University of Groningen

Bachelor Project Mathematics

Coupling of functionals of Determinantal point processes and Poisson point processes

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Contents

1 Introduction

In this thesis we will explore the article written by Moritz Otto titled " Couplings and Poisson approximation fot stabelizing functionals of determinantal point process" [18]. The goal of this thesis we be to explain the main results of the paper and explain the proof of this paper for readers of a bachelor's level of mathematics. Consequently, we will cover basic concepts such as measurable sets and point processes, also we will look at the the specific Poisson point process ,which is a very widely reconigezed point process. We will also look at the determinantal point process (DPP)that was introduced in quantum mechanics to study the arrangement of fermions [16]. A key characteristic of a DPP is that it is a repulsive point process, repulsion just means that points want to move away from each other this makes a DPP useful in applied sciences. For example, they can be used as a model for base stations in wireless network [17]. In mathematics DPPs arise naturally in different fields, such as an random spanning tree [5]. DDPs have important probabilistic properties. Notably, the (reduced) Palm process is again a determinantal process. A DPP on $\mathbb{R}^d$ is determined by its correlation kernel K, which is hermitian function from $\mathbb{R}^d \times \mathbb{R}^d$ to $\mathbb{C}$, what means that we take to point and returns a complex element. An well know DPP is the Ginibre process on $\mathbb{R}^2$ with Gaussian kernel explained in section 3. In this thesis we will look at the functionals of DPPs. But what does that functionals of DPP mean? This model is a stationary determinantal point process ξ on $\mathbb{R}^d$ and let g be measurable function $\mathbb{R}^d \times \boldsymbol{N}$ to $\{0, 1\}$, where we write $\boldsymbol{N}$ is the σ-finite set with all possible point configurations in $\mathbb{R}^d$ (will be explained in section 2). For some measurable $W \subset \mathbb{R}^d$, let

$$\Xi[\xi] := \sum_{x \in \xi \cap W} g(x, \xi)\delta_x,$$

where δ_x denotes the Dirac measure at x. The function g will remove points away from the DPP of ξ. This means that Ξ has a point x that is in ξ and $g(x, \xi) = 1$ holds. With an idea of what the functional of a determinantal point process is, we finally want to take the distance in some appropriate metric of point processes because this processes are random we cannot take a simple distance between points. So, the idea is to find a distance between Ξ and Poisson point process. This continuous the studies for stabilizing functionals of Poisson point process [3] ,[19] . However , the repulsive of DPPs requires different tools then if we would use a Poisson input as the ξ in the functional of Ξ. The main result of this paper will be,

- If the correlation kernel K is fast decaying and if the function g is stabilizing and satisfies certain assumptions , then the distance between Ξ and a Poisson process can be bounded is comparable to a bound seen from the thinned Poisson point processes [3].
- If ξ is a Ginibre process and if Ξ is the point process of elements in $\xi \cap W$, and if we take the thinning function such that we get a large distance

between the nearest neighbor, we will proof that if the volume W tends to infinity, then with an appropriate scaling of Ξ will asymptotically converge to a Poisson point process.

This thesis will start with section 2 covering basic concepts in measure theory. In section 3, we will introduce point processes (Poisson, determinantal, Ginibre and Palm processes) and the distance we taking such that we can take a distance between point processes. In section 4 we will discuss the main result in this paper, section 5 we will see the preliminaries of the main theorems, followed by section 6, where we will present the proof of the main theorems in this paper.

2 Measure theory

2.1 σ-algebra

We first need some basic notation from measure theory so we can explain the point processes later. Therefore we start with the notions of algebras.

Definition 2.1. *A collection $\mathcal{F}$ of subsets of a set Ω is a σ-algebra if,*

1. $\Omega \in \mathcal{F}$
2. $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$
3. $A_n \in \mathcal{F}, n \in \mathbb{N} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$

We denote a σ-algebra as a measurable set $(\Omega, \mathcal{F})$. This notation will be used throughout the paper.

Examples of a σ-algebra of the set Ω are $\mathcal{F} = (\emptyset, \Omega)$ or $\mathcal{F} = P(\Omega)$. The power set of Ω. This two set are the smallest and biggest σ- algebras of any set. Now we want to define the measure for any given σ-algebra.

Definition 2.2. *A measure μ on a σ-algebra $\mathcal{F}$ is an extended real-valued function $\mu : \mathcal{F} \to [0, \infty]$ which satisfies:*

1. $\mu(\emptyset) = 0$
2. $A_n \in \mathcal{F}, n \in \mathbb{N}$ *pairwise disjoint* $\Rightarrow \mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$

With this last definition we can define a measure space with the triplet of $(\Omega, \mathcal{F}, \mu)$. Very important point: a measure space is a probability space if the measure $\mu(\Omega) = 1$. This triple gets distinguished with the notation $(\Omega, \mathcal{F}, \mathbb{P})$. Where we change μ by the well known $\mathbb{P}$ from probability. Now we want to look at some important measures for point processes.

Definition 2.3. *(Dirac measure) Let Ω be set and let $\mathcal{F}$ be a σ-algebra. For $x \in \Omega$ and $A \in \mathcal{F}$. Then define the function:*

$$\delta_x(A) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{otherwise} \end{cases}$$

Definition 2.4. *(counting measure) Let Ω be a set and let $\mathcal{F} = P(\Omega)$ (the power set). Let $A \in \mathcal{F}$. Then we can define μ as:*

$$\mu(A) = \begin{cases} |A| & \text{if } A \text{ is a finite set} \\ \infty & \text{if } A \text{ is an infinite set} \end{cases}$$

Here the measure counts the amount of point in the set. Also the σ-algebra needs to be the power set (which is the biggest σ-algebra in the set).

2.2 Generators of σ-algebra

Important idea is how can we create a σ-algebra from a given subset from Ω. Also can we make new algebras from different algebras. The second point will be shown by the lemma below.

Lemma 2.5. *The intersection of a nonempty family of σ-algebras on a set Ω is a σ-algebra.*

Proof. we going to check the definition of the σ-algebra. Let us define $\mathcal{F}_\alpha$ with $\alpha \in I$ some index set, be a collection of σ- algebras. The first point we know that $\Omega \in \mathcal{F}_\alpha$ for all α This means it is also in the intersection of these sets. second point take a element A and let A$\in \bigcap_{\alpha \in I} \mathcal{F}_\alpha$. This means that A is an element in all of the $\mathcal{F}_\alpha$. Because the $\mathcal{F}_\alpha$ is a σ-algebra. We can say that $A^c \in \mathcal{F}_\alpha$, for all α. This means that $A^c \in \bigcap_{\alpha \in I} \mathcal{F}_\alpha$. This means statement two holds. The third statement let use take $A_n, n \in \mathbb{N}$ be in $\bigcap_{\alpha \in I} \mathcal{F}_\alpha$. Then for all $\alpha \in I$ A_n belongs in $\mathcal{F}_\alpha$ and by definition of σ-algebra this means $\bigcup_{n=1}^{\infty} A_n$ This union also belongs to $\mathcal{F}_\alpha$. thus this means also for $\bigcap_{\alpha \in I} \mathcal{F}_\alpha$. □

This proof tells use that if we have two of more σ-algebras we can always have the smallest σ-algebra by taking the intersection of them.

Proposition 2.6. *Let ε be a collection of subsets of a set Ω. Then there is precisely one σ-algebra $\mathcal{F}$ such that:*

1. $\varepsilon \subset \mathcal{F}$
2. *If $\mathcal{A}$ is a σ-algebra with $\varepsilon \subset \mathcal{A}$ then $\mathcal{F} \subset \mathcal{A}$*

Proof. The set is never empty by just producing the power set we know ε will be in the set. statement one follows from the idea of the lemma. If the second statement holds for another $\mathcal{F}'$ then we can show easily that $\mathcal{F}' \subset \mathcal{F}$ and $\mathcal{F} \subset \mathcal{F}'$. This means it is unique σ-algebra.

□

This proposition tells use there exist a smallest σ-algebra. from any subset in Ω. Also now we want to construct this set.

Definition 2.7. *Let ε be a collection of subsets of a set Ω. The unique σ-algebra in proposition above is generated by ε, denoted by $\sigma(\varepsilon)$, and ε is said to be generator of this σ-algebra.*

Now we gonna use by defining a specific subset and define the Borel σ-algebra, this algebra will be used in the next chapter to define the space we will work in.

Definition 2.8. *Borel σ-algebra Let Ω be a topological space. The σ-algebra generated by all open sets of Ω in the Borel σ-algebra $\mathcal{B}(\Omega)$. Its elements are called Borel measurable subsets*

This definition tells use what a Borel σ-algebra is. It is the set of subset generated by open sets in the topology. This definition is important but we want to work for this theory in a $\mathbb{R}^d$ space. Thus we look at the next proposition.

Proposition 2.9. *The Borel σ-algebra $\mathcal{B}^d$ on $\mathbb{R}^d$ where d is the dimensions is generated by:*

1. *the collection of closed subsets;*
2. *the collection of half-spaces $(x_1, \ldots, x_d) : x_i \leq b$ for some index i and $b \in \mathbb{R}$*
3. *the collection of rectangles $(a_1, b_1] \times \ldots \times (a_d, b_d]$ where $a_i, b_i \in \mathbb{R}, a_i \leq b_i$ and $1 \leq i \leq d$.*

Now we can create a Borel σ-algebra, given a set of closed subsets in $\mathbb{R}^d$. Now we will define a measure that will be used later in chapter two to define a measure of our space we are in.

Definition 2.10. *Let Ω be a set and let $P(\Omega)$ be the power set. the real-valued function $\mu^* : P(\Omega) \rightarrow [0, \infty]$ is an outer measure on Ω if:*

1. $\mu^*(\emptyset) = 0$
2. $A \subset B \Rightarrow \mu^*(A) \leq \mu^*(B)$
3. $A_n \subset \Omega, n \in \mathbb{N} \Rightarrow \mu^*(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$

Definition 2.11. *Let μ^* be an outer measure on Ω. A set $A \subset \Omega$ is measurable with respect to μ^* if for any set $Z \subset \Omega$.*

$$\mu^*(Z) = \mu^*(Z \cap A) + \mu^*(Z \cap A^c)$$

The outer measure is a measure were we can split the measure in smaller sub-measures so it can be easier solved. This property is an important point for the next measure that we will see. This measure is the Lebesgue measure.

We can take the volume of a $\mathbb{R}$ rectangle R of the form

$$R = [a_1, b_1] \times \ldots \times [a_d, b_d], a_i, b_i \in \mathbb{R}, a_i \leq b_i, i = 1, \ldots, d.$$

This volume is defined as

$$l(R) = (b_1 - a_1) \times \ldots \times (b_d - a_d). \tag{1}$$

This measure can be used to calculate rectangles and has defining elements like if the rectangle is dimension $\mathbb{R}^{d-1}$ the measure is zero and the unit box(defined by all $R = [0, 1] \times \ldots \times [0, 1]$) has measure 1.

Definition 2.12. *Let $A \subset \mathbb{R}^d$. Then $m^*(A) \in [0, \infty]$ is defined by*

$$m^*(A) = inf\{\sum_{N=1}^{\infty} l(R_n) : R_n \subset \mathbb{R}^d \textit{ closed rectangle}, A \subset A \subset \bigcup_{n=1}^{\infty} R_n\} \tag{2}$$

This is called the Lebesgue outer measure.

Theorem 2.13. *[21, Theorem 2.7] Let the Lebesgue outer measure* m^* *an let* $R \subset \mathbb{R}^d$ *be a closed rectangle with volume l(R). then* $M^*(R) = l(R)$

Definition 2.14. *The Lebesgue measurable sets of* $\mathbb{R}^d$ *are the measurable sets defined by the Lebesgue outer measure* m^**, and Lebesgue measure is the restriction of* m^* *to the Lebesgue measurable sets. The measure space is denoted by* $(\mathbb{R}^d, \mathcal{M}^d, m)$.

Remark we can now take the volume of some Borel-set in $\mathbb{R}^d$. By using the lebesgue measure, defined by (1). This will be used in the section about point processes.

3 Point processes

3.1 Point process

A point process is a random collection of countable many points, with multiplicities meaning that not all points have the same integer measure. We first want to define a counting measure on $\mathbb{R}^d$.

Definition 3.1. *A counting measure ξ on $\mathbb{R}^d$ is a map $\xi : \mathcal{B}^d \to \mathbb{N}_0 \cup \infty, B \mapsto \xi(B), B \in \mathcal{B}^d$.*

This is a function that takes the set of Borel-sets in $\mathbb{R}^d$ and gives you the amount of points with multiplicities in this Borel-set. We can define a simple counting measure as a counting measure that counts points with a measure 1. We can write this statement above as take a counting measure ξ, with for all $x \in \mathbb{R}^d$, such that $\xi(\{x\}) \in \{0, 1\}$.

The counting measure ξ counts the amount of points. However as we wrote above, it should be countable. How can we making this counting measure countable and have a collection of random counting measure? For this we want a set of bounded Borel-set that keeps them from being infinite.

Definition 3.2. *A counting measure ξ on $\mathbb{R}^d$ is σ-finite if for all $B \in \mathcal{B}^d$ where B is a bounded set, then $\xi(B) < \infty$.*

Now we have a definition of a σ-finite but now we want this ξ counting measure to be random and not determined.

Definition 3.3. *We define $\boldsymbol{N}$ as the set of all σ-finite counting measure on $\mathbb{R}^d$.*

Definition 3.4. *We define $\hat{\boldsymbol{N}}$ as the set of all finite counting measures on $\mathbb{R}^d$*

The difference between this set is that for all finite counting measure we mean that for any $B \in \mathcal{R}^d$, then the $\mu(B) < \infty$. We defined two sets of functions but now we want to find an σ-algebra $\mathcal{N}$ and $\hat{\mathcal{N}}$. We want to generate this by looking at all counting measure that are measurable from a map from the space $(\Omega, \mathcal{F})$.

Definition 3.5. *A map between measure spaces f:$(\Omega, \mathcal{F}) \to (\Omega', \mathcal{F}')$ is measurable if $f^{-1}(A) \in \mathcal{F}$ for any $A \in \mathcal{F}'$.*

Definition 3.6. *A σ-algebra for $\boldsymbol{N}$ is induced as $\mathcal{N} := \{f_B : (\Omega, \mathcal{F}) \to (\mathbb{N}_0, \mathcal{F}(\mathbb{N}_0)$: $f_B^{-1}(n) \in \mathcal{F}$ for all $n \in \mathbb{N}_0\}$, where f_B takes all bounded Borel sets.*

Definition 3.7. *A σ-algebra for $\hat{\boldsymbol{N}}$ is induced as $\hat{\mathcal{N}} := \{f : (\Omega, \mathcal{F}) \to (\mathbb{N}_0, \mathcal{F}(\mathbb{N}_0)$: $f_B^{-1}(n) \in \mathcal{F}$ for all $n \in \mathbb{N}_0\}$.*

This mean we can generate the σ-algebra for this set as $\sigma(\{f_B^{-1}(\{n\}\}) : B \in \mathcal{B}, n \in \mathbb{N}_0 \cup \infty\})$. This is the smallest σ-algebra on N that contains all sets of the pre-image of a number of points for all Borel-sets in $(\Omega, \mathcal{F}, \mathbb{P})$. This makes $(\boldsymbol{N}, \mathcal{N})$ and $(\hat{\boldsymbol{N}}, \hat{\mathcal{N}})$ are measurable sets of functions, We can finally give the general definition for a point processes.

Definition 3.8. *A point process on* $\mathbb{R}^d$ *is a random element* ξ *of* $(\boldsymbol{N}, \mathcal{N})$*, that is a measurable mapping* $\xi : \Omega \to \boldsymbol{N}$*, on the probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*.*

Now some example for some point processes. Take a point processes on $\mathbb{R}^d$. If $Z : \Omega \to \boldsymbol{N}$, $\omega \to Z(w)$ is such that $(Z(\omega)(\{x\})) \in \{0, 1\}$ for all $\omega \in \Omega$ and all $x \in \mathbb{R}^d$. We can define then we can identify $z(\omega)$ with the support in the case we write,

$$(Z(\omega))(B) = card(Z(\omega) \cap B), B \in \mathbb{R}^d.$$

where card means the cardinality of the set, thus counting how many points are in the set. This measure is just the counting measure defined earlier for a set. This is also a simple counting measure because every point is counted only once.

An important question of point processes is the amount of points we expect to see in a Borel set B. This is called the intensity measure and we define it like this.

Definition 3.9. *The intensity measure of a point processes* ξ *on* $\boldsymbol{N}$ *is the measure on* $(\mathbb{R}^d, \mathcal{B}^d)$ *defined by:*
$\mathbb{E}[\xi](B) := \mathbb{E}[\xi(B)]$ *for* $B \in \mathcal{B}^d$*.*

The intensity measure is defined on a Borel-set for the point process.

Now we want to talk stationary point processes. In probability we encounter a lot of stationary distributions. It just means that under translation the distribution does not change. let us first define the shifts: $\theta_y : \boldsymbol{N} \to \boldsymbol{N}, y \in \mathbb{R}^d$, defined by,

$$\theta_y \mu(B) := \mu(B + y), \mu \in \boldsymbol{N}, B \in \mathcal{B}^d \tag{3}$$

Definition 3.10. *Two point processes* η *and* ξ *are equal in distribution if* $\mathbb{P}(\eta \in E) = \mathbb{P}(\xi \in E)$ *for any* $E \in \mathcal{N}$ *this we can write as* $\eta \overset{d}{=} \xi$*.*

Now we can define this shift map for point processes and distribution.

Definition 3.11. *A point process* ξ *on* $\mathbb{R}^d$ *is said to be stationary if* $\theta_x \xi \overset{d}{=} \xi$ *for all* $x \in \mathbb{R}^d$*.*

If we have a stationary distribution we can say something about the intensity measure. Therefore there is also a term called the intensity this is defined as $\lambda \in \mathbb{R}_+$ such that the intensity measure could be written like this :

$$\mathbb{E}[\xi(B)] = \lambda |B|, \tag{4}$$

where the $|B|$ is the Lebesgue measure on the set B. We also want to look at the correlation function.This will tell us if the point processes have points being attractive or repulsive to each other. We first will look at the notation of locally square integrable .

Definition 3.12. *We say a point processes ξ on $\mathbb{R}^d$ is locally square integrable if:*

$$\mathbb{E}[(\xi(B))^2] < \infty, B \in \mathcal{B}_b^d.$$

Here $\mathcal{B}_b^d$ is all bounded Borel subsets.

We will define the correlation function for order m. This means we look at m points x_i in $\mathbb{R}^d$ where $i \in I$. This tells us if the points are repulsive or attractive to each other. We will define this correlation function as:

Definition 3.13. *The function $\rho^m : (\mathbb{R}^d)^m \to [0, \infty]$ is the correlation function of a point process ξ if*

$$\mathbb{E}[\xi(A_1) \cdots \xi(A_m)] = \int_{A_1 \times \ldots \times A_m} \rho^m(x_1, \ldots, x_m) d(x_1, \ldots, x_m)$$

For pairwise disjoint $A_1, \ldots, A_m \in \mathcal{B}^d$, and for $x_1, \ldots, x_m \in \mathbb{R}^d, m \in \mathbb{N}$

Remark: We could take this definition above that we can write this correlation function for $m = 1$ we get:

$$\mathbb{E}[\xi](B) = \int_B \rho^1(x_1) dx_1 \tag{5}$$

Proposition 3.14. *[12]Let ξ be a locally square integrable stationary point processes on $\mathbb{R}^d$ with positive intensity. Then there exists a correlation function ρ^m for the point processes.*

Last property we want look at is negatively associated.

Definition 3.15. *Let's take a point process ξ. Then the point process is negatively associated if for each collection of disjoint sets $B_1, \ldots, B_m \in \mathcal{B}^d$ and each subset $I \subset \{1, \ldots m\}$ we have that*

$$Cov(F(\xi(B_i), i \in I), G(\xi(B_i), i \in I^c)) \leq 0, \tag{6}$$

where F, G are real bounded and increasing functions

Now we can look at point processes, like determinantal, Poisson, Ginibre, and Palm point processes.

3.2 Determinantal point processes

We first define the determinantal point processes (DPP). Let $K : (\mathbb{R}^d)^2 \to \mathbb{C}$ be a complex function. We say that ξ is a determinantal point processes with correlation kernel K, if for every $n \in \mathbb{N}$ and pairwise disjoint $A_1, \ldots, A_n \in \mathcal{B}^d$ we have that

$$\mathbb{E}[\xi(A_1) \cdots \xi(A_n)] = \int_{A_1 \times \ldots \times A_n} det[(K(x_i, x_j))_{i,j=1}^n] d(x_1, \ldots, x_n), \tag{7}$$

where $d..$ denotes integration with respect to the Lebesgue measure on $\mathbb{R}^d$, the $(K(x_i,x_j)_{i,j=1}^m)$ is the $m \times m$- matrix with entry $K(x_i,x_j)$ at position (i,j) and $det(M)$ is the determinant of the complex value matrix. If we look back at definition 3.13 we can say the correlation is:

$$\rho^m(x_1,\cdots,x_m) = det(K(x_i,x_j))_{i,j=1}^m, x_1,\ldots,x_m \in \mathbb{R}^d, m \in \mathbb{N} \tag{8}$$

In this article we will assume K to have the following four assumptions.

1. K is hermitian i.e. $K(x,y) = \overline{K(y,x)}, x,y \in \mathbb{R}^d$, where the bar means the complex conjugate of the element.

2. K is locally square integrable like in the definition 3.12 for every compact $B \in \mathcal{B}^d$ the integral

$$\int_B \int_B |K(x,y)|^2 dydx < \infty.$$

3. K is locally of the trace class, i.e. for every compact $B \in \mathcal{B}^d$ the integral $\int_B K(x,x) < \infty$.

   Under the assumptions 1-3 we can use Mercer's theorem that states:

   Theorem 3.16. *Mercer's theorem [13, Theorem B.18] Suppose that $\mathcal{B} \subset \mathbb{R}^d$ is a compact Borel set. Let $K : B \times B \to \mathbb{R}$ be a symmetric non-negative definite and continuous function. Let η be a finite measure on $\mathbb{R}^d$ then there exist $\lambda_k^B \geq 0$ and $\phi_K^B, \phi_m^B \in L^2(B), k,m \in \mathbb{N}$ s.t*

$$\int_B \phi_k^B(x)\overline{\phi_m^B(x)}\eta(dx) = \mathbf{1}\{\lambda_k > 0\}\mathbf{1}\{k = m\}$$

   and

$$K(x,y) = \sum_{k=1}^{\infty} \lambda_k^B \phi_k^B(x)\overline{\phi_k^B(y)}, (x,y) \in B \times B.$$

   This shows us that we can write K as the last equation above. Now the last assumption

4. $0 \leq \lambda_k^B \leq 1$ for all $k \in \mathbb{N}$ and all compact $B \in \mathcal{B}^d$.

Now with the four assumptions we can say we have a unique determinantal point processes with correlation kernel K. [22, Theorem 3].

We now want to define a functionals we will use:

Definition 3.17. *Let ξ be a stationary determinantal processes with intensity $\rho > 0$. Let $g : \mathbb{R}^d \times \boldsymbol{N} \to \{0,1\}$ be a measurable function and let $W \in \mathcal{B}^d$. We define:*

$$\Xi[\xi] = \sum_{x \in \xi \cap W} g(x,\xi)\delta_x. \tag{9}$$

We call g the score function.

We can see that at the end of the function there is a Dirac measure on x, that sums points that are in the the intersection of ξ and W. For any score function g. We can say this is a thinning function if g takes values in $\{0,1\}$.

The last property that we will be :

Proposition 3.18. *[15, Theorem 3.7] Determinantal point processes are negatively associated, as seen in definition 3.15 .*

3.3 Poisson processes

In the paper we will look at the Poisson processes that will give us the point processes on $\mathbb{R}^d$.

Definition 3.19. *Let λ be a measure on $(\mathbb{R}^d, \mathcal{B}^d)$. A Poisson processes with intensity λ is a point process ζ on $\mathbb{R}^d$ with the following properties:*

1. *For every $B \in \mathcal{B}^d$ the distribution of $\zeta(B)$ is a Poisson random variable with parameter $\lambda(B)$, that is to say $\mathbb{P}(\zeta(B) = k) = PO(\lambda(B); k)$ for all $k \in \mathbb{N}_0$*
2. *For every $m \in \mathbb{N}$ and pairwise disjoint sets $B_1, \ldots, B_m \in \mathcal{B}^d$ then the random variables $\zeta(B_1), \ldots, \zeta(B_m)$ are independent.*

We want to write the correlation in the form of a determinantal point process.

Proposition 3.20. *For any given finite stationary Poisson process with intensity $\lambda \in (0, \infty)$ we can write it as a DPP with correlation kernel:*

$$K(x,y) = \begin{cases} \lambda & \text{for } x = y \\ 0 & \text{otherwise} \end{cases} \text{for all } x, y \in \mathbb{R}^d$$

Proof. First note that if we take the pairwise disjoint sets are independent. Makes that if we look back at the intensity measure for every $n \in \mathbb{N}$ and pairwise disjoint $A_1, \ldots, A_n \in \mathcal{B}^d$.

$$\int_{A_1 \times \ldots \times A_n} \rho^m(x_1, \ldots, x_n) d(x_1, \ldots, x_n) = \mathbb{E}[\zeta(A) \cdots \zeta(A_n)] = \mathbb{E}[\zeta(A_1)] \cdots \mathbb{E}[\zeta(A_n)].$$

Now we know for all the intensity there exist a

$$\mathbb{E}[\zeta(A_1)] \cdots \mathbb{E}[\zeta(A_n)] = \lambda|A_1| \cdots \lambda|A_n| =$$

$$\lambda^m \Pi_{i=1}^m |A_i| = \int_{A_1 \times \ldots \times A_m} \lambda^m d(x_1, \ldots, dx_n)$$

where $\lambda^m = \rho^m$ are the same. intensity measure with the intensity being λ as in (4). Now with this knowledge we can write the correlation kernel of this set as.

$$K(x,y) = \begin{cases} \lambda & \text{for } x = y \\ 0 & \text{otherwise} \end{cases} \text{ for all } i, j \in I.$$

□

This is the Poisson processes, this will be useful for both main theorem and the smaller theorem.

3.4 Ginibre processes

The Ginibre process is a simple example of a determinantal point process in $\mathbb{R}^2$. The Ginibre process is a stationary determinantal point process with a correlation kernel given by:

$$K(z,w) = \pi^{-1} e^{-(|z|^2+|w|^2)/2} e^{z\overline{w}} \tag{10}$$

for all $z, w \in \mathbb{C}$ see, [18].

We want to calculate the intensity of this point process. We take the function by Definition 3.9 we get :

$$\mathbb{E}[\xi](B) = \int_B K(x,x)dx, B \in \mathcal{B}^2.$$

If we substitute (10) and use of the Definition 3.13 with $m = 1$ we get:

$$\mathbb{E}[\xi(B)] = \int_B \pi^{-1} e^{-(|x|^2+|x|^2)/2} e^{x\overline{x}} dx.$$

Looking at $x\overline{x} = |x|^2$ by definition.

$$\mathbb{E}[\xi](B) = \int_B \pi^{-1} e^{-|x|^2} e^{|x|^2} dx = \int_B \pi^{-1} = \pi^{-1} \int_B dx = \pi^{-1}|B|.$$

Thus the Ginibre process has intensity π^{-1}.

We also need for the proof an important statement about the Ginibre process.

Theorem 3.21. *[11]Take a (infinite) Ginibre process ξ and take the set of all point $x_i \in \mathbb{C}$ $i \in I$ take the absolute value of this points. this is the same distribution as a sequence $(X_i)_{i\in\mathbb{N}}$ of independent random variables with distribution $X_i^2 \sim$ Gamma$(i, 1)$*

The statement comes from the idea that the correlation function is a Gaussian kernel.

3.5 Palm point processes

Defining a Palm process we first need to give a Campbell Theorem.

Theorem 3.22. *(Refined Campbell theorem) [13] Suppose that ξ is a stationary point process on $\mathbb{R}^d$ with finite strictly positive intensity λ, and a measurable function $f : (\mathbb{R}^d \times \boldsymbol{N}) \to \mathbb{R}_+$. Then there exists a unique probability measure ξ^x on $\boldsymbol{N}$ such that:*

$$\mathbb{E}[\int f(x,\xi)\xi(dx)] = \int \mathbb{E}[f(x,\xi^x)]\lambda dx.$$

The element defined as ξ^x in the theorem above can be called a Palm version of the point process ξ seen at $x \in \mathbb{R}^d$. This can be seen as a new point process with the condition that you look at $x \in \mathbb{R}^d$. In a simple point process, this can be seen as point process ξ that is conditioned to have a point at x, then we get the Palm version ξ^x. Additionally, the Campbell theorem tells us that we can calculate the intensity measure for a measurable function f and a point process ξ by using the Palm version.

Now we can write the intensity measure $\boldsymbol{L}$ of $\Xi[\xi]$ defined in (9). Using the Campbell theorem we get,

$$L(A) = \rho \int_{W \cap A} \mathbb{E}[g(x,\xi^x)]dx, A \in \mathcal{B}^d. \tag{11}$$

We can state the Campbell's Theorem for stationary point process ξ^x. But what if we take the condition not on a single point x but on more random finite points? This generalize this statement we call this a Palm measure.

Definition 3.23. *Let ξ, Ξ be point processes on $(\mathbb{R}^d, \mathcal{B}^d)$, $f : \mathbb{R}^d \times \boldsymbol{N} \to \mathbb{R}_+$ be a measurable function that assume Ξ has σ-finite intensity measure $\boldsymbol{L}$. Then there exists a point process $\xi^{x,\Xi}, x \in \mathbb{R}^d$ such that,*

$$\mathbb{E}[\int f(x,\xi)\Xi(dx)] = \int \mathbb{E}[f(x,\xi^{x,\Xi})]\boldsymbol{L}(dx)$$

The process $\xi^{x,\Xi}, x \in \mathbb{R}^d$, is a Palm process of ξ with the condition that Ξ has a point at x. The distribution $P^{x,\Xi}$ is called a Palm measure. If it is simple point process $\xi^{x,\Xi}$ can be interpreted as the process where the points x from Ξ are conditioned on the point process. Now if we define ξ to be a determinantal point process with the four assumptions made in Section 3.2 and Ξ the equation (9) with a thinning function g for ξ, This gives us the next property.

Proposition 3.24. *[9, Lemma 6.2] Let's ξ be a DPP. Let Ξ be defined as in (9).Then their exist the Palm process $\xi^{x,\Xi}$ on $\mathbb{R}^d$. Let $n \in \mathbb{N}$ and $x_i \in \mathbb{R}^d$. We a.s. have the following property:*

1. $\mathbb{P}(\Pi_{k<n}\xi^{x,\Xi}\{x_k\} = 0) = 0, x_k \in \mathbb{R}^d \setminus \{x\}$

2. $\delta_x \in \xi^{x,\Xi}$

With the last property only holding because the Ξ is a sub process of the ξ. Let now define the reduced Palm process for ξ^x such that we omitted the specific point from the point processes.

Definition 3.25. *The reduced Palm process on ξ and $x \in \mathbb{R}^d$ is $\xi^{x!} := \xi - \delta_x$*

We can define a reduced Palm process, on this specific point process.

Definition 3.26. *Let ξ, Ξ be point process as in* (9)*, and Palm process $\xi^{x,\Xi}$ then we can define a reduced Palm processes as :*

$$\xi^{x!,\Xi} := \xi^{x,\Xi} - \delta_x \, a.s.$$

Now we finally can define the correlation function of this reduced Palm process for ξ. Important to know this can only happen because all reduced palm processes of a DPP is also a new DPP.

Theorem 3.27. *[20, Theorem 1.7] Let ξ be a DPP distributed by P satisfying conditions 1-4 in Section 2.2, with correlation kernel K.Then the reduced Palm process $\xi^{x!}$, $x \in \mathbb{R}^d$ distributed by $P^{x!}$, has a correlation kernel K^x given by,*

$$K^x(z,w) = K(z,w) - \frac{K(z,x)K(x,w)}{K(x,x)}, z, w \in \mathbb{R}^d, \tag{12}$$

whenever $K(x,x) > 0$

The next theorem we will state is that $\xi^{x!}$ stochastically dominated by ξ .

Theorem 3.28. *Let ξ be a DPP on $\mathbb{R}^d$, and a point $x \in \mathbb{R}^d$, then the reduced Palm process $\xi^{x!}$ is stochastically dominated by ξ which means that:*

$$\mathbb{E}[F(\xi^{x!})] \leq \mathbb{E}[F(\xi)] \tag{13}$$

for each measurable $F : \boldsymbol{N} \to \mathbb{R}$ which is bounded and increasing, and denoted by $\xi^{x!} \leq \xi$.

We mean be increasing that for all $F(\omega_1) \leq F(\omega_2)$ if $\omega_1 \subset \omega_2$

Now we can use the refined Campbells Theorem. We can write for $x \in \mathbb{R}^d$ let ξ^x be a Palm process of ξ at x and $\xi^{x,\Xi}$ a Palm process of ξ with respect to Ξ at x. Then we get :

$$\begin{aligned}\mathbb{E}[\int f(x,\xi)\Xi(dx)] = \mathbb{E}[\int f(x,\xi)g(x,\xi)\xi]dx = \int \mathbb{E}[f(x,\xi^x)g(x,\xi^x)]\rho(x)\lambda dx \\ = \int \mathbb{E}[f(x,\xi^{x,\Xi}]\mathbb{E}[g(x,\xi^x)]\rho(x)\lambda dx\end{aligned} \tag{14}$$

Now we defined the four point processes we will talk about in this paper.

3.6 Kantorovich-Rubinstein distance

The Kantorovich-Rubinstein distance is a distance that measures the closeness between point processes. we are going to use it to show that the the Ξ approximated the Poisson processes ζ. First, we need to look back at some basic notations, such as total variation. Total variation distance is defined as follow: let $(\Omega, \mathcal{F})$ be a probability space, and let $\mathbb{P}$ and $\mathbb{Q}$ be two probability measures. Then the total variation distance is:

$$\boldsymbol{d_{TV}}(\mathbb{P}, \mathbb{Q}) := sup_{A \in \mathcal{F}} |\mathbb{P}(A) - \mathbb{Q}(A)| \tag{15}$$

The second distance we have to denote is called the wasserstein distance with a Lip(1) as the set of all $h : \mathbb{R} \to \mathbb{R}$ whose Lipschitz constant is at most 1. Then for two real valued random variable Y_1 and Y_2 by:

$$\boldsymbol{d_W}(Y_1, Y_2) := sup_{h \in Lip(1)} |\mathbb{E}(h(Y_1)) - E(h(Y_2))|. \tag{16}$$

With the two distances above we can define the Kantorovich-Rubinstein (KR) distance as:

Definition 3.29. *For finite point processes ζ and ξ on $\mathbb{R}^d$ the KR distance is given by,*

$$\boldsymbol{d_{KR}}(\zeta, \xi) := sup_{h \in Lip} |\mathbb{E}h(\zeta) - \mathbb{E}h(\xi)|$$

where Lip is defined as the set of all measurable 1-Lipschitz functions $h : \hat{\boldsymbol{N}} \to \mathbb{R}$ with respect to the total variation between the measure ω_1, ω_2 on $\mathbb{R}^d$ given by:

$$\boldsymbol{d_{TV}}(\omega_1, \omega_2) = sup_{A \in \mathcal{B}^d} |\omega_1(A) - \omega_2(A)|,$$

where $\omega_1(A), \omega_2(A) < \infty$

The set Lip is then defined as the set where, the map $h : \hat{\boldsymbol{N}} \to \mathbb{R}$ is 1-lipschitz with respect to total variation. This means:

$$|h(\omega_1) - h(\omega_2)| \leq \boldsymbol{d_{TV}}(\omega_1, \omega_2) \text{ for all } \omega_1, \omega_2 \in \hat{\boldsymbol{N}}.$$

The question is when can we can say that two point process are convergence to eachother.

Proposition 3.30. *[6] Assume that $(\xi_n)_{n \in \mathbb{N}}$ is a sequence of locally finite point processes on $\mathbb{R}^d$ such that $\boldsymbol{d_{KR}}(\xi_N, \zeta) \to 0$ as $n \to \infty$. Then ξ_n converges in distribution to ζ, as n goes to ∞.*

4 Main theorem

In this part, we want to explain the main theorem of this paper. We are going to show that for a stationary determinantal point processes ξ, with appropriate conditions on ξ and g (the thinning function), we can prove that the functionals Ξ in equation (9) can be approximated by the Poisson processes. In this section, we will explain which conditions we need, and we look at the main theorem.

4.1 Main theorem

We first need to give some conditions for the determinantal point process ξ and for Ξ as in equation (9). We first want to put some assumptions on the thinning function g. Suppose that there exist $\alpha \in (0, \infty)$ such that for all $A \in \mathcal{B}^d$ and for all counting measures $\omega \in \boldsymbol{N}$,

$$\sum_{x \in W \cap A} g(x, \omega) < \alpha |A| \tag{17}$$

where $|A|$ denotes the Lebesgue measure as in (1) of the set A.

We also want to assume that g is monotonic in the sense that for all $x \in W$, it holds that

$$g(x, \omega_1) \leq g(x, \omega_2) \text{ or } g(x, \omega_1) \geq g(x, \omega_2), \omega_1 \subset \omega_2. \tag{18}$$

We also want g to be stabilizing, by which we mean that there is a measurable function $\mathcal{S} : \mathbb{R}^d \times \boldsymbol{N} \to \mathcal{F}$ such that

$$g(x, \omega) = g(x, \omega \cap \mathcal{S}(x, \omega)) \tag{19}$$

where for any $\omega \in \boldsymbol{N}$ and any $x \in \mathbb{R}^d$, where $\mathcal{F}$ is the set of closed subsets in $\mathbb{R}^d$. Further suppose that $\mathcal{S}$ is a stopping set which says that

$$\{\omega \in \boldsymbol{N} : \mathcal{S}(x, \omega) \subset S\} = \{\omega \in \boldsymbol{N} : \mathcal{S}(x, \omega \cap S) \subset S\} \tag{20}$$

where $S \subset \mathbb{R}^d$.

The last assumption we have to talk about the determinantal point process and look at the kernel $K : (\mathbb{R}^d)^2 \to \mathbb{C}$ satisfies:

$$|K(x, y)| \leq \phi(||x - y||), x, y \in \mathbb{R}^d, \tag{21}$$

for some decreasing function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ with $lim_{r \to \infty} \phi(r) = 0$. With the absolute value of the $K(x, y)$ value. This makes the correlation decreasing over a distance between points

Now we finally have all the information needed to look at the main theorem

Theorem 4.1. *Main theorem*

Let ξ be a stationary determinantal point process with kernel K satisfying (21) *and intensity $\rho \in (0, \infty)$. Let Ξ be defined as in* (9) *with intensity measure $\boldsymbol{L}$ and suppose that g satisfies* (17) *and* (18) *and is stabilizing with respect to the* *stopping set $\mathcal{S}$. Let S and T be Borel sets with $0 \in S \subset T$. Set $S_x := x + S$ and $T_x := x + T$, $x \in W$, and define*

$$\tilde{g}(x,\omega) := g(x,\omega)\mathbb{1}\{\mathcal{S}(x,\omega) \subset S_x\}, x \in \mathbb{R}^d, \omega \in \boldsymbol{N}.$$

Let ζ be a finite Poisson process on $\mathbb{R}^d$ with intensity measure $\boldsymbol{M}$. Then ,

$$\boldsymbol{d_{KR}}(\Xi,\zeta) \leq \boldsymbol{d_{TV}}(\boldsymbol{L},\boldsymbol{M}) + 2(E_1 + E_2 + E_3) + F,$$

where

$$\begin{aligned}
E_1 &:= \rho \int_W \mathbb{P}(\mathcal{S}(x,\xi^x) \not\subset S)dx,\\
E_2 &:= \rho^2 \int_W \int_{W \cap T_x} \mathbb{E}[\tilde{g}(x,\xi^x)]\mathbb{E}[\tilde{g}(y,\xi^y)]dydx,\\
E_3 &:= \int_W \int_{W \cap T_x} \mathbb{E}[\tilde{g}(x,\xi^{x,y})\tilde{g}(y,\xi^{x,y})]\rho^{(2)}(x,y)dydx,\\
F &:= c\|K\|max(|S|,1)|W \oplus S|^2\phi(d(T,S)),
\end{aligned}$$

where $||K|| = sup_{x,y\in\mathbb{R}^d}|K(x,y)|$, $W \oplus S := \{x + s : x \in W, s \in S\}$ is the Minkowski sum of W and S, $d(T,S) := max\{sup_{t\in T}d(t,S), sup_{s\in S}d(T,s)\}$ is Hausdorff distance of T and S and the constant $c > 0$ does not depend on K, g, W, S and T

Now we want to look at the theorem that is an application of theorem above. This will look at the Ginibre process. This theorem is a application of the study of the largest nearest neighbor balls. In this project we generalize the Borel sets as a closed ball at the origin in dimension 2. We can define this as $B_n := B_n(0)$ the closed ball with radius n. Additionally, we want that $|K(z,w)| \leq \phi(\|z-w\|)$ with $\phi(r) := \pi^{-1}exp(\frac{-r^2}{2})$ Now we want to define a process needed for the proof:

Definition 4.2. *let ξ be a Ginibre point process on $\mathbb{R}^2$. Let $B_n := B_n(0)$ the closed ball with radius $n > 0$ in $\mathbb{R}^d$ on the origin. We consider then the functional process as:*

$$\Xi_n := \sum_{x\in\xi\cap B_n} \mathbb{1}\{\xi(B_n(x) \setminus \{x\}) = 0\}\delta_x.$$

We also define the scaled Functional processes:

$$\Psi_n := \sum_{y\in\Xi} \delta_{y/n} = \sum_{x\in\xi\cap B_n} \mathbb{1}\{\xi(B_n(x) \setminus \{x\}) = 0\}\delta_{x/n} \tag{22}$$

This scales the point x/n what changes the point place. This will be the theorem.

Theorem 4.3. *Let take the scaled functional process Ψ_n and let v be a stationary Poisson process on $\mathbb{R}^d$ with intensity $\lambda > 0$. There exist a sequence $(v_n)_{n\in\mathbb{N}}$ with $v_n^4 \sim 8log(n)$ as $n \to \infty$ such that for all $n \in \mathbb{N}$ and any $\epsilon > 0$ and $C \in \mathbb{R}_+$,*

$$\boldsymbol{d_{KR}}(\Psi_n, v \cap B_1) \leq Cn^{\epsilon - \frac{1}{8}}.$$

We can also define the largest nearest neighbor (nn) for any $\xi \in \mathbb{R}^2$ and for point process ξ such that we get:

$$nn(x,\xi) := argmin_{y\in\xi\setminus\{x\}}|x-y| \tag{23}$$

Now we want this largest neighbor convergence

Corollary 4.4. *[19, Corollary 4.2] We have as $n \to \infty$,*

$$\frac{1}{2\pi\sqrt{log(n)}} max_{x\in\xi\cap B_n}|B_{nn}(x,\xi)| \xrightarrow{\mathbb{P}} 1.$$

These are the three statements we will prove in this paper.

5 Preliminaries

Here we will talk about the some bounds that we will need to bound the statement 4.1. For the first bound we have to look back at the determinantal point process and its Palm distribution and version. Also recall the definition of stochastically dominated, in equation (13) to get this properties.

Proposition 5.1. *Let's take ξ as a DPP holding to the assumptions in section 3.2 and Ξ as in equation* (9). *Let $F : \boldsymbol{N} \to \mathbb{R}$ be bounded and increasing, then we have for almost all $x \in \mathbb{R}^d$ that ,*

$$\xi^{x,\Xi} \leq \xi^x \textit{ if } g \textit{ is increasing.}$$
$$\xi^x \leq \xi^{x,\Xi} \textit{ if } g \textit{ is decreasing}$$

Proof. The first step is to write the left hand side of 1 and 2 from a Palm process in the function F such that it is a Palm version at only x. For this we can use (14) (look at the last equations signs and use it the opposite direction). We then get this,

$$\mathbb{E}[F(\xi^{x,\Xi})]\mathbb{E}[g(x,\xi^x)] = \mathbb{E}[F(\xi^x)g(x,\xi^x)]$$

Now we want to use the statement that ξ and ξ^x are determinantal point process and use theorem 3.18 and the definition3.15. We can write the covariance for ξ^x as,

$$cov(F(\xi^x), g(x,\xi^x)) \leq 0$$

Now we have to make two case distinction one for g increasing and decreasing.

- Now assume g is increasing. We can write the covariance as in expected value,
$$\mathbb{E}[F(\xi^x)g(x,\xi^x)] - \mathbb{E}[F(\xi^x)]\mathbb{E}[g(x,\xi^x)] \leq 0.$$
Now take the right hand side to the right we get,
$$\mathbb{E}[F(\xi^x)g(x,\xi^x)] \leq \mathbb{E}[F(\xi^x)]\mathbb{E}[g(x,\xi^x)]$$
Now we can write the left hand side as the equation above as in (14) we have written above,
$$\mathbb{E}[F(\xi^{x,\Xi})]\mathbb{E}[g(x,\xi^x)] \leq \mathbb{E}[F(\xi^x)]\mathbb{E}[g(x,\xi^x)]$$
Now divide by $\mathbb{E}[g(x,\xi^x)]$ we get
$$\mathbb{E}[F(\xi^{x,\Xi})] \leq \mathbb{E}[F(\xi^x)]$$
What proves statement 1.

- Now assume g is decreasing. Now we can rewrite as $-g$. Because for the definition 3.15 to hold g has to be increasing.Take the expected value of the covariance,

  $$\mathbb{E}[-F(\xi^x)g(x,\xi^x)] - \mathbb{E}[F(\xi^x)]\mathbb{E}[-g(x,\xi^x)] \leq 0.$$

  The minus sign can be put out of the expected values so we get,

  $$-\mathbb{E}[F(\xi^x)g(x,\xi^x)] + \mathbb{E}[F(\xi^x)]\mathbb{E}[g(x,\xi^x)] \leq 0.$$

  Now the left side we can rewrite this with (14) as above and take away the negative sign,

  $$\mathbb{E}[F(\xi^x)g(x,\xi^x)] \geq \mathbb{E}[F(\xi^x)]\mathbb{E}[g(x,\xi^x)]$$

  Now we want to write the first equation with the left hand side.

  $$\mathbb{E}[F(\xi^{x,\Xi})]\mathbb{E}[g(x,\xi^x)] \geq \mathbb{E}[F(\xi^x)]\mathbb{E}[g(x,\xi^x)].$$

  Then divide by $\mathbb{E}[g(x,\xi^x)]$ then we get,

  $$\mathbb{E}[F(\xi^{x,\Xi})] \geq \mathbb{E}[F(\xi^x)].$$

Now we have proven the statement above. □

We want one last theorem to be stated:

Theorem 5.2. *[14] Let take two distribution P and P' and $P \leq P'$ (stochastically dominated) respectively, on $(\Omega, \mathcal{F}, \mathbb{P})$, Then there exist a ξ and ξ' : $(\Omega, \mathcal{F}, \mathbb{P}) \to (\boldsymbol{N}, \mathcal{N})$, with distribution P and P' such that $\xi(\omega) \subset \xi'(\omega)$*

5.1 Fast decay of correlation

The theorem that states that we can let the covaraince kernel decays fast.

Lemma 5.3. *[2, lemma 3.1] Let ξ be a stationary Determinantal process on $\mathbb{R}^d$ with covariance kernel K that satisfies the condition in section 3.2 and $|K(x,y)| \leq \phi(||x-y||)$ for some expontially decreasing function ϕ as seen in (21). Then we have that the correlation functions $\rho^{(m)}, m \in \mathbb{N}$ of ξ satisfy,*

$$|\rho^{(p+q)}(x_1,\ldots,x_{p+q}) - \rho^{(p)}(x_1,\ldots,x_p)\rho^{(q)}(x_{p+1},\ldots,x_{p+q}| \leq m^{1+\frac{m}{2}}\phi(s)||K||^{m-1} \tag{24}$$

where,

$$s := d(\{x_1,\ldots,x_p\},\{x_{p+1},\ldots x_{p+q}\}) = inf_{i\in\{1,\ldots,p\},j\in\{p+1,\ldots p+q\}}|x_i - x_j|,$$

$||K|| := sup_{x,y\in\mathbb{R}^d}|K(x,y)|$ *and* $m := p+q$.

Now we want to state another lemma:

Lemma 5.4. *[20, Lemma 6.4] Let ξ be a DPP as in section 3.2 with distribution P. Let the correlation kernel $\{\rho^n\}_{n\geq 1}$. For $x \in \mathbb{R}^d$, the Palm measure ξ^x admits the correlation function$\{\rho^n_x\}_{n\geq 1}$. We get:*

$$\rho(x) \cdot \rho_x^{(n)}(y_1, \dots, y_n) = \rho^{(n+1)}(y_1, \dots, y_n, x)$$

holds for all $y_1, \dots, y_n \in \mathbb{R}^d \backslash \{x\}$.

5.2 Poisson process approximation

We want to bound the Kantorovich-Rubinstein distance for the Ξ as seen in (9) and the finite Poisson process. We just need to define what a symmetric difference is. We can define this as $A\Delta B = (A \cap B^c) \cup (A^c \cap B)$. This will be used in the next theorem.

Theorem 5.5. *[4, Theorem 3.1] Let $\boldsymbol{L}$ be the (finite) intensity measure of the point process Ξ as defined at (9). Suppose for $x \in W$ that Ξ^x is a Palm version of Ξ at x. Take the coupled point process Ξ and $\tilde{\Xi}$ such that $\Xi \stackrel{d}{=} \Xi^x$ and $\tilde{\Xi}^x \stackrel{d}{=} \Xi^{x!}$. Then the Kantorovich distance of Ξ and a finite Poisson process ζ with intensity measure $\boldsymbol{M}$ is bounded by :*

$$\boldsymbol{d_{KR}}(\Xi \cap W, \zeta) \leq d_{TV}(\boldsymbol{L}, \boldsymbol{M}) + 2\int_W \mathbb{E}[(\Xi^x \Delta \Xi^{x!})(W)]\boldsymbol{L}(dx) \tag{25}$$

For the proof of 4.1 note that if $\xi^{x,\Xi}$ is a Palm version of ξ at x with respect to Ξ, then the reduced Palm version of Ξ is $\Xi[\xi^{x,\Xi}] - \delta_x$. This means we can reduce the coupling of Ξ and it reduced Palm measure to construct a coupling of ξ and its Palm measure with respect to Ξ. We can now split the expected symmetric difference $\mathbb{E}[(\Xi^x \Delta \tilde{\Xi}^x)(W)]$ for all $x \in W$ and $W \in \mathbb{R}^d$ into a part that represent the local points around x and the rest. Now, we can split this to get a better bound, written out as follows:

Proposition 5.6. *Let ξ be a DPP such in section 3.2 and let $\boldsymbol{L}$ be the finite intensity measure of the point process Ξ defined as (9), and a finite Poisson process ζ with intensity measure $\boldsymbol{M}$ bounded. Then For $x, y \in W$ let Ξ^x be a Palm version of ξ at x, let $\xi^{x,y}$ be a Palm version of ξ^x at y and let $\xi^{x,\Xi}$ be a Palm process of ξ with respect to Ξ at x then we have for some T_x defined such that $x \in T_x$, $T_x \subset W$ and $T_x \in \mathcal{B}^d$ we get this,*

$$\boldsymbol{d_{KR}}(\Xi \cap W, \zeta) \leq d_{TV}(\boldsymbol{L}, \boldsymbol{M}) + 2(T_1 + T_2 + T_3 + T_4),$$

where

$$T_1 := \int_W \int_{T_x} \mathbb{E}[g(x,\xi^x)]\mathbb{E}[g(y,\xi^y]\rho(x)\rho(y)dydx,$$
$$T_2 := \int_W \int_{T_x} \mathbb{E}[g(x,\xi^{x,y})g(y,\xi^{x,y})]\rho^{(2)}(x,y)dydx,$$
$$T_3 := \int_W \mathbb{E}[\xi^x \Delta \xi^{x,\Xi})(W \backslash T_x)]\rho(x)dx,$$
$$T_4 := \rho \int_W \mathbb{E}[\sum_{y\in\xi^x\cap\xi^{x,\Xi}\cap W\backslash T_x} |g(y,\xi^x) - g(y,\xi^{x,\Xi})|]\rho(x)dx$$

Where $\rho(x) := \rho^{(1)}(x) = K(x,x), x \in \mathbb{R}^d$.

Proof. We can start with the Theorem as in 5.5. This then only rewritten in $\Xi[\xi], \xi[\xi^{x,\Xi}]$ such we get this.

$$\boldsymbol{d_{KR}}(\Xi,\zeta) \le d_{TV}(\boldsymbol{L},\boldsymbol{M}) + 2\int_W \mathbb{E}[(\Xi[\xi]\Delta(\Xi[\xi^{x,\Xi}] - \delta_x))(W)]\boldsymbol{L}(dx).$$

Now for $x \in W$ recall that T_x is a Borel set with $x \in T_x$. Then we can split the set in the set with T_x and $W\backslash T_x$,

$$(\Xi[\xi]\Delta(\Xi[\xi^{x,\Xi}]-\delta_x))(W) = (\Xi[\xi]\Delta(\Xi[\xi^{x,\Xi}]-\delta_x))(T_x)+(\Xi[\xi]\Delta(\Xi[\xi^{x,\Xi}]-\delta_x))(W\backslash T_x).$$

We can bound the symmetric difference of the set of T_x as the sum of all points in T_x.

$$(\Xi[\xi]\Delta(\Xi[\xi^{x,\Xi}] - \delta_x))(T_x) \le \Xi[\xi](T_x) + (\Xi[\xi^{\xi,\Xi}] - \delta_x)$$

a.s.

We can bound the symmetric difference with the set of $W\backslash T_x$.we have to look back at equation (9) and we can say that if we take the symmetric difference of the ξ and $\xi^{x,\Xi}$, but by using the thinning function g there are point that are in both ξ and $\xi^{x,\Xi}$ that will dissapear in one of the point sets. We will write this points as the sum $\sum_{y\in\xi\cap\xi^{x,\Xi}\cap W\backslash T_x} |g(y,\xi) - g(\xi^{x,\Xi})|$. Using this we can write a full bound for this set.

$$(\Xi[\xi]\Delta\Xi[\xi^{x,\Xi}])(W\backslash T_x) \le (\xi\Delta\xi^{x,\Xi})(W\backslash T_x) + \sum_{y\in\xi\cap\xi^{x,\Xi}\cap W\backslash T_x} |g(y,\xi) - g(\xi^{x,\Xi})|$$

We finally can rewrite (25) and change the symmetric difference with the four bounds we found we can now take four expectation of this four different sets and get T_1, T_2, T_3, T_4 We also know that $\boldsymbol{L}(dx) = \mathbb{E}[g(x, \xi^x)]\rho(x)dx$ then we get for

$$
\begin{aligned}
T_1 &= \int_W \mathbb{E}[\Xi[\xi](T_1)]\boldsymbol{L}(dx) \\
&= \int_W \int_{T_x} \mathbb{E}[g(y, \xi^y)]\rho(y)dy\boldsymbol{L}(dx) && \text{by using (3.9)} \\
&= \int_W \int_{T_x} \mathbb{E}[g(x, \xi^x]\mathbb{E}[g(x, \xi^y]\rho(y)\rho(x)dydx && \text{by using the definition of } \boldsymbol{L}(\mathrm{dx})
\end{aligned}
$$

Now we can write T_2

$$
\begin{aligned}
T_2 = &\int_W \mathbb{E}[\Xi[(\xi^{x,\Xi} - \delta_x)(T_x)]\boldsymbol{L}(dx) \\
=&\mathbb{E}[\int_W (\Xi[\xi] - \delta_x)(T_x)\Xi dx] && \text{by Definition 3.23} \\
= &\int_W \int_{T_x} \mathbb{E}[g(x, \xi^{x,y})g(y, \xi^{x,y})]\rho^{(2)}(x, y)dydx
\end{aligned}
$$

Now we can use the fact that we can bound $\mathbb{E}[g(x, \xi^x)] \leq 1$. So we can write T_3 as,

$$
\begin{aligned}
&\int_W \mathbb{E}[(\xi\Delta\xi^{x,\Xi})(W\backslash T_x)]\boldsymbol{L}(dx) \\
\leq &\int_W \mathbb{E}[(\xi\Delta\xi^{x,\Xi})(W\backslash T_x)]\rho(x)dx = T_3
\end{aligned}
$$

Now we can write T_4 as,

$$
\begin{aligned}
&\int_W \mathbb{E}[\sum_{y\in\xi\cap\xi^{x,\Xi}\cap W\backslash T_x} |g(y, \xi) - g(\xi^{x,\Xi})|]\boldsymbol{L}(dx) \\
\leq &\int_W \mathbb{E}[\sum_{y\in\xi\cap\xi^{x,\Xi}\cap W\backslash T_x} |g(y, \xi) - g(\xi^{x,\Xi})|]\rho(x)dx = T_4
\end{aligned}
$$

□

6 Proof

6.1 Proof Theorem 4.1

Proof. theorem 4.1 Let $x \in W$ let $S_x := x + S$ and $T_x := x + T$. We let $\tilde{g}(x,\omega) := g(x,\omega)\mathbb{1}\{\mathcal{S}(x,\omega) \subset S_x\}$, $\omega \in \boldsymbol{N}$, and consider a truncated function of (9). This mean we restricted the domain to the stopping set $\mathcal{S}(x,\omega)$ to be inside the subset S_x such we get:

$$\Xi_{tr} := \sum_{x \in \xi \cap W} \tilde{g}(x,\xi)\delta_x. \tag{26}$$

Note that because of the stopping set property of $\mathcal{S}(x,\omega)$ of the $\tilde{g}$ is measurable with respect to $\xi \cap S_x$.

we start with step 1: Here we assume that $g = \tilde{g}$ and therefore we get $\Xi = \Xi_{tr}$. Let $\boldsymbol{L}$ be the intensity measure of Ξ. We then use Proposition 5.6. This immediately gives me that T_1, T_2 from Proposition 5.6 are equal to E_2, E_3 from the Theorem 4.1. So we just have to bound T_3 and T_4. Thus we start By first ***bounding*** T_3. Let take P as the distribution of ξ with correlation kernel K, Then also take P^x is it Palm measure for ξ^x and take $P^{x,\Xi}$ a Palm measure with respect to Ξ. The idea of the this construction a coupling of $(\xi, \xi^{x,\Xi})$ of the distribution P and $P^{x,\Xi}$ for each $x \in W$ such that the symmetric difference $(\xi \Delta \xi^{x,\Xi})(W \backslash T_x)$ becomes small. We first have to look at the score function g is increasing and decreasing

- Increasing score function means that $g(x,\omega_1) \leq g(x,\omega_2)$ for $\omega_1 \subset \omega_2$, by (13) we get $P^{x!} \leq P$ and take the Proposition (5.1) we get $P^{x!,\Xi} \leq P^{x!}$ with this we can imply that $P^{x!,\Xi} \leq P$ where $P^{x!}$ and $P^{x!,\Xi}$ are the Palm measures P^x and $P^{x,\Xi}$ reduced by the point at x. Now using Theorem 5.2 this implies that the random element ξ from distribution P and then their exist a random element $\xi^{x!,\Xi}$ from distribution $P^{x!,\Xi}$, such that we can conclude that $\xi^{x!,\Xi} \subset \xi$ a.s. now we can write T_3 as,

$$\begin{aligned}\mathbb{E}[(\xi \Delta \xi^{x,\Xi})(W \backslash T_x)] =& \mathbb{E}[\xi(w \backslash)T_x] - \mathbb{E}[\xi^{x!,\Xi}(W \backslash T_x)] \\ =& \mathbb{E}[\xi(W \backslash T_X)] - \mathbb{E}[\xi^{x!}(W \backslash T_x)] + \mathbb{E}[\xi^{x!}(W \backslash T_x)] - \mathbb{E}[\xi^{x!,\Xi}(W \backslash T_x)].\end{aligned} \tag{27}$$

The term $\mathbb{E}[\xi(W \backslash T_X)] - \mathbb{E}[\xi^{x!}(W \backslash T_X)]$ by using (12) we can write the correlation of both sides and use the fact that $K(x,x) = \rho$ for all $x \in \mathbb{R}^d$, correlation of $K(y,y)$ for $y \in \mathbb{R}^d$.

$$\begin{aligned}\int_{W \backslash T_x} K(y,y) - K^x(y,y)dy =& \int_{W \backslash T_x} K(y,y) - K(y,y) + \frac{k(y,x)K(x,y)}{K(x,x)}dy \\ =& \rho^{-1} \int_{W \backslash T_x} |K(x,y)|^2 dy.\end{aligned} \tag{28}$$

The equation above we can bound by using (21) This gives us.

$$\rho^{-1}\int_{W\setminus T_x}|K(x,y)|^2dy \leq \rho^{-1}\int_{W\setminus T_x}\phi(\|x-y\|)^2dy \leq \rho^{-1}|W|sup_{y\in T}\phi(\|y\|)^2 \tag{29}$$

Now we look at the right-hand side above of (27). If we multiply this element with the $\mathbb{E}[\tilde{g}(x,\xi^x)]$ and using equation (14) we can write this out like this for reduced Palm process,

$$\begin{aligned}&\mathbb{E}[\tilde{g}(x,\xi^x)](\mathbb{E}[\xi^{x!}(W\setminus T_X)]-\mathbb{E}[\xi^{x!,\Xi}(W\setminus T_X)])\\ =&\mathbb{E}[\xi^{x!}(w\setminus T_x)]\mathbb{E}[\tilde{g}(x,\xi^x)]-\mathbb{E}[\xi^{x!}(W\setminus T_x)\tilde{g}(x,\xi^x)]\\ =&-Cov(\xi^{x!}(W\setminus T_x),\tilde{g}(x,\xi^x))\end{aligned} \tag{30}$$

We know the reduced Palm process $\xi^{x!}$ is a determinantal process itself and therefore negatively associated. For $k\in\mathbb{N}$ we consider the auxiliary functions

$$f^{(k)}(\omega) := min\{k,\omega(S_x)-\tilde{g}(x,\omega)\}, f(\omega) := \omega(S_x)-\tilde{g}(x,\omega), \omega\in\boldsymbol{N}. \tag{31}$$

It is easy that $f^{(k)}$, $k\in\mathbb{N}$ and f are bounded and increasing. This gives us we can say:

$$Cov(min\{k,\xi^{x!}(W\setminus T_x)\}, f^{(k)}(\xi^{x!}))\leq 0 \tag{32}$$

By monotone convergence we can write :

$$\begin{aligned}lim_{k\to\infty}Cov(min\{k,\xi^{x!}(W\setminus T_x)\}, f^{(k)}(\xi^{x!})) &= Cov(\xi^{x!}(W\setminus T_X), f(\xi^{x!}))\\ &= Cov(\xi^{x!}(W\setminus T_x),\xi^{x!}(S_x))-Cov(\xi^{x!}(W\setminus T_x),\tilde{g}(x,\xi^x))\leq 0\end{aligned}$$

Now we can bound (30) by the statement above,

$$-Cov(\xi^{x!}(W\setminus T_x),\tilde{g}(x,\xi^x))\leq -Cov(\xi^{x!}(W\setminus T_x),\xi^{x!}(S_x))$$

Now we can write the definition of covariance down we get this

$$\begin{aligned}&-Cov(\xi^{x}(W\setminus T_x),\xi^{x!}(S_x))\\ =&\mathbb{E}[\xi^{x!}(W\setminus T_x)]\mathbb{E}[\xi^{x!}(S_x)]-\mathbb{E}[\xi^{x!}(W\setminus T_x)\xi^{x!}(S_x)]\\ =&\mathbb{E}[\xi^{x!}(W\setminus T_x)]\mathbb{E}[\xi^{x!}(S_x)]-\mathbb{E}[\xi^{x!}(W\setminus T_x)\xi^{x!}(S_x)]\\ +&\mathbb{E}[\xi(W\setminus T_x)]\mathbb{E}[\xi^{x!}(S_x)]-\mathbb{E}[\xi(W\setminus T_x)]\mathbb{E}[\xi^{x!}(S_x)]\\ \leq&|\mathbb{E}[\xi^{x!}(W\setminus T_x)]-\mathbb{E}[\xi(W\setminus T_x)]|\mathbb{E}[\xi^{x!}(S_x)]+|\mathbb{E}[\xi^{x!}(W\setminus T_x)\xi^{x!}(S_x)]-\mathbb{E}[\xi(W\setminus T_x)]|\end{aligned} \tag{33}$$

We want to bound this part of the equation above, $|\mathbb{E}[\xi^{x!}(W\backslash T_x)] - \mathbb{E}[\xi(W\backslash T_x)]|\mathbb{E}[\xi^{x!}(S_x)]|$. The part in absolute value is the same as in (29) and we can bound $\mathbb{E}[\xi^{x!}(S_x)] \leq \mathbb{E}[\xi(S_x)]$ by knowing that it is stochastically dominated. We also know that $\mathbb{E}[\xi(S_x)] = \rho|S|$ the equal sign holds because it is a stationary determinantal process and we know that S_x is only a translation from the original S, means we can write it like this,

$$|\mathbb{E}[\xi^{x!}(W\backslash T_x)] - \mathbb{E}[\xi^{x}(W\backslash T_x)]|\mathbb{E}[\xi^{x!}(S_x)]| \leq |W||S|sup_{y\in T}(\phi(\|y\|))^2 \quad (34)$$

Now we want to bound the other side of (33). We write $\rho_x^{(m)}$ for the m-th correlation function of $\xi^{x!}$ and by using Lemma 5.4 and Lemma 5.3 we can get this,

$$\begin{aligned}
&\rho\int_W (\mathbb{E}[\xi^{x!}(W\backslash T_x)\xi^{x!}(S_x)] - \mathbb{E}[\xi(W\backslash T_x)]\mathbb{E}[\xi^{x!}(S_x)])dx \\
=&\rho\int_W \int_{S_x}\int_{W\backslash T_x} (\rho_x^{(2)}(y,z) - \rho_x(y)\rho)dzdydx \\
=&\int_W \int_{S_x}\int_{W\backslash T_x} (\rho^{(3)}(x,y,z) - \rho^{(2)}(x,y)\rho)dzdydx \\
\leq& 3^{\frac{5}{2}}\|K\|^2\int_W \int_{S_x}\int_{W\backslash T_x} \phi(d(\{x,y\},\{z\}))dzdydx \\
\leq& 3^{\frac{5}{2}}\|K\||W|^2|S|\phi(d(T,S)).
\end{aligned} \quad (35)$$

Thus we can finally take (29), (34) and take the integral over W and multiply by ρ and (35) we can finally bound T_3

$$T_3 = \int_W \mathbb{E}[(\xi\Delta\xi^{x,\Xi})(W\backslash T_x)]\rho(x)dx \leq (1+\rho|S|)|W|^2 sup_{y\in T}\phi(\|y\|)^2 + 3^{\frac{5}{2}}\|K\||W|^2|S|\phi(d(T,S)). \quad (36)$$

- decrease scores. If $g(x,\omega_1) \geq g(x,\omega_2)$ for $\omega_1 \subset \omega_2$ we get $P^{x!} \leq P$, and by Proposition 5.1 we get that $P^{x!} \leq P^{x,\Xi}$. Then using theorem 5.2 their exist a ξ and $\xi^{x,\Xi}$ such that $\xi^{x!} \subset \xi$ and $\xi^{x!} \subset \xi^{x!,\Xi}$, This with [10] gives us

$$\begin{aligned}
\mathbb{E}[(\xi\Delta\xi^{x!,\Xi})(W\backslash T_x)] &\leq \mathbb{E}[(\xi\backslash\xi^{x!})(W\backslash T_x)] + \mathbb{E}[(\xi^{x!,\Xi}\backslash\xi^{x!})(W\backslash T_x)] \\
&= \mathbb{E}[\xi(W\backslash T_x)] - \mathbb{E}[\xi^{x!}(W\backslash T_x)] + \mathbb{E}[\xi^{x!,\Xi}(W\backslash T_x)] - \mathbb{E}[\xi^{x!}(W\backslash T_x)]
\end{aligned} \quad (37)$$

Now if we look at the last eqaution above on the left side element $\mathbb{E}[\xi(W\backslash T_x)] - \mathbb{E}[\xi^{x!}(W\backslash T_x)]$ This is the same as we used in (27). This means we can write this as (29) as in the increased section.

Now we take the same step with the right hand side by multiplying it with,

$$\begin{aligned} &\mathbb{E}[\tilde{g}(x,\xi)](\mathbb{E}[\xi^{x!,\Xi}(W\backslash T_x)] - \mathbb{E}[\xi^{x!}(W\backslash T_x)]) \\ &= \mathbb{E}[\xi^{x}(W\backslash T_x)\tilde{g}(x,\xi^{x})] - \mathbb{E}[\xi^{x}(W\backslash T_x)]\mathbb{E}[\tilde{g}(x,\xi^{x})] \\ &= Cov(\xi^{x}(W\backslash T_x), \tilde{g}(x,\xi^{x})). \end{aligned} \tag{38}$$

Now we use that a reduced palm process $\xi^{x!}$ is a determinantal process itself therefore negatively associated by Proposition 3.15. we want to consider a auxiliary function as in (31) We also know that

$$\mathrm{Cov}(\min\{k, \xi^{x!}(W\backslash T_x))\}, f^{(k)}(\xi^{x!}) \leq 0$$

Hence by monotone convergence we have,

$$\begin{aligned} &\lim_{k\to\infty}\mathrm{Cov}(\min\{k, \xi^{x!}(W\backslash T_x))\}, f^{(k)}(\xi^{x!}) \\ &= \mathrm{Cov}(\xi^{x!}(W\backslash T_x), f(\xi^{x!})) \\ &= \mathrm{Cov}(\xi^{x!}(W\backslash T_x), \xi^{x!}(S_x) + \mathrm{Cov}(\xi^{x!}(W\backslash T_x), \tilde{g}(x,\xi^{x!})) \end{aligned}$$

This means that we can bound (37) by $-\mathrm{Cov}(\xi^{x!}(W\backslash T_x), \xi^{x!}(S_x))$ Therfore we can proceed as in the increasing part and get the same bound as in (36)

Bounding$\mathbf{T_4}$. For each $x \in W$ we have

$$\begin{aligned} &\mathbb{E}[\sum_{y\in\xi^{x,\Xi}\cap\xi\cap W\backslash T_x} |\tilde{g}(y,\xi^{x,\Xi}) - \tilde{g}(y,\xi)|] \\ &= \mathbb{E}[\sum_{y\in\xi^{x,\Xi}\cap\xi\cap W\backslash T_x} \mathbb{1}\{(\xi^{x,\Xi}\Delta\xi)\cap S_x \neq \emptyset\}|\tilde{g}(y,\xi^{x,\Xi}) - \tilde{g}(y,\xi)|] \\ &\leq \mathbb{E}[\sum_{z\in(\xi^{x,\Xi}\Delta\xi)\cap(W\oplus S_x)\backslash T_X}\ \sum_{y\in\xi^{x,\Xi}\cap\xi\cap S_Z} |\tilde{g}(y,\xi^{x,\Xi}) - \tilde{g}(y,\xi)|] \\ &\leq \mathbb{E}[\sum_{z\in(\xi^{x,\Xi}\Delta\xi)\cap(W\oplus S_x)\backslash T_X} \max_{\{\omega\in\{\xi^{x,\Xi}\},\xi\}} \sum_{y\in\omega\cap S_z} \tilde{g}(y,\xi^{x,\Xi}) - \tilde{g}(y,\xi)|] \end{aligned} \tag{39}$$

Here we can use equation (17) to bound the sum inside we bound it by,

$$\alpha|S|\mathbb{E}[(\xi^{x,\Xi})((W\oplus S)\backslash T_x)].$$

Hence if we use (36) with the change W by $W \oplus S$ this gives us,

$$\begin{aligned} &T_4 := \rho\int_W \mathbb{E}[\sum_{y\in\xi^{x}\cap\xi^{x,\Xi}\cap W\backslash T_x} |g(y,\xi^{x}) - g(y,\xi^{x,\Xi})|]\rho(x)dx \\ &\leq \alpha|S|(1+\rho\sup_{x\in W}|S|)|W\oplus S|^2 sup_{y\in T}\phi(\|y\|)^2 + 3^{\frac{5}{\alpha}2}\|K\||W\oplus S|^2|S|^2\phi(d(T,S)) \end{aligned} \tag{40}$$

Now we can take step 2: We will complete the proof in the last step. Let $\boldsymbol{L}$ and $\boldsymbol{L}_{tr}$ denote the intensity measures of the truncated process Ξ_{tr} from Ξ and and Ξ_{tr}. by (11) we can write the truncated intensity measure,

$$\boldsymbol{L}_{tr}(A) = \rho \int_{A \cap W} \mathbb{E}[g(x, \xi^x) \mathbb{1}\{\mathcal{S}(x, \xi^x) \subset S_x\}] dx, A \in \mathcal{B}^d.$$

Let ζ_{tr} be a Poisson process with intensity measure $\boldsymbol{L}_{tr}$ we can use the triangle inequality to write the KR- distance of Ξ and ζ ,

$$\boldsymbol{d_{KR}}(\Xi, \zeta) \leq \boldsymbol{d_{KR}}(\Xi, \Xi_{tr}) + \boldsymbol{d_{KR}}(\Xi_{tr}, \zeta_{tr}) + \boldsymbol{d_{KR}}(\zeta_{tr}, \zeta)$$

From section 3.6 we know that $\boldsymbol{d_{KR}}(\zeta_{tr}, \zeta) \leq \boldsymbol{d_{TV}}(\boldsymbol{L}, \boldsymbol{L}_{tr})$ and $\boldsymbol{d_{KR}}(\Xi, \Xi_{tr}) \leq \boldsymbol{d_{TV}}(\Xi, \Xi_{tr})$, where

$$\boldsymbol{d_{TV}}(\Xi, \Xi_{tr}) = \mathbb{E}[\Xi(W)] - \mathbb{E}[\Xi_{tr}(W)] = \boldsymbol{d_{TV}}(\boldsymbol{L}, \boldsymbol{L}_{tr})$$

and

$$\boldsymbol{d_{TV}}(\boldsymbol{L}, \boldsymbol{L}_{tr}) = \rho \int_{W} \mathbb{E}[g(x, \xi^x) \mathbb{1}\{\mathcal{S}(x, \xi^x) \not\subset S_x\}] dx$$

This gives us that,

$$\boldsymbol{d_{KR}}(\Xi, \zeta) \leq \boldsymbol{d_{KR}}(\Xi_{tr}, \zeta_{tr}) + 2\rho \int_{W} \mathbb{E}[g(x, \xi^x) \mathbb{1}\{\mathcal{S}(x, \xi^x) \not\subset S_x\}] dx. \tag{41}$$

Combining (41) with (36) and (40) proof statement above. □

6.2 Proof Theorem 4.3

In the proof we gonna use Theorem 3.21 This implies that the mapping $r \mapsto \mathbb{P}(\xi(B_r) = 0)$ is continuous such that $\mathbb{P}(\xi(B_r) = 0) \downarrow 0$ as $r \to \infty$. Now we can write the Definition 3.9 and Definition 4.2 knowing that the intensity of Ginibre process is π^{-1} , then for all $\tau > 0$ there exist an increasing sequence $(v_n)_{n \in \mathbb{N}}$ such that:

$$\boldsymbol{L}_n(A) := \mathbb{E}[\Xi_n(A)] = \frac{|A \cap B_n|}{\pi} \mathbb{P}(\xi^{0!}(B_{V_n}) = 0) = \frac{\tau |A \cap B_n|}{\pi n^2}, n \in \mathbb{N}, A \in \mathcal{B}^2 \tag{42}$$

To determine the asomptotic behavior of v_n as $n \to \infty$, we use that by [1, theorem 26]

$$\mathbb{P}(\xi^{0!}(B_{v_n}) = 0) = e^{v_n^2} \mathbb{P}(\xi(B_{v_n}) = 0) \tag{43}$$

Also by [1, proposition 7.3.1],

$$lim_{r \to \infty} \frac{1}{r^4} log \mathbb{P}(\xi(B_r) = 0) = -\frac{1}{4} \tag{44}$$

Here we can re-write $\log(\mathbb{P}(\xi(B_{v_n})=0))$as

$$\log(n^2 e^{v_n^2}\mathbb{P}(\xi(B_{v_n})=0)) - 2\log n - v_n^2$$

From (42) and (43) we can write the the equation as,

$$\log((\mathbb{P}(\xi(B_{v_n})=0)) = \log(\tau) - 2\log n - v_n^2.$$

Now multiply both sides with $\frac{1}{v_n^4}$. We get,

$$\frac{1}{v_n^4}\log((\mathbb{P}(\xi(B_{v_n})=0)) = \frac{1}{v_n^4}\log(\tau) - \frac{2\log n}{v_n^4} - \frac{1}{v_n^2}.$$

Take the limit where we let $n \to \infty$ we know that v_n also goes to infinity then, this will look like this,

$$\lim_{n\to\infty}\frac{1}{v_n^4}\log((\mathbb{P}(\xi(B_{v_n})=0)) = \lim_{n\to\infty}(\frac{1}{v_n^4}\log(\tau) - \frac{2\log n}{v_n^4} - \frac{1}{v_n^2})$$

use equation (44) for the left hand side and the idea $\lim_{n\to\infty}\frac{1}{v_n^2} \to 0$ holds,

$$-\frac{1}{4} = 0 - \lim_{n\to\infty}\frac{2\log n}{v_n^4} - 0$$

$$\frac{v_n^4}{\log n} \to 8 \text{ as } n \to \infty$$

With this statement we can start the proof of 4.3

Proof. We first look back at the Theorem 4.1, and write Given $n \in \mathbb{N}$, we choose ξ as the Ginibre process, let $g(x,\omega) := \mathbb{1}\{\omega(B_{v_n}\backslash\{x\}) = 0\}$ what means that we look around if there are no point in a radius v_n around the point, $S_x := B_{v_n}(x)$ and $T_x := B_{\log n}, x \in \mathbb{R}^2$. We can easily say that g is stabilizing with respect to the stopping set $\mathcal{S}(x,\omega) := S_x, \omega \in \boldsymbol{N}$. Now apply Theorem 4.1 to the process Ξ_n as in 4.2, where we choose ζ as a stationary Poisson process with intensity $\frac{\tau}{\pi n^2}$. Then we can define the intensity measure of $\zeta \cap B_n$ as $\boldsymbol{L}_n$ given as (42). We have to first check conditions (17) and (18). Note that for all $\omega \in \boldsymbol{N}$ and $n \in \mathbb{N}$,

$$\bigcap_{x\in\Xi_n[\omega]} B_{v_2/2}(x) = \emptyset.$$

If we take $\alpha := \mathcal{K}_d 2^{-d} v_n^{-d} \leq \mathcal{K}_d 2^{-d} v_1^{-d}$ for $\mathcal{K}_d \in \mathbb{R}, d \in \mathbb{N}$ we now it holds for (17), also it is clear that $g(x,\omega_1) \geq g(x,\omega_2)$ for $\omega_1 \subset \omega_2$. Because ω_2 has more points such that there are more points such that the distance between points get smaller.

Since g is deterministically stabilizing, we have that $E_1 = 0$ because we have no elements in the stopping set outside the ball around x. We will also know that if we look at E_2 we can bound this by taking the $\mathbb{E}[g(x,\omega)] \leq 1$ so bound this by 1. this means we can take a integrate 1 over the two set times the intensity $\frac{\tau}{\pi n^2}$. this is just a constant value determined by n , so we can bound this by a constant multiplied by $\frac{(\log n)^2}{n}$. This can also be done for the element in F. Now we can by contractivity property of the KR distance and use 4.1 we can also say that their intensity is the same makes them total variation also zero, we have

$$\begin{aligned} \boldsymbol{d_{KR}}(\Psi_n, \nu \cap B_1) \leq& \boldsymbol{d_{KR}}(\Xi_n, \zeta \cap B_n) \\ &\leq \int_{B_n} \int_{T_x} \mathbb{P}(\xi^{x!,y!}(B_{v_n}(x) \cup Bv_n(y)) = 0) \rho^{(2)}(x,y) dy dx + \beta \frac{(\log n)^2}{n}, \end{aligned} \tag{45}$$

for some $\beta > 0$. Thus we have to bound the integral above. By Theorem 3.27 we know the reduced Palm process $\xi^{x!,y!}$ is a determinantal process itself. With this we can conclude with Theorem 3.15 that this is also negatively associations. Recall that a point process ζ has negative associations if $\mathbb{E}[f(\zeta)g(\zeta)] \leq \mathbb{E}[f(\zeta)]\mathbb{E}[g(\zeta)]$ for every pair f, g of a real bounded increasing or decreasing functions that are measurable with respect to complementary subsets A, A^c of $\mathbb{R}^d$, meaning that a function is measurable with respect to A if it is measurable with respect to $\mathcal{N}_A$. We will apply this with a specific decreasing function $f(\mu) = \mathbb{1}\{\mu(B_{v_n}(x) = 0)\}$ and $f(\mu) = \mathbb{1}\{\mu(B_{v_n}(y) \backslash B_{v_n}(x) = 0)\}$.. If we fill this in the definition of the negative associations with the idea that the expected value of a indicator function is a probability of the set happening, this gives us,

$$\mathbb{P}(\xi^{x!,y!}(B_{v_n}(x) \cup B_{v_n}(y)) = 0) \leq \mathbb{P}(\xi^{x!,y!}(B_{v_n}(x)) = 0) \mathbb{P}(\xi^{x!,y!}(B_{v_n}(y) \backslash B_{v_n}(x)) = 0). \tag{46}$$

To bound the first probability on the right side we look at [8, Theorem 3] This states that for a reduced Palm process $\xi^{x!}$ of ξ such that their exist a $\xi^{x!,y!} \subset \xi^{x!}$ and $|\xi^{x!} \backslash \xi^{x!,y!}| \leq 1$ a.s. This tells us that their can only be one extra point in this specific set $\xi^{x!}$.This gives,

$$\mathbb{P}(\xi^{x!,y!}(B_{v_n}(x)) = 0) \leq \mathbb{P}(\xi^{x!}(B_{v_n}) \leq 1)$$

Now we can use the same to the determinantal process $\xi^{x!}$ and obtain the bound

$$\mathbb{P}(\xi^{x!}(B_{v_n}(x)) \leq 1) \leq \mathbb{P}(\xi(B_{v_n}(x)) \leq 2) = \mathbb{P}(\xi(B_{v_n}) \leq 2),$$

where the last equation holds by stationary distribution of ξ. As we know from the begining of the proof we talked about that, the set of absolute values of the points of the Ginibre process ξ has the same distribution as the sequence $(X_i)_{i \in N}$ of independent random variables with $X_i^2 \sim \text{Gamma}(i, 1)$. This gives us,

$$
\begin{aligned}
\mathbb{P}(\xi(B_{v_n}) \leq 2) =& \mathbb{P}(\#\{j \in \mathbb{N} : X_j \leq v_n\} \leq 2) \\
\leq& \mathbb{P}(\{j \in \{1, \ldots, v_n^2\} : X_j \leq v_n\} \leq 2) \\
=& \mathbb{P}(\bigcup_{i=1}^{v_n^2} \bigcup_{j=1 j\neq i}^{v_n^2} \{\forall k \in \{1, \ldots, v_n^2\} \backslash i, j : X_k > v_n\})
\end{aligned}
\tag{47}
$$

we have put slight abuse of notation because v_n^2 has to be written down as $\lfloor v_n^2 \rfloor$. But with this we can write the union as a sum so we get:

$$
\sum_{i=1}^{v_n^2} \sum_{j=1,j\neq i}^{v_n^2} \mathbb{P}(\forall k \in \{1, \ldots, v_n^2 \backslash i, j : X_k > v_n\}) = \sum_{i=1}^{v_n^2} \sum_{j=1,j\neq i}^{v_n^2} \prod_{k=1,k\neq i,j}^{v_n^2} \mathbb{P}(x_k^2 > v_n^2).
$$

Let $t < 1$. The moment generating function $M_{x_k^2}(t) = \mathbb{E}[e^{-tr^2}] = (1-t)^{-k}$ of X_k^2 exists and we obtain from the Chernoff bound that,

$$
\mathbb{P}(x_k^2 > r^2) \leq e^{-tr^2}(1-t)^{-k}.
$$

For $k < r^2$, this bound is maximized for $t = 1 - \frac{k}{r^2}$, which gives

$$
\mathbb{P}(\xi(B_{v_n}) \leq 2) \leq \sum_{i=1}^{v_n^2} \sum_{j=1,j\neq i}^{v_n^2} \prod_{k=1,k\neq i,j}^{v_n^2} e^{-(1-\frac{k}{v_n^2})v_n^2 - k\log(\frac{k}{v_n^2})} = \sum_{i=1}^{v_n^2} \sum_{j=1,j\neq i}^{v_n^2} \prod_{k=1,k\neq i,j}^{v_n^2} e^{-(v_n^2 - k - k\log(\frac{k}{v_n^2})}.
$$

Using here that $u \mapsto u - u\log(\frac{u}{r^2})$ is increasing for $u \leq r^2$, we find this,

$$
\begin{aligned}
\mathbb{P}(X_k^2 > r^2) \leq& \sum_{i=1}^{v_n^2} \sum_{j=1,j\neq i}^{v_n^2} \prod_{k=1,k\neq i,j}^{v_n^2} e^{-(v_n^2 - k - k\log(\frac{k}{v_n^2})} \\
\leq& v_n^4 \prod_{k=3}^{v_n^2} e^{-(v_n^2 - k - k\log(\frac{k}{v_n^2})} \\
=& v_n^2 e^{-\frac{1}{2}(v_n^2-3)(v_n^2-2) - v_n^4 \int_{3/v_n^2}^1 ulog(u)dx + O(v_n^2 \log v_n^2)} \\
=& e^{-\frac{1}{4}v_n^4(1+o(1))}(1+o(1))
\end{aligned}
\tag{48}
$$

as $n \to \infty$ where we have that $\int_0^1 ulog(u)dx = -\frac{1}{4}$. Next we bound the right side in probability (46). By coupling with the same argument above,

$$
\mathbb{P}(\xi^{x!,y!}(B_{v_n}(y) \backslash B_{v_n}(x)) = 0) \leq \mathbb{P}(\xi(B_{v_n}(y) \backslash B_{v_n}(x)) \leq 2).
$$

Next we note that $B_{v_{n/2}}(y + \frac{v_n(y-x)}{2|y-x|}) \subset B_{v_n}(y) \backslash B_{v_n}(x)$ holds this only works because we know that x is not a point in $B_{v_n}(y)$. Hence the last probability is bounded by,

$$
\mathbb{P}(\xi(B_{v_{n/2}}(y + \frac{v_n(y-x)}{2|y-x|})) \leq 2) = \mathbb{P}(\xi(B_{v_n} \leq 2) \leq e^{-\frac{1}{4}(v_n/2)^4(1+o(1))}
$$

By the stationary distribution and the argument above. Using the estimates as above with $v_n/2$ instead of v_n. also we know that the correlation function can be bounded by the two intensity $p^2(x,y) \leq 1/\pi^2$ for all $x, y \in \mathbb{R}$, we then can for all $\varepsilon > 0$ at the bound,

$$\int_{B_n}\int_{T_x} \mathbb{P}(\xi^{x!,y!}(B_{v_n}(x) \cup Bv_n(y)) = 0)\rho^{(2)}(x,y)dydx + \beta\frac{(\log n)^2}{n} \leq \frac{n(\log n)^2)}{\pi}e^{-\frac{1}{4}v_n^4}e^{-\frac{1}{64}v_n^4},$$

We used (42) to bound it to the intensity measure, also We can rewrite the a small part above like this ,

$$\begin{aligned} ne^{-\frac{1}{4}v_n^4}e^{-\frac{1}{64}v_n^4} =& n^{\log_n(n)}e^{-\frac{1}{4}v_n^4}e^{-\frac{1}{64}v_n^4} \\ =& n^{\log_n(n)}n^{-\frac{1}{4}v_n^4\log_n(e)}n^{-\frac{1}{64}v_n^4\log_n(e)} \\ =& n^{\log_n(n)-\frac{1}{4}v_n^4\log_n(e)-\frac{1}{64}v_n^4\frac{\log(e)}{\log(n)}} \\ =& n^{\log_n(n)-\frac{1}{4}v_n^4\log_n(e)-\frac{1}{64}\frac{v_n^4}{\log(n)}} \end{aligned} \tag{49}$$

We know that $\frac{v_n^4}{\log(n)} \to 8$ as $n \to \infty$, hence we can bound the statement above by a $\varepsilon > 0$ we get,

$$n^{\log_n(n)-\frac{1}{4}v_n^4\log_n(e)-\frac{1}{64}\frac{v_n^4}{\log(n)}} \leq n^{\varepsilon-\frac{1}{8}},$$

Put it all together we get,

$$\int_{B_n}\int_{T_x} \mathbb{P}(\xi^{x!,y!}(B_{v_n}(x) \cup Bv_n(y)) = 0)\rho^{(2)}(x,y)dydx + \beta\frac{(\log n)^2}{n} \leq \frac{(\log(n))^2}{\pi}n^{\varepsilon-\frac{1}{8}}.$$

What finish the proof such we can finally write (45) as

$$\boldsymbol{d_{KR}}(\Psi_n, \nu \cap B_1) \leq \beta\frac{(\log n)^2}{n} + \frac{(\log(n))^2}{\pi}n^{\varepsilon-\frac{1}{8}} = Cn^{\varepsilon-\frac{1}{8}}$$

For some constant $C > 0$.

$\square$

7 Conclusion

In this thesis we proved the theorem 4.1 and 4.3. The first proof states that the Kantorovich-Rubinstein distance for a stationary Poisson process with a functional of a determinantal point process, as we defined in section 3, Also we want the correlation kernel K to decay fast and the thinning function being stable. This distance is comparable to other distances between point processes. The proof mainly utilizes the idea that the Palm distribution of a DPP is a DPP with different correlation function. Also we utilizes the idea that a DPP is negatively associated property.

The second proof states that the Kantorovich-Rubenistein distance between the Poisson process and the Ginibre process, under the conditions specified in theorem 4.1, can be bounded.Here, the thinning function is taken to be a deterministic function that removes all points having a neighbor closer than 1 unit away. We scale the functional of the Ginibre process by a factor n, Then consider the limit as $n \to \infty$. Then the distance can asymptotically get bounded. With this proof we can explore application such as the largest distance to the nearest neighbor. By Corollary 4.4 states that as $n \to \infty$ we know the $max_{x \in \xi \cap B} |B_{nn}(x, \xi)| \to 2\pi\sqrt{logn}$.

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