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The Gibbs phenomenon

Bachelor's Project Mathematics

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1 Introduction

We can approximate a periodic function by using Fourier series. This approximation encounters some problems whenever we try to approximate a discontinuous function. Namely, whenever we approximate a discontinuous function, an over- or undershoot happens near the jump discontinuities of this function. This over- or undershoot has a limit of about 9% of the size of the jump. This phenomenon is called the Gibbs phenomenon and it can be seen in some aspects of our daily life. Some of these aspects can be seen as a disadvantage and some of them can be seen as an advantage. We will look at some examples of both cases.

Example 1.1. Almost everyone has used an amplifier once in their life. These amplifiers sometimes experience clipping. Clipping is the event where the amplifier tries to deliver a voltage output that is higher than its maximum capacity. In doing so a waveform distortion is created. In other words, the sounds produced by the amplifier are not as desired. This clipping is one of the undesired results of the Gibbs phenomenon (Esqueda et al., 2016).

Example 1.2. The Gibbs phenomenon can also be used as an advantage. One of these advantages is the use of the Gibbs phenomenon for so called edge detection (Jerri, 1998, p. 80). In this edge detection we have an image with its contrasts that represent edges or contours. These edges can be modeled as a jump discontinuity in a Fourier representation and thus we are dealing with the Gibbs phenomenon. What stands out when using this edge detection is that the presence of the Gibbs phenomenon, and thus the presence of an over- or undershoot, results in clearer images. The improvement of the images is greater when the overshoots have a steeper rise. Hence, we can use the Gibbs phenomenon in creating clearer images. This result is of great importance in two applications. We will enlighten both applications briefly.

1. Magnetic Resonance Imaging (MRI) uses images to detect defects or diseases. It has been shown in the research of Roerdink and Zwaan that the Gibbs phenomenon enhances the image of the cross section of the heart (Roerdink & Zwaan, 1993; Zwaan, 1990). In this specific case we can thus detect heart defects or heart diseases better.

2. If a signal with a shock is analyzed, we can use the Gibbs phenomenon to locate these shocks. We can locate these shocks by locating the over-/undershoots. An estimate of the location of the shocks is used to get an advantage in signaling (Jerri, 1998, p. 82).

These examples show us that the Gibbs phenomenon does not only appear as a problem, but also as an aid in daily problems.

Since it is of importance to solve the problems and to achieve a bigger advantage from the Gibbs phenomenon, we strive to understand it better. In this paper, we will do so by looking at the Gibbs phenomenon for the Fourier approximation of two functions. However, we will also look at the Gibbs phenomenon in other approximations. We will look at the similarities and the differences for both approximations.

2 Approximation in normed spaces

As we have stated in the introduction, the Gibbs phenomenon can be seen when approximating a function using Fourier series. This approximation is based on the theory of best approximation. In this section, we will therefore focus on the theory of the best approximation. This theory is based on our findings in (De Snoo & Sterk, 2023; Shadrin, 2005).

Definition 2.1 (Best approximation)**.** Let X be a normed linear space and let $V \subset X$ be a subset. A point $v_0 \in V$ is called a best approximation of $x \in X$ if

$$||x - v_0|| = d(x_0, V) := \inf\{||x - v|| : v \in V\}.$$

Example 2.1. Let $X = \mathbb{R}^2$ and $V = \{(v, v) : v \in \mathbb{R}\}$. If we take $x = (0, 1)$, we can compute the best approximation $v_0 \in V$ using the L_1 norm. We compute

$$\begin{aligned} ||x - v_0||_1 &= \inf\{||x - v||_1 : v \in V\} \\ &= \inf\{|x_1 - v_1| + |x_2 - v_2| : v \in V\} \\ &= \inf\{|0 - v_1| + |1 - v_1| : v_1 \in \mathbb{R}\} \\ &= \inf\{|v_1| + |1 - v_1| : v_1 \in \mathbb{R}\}. \end{aligned}$$

If we then take $0 \leq v_1 \leq 1$, we obtain that $v_0 = (v_1, v_1)$ is a best approximation with $||x - v_0||_1 = 1$.
If we take $v_1 < 0$, we obtain that $v_0 = (v_1, v_1)$ with $||x - v_0||_1 = 1 - 2v_1 > 1$. Hence, this will not give us a best approximation.
If we take $v_1 > 1$, we obtain that $v_0 = (v_1, v_1)$ with $||x - v_0||_1 = 2v_1 - 1 > 1$. Hence, this will not give us a best approximation either.
In conclusion, we have infinitely many best approximations, namely the points $v_0 = (v_1, v_1)$ with $0 \leq v_1 \leq 1$.

Example 2.2. Let $X = \mathbb{R}^2$ and $V = \{(v, v) : v \in \mathbb{R}\}$. If we take $x = (0, 1)$, we can compute the best approximation $v_0 \in V$ using the L_∞ norm. We compute

$$\begin{aligned} ||x - v_0||_\infty &= \inf\{||x - v||_\infty v \in V\} \\ &= \inf\{\max\{|x_1 - v_1|, |x_2 - v_2|\} : v \in V\} \\ &= \inf\{\max\{|0 - v_1|, |1 - v_1|\} : v_1 \in \mathbb{R}\} \\ &= \inf\{\max\{|v_1|, |1 - v_1|\} : v_1 \in \mathbb{R}\}. \end{aligned}$$

If we then take $v_1 = \frac{1}{2}$, we obtain that $v_0 = (v_1, v_1)$ is a best approximation with $||x - v_0||_\infty = \frac{1}{2}$.
If we take $v_1 < \frac{1}{2}$, we obtain that $v_0 = (v_1, v_1)$ with $||x - v_0||_\infty = 1 - v_1 > \frac{1}{2}$. Hence, this will not give us a best approximation.
If we take $v_1 > \frac{1}{2}$, we obtain that $v_0 = (v_1, v_1)$ with $||x - v_0||_\infty = v_1 > \frac{1}{2}$. Hence, this will not give us a best approximation either.
In conclusion, we have a unique best approximation, namely the point $v_0 = (v_1, v_1) = (\frac{1}{2}, \frac{1}{2})$.

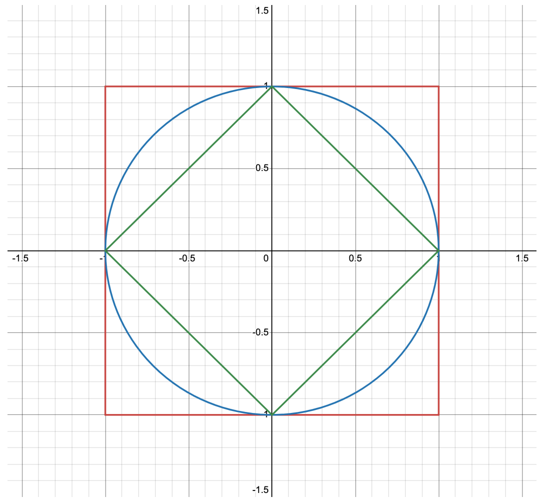

Figure 1: The unit spheres for respectively the L_1 norm (green), the L_∞ norm (red) and the L_2 norm (blue).

Example 2.3. Let $X = \mathbb{R}^2$ and $V = \{(v, v) : v \in \mathbb{R}\}$. If we take $x = (0, 1)$, we can compute the best approximation $v_0 \in V$ using the L_2 norm. We compute

$$\begin{aligned}
||x - v_0||_2 &= \inf\{||x - v||_2 : v \in V\} \\
&= \inf\{\sqrt{|x_1 - v_1|^2 + |x_2 - v_2|^2} : v \in V\} \\
&= \inf\{\sqrt{|0 - v_1|^2 + |1 - v_1|^2} : v_1 \in \mathbb{R}\} \\
&= \inf\{\sqrt{|v_1|^2 + |1 - v_1|^2} : v_1 \in \mathbb{R}\} \\
&= \inf\{\sqrt{2v_1^2 - 2v_1 + 1} : v_1 \in \mathbb{R}\} \\
&= \inf\{\sqrt{2\Big(v_1^2 - 2 \cdot \frac{1}{2}v_1 + \frac{1}{4} + \frac{1}{4}\Big)} : v_1 \in \mathbb{R}\} \\
&= \inf\{\sqrt{2\Big((v_1 - \frac{1}{2})^2 + \frac{1}{4}\Big)} : v_1 \in \mathbb{R}\} \\
&= \{\sqrt{2\Big((v_1 - \frac{1}{2})^2 + \frac{1}{4}\Big)} : v_1 = \frac{1}{2}\} \\
&= \sqrt{\frac{1}{2}}.
\end{aligned}$$

Hence, we see that we have that $v_0 = (v_1, v_1) = (\frac{1}{2}, \frac{1}{2})$ is the best approximation with $||x - v_0||_2 = \sqrt{\frac{1}{2}}$.

We thus see, that using different norms, we obtain different best approximations. Therefore, we can say that the existence and uniqueness depends on the choice of the norm.

Definition 2.2. (Strictly convex normed linear space) A normed linear space X is called strictly convex if

$$||x|| = ||y|| = 1, \qquad x \neq y \implies ||\frac{1}{2}(x + y)|| < \frac{1}{2}||x|| + \frac{1}{2}||y|| = 1 \qquad \forall x, y.$$

We can rephrase this definition such that a norm is strictly convex when given any two distinct points x and y on the unit sphere, the line segment between these points is not (partially) contained on the boundary itself. Looking at figure 1, we see that this is however the case for the L_1 and the L_∞ norms. Hence, these norms are not strictly convex. The boundary of the L_2 norm does not contain any line segments and thus we can say that the L_2 norm is strictly convex. We can also come to this conclusion using a lemma.

Lemma 2.1. Every norm that comes from an inner product space is strictly convex.

Proof. Let us set $||x|| = ||y|| = 1$ and $x \neq y$. Then by the parallelogram law and absolute homogeneity we obtain

$$\begin{aligned}||\frac{1}{2}(x+y)||^2 + ||\frac{1}{2}(x-y)||^2 &= 2||\frac{1}{2}x||^2 + 2||\frac{1}{2}y||^2 \\ &= 2\cdot\frac{1}{4}||x||^2 + 2\cdot\frac{1}{4}||y||^2 \\ &= \frac{1}{2}\cdot 1^2 + \frac{1}{2}\cdot 1^2 \\ &= 1\end{aligned}$$

and thus

$$||\frac{1}{2}(x+y)||^2 = 1 - ||\frac{1}{2}(x-y)||^2.$$

Since $x \neq y$, this latter inner product will not be equal to zero and by definition of an inner product it is nonnegative. Hence, we conclude

$$||\frac{1}{2}(x+y)||^2 < 1,$$

and thus by definition 2.2 we can say that the norm is strictly convex. □

Lemma 2.2. Let X be a strictly convex normed linear space and let $V \subset X$ be a subspace. Then, for each element $x \in X$, there is at most one element of best approximation.

Proof. Suppose that v_1 and v_2 are two distinct best approximations from V to x and $||x - v_i|| = \alpha$. Then we obtain

$$||x - \frac{1}{2}(v_1+v_2)|| = ||\frac{1}{2}(x-v_1) + \frac{1}{2}(x-v_2)|| < \frac{1}{2}||x-v_1|| + \frac{1}{2}||x-v_2|| = \alpha.$$

Then by definition v_1 and v_2 are not the best approximations. Thus, we obtain a contradiction and therefore we have that $v_1 = v_2$. □

Since the L_2 norm is strictly convex, as stated before, we can conclude that its best approximations are always unique. Therefore, we will use the L_2 norm in the rest of our approximations.

Theorem 2.1. Let X be an inner product space and let $V \subset X$ be a linear subspace with $\dim V < \infty$. Then there exists a unique best approximation $v_0 \in V$ for all $x \in X$. If $\{e_1, e_2, \ldots, e_n\}$ is an orthonormal basis for V, this best approximation is given by

$$v_0 = \sum_{j=1}^{n} < x, e_j > e_j$$

Proof. Let us set $c_j = \langle x, e_j\rangle$, then

$$\begin{aligned}\Big|\Big|x - \sum_{j=1}^{n}\lambda_j e_j\Big|\Big|^2 &= ||x||^2 - \sum_{j=1}^{n}\overline{\lambda_j}c_j - \sum_{j=1}^{n}\lambda_j\overline{c_j} + \sum_{j=1}^{n}|\lambda_j|^2 \\ &= ||x||^2 + \sum_{j=1}^{n}|\lambda_j - c_j|^2 - \sum_{j=1}^{n}|c_j|^2,\end{aligned}$$

where the minimum is attained if and only if $\lambda_j = c_j$ for all j. □

3 The Gibbs phenomenon for Fourier series

3.1 Fourier series

The concept of the Fourier series has been named after Joseph Fourier (1768-1830). The Fourier series arose out of two physical problems, the heat conduction of solids and the motion of a vibrating string. In his work *Théorie Analytique de la Chaleur*, which can be translated to *The Analytical Theory of Heat*, Fourier looked at one of those physical problems. He stated that 'there is no function $f(x)$, or part of a function, which cannot be expressed by a trigonometric series' in this work (Abbott, 2015; Angelidis, 1996).

Fourier initially looked at even functions. After a while, he stated that a function $f(x)$ can be formulated as

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx) \tag{1}$$

given the suitable coefficients (a_n) and (b_n), which we will define in this chapter. The functions $f(x)$ need not be even in this approximation, as long as $f(x)$ is continuous. For all of the deductions that will follow, we use information from (Abbott, 2015).

The first coefficient we want to define is a_0. Before we will be able to define this coefficient, we consider the following integrals:

$$\int_{-\pi}^{\pi} \cos(nx)dx = 0 \tag{2}$$

and

$$\int_{-\pi}^{\pi} \sin(nx)dx = 0. \tag{3}$$

These integrals hold for all $n \in \mathbb{N}$. To be able to define a_0, we will now integrate both sides of our initial formulated function (1)

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)dx &= \int_{-\pi}^{\pi} \Big[a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)\Big]dx \\ &= \int_{-\pi}^{\pi} a_0 dx + \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)dx.\end{aligned}$$

Let us assume that

$$a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

converges uniformly to $f(x)$ on the interval $[-\pi, \pi]$. Then, we can continue with our derivation of a_0. We can interchange the sum and integral, simply because of the assumption. Then, taking the values of our integrals (2) and (3) we get

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)dx &= \int_{-\pi}^{\pi} a_0 dx + \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} a_n \cos(nx) + b_n \sin(nx)dx \\ &= a_0 \int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos(nx)dx + b_n \int_{-\pi}^{\pi} \sin(nx)dx \\ &= a_0 2\pi.\end{aligned}$$

Hence, we can define

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx. \tag{4}$$

In the same way, we can derive a_n and b_n. Let us start with the derivation of a_n. Before we will be able to define this coefficient, we must calculate the following two integrals. By using the sum and difference formula for cosine and by rewriting the square of cosine using the double angle formula for cosine, we obtain the following two cases for the first integral

$$\int_{-\pi}^{\pi} \cos(nx)\cos(mx)dx = \begin{cases} 0 & \text{if}\, n \neq m \\ \pi & \text{if}\, n = m \end{cases} \qquad \forall n, m \in \mathbb{N}. \tag{5}$$

By rewriting $\cos(nx)$ as a form of $\sin(nx)$ we can define the second integral

$$\int_{-\pi}^{\pi} \cos(mx)\sin(nx)dx = 0 \qquad \forall n, m \in \mathbb{N}. \tag{6}$$

To be able to define a_n, we will look at the case where $n = m$ and multiply both sides of our function (1) with $\cos(mx)$, to get

$$f(x)\cos(mx) = a_0\cos(mx) + \sum_{n=1}^{\infty} a_n\cos(nx)\cos(mx) + b_n\cos(mx)\sin(nx).$$

Then by integrating both sides of this equation, we obtain

$$\int_{-\pi}^{\pi} f(x)\cos(mx)dx = \int_{-\pi}^{\pi} \Big[a_0\cos(mx) + \sum_{n=1}^{\infty} a_n\cos(nx)\cos(mx) + b_n\cos(mx)\sin(nx)\Big]dx.$$

Still assuming that

$$a_0 + \sum_{n=1}^{\infty} a_n\cos(nx) + b_n\sin(nx)$$

converges uniformly to $f(x)$ on the interval $[-\pi, \pi]$, we can continue with our derivation of a_n. We can interchange the sum and integral again and by taking the values of our integrals (5) , (6) and $n = m$ we get

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\cos(mx)dx &= \int_{-\pi}^{\pi} a_0\cos(mx)dx + \sum_{n=1}^{\infty}\int_{-\pi}^{\pi} a_n\cos(nx)\cos(mx) + b_n\cos(mx)\sin(nx)dx \\ &= a_0\int_{-\pi}^{\pi}\cos(mx)dx + \sum_{n=1}^{\infty} a_m\int_{-\pi}^{\pi}\cos^2(mx)dx + b_m\int_{-\pi}^{\pi}\cos(mx)\sin(mx)dx \\ &= a_m\pi.\end{aligned}$$

Since we still look at the case where $n = m$, we can define

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx. \tag{7}$$

Note that this result can be seen as an application of theorem 2.1.
We will derive our last Fourier coefficient b_n in the same way as we derived a_n. However, instead of multiplying by $\cos(mx)$ we will now multiply by $\sin(mx)$. For this to work we define one more integral first, namely

$$\int_{-\pi}^{\pi} \sin(nx)\sin(mx)dx = \begin{cases} 0 & \text{if}\, n \neq m \\ \pi & \text{if}\, n = m \end{cases} \qquad \forall n, m \in \mathbb{N}. \tag{8}$$

As stated before, we will now multiply by $\sin(mx)$, which gives us

$$f(x)\sin(mx) = a_0\sin(mx) + \sum_{n=1}^{\infty} a_n\cos(nx)\sin(mx) + b_n\sin(nx)\sin(mx).$$

Then, by integrating both sides of this equation, we obtain

$$\int_{-\pi}^{\pi} f(x)\sin(mx)dx = \int_{-\pi}^{\pi} \Big[a_0 \sin(mx) + \sum_{n=1}^{\infty} a_n \cos(nx)\sin(mx) + b_n \sin(nx)\sin(mx)\Big] dx.$$

Still assuming that

$$a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

converges uniformly to $f(x)$ on the interval $[-\pi, \pi]$, we can continue with our derivation of b_n. We can interchange the sum and integral again. Then, by taking the values of our integrals (6) , (8) and $n = m$ we get

$$\begin{aligned}
\int_{-\pi}^{\pi} f(x)\sin(mx)dx &= \int_{-\pi}^{\pi} a_0 \sin(mx)dx + \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} a_n \cos(nx)\sin(mx) + b_n \sin(nx)\sin(mx)dx \\
&= a_0 \int_{-\pi}^{\pi} \sin(mx)dx + \sum_{n=1}^{\infty} a_m \int_{-\pi}^{\pi} \cos(mx)\sin(mx)dx + b_m \int_{-\pi}^{\pi} \sin^2(mx)dx \\
&= a_0 0 + \sum_{n=1}^{\infty} a_m 0 + b_m \pi \\
&= b_m \pi.
\end{aligned}$$

Since we still look at the case $n = m$, we obtain

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\sin(nx)dx. \tag{9}$$

Note that this can be seen as an application of theorem 2.1 again.
Altogether, we see that the Fourier series gives us an expansion of a function $f(x)$ of the form (1) in which the Fourier coefficients a_0, a_n and b_n are defined as (4), (7) and (9).

3.2 The Gibbs-Wilbraham constant

Now we look into the Gibbs phenomenon. Let us define a function f to be a piecewise smooth function that is defined on the interval $[-\pi, \pi)$, extended to the real line via periodicity. Moreover, we let f have at most finitely many discontinuities on this interval, where the discontinuities are all finite jumps. The Gibbs phenomenon then describes the overshoot of the partial sum of the Fourier series near the discontinuity of this function f. This phenomenon was first discovered by H. Wilbraham in 1848. The phenomenon was then rediscovered by Gibbs in 1898. Since Wilbrahams discovery was unknown from 1848 till 1914, the phenomenon was named after Gibbs.

Taking $x = c$ as one of the discontinuities of our function f as defined before, we define

$$\delta = \lim_{x \to c^+} f(x) - \lim_{x \to c^-} f(x) \tag{10}$$

and

$$\mu = \frac{1}{2}\Big[\lim_{x \to c^-} f(x) + \lim_{x \to c^+} f(x)\Big], \tag{11}$$

where we assume that $\delta > 0$ without loss of generality. Moreover, let us denote the n-th partial sum of the Fourier series of f as

$$S_n(f, x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos(kx) + b_k \sin(kx)). \tag{12}$$

Using (10) and (11) we denote the first local maximum of $S_n(f,x)$ to the left of the discontinuity at the point $x = c$ as $x_n < c$ such that

$$\lim_{n\to\infty} S_n(f, x_n) = \mu + \frac{\delta}{\pi} G$$

and denote the first local minimum of $S_n(f,x)$ to the right of the discontinuity at the point $x = c$ as $\xi_n > c$ such that

$$\lim_{n\to\infty} S_n(f, \xi_n) = \mu - \frac{\delta}{\pi} G.$$

In both the definition of x_n and of ξ_n we see the **Gibbs-Wilbraham constant**, which is denoted by G. This constant is defined in (Finch, 2003) as

$$\begin{aligned} G &= \int_0^\pi \frac{\sin(\theta)}{\theta} d\theta = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{(2n+1)(2n+1)!} = 1.8519370519... \\ &= \frac{\pi}{2}(1.1789797444...). \end{aligned}$$

Definition 3.1. (Sine integral function) We define the sine integral function as

$$Si(x) = \int_0^x \frac{\sin u}{u} du. \tag{13}$$

This function can be represented by the series expansion

$$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^{2k-1}}{(2k-1)(2k-1)!}$$

(Havil, 2010, p. 106). If we approximate this sum, by taking k up till 100, we have

$$\sum_{k=1}^{100} (-1)^{k-1} \frac{x^{2k-1}}{(2k-1)(2k-1)!} \approx 1.85193705198.$$

Note that this sum is thus the exact same as the one in the computation of the Gibbs-Wilbraham constant. We can thus conclude that $G = Si(\pi)$.

3.3 Two examples

This section will show the Gibbs phenomenon for some explicit functions.

3.3.1 Square wave

For this example, we used the work in (Jerri, 1998, p. 40-43). We take a look at the Gibbs phenomenon in the square wave of unit amplitude on the interval $(-\pi, \pi)$. We define this function as

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < \pi, \\ -1 & \text{if } -\pi < x < 0, \end{cases} \tag{14}$$

where there is a jump-discontinuity at $x = 0$. We denote the n-th partial sum of the Fourier series of this square wave as

$$S_n(f, x) = \frac{4}{\pi} \sum_{k=1}^{n} \frac{1}{2k-1} \sin((2k-1)x). \tag{15}$$

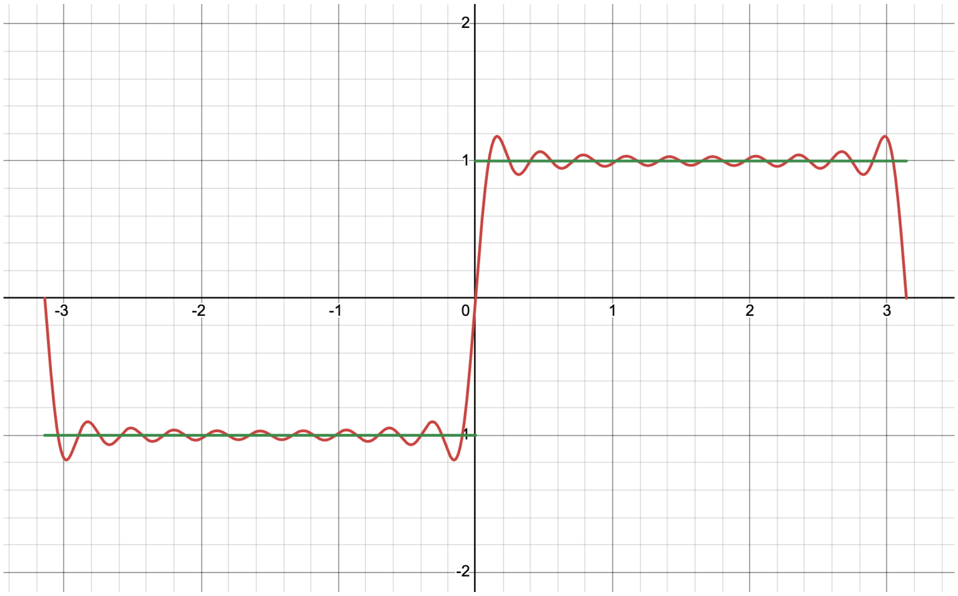

Figure 2: The square wave function and its Fourier series on the interval $(-\pi, \pi)$

Graphing both (21) and (15) for $n = 10$, we obtain the plot in figure 2. Note that we see the overshoots of S_n near the discontinuities of $f(x)$. To be able to locate these overshoots or extrema of $S_n(f, x)$, we need to rewrite the partial sum. First, we note that

$$\frac{1}{(2k-1)} \sin((2k-1)x) = \int_0^x \cos((2k-1)t)dt. \tag{16}$$

Then, by using (16), we can rewrite our partial sum as

$$\begin{aligned} S_n(f,x) &= \frac{4}{\pi} \sum_{k=1}^{n} \frac{1}{2k-1} \sin((2k-1)x) \\ &= \frac{4}{\pi} \sum_{k=1}^{n} \int_0^x \cos((2k-1)t)dt \\ &= \frac{4}{\pi} \int_0^x \Big[\sum_{k=1}^{n} \cos((2k-1)t) \Big] dt. \end{aligned}$$

Moreover, it follows that

$$\begin{aligned} 2\sin t \sum_{k=1}^{n} \cos((2k-1)t) &= \sum_{k=1}^{n} 2\sin t \cos((2k-1)t) \\ &= \sum_{k=1}^{n} [\sin((2k-1+1)t) + \sin((1-2k+1)t)] \\ &= \sum_{k=1}^{n} [\sin 2kt - \sin((2k-2)t)] \\ &= \sin 2t + \sin 4t - \sin 2t + \cdots + \sin 2nt - \sin((2n-2)t) \\ &= \sin 2nt. \end{aligned}$$

In other words, we have

$$\sum_{k=1}^{n} \cos((2k-1)t) = \frac{\sin 2nt}{2\sin t}. \tag{17}$$

This fraction is not defined for $t = 0$. However, we note

$$\lim_{t\to 0} \frac{\sin 2nt}{2\sin t} = \lim_{t\to 0} \frac{2n\cos 2nt}{2\cos t} = n.$$

Then, by using (17) in our partial sum containing the integral, we obtain

$$\begin{aligned} S_n(f,x) &= \frac{4}{\pi}\int_0^x \Big[\sum_{k=1}^{n} \cos((2k-1)t)\Big] dt \\ &= \frac{4}{\pi}\int_0^x \left[\frac{\sin 2nt}{2\sin t}\right] dt \\ &= \frac{2}{\pi}\int_0^x \left[\frac{\sin 2nt}{\sin t}\right] dt. \end{aligned}$$

This partial sum has its extrema at the points $x_a = \frac{a\pi}{2n}$ with $a \in \mathbb{Z}$. The maximum that we want to look at, namely the first maximum where $x > 0$ is $x_a = \frac{\pi}{2n}$. This yields

$$S_n\Big(f, \frac{\pi}{2n}\Big) = \frac{2}{\pi}\int_0^{\frac{\pi}{2n}} \left[\frac{\sin 2nt}{\sin t}\right] dt.$$

Note that if we take $n \to \infty$, the interval $\Big(0, \frac{\pi}{2n}\Big)$ that we are integrating over will get really small. Moreover, our value of t will therefore get small enough to approximate $\sin t \sim t$. This yields

$$\begin{aligned} S_n\Big(f, \frac{\pi}{2n}\Big) &\sim \frac{2}{\pi}\int_0^{\frac{\pi}{2n}} \left[\frac{\sin 2nt}{t}\right] dt = \frac{2}{\pi}\int_0^{\pi} \left[\frac{\sin y}{y}\right] dy \\ &= \frac{2}{\pi}\mathrm{Si}(\pi) = \frac{2}{\pi}\frac{\pi}{2}(1.1789797444...) = 1.1789797444..., \end{aligned}$$

where $\mathrm{Si}(\pi)$ is the sine integral function. Thus, we see that we have a maximum overshoot of about $\frac{1.1789797444...-1}{2} \cdot 100\% \approx 8.9\%$ of the jump as $n \to \infty$.

3.3.2 Saw-tooth function

Moreover, we take a look at the Gibbs phenomenon in the saw-tooth function defined by

$$f(x) = x \tag{18}$$

with $-\pi < x < \pi$ and a period of 2π, where there is thus a jump-discontinuity at $x = 2 + 4a$ for $a \in \mathbb{Z}$. We denote the n-th partial sum of the Fourier series of this saw-tooth function as

$$S_n(f,x) = 2\sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} \sin(kx). \tag{19}$$

Graphing both (18) and (19) for $n = 10$, we obtain the plot in figure 3.
Note that we see the overshoots of S_n near the discontinuities of $f(x)$. To be able to locate these overshoots or extrema of $S_n(f,x)$, we need to rewrite the partial sum. First, we note that

$$\frac{1}{k}\sin(kx) = \int_0^x \cos(kt)dt. \tag{20}$$

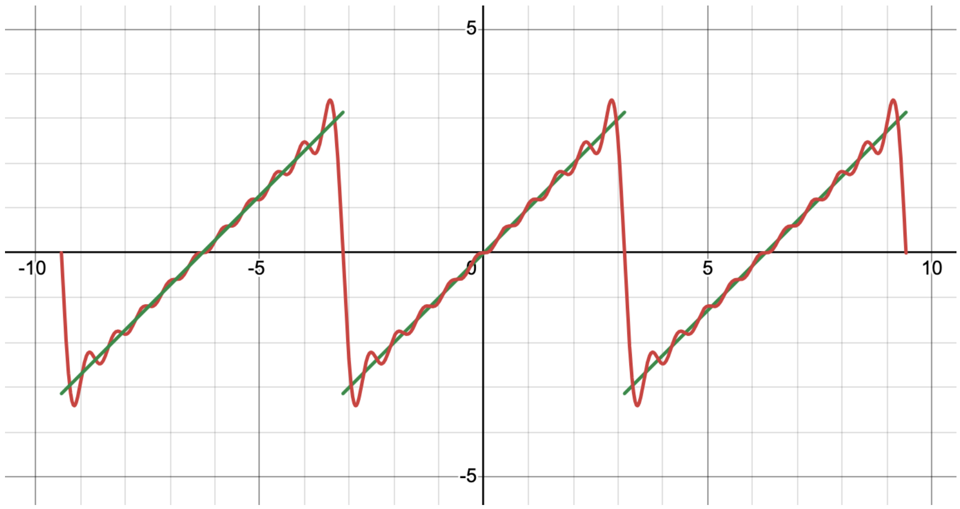

Figure 3: The saw-tooth function and its Fourier series on the interval $(-3\pi, 3\pi)$

Then, by using (20), we can rewrite our partial sum as

$$\begin{aligned}
S_n(f,x) &= 2\sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}\sin(kx) \\
&= 2\sum_{k=1}^{n}(-1)^{k+1}\int_0^x \cos(kt)dt \\
&= 2\int_0^x \Big[\sum_{k=1}^{n}(-1)^{k+1}\cos(kt)\Big]dt.
\end{aligned}$$

Moreover, it follows that

$$\begin{aligned}
\sum_{k=1}^{n}(-1)^{k+1}\cos(kt) &= \sum_{k=1}^{n}(-1)^{k+1}\frac{e^{ikt}+e^{-ikt}}{2} \\
&= -\frac{1}{2}\sum_{k=1}^{n}\Big((-e^{it})^k + (-e^{-it})^k\Big) \\
&= -\frac{1}{2}\left(\frac{r-r^{n+1}}{1-r} + \frac{r^{-1}-r^{-(n+1)}}{1-r^{-1}}\right) \\
&= -\frac{1}{2}\left(\frac{(r-r^{n+1})(1-r^{-1}) + (r^{-1}-r^{-(n+1)})(1-r)}{(1-r)(1-r^{-1})}\right) \\
&= -\frac{1}{2}\left(\frac{(r-1-r^{n+1}+r^{n}) + (r^{-1}-1-r^{-(n+1)}+r^{-n})}{2-(r+r^{-1})}\right) \\
&= -\frac{1}{2}\left(\frac{-2+r+r^{-1}+r^{n}+r^{-n}-(r^{n+1}+r^{-(n+1)})}{2-(r+r^{-1})}\right) \\
&= -\frac{1}{2}\left(\frac{-2-2\cos(t)+2(-1)^n\cos(nt)-2(-1)^{n+1}\cos((n+1)t)}{2+2\cos(t)}\right) \\
&= \frac{1+\cos(t)-(-1)^n\cos(nt)+(-1)^{n+1}\cos((n+1)t)}{2+2\cos(t)},
\end{aligned}$$

where we substitute $r = -e^{it}$ in our computation.
Then, by using this result, we obtain

$$\begin{aligned}
S_n(f,x) &= 2\int_0^x \Big[\sum_{k=1}^{n}(-1)^{k+1}\cos(kt)\Big]dt \\
&= 2\int_0^x \frac{1+\cos(t)-(-1)^n\cos(nt)+(-1)^{n+1}\cos((n+1)t)}{2+2\cos(t)}dt \\
&= \int_0^x \frac{1+\cos(t)-(-1)^n\cos(nt)+(-1)^{n+1}\cos((n+1)t)}{1+\cos(t)}dt \\
&= \int_0^x 1+\frac{-(-1)^n\cos(nt)+(-1)^{n+1}\cos((n+1)t)}{1+\cos(t)}dt \\
&= \int_0^x 1+(-1)^{n+1}\frac{\cos(nt)+\cos((n+1)t)}{\cos(0)+\cos(t)}dt \\
&= \int_0^x 1+(-1)^{n+1}\frac{\cos((2n+1)\frac{t}{2})\cos(-\frac{t}{2})}{\cos(\frac{t}{2})\cos(-\frac{t}{2})}dt \\
&= \int_0^x 1+(-1)^{n+1}\frac{\cos((2n+1)\frac{t}{2})}{\cos(\frac{t}{2})}dt
\end{aligned}$$

We plot $S_n'(f,x)$ for $n=5, n=10$ and $n=20$. To find the x-value of the maximum overshoot, we locate the intersection of the graph with the x-axis that is closest to the jump-discontinuity at $x=\pi$. For our chosen n values, we find respectively $x=\frac{5\pi}{6}$, $x=\frac{10\pi}{11}$ and $x=\frac{20\pi}{21}$. Hence, we see a pattern and we guess $x=\frac{n\pi}{n+1}$. Let us substitute this guess into $S_n'(f,x)$ to obtain

$$\begin{aligned}
S_n'(f,x) &= 1+(-1)^{n+1}\frac{\cos((2n+1)\frac{x}{2})}{\cos(\frac{x}{2})} \\
&= 1+(-1)^{n+1}\frac{\cos((2n+1)\frac{n}{n+1}\frac{\pi}{2})}{\cos(\frac{n}{n+1}\frac{\pi}{2})} \\
&= 1+(-1)^{n+1}\frac{\cos(n\pi-\frac{n}{n+1}\frac{\pi}{2})}{\cos(\frac{n}{n+1}\frac{\pi}{2})} \\
&= 1+(-1)^{n+1}\frac{\cos(n\pi)\cos(\frac{n}{n+1}\frac{\pi}{2})-\sin(2\pi)\sin(\frac{n}{n+1}\frac{\pi}{2})}{\cos(\frac{n}{n+1}\frac{\pi}{2})} \\
&= 1+(-1)^{n+1}\frac{\cos(n\pi)\cos(\frac{n}{n+1}\frac{\pi}{2})}{\cos(\frac{n}{n+1}\frac{\pi}{2})} \\
&= 1+(-1)^{n+1}\cos(n\pi) \\
&= 1+(-1)^{n+1}\cdot(-1)^n \\
&= 1-1 \\
&= 0
\end{aligned}$$

Indeed, our guess is correct and we have found that the intersection of the graph with the x-axis has $x=\frac{n\pi}{n+1}$. Hence, this is where the maximum overshoot takes place. Filling in our x-value into (19) for large n, say $n=1000$, gives us

$$S_{1000}(f,\frac{1000\pi}{1001}) = 2\sum_{k=1}^{1000}\frac{(-1)^{k+1}}{k}\sin(k\cdot\frac{1000\pi}{1001}) = 3.70073512721\ldots$$

Thus, we see that we have an overshoot of about $\frac{3.70073512721\cdots-\pi}{2\pi}\cdot 100\% \approx 8.9\%$ of the jump.

4 The Gibbs phenomenon for Piecewise-Linear approximation

Even though the Gibbs phenomenon focuses on Fourier series, the same effect can be seen when using a Piecewise-Linear approximation. We will look at this approximation now.

4.1 An example: the square wave

In this section, we will focus on the square wave function again when using the Piecewise-Linear approximation. This way we can compare the results and see if the Gibbs phenomenon is also present in other approximations. For this example, we used the work of (Foster & Richards, 1991). We take a look at the Gibbs phenomenon in the square wave of unit amplitude on the interval $(-1, 1)$. We define this function as

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < 1, \\ -1 & \text{if } -1 < x < 0, \end{cases} \tag{21}$$

where there is a jump-discontinuity at $x = 0$. We will approximate the square wave function by a unique periodic continuous function $y^{(n)}$. This function is linear on each interval $[\frac{k}{n}, \frac{k+1}{n}]$, with $n \in \mathbb{Z}$ fixed and where we call $\frac{k}{n}$ the nodes of $y^{(n)}$. Moreover, the function $y^{(n)}$ is the function that approximates the square wave function best in the L_2 norm on the given interval $[-1, 1]$. Let us denote $y_k = y^{(n)}(\frac{k}{n})$, then we need to find y_i for $i \in \{0, 1, \dots, n\}$ such that

$$\int_{-1}^{1} \left(y^{(n)}(x) - 1\right)^2 dx \tag{22}$$

gets minimized. Using this definition of y_k, we see that the y_k are symmetric with respect to the origin. Namely, we see that $y_{-k} = -y_k$. Moreover, we note that $y_{-n} = y_0 = y_n = 0$, due to periodicity of the square wave. All in all, this shows that it suffices to look at the interval $[0, 1]$ when integrating. Let us then take $x \in [\frac{k}{n}, \frac{k+1}{n}]$ to obtain

$$y^{(n)}(x) = (nx - k)(y_{k+1} - y_k) + y_k. \tag{23}$$

Note that for $n = 1$, we obtain $y^{(n)}(x) = xy_1 = xy_n = x \cdot 0 = 0$. Now, we will look at the cases where $n \in \{2, 3\}$.

Example 4.1. We take $n = 2$, this gives us

$$y^{(2)}(x) = \begin{cases} 2xy_1 & \text{if } 0 \leq x \leq \frac{1}{2}, \\ (2x - 1)(y_2 - y_1) + y_1 & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

We substitute this into (22) to obtain

$$
\begin{aligned}
M_2 &= \int_0^1 \Big(y^{(2)}(x)-1\Big)^2 dx \\
&= \int_0^{\frac{1}{2}} \Big(2xy_1-1\Big)^2 dx + \int_{\frac{1}{2}}^1 \Big((2x-1)(y_2-y_1)+y_1-1\Big)^2 dx \\
&= \Big[\frac{1}{6y_1}(2xy_1-1)^3\Big]_0^{\frac{1}{2}} + \Big[\frac{1}{6(y_2-y_1)}\Big((2x-1)(y_2-y_1)+y_1-1\Big)^3\Big]_{\frac{1}{2}}^1 \\
&= \frac{(y_1-1)^3+1}{6y_1} + \frac{\Big((y_2-y_1)+y_1-1\Big)^3-\Big(y_1-1\Big)^3}{6(y_2-y_1)} \\
&= \frac{y_1^3-3y_1^2+3y_1}{6y_1} + \frac{(y_2-1)^3-(y_1-1)^3}{6(y_2-y_1)} \\
&= \frac{y_1^2-3y_1+3}{6} + \frac{(y_2-y_1)\Big((y_2-1)^2+(y_2-1)(y_1-1)+(y_1-1)^2\Big)}{6(y_2-y_1)} \\
&= \frac{y_1^2-3y_1+3}{6} + \frac{(y_2-1)^2+(y_2-1)(y_1-1)+(y_1-1)^2}{6} \\
&= \frac{y_1^2-3y_1+3}{6} + \frac{y_2^2-2y_2+1+y_2y_1-y_2-y_1+1+y_1^2-2y_1+1}{6} \\
&= \frac{2y_1^2+y_2^2-6y_1-3y_2+y_1y_2+6}{6}.
\end{aligned}
$$

Since $y_2 = y_n = 0$, we can reduce this to

$$M_2 = \frac{2y_1^2-6y_1+6}{6} = \frac{y_1^2-3y_1+3}{3}. \tag{24}$$

To minimize M_2, we need

$$\frac{\partial M_2}{\partial y_1} = \frac{2y_1-3}{3} = 0.$$

In other words, we need $2y_1 - 3 = 0$. Hence, the integral is minimized for $y_1 = \frac{3}{2}$ and $y_2 = 0$. Thus, the overshoot happens at $x = x_1$ with a size of $\frac{1}{2}$.

Example 4.2. We take $n = 3$, this gives us

$$y^{(3)}(x) = \begin{cases} 3xy_1 & \text{if } 0 \le x \le \frac{1}{3}, \\ (3x-1)(y_2-y_1)+y_1 & \text{if } \frac{1}{3} \le x \le \frac{2}{3}, \\ (3x-2)(y_3-y_2)+y_2 & \text{if } \frac{2}{3} \le x \le 1. \end{cases}$$

We substitute this into (22) to obtain

$$\begin{aligned}
M_3 &= \int_0^1 \Big(y^{(3)}(x) - 1\Big)^2 dx \\
&= \int_0^{\frac{1}{3}} \Big(3xy_1 - 1\Big)^2 dx + \int_{\frac{1}{3}}^{\frac{2}{3}} \Big((3x-1)(y_2-y_1) + y_1 - 1\Big)^2 dx \\
&\quad + \int_{\frac{2}{3}}^{1} \Big((3x-2)(y_3-y_2) + y_2 - 1\Big)^2 dx \\
&= \Big[\frac{1}{9y_1}(3xy_1-1)^3\Big]_0^{\frac{1}{3}} + \Big[\frac{1}{9(y_2-y_1)}\Big((3x-1)(y_2-y_1)+y_1-1\Big)^3\Big]_{\frac{1}{3}}^{\frac{2}{3}} \\
&\quad + \Big[\frac{1}{9(y_3-y_2)}\Big((3x-2)(y_3-y_2)+y_2-1\Big)^3\Big]_{\frac{2}{3}}^{1} \\
&= \frac{(y_1-1)^3+1}{9y_1} + \frac{\Big((y_2-y_1)+y_1-1\Big)^3 - \Big(y_1-1\Big)^3}{9(y_2-y_1)} \\
&\quad + \frac{\Big((y_3-y_2)+y_2-1\Big)^3 - \Big(y_2-1\Big)^3}{9(y_3-y_2)} \\
&= \frac{y_1^3 - 3y_1^2 + 3y_1}{9y_1} + \frac{(y_2-1)^3 - (y_1-1)^3}{9(y_2-y_1)} + \frac{(y_3-1)^3-(y_2-1)^3}{9(y_3-y_2)} \\
&= \frac{y_1^2 - 3y_1 + 3}{9} + \frac{(y_2-y_1)\Big((y_2-1)^2 + (y_2-1)(y_1-1) + (y_1-1)^2\Big)}{9(y_2-y_1)} \\
&\quad + \frac{(y_3-y_2)\Big((y_3-1)^2 + (y_3-1)(y_2-1) + (y_2-1)^2\Big)}{9(y_3-y_2)} \\
&= \frac{y_1^2 - 3y_1 + 3}{9} + \frac{(y_2-1)^2 + (y_2-1)(y_1-1) + (y_1-1)^2}{9} \\
&\quad + \frac{(y_3-1)^2 + (y_3-1)(y_2-1) + (y_2-1)^2}{9} \\
&= \frac{y_1^2 - 3y_1 + 3}{9} + \frac{y_2^2 - 2y_2 + 1 + y_2y_1 - y_2 - y_1 + 1 + y_1^2 - 2y_1 + 1}{9} \\
&\quad + \frac{y_3^2 - 2y_3 + 1 + y_3y_2 - y_3 - y_2 + 1 + y_2^2 - 2y_2 + 1}{9} \\
&= \frac{2y_1^2 + 2y_2^2 + y_3^2 - 6y_1 - 6y_2 - 3y_3 + y_2y_1 + y_3y_2 + 9}{9}.
\end{aligned}$$

Since $y_3 = y_n = 0$, we can reduce this to

$$M_2 = \frac{2y_1^2 + 2y_2^2 - 6y_1 - 6y_2 + y_2y_1 + 9}{9} \tag{25}$$

To minimize M_3, we need

$$\frac{\partial M_3}{\partial y_1} = \frac{4y_1 - 6 + y_2}{9} = 0$$

and

$$\frac{\partial M_3}{\partial y_2} = \frac{4y_2 - 6 + y_1}{3} = 0.$$

Hence, we obtain the system of equations

$$\begin{cases} 4y_1 - 6 + y_2 = 0 \\ 4y_2 - 6 + y_1 = 0 \end{cases}.$$

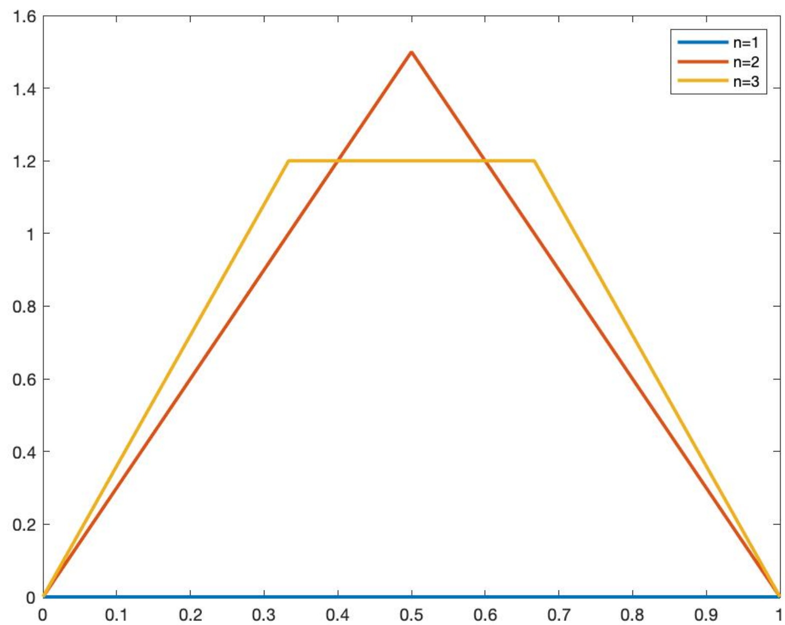

Figure 4: The linear approximation of the square wave function for $n = 1, 2, 3$.

When solving this, we obtain that the integral is minimized for $y_1 = y_2 = \frac{6}{5}$. Thus, the overshoot happens at $x = x_1$ and $x = x_2$ with a size of $\frac{1}{5}$.

We plot all three examples for our linear approximation in figure 4. We see that the computed values for our y_k are indeed correct.
Now, if we set $z_k = y_k - 1$ for simplicity later on, we get

$$y^{(n)}(x) - 1 = (nx - k)(z_{k+1} - z_k) + z_k \tag{26}$$

and moreover

$$\begin{aligned}
\int_{\frac{k}{n}}^{\frac{k+1}{n}} \left(y^{(n)}(x) - 1\right)^2 dx &= \int_{\frac{k}{n}}^{\frac{k+1}{n}} \Big((nx-k)(z_{k+1} - z_k) + z_k\Big)^2 dx \\
&= \left[\frac{1}{3n(z_{k+1} - z_k)} \Big((nx-k)(z_{k+1} - z_k) + z_k\Big)^3\right]_{\frac{k}{n}}^{\frac{k+1}{n}} \\
&= \frac{\Big((\frac{n(k+1)}{n} - k)(z_{k+1} - z_k) + z_k\Big)^3 - \Big((\frac{nk}{n} - k)(z_{k+1} - z_k) + z_k\Big)^3}{3n(z_{k+1} - z_k)} \\
&= \frac{\Big(z_{k+1} - z_k + z_k\Big)^3 - \Big(z_k\Big)^3}{3n(z_{k+1} - z_k)} \\
&= \frac{z_{k+1}^3 - z_k^3}{3n(z_{k+1} - z_k)} \\
&= \frac{(z_{k+1}^2 + z_{k+1}z_k + z_k^2)(z_{k+1} - z_k)}{3n(z_{k+1} - z_k)} \\
&= \frac{z_{k+1}^2 + z_{k+1}z_k + z_k^2}{3n}.
\end{aligned}$$

Taking 1 as our upper bound and 0 as our lower bound (since this is the integral we want to minimize), this changes to

$$\begin{aligned}
M_n &= \int_0^1 \left(y^{(n)}(x) - 1\right)^2 dx \\
&= \sum_{k=0}^{n-1} \int_{\frac{k}{n}}^{\frac{k+1}{n}} \left(y^{(n)}(x) - 1\right)^2 dx \\
&= \frac{\sum_{k=0}^{n-1}(z_{k+1}^2 + z_{k+1}z_k + z_k^2)}{3n} \\
&= \frac{\sum_{k=0}^{n-1} z_{k+1}^2 + \sum_{k=0}^{n-1} z_{k+1}z_k + \sum_{k=0}^{n-1} z_k^2}{3n} \\
&= \frac{\sum_{k=1}^{n} z_k^2 + \sum_{k=0}^{n-1} z_{k+1}z_k + \sum_{k=0}^{n-1} z_k^2}{3n} \\
&= \frac{2\sum_{k=1}^{n-1} z_k^2 + z_0^2 + z_n^2 + \sum_{k=0}^{n-1} z_{k+1}z_k}{3n} \\
&= \frac{2\sum_{k=1}^{n-1} z_k^2 + z_0^2 + z_n^2 + \sum_{k=1}^{n-2} z_{k+1}z_k + z_0z_1 + z_{n-1}z_n}{3n} \\
&= \frac{2\sum_{k=1}^{n-1} z_k^2 + \sum_{k=1}^{n-2} z_{k+1}z_k + 2 - z_1 - z_{n-1}}{3n}.
\end{aligned}$$

To arrive at our last step we used the fact that $y_0 = y_n = 0$ gives that $z_0 = z_n = -1$. Let us now take $\frac{\partial M_n}{\partial z_i} = 0$ with $i \in \{1, 2, \dots, n-1\}$. By solving these equations, we will be able to derive the equations that minimize the sum of the square differences between the left and right sides of this equation. The equations minimize the sum (and not maximize it) since the sum of the square differences is unbounded above as the parameters grow to infinity in magnitude. The sum of squares is always nonnegative, and hence has a minimum. We compute that the equations for $n \geq 4$ are given by

$$4z_1 + z_2 = 1 \quad \text{for } k = 1, \tag{27}$$

$$z_{k-1} + 4z_k + z_{k+1} = 0 \quad \text{for } 2 \leq k \leq n-2, \tag{28}$$

$$z_{n-2} + 4z_{n-1} = 1 \quad \text{for } k = n-1. \tag{29}$$

We focus on the cases $n \geq 4$ since we have already seen the cases where $n \in \{1, 2, 3\}$. We note that our general equation in this system of our normal equations is a second-order linear homogeneous equation. Therefore it has the general solution of the form $z_k = c_1 r_1^k + c_2 r_2^k$. Moreover, the characteristic equation is $r^2 + 4r + 1 = 0$. This characteristic equation has roots $r_1 = -2 + \sqrt{3}$ and $r_2 = -2 - \sqrt{3}$. Then taking (27) and (29), we obtain

$$\begin{aligned}
4z_1 + z_2 &= 4(c_1 r_1 + c_2 r_2) + c_1 r_1^2 + c_2 r_2^2 = 1, \\
z_{n-2} + 4z_{n-1} &= c_1 r_1^{n-2} + c_2 r_2^{n-2} + 4(c_1 r_1^{n-1} + c_2 r_2^{n-1}) = 1.
\end{aligned}$$

From this, we can compute c_1 and c_2 in terms of r_1 and r_2.

$$\begin{aligned}
c_1 &= \frac{r_2^n - 1}{r_1^n - r_2^n}, \\
c_2 &= \frac{r_1^n - 1}{r_2^n - r_1^n}.
\end{aligned}$$

This gives us

$$\begin{aligned} z_k &= c_1 r_1^k + c_2 r_2^k \\ &= \frac{r_1^k(r_2^n - 1)}{r_1^n - r_2^n} + \frac{r_2^k(r_1^n - 1)}{r_2^n - r_1^n} \\ &= \frac{-r_1^k - r_1^{n-k}}{1 + r_1^n}. \end{aligned}$$

Since $r_1 = -2 + \sqrt{3}$, we have $0 < -r_1 < 1$. Using this together with the latter equation for z_k we can thus see that the following sequence is positive for $1 \leq k < \frac{n}{2}$

$$(-1)^{k+1} z_k = \frac{(-r_1)^k (r_1^{n-2k} + 1)}{1 + r_1^n} \tag{30}$$

Moreover, the sequence $(-1)^{k+1} z_k$ is decreasing for $1 \leq k < \frac{n}{2}$, since

$$(-1)^{k+1} z_k - (-1)^{k+2} z_{k+1} = \frac{(-r_1)^k (r_1 + 1)(r_1^{n-2k-1} + 1)}{(1 + r_1^n)} > 0. \tag{31}$$

Therefore, we see that $(-1)^{k+1} z_k > (-1)^{k+2} z_{k+1}$. Then by putting (30) and (31) together, we can conclude that $(-1)^{k+1} z_k$ is a positive decreasing sequence for $1 \leq k < \frac{n}{2}$. And thus the maximum overshoot happens at $k = 1$. This then tells us that the maximum overshoot is given by z_1, since this is the difference between our computed value y_1 and the actual value of the square wave. If we then compute the limit of z_1, we can compute the limit of the overshoot. We compute

$$\begin{aligned} \lim_{n \to \infty} z_1 &= \lim_{n \to \infty} \frac{-r_1^1 - r_1^{n-1}}{1 + r_1^n} \\ &= \lim_{n \to \infty} \frac{-(-2+\sqrt{3})^1 - (-2+\sqrt{3})^{n-1}}{1 + (-2+\sqrt{3})^n} \\ &= \frac{-(-2+\sqrt{3}) - 0}{1 + 0} \\ &= 2 - \sqrt{3} \\ &= 0.26794919243... \end{aligned}$$

Thus, we see that we have a maximum overshoot of about $\frac{0.26794919243...}{2} \cdot 100\% \approx 13\%$ of the jump when using the Piecewise-Linear approximation. We thus see that the Gibbs phenomenon also appears when using the Piecewise-Linear approximation, but the error of the overshoot is larger in this case.

5 Concluding remarks

As stated before, the Gibbs phenomenon is usually linked to the Fourier series. However, as we have seen, this phenomenon can be seen in multiple approximations using the same norm. The Gibbs phenomenon is thus dependent on the norm and not on the approximation. However, the way of approximating does have an impact on the error percentage.

In this paper, we looked at two ways of approximating a function, namely using Fourier series and the Piecewise-Linear approximation. During this research, we have also come across the Gibbs phenomenon in wavelets (Jerri, 1998). However, since this implementation requires a lot of involvement in this field, this could not be included in this research. For further research, this could be interesting to look at.

References

Abbott, S. (2015). *Understanding analysis*. Springer.

Angelidis, P. A. (1996). The life of JBJ Fourier. *Concepts in Magnetic Resonance*, *8*(5), 383.

De Snoo, H. S. V., & Sterk, A. E. (2023). Functional Analysis, Lecture notes, Rijksuniversiteit Groningen.

Esqueda, F., Bilbao, S., & Välimäki, V. (2016). Aliasing reduction in clipped signals. *IEEE Transactions on Signal Processing*, *64*(20), 5255–5267.

Finch, S. R. (2003). *Mathematical constants*. Cambridge university press.

Foster, J., & Richards, F. B. (1991). The Gibbs phenomenon for Piecewise-Linear Approximation. *The American Mathematical Monthly*, *98*(1), 47–49.

Havil, J. (2010). *Gamma: exploring Euler's constant*. Princeton University Press.

Jerri, A. J. (1998). *The Gibbs phenomenon in Fourier analysis, splines and wavelet approximations* (Vol. 446). Springer Science & Business Media.

Roerdink, J. B. T. M., & Zwaan, M. (1993). Cardiac magnetic resonance imaging by retrospective gating: Mathematical modelling and reconstruction algorithms. *European Journal of Applied Mathematics*, *4*(3), 241–270.

Shadrin, A. (2005). Approximation Theory, Lecture notes, Cambridge University.

Zwaan, M. (1990). Approximation of the solution to the moment problem in a Hilbert space. *Numerical functional analysis and optimization*, *11*(5-6), 601–608.