})$ with $i, j = 1, 2, \ldots, N^2$ is Hermitian and positive.

Proof. For this proof, we will drop the labels with time dependence. Let us first prove hermiticity. We only need to show:

$$c^\dagger = c \iff c^*_{ji} = c_{ij}.$$

Then,

$$\begin{aligned} c_{ij} &= \sum_k \langle\!\langle F_i | V_k \rangle\!\rangle \langle\!\langle V_k | F_j \rangle\!\rangle = \sum_k \operatorname{Tr}(F_i^\dagger V_k) \cdot \operatorname{Tr}(V_k^\dagger F_j) \\ &= \sum_k \operatorname{Tr}(F_i^T V_k^*)^* \cdot \operatorname{Tr}(V_k^T F_j^*)^*, \end{aligned}$$

where in the third equality we have used the property $\operatorname{Tr}(A^*) = (\operatorname{Tr} A)^*$ for any matrix A, and with A^T simply being the transpose of A. Now, we use that $\operatorname{Tr} A^T = \operatorname{Tr} A$ and $(AB)^T = B^T A^T$ (for two matrices A, B), which yields

$$\begin{aligned} c_{ij} &= \sum_k \operatorname{Tr}(V_k^\dagger F_i)^* \cdot \operatorname{Tr}(F_j^\dagger V_k)^* = \left(\sum_k \operatorname{Tr}(F_j^\dagger V_k) \cdot \operatorname{Tr}(V_k^\dagger F_i) \right)^* \\ &= \left(\sum_k \langle\!\langle F_j | V_k \rangle\!\rangle \langle\!\langle V_k | F_i \rangle\!\rangle \right)^* = c^*_{ji}, \end{aligned}$$

where in the second equality we have commuted the traces as they are just numbers. This proves hermiticity. For positivity, it suffices to show that $\langle v | \, c \, | v \rangle \geq 0$ for any N^2 dimensional complex vector $|v\rangle$. In index notation, we need to show:

$$\sum_{i,j} v_i^* c_{ij} v_j \geq 0.$$

We have that:

$$\begin{aligned} \sum_{i,j} v_i^* c_{ij} v_j &= \sum_{i,j} \sum_k v_i^* \langle\!\langle F_i | V_k \rangle\!\rangle \langle\!\langle V_k | F_j \rangle\!\rangle v_j \\ &= \sum_{i,j} \sum_k v_i^* \operatorname{Tr}(F_i^\dagger V_k) \cdot \operatorname{Tr}(V_k^\dagger F_j) v_j \\ &= \sum_{i,j} \sum_k \operatorname{Tr}(v_i^* F_i^\dagger V_k) \cdot \operatorname{Tr}(V_k^\dagger F_j v_j), \end{aligned}$$

where in the last line we have moved v_i and v_j into the traces are they are simply complex numbers. Moreover, since they are numbers we have that $(v_i F_i)^\dagger = v_i^* F_i^\dagger$. It follows that:

$$\sum_{i,j} v_i^* c_{ij} v_j = \sum_k \left(\sum_i \langle\!\langle v_i F_i | V_k \rangle\!\rangle \right) \left(\sum_j \langle\!\langle V_k | v_j F_j \rangle\!\rangle \right)$$

$$= \sum_k \left(\sum_i \langle\langle v_i F_i | V_k \rangle\rangle \right) \left(\sum_j \langle\langle v_j F_j | V_k \rangle\rangle \right)^\dagger$$
$$= \sum_k \left| \left\langle\!\!\left\langle \sum_i v_i F_i \middle| V_k \right\rangle\!\!\right\rangle \right|^2 \geq 0.$$

This shows that the coefficient matrix c is positive. ☐

Since the CPTP map $\varphi(t)$ takes a density matrix ρ_S and describes its evolution over a time t, we can use the map $\varphi(\Delta t)$ represented by:

$$\rho_S(t + \Delta t) = \varphi(\Delta t)\rho_S(t) = \sum_k V_k(\Delta t)\rho_S(t)V_k^\dagger(\Delta t), \tag{3.11}$$

to describe the density matrix at a later time $t + \Delta t$. Then, we can express the time derivative of ρ_S as:

$$\begin{aligned}\mathcal{L}\rho_S = \frac{d\rho_S}{dt} &= \lim_{\Delta t \to 0} \frac{\rho_S(t+\Delta t) - \rho_S(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\varphi(\Delta t)\rho_S(t) - \rho_S(t)}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left(\sum_{i,j=1}^{N^2} c_{ij}(\Delta t) F_i \rho_S F_j^\dagger - \rho_S \right),\end{aligned}$$

where in the final equality we have simply used the representation (3.10). We once again emphasize the importance of the Markovian assumption: It is exactly the Markovian assumption that implies the value of the density matrix at a time $t + \Delta t$ is determined by its value at an earlier time t, hence allowing us to express the derivative as in equation (3.2). Now, recalling that $F_{N^2} = 1/\sqrt{N}\mathbb{1}_S$, we write:

$$\begin{aligned}\mathcal{L}\rho_S = \lim_{\Delta t \to 0} \frac{1}{\Delta t} &\left(\sum_{i,j=1}^{N^2-1} c_{ij}(\Delta t) F_i \rho_S F_j^\dagger + \sum_{i=1}^{N^2-1} c_{iN^2}(\Delta t) F_i \rho_S F_{N^2}^\dagger \right. \\ &\left. + \sum_{j=1}^{N^2-1} c_{N^2 j}(\Delta t) F_{N^2} \rho_S F_j^\dagger + c_{N^2N^2}(\Delta t) F_{N^2} \rho_S F_{N^2}^\dagger - \rho_S \right) \\ = \lim_{\Delta t \to 0} \frac{1}{\Delta t} &\left(\sum_{i,j=1}^{N^2-1} c_{ij}(\Delta t) F_i \rho_S F_j^\dagger + \frac{1}{\sqrt{N}} \sum_{i=1}^{N^2-1} c_{iN^2}(\Delta t) F_i \rho_S \right. \\ &\left. + \frac{1}{\sqrt{N}} \sum_{j=1}^{N^2-1} c_{N^2 j}(\Delta t) \rho_S F_j^\dagger + \frac{1}{N} c_{N^2N^2}(\Delta t) \rho_S - \rho_S \right) \\ = \lim_{\Delta t \to 0} &\left[\sum_{i,j=1}^{N^2-1} \frac{c_{ij}(\Delta t)}{\Delta t} F_i \rho_S F_j^\dagger + \frac{1}{\sqrt{N}} \sum_{i=1}^{N^2-1} \left(\frac{c_{iN^2}(\Delta t)}{\Delta t} F_i \rho_S + \frac{c_{N^2 i}(\Delta t)}{\Delta t} \rho_S F_i^\dagger \right) \right. \\ &\left. + \frac{1}{N} \frac{c_{N^2N^2}(\Delta t) - N}{\Delta t} \rho_S \right]\end{aligned}$$

Next, we define the complex coefficients a_{ij} to get rid of the explicit time dependence:

$$a_{ij} \coloneqq \lim_{\Delta t \to 0} \frac{c_{ij}(\Delta t)}{\Delta t}, \qquad i, j = 1, 2, ..., N^2 - 1$$

$$
\begin{aligned}
a_{iN^2} &:= \lim_{\Delta t\to 0} \frac{c_{iN^2}(\Delta t)}{\Delta t}, \qquad j = 1, 2, ..., N^2-1 \\
a_{N^2 i} &:= \lim_{\Delta t\to 0} \frac{c_{N^2 i}(\Delta t)}{\Delta t}, \qquad i = 1, 2, ..., N^2-1 \\
a_{N^2 N^2} &:= \lim_{\Delta t\to 0} \frac{c_{N^2 N^2}(\Delta t) - N}{\Delta t}.
\end{aligned}
$$

Since the coefficient matrix $c = (c_{ij})$ is positive and hermitian, it follows that the coefficient matrix $a = (a_{ij})$ is also positive and hermitian. Substituting the coefficients a_{ij}:

$$
\mathcal{L}\rho_S = \sum_{i,j=1}^{N^2-1} a_{ij} F_i \rho_S F_j^\dagger + \frac{1}{\sqrt{N}} \sum_{i=1}^{N^2-1} \left(a_{iN^2} F_i \rho_S + a_{N^2 i} \rho_S F_i^\dagger \right) + \frac{a_{N^2N^2}}{N}\rho_S. \tag{3.12}
$$

Since we are summing over the operators F_i, it is also useful to define

$$
F := \frac{1}{\sqrt{N}} \sum_{i=1}^{N^2-1} a_{iN^2} F_i \tag{3.13}
$$

$$
\Rightarrow F^\dagger = \frac{1}{\sqrt{N}} \sum_{i=1}^{N^2-1} a_{iN^2}^* F_i^\dagger = \frac{1}{\sqrt{N}} \sum_{i=1}^{N^2-1} a_{N^2 i} F_i^\dagger, \tag{3.14}
$$

where in the last equality we have used that $a = (a_{ij})$ is hermitian, implying that $a_{iN^2}^* = a_{N^2 i}$. Substituting equations (3.13) and (3.14) into (3.12) yields

$$
\mathcal{L}\rho_S = \sum_{i,j=1}^{N^2-1} a_{ij} F_i \rho_S F_j^\dagger + F\rho_S + \rho_S F^\dagger + \frac{a_{N^2N^2}}{N}\rho_S.
$$

We now split the operator F into a hermitian and an anti-hermitian part

$$
F = \frac{F + F^\dagger}{2} + i\frac{F - F^\dagger}{2i} =: G - \frac{i}{\hbar} H. \tag{3.15}
$$

Noting that both G and H are hermitian, we have that

$$
\begin{aligned}
\mathcal{L}\rho_S &= \sum_{i,j=1}^{N^2-1} a_{ij} F_i \rho_S F_j^\dagger + (G\rho_S + \rho_S G) + -\frac{i}{\hbar}(H\rho_S - \rho_S H) + \frac{a_{N^2N^2}}{N}\rho_S \\
&= -\frac{i}{\hbar}[H, \rho_S] + \{G, \rho_S\} + \frac{a_{N^2N^2}}{N}\rho_S + \sum_{i,j=1}^{N^2-1} a_{ij} F_i \rho_S F_j^\dagger,
\end{aligned}
$$

where the curly brackets represent the anti-commutator given by $\{A, B\} = AB + BA$ for two operators A, B. Finally, we define

$$
\tilde{G} := G + \frac{a_{N^2N^2}}{2N}\mathbb{1}_S
$$

From which, we can write the superoperator as

$$
\mathcal{L}\rho_S = -\frac{i}{\hbar}[H, \rho_S] + \{\tilde{G}, \rho_S\} + \sum_{i,j=1}^{N^2-1} a_{ij} F_i \rho_S F_j^\dagger. \tag{3.16}
$$

So far we have only used the complete positivity of our map $\phi(\Delta t)$, and we are yet to use the trace preserving property. The trace preserving property requires that the trace of ρ_S does not change over time:

$$\begin{aligned} \operatorname{Tr}\left(\frac{d\rho_S}{dt}\right) &= \operatorname{Tr}(\mathcal{L}\rho_S) = 0 \\ \Rightarrow \operatorname{Tr}\left(-\frac{i}{\hbar}[H,\rho_S] + \{\tilde{G},\rho_S\} + \sum_{i,j=1}^{N^2-1} a_{ij} F_i \rho_S F_j^\dagger\right) &= 0. \end{aligned} \tag{3.17}$$

Using the property that $\operatorname{Tr}(A+B) = \operatorname{Tr} A + \operatorname{Tr} B$ (for two matrcies A and B of appropriate size), let us compute the terms of equation (3.17) separately. For the commutator, we get:

$$\begin{aligned} \operatorname{Tr}\left(-\frac{i}{\hbar}[H,\rho_S]\right) &= -\frac{i}{\hbar}\operatorname{Tr}(H\rho_S) + \frac{i}{\hbar}\operatorname{Tr}(\rho_S H) \\ &= -\frac{i}{\hbar}\operatorname{Tr}(H\rho_S) + \frac{i}{\hbar}\operatorname{Tr}(H\rho_S) = 0, \end{aligned}$$

and for the anti-commutator,

$$\operatorname{Tr}(\{\tilde{G}\rho_S\}) = \operatorname{Tr}(\tilde{G}\rho_S) + \operatorname{Tr}(\rho_S\tilde{G}) = 2\operatorname{Tr}(\tilde{G}\rho_S),$$

and finally for the last summation term,

$$\begin{aligned} \operatorname{Tr}\left(\sum_{i,j=1}^{N^2-1} a_{ij} F_i \rho_S F_j^\dagger\right) &= \sum_{i,j=1}^{N^2-1} a_{ij}\operatorname{Tr}(F_i\rho_S F_j^\dagger) \\ &= \sum_{i,j=1}^{N^2-1} a_{ij}\operatorname{Tr}(F_j^\dagger F_i \rho_S) \\ &= \operatorname{Tr}\left(\sum_{i,j=1}^{N^2-1} a_{ij} F_j^\dagger F_i \rho_S\right). \end{aligned}$$

Putting this back into equation (3.17):

$$\operatorname{Tr}\left[\left(2\tilde{G} + \sum_{i,j=1}^{N^2-1} a_{ij} F_j^\dagger F_i\right)\rho_S\right] = 0, \qquad \forall\ \rho_S \in \mathcal{B}(\mathcal{H}_S).$$

From which we deduce that

$$\tilde{G} = -\frac{1}{2}\sum_{i,j=1}^{N^2-1} a_{ij} F_j^\dagger F_i.$$

Which allows us to write equation (3.16) as:

$$\mathcal{L}\rho_S = -\frac{i}{\hbar}[H,\rho_S] + \sum_{i,j=1}^{N^2-1} a_{ij}\left(F_i\rho_S F_j^\dagger - \frac{1}{2}\{F_j^\dagger F_i, \rho_S\}\right). \tag{3.18}$$

Since the coefficient matrix $a = (a_{ij})$ is positive, it's eigenvalues must be non-negative. Therefore, we can diagonalise it with an appropriate unitary matrix u:

$$uau^\dagger = \begin{pmatrix} \gamma_1 & 0 & \cdots & 0 \\ 0 & \gamma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \gamma_{N^2-1}, \end{pmatrix} \tag{3.19}$$

where the eigenvalues γ_i are non-negative. Note that the indices i, j usually run from 1 to N^2. However, since we are only summing until $N^2 - 1$ in equation (3.18) we have reduced the coefficient matrix a to a $(N^2-1) \times (N^2-1)$ matrix by removing the last row and column. We can now introduce the set of operators L_k to get equation 3.18 into diagonal form:

$$F_i = \sum_{k=1}^{N^2-1} u_{ki} L_k \Rightarrow F_i^\dagger = \sum_{k=1}^{N^2-1} u_{ki}^* L_k^\dagger.$$

Note that in the above expression u_{ki} are matrix elements of the unitary matrix u, so they are simply complex numbers. Therefore, we can write the expression for the Lindblad superoperator in terms of the operators L_k:

$$\begin{aligned} \mathcal{L}\rho_S &= -\frac{i}{\hbar}[H, \rho_S] + \sum_{i,j=1}^{N^2-1} \sum_{k,k'=1}^{N^2-1} a_{ij} u_{ki} u_{k'j}^* \left(L_k \rho_S L_{k'}^\dagger - \frac{1}{2}\{L_k^\dagger L_{k'}, \rho_S\} \right) \\ &= -\frac{i}{\hbar}[H, \rho_S] + \sum_{i,j=1}^{N^2-1} \sum_{k,k'=1}^{N^2-1} u_{ki} a_{ij} (u^\dagger)_{jk'} \left(L_k \rho_S L_{k'}^\dagger - \frac{1}{2}\{L_k^\dagger L_{k'}, \rho_S\} \right). \end{aligned} \tag{3.20}$$

By writing equation (3.19) in index notation, we have that:

$$\sum_{i,j=1}^{N^2-1} u_{ki} a_{ij} (u^\dagger)_{jk'} = (uau^\dagger)_{kk'} = \gamma_k \delta_{kk'}.$$

Thus, we can write equation (3.20) as

$$\mathcal{L}\rho_S = \frac{d\rho_S}{dt} = -\frac{i}{\hbar}[H, \rho_S] + \sum_{k=1}^{N^2-1} \gamma_k \left(L_k \rho_S L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho_S\} \right). \tag{3.21}$$

The operator H in the Lindblad equation (3.21) —which was the anti-hermitian part of the operator F in (3.15)— is identified with the Hamiltonian of our interacting system. However, it is important to note that H is not equivalent to the free Hamiltonian H_S, it can contain additional coupling terms signifying the interaction of the system with the environment. The first term in the Lindblad equation (3.21) is the von Neumann equation, which represents the unitary evolution governed by the Hamiltonian H. The second term is referred to as the *dissipator*, and the operators L_k are the so-called *quantum jump operators*. The total term inside the summation represents the potential transitions – or "jumps" – that can occur within the system due to its interaction with the environment. Finally, the non-negative eigenvalues γ_k represent the strength of the coupling with the environment and are attributed to the relaxation rates of different decay modes of the open system[7]. In the case where all γ_k are zero, we recover the unitary dynamics given by the von Neumann equation.

In the above derivation, we have presupposed that both the Lindblad superoperator $\mathcal{L}$ and the jump operators L_k are bounded. However, in general this is not the case; the Hamiltonian H_S and the jump operators L_k are usually unbounded. Yet, all known examples of the dynamics of open quantum systems are either in Lindblad form, or can be turned into it by modifications. Therefore, the Lindblad equation is considered the most general form for the generator of the dynamics of Open Quantum Systems. Namely, the works of Gorini, Kossakowski and Sudarshan [23] details the mathematical proof that (3.18) defines such a generator in a finite Hilbert space; while the work of Lindblad [24] provides a theorem stating (3.21) describes the most general bounded generator for a seperable Hilbert space, where the index k belongs to a countably finite set.

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[7] The non-negative eigenvalues γ_k are sometimes absorbed into the operators L_k in literature

4 Quantum Integrable Spin Chains

In the previous sections, we have explained Open Quantum Systems and their dynamics in detail, bringing us one step closer to *Integrable Open Quantum Systems*. We now divert our attention to *Quantum Integrable Systems*; systems most of which are isolated and have numerous conserved quantities, allowing them to be exactly solvable. The aim of this chapter is to introduce the reader to the concept of integrability through the famous *Heisenberg Spin Chain*. The model was proposed by Heisenberg in 1928 [25], and was later solved by Bethe in 1931 [8]. Bethe's ansatz, referred to as the *Coordinate Bethe Ansatz*, now serves as the cornerstone of the field of Integrable Systems. The following discussion of the Heisenberg Spin Chain and its various approaches are based on the works [26, 27, 28], while the classification of quantum integrable systems is based on [1, 17].

4.1 The Heisenberg Spin Chain

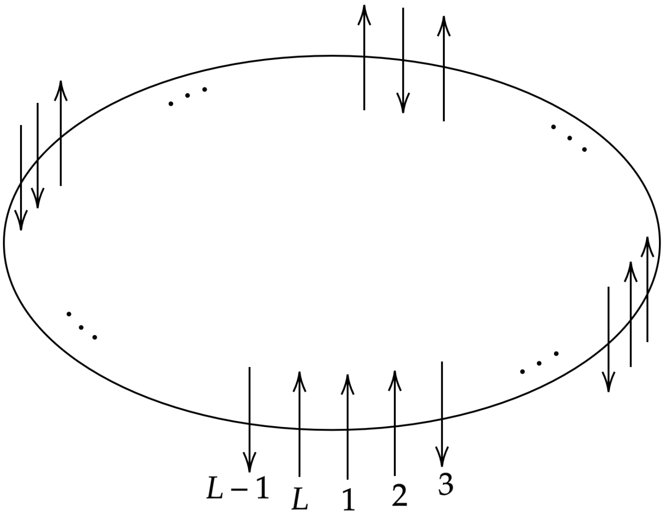

Figure 3: *A diagram of the Heisenberg spin chain with a random configuration of spins, the periodicity can be seen as the site* 1 *comes after the site* L.

The Heisenberg Spin Chain, sometimes referred to as the $\textsc{xxx}_{1/2}$ spin chain, is a chain of L electrons that are equally separated on a one-dimensional periodic lattice. The electrons only have nearest neighbour spin interactions. An illustration of the Heisenberg Spin Chain can be found in Figure 3. The location of each electron is referred to as a 'site', and each electron can either be spin-up or spin-down. Therefore, the local Hilbert space of a site is $\mathbb{C}^2$, with the total Hilbert space of the spin chain being:

$$\mathcal{H} = \bigotimes_{n=1}^{L} \mathbb{C}^2. \tag{4.1}$$

It follows that $\dim\mathcal{H} = 2^L$. The Hamiltonian of the spin chain is given by:

$$H = \frac{JL}{4}\mathbb{1} - J\sum_{n=1}^{L} \vec{S}_n \cdot \vec{S}_{n+1}, \quad \vec{S}_{L+1} = \vec{S}_1, \tag{4.2}$$

where $J > 0$ is the exchange integral representing the strength of the spin-spin interaction. This is called the Heisenberg ferromagnet, where it is more energetically favorable for the spins to be aligned. Conversely, $J < 0$ is the Heisenberg antiferromagnet, where the spins prefer to be anti-parallel. The antiferromagnetic case gives rise to a complex ground state and frustrated excited states, and is not preferred as a consequence. For this reason, we only work with $J > 0$. The first term in the Hamiltonian (4.2) is purely for normalisation, and the expression on the right is the periodicity condition. The vector $\vec{S}$ is simply the vector of the regular spin matrices:

$$\vec{S} = (S^x, S^y, S^z) = \frac{1}{2}(\sigma^x, \sigma^y, \sigma^z) = \frac{1}{2}\vec{\sigma},$$

where the $\sigma^{x,y,z}$ are the Pauli matrices, and we have dropped the factor of $\hbar$ in the second equality. The subscript n in the Hamiltonian (4.2) denotes which position of the spin chain the operator acts on:

$$S_n^a = \mathbb{1} \otimes \mathbb{1} \otimes \cdots \otimes \underbrace{S^a}_{n^{th}\text{ site}} \otimes \cdots \otimes \mathbb{1}, \quad a = x, y, z \tag{4.3}$$

from which it is easy to see that:

$$S_n^a S_{n+1}^a = \mathbb{1} \otimes \mathbb{1} \otimes \cdots \otimes \underbrace{S^a}_{n} \otimes \underbrace{S^a}_{n+1} \otimes \cdots \otimes \mathbb{1}.$$

Then, the Hamiltonian may be re-written:

$$H = \frac{JL}{4}\mathbb{1} - \frac{J}{4}\sum_{n=1}^{L} \sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \sigma_n^z \sigma_{n+1}^z.$$

Introducing the raising and lowering operators $\sigma^{\pm} = \frac{1}{2}(\sigma^x \pm i\sigma^y)$, the Hamiltonian becomes:

$$H = \frac{JL}{4}\mathbb{1} - \frac{J}{4}\sum_{n=1}^{L} 2(\sigma_n^+ \sigma_{n+1}^- + \sigma_n^- \sigma_{n+1}^+) + \sigma_n^z \sigma_{n+1}^z.$$

By substituting the explicit matrix forms of the Pauli spin matrices, the term inside of the sum simplifies to:

$$2(\sigma_n^+ \sigma_{n+1}^- + \sigma_n^- \sigma_{n+1}^+) + \sigma_n^z \sigma_{n+1}^z = 2P_{n,n+1} - \mathbb{1}, \tag{4.4}$$

where $P_{n,n+1}$ is the permutation operator acting on the sites n and $n+1$, similar to definition (4.3):

$$P_{n,n+1} = \mathbb{1} \otimes \cdots \otimes \underbrace{\mathbb{1}}_{n-1} \otimes \underbrace{P}_{n \text{ and } n+1} \otimes \underbrace{\mathbb{1}}_{n+2} \cdots \otimes \mathbb{1},$$

with P given by

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

P is called the permutation matrix as for any two vectors $|a\rangle, |b\rangle \in \mathbb{C}^2$:

$$P(|a\rangle \otimes |b\rangle) = |b\rangle \otimes |a\rangle.$$

It can be seen that $P^2 = \mathbb{1}$. Moreover, for any operator A that acts on two sites j and k, one has that

$$P_{ij} A_{jk} P_{ij} = A_{ik}, \tag{4.5}$$

where P_{ij} is the permutation operator acting on sites i and j. Together with the permutation operator in (4.4), the Hamiltonian is:

$$H = \frac{JL}{4}\mathbb{1} - \frac{J}{4}\sum_{n=1}^{L}(2P_{n,n+1} - \mathbb{1}) = \frac{JL}{2}\mathbb{1} - \frac{J}{2}\sum_{n=1}^{L} P_{n,n+1}. \tag{4.6}$$

We note that the Hamiltonian is a $2^L \times 2^L$ matrix, as it acts on the Hilbert space $\mathcal{H}$ of dimension 2^L. Therefore, directly computing the eigenvalues and eigenvectors of the Hamiltonian by brute force is practically impossible for large L. For this reason, we turn our attention to the *Coordinate Bethe Ansatz*, the same method Bethe used in his article [8] to solve the spectral problem of the Heisenberg spin chain.

4.1.1 The Coordinate Bethe Ansatz: a Summary

This section will give a summary of Coordinate Bethe Ansatz, as the full discussion is very involved and not strongly related to our goal of classifying quantum integrable systems. For a full treatment, we suggest the lecture notes [27, 28] to the reader. The summary is largely based on the article of Batchelor [29].

The Coordinate Bethe Ansatz starts from the ground state of the Hamiltonian (4.6). Typically, for the ferromagnetic case, the ground state is the state where all the spins are pointed up:

$$|0\rangle = |\uparrow\uparrow \ldots \uparrow\rangle \,,$$

where acting on the ground state with the Hamiltonian (4.6) yields:

$$H\,|0\rangle = 0.$$

The next step is to consider states where some spins are overturned, these are referred to as “magnons”. The state with M overturned spins is denoted by:

$$|n_1, \ldots, n_M\rangle = \sigma^-_{n_1} \cdots \sigma^-_{n_M} |0\rangle \,,$$

where the integers $1 \le n_1, \ldots, n_M \le L$ refer to the locations of the overturned spins. The set of all such vectors consists a basis in the Hilbert space $\mathcal{H}$. Naturally, the eigenstates of the Hamiltonian are linear combinations of the basis vectors:

$$|\psi\rangle = \sum_{1 \le n_1 \le \cdots \le n_M \le L} a(n_1, \ldots, n_M)\,|n_1, \ldots, n_M\rangle \,.$$

Bethe’s remarkable insight was in noticing that the amplitudes $a(n_1, \ldots, n_m)$ were of the form

$$a(n_1, \ldots, n_M) = \sum_{P \in S_n} A_P \exp\left(\sum_{j=1}^{L} i k_{p_j} n_j\right),$$

where $P \in S_n$ are all possible $n!$ permutations of the set of integers $\{1, 2, \dots, n\}$. Working with this Ansatz, Bethe realised that he could factorize the interactions of the Heisenberg spin chain into two-body interactions. Namely, he found the coefficients A_p to be

$$A_p = \varepsilon_P \prod_{1 \le i \le j \le M} s_{P_i} s_{P_j},$$

where $s_{m\ell} = 1 - 2e^{ik_\ell} + e^{i(k_m + k_\ell)}$ are the two-body interaction terms, and ε_P is the sign of the permutation. The wavenumbers k_j are required to satisfy the Bethe equations

$$e^{ik_j L} = (-1)^{M-1} \prod_{\ell \neq j} \frac{s_{\ell j}}{s_{j\ell}}, \quad j = 1, \dots, M$$

By solving the Bethe equations for k_j, the energy eigenvalues of the Hamiltonian are found to be:

$$E = 2J \sum_{j=1}^{M} \cos k_j - 1.$$

At the time, Bethe's result was deemed remarkable, but was mostly viewed as "academic exercise". However, in later years, people came across many more one-dimensional systems with two-body reducible interactions, allowing them to be exactly solvable. These models were classified as *Integrable Systems*. The astounding part was that, almost without exception, it was possible to solve these systems in terms of the Coordinate Bethe Ansatz, making Bethe's work the staple of Integrable systems.

4.1.2 The Algebraic Bethe Ansatz: an Introduction

Following the Coordinate Bethe Ansatz, we now introduce the Algebraic Bethe Ansatz: an alternative and algebraic way of studying the Heisenberg spin chain. This way, the integrable nature of the model will be made much clearer. In this section, we only present the ingredients required to solve the spectral problem of the Heisenberg spin chain as the actual method of solving the spectral problem is quite involved. For a full treatment, we refer the reader to the excellent works [26, 27]. Later, we will comment on the generalisation of this approach to introduce quantum integrability.

We start our approach with the **Lax Operator**.

Definition 4.1. (Lax Operator) Consider a spin chain with N sites[a] and the corresponding Hilbert space $\mathcal{H} = \bigotimes_{n=1}^{N} \mathbb{C}^2 = \bigotimes_{n=1}^{N} V_n$, where $V_n \equiv \mathbb{C}^2$ is referred to as the (local) physical space. We now add an auxiliary space V_a to this spin chain, which we again identify with $\mathbb{C}^2$. The Lax operator is an operator $L : V_n \otimes V_a \to V_n \otimes V_a$. For the Heisenberg Spin Chain, the Lax operator is of the form:

$$L_{n,a}(u) = u(\mathbb{1}_n \otimes \mathbb{1}_a) + i \sum_{\alpha=1}^{3} S_n^\alpha \otimes \sigma^\alpha, \tag{4.7}$$

where S_n^α are the spin matrices related to the Pauli matrices through $S_n^\alpha = \frac{1}{2}\sigma_n^\alpha$ with

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The parameter u in (4.7) is the so called “spectral parameter”.

---

[a] In the next couple of sections, we will use N as the number of sites before we switch back to L. This is to avoid confusion with the Lax operator L.

It is possible to view the Lax matrix defined by (4.7) as a 2×2 matrix acting on the auxiliary space with its entries acting on the physical space V_n:

$$L_{n,a}(u) = \begin{pmatrix} u + iS_n^z & iS_n^- \\ iS_n^+ & u - iS_n^z. \end{pmatrix} \tag{4.8}$$

To proceed further, we need the commutation relations of the Lax operator. We have already introduced the permutation operator P:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \frac{1}{2}\left(\mathbb{1} + \sum_{\alpha=1}^{3} \sigma^\alpha \otimes \sigma^\alpha\right).$$

From which we can rewrite (4.7) as

$$\begin{aligned} L_{n,a}(u) &= u\mathbb{1}_{n,a} + \frac{i}{2}\sum_{\alpha=1}^{3} \sigma_n^\alpha \otimes \sigma^\alpha = u\mathbb{1}_{n,a} + \left(iP_{n,a} - \frac{i}{2}\mathbb{1}_{n,a}\right) \\ &= \left(u - \frac{i}{2}\right)\mathbb{1}_{n,a} + iP_{n,a}. \end{aligned} \tag{4.9}$$

Now, let us make the following claim regarding the commutation relations of the Lax matrix:

Claim. Consider two Lax operators $L_{n,a}(u_1)$ and $L_{n,b}(u_2)$ acting on the same physical space but different auxiliary spaces V_a and V_b, both of which are equivalent to $\mathbb{C}^2$. We claim that there exists an operator $R_{ab}(u_1 - u_2) : V_a \otimes V_b \to V_a \otimes V_b$ such that:

$$R_{ab}(u_1 - u_2)L_{n,a}(u_1)L_{n,b}(u_2) = L_{n,b}(u_2)L_{n,a}(u_1)R_{ab}(u_1 - u_2), \tag{4.10}$$

with the explicit form of R_{ab} given by:

$$R_{ab}(u) = u\mathbb{1}_{ab} + iP_{ab}. \tag{4.11}$$

Proof. The proof is done by direct computation through substituting the expressions (4.9) and (4.11) and using the following properties of the permutation matrix:

$$P_{n,a}P_{n,b} = P_{a,b}P_{n,a} = P_{n,b}P_{b,a}, \qquad P_{a,b} = P_{b,a}. \tag{4.12}$$

The commutation relation (4.10) is called the **Fundamental Commutation Relation (FCR)** or the **RLL** relation, and the matrix (4.11) is called he **R-matrix**. The Lax operator $L_{a,n}$ is an object that acts on the two vector spaces V_a and V_n, and it is graphically represented as two lines crossing. The n-line depicts the physical space V_n, while the a-line depicts the auxiliary space V_a. Similarly, this can be done for the R-matrix R_{ab}, where one draws the a and b-lines instead. These graphical portrayals are illustrated in Figure 4.

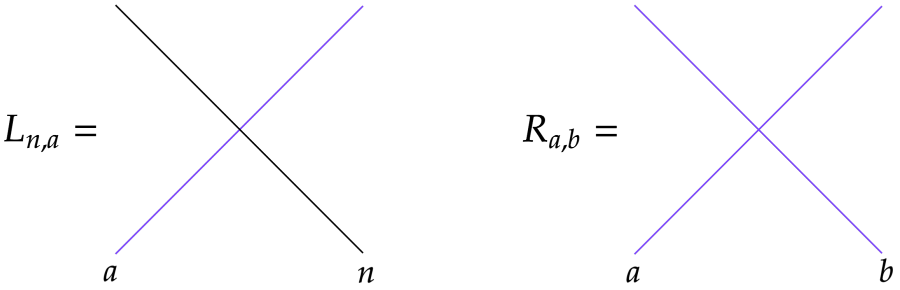

Figure 4: *Graphical representation of the Lax operator (left) and R-matrix (right). The auxiliary spaces a, b are drawn in purple to distinguish from the physical space n.*

Following the graphical representation in Figure 4, the RLL relation (4.10) can also be drawn in pictorial form:

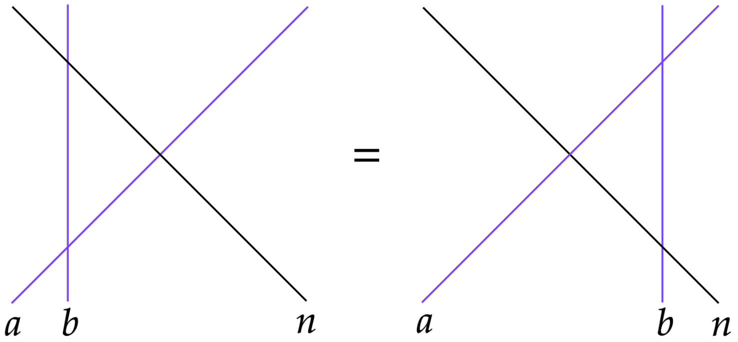

Figure 5: *The graphical representation of the RLL relation.*

We note that the R-matrix (4.11) and the Lax operator (4.9) have the same form, they only differ by a shift in the spectral parameter. Using the RLL relation (4.10), dropping the label for the physical space for now, and considering three auxiliary spaces V_1, V_2, V_3, we get that:

$$\begin{aligned} L_1L_2L_3 &= (R_{12}^{-1}L_2L_1R_{12})L_3 = R_{12}^{-1}L_2L_1L_3R_{12} \\ &= R_{12}^{-1}L_2(R_{13}^{-1}L_3L_1R_{13})R_{12} \\ &= R_{12}^{-1}R_{13}^{-1}L_2L_3L_1R_{13}R_{12} \\ &= R_{12}^{-1}R_{13}^{-1}(R_{23}^{-1}L_3L_2R_{23})L_1R_{13}R_{12} \\ &= R_{12}^{-1}R_{13}^{-1}R_{23}^{-1}L_3L_2L_1R_{23}R_{13}R_{12}, \end{aligned} \tag{4.13}$$

where we have repeatedly made use of the RLL relation and have commuted e.g. R_{12} and L_3 as they act on different auxiliary spaces. Similarly, we have that:

$$L_1L_2L_3 = R_{23}^{-1}L_1L_3L_2R_{23} = R_{23}^{-1}R_{13}^{-1}L_3L_1L_2R_{13}R_{23}$$

$$= R_{23}^{-1} R_{13}^{-1} R_{12}^{-1} L_3 L_2 L_1 R_{12} R_{13} R_{23}. \tag{4.14}$$

For equations (4.13) and (4.14) to coincide, we require that:

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}. \tag{4.15}$$

This is known as the **Yang-Baxter Equation** (YBE), also known as the star-triangle relation. The pictorial form of the YBE is depicted in Figure 6.

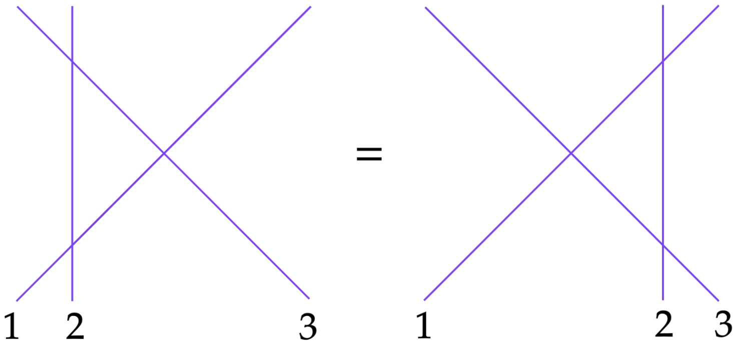

Figure 6: *The graphical representation of the Yang-Baxter Equation.*

Explicitly including the spectral parameters in the YBE:

$$R_{12}(u_1 - u_2) R_{13}(u_1 - u_3) R_{23}(u_2 - u_3) = R_{23}(u_2 - u_3) R_{13}(u_1 - u_3) R_{12}(u_1 - u_2). \tag{4.16}$$

So far we have only considered an arbitrary physical site n and an auxiliary space a on the Heisenberg spin chain. However, our spin chain consists of N sites, and we desire an operator that acts on the whole spin chain and is related to the Lax operator. For this, we define the **Monodromy** and **Transfer** matrices.

> *Definition 4.2.* (Monodromy Matrix) The Monodromy matrix is an operator $T_a : \mathcal{H} \otimes V_a \to \mathcal{H} \otimes V_a$ given by
>
> $$T_a(u) = L_{N,a}(u) L_{N-1,a}(u) \cdots L_{1,a}(u). \tag{4.17}$$

The graphical representation of the Monodromy matrix is:

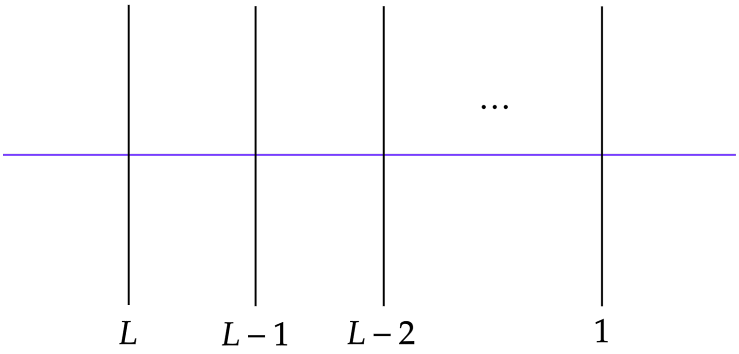

Figure 7: *The graphical representation of the Monodromy matrix.*

Similar to the Lax matrix in (4.8), we can also represent the Monodromy matrix as a 2×2 matrix acting on the auxiliary spaces with its entries acting on the physical Hilbert space $\mathcal{H}$. We typically write:

$$T_a(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}. \tag{4.18}$$

Through the Monodromy matrix, we will now see how a tower of conserved charges arise. This exactly the property that defines integrable systems.

First, we again desire the commutation relations of the Monodromy matrix:

Claim. The commutation relations of the Monodromy matrix are governed by

$$R_{ab}(u_1 - u_2) T_a(u_1) T_b(u_2) = T_b(u_2) T_a(u_1) R_{ab}(u_1 - u_2). \tag{4.19}$$

Proof. From the definition of the Monodromy matrix, we have that:

$$T_a(u_1)T_b(u_2) = L_{N,a}(u_1)L_{N-1,a}(u_1)\cdots L_{1,a}(u_1)L_{N,b}(u_2)L_{N-1,b}(u_2)\cdots L_{1,b}(u_2).$$

Let us now collect the terms with the same physical spaces to use the RLL relation.

$$T_a(u_1)T_b(u_2) = (L_{N,a}(u_1)L_{N,b}(u_2))(L_{N-1,a}(u_1)L_{N-1,b}(u_2))\cdots(L_{1,a}(u_1)L_{1,b}(u_2)).$$

Note that we are able to commute the Lax operators through in this way as they act on different physical and auxiliary spaces. Now, we apply the RLL relation to commute the lax operators:

$$\begin{aligned} T_a(u_1)T_b(u_2) = {} & (R^{-1}_{ab}(u_1 - u_2)L_{N,b}(u_2)L_{N,a}(u_1)\cancel{R_{ab}(u_1 - u_2)}) \\ & \cdot (\cancel{R^{-1}_{ab}(u_1 - u_2)}L_{N-1,b}(u_2)L_{N-1,a}(u_1)\cancel{R_{ab}(u_1 - u_2)}) \\ & \cancel{\cdots}(\cancel{R^{-1}_{ab}(u_1 - u_2)}L_{1,b}(u_2)L_{1,a}(u_1)R_{ab}(u_1 - u_2)) \\ = {} & R^{-1}_{ab}(u_1 - u_2)L_{N,b}(u_2)L_{N,a}(u_1)\cdots L_{1,b}(u_2)L_{1,a}(u_1)R_{ab}(u_1 - u_2). \end{aligned}$$

We can once again re-arrange the terms accordingly to get:

$$\begin{aligned} T_a(u_1)T_b(u_2) &= R^{-1}_{ab}(u_1 - u_2)L_{N,b}(u_2)\cdots L_{1,b}(u_2)L_{N,a}(u_1)\cdots L_{1,a}(u_1)R_{ab}(u_1 - u_2) \\ &= R^{-1}_{ab}(u_1 - u_2)T_b(u_2)T_a(u_1)R_{ab}(u_1 - u_2). \end{aligned}$$

From which it follows that:

$$R_{ab}(u_1 - u_2)T_a(u_1)T_b(u_2) = T_b(u_2)T_a(u_1)R_{ab}(u_1 - u_2).$$

This completes the proof. □

The commutation relation (4.19) is known as the RTT relation, and is graphically represented in Figure 8. We can now define the transfer matrix through the Monodromy matrix:

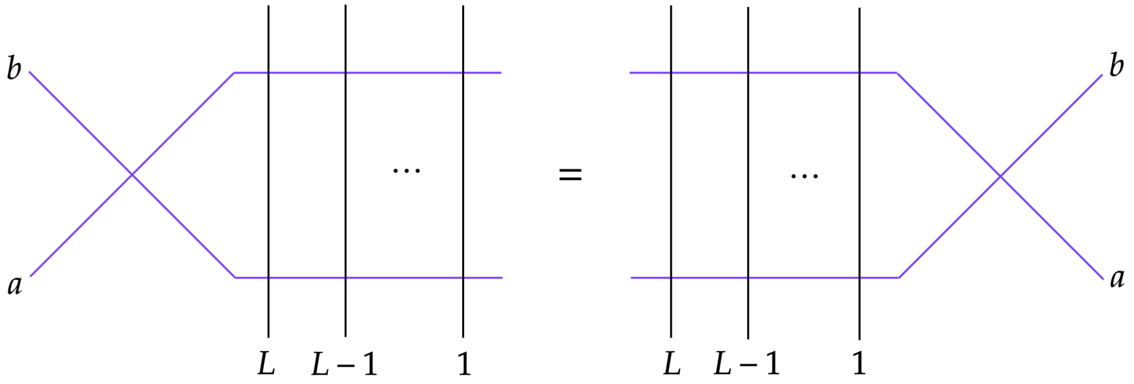

Figure 8: *The graphical representation of the RTT relation.*

Definition 4.3. (Transfer Matrix) The Transfer matrix is an operator $t : \mathcal{H} \to \mathcal{H}$ given by taking the partial trace of the Monodromy the matrix:

$$t(u) = \mathrm{Tr}_a\, T_a(u) = A(u) + D(u). \tag{4.20}$$

The Transfer matrix can also be represented graphically as in Figure 9.

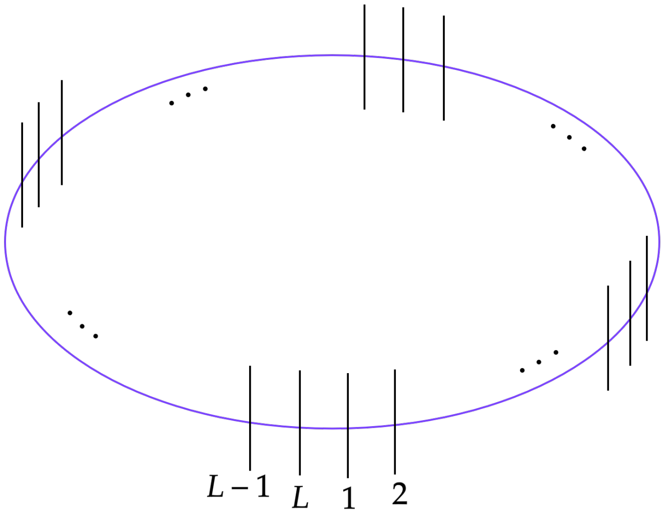

Figure 9: *The graphical representation of the transfer matrix.*

Writing the RTT relation (4.19) in the following way:

$$T_a(u_1)T_b(u_2) = R_{ab}^{-1}(u_1-u_2)T_b(u_2)T_a(u_1)R_{ab}(u_1-u_2),$$

and subsequently the partial trace over both auxiliary spaces and using the cyclice property of the trace yields:

$$\begin{aligned}\mathrm{Tr}_b\left(\mathrm{Tr}_a\left(T_a(u_1)T_b(u_2)\right)\right) &= \mathrm{Tr}_b\left(\mathrm{Tr}_a\left(R_{ab}^{-1}(u_1-u_2)T_b(u_2)T_a(u_1)R_{ab}(u_1-u_2)\right)\right)\\ &= \mathrm{Tr}_b\Big(\mathrm{Tr}_a\Big(\cancel{R_{ab}(u_1-u_2)R_{ab}^{-1}(u_1-u_2)}T_b(u_2)T_a(u_1)\Big)\Big)\\ &= \mathrm{Tr}_b\left(\mathrm{Tr}_a\left(T_b(u_2)T_a(u_1)\right)\right),\end{aligned}$$

which further simplifies to:

$$t(u_1)t(u_2) = t(u_2)t(u_1) \iff [t(u_1), t(u_2)] = 0. \tag{4.21}$$

Since the transfer matrix depends on a spectral parameter, we can simply write power series in u_1 and u_2. The commutator (4.21) then tells us that there is a family of commuting operators, each corresponding to a power in the expansion. For the Heisenberg spin chain, we would like the momentum operator and the Hamiltonian to be in this set of commuting charges. To this end, we look at the logarithmic derivatives of the transfer matrix.

In order to see how the momentum operator arises, let us expand $t(u)$ around $u = i/2$. To first order, we have that:

$$T_a\left(\frac{i}{2}\right) = L_{N,a}\left(\frac{i}{2}\right)\cdots L_{1,a}\left(\frac{i}{2}\right) = i^N P_{N,a}\cdots P_{1,a},$$

where we have simply substituted the definition of the Lax operator (4.9) for $u = i/2$ in the second equality. The properties (4.12) of the permutation matrix allows us to write:

$$T_a\left(\frac{i}{2}\right) = i^N P_{12}P_{23}\cdots P_{N-1,N}P_{N,a}.$$

Taking the partial trace and noting that $\mathrm{Tr}_a\, P_{N,a} = \mathbb{1}_N$:

$$t\left(\frac{i}{2}\right) = i^N P_{12}P_{23}\cdots P_{N-1,N} = \mathcal{U}, \tag{4.22}$$

where $\mathcal{U}$ is the shift operator – the operator which shifts each site to the right by one slot. By definition, the shift operator is identified with the momentum $\mathbb{P}$ in the following way:

$$\mathcal{U} = e^{i\mathbb{P}}.$$

Thus, we see that:

$$\mathbb{P} = -i\log\mathcal{U} = -i\log t(u)\big|_{u=i/2}. \tag{4.23}$$

To acquire the Hamiltonian we look at the next order in the expansion. This is given by:

$$\begin{aligned}
\frac{d}{du}T_a(u)\bigg|_{u=i/2} &= \frac{d}{du}\left(L_{N,a}(u)L_{N-1,a}(u)\cdots L_{1,a}(u)\right)\bigg|_{u=i/2} \\
&= \left(\frac{d}{du}L_{N,a}(u)\right)_{u=i/2} L_{N-1,a}\left(\frac{i}{2}\right)\cdots L_{1,a}\left(\frac{i}{2}\right) \\
&\quad + L_{N,a}\left(\frac{i}{2}\right)\left(\frac{d}{du}L_{N-1,a}(u)\right)_{u=i/2}\cdots L_{1,a}\left(\frac{i}{2}\right) \\
&\quad + \cdots + L_{N,a}\left(\frac{i}{2}\right)\cdots\left(\frac{d}{du}L_{1,a}(u)\right)_{u=i/2}.
\end{aligned}$$

Using the definition (4.9) we can write

$$L_{n,a}\left(\frac{i}{2}\right) = iP_{n,a} \quad \text{and} \quad \frac{d}{du}L_{n,a}(u) = \mathbb{1}_{n,a}\ \forall\ u,$$

from which it follows that

$$\frac{d}{du}T_a(u)\bigg|_{u=i/2} = i^{N-1}\sum_{n=1}^{N} P_{N,a}\cdots \hat{P}_{n,a}\cdots P_{1,a},$$

where the hat $\hat{P}_{n,a}$ indicates that the term is missing from the sum. Through the properties (4.12), we can write:

$$\frac{d}{du}T_a(u)\bigg|_{u=i/2} = i^{N-1}\sum_{n=1}^{N} P_{12}P_{23}\cdots P_{n-1,n+1}\cdot P_{N-1,N}P_{N,a}.$$

Taking the partial trace over the auxiliary space yields

$$\frac{d}{du}t(u)\bigg|_{u=i/2} = i^{N-1}\sum_{n=1}^{N} P_{12}P_{23}\cdots P_{n-1,n+1}\cdots P_{N-1,N}.$$

Now, multiplying by $t^{-1}\left(\frac{i}{2}\right) = \frac{1}{i^N}P_{N-1,N}\cdots P_{23}P_{12}$[8]:

$$\left(t^{-1}(u)\frac{d}{du}t(u)\right)_{u=i/2} = -i\sum_{n=1}^{N}(P_{N-1,N}\cdots P_{n-1,n}P_{n-2,n-1}\cdots P_{12}) \\ \cdot(P_{12}\cdots P_{n-2,n-1}P_{n-1,n+1}\cdots P_{N-1,N}).$$

We see that the term on the left is the logarithmic derivative of $t(u)$ evaluated at $u = i/2$, and that the terms in the middle cancel up to

$$\begin{aligned}\left(\frac{d}{du}\log t(u)\right)_{u=i/2} &= -i\sum_{n=1}^{N} P_{N-1,N}\cdots P_{n,n+1}\underbrace{P_{n-1,n}P_{n-1,n+1}}_{=P_{n,n+1}P_{n-1,n}}\cdots P_{N-1,N}\\ &= -i\sum_{n=1}^{N} P_{N-1,N}\cdots \cancel{P_{n,n+1}}\cancel{P_{n,n+1}}P_{n-1,n}\cdots P_{N-1,N}\\ &= -i\sum_{n=1}^{N} P_{N-1,N}\cdots P_{n+1,n+2}P_{n-1,n}P_{n+1,n+2}\cdots P_{N-1,N},\end{aligned}$$

where we have once again used the properties (4.12) for the term in the braces. Since no other terms in the sum contain the subscript n or $n-1$, we can commute $P_{n-1,n}$ through, leaving the other terms to collapse in the middle and cancel. Thus, we are left with:

$$\left(\frac{d}{du}\log t(u)\right)_{u=i/2} = -i\sum_{n=1}^{N} P_{n-1,n} = -i\sum_{n=1}^{N} P_{n,n+1}.$$

In the last equality, we have used the periodicity of the spin chain $P_{N,N+1} = P_{N,1}$. We can now identify the Hamiltonian (4.6) with:

$$H = \frac{JL}{2}\mathbb{1} - i\frac{J}{2}\frac{d}{du}\log t(u)\bigg|_{u=i/2}.$$

---

[8] The inverse of $t\left(\frac{i}{2}\right)$ is easy to see simply by looking at expression (4.22).

What remains now is finding the energy eigenstates and eigenvalues of the Hamiltonian. In principle, instead of directly finding the eigenstates of the Hamiltonian, one finds the eigenstates of the transfer matrix instead. This is achieved by viewing the entries A, C, D of the Monodromy matrix in (4.18) as annihilation and creation operators acting on the spin chain, and identifying B as an operator that flips spins. Then, through using the RTT relation (4.19), one can derive explicit forms of eigenstates and eigenvalues of the transfer matrix. It is then a simple task to find the energy eigenvalues of the Hamiltonian, as it is simply the logarithmic derivative of the transfer matrix. Again, for the complete treatment we refer to the works [26, 27, 28].

In the above discussion, it was exactly the existence of a set of conserved charges (which includes the momentum and Hamiltonian) that makes the Heisenberg spin chain exactly solvable, or in other words quantum integrable. To this end, we have seen that both the RLL relation (4.10) and the YBE (4.15) were necessary in discovering this family of conserved charges underlying the Heisenberg spin chain. More generally, the YBE is known as the cornerstone of quantum integrability, and if a model has an associated R-matrix that solves the YBE it is said to be *quantum integrable* or *Yang-Baxter integrable*. In actuality, the statement that an R-matrix satisfies the YBE is an even stronger assertion, as it is always possible find an associated quantum integrable spin chain. We will derive this result in the following parts of this section after making some general comments about the R-matrix.

4.1.3 General Comments on the R-matrix

In the preceding section, we have introduced the R-matrix as an intermediary that acts on the auxiliary spaces in the RLL relation. However, it is also possible to view the R-matrix as an object that acts on the physical space if we identify the auxiliary space to be equivalent to the physical space. In the context of spin chains, both spaces are equivalent to $\mathbb{C}^2$. In such a situation, the Lax operator and the R-matrix are related to each other by a shift in the spectral parameter, as seen in (4.9) and (4.11). Since both the auxiliary and physical spaces are identical to $\mathbb{C}^2$, we define the R-matrix as an operator that acts on $\mathbb{C}^2 \otimes \mathbb{C}^2$, or in mathematical terms $R \in \mathrm{End}(\mathbb{C}^2 \otimes \mathbb{C}^2)$. From now on, we adapt the description that the R-matrix acts on the physical space, unless explicitly stated otherwise. With this definition of the R-matrix, the Yang-Baxter equation (4.15) reads

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}, \tag{4.24}$$

where the matrices R_{12}, R_{13} and R_{23} all act on $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ — these vector spaces are identified with the first three sites of the spin chain. The explicit forms of R_{12} and R_{23} are:

$$R_{12} = R \otimes \mathbb{1} \qquad\qquad R_{23} = \mathbb{1} \otimes R, \tag{4.25}$$

and the matrix R_{13} is viewed as a map

$$\begin{aligned} R_{13} : \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 &\to \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \\ |\phi_1\rangle \otimes |\phi_2\rangle \otimes |\phi_3\rangle &\mapsto |\psi_1\rangle \otimes |\phi_2\rangle \otimes |\psi_3\rangle\,, \end{aligned}$$

where $R(|\phi_1\rangle \otimes |\phi_3\rangle) = |\psi_1\rangle \otimes |\psi_3\rangle$. Or, equivalently we can use the permutation operator with the property (4.5) to write

$$R_{13} = P_{12}R_{23}P_{12}. \tag{4.26}$$

It is also important to note that there are two distinct types of R-matrices, these are:

- R-matrices of difference form: $R_{ij}(u_i, u_j) = R_{ij}(u_i - u_j)$,
- R-matrices of non-difference form: $R_{ij}(u_i, u_j) \neq R_{ij}(u_i - u_j)$.

The R-matrix in the Heisenberg spin chain is of difference form, as seen in the RLL relation (4.10) and (4.16). In future sections we also consider R-matrices of non-difference type.

4.2 From Spin Chains to Integrability

In what follows, we generalise the concept of the "tower of commuting charges" seen in the Heisenberg spin chain, laying the framework of identifying integrable models. To this end, we first answer the following question: *Given an R-matrix that solves the YBE, is it possible to find a corresponding quantum integrable spin chain?* To show that the answer is positive, we prove the following claim:

Claim. Let $R_{ij} \coloneqq R_{ij}(u_i, u_j)$ be an R-matrix of difference form[a] that satisfies the YBE

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}, \tag{4.27}$$

and at some point λ is equal to the permutation matrix

$$R(\lambda) = P. \tag{4.28}$$

The property (4.28) is called "regularity". Then, this R-matrix defines a quantum integrable spin chain with the nearest neighbour Hamiltonian

$$H = \sum_n \mathcal{H}_{n,n+1}, \qquad \mathcal{H}_{n,n+1} = \left[R^{-1}_{n,n+1} \frac{d}{du} R_{n,n+1} \right]_{u=\lambda},$$

where $\mathcal{H}_{n,n+1}$ is the nearest neighbour Hamiltonian density.

Proof. We emphasized in the Heisenberg spin chain that the existence of the RLL relation and the YBE was crucial in acquiring the family of conserved charges. Therefore, we wish to construct a Lax operator that is related to the R-matrix, from which it becomes evident that the Lax operator satisfies the RLL relation. Moreover, we have previously mentioned that in the case where the auxiliary space and physical space are equivalent, the Lax operator and the R-matrix are related by a shift in the spectral parameter. This is precisely also the case here, as they are both equivalent to $\mathbb{C}^2$. Then, we define the Lax operator as:

$$L_{n,a}(u) = R_{a,n}(u - \mu), \tag{4.29}$$

for some constant μ. Then, by the YBE (4.27)

$$R_{12}(u_1 - u_2)L_{31}(u_1)L_{32}(u_2) = L_{32}(u_2)L_{31}(u_1)R_{12}(u_1 - u_2),$$

by identifying $1 \mapsto a$, $2 \mapsto b$, and $3 \mapsto n$ we get

$$R_{ab}(u_1 - u_2)L_{n,a}(u_1)L_{n,b}(u_2) = L_{n,b}(u_2)L_{n,a}(u_1)R_{ab}(u_1 - u_2),$$

which is precisely the RLL relation (4.10). Next, we define the Monodromy and Transfer matrices as (4.17) and (4.20):

$$T_a(u) = L_{N,a}(u)\cdots L_{1,a}(u), \qquad t(u) = \operatorname{Tr}_a T_a(u).$$

Similar to the Heisenberg spin chain, the Monodromy matrix satisfies the RTT relation (4.19), and thus we recover the commutator (4.21):

$$[t(u), t(v)] = 0 \quad \forall\ u, v. \tag{4.30}$$

As we have seen, this is the exact commutation relation that defines the family of conserved charges. This was done by looking at the power series expansion of the natural logarithm of the transfer matrix around the point where the R-matrix reduces to permutation matrix. Then, let us define the point $\nu := \lambda + \mu$, at which

$$L_{n,a}(\nu) = R_{a,n}(\nu - \mu) = R_{a,n}(\lambda) = P_{a,n} = P_{n,a}.$$

From which, it follows that

$$\begin{aligned}
&T_a(\nu) = P_{N,a}\cdots P_{1,a} = P_{12}P_{23}\cdots P_{N-1,N}P_{N,a}\\
&\Rightarrow t(\nu) = \operatorname{Tr}_a T_a(\nu) = P_{12}\cdots P_{N-1,N} = \mathcal{U} = e^{i\mathbb{P}}\\
&\Rightarrow \mathbb{P} = -i\log t(\nu)
\end{aligned}$$

as before. To get the Hamiltonian, we again look at the first logarithmic derivative of the transfer matrix. We have that:

$$\begin{aligned}
\left.\frac{dT_a(u)}{du}\right|_{u=\nu} &= \sum_{n=1}^{N} P_{N,a}\cdots \underbrace{P_{n+1,a}\cdot \left.\frac{dL_{n,a}(u)}{du}\right|_{u=\nu}}_{\left.\frac{dL_{n,n+1}(u)}{du}\right|_{u=\nu}\cdot P_{n+1,a}} P_{n-1,a}\cdots P_{1,a}\\
&= \sum_{n=1}^{N} \left.\frac{dL_{n,n+1}(u)}{du}\right|_{u=\nu} \cdot P_{N,a}\cdots P_{n+1,a}\cdot P_{n-1,a}\cdots P_{1,a}\\
&= \sum_{n=1}^{N} \left.\frac{dL_{n,n+1}(u)}{du}\right|_{u=\nu} \cdot P_{12}\cdots P_{n-1,n+1}\cdots P_{N-1,N}P_{N,a},
\end{aligned}$$

where in the first brace we have used the property (4.5). Taking the partial trace over the auxiliary space yields

$$\left.\frac{dt(u)}{du}\right|_{u=\nu} = \sum_{n=1}^{N} \left.\frac{dL_{n,n+1}(u)}{du}\right|_{u=\nu} \cdot P_{12}\cdots P_{n-1,n+1}\cdots P_{N-1,N}.$$

Multiplying from the right by $t^{-1}(v) = P_{N-1,N} \cdots P_{12}$:

$$\begin{aligned} \left.\frac{d\log t(u)}{du}\right|_{u=\nu} &= \sum_{n=1}^{N} \left.\frac{dL_{n,n+1}(u)}{du}\right|_{u=\nu} \cdot \underbrace{P_{12} \cdots P_{n-1,n+1} \cdots P_{12}}_{=P_{n,n+1} \text{ as before}} \\ &= \sum_{n=1}^{N} \left.\frac{dL_{n,n+1}(u)}{du}\right|_{u=\nu} P_{n,n+1}, \end{aligned}$$

but from the property (4.5) we have that:

$$\left.\frac{dL_{n,n+1}(u)}{du}\right|_{u=\nu} P_{n,n+1} = P_{n,n+1} \left.\frac{dL_{n+1,n}(u)}{du}\right|_{u=\nu},$$

and finally recalling the regularity (4.28) and the relationship (4.29) between the Lax operator and the R-matrix:

$$\begin{aligned} P_{n,n+1} \left.\frac{dL_{n+1,n}(u)}{du}\right|_{u=\nu} &= R_{n,n+1}(\lambda) \left.\frac{dR_{n,n+1}(u-\mu)}{du}\right|_{u=\lambda+\mu} \\ &= R_{n,n+1}(\lambda) \left.\frac{dR_{n,n+1}(u)}{du}\right|_{u=\lambda}. \end{aligned}$$

Thus:

$$H := \left.\frac{d\log t(u)}{du}\right|_{u=\nu} = \sum_{n=1}^{N} \left[R_{n,n+1}(u)\frac{d}{du}R_{n,n+1}(u)\right]_{u=\lambda}.$$

Noting that

$$P_{n,n+1} = P^{-1}_{n,n+1} \Rightarrow R_{n.n+1}(\lambda) = R^{-1}_{n,n+1}(\lambda),$$

we get the desired result

$$H = \sum_{n=1}^{N} \mathcal{H}_{n,n+1} = \left[R^{-1}_{n,n+1}(u)\frac{d}{du}R_{n,n+1}(u)\right]_{u=\lambda}.$$

□

---

[a] We give the the proof for only the difference form. For the non-difference form, the proof goes similarly, with the regularity condition being changed to $R(\lambda,\lambda) = P$ and the Lax operator being given by $L_{n,a}(u,v) = R_{a,n}(u-\mu, v)$.

To summarize, if a regular[9] R-matrix of difference form satisfies the YBE, there is an associated quantum integrable spin chain with nearest neighbour Hamiltonian density[10]

$$\mathcal{H}_{n,n+1} = P_{n,n+1} \left.\frac{dR_{n,n+1}(u)}{du}\right|_{u=\lambda}. \tag{4.31}$$

---

[9] In this thesis, all R-matrices presented will be regular.

[10] The Hilbert space and the Hamiltonian density are both denoted $\mathcal{H}$, but it will be clear which one we refer to from the context.

Typically, the point λ is taken to be zero. Similarly, an R-matrix of non-difference form that satisfies the YBE defines a quantum integrable spin chain with Hamiltonian density

$$\mathcal{H}_{n,n+1} = P_{n,n+1} \left. \frac{dR_{n,n+1}(u,v)}{du} \right|_{u=v}, \tag{4.32}$$

where the regularity is now given by $R(u,u) = P$. This description of the Hamiltonian arises from the first logarithmic derivative of the transfer matrix. Therefore, to generalise the entire family of conserved charges, we define the following:

Definition 4.4. (Conserved Charges) Given a transfer matrix, its logarithmic derivative in series is

$$\log t(u,v) = \mathbb{Q}_1\left(\frac{u+v}{2}\right) + (u-v)\mathbb{Q}_2\left(\frac{u+v}{2}\right) + \frac{(u-v)^2}{2}\mathbb{Q}_3\left(\frac{u+v}{2}\right) + \mathcal{O}(u-v)^3. \tag{4.33}$$

The charges $\mathbb{Q}_i$ are called the conserved charges, and as a consequence of the commutation (4.30) they satisfy

$$[\mathbb{Q}_i(v), \mathbb{Q}_j(v)] = 0 \quad \forall\, i,j \in \mathbb{N}. \tag{4.34}$$

The existence of these conserved charges is of great importance, as this is what allows models such as the Heisenberg spin chain to be integrable. Moreover, from the above proof, we see that given an R-matrix satisfying the YBE, there is always a corresponding set of conserved charges. For this reason, if a system a model has an associated R-matrix satisfying the YBE, the system is said to be *quantum integrable.*

Consider again a spin chain of length L. The charges $\mathbb{Q}_i$ are local and have interaction range i. In other words, the index indicates how many sites of the spin chain the charge is acting on. For instance, the charge $\mathbb{Q}_2$ acts on two sites and $\mathbb{Q}_3$ acts on three. Both of these charges can be written as a sum of range-2 and range-3 densities, respectively:

$$\mathbb{Q}_2 = \sum_{n=1}^{L} \mathcal{D}_{n,n+1}, \qquad \mathbb{Q}_3 = \sum_{n=1}^{L} \mathcal{D}_{n,n+1,n+2} \tag{4.35}$$

where $\mathcal{D}$ are the densities. Since we will always work with periodic spin chains in this thesis, we impose the periodicity condition

$$\mathcal{D}_{L,L+1} = \mathcal{D}_{L,1},$$

on the densities. It should be noted that for a spin chain of length L, the index i should be strictly less than L. Taking $i \geq L$ results in an undesired wrapping effect [30]. Additionally, the charges $\mathbb{Q}_i$ are related to the transfer matrix by:

$$\mathbb{Q}_i(v) = \left. \frac{\partial^{i-1}}{\partial u^{i-1}} \log t(u,v) \right|_{u=v}, \tag{4.36}$$

which is evident seen from (4.33). Let us now comment on the first two charges. As in the Heisenberg spin chain, the first charge is associated with the momentum:

$$\mathbb{P} = -i \log t(v,v) = -i\mathbb{Q}_1,$$

and the second charge is associated with the Hamiltonian:

$$\mathbb{Q}_2 = H = \sum_{n=1}^{L} \mathcal{H}_{n,n+1}, \qquad \mathcal{H}_{L,L+1} = \mathcal{H}_{L,1}, \tag{4.37}$$

where the Hamiltonian densities $\mathcal{H}_{n,n+1}$ are given by (4.32). The charge $\mathbb{Q}_2$ is directly related to the transfer matrix by equation (4.36) with $i = 2$.

4.3 Classifying Integrable Systems

Even though quantum integrable models have been known for many years, the methods used in classifying them are relatively new. In this section, we will introduce *the Boost Automorphism Method* — a bottom-up approach in classifying quantum integrable systems.

4.3.1 The Boost Automorphism Method: A Brief Description

The Boost Automorphism Method was first applied to classifying integrable models with R-matrices of difference form in [16, 14], and was later generalised to non-difference form in [17, 15]. The idea behind the method is as follows: One starts from a general ansatz for the Hamiltonian, and aims to construct a corresponding R-matrix that solves the Yang-Baxter Equation. First, a general ansatz is made for the Hamiltonian. Since the Hamiltonian corresponds to the charge $\mathbb{Q}_2$ (4.37), one constructs the next conserved charge $\mathbb{Q}_3$ by the means of a "boost operator". Considering that the charges are required to commute for integrability, one imposes the commutation $[\mathbb{Q}_2, \mathbb{Q}_3] = 0$. This restricts the entries of the Hamiltonian. However, computing a large set of conserved charges through the boost operator is a difficult task. As a result, one aims to find the corresponding R-matrix instead and checks that it is a solution to the YBE. This ensures the integrability of the model. The method in more detailed steps is as follows:

1. Start from the general ansatz for the Hamiltonian density $\mathcal{H}$, which defines the Hamiltonian density acting on two sites of the spin chain. For example $\mathcal{H}_{12} = \mathcal{H} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}$. The Hamiltonian density depends on some functions $h_1(\theta), h_2(\theta), \ldots$

2. Construct the charge $\mathbb{Q}_2$ using the Hamiltonian

   $$H = \mathbb{Q}_2 = \sum_{n=1}^{L} \mathcal{H}_{n,n+1}.$$

   Following the Hamiltonian, generate the charge $\mathbb{Q}_3$ with the boost operator.

3. Impose the commutation $[\mathbb{Q}_2, \mathbb{Q}_3] = 0$. This makes the Hamiltonian potentially integrable.

4. Write down the expansion of the R-matrix in terms of the Hamiltonian density $\mathcal{H}$ in order to propose an ansatz for the R-matrix. Subsequently, use the Sutherland equation together with this ansatz to find the exact form of the R-matrix[11].

5. Check that the R-matrix satisfies the YBE.

6. Examine if the resulting R-matrix can be identified by an existing model or if it is new.

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[11] The Sutherland Equation will be defined in the next section. At this point, the reader should simply think of it as an expression that relates $\mathcal{H}_{12}$ and the R-matrix, restricting the entries of the R-matrix.

4.3.2 The Boost Operator and Sutherland Equations

Instead of constructing the Boost Operator by hand, we will directly start with its definition. For a detailed derivation, we refer to the works [31, 32]. The Boost Operator is defined as:

$$\mathcal{B}[\mathbb{Q}_2] \coloneqq \frac{\partial}{\partial\theta} + \sum_{k=-\infty}^{\infty} k\mathcal{H}_{k,k+1}, \tag{4.38}$$

and generates the next conserved charge through the commutator

$$\mathbb{Q}_{r+1} = [\mathcal{B}[\mathbb{Q}_2], \mathbb{Q}_r], \quad r \geq 1. \tag{4.39}$$

Let us now generate the charge $\mathbb{Q}_3$ using (4.39). Recalling that $\mathbb{Q}_2$ is given by (4.37), we have that

$$\begin{aligned}\mathbb{Q}_3 = [\mathcal{B}[\mathbb{Q}_2], \mathbb{Q}_2] &= \left[\frac{\partial}{\partial\theta} + \sum_{k=-\infty}^{\infty} k\mathcal{H}_{k,k+1}, \sum_{n=1}^{L}\mathcal{H}_{n,n+1}\right] \\ &= \frac{\partial}{\partial\theta}\mathbb{Q}_2 + \sum_{k=-\infty}^{\infty}\sum_{n=1}^{L} k\left[\mathcal{H}_{k,k+1}, \mathcal{H}_{n,n+1}\right].\end{aligned}$$

For $n = k$ and all $n \neq k \pm 1$, the commutator $[\mathcal{H}_{k,k+1}, \mathcal{H}_{n,n+1}]$ vanishes[12]. Thus, we are simply left with

$$\mathbb{Q}_3 = \frac{\partial}{\partial\theta}\mathbb{Q}_2 + \sum_{k=-\infty}^{\infty} k\left[\mathcal{H}_{k,k+1}, \mathcal{H}_{k-1,k} + \mathcal{H}_{k+1,k+2}\right].$$

The factor of k before the commutator is undesired, so we do the following: first, we split the sum as

$$\sum_{k=-\infty}^{\infty} k\left[\mathcal{H}_{k,k+1}, \mathcal{H}_{k-1,k}\right] + \sum_{k=-\infty}^{\infty} k\left[\mathcal{H}_{k,k+1}, \mathcal{H}_{k+1,k+2}\right].$$

Next, we introduce the shift $k \mapsto k-1$ in the second sum:

$$\Rightarrow \cancel{\sum_{k=-\infty}^{\infty} k\left[\mathcal{H}_{k,k+1}, \mathcal{H}_{k-1,k}\right]} + \cancel{\sum_{k=-\infty}^{\infty} k\left[\mathcal{H}_{k-1,k}, \mathcal{H}_{k,k+1}\right]} - \sum_{k=-\infty}^{\infty}\left[\mathcal{H}_{k-1,k}, \mathcal{H}_{k,k+1}\right].$$

This allows us to write $\mathbb{Q}_3$ as:

$$\mathbb{Q}_3 = \frac{\partial}{\partial\theta}\mathbb{Q}_2 - \sum_{k=-\infty}^{\infty}\left[\mathcal{H}_{k-1,k}, \mathcal{H}_{k,k+1}\right].$$

Once again introducing the shift $k \mapsto k+1$ and considering that the spin chain is of length L, we get:

$$\mathbb{Q}_3 = \frac{\partial}{\partial\theta}\mathbb{Q}_2 - \sum_{k=1}^{L}\left[\mathcal{H}_{k,k+1}, \mathcal{H}_{k+1,k+2}\right]. \tag{4.40}$$

We remark that equation (4.40) holds for systems with R-matrices of non-difference form. For the difference form, the first derivative term is absent. This is because difference form R-matrices correspond to constant conserved charges $\mathbb{Q}_i$.

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[12] In this case, the two Hamiltonians act on uncorrelated sites of the spin chain, and thus they commute.

We shall now derive the Sutherland equations – equations that relate the Hamiltonian density $\mathcal{H}_{12}$ and the R-matrix. In the next section, we will see with a concrete example that these equations result in an R-matrix that satisfies the YBE. The starting point of the derivation is the YBE (4.15) in non-difference form:

$$R_{12}(u_1,u_2)R_{13}(u_1,u_3)R_{23}(u_2,u_3) = R_{23}(u_2,u_3)R_{13}(u_1,u_3)R_{12}(u_1,u_2). \tag{4.41}$$

Derivating the YBE with respect to u_1 yields:

$$\begin{aligned}&\dot{R}_{12}(u_1,u_2)R_{13}(u_1,u_3)R_{23}(u_2,u_3) + R_{12}(u_1,u_2)\dot{R}_{13}(u_1,u_3)R_{23}(u_2,u_3)\\ &= R_{23}(u_2,u_3)\dot{R}_{13}(u_1,u_3)R_{12}(u_1,u_2) + R_{23}(u_2,u_3)R_{13}(u_1,u_3)\dot{R}_{12}(u_1,u_2),\end{aligned} \tag{4.42}$$

where we have defined $\dot{R}_{ij}(u_i,u_j) \coloneqq \frac{\partial}{\partial u_i}R_{ij}(u_i,u_j)$. We now take the limit $u_1 \to u_2$ of both sides, noting that

$$R_{ij}(u,u) = P_{ij}, \qquad \dot{R}_{ij}(u_i,u_j)\big|_{u_i\to u_j} = \frac{\partial}{\partial u_i}R_{ij}(u_i,u_j)\bigg|_{u_i\to u_j} = P_{ij}\mathcal{H}_{ij}(u_j),$$

where the second expression follows from (4.32). Then, in the limit $u_1 \to u_2$, the terms in (4.42) each become:

$$\begin{aligned}&R_{12}(u_1,u_2) \mapsto R_{12}(u_2,u_2) = P_{12}, && \dot{R}_{12}(u_1,u_3) \mapsto P_{12}\mathcal{H}_{12}(u_2),\\ &R_{13}(u_1,u_3) \mapsto R_{13}(u_2,u_3), && \dot{R}_{13}(u_1,u_3) \mapsto \dot{R}_{13}(u_2,u_3),\\ &R_{23}(u_2,u_3) \mapsto R_{23}(u_2,u_3).\end{aligned}$$

From which it follows that:

$$\begin{aligned}&P_{12}\mathcal{H}_{12}(u_2)R_{13}(u_2,u_3)R_{23}(u_2,u_3) + P_{12}\dot{R}_{13}(u_2,u_3)R_{23}(u_2,u_3)\\ &= R_{23}(u_2,u_3)\dot{R}_{13}(u_2,u_3)P_{12} + R_{23}(u_2,u_3)R_{13}(u_2,u_3)P_{12}\mathcal{H}_{12}(u_2).\end{aligned} \tag{4.43}$$

We now multiply by P_{12} from the left. Since $P_{12}^2 = \mathbb{1}_{12}$, the permutation operators on the left hand side cancel and the terms on the right hand side are equal to:

$$\begin{aligned}P_{12}R_{23}(u_2,u_3)\dot{R}_{13}(u_2,u_3)P_{12} &= [P_{12}R_{23}(u_2,u_3)P_{12}][P_{12}\dot{R}_{13}(u_2,u_3)P_{12}]\\ &= R_{13}(u_2,u_3)\dot{R}_{23}(u_2,u_3)\\ P_{12}R_{23}(u_2,u_3)R_{13}(u_2,u_3)P_{12}\mathcal{H}_{12}(u_2) &= R_{13}(u_2,u_3)R_{23}(u_2,u_3)\mathcal{H}_{12}(u_2),\end{aligned}$$

where we have made use of the property (4.5). Defining $R_{ij} \coloneqq R_{ij}(u,v)$, with $u \coloneqq u_2$, $v \coloneqq u_3$, equation (4.43) becomes:

$$\mathcal{H}_{12}(u)R_{13}R_{23} + \dot{R}_{13}R_{23} = R_{13}\dot{R}_{23} + R_{13}R_{23}\mathcal{H}_{12}(u).$$

This equation can simply be rearranged into the more compact form:

$$[R_{13}R_{23}, \mathcal{H}_{12}(u)] = \dot{R}_{13}R_{23} - R_{13}\dot{R}_{23}, \tag{4.44}$$

which is the first Sutherland equation. It is also possible to write the second Sutherland equation by derivating the YBE (4.42) by u_3, and then taking the limit $u_2 \to u_3$. This procedure results in:

$$[R_{13}R_{12}, \mathcal{H}_{23}(v)] = R_{13}R'_{12} - R'_{13}R_{12}, \tag{4.45}$$

where $R'_{ij} \coloneqq \frac{\partial}{\partial u_j}R_{ij}(u_i,u_j)$. The derivation is very similar[13], so we present it in Appendix B instead.

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[13] For the second Sutherland equation, the definitions of R_{ij}, u and v are different.

4.4 Detailed Example

Together with the boost operator and the Sutherland equations, we are now ready to demonstrate the Boost Automorphism Method with a detailed example. The example is from the works of Marius de Leeuw and Chiara Paletta [1, 17], and we reproduce their result and give the extra steps in between. All computations regarding matrices in this example have been done in Python using the Numpy and Sympy libraries. The code used can be found attached in Appendix A.

Step 1. Ansatz for the Hamiltonian density
We start with the Hamiltonian density which acts on two neighbouring sites $\mathbb{C}^2 \otimes \mathbb{C}^2$. The ansatz is:

$$\mathcal{H}(\theta) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_1(\theta) & h_3(\theta) & 0 \\ 0 & h_4(\theta) & h_2(\theta) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.46}$$

In this example, we work with a spin chain of length $L = 4$. Thus, we have

$$\mathcal{H}_{12} = \mathcal{H} \otimes \mathbb{1} \otimes \mathbb{1}, \quad \mathcal{H}_{23} = \mathbb{1} \otimes \mathcal{H} \otimes \mathbb{1}, \quad \mathcal{H}_{34} = \mathbb{1} \otimes \mathbb{1} \otimes \mathcal{H} \tag{4.47}$$

$$\mathcal{H}_{41} = P_{13} P_{24} \mathcal{H}_{23} P_{24} P_{13}, \tag{4.48}$$

where for $\mathcal{H}_{41}$ we have made repeated use of the property (4.5). The Hamiltonian densities in (4.47) are easily computed, which are then summed to get the total Hamiltonian:

$$H = \mathbb{Q}_2 = \sum_{n=1}^{4} \mathcal{H}_{n,n+1}, \quad \mathcal{H}_{45} = \mathcal{H}_{41}.$$

Since the Hamiltonian acts on $\bigotimes_{n=1}^{4} \mathbb{C}^2$, it is a 16 × 16 matrix, so we do not present its full form here.

Step 2. Constructing $\mathbb{Q}_3$
Using the Boost operator, the charge $\mathbb{Q}_3$ is constructed by (4.40):

$$\mathbb{Q}_3 = \frac{\partial}{\partial \theta} \mathbb{Q}_2 - \sum_{k=1}^{4} [\mathcal{H}_{k,k+1}, \mathcal{H}_{k+1,k+2}].$$

Similar to the Hamiltonian, we do not present the explicit form of $\mathbb{Q}_3$ as it is a 16 × 16 matrix. However, we can still look at the individual densities that make up $\mathbb{Q}_3$, which are given in (4.35):

$$\mathbb{Q}_3 = \sum_{k=1}^{4} \mathcal{D}_{k,k+1,k+2} = \sum_{k=1}^{4} \frac{\partial}{\partial \theta} \mathcal{H}_{k,k+1} - [\mathcal{H}_{k,k+1}, \mathcal{H}_{k+1,k+2}].$$

Setting each term of the sums being equal and defining $\mathcal{D}_{123} := \mathcal{Q}_{123} \otimes \mathbb{1}$, we get that:

$$\mathcal{D}_{123} = \partial_\theta \mathcal{H}_{12} - [\mathcal{H}_{12}, \mathcal{H}_{23}]$$
$$\Rightarrow \mathcal{Q}_{123} = \frac{\partial}{\partial \theta} (\mathcal{H} \otimes \mathbb{1}) - [\mathcal{H} \otimes \mathbb{1}, \mathbb{1} \otimes \mathcal{H}].$$

Using the ansatz (4.46), the density $\mathcal{Q}_{123}$ is found to be:

$$Q_{123} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & h_1 h_3 & 0 & h_3^2 & 0 & 0 & 0 \\ 0 & -h_1 h_4 & \dot{h}_1 & 0 & \dot{h}_3 + h_2 h_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dot{h}_1 & 0 & \dot{h}_3 - h_1 h_3 & -h_3^2 & 0 \\ 0 & -h_4^2 & \dot{h}_4 - h_2 h_4 & 0 & \dot{h}_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dot{h}_4 + h_1 h_4 & 0 & \dot{h}_2 & -h_2 h_3 & 0 \\ 0 & 0 & 0 & h_4^2 & 0 & h_2 h_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where we have suppressed the dependence on the parameter θ and have defined the shorthand notation $\dot{h}_i := \frac{\partial h_i}{\partial \theta}$. The density $\mathcal{Q}_{123}$ is only one of the four range three densities that make up $\mathbb{Q}_3$.

Step 3. Imposing the integrability constraint

Following the construction of $\mathbb{Q}_3$, we impose the commutation

$$[\mathbb{Q}_2, \mathbb{Q}_3] = 0$$

in order for our Hamiltonian to be (potentially) integrable. This constraint gives the following set of ordinary differential equations:

$$h_3(\dot{h}_1 + \dot{h}_2) = \dot{h}_3(h_1 + h_2), \qquad \frac{\dot{h}_3}{h_3} = \frac{\dot{h}_4}{h_4},$$

which can be written as:

$$h_3(\dot{h}_1 + \dot{h}_2) = \dot{h}_3(h_1 + h_2), \qquad h_4(\dot{h}_1 + \dot{h}_2) = \dot{h}_4(h_1 + h_2).$$

These are solved by:

$$\begin{aligned} \frac{\dot{h}_3}{h_3} = \frac{\dot{h}_1 + \dot{h}_2}{h_1 + h_2} \Rightarrow \frac{d \log h_3}{d\theta} &= \frac{d \log(h_1 + h_2)}{d\theta} \\ \Rightarrow \log h_3 &= \log(h_1 + h_2) + C, && C \in \mathbb{C} \\ \Rightarrow h_3 &= (h_1 + h_2) e^C = \frac{c_3}{2}(h_1 + h_2), && c_3 := 2e^C, \end{aligned}$$

and similarly

$$\frac{\dot{h}_4}{h_4} = \frac{\dot{h}_1 + \dot{h}_2}{h_1 + h_2} \Rightarrow h_4 = \frac{c_4}{2}(h_1 + h_2),$$

for some constant c_4. Substituting these back into the Hamiltonian density (4.46), we get:

$$\mathcal{H} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_1 & \frac{c_3}{2}(h_1 + h_2) & 0 \\ 0 & \frac{c_4}{2}(h_1 + h_2) & h_2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.49}$$

Step 4. Ansatz for the R-matrix and the Sutherland equation

The next step is to make an ansatz for the R-matrix. Since the R-matrix acts on $\mathbb{C}^2 \otimes \mathbb{C}^2$, we could make a general ansatz with all 16 entries, but we can already put constraints on some of the entries. To this end, we look at the expansion of the R-matrix around $u = v$ up to second order:

$$R_{12}(u,v) = P_{12}\left(\mathbb{1}_{12} + (u-v)\mathcal{H}_{12}\left(\frac{u-v}{2}\right) + \frac{(u-v)^2}{2}\mathcal{H}_{12}^2\left(\frac{u-v}{2}\right) + \mathcal{O}(u-v)^3\right). \tag{4.50}$$

Note that this expansion agrees with (4.32) and the regularity condition $R_{12}(u,u) = P_{12}$. Recalling that R_{12} acts on $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$, which we identify with the first three sites of our spin chain, we now view H_{12} as also an object that acts on the first three sites. Explicitly, we have that

$$R_{12} = R \otimes \mathbb{1}, \quad \mathcal{H}_{12} = \mathcal{H} \otimes \mathbb{1}, \quad P_{12} = P \otimes \mathbb{1}.$$

Substituting this and the Hamiltonian density (4.49) into the expansion yields shows that the non-zero entries are:

$$\begin{aligned} R &= P\left(\mathbb{1} + (u-v)\mathcal{H}\left(\frac{u-v}{2}\right) + \frac{(u-v)^2}{2}\mathcal{H}^2\left(\frac{u-v}{2}\right) + \mathcal{O}(u-v)^3\right) \\ &= \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & * & 0 \\ 0 & * & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix}, \end{aligned}$$

with the entries in positions $(1,1)$ and $(4,4)$ being equal. Thus, the ansatz for the R-matrix is:

$$R(u,v) = \begin{pmatrix} r_1 & 0 & 0 & 0 \\ 0 & r_2 & r_3 & 0 \\ 0 & r_4 & r_5 & 0 \\ 0 & 0 & 0 & r_1 \end{pmatrix}, \tag{4.51}$$

with $r_i := r_i(u,v)$. Recalling the regularity condition $R(u,u) = P$ and the relationship between R and $\mathcal{H}$ (4.32), we get the following boundary conditions for r_i:

$$r_1(u,u) = 1, \qquad r_2(u,u) = 0, \qquad r_3(u,u) = 1, \tag{4.52}$$

$$r_4(u,u) = 1, \qquad r_5(u,u) = 0, \tag{4.53}$$

$$\dot r_1(u,v) = 0, \qquad \dot r_2(u,v) = \frac{c_4}{2}(h_1(u) + h_2(u)), \qquad \dot r_3(u,v) = h_2(u), \tag{4.54}$$

$$\dot r_4(u,v) = h_1(u), \qquad \dot r_5(u,v) = \frac{c_3}{2}(h_1(u) + h_2(u)), \tag{4.55}$$

where $\dot r_i(u,u) := \frac{\partial}{\partial u} r(u,v)\big|_{v=u}$. Substituting $\mathcal{H}$ and R into the second Sutherland equation (4.44) gives the following set of partial differential equations:

$$\begin{aligned} &\frac{\dot r_2}{r_2} = \frac{\dot r_5}{r_5}, && c_3 r_2 = c_4 r_5, \\ &\frac{\dot r_1}{r_1} = \frac{\dot r_5}{r_5} - (h_1 + h_2)\left(\frac{c_4 r_1}{2 r_2} + \frac{c_4 r_5}{2 r_1} + 1\right), && \frac{\dot r_3}{r_3} = \frac{2\dot r_5 r_2 - c_4 r_1 r_5 (h_1 + h_2) - 2 h_1 r_2 r_5}{2 r_2 r_5}, \end{aligned}$$

$$\frac{\dot{r}_4}{r_4}=\frac{2\dot{r}_5r_2-c_4r_1r_5(h_1+h_2)-2h_2r_2r_5}{2r_2r_5},\qquad r_3=\frac{c_4r_1^2+c_4r_2r_5+2r_1r_2}{c_4r_4},$$

These can be cast into:

$$c_3r_2=c_4r_5,\qquad \frac{\dot{r}_2}{r_2}=\frac{\dot{r}_5}{r_5},\tag{4.56}$$

$$\frac{\dot{r}_2}{r_2}=\frac{\dot{r}_3}{r_3}+h_1+\frac{c_3}{2}(h_1+h_2)\frac{r_1}{r_5},\qquad \frac{\dot{r}_3}{r_3}=\frac{\dot{r}_1}{r_1}+h_2+\frac{c_3}{2}(h_1+h_2)\frac{r_2}{r_1},\tag{4.57}$$

$$\frac{\dot{r}_4}{r_4}=\frac{\dot{r}_3}{r_3}+h_1-h_2,\qquad \frac{c_3}{2}\left[\frac{r_3r_4}{r_1r_5}-\frac{r_1}{r_5}-\frac{r_2}{r_1}\right]=1.\tag{4.58}$$

We give the explicit steps in Appendix C.1. Let us now solve these by brute force. From the fifth equation, we derive that:

$$\begin{aligned}\frac{\dot{r}_4}{r_4}=\frac{\dot{r}_3}{r_3}+h_1-h_2 &\Rightarrow \frac{\partial\log r_4(u,v)}{\partial u}=\frac{\partial\log r_3(u,v)}{\partial u}+h_1(u)-h_2(u)\\ &\Rightarrow \int_v^u\frac{\partial\log r_4(\theta,v)}{\partial\theta}d\theta=\int_v^u\frac{\partial\log r_3(\theta,v)}{\partial\theta}d\theta+\underbrace{\int_v^u h_1(\theta)-h_2(\theta)d\theta}_{=2H_-}\\ &\Rightarrow \log r_4(u,v)-\underbrace{\log r_4(v,v)}_{=0}=\log r_3(u,v)-\underbrace{\log r_3(v,v)}_{=0}+2H_-\\ &\Rightarrow r_4(u,v)=r_3(u,v)e^{2H_-},\end{aligned}\tag{4.59}$$

where in the third line we have used the boundary conditions (4.52) and have defined $H_\pm=\frac{1}{2}\int_v^u h_1(\theta)\pm h_2(\theta)d\theta$. The first and second equations in (4.56) already relate r_2 and r_5 to each other, so we are left the remaining three equations:

$$\frac{\dot{r}_2}{r_2}=\frac{\dot{r}_3}{r_3}+h_1+\frac{c_3}{2}(h_1+h_2)\frac{r_1}{r_5},\qquad \frac{\dot{r}_3}{r_3}=\frac{\dot{r}_1}{r_1}+h_2+\frac{c_3}{2}(h_1+h_2)\frac{r_2}{r_1},\tag{4.60}$$

$$\frac{c_3}{2}\left[\frac{r_3r_4}{r_1r_5}-\frac{r_1}{r_5}-\frac{r_2}{r_1}\right]=1.\tag{4.61}$$

To solve these, we introduce $\tilde{r}_1$ and $\tilde{r}_2$ to redefine r_1 and r_2 as:

$$r_1\coloneqq r_3\left(\tilde{r}_1-\frac{\tilde{r}_2}{c_4}\right),\qquad r_2\coloneqq r_3\tilde{r}_2.\tag{4.62}$$

Note that the boundary conditions (4.52) imply that $\tilde{r}_1(u,u)=1$ and $\tilde{r}_2(u,u)=0$. With this identification, the last equation becomes:

$$\frac{c_3}{2}\left[\frac{r_4}{r_5\left(\tilde{r}_1-\frac{\tilde{r}_2}{c_4}\right)}-\frac{r_3}{r_5}\left(\tilde{r}_1-\frac{\tilde{r}_2}{c_4}\right)-\frac{\tilde{r}_2}{\left(\tilde{r}_1-\frac{\tilde{r}_2}{c_4}\right)}\right]=1.$$

Multiplying both sides by $\left(\tilde{r}_1-\frac{\tilde{r}_2}{c_4}\right)$, and noting that $r_5=\frac{c_3}{c_4}r_2=\frac{c_3}{c_4}r_3\tilde{r}_2$:

$$\frac{c_3}{2}\left[\frac{c_4r_4}{c_3r_3\tilde{r}_2}-\frac{c_4}{c_3}\left(\tilde{r}_1-\frac{\tilde{r}_2}{c_4}\right)^2-\tilde{r}_2\right]=\left(\tilde{r}_1-\frac{\tilde{r}_2}{c_4}\right).$$

Then, substituting (4.59) and expanding:

$$\frac{c_4 e^{2H_-}}{2\tilde{r}_2} - \frac{c_4}{2\tilde{r}_2}\left(\tilde{r}_1^2 - \frac{2\tilde{r}_1\tilde{r}_2}{c_4} + \frac{\tilde{r}_2^2}{c_4^2}\right) - \frac{c_3\tilde{r}_2}{2} = \left(\tilde{r}_1 - \frac{\tilde{r}_2}{c_4}\right).$$

Finally, multiplying both sides by $2c_4\tilde{r}_2$ we get that:

$$\begin{aligned} c_4^2 e^{2H_-} - c_4^2\tilde{r}_1^2 + \cancel{2c_4\tilde{r}_1\tilde{r}_2} - \tilde{r}_2^2 - c_3c_4\tilde{r}_2^2 &= \cancel{2c_4\tilde{r}_1\tilde{r}_2} - 2\tilde{r}_2^2 \\ \Rightarrow c_4 e^{2H_-} &= c_4^2\tilde{r}_1^2 + (c_3c_4 - 1)\tilde{r}_2^2 \\ &= c_4^2\tilde{r}_1^2 + \omega^2\tilde{r}_2^2, \end{aligned}$$

where $\omega \coloneqq (c_3c_4 - 1)$. This equation can be solved by using polar coordinates:

$$\tilde{r}_1 = e^{H_-}\cos\phi, \qquad \tilde{r}_2 = e^{H_-}\frac{c_4}{\omega}\sin\phi, \tag{4.63}$$

for some function $\phi = \phi(u,v)$ that satisfies the boundary condition $\phi(u,u) = 0$. Substituting the polar equivalent of $\tilde{r}_1$ and $\tilde{r}_2$ in (4.62), and then the first (or second) equation in (4.60) yields the condition:

$$\frac{\partial\phi(u,v)}{\partial u} = \omega\left(\frac{h_1(u) + h_2(u)}{2}\right). \tag{4.64}$$

Once again, we give the explicit steps of deriving this in Appendix C.2. Integrating both sides from v to u, we get that ϕ is simply given by:

$$\phi(u,v) = \omega H_+.$$

Substituting this result back into the polar forms and then into r_1 and r_2, we end up with the solution:

$$\begin{aligned} r_1 &= r_3 e^{H_-}\left(\cos\omega H_+ - \frac{1}{\omega}\sin\omega H_+\right), & r_2 &= r_3 e^{H_-}\frac{c_4}{\omega}\sin\omega H_+, \\ r_4 &= r_3 e^{2H_-}, & r_5 &= \frac{c_3 r_2}{c_4} = r_3 e^{H_-}\frac{c_3}{\omega}\sin\omega H_+. \end{aligned}$$

Therefore, our R-matrix in (4.51) is given by:

$$R(u,v) = r_3 e^{H_-}\begin{pmatrix} \cos\omega H_+ - \frac{1}{\omega}\sin\omega H_+ & 0 & 0 & 0 \\ 0 & \frac{c_4}{\omega}\sin\omega H_+ & e^{-H_-} & 0 \\ 0 & e^{H_-} & \frac{c_3}{\omega}\sin\omega H_+ & 0 \\ 0 & 0 & 0 & \cos\omega H_+ - \frac{1}{\omega}\sin\omega H_+ \end{pmatrix}.$$

Keeping the boundary conditions (4.52) in mind, we are allowed to choose the overall normalization factor r_3 to be[14]:

$$r_3 = e^{H_+ - H_-} = e^{\int_v^u h_2(\theta)d\theta}.$$

Thus, the integrable R-matrix corresponding to the Hamiltonian density (4.46) is:

$$R(u,v) = e^{H_+}\begin{pmatrix} \cos\omega H_+ - \frac{1}{\omega}\sin\omega H_+ & 0 & 0 & 0 \\ 0 & \frac{c_4}{\omega}\sin\omega H_+ & e^{-H_-} & 0 \\ 0 & e^{H_-} & \frac{c_3}{\omega}\sin\omega H_+ & 0 \\ 0 & 0 & 0 & \cos\omega H_+ - \frac{1}{\omega}\sin\omega H_+ \end{pmatrix}. \tag{4.65}$$

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[14] It is clear that this choice does not affect the Yang-Baxter equation.

It should be noted that due to the integrals $H_\pm$ present, the R-matrix is of non-difference form.

Step 5. Check that the R-matrix solves the YBE
It is easily checked that the R-matrix (4.65) solves the YBE (4.41).

Step 6. Does the Hamiltonian and R-matrix correspond to a new integrable model?
Using transformations that preserve the integrability of the model, it is possible to map the Hamiltonian density (4.46) to the Hamiltonian density of the XXZ spin chain[15]. We will not discuss this identification, as it is very involved and goes beyond the scope of this thesis. For a detailed explanation, we once again refer to the works [1, 17]. In conclusion, the model and the R-matrix discussed in this example is **not** new, and corresponds to the XXZ spin chain.

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[15] The XXZ spin chain is similar to the Heisenberg Spin Chain, with the key difference being that the interaction strength J_z being different for the term $S^z_n S^z_{n+1}$ in the Hamiltonian (4.2). The Heisenberg spin chain considers the case $J = J_x = J_y = J_z$.

5 Integrable Open Quantum Systems

Throughout the preceding sections, we have treated the dynamics of Open Quantum Systems and Quantum Integrable Systems separately. This section aims to connect the two fields, and introduce the reader to *Integrable Open Quantum Systems*. It was previously emphasised that integrability is a delicate property, as adding interactions term to an integrable Hamiltonian usually breaks this attribute. Despite this intricacy, in recent years *Integrable Open Quantum Systems* and methods in classifying them have been constructed. While some research groups have investigated the existence of such systems in the past [33, 34], the works of Chiara Paletta and Marius de Leeuw in [1, 2] detail **the first systematic approach** in classifying integrable open quantum systems. We present the reader with an explanation on the classification of integrable open quantum systems utilising the boost automorphism method introduced in Section 4. The arguments and the Model A1 investigated in this chapter are based on Paletta's PhD thesis [1] and the review letter [2].

5.1 The Lindblad Superoperator

In order to classify integrable open quantum systems using the boost automorphism method, we need to view the Lindblad superoperator $\mathcal{L}$ in the Lindblad equation (3.3) as a Hamiltonian density acting on neighbouring sites of a spin chain.

5.1.1 Vectorizing the Density Matrix

We begin with the Lindblad equation (3.3) with only one family of jump operators:

$$\mathcal{L}\rho_S = -i[H, \rho_S] + \left(L\rho_S L^\dagger - \frac{1}{2}\{L^\dagger L, \rho_S\} \right), \tag{5.1}$$

where $\hbar$ has now been dropped, and the coupling strengths γ_k have been absorbed into the jump operators. The reason for considering only one family is that we take each jump operator as acting on nearest neighbours of a spin chain in the following sub-sections. To formally view $\mathcal{L}$ as a superoperator, we need an operator-state correspondence. This is achieved by "vectorizing" the density matrix: effectively converting the density matrix (operator) to a vector (state). The density matrix ρ_S was defined as a bounded operator that acts on the Hilbert space of the system[16] $\mathcal{H}$. Recalling the definition 3.1 of the Liouville space, the previous statement is equivalent to stating that the density matrix is an element of the Liouville space. In fact, the Liouville space $\mathcal{B}(\mathcal{H})$ is equivalent (isometrically isomorphic) to the tensor product of $\mathcal{H}$ and its dual $\mathcal{H}^*$ [35], from which, it follows that:

$$\rho_S \in \mathcal{B}(\mathcal{H}) \equiv \mathcal{H} \otimes \mathcal{H}^*. \tag{5.2}$$

If the 'kets' (column vectors) belong to $\mathcal{H}$, the dual space $\mathcal{H}^*$ is where the 'bras' (the row vectors) live. With this identification of the density matrix, it is effectively regarded as a vector in the tensor product space $\mathcal{H} \otimes \mathcal{H}^*$. The vectorization of the density matrix should be thought of as taking the row vectors of the matrix, and placing them as column vectors on top of each other. We denote the vectorized density matrix by $|\rho_S\rangle\rangle$. For instance, in the case of a 2×2

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[16] The subscript S for the Hilbert space of the system is dropped.

density matrix:

$$\rho_S = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \to \begin{pmatrix} \rho_{11} \\ \rho_{12} \\ \rho_{21} \\ \rho_{22} \end{pmatrix} = |\rho_S\rangle\!\rangle$$

The next step is to rewrite the Lindblad equation (5.1) using the vectorized form of the density matrix. Consequently, the Lindblad superoperator $\mathcal{L}$ is viewed as a matrix that acts on the space of density matrices. To this end, we utilise the following lemma [36]:

Lemma. For two matrices A and B, of size $m \times n$ and $n \times k$ respectively, their vectorized product is given by:

$$|AB\rangle\!\rangle = (A \otimes \mathbb{1}_k)|B\rangle\!\rangle = (\mathbb{1}_m \otimes B^T)|A\rangle\!\rangle. \tag{5.3}$$

For three matrices A, B and C, of size $m \times n$, $n \times k$ and $k \times l$ respectively, the vectorized triple product is:

$$|ABC\rangle\!\rangle = (A \otimes C^T)|B\rangle\!\rangle. \tag{5.4}$$

Expanding the Lindblad equation (5.1):

$$\mathcal{L}\rho_S = -iH\rho_S + i\rho_S H + \left(L\rho_S L^\dagger - \frac{1}{2}L^\dagger L\rho_S - \frac{1}{2}\rho_S L^\dagger L\right),$$

and using the lemma above for each term:

$$\begin{aligned} H\rho_S &\to (H \otimes \mathbb{1})|\rho_S\rangle\!\rangle \\ \rho_S H &\to (\mathbb{1} \otimes H)|\rho_S\rangle\!\rangle \\ L\rho_S L^\dagger &\to (L \otimes (L^\dagger)^T)|\rho_S\rangle\!\rangle = (L \otimes L^*)|\rho_S\rangle\!\rangle = (L \otimes \mathbb{1})(\mathbb{1} \otimes L^*)|\rho_S\rangle\!\rangle \\ (L^\dagger L)\rho_S &\to (L^\dagger L \otimes \mathbb{1})|\rho_S\rangle\!\rangle = (L^\dagger \otimes \mathbb{1})(L \otimes \mathbb{1})|\rho_S\rangle\!\rangle \\ \rho_S(L^\dagger L) &\to (\mathbb{1} \otimes (L^\dagger L)^T)|\rho_S\rangle\!\rangle = (\mathbb{1} \otimes L^T L^*)|\rho_S\rangle\!\rangle = (\mathbb{1} \otimes L^T)(\mathbb{1} \otimes L^*)|\rho_S\rangle\!\rangle. \end{aligned}$$

Thus, the Lindblad equation becomes:

$$\mathcal{L}|\rho_S\rangle\!\rangle = \left[-iH^{(1)} + iH^{T^{(2)}} + \left(L^{(1)}L^{*(2)} - \frac{1}{2}L^{\dagger^{(1)}}L^{(2)} - \frac{1}{2}L^{T^{(2)}}L^{*(2)}\right)\right]|\rho_S\rangle\!\rangle, \tag{5.5}$$

where we have defined $A^{(1)} := A \otimes \mathbb{1}$ and $B^{(2)} := \mathbb{1} \otimes B$, for any $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{H}^*)$. From (5.5) it follows that the Lindblad superoperator $\mathcal{L}$ is given by:

$$\mathcal{L} = -iH^{(1)} + iH^{T^{(2)}} + \left(L^{(1)}L^{*(2)} - \frac{1}{2}L^{\dagger^{(1)}}L^{(2)} - \frac{1}{2}L^{T^{(2)}}L^{*(2)}\right). \tag{5.6}$$

We note that $\mathcal{L} \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H}^*)$.

5.1.2 The Lindblad Superoperator as a Nearest-Neighbour Hamiltonian

So far we have worked with the general Lindblad equation and have not adapted it to spin chains. To this end, we consider H and L in the above as bounded operator densities acting on nearest neighbours of a spin chain. Following the notation in [1], we denote these densities by $H_{j,j+1}$ and $L_{j,j+1}$. For a spin chain of length L, the total hilbert space is $\mathcal{H} = \bigotimes_{n=1}^{L} V = \bigotimes_{n=1}^{L} \mathbb{C}^2$, with $V \equiv \mathbb{C}^2$. Then, the densities $H_{j,j+1}$ and $L_{j,j+1}$ act on the nearest neighbour local Hilbert space $V \otimes V$:

$$H_{j,j+1} \in \mathcal{B}(V \otimes V), \qquad L_{j,j+1} \in \mathcal{B}(V \otimes V).$$

Consider the vectorized density matrix $|\rho_S\rangle\rangle_{j,j+1}$ of nearest-neighbours on the spin chain. The local Hilbert space is $V \otimes V$. By the identification (5.2) we have that:

$$|\rho_S\rangle\rangle_{j,j+1} \in (V \otimes V) \otimes (V \otimes V)^* = V \otimes V \otimes V^* \otimes V^*.$$

The Lindblad superoperator $\mathcal{L}_{j,j+1}$ is viewed as a matrix that acts on this density matrix, and defines its time derivative. In other words, it is a bounded operator acting on the space of density matrices:

$$\mathcal{L}_{j,j+1} \in \mathcal{B}(V \otimes V \otimes V^* \otimes V^*).$$

As a result of this identification, (5.6) becomes:

$$\mathcal{L}_{j,j+1} = -iH^{(1)}_{j,j+1} + iH^{T(2)}_{j,j+1} + \left(L^{(1)}_{j,j+1} L^{*(2)}_{j,j+1} - \frac{1}{2} L^{\dagger(2)}_{j,j+1} L^{(2)}_{j,j+1} - \frac{1}{2} L^{T(2)}_{j,j+1} L^{*(2)}_{j,j+1} \right), \tag{5.7}$$

where now the notation $A^{(1)}$ and $A^{(2)}$ explicitly signify[17]:

$$A^{(1)} = A \otimes \mathbb{1} \otimes \mathbb{1}, \qquad A^{(2)} = \mathbb{1} \otimes \mathbb{1} \otimes A.$$

We note that the superscript (1) means that the operator acts on $V \otimes V$, and leaves the dual $V^* \otimes V^*$ invariant; while the superscript (2) implies the opposite. To finally be able to interpret $\mathcal{L}_{j,j+1}$ as acting on neighbouring sites of the spin chain, we have to switch the two middle spaces of the Liouville Space. This is explicitly done by

$$V \otimes V \otimes V^* \otimes V^* \to V \otimes V^* \otimes V \otimes V^*,$$

and is allowed because the two are isomorphic to each other. With this, the identification in which $\mathcal{L}_{j,j+1}$ acts on neighbouring sites becomes more transparent:

$$\mathcal{L} \in \mathcal{B}(\underbrace{V \otimes V^*}_{\text{site } j} \otimes \underbrace{V \otimes V^*}_{\text{site } j+1}). \tag{5.8}$$

To make sense of the terms in (5.7), following the notations in [1, 32, 34], we denote the matrices acting on V by σ, and the matrices acting on V^* by τ:

$$\underbrace{V}_{\sigma_j} \otimes \underbrace{V^*}_{\tau_j} \otimes \underbrace{V}_{\sigma_{j+1}} \otimes \underbrace{V^*}_{\tau_{j+1}}. \tag{5.9}$$

Now, the notation $A^{(1)}$ signifies operators only containing σs, and $A^{(2)}$ only containing τs. To summarize, the expression (5.7) together with the identification (5.8) defines a Lindblad superoperator that acts on neighbouring sites of a spin chain. The spin chain has local Hilbert space $V \otimes V^*$ for each site, and the total Hilbert space is:

$$\mathcal{H} = \bigotimes_{n=1}^{L} (V \otimes V^*). \tag{5.10}$$

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[17] From this point onwards, the identity $\mathbb{1}$ denotes the 2×2 identity matrix.

We interpret the Lindblad superoperator $\mathcal{L}_{j,j+1}$ as a (non-Hermitian) Hamiltonian acting on neighbouring sites. The total Lindblad superoperator $\mathcal{L}$ is the sum of all such densities[18]

$$\mathcal{L} = \sum_{j=1}^{L} \mathcal{L}_{j,j+1}, \qquad \mathcal{L}_{L,L+1} = \mathcal{L}_{L,1}, \tag{5.11}$$

and acts on the entire Hilbert space (5.10). It should be noted that the total Hilbert space has dimension 2^{L^2}, as the dimension of each local Hilbert space has been squared compared to (4.1). As a result, the matrix form of $\mathcal{L}$ is $2^{L^2} \times 2^{L^2}$ and computing it is computationally expensive for large L. Similar to the Hamiltonian density in (4.31), $\mathcal{L}$ is integrable if it can be written as the logarithmic derivative of an R-matrix that satisfies the YBE.

5.1.3 Consequences on the R-matrix

Before turning to classifying integrable Lindblad superoperators, we will discuss the consequences of the identification (5.10) on the R-matrix. We had previously mentioned that the R-matrix acts on two sites of the spin chain, which was given by $\mathbb{C}^2 \otimes \mathbb{C}^2$. However, together with (5.10), each site of the spin chain now becomes $V \otimes V^*$. This implies that the R-matrix must now act on $V \otimes V^* \otimes V \otimes V^*$. It follows that the R-matrix is a 16×16 matrix, as $V \equiv \mathbb{C}^2$. Recalling that the matrices R_{12}, R_{23} and R_{13} in the YBE (4.41) all act on the first three sites $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$, they now become:

$$R_{12} = R \otimes \mathbb{1} \otimes \mathbb{1}, \qquad R_{23} = \mathbb{1} \otimes \mathbb{1} \otimes R, \qquad P_{13} = P_{12} R_{23} P_{12}, \tag{5.12}$$

where the permutation operator P that permutes two sites also differs from its original definition. It is now defined by the map:

$$\begin{aligned} P : V \otimes V^* \otimes V \otimes V^* &\to V \otimes V^* \otimes V \otimes V^* \\ |a\rangle \otimes |b\rangle \otimes |c\rangle \otimes |d\rangle &\mapsto |c\rangle \otimes |d\rangle \otimes |a\rangle \otimes |b\rangle \end{aligned}$$

The explicit matrix form is found by acting on the 16 basis vectors of $V \otimes V^* \otimes V \otimes V^* \cong \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$, and is given by:

$$P = \begin{pmatrix} 1&&&&&&&&&&&&&&& \\ &&&&1&&&&&&&&&&& \\ &&&&&&&&1&&&&&&& \\ &&&&&&&&&&&&1&&& \\ &1&&&&&&&&&&&&&& \\ &&&&&1&&&&&&&&&& \\ &&&&&&&&&1&&&&&& \\ &&&&&&&&&&&&&1&& \\ &&1&&&&&&&&&&&&& \\ &&&&&&1&&&&&&&&& \\ &&&&&&&&&&1&&&&& \\ &&&&&&&&&&&&&&1& \\ &&&1&&&&&&&&&&&& \\ &&&&&&&1&&&&&&&& \\ &&&&&&&&&&&1&&&& \\ &&&&&&&&&&&&&&&1 \end{pmatrix}. \tag{5.13}$$

Following this, the permutation operator P_{12} in (5.12) is related to P by $P_{12} = P \otimes \mathbb{1} \otimes \mathbb{1}$.

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[18] In the expression for $\mathcal{L}$, the summand $\mathcal{L}_{j,j+1}$ should not be interpreted as acting on neighbouring sites, but rather the entire Hilbert space. To achieve this, we tensor product $\mathcal{L}_{j,j+1}$ with the identity for the sites it acts on trivially.

5.2 Discovering Integrable Lindblad Superoperators

Now, we have constructed the necessary framework to find and classify integrable open quantum systems. We once again remark that the classification is from the works of Marius de Leeuw and Chiara Paletta [1, 2], and details **the first systematic approach in classifying** integrable open quantum systems. The method is equivalent to the boost automorphism method in section 4.3.1, but now the charge $\mathbb{Q}_2$ is associated with the superoperator $\mathcal{L}$:

$$\mathbb{Q}_2 = \mathcal{L} = \sum_{j=1}^{L} \mathcal{L}_{j,j+1}, \qquad \mathcal{L}_{L,L+1} = \mathcal{L}_{1,L}. \tag{5.14}$$

It should be noted that $\mathcal{L}_{j,j+1}$ is not hermitian by construction, even though it is associated with the Hamiltonian density. The hermitian operator describing the system S and its interaction with the environment is H in (5.7). The detailed steps of the identification are as follows:

1. Start with the ansatz for the densities $H_{j,j+1}$ and $L_{j,j+1}$. Since they both act on $V \otimes V^*$, they are given by 4×4 matrices. The density $H_{j,j+1}$ depends on some functions $h_1(\theta), h_2(\theta), \dots$; while $L_{j,j+1}$ depends on some functions $l_1(\theta), l_2(\theta), \dots$
2. Using the ansatz, construct the Hamiltonian density $\mathcal{L}_{j,j+1}$ through (5.7), and subsequently use (5.14) to construct $\mathbb{Q}_2$. Then, replace $\mathcal{H}_{k,k+1} \mapsto \mathcal{L}_{k,k+1}$ in the boost operator (4.40) and generate $\mathbb{Q}_3$.
3. Impose the commutation $[\mathbb{Q}_2, \mathbb{Q}_3] = 0$ for the Lindblad superoperator to be potentially integrable.
4. In the expansion (4.50), replace the Hamiltonian density $\mathcal{H}_{12} \mapsto \mathcal{L}_{12}$ to create an ansatz for the R-matrix. Solve the Sutherland equation (4.44) after making the same substitution to determine the precise form of the R-matrix.
5. Check that the R-matrix solves the YBE.
6. Examine if the R-matrix is new.

5.3 Model A1

Following the above description, we examine the classification of an integrable open quantum system in detail. The system investigated is referred to as Model A1, and is presented in the works [1, 2]. Similar to the detailed example in section 4.4, we work with a spin chain of length $L = 4$. All computations regarding matrices in this example have been done in Python using the Numpy and Sympy libraries. The code used can be found attached in Appendix A.

Step 1. Ansatz for the densities $H_{j,j+1}$ and $L_{j,j+1}$
We start with the ansatz for the densities $H_{j,j+1}$ and $L_{j,j+1}$, which both act on $V \otimes V$:

$$H_{j,j+1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & h_1 & 0 \\ 0 & h_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = h_1 \sigma_j^+ \sigma_{j+1}^- + h_2 \sigma_j^- \sigma_{j+1}^+, \tag{5.15}$$

$$L_{j,j+1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & \ell_1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \sigma_j^- \sigma_{j+1}^+ + \frac{\ell_1}{4}(\mathbb{1} - \sigma^z)_j (\mathbb{1} + \sigma^z)_{j+1}, \tag{5.16}$$

where h_1, h_2, ℓ_1 are functions of θ, $\sigma^\pm$ and σ^z are the Pauli raising, lowering and z matrices, respectively. The subscript denotes which site the matrices act on. Currently, we have not initiated the identification (5.8), and our Hilbert space still remains as:

$$\mathcal{H} = V \otimes V \otimes V \otimes V.$$

Then, for instance, the term $\sigma_j^+ \sigma_{j+1}^-$ in our ansatz means:

$$\sigma_j^+ \sigma_{j+1}^- = \sigma^+ \otimes \sigma^-.$$

We remark that this notation excludes the tensor products with the identity $\mathbb{1}$ for the sites that are left invariant. We consider these products only when referring to the indices with numbers. Conversely, we view the object as a density if it includes the index j instead. For example, we say:

$$H_{12} = h_1(\sigma^+ \otimes \sigma^- \otimes \mathbb{1} \otimes \mathbb{1}) + h_2(\sigma^- \otimes \sigma^+ \otimes \mathbb{1} \otimes \mathbb{1}).$$

considering that our spin chain is of length 4. Commencing the identification (5.10), the total Hilbert space becomes:

$$\mathcal{H} = \underbrace{V \otimes V^*}_{\text{site 1}} \otimes \underbrace{V \otimes V^*}_{\text{site 2}} \otimes \underbrace{V \otimes V^*}_{\text{site 3}} \otimes \underbrace{V \otimes V^*}_{\text{site 4}}, \tag{5.17}$$

from which we can construct the Lindblad superoperator using (5.7). Explicitly, some of the terms in (5.7) are given by:

$$\begin{aligned} H_{12}^{(1)} &= h_1(\underbrace{\sigma^+ \otimes \mathbb{1}}_{1} \otimes \underbrace{\sigma^- \otimes \mathbb{1}}_{2} \otimes \underbrace{\mathbb{1} \otimes \mathbb{1}}_{3} \otimes \underbrace{\mathbb{1} \otimes \mathbb{1}}_{4}) \\ &\quad + h_2(\underbrace{\sigma^- \otimes \mathbb{1}}_{1} \otimes \underbrace{\sigma^+ \otimes \mathbb{1}}_{2} \otimes \underbrace{\mathbb{1} \otimes \mathbb{1}}_{3} \otimes \underbrace{\mathbb{1} \otimes \mathbb{1}}_{4}) = h_1 \sigma_1^+ \sigma_2^- + h_2 \sigma_1^- \sigma_2^+ \\ H_{12}^{(2)} &= h_1(\underbrace{\mathbb{1} \otimes \sigma^+}_{1} \otimes \underbrace{\mathbb{1} \otimes \sigma^-}_{2} \otimes \underbrace{\mathbb{1} \otimes \mathbb{1}}_{3} \otimes \underbrace{\mathbb{1} \otimes \mathbb{1}}_{4}) \\ &\quad + h_2(\underbrace{\mathbb{1} \otimes \sigma^-}_{1} \otimes \underbrace{\mathbb{1} \otimes \sigma^+}_{2} \otimes \underbrace{\mathbb{1} \otimes \mathbb{1}}_{3} \otimes \underbrace{\mathbb{1} \otimes \mathbb{1}}_{4}) = h_1 \tau_1^+ \tau_2^- + h_2 \tau_1^- \tau_2^+, \end{aligned}$$

where we have made use of the notation (5.9):

$$\sigma_1^a = \sigma^a \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1},$$

$$\tau_1^a = \mathbb{1} \otimes \sigma^a \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1},$$

with $a = \pm, z$. We recall that σ denotes operators only acting on V, and τ denotes operators only acting on V^*; while the subscript signifies which site is acted on. This allows us to write our ansatz (5.15) as:

$$H_{j,j+1}^{(1)} = h_1 \sigma_j^+ \sigma_{j+1}^- + h_2 \sigma_j^- \sigma_{j+1}^+, \qquad L_{j,j+1}^{(1)} = \sigma_j^- \sigma_{j+1}^+ + \frac{\ell_1}{4}(\mathbb{1} - \sigma^z)_j (\mathbb{1} + \sigma^z)_{j+1},$$

$$H^{(2)}_{j,j+1} = h_1 \tau^+_j \tau^-_{j+1} + h_2 \tau^-_j \tau^+_{j+1}, \qquad L^{(2)}_{j,j+1} = \tau^-_j \tau^+_{j+1} + \frac{\ell_1}{4}(\mathbb{1} - \tau^z)_j (\mathbb{1} + \tau^z)_{j+1}.$$

Substituting these into (5.7) and simplifying yields:

$$\begin{aligned}\mathcal{L}_{j,j+1} =& -i(h_1 \sigma^+_j \sigma^-_{j+1} + h_2 \sigma^-_j \sigma^+_{j+1}) + i(h_1 \tau^+_j \tau^-_{j+1} + h_2 \tau^-_j \tau^+_{j+1}) \\ &+ \left[\sigma^-_j \sigma^+_{j+1} + \frac{\ell_1}{4}(\mathbb{1} - \sigma^z)_j (\mathbb{1} + \sigma^z)_{j+1}\right] \left[\tau^-_j \tau^+_{j+1} + \frac{\overline{\ell_1}}{4}(\mathbb{1} - \tau^z)_j (\mathbb{1} + \tau^z)_{j+1}\right] \\ &- \frac{1}{2}\left[\ell_1 \sigma^+_j \sigma^-_{j+1} + \overline{\ell_1} \sigma^-_j \sigma^+_{j+1} + \frac{|\ell_1|^2}{4}(\mathbb{1} - \sigma^z)_j (\mathbb{1} + \sigma^z)_{j+1} + \frac{1}{4}(\mathbb{1} + \sigma^z)_j (\mathbb{1} - \sigma^z)_{j+1}\right] \\ &- \frac{1}{2}\left[\overline{\ell_1} \tau^+_j \tau^-_{j+1} + \ell_1 \tau^-_j \tau^+_{j+1} + \frac{|\ell_1|^2}{4}(\mathbb{1} - \tau^z)_j (\mathbb{1} + \tau^z)_{j+1} + \frac{1}{4}(\mathbb{1} + \tau^z)_j (\mathbb{1} - \tau^z)_{j+1}\right],\end{aligned} \tag{5.18}$$

where $\overline{\ell_1}$ is the complex conjugate of ℓ_1. The Lindblad superoperator (5.18) defines a Hamiltonian density acting on the neighbouring sites $V \otimes V^* \otimes V \otimes V^*$. Thus, the charge $\mathbb{Q}_2$ is simply[19]:

$$\mathbb{Q}_2 = \sum_{j=1}^{4} \mathcal{L}_{j,j+1}, \qquad \mathcal{L}_{4,5} = \mathcal{L}_{4,1}.$$

We remark that the charge $\mathbb{Q}_2$ is a 256×256 matrix, as the total Hilbert space is given by (5.17).

Step 2. Constructing $\mathbb{Q}_3$
Using the Boost operator (4.40), the charge $\mathbb{Q}_3$ is constructed:

$$\mathbb{Q}_3 = \frac{\partial}{\partial \theta} \mathbb{Q}_2 - \sum_{j=1}^{4} [\mathcal{L}_{j,j+1}, \mathcal{L}_{j+1,j+2}].$$

Evidently, the charge $\mathbb{Q}_3$ is also a 256×256 matrix. Thus, we do not give its explicit form.

Step 3. Imposing the integrability constraint We now impose the commutation

$$[\mathbb{Q}_2, \mathbb{Q}_3] = 0,$$

so that our superoperator $\mathbb{L}$ is potentially integrable. Solving the resultant ordinary differential equations, the result is found to be:

$$\dot{\ell}_1 = \dot{h}_1 = \dot{h}_2 = 0, \qquad |\ell_1|^2 = 1, \qquad h_1 = \frac{i}{2}\ell_1, \qquad h_2 = \frac{i}{2}\frac{1}{\ell_1}.$$

Thus, the coefficients ℓ_1, h_1 and h_2 are all constants. Since $|\ell_1|^2 = 1$, we are allowed to choose

$$\ell_1 = -ie^{i\phi},$$

for some $\phi \in \mathbb{R}$. The second and third equations then tell us:

$$h_1 = \frac{1}{2}e^{i\phi}, \qquad h_2 = -\frac{1}{2}e^{-i\phi}.$$

---

[19] In this sum, the superoperators $\mathcal{L}_{j,j+1}$ should not be viewed as densities. Rather, they are the full 256×256 matrices.

Therefore, our ansatz (5.15) becomes:

$$H_{j,j+1} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & e^{i\phi} & 0 \\ 0 & e^{-i\phi} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad L_{j,j+1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -ie^{i\phi} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We notice that the Hamiltonian $H_{j,j+1}$ is hermitian, even though we did not impose it from the beginning. We now plug the expressions for ℓ_1, h_1 and h_2 into (5.18). This makes the Lindblad superoperator potentially integrable.

Step 4. Ansatz for the R-matrix and the Sutherland equation

The next step is to find an R-matrix that solves the YBE, and is associated with the superoperator $\mathcal{L}$. First, we note that the conserved charge $\mathbb{Q}_2$ is constant, as the parameters ℓ_1, h_1 and h_2 were all found to be constant. Since constant conserved charges correspond to R-matrices of difference form, we have that our R-matrix must also be of difference form. To make an ansatz, we consider again the expansion (4.50) replacing the Hamiltonian density $\mathcal{H}_{12} \mapsto \mathcal{L}_{12}$. Considering which entries are zero and fixing the entries that are the same, we make the ansatz:

$$R(u) = \begin{pmatrix}
r_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & r_3 & 0 & 0 & r_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & r_7 & 0 & 0 & 0 & 0 & 0 & r_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & r_8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & r_1 & 0 & 0 & 0 \\
0 & r_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & r_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & r_5 & 0 & 0 & 0 & 0 & 0 & r_4 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & r_7 & 0 & 0 & 0 & 0 & 0 & r_2 & 0 & 0 \\
0 & 0 & r_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & r_6 & 0 & 0 & r_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & r_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & r_3 & 0 & 0 & r_2 & 0 \\
0 & 0 & 0 & r_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & r_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & r_2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & r_1
\end{pmatrix} \tag{5.19}$$

where $r_i := r_i(u)$. This R-matrix must satisfy the boundary conditions[20]

$$R(0) = P, \qquad \dot{R}_{12}(u)\big|_{u=0} := \frac{d}{du} R_{12}(u)\Big|_{u=0} = P_{12}\mathcal{L}_{12}.$$

---

[20] The superoperator $\mathcal{L}_{12}$ should be taken as a density acting on the first three sites for the matrices to be compatible. This simply means substituting $j = 1$ in (5.18) and taking, for example, $\sigma_1^a = \sigma^a \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}$, $\tau_1^a = \mathbb{1} \otimes \sigma^a \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}$.

with the permutation matrix P given in (5.13). It follows that the entries r_i satisfy the boundary conditions:

$$r_1(0) = 1, \qquad r_2(0) = 1, \qquad r_3(0) = 0, \qquad r_4(0) = 1, \tag{5.20}$$

$$r_5(0) = 0, \qquad r_6(0) = 0, \qquad r_7(0) = 0, \qquad r_8(0) = 0, \tag{5.21}$$

$$\dot{r}_1(0) = 0, \qquad \dot{r}_2(0) = -\frac{1}{2}, \qquad \dot{r}_3(0) = ie^{i\phi}, \qquad \dot{r}_4(0) = -1, \tag{5.22}$$

$$\dot{r}_5(0) = -ie^{-i\phi}, \qquad \dot{r}_6(0) = ie^{i\phi}, \qquad \dot{r}_7(0) = -ie^{-i\phi}, \qquad \dot{r}_8(0) = 1, \tag{5.23}$$

where $\dot{r}_i(0) := \frac{dr_i(u)}{du}\Big|_{u=0}$. Keeping these boundary conditions in mind, we are allowed to choose the normalisation $r_1(u) = 1$. In order to get the expressions for the remaining r_i, we need to solve the Sutherland equation (4.44). Replacing the Hamiltonian density $\mathcal{H}_{12} \mapsto \mathcal{L}_{12}$ in (4.44) and adapting it to the difference form yields:

$$[R_{13}R_{23}, \mathcal{L}_{12}(u)] = \dot{R}_{13}R_{23} - R_{13}\dot{R}_{23},$$

where $R_{ij} := R_{ij}(u)$ and $\dot{R}_{ij} := \frac{d}{du}R_{ij}(u)$. Substituting $\mathcal{L}_{12}$ and R into the Sutherland equations yields the following set of ordinary differential equations:

$$r_1 = 1, \qquad \dot{r}_2 = -\frac{r_2}{2}, \qquad \dot{r}_3 = ie^{i\phi} - r_3, \tag{5.24}$$

$$\dot{r}_4 = -r_4, \qquad \dot{r}_5 r_7 - \dot{r}_7 r_5 + \frac{1}{2} r_5 r_7 = 0, \qquad -\dot{r}_3 r_6 + \dot{r}_6 r_3 + \frac{1}{2} r_3 r_6 = 0, \tag{5.25}$$

$$\dot{r}_7 = -r_7 - ie^{-i\phi}, \qquad r_8 = 1 - r_4. \tag{5.26}$$

Imposing the boundary conditions (5.20), these are solved by:

$$r_1 = 1, \qquad r_2 = e^{-u/2}, \qquad r_3 = ie^{i\phi}(1 - e^{-u}), \tag{5.27}$$

$$r_4 = e^{-u}, \qquad r_5 = ie^{-u/2}e^{-i\phi}(e^{-u} - 1), \qquad r_6 = ie^{-u/2}e^{i\phi}(1 - e^{-u}), \tag{5.28}$$

$$r_7 = ie^{-i\phi}(e^{-u} - 1), \qquad r_8 = 1 - e^{-u}. \tag{5.29}$$

We give the explicit steps in Appendix D. Thus, we find that the R-matrix associated with $\mathcal{L}$ is:

$$R(u) =$$

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & ie^{i\phi}(1-e^{-u}) & 0 & 0 & e^{-u/2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & ie^{-i\phi}(e^{-u}-1) & 0 & 0 & 0 & 0 & 0 & e^{-u/2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1-e^{-u} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & e^{-u/2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & ie^{-u/2}e^{-i\phi}(e^{-u}-1) & 0 & 0 & 0 & 0 & 0 & e^{-u} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & ie^{-i\phi}(e^{-u}-1) & 0 & 0 & 0 & 0 & 0 & e^{-u/2} & 0 & 0 \\
0 & 0 & e^{-u/2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & ie^{-u/2}e^{i\phi}(1-e^{-u}) & 0 & 0 & e^{-u} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & ie^{i\phi}(1-e^{-u}) & 0 & 0 & e^{-u/2} & 0 \\
0 & 0 & 0 & e^{-u} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-u/2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-u/2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} \tag{5.30}$$

Step 5. Check that the R-matrix satisfies the YBE

We need to check that the R-matrix (5.30) solves the YBE of difference form (4.16):

$$R_{12}(u_1 - u_2)R_{13}(u_1 - u_3)R_{23}(u_2 - u_3) = R_{23}(u_2 - u_3)R_{13}(u_1 - u_3)R_{12}(u_1 - u_2)$$

To write the YBE in simpler terms, we define $u \coloneqq u_1 - u_2$, $v \coloneqq u_2 - u_3$. This yields:

$$R_{12}(u)R_{13}(u + v)R_{23}(v) = R_{23}(v)R_{13}(u + v)R_{12}(u).$$

It was confirmed in Python that the R-matrix (5.30) satisfies the above YBE.

Step 6. Examine if the found R-matrix is new

Paletta discusses that the R-matrix (5.30) seems to be new, as it can not be successfully mapped onto a known R-matrix by using transformations that preserve the integrability of $\mathcal{L}$ [1]. Precisely, the model is thought to detail the first description of the *Totally Asymmetric Simple Exclusion Process* (TASEP) through an integrable Lindblad superoperator [2]. The TASEP is a Markovian stochastic process in which particles diffuse on a one-dimensional lattice only in one direction (hence asymmetric), and are subject to an exclusion process – i.e. at most one particle can occupy a site [37, 38].

6 Conclusion

Throughout this thesis, we have successfully presented the classification of integrable open quantum systems using the boost automorphism method. This approach was first realised by Chiara Paletta and Marius de Leeuw, and details the first systematic method in classifying integrable open quantum systems [1, 2].

In order to achieve the classification of integrable open quantum systems, first, the theory of *Open Quantum Systems* was discussed in detail. This included key concepts such as the density matrix formalism and the partial trace. Along with the mathematical formulation of open quantum systems, their dynamics were also discussed in detail. From considering the most general way of mapping a density matrix onto a density matrix, it was seen that completely positive trace preserving (CPTP) maps were necessary – from which, *the Lindblad equation* was readily derived. The Lindblad equation details the time evolution of an open quantum system, and is equipped with the Lindblad superoperator $\mathcal{L}$ – the generator of the dynamics of open quantum systems.

We then turned our attention to non-interacting models in Section 4, specifically *Quantum Integrable Spin Chains*. In this section, the representative example of the Heisenberg Spin Chain was used in introducing the *Algebraic Bethe Ansatz* framework, whereby the key ingredients such as the Lax Operator, R-matrix, Yang-Baxter Equation (YBE), Monodromy and Transfer matrices were established. In fact, it was shown that given an R-matrix that satisfies the Yang-Baxter equation, it is always possible to find an associated quantum integrable spin chain – resulting in the YBE becoming the cornerstone of quantum integrability. It was emphasised that quantum integrable systems possess the key property of having a family of commuting charges, denoted $\mathbb{Q}_i$, which allows them to be exactly solvable. The family of conserved charges includes the Hamiltonian – typically associated with the second charge $\mathbb{Q}_2$.

Following the mathematical framework of quantum integrable spin chains, classifying such systems were discussed. To this end, *the Boost Automorphism Method* was introduced [14, 15, 16, 17]. The approach was initiated with the ansatz for the Hamiltonian $\mathbb{Q}_2$, and subsequently utilised a boost operator to construct the next charge $\mathbb{Q}_3$. Then, imposing the commutation $[\mathbb{Q}_2, \mathbb{Q}_3] = 0$ restricted the entries of the Hamiltonian, effectively making it potentially integrable. Next, the an ansatz for the associated R-matrix was constructed by the means of an expansion. The exact expressions for the entries of the R-matrix were determined by the Sutherland equations , and was confirmed to satisfy the YBE. To demonstrate this method, a detailed example was presented in Section 4.4.

In the final section *Integrable Open Quantum Systems*, all notions in the preceding sections were connected, leading to the classification of integrable open quantum systems. This was possible by viewing the Lindblad superoperator $\mathcal{L}$ as a (non-hermitian) Hamiltonian density acting on neighbouring sites of a spin chain. Together with this identification of $\mathcal{L}$, the charge $\mathbb{Q}_2$ was associated with the Lindblad superoperator. The boost automorphism method was once again applied to construct $\mathbb{Q}_3$, impose $[\mathbb{Q}_2, \mathbb{Q}_3] = 0$, and find the associated R-matrix which solved the YBE. Once again, all of the steps were carried out in detail for a specific example – referred to as Model A1. The resultant R-matrix (5.30), which Paletta identifies to be **new**, corresponds to the first realisation of the *Totally Asymmetric Simple Exclusion Process (TASEP)* with an integrable Lindblad superoperator [1, 2]. However, recently discovered integrable open quantum systems are not only limited to Model A1, there also exist other instances of integrable Lindblad superoperators such as the Hubbard model with imaginary coupling [1, 2].

Along with the TASEP identification of Model A1, we leave the classification of such models as future work. As a final remark, it is import to state that in all models, the commutation $[\mathbb{Q}_2, \mathbb{Q}_3] = 0$ sufficed in finding a corresponding R-matrix, ensuring integrability. According to Paletta [1], this relates to an old conjecture [39] which remains unproven, and as an open question.

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Appendix

A Python Code

The Python code used in solving both the detailed example in section 4.4 and Model A1 can be found in the following:
`https://colab.research.google.com/drive/1RJlbgGYUxhS7qfdDvv9EAd2iMKDe2rtM?usp=sharing`

B Derivation of the Second Sutherland Equation

Here, we derive the second Sutherland equation (4.45).

Derivation. The starting point of the derivation is the YBE (4.15) in non-difference form:

$$R_{12}(u_1,u_2)R_{13}(u_1,u_3)R_{23}(u_2,u_3) = R_{23}(u_2,u_3)R_{13}(u_1,u_3)R_{12}(u_1,u_2). \tag{.1}$$

Derivating the YBE with respect to u_3 yields:

$$\begin{aligned} &R_{12}(u_1,u_2)R'_{13}(u_1,u_3)R_{23}(u_2,u_3) + R_{12}(u_1,u_2)R_{13}(u_1,u_3)R'_{23}(u_2,u_3) \\ &= R'_{23}(u_2,u_3)R_{13}(u_1,u_3)R_{12}(u_1,u_2) + R_{23}(u_2,u_3)R'_{13}(u_1,u_3)R_{12}(u_1,u_2), \end{aligned} \tag{.2}$$

where we have defined $R'_{ij}(u_i,u_j) := \frac{\partial}{\partial u_j}R_{ij}(u_i,u_j)$. We now take the limit $u_2 \to u_3$ of both sides, noting that

$$R_{ij}(u,u) = P_{ij}, \qquad R'_{ij}(u_i,u_j)\big|_{u_i\to u_j} = \frac{\partial}{\partial u_j}R_{ij}(u_i,u_j)\bigg|_{u_i\to u_j} = -P_{ij}\mathcal{H}_{ij}(u_j),$$

where the second expression follows from (4.32). There is an extra minus sign present as the derivative is taken with respect to the second spectral parameter. Then, in the limit $u_2 \to u_3$, the terms in (.2) each become:

$$\begin{aligned} R_{12}(u_1,u_2) &\mapsto R_{12}(u_1,u_3) & R'_{23}(u_2,u_3) &\mapsto -P_{23}\mathcal{H}_{23}(u_3) \\ R_{13}(u_1,u_3) &\mapsto R_{13}(u_1,u_3) & R'_{13}(u_1,u_3) &\mapsto R'_{13}(u_1,u_3) \\ R_{13}(u_2,u_3) &\mapsto R_{23}(u_3,u_3) = P_{23}. \end{aligned}$$

From which it follows that:

$$\begin{aligned} &R_{12}(u_1,u_3)R'_{13}(u_1,u_3)P_{23} - R_{12}(u_1,u_3)R_{13}(u_1,u_3)P_{23}\mathcal{H}_{23}(u_3) \\ &= -P_{23}\mathcal{H}_{23}(u_3)R_{13}(u_1,u_3)R_{12}(u_1,u_3) + P_{23}R'_{13}(u_1,u_3)R_{12}(u_1,u_2). \end{aligned} \tag{.3}$$

We now multiply by P_{23} from the left. Since $P_{23}^2 = \mathbb{1}_{23}$, the permutation operators on the right hand side cancel and the terms on the left hand side are equal to:

$$\begin{aligned} P_{23}R_{12}(u_1,u_3)R'_{13}(u_1,u_3)P_{23} &= [P_{23}R_{12}(u_1,u_3)P_{23}][P_{23}R'_{13}(u_1,u_3)P_{23}] \\ &= R_{13}(u_1,u_3)R'_{12}(u_1,u_3) \\ P_{23}R_{12}(u_1,u_3)R_{13}(u_1,u_3)P_{23}\mathcal{H}_{23}(u_3) &= R_{13}(u_1,u_3)R_{12}(u_1,u_3)\mathcal{H}_{23}(u_3), \end{aligned}$$

where we have made use of the property (4.5). Defining $R_{ij} := R_{ij}(u,v)$, with $u \equiv u_i$, $v \equiv u_3$, equation (4.43) becomes:

$$R_{13}R'_{12} - R_{13}R_{12}\mathcal{H}_{23}(v) = -\mathcal{H}_{23}(v)R_{13}R_{12} + R'_{13}R_{12}.$$

This equation can simply be rearranged into the more compact form:

$$[R_{13}R_{12}, \mathcal{H}_{23}(v)] = R_{13}R'_{12} - R'_{13}R_{12}. \tag{.4}$$

This is the second Sutherland equation (4.45).

C Example 4.4

This section refers to the appendix of 4.4 - Detailed Example.

C.1 Equivalence of Equations

We now show that the set of equations:

$$\begin{aligned}
&\frac{\dot r_2}{r_2} = \frac{\dot r_5}{r_5}, && c_3 r_2 = c_4 r_5, && (.5)\\
&\frac{\dot r_1}{r_1} = \frac{\dot r_5}{r_5} - (h_1 + h_2)\left(\frac{c_4 r_1}{2r_2} + \frac{c_4 r_5}{2r_1} + 1\right), && \frac{\dot r_3}{r_3} = \frac{2\dot r_5 r_2 - c_4 r_1 r_5 (h_1 + h_2) - 2h_1 r_2 r_5}{2r_2 r_5}, && (.6)\\
&\frac{\dot r_4}{r_4} = \frac{2\dot r_5 r_2 - c_4 r_1 r_5 (h_1 + h_2) - 2h_2 r_2 r_5}{2r_2 r_5}, && r_3 = \frac{c_4 r_1^2 + c_4 r_2 r_5 + 2r_1 r_2}{c_4 r_4}, && (.7)
\end{aligned}$$

are equivalent to (4.56):

$$\begin{aligned}
&c_3 r_2 = c_4 r_5, && \frac{\dot r_2}{r_2} = \frac{\dot r_5}{r_5}, && (.8)\\
&\frac{\dot r_2}{r_2} = \frac{\dot r_3}{r_3} + h_1 + \frac{c_3}{2}(h_1 + h_2)\frac{r_1}{r_5}, && \frac{\dot r_3}{r_3} = \frac{\dot r_1}{r_1} + h_2 + \frac{c_3}{2}(h_1 + h_2)\frac{r_2}{r_1}, && (.9)\\
&\frac{\dot r_4}{r_4} = \frac{\dot r_3}{r_3} + h_1 - h_2, && \frac{c_3}{2}\left[\frac{r_3 r_4}{r_1 r_5} - \frac{r_1}{r_5} - \frac{r_2}{r_1}\right] = 1. && (.10)
\end{aligned}$$

Proof. We see that the equalities

$$\boxed{c_3 r_2 = c_4 r_5 \quad \text{and} \quad \frac{\dot r_2}{r_2} = \frac{\dot r_5}{r_5}} \tag{.11}$$

are already present in both sets. We start from the fourth equation in (.5):

$$\begin{aligned}
\frac{\dot r_3}{r_3} &= \frac{2\dot r_5 r_2 - c_4 r_1 r_5 (h_1 + h_2) - 2h_1 r_2 r_5}{2r_2 r_5}\\
&= \frac{\dot r_5}{r_5} - \frac{c_4 r_1 (h_1 + h_2)}{2r_2} - h_1.
\end{aligned} \tag{.12}$$

But from the third equation in (.5), we have that:

$$\frac{\dot r_1}{r_1} = \frac{\dot r_5}{r_5} - \frac{c_4 r_1 (h_1 + h_2)}{2r_2} - \frac{c_4 r_5 (h_1 + h_2)}{2r_1} - h_1 h - h_2. \tag{.13}$$

By comparison of (.12) and (.13):

$$\frac{\dot r_3}{r_3} = \frac{\dot r_1}{r_1} + h_2 + \frac{c_4 r_5 (h_1 + h_2)}{2r_1}.$$

Since $c_4 r_5 = c_3 r_2$, this simplifies to:

$$\boxed{\frac{\dot r_3}{r_3} = \frac{\dot r_1}{r_1} + h_2 + \frac{c_3}{2}(h_1 + h_2)\frac{r_2}{r_1},} \tag{.14}$$

which is indeed the fourth equation in (.8). We then move onto the fifth equation in (.5):

$$\begin{aligned}\frac{\dot{r}_4}{r_4} &= \frac{2\dot{r}_5 r_2 - c_4 r_1 r_5 (h_1 + h_2) - 2h_2 r_2 r_5}{2r_2 r_5} \\ &= \frac{\dot{r}_5}{r_5} - \frac{c_4 r_1 (h_1 + h_2)}{2r_2} - h_2.\end{aligned}$$

By comparison with (.12), it is seen that:

$$\boxed{\frac{\dot{r}_4}{r_4} = \frac{\dot{r}_3}{r_3} + h_1 - h_2.} \tag{.15}$$

This is identical to the fifth equation in (.8). Let us now look at the third equation in (.5):

$$\frac{\dot{r}_1}{r_1} = \frac{\dot{r}_5}{r_5} - (h_1 + h_2)\left(\frac{c_4 r_1}{2r_2} + \frac{c_4 r_5}{2r_1} + 1\right) \tag{.16}$$

Re-arranging and noting that $\dot{r}_5/r_5 = \dot{r}_2/r_2$:

$$\begin{aligned}\frac{\dot{r}_2}{r_2} &= \frac{\dot{r}_1}{r_1} + \frac{c_4 r_1 (h_1 + h_2)}{2r_2} - + \frac{c_4 r_5 (h_1 + h_2)}{2r_1} + h_1 + h_2 \\ &= \left(\frac{\dot{r}_1}{r_1} + h_2 + \frac{c_3}{2}(h_1 + h_2)\frac{r_2}{r_1}\right) + \frac{c_3}{2}(h_1 + h_2)\frac{r_1}{r_5} + h_1\end{aligned}$$

We notice that the term inside the brackets is identical to (.14), and thus we recover the third equation in (.8):

$$\boxed{\frac{\dot{r}_2}{r_2} = \frac{\dot{r}_3}{r_3} + h_1 + \frac{c_3}{2}(h_1 + h_2)\frac{r_1}{r_5}.} \tag{.17}$$

Finally, we look a the final equation in (.5):

$$r_3 = \frac{c_4 r_1^2 + c_4 r_2 r_5 + 2r_1 r_2}{c_4 r_4}.$$

Multiplying both sides by $c_4 r_4$:

$$c_4 r_3 r_4 = c_4 r_1^2 + c_4 r_2 r_5 + 2r_1 r_2,$$

and re-arranging:

$$c_4 r_3 r_4 - c_4 r_1^2 - c_4 r_2 r_5 = 2r_1 r_2.$$

Dividing both sides by $2r_1 r_2$:

$$\frac{c_4}{2}\left[\frac{r_3 r_4}{r_1 r_2} - \frac{r_1}{r_2} - \frac{r_5}{r_1}\right] = 1,$$

and noting that $c_4 = c_3 r_2/r_5$ yields:

$$\boxed{\frac{c_3}{2}\left[\frac{r_3 r_4}{r_1 r_5} - \frac{r_1}{r_5} - \frac{r_2}{r_1}\right] = 1,} \tag{.18}$$

which is indeed the final equation in (.8). The set of boxed equations in the above, which were derived from the set of equations (.5), are identical to (.8). This completes the proof. ☐

C.2 The Function ϕ

We also derive the condition (4.64):

$$\frac{\partial\phi(u,v)}{\partial u} = \omega\left(\frac{h_1+h_2}{2}\right) \tag{.19}$$

for the function ϕ introduced in (4.63).

Derivation. We start our derivation by substituting the polar forms (4.63) into the expressions for r_1 and r_2 (4.62):

$$\begin{aligned} r_1 &= r_3\left(\tilde{r}_1 - \frac{\tilde{r}_2}{c_4}\right) = r_3 e^{H_-}\left(\cos\phi - \frac{1}{\omega}\sin\phi\right) \\ r_2 &= r_3\tilde{r}_2 = r_3 e^{H_-}\frac{c_4}{\omega}\sin\phi \end{aligned}$$

We take their derivatives with respect to u and adapt the shorthand notation $\dot{\phi} = \partial_u\phi$:

$$\begin{aligned} \dot{r}_1 = \dot{r}_3 e^{H_-}&\left(\cos\phi - \frac{1}{\omega}\sin\phi\right) + r_3 e^{H_-}\left(\frac{h_1-h_2}{2}\right)\left(\cos\phi - \frac{1}{\omega}\sin\phi\right) \\ &- r_3 e^{H_-}\dot{\phi}\left(\cos\phi - \frac{1}{\omega}\sin\phi\right) \\ \dot{r}_2 = \dot{r}_3 e^{H_-}&\frac{c_4}{\omega}\sin\phi + r_3 e^{H_-}\left(\frac{h_1-h_2}{2}\right)\frac{c_4}{\omega}\sin\phi + r_3 e^{H_-}\frac{c_4}{\omega}\dot{\phi}\cos\phi \end{aligned}$$

We can now take two approaches in deriving (.19): either substitute the above into the first equation in (4.60), or the second. We will go with the approach of using the first equation, but the other option also yields the same result. From the above, we see that:

$$\begin{aligned} \frac{\dot{r}_2}{r_2} &= \frac{1}{r_3 e^{H_-}\frac{c_4}{\omega}\sin\phi}\left[\dot{r}_3 e^{H_-}\frac{c_4}{\omega}\sin\phi + r_3 e^{H_-}\left(\frac{h_1-h_2}{2}\right)\frac{c_4}{\omega}\sin\phi + r_3 e^{H_-}\frac{c_4}{\omega}\dot{\phi}\cos\phi\right] \\ &= \frac{\dot{r}_3}{r_3} + \left(\frac{h_1-h_2}{2}\right) + \dot{\phi}\frac{\cos\phi}{\sin\phi} \end{aligned} \tag{.20}$$

and from the first equation in (4.60):

$$\begin{aligned} \frac{\dot{r}_2}{r_2} &= \frac{\dot{r}_3}{r_3} + h_1 + \frac{c_3}{2}(h_1+h_2)\frac{r_1}{r_5} \\ &= \frac{\dot{r}_3}{r_3} + h_1 + \frac{c_3}{2}(h_1+h_2) r_3 e^{H_-}\left(\cos\phi - \frac{1}{\omega}\sin\phi\right)\frac{1}{r_5} \\ &= \frac{\dot{r}_3}{r_3} + h_1 + \frac{c_4}{2}(h_1+h_2) r_3 e^{H_-}\left(\cos\phi - \frac{1}{\omega}\sin\phi\right)\frac{1}{r_2} \\ &= \frac{\dot{r}_3}{r_3} + h_1 + \frac{\omega}{2}(h_1+h_2)\left(\frac{\cos\phi - \frac{1}{\omega}\sin\phi}{\sin\phi}\right) \end{aligned} \tag{.21}$$

where we have substituted the expression for r_1 in the first equality, and used $c_3/r_5 = c_4/r_2$ together with the expression for r_2 in the third and fourth. Now, equation (.20) and (.21) yields:

$$\cancel{\frac{\dot{r}_3}{r_3}} + \left(\frac{h_1-h_2}{2}\right) + \dot{\phi}\frac{\cos\phi}{\sin\phi} = \cancel{\frac{\dot{r}_3}{r_3}} + h_1 + \frac{\omega}{2}(h_1+h_2)\left(\frac{\cos\phi - \frac{1}{\omega}\sin\phi}{\sin\phi}\right)$$

This is re-arranged into:

$$\dot{\phi}\frac{\cos\phi}{\sin\phi} = \left(\frac{h_1+h_2}{2}\right)\left(1+\omega\frac{\cos\phi}{\sin\phi}-1\right)$$

which simplifies to:

$$\dot{\phi} = \frac{\partial\phi(u,v)}{\partial u} = \omega\left(\frac{h_1+h_2}{2}\right)$$

This completes the derivation.

D Solving the Differential Equations in Model A1

In this section, we solve the differential equations:

$$r_1 = 1, \qquad \dot{r}_2 = -\frac{r_2}{2}, \qquad \dot{r}_3 = ie^{i\phi} - r_3, \tag{.22}$$

$$\dot{r}_4 = -r_4, \qquad \dot{r}_5 r_7 - \dot{r}_7 r_5 + \frac{1}{2} r_5 r_7 = 0, \qquad -\dot{r}_3 r_6 + \dot{r}_6 r_3 + \frac{1}{2} r_3 r_6 = 0, \tag{.23}$$

$$\dot{r}_7 = -r_7 - ie^{-i\phi}, \qquad r_8 = 1 - r_4. \tag{.24}$$

that are present in Model A1.

Solution. We immediately see that the solution

$$\boxed{r_1 = 1}$$

is trivially present. Moving onto the second equation, we have that:

$$\begin{aligned}
\dot{r}_2 = -\frac{r_2}{2} &\Rightarrow \frac{\dot{r}_2}{r_2} = -\frac{1}{2} \\
&\Rightarrow \frac{d\log r_2}{du} = -\frac{1}{2} \\
&\Rightarrow \log r_2 = -\frac{u}{2} + C, \qquad C = 0 \text{ as } r_2(0) = 1, \\
&\Rightarrow \boxed{r_2 = e^{-u/2},}
\end{aligned}$$

where we have made use of the boundary conditions (5.20). The fourth equation is also easily solved:

$$\begin{aligned}
\dot{r}_4 = -r_4 &\Rightarrow \frac{\dot{r}_4}{r_4} = -1 \\
&\Rightarrow \log r_4 = -u + C, \qquad C = 0 \text{ as } r_4(0) = 1, \\
&\Rightarrow \boxed{r_4 = e^{-u}.}
\end{aligned}$$

From which, the eighth equation simply becomes

$$\boxed{r_8 = 1 - e^{-u}.}$$

We now move onto the third equation, which is slightly more tricky:

$$\dot{r}_3 = ie^{i\phi} - r_3. \tag{.25}$$

We derivate both sides once more:

$$\ddot{r}_3 = -\dot{r}_3 \Rightarrow \ddot{r}_3 + \dot{r}_3 = 0. \tag{.26}$$

This is a second order differential equation, and is solved by a function of the form $r_3(u) = ce^{\lambda u}$, for some constants c and λ. Substituting this into the above yields the quadratic equation:

$$\lambda^2 c + \lambda c = 0 \Rightarrow \lambda(\lambda + 1) = 0,$$

which is easily solved by $\lambda = 0, -1$. Since the sum of solutions is also a solution to a differential equation, we have that r_3 is of the form:

$$r_3 = c_1 + c_2 e^{-u},$$

for some constants c_1, c_2. Substituting this back into the original expression (.25):

$$\cancel{-c_2 e^{-u}} = ie^{i\phi} - c_1 \cancel{-c_2 e^{-u}} \Rightarrow c_1 = ie^{i\phi}.$$

Moreover, from the boundary condition $r_3(0) = 0$, we have that:

$$r_3(0) = 0 = ie^{i\phi} + c_2 \Rightarrow c_2 = -ie^{i\phi}.$$

Therefore, the expression for r_3 is:

$$\boxed{r_3 = ie^{i\phi}(1 - e^{-u}).} \tag{.27}$$

We apply the same method to the seventh equation

$$\dot{r}_7 = -r_7 - ie^{-i\phi} \Rightarrow \ddot{r}_7 = -\dot{r}_7$$

This is the same form as (.26), and is solved by:

$$r_7 = c_1 + c_2 e^{-u}$$

for some constants c_1, c_2. By substituting this into the original expression, we get:

$$\cancel{-c_2 e^{-u}} = -c_1 \cancel{-c_2 e^{-u}} - ie^{-i\phi} \Rightarrow c_1 = -ie{-}i\phi$$

Moreover, from the boundary condition $r_7(0) = 0$:

$$r_7(0) = 0 = -ie^{-i\phi} + c_2 \Rightarrow c_2 = ie{-}i\phi$$

Thus:

$$\boxed{r_7 = ie^{-i\phi}(e^{-u} - 1)} \tag{.28}$$

We are only left to solve the fifth and sixth equations. Starting with the fifth equation:

$$\dot{r}_5 r_7 - \dot{r}_7 r_5 + \frac{1}{2} r_5 r_7 = 0$$

we substitute the expression for r_7 (.28):

$$\cancel{ie^{-i\phi}}(e^{-u} - 1)\dot{r}_5 + \cancel{ie^{-i\phi}} e^{-u} r_5 + \frac{1}{2}\cancel{ie^{-i\phi}} r_5 (e^{-u} - 1) = 0$$

$$\Rightarrow (e^{-u}-1)\dot{r}_5 + e^{-u}r_5 + \frac{1}{2}r_5(e^{-u-1}) = 0$$

$$\Rightarrow \dot{r}_5(1-e^{-u}) = r_5\left(\frac{3}{2}e^{-u} - \frac{1}{2}\right)$$

This is a seperable differential equation, and can be solved as follows:

$$\frac{\dot{r}_5}{r_5} = \frac{d\log r_5}{du} = \frac{1}{2}\frac{3e^{-u}-1}{1-e^{-u}} \tag{.29}$$

$$\begin{aligned}
\Rightarrow \log r_5 &= \frac{1}{2}\int \frac{3e^{-u}-1}{1-e^{-u}}du \\
&= \frac{1}{2}\int\left(\frac{e^{-u}-1}{1-e^{-u}} + \frac{2e^{-u}}{1-e^{-u}}\right)du \\
&= \frac{1}{2}\int\left(-1 + \frac{2e^{-u}}{1-e^{-u}}\right)du \\
&= -\frac{u}{2} + \int\frac{e^{-u}}{1-e^{-u}}du, \qquad\qquad v = 1-e^{-u} \Rightarrow dv = e^{-u}du \\
&= -\frac{u}{2} + \int\frac{1}{v}dv = \frac{-u}{2} + \log v + C, \qquad\qquad C \in \mathbb{C} \\
&= -\frac{u}{2} + \log(1-e^{-u}) + C
\end{aligned}$$

Taking the exponential of both sides:

$$r_5 = C_1e^{-u/2}(1-e^{-u}), \qquad C_1 = e^C$$

To find the constant C_1, we need to use the boundary condition $\dot{r}_5 = -ie^{-\phi}$, as the boundary condition $r_5(0) = 0$ trivially vanishes. Derivating the expression for r_5:

$$\dot{r}_5 = -\frac{C_1}{2}e^{-u/2}(1-e^{-u}) + C_1e^{-u/2}e^{-u}$$

$$\Rightarrow \dot{r}_5(0) = C_1 = -ie^{-i\phi}$$

Therefore:

$$\boxed{r_5 = ie^{-u/2}e^{-i\phi}(e^{-u}-1).}$$

Similarly for the sixth equation

$$-\dot{r}_3r_6 + \dot{r}_6r_3 + \frac{1}{2}r_3r_6 = 0$$

we substitute the expression for r_3 (.27):

$$-r_6\cancel{ie^{i\phi}}e^{-u} + \dot{r}_6\cancel{ie^{i\phi}}(1-e^{-u}) + \frac{1}{2}r_6\cancel{e^{i\phi}}(1-e^{-u}) = 0$$

$$-r_6e^{-u} + \dot{r}_6(1-e^{-u}) + \frac{1}{2}r_6(1-e^{-u}) = 0$$

$$\dot{r}_6(1-e^{-u}) = r_6\left(\frac{3}{2}e^{-u} - \frac{1}{2}\right)$$

This is once again a seperable differential equation:

$$\frac{\dot{r}_6}{r_6} = \frac{d\log r_6}{du} = \frac{1}{2}\frac{3e^{-u}-1}{1-e^{-u}}$$

and has an identical form to (.29). As a result it is solved by the same expression:

$$r_6 = C_2 e^{-u/2}(1-e^{-u}),$$

for some constant C_2. The boundary condition $r_6(0) = 0$ once again trivially vanishes, leaving us to use $\dot{r}_6(0) = ie^{i\phi}$ to find C_2:

$$\dot{r}_6 = -\frac{C_2}{2}e^{-u/2}(1-e^{-u}) + C_2 e^{-u/2}e^{-u},$$

$$\dot{r}_6(0) = C_2 = ie^{i\phi},$$

and thus:

$$\boxed{r_6 = ie^{-u/2}e^{i\phi}(1-e^{-u}).}$$

In the above, the set of boxed equations is exactly the set of solutions (5.27). This concludes our derivation.