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A Mathematical Analysis of Cosmic Inflation

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Contents

Abstract

This paper provides a comprehensive mathematical analysis of cosmic inflation, demonstrating its effectiveness in addressing several fundamental puzzles in cosmology and its robustness against current observational constraints. The thesis begins with an overview of the Big Bang model and the Cosmic Microwave Background (CMB), supported by the mathematical framework of the Friedmann-Robertson-Walker (FRW) metric and Friedmann equations. The paper then delves into the primary shortcomings of the Big Bang model, such as the horizon, flatness, and monopole problems, and presents initial conditions leading to their resolution through inflation. The theory of inflation is introduced, focusing on the single-field slow-roll model based on the inflation field ϕ and its parameters, ϵ and η, to explore the model's limitations. Cosmological perturbations $\delta\chi_{\mathbf{k}}$ are then introduced which are followed by quantum fluctuations of both massless and massive generic scalar fields. With the derivation of the power spectrum $\mathcal{P}_{\delta\chi}$ and spectral index n_s, we illustrate how inflation provides a robust framework for explaining the observed large-scale structure and temperature anisotropies in the CMB. The paper concludes by affirming the success of the inflationary paradigm in solving the major issues of the Big Bang model and suggesting avenues for further research to refine and test the predictions of inflationary models.

Acknowledgements

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1 Introduction

The Hot Big Bang model, first proposed by Georges Lemaître in 1927, is currently the most widely accepted theory describing the early universe [1]. This model traces the universe back to an incredibly dense and hot initial state. From this primordial condition, the universe expanded and cooled, eventually evolving into the complex cosmos we observe today. While the Big Bang model successfully predicted phenomena such as the existence of the Cosmic Microwave Background (CMB), it also introduced several significant problems requiring resolution.

One of the most profound solutions to these issues is the theory of inflation. Proposed by Alan Guth in 1981, inflation posits a period of extremely rapid exponential expansion in the early universe, solving the horizon and flatness problems that the Big Bang model alone could not address [2]. Inflation, by predicting a period of accelerated expansion, not only addresses the classical problems of cosmology but also provides a framework for understanding the quantum fluctuations that give rise to the large-scale structure of the universe. These fluctuations are the seeds for galaxy formation and are observable in the anisotropies of the CMB. In this thesis, we will delve into the mathematics of inflationary cosmology, and provide solutions to the problems of the Big Bang model.

We start with a comprehensive review of the standard cosmology, focusing on the Friedmann-Robertson-Walker (FRW) metric and Friedmann equations. We then introduce the problems that arise with the Big Bang theory such as the horizon, monopole, and flatness issues. To solve these problems, we then introduce specific conditions which leads to the inflationary theory. We look at the slow-roll model of inflation, which simplifies the inflationary process using specific parameters. A significant part of this thesis is the study of quantum fluctuations in scalar fields during inflation. There, we delve deeper into the mathematical analysis for both the massless and massive case using Mathematica. We then find an expression for the power spectrum and derive the spectral index, creating a framework to compare our predictions with observational constraints, such as that from the Wilkinson Microwave Anisotropy Probe (WMAP) and the CMB.

2 The Big Bang Model

According to the Big Bang theory, the universe started out incredibly hot and dense and has subsequently expanded and cooled [1]. General Relativity (GR), which was developed by Albert Einstein in 1915 and defines gravity as the bending of spacetime brought about by mass and energy, is the foundation of this paradigm. To describe the universe within this framework, a metric is needed to represent the geometric and causal structure of spacetime. In the 1920s and 1930s, Alexander Friedmann, Howard Robertson, Arthur Walker, and Georges Lemaître independently discovered a metric for an expanding, isotropic, and homogeneous universe [3]. This metric is based on the cosmological principle, which postulates that the universe is homogenous and isotropic on large-scales. When it is inserted into Einstein's field equations under the assumption of the universe behaving like a perfect fluid, leads to the Friedmann equations governing the universe's expansion.

The first observational evidence for an expanding universe came from Edwin Hubble in 1929, who discovered that distant galaxies are moving away from us at speeds proportional to their distance:

$$v = H_0 d$$

where v is the velocity of a receding galaxy, d is the distance to the galaxy, and H_0 is Hubble's constant [4]. When distance is measured in meters and velocity in meters per second, Hubble's constant is about $2.3 \times 10^{-18}\,\mathrm{s}^{-1}$. This relationship implies that the further a galaxy is from Earth, the faster it is moving away.

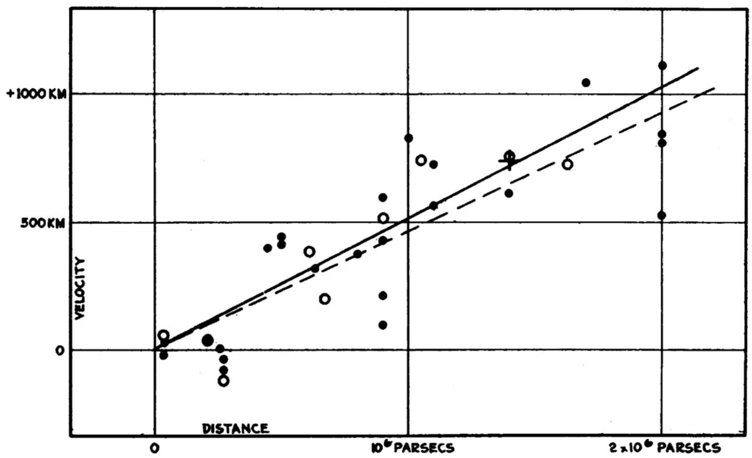

Figure 1: *Velocity-Distance Relation among Extra-Galactic Nebulae. Observation of the expansion of the universe as presented in the original publication by Hubble [4].*

Hubble's Law indicates that the universe is expanding in all directions. This supports the Big Bang theory, which posits that all matter in the universe originated from a single point. Most galaxies are moving away from us, showing redshift, while a few are moving toward us, showing blueshift. This redshift provides strong evidence for an expanding universe.

The Big Bang theory was gradually becoming more supported by the data over time. The strongest evidence for this comes from the 1964 discovery of the Cosmic Microwave Background (CMB) radiation by Robert Wilson and Arno Penzias [5]. The remaining heat from the early universe is represented by this faint glow of microwave radiation, which offers a glimpse of the universe at the moment of photon decoupling, when it was just 380,000 years old. The early universe was a dense, heated plasma in which free electrons regularly dispersed light. Photons could move freely because of the neutral hydrogen formed when protons and electrons combined during the universe's expansion and cooling. The CMB, which has a constant temperature throughout the sky and is presently visible as these photons, suggests that the early universe was homogenous.

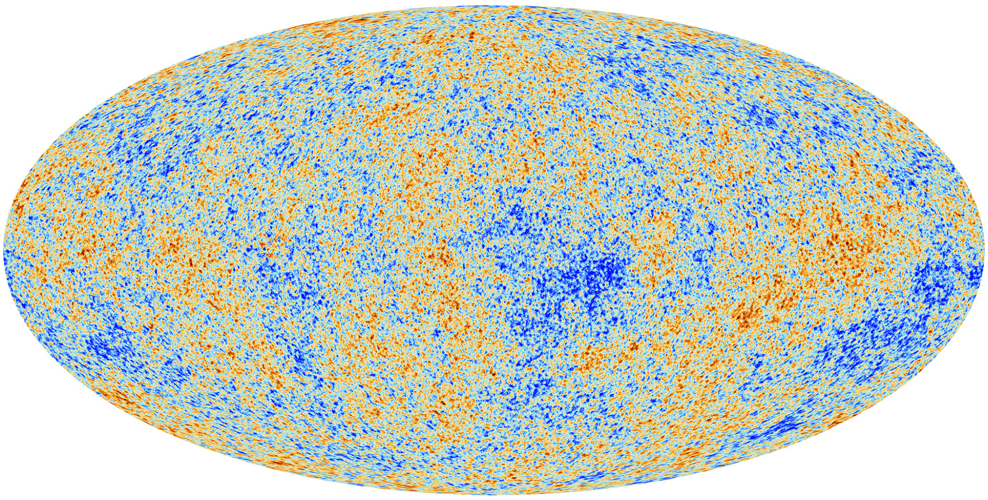

Figure 2: *The European Space Agency (ESA) and Planck's joint observations of the CMB anisotropies [6]. The tiny temperature variations of areas with slightly differing densities are shown by diverse hues, which stand for the seeds of all future structures, including today's galaxies and stars.*

The Big Bang theory was also successful in explaining other key observations such as the relative abundances of light elements (predicted by nucleosynthesis), the predicted age of the universe and oldest known objects, and the formation of large-scale structures through gravitational collapse. Despite its successes, the Big Bang model left some critical questions unanswered, such as the horizon and flatness problems. These challenges led to the development of the inflationary theory, which will be explored in subsequent sections. First, we will delve deeper into the underlying physics of the Big Bang model itself.

2.1 The Friedmann-Robertson-Walker Universe

Our universe is roughly homogenous and isotropic on sizes larger than 100 Mpc, which is consistent with the "cosmological principle" [7]. It claims that, at large scales, all comoving observers see the universe as looking the same in every direction. The Einstein field equations, which explain the relationship between matter and spacetime geometry as well as the universe's subsequent development, are made simpler by this idea. Here is an expression for the Einstein field equations:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu} - g_{\mu\nu}\Lambda, \tag{2.1}$$

where $g_{\mu\nu}$ is the spacetime metric, $R_{\mu\nu}$ is the Ricci tensor, R is the Ricci scalar, $T_{\mu\nu}$ is the energy-momentum tensor for all fields present, and Λ is the cosmological constant. Assuming an ansatz for the metric allows solving for the matter components. For simplicity, we consider a single matter component, assuming it is a perfect fluid with an energy-momentum tensor:

$$T_\mu^\nu = (\rho + P)u_\mu u^\nu + P\delta_\mu^\nu = \mathrm{diag}(-\rho, P, P, P),$$

where u_μ=$(1, 0, 0, 0)$ is the fluid's four-velocity, ρ its energy density, and P its pressure.
We can then find the trace of the metric:

$$T_\mu^\nu = (-\rho + 3P) \tag{2.2}$$

The spatial metric is homogeneous and isotropic and is described by:

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right),$$

where k indicates the spatial curvature $(-1, 0, +1)$ corresponding to the three-dimensional hyperbolic surfaces with negative curvature, flat Euclidean surfaces with zero curvature, or three-spheres with positive curvature. All distances in a flat spacetime are multiplied by the scale factor $a(t)$. Comoving observers, stationary at fixed coordinates (t, r, θ, ϕ), remain at rest regardless of the universe's expansion. We consider a flat universe for the calculations for the reasons which will be explained later. Therefore, for a flat FRW metric, we have:

$$ds^2 = -dt^2 + a^2(t)\gamma_{ij}dx^i dx^j$$

where γ_{ij} denotes the metric of a maximally symmetric 3-space restricted to flat spatial slices, $\gamma_{ij} = \delta_{ij}$. Defining $d\boldsymbol{x}^2 \equiv \delta_{ij}dx^i dx^j$ with conformal time $d\tau = \frac{dt}{a(t)}$, we then can rewrite the metric as:

$$ds^2 = a^2(\tau)\left(-d\tau^2 + d\boldsymbol{x}^2\right) \tag{2.3}$$

It is worth noting that for an expanding universe $da/dt > 0$.

2.2 Friedmann Equations

The rate at which the universe's scale factor increases is determined by its content, which is described by its energy density (ρ) and pressure (p). The influence of these factors on the expansion rate is revealed by applying the FRW metric to the Einstein equation, without considering the cosmological constant term that was initially introduced to achieve a static universe solution. This application yields the Friedmann equations for a perfect fluid [1].

Let's derive the Friedmann equations from the Einstein field equations that were introduced before. Due to isotropy, we have only two independent components to consider: one of the ij-components and the 00-component. The Christoffel symbols may be computed using $a(t)$. Remembering from [8] general relativity:

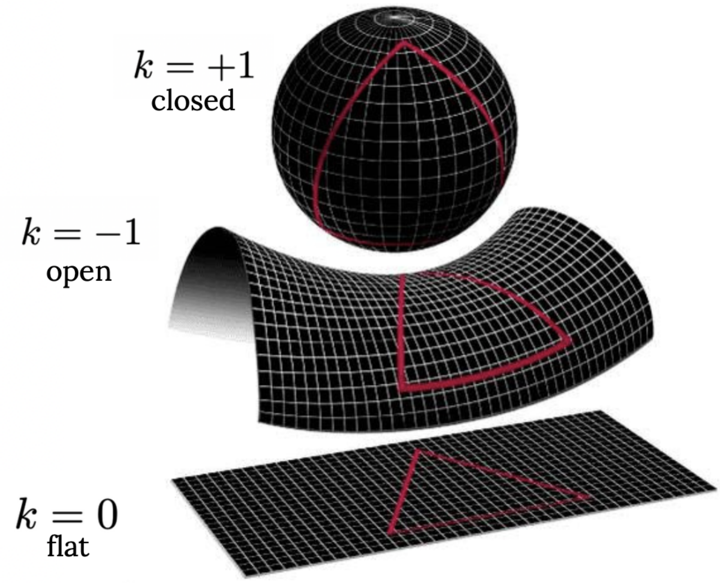

Figure 3: *Analogy to what is meant by closed, open and flat universe with the corresponding curvature values k. Similar picture from NASA.*

$$\Gamma^{\mu}_{\nu\lambda} = \frac{1}{2} g^{\mu\rho} \left(\frac{\partial g_{\rho\nu}}{\partial x^{\lambda}} + \frac{\partial g_{\rho\lambda}}{\partial x^{\nu}} - \frac{\partial g_{\nu\lambda}}{\partial x^{\rho}} \right),$$

The non-zero connection coefficients associated with the metric (2.3) are:

$$\Gamma^{0}_{00} = \frac{\dot{a}}{a}, \qquad \Gamma^{0}_{ij} = \frac{\dot{a}}{a}\delta_{ij}, \qquad \Gamma^{i}_{0j} = \frac{\dot{a}}{a}\delta^{i}_{j}, \tag{2.4}$$

and by inserting these into the Riemann curvature tensor:

$$R^{\lambda}_{\mu\nu\sigma} = \partial_{\nu}\Gamma^{\lambda}_{\mu\sigma} - \partial_{\sigma}\Gamma^{\lambda}_{\mu\nu} + \Gamma^{\lambda}_{\nu\rho}\Gamma^{\rho}_{\mu\sigma} - \Gamma^{\lambda}_{\sigma\rho}\Gamma^{\rho}_{\mu\nu},$$

we get:

$$R_{00} = -3\frac{\ddot{a}}{a}, \qquad R_{ij} = \left(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} \right) \delta_{ij}$$

leading to

$$a^2 R \equiv a^2 g^{\mu\nu} R_{\mu\nu} = -6(\dot{H} + H^2)$$

where $H = \frac{\dot{a}}{a}$ is defined to be the conformal Hubble parameter, describing the expansion of the universe. Hence, the non-zero components of the Einstein field equation 2.1 are

$$-3\frac{\ddot{a}}{a} = 8\pi G \left(\rho + \frac{1}{2}(-\rho + 3P) \right) + \Lambda \qquad \Longrightarrow \qquad -3\frac{\ddot{a}}{a} = 4\pi G(\rho + 3P) - \Lambda,$$

and

$$\left(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2 + k}{a^2} \right) = 8\pi G \left(\frac{1}{2}(\rho - 3P) \right) + \Lambda.$$

Using the 00-component to eliminate $\ddot{a}$, we get:

$$\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2 + k}{a^2} = 4\pi G(\rho - P) + \Lambda.$$

Combining this with the first equation, we can then solve for $\dot{a}$ and $\ddot{a}$:

$$H^2 + \frac{k}{a^2} = \frac{8\pi G\rho}{3} + \frac{\Lambda}{3} \tag{2.5}$$

which is the first Friedmann equation. The second Friedmann or acceleration equation can also be derived similarly to give:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) + \frac{\Lambda}{3} \tag{2.6}$$

Since we assume that the universe is homogenous and isotropic, the first two Friedmann equations can be rewritten as:

$$H^2 = \frac{8\pi G\rho}{3} - \frac{k}{a^2},$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P),$$

Lastly, we derive the continuity equation from the conservation law $\nabla_\mu T^{\mu\nu} = 0$:

$$\nabla_\mu T^{\mu 0} = \partial_\mu T^{\mu 0} + \Gamma^\mu_{\mu\lambda} T^{\lambda 0} + \Gamma^0_{\mu\lambda} T^{\mu\lambda} = 0.$$

For a perfect fluid, the non-zero components of the energy-momentum tensor in a comoving frame are $T^{00} = \rho$ and $T^{ij} = Pg^{ij}$. Therefore,

$$\nabla_\mu T^{\mu 0} = \partial_0 T^{00} + \partial_i T^{i0} + \Gamma^\mu_{\mu 0} T^{00} + \Gamma^0_{\mu\lambda} T^{\mu\lambda} = 0.$$

Since $\partial_i T^{i0} = 0$ in a homogeneous universe and $T^{00} = \rho$,

$$\partial_0 \rho + \Gamma^\mu_{\mu 0}\rho + \Gamma^0_{ij} T^{ij} = 0.$$

Using the Christoffel symbols 2.4,

$$\partial_0 \rho + 3H\rho + H(\rho + P) = 0 \qquad \Rightarrow \qquad \dot{\rho} + 3H(\rho + P) = 0. \tag{2.7}$$

We refer to this as the continuity equation. It guarantees energy conservation in a universe that is expanding. We present an equation of state:

$$P = w\rho \tag{2.8}$$

where w is a constant that depends on the type of fluid. For a perfect fluid, we then have:

$$\frac{d\rho}{\rho} = -3(1+\omega)\frac{da}{a} \qquad \Rightarrow \qquad \rho(a) = \rho_0 a^{-3(1+\omega)} \tag{2.9}$$

By substituting this relation into the first Friedmann equation 2.5 we get:

$$a(t) \propto t^{\frac{2}{3(1+\omega)}} \tag{2.10}$$

We assumed that in the early universe, the scale factor $a(t)$ scales as $a \propto t^n$ with $n < 1$, and solving the conformal time $\tau = dt/a$ we find:

$$a(\tau) \propto \tau^{\frac{n}{1-n}}$$

for matter- and radiation-dominated phases. It follows that:

- The energy-momentum tensor 2.2 for radiation, which includes all relativistic particles, is traceless, implying that $P_r = \rho_r/3$ and thus $\omega_r = 1/3$. The energy density, which scales proportionately to the inverse of the volume and has a size of $\propto a^3$, can be used to compute this. Furthermore, it follows from 2.9 and 2.10 that $\rho_r \propto a^{-4}$ and $a \propto t^{1/2}$ hold true if radiation dominates the universe.

- For non-relativistic matter (dust) or cold matter -including baryons, dark matter, and neutrinos- we require $\omega_m = 0$ and thus $\rho_m \propto a^{-3}$. This is because the pressure of cosmic matter is negligible $P_m = 0$, like at the present day. Therefore, if the universe is dominated with matter, then $a \propto t^{2/3}$.

- Vacuum energy (cosmological constant) $\rho_\Lambda \propto \textit{constant}$ with $\omega_\Lambda = -1$. This is an indication of the energy density of the cosmological constant not decreasing due to the expansion of the universe. From the first two Friedmann equations 2.5 and 2.6, one can derive the relation $a \propto e^{Ht}$ which will be revisited in the following sections.

Assuming $\Omega = \rho/\rho_c$ where $\rho_c = 3H^2/8\pi G$, we may define Ω as the ratio of energy density ρ to critical density ρ_c. The first Friedmann equation, 2.5, can therefore be rewritten as follows:

$$\Omega - 1 = \frac{k}{a^2 H^2},$$

such that the sign of k corresponds to different values of Ω:

$$k = +1 \quad \Rightarrow \quad \Omega > 1 \quad \text{(closed)}$$

$$k = 0 \quad \Rightarrow \quad \Omega = 1 \quad \text{(flat)}$$

$$k = -1 \quad \Rightarrow \quad \Omega < 1 \quad \text{(open)}$$

These relations hold at all times, with Ω and ρ_c varying over time. Early universe phases show $\Omega - 1$ scaling with a^2 during radiation dominance and a during matter dominance, which will be crucial in the following chapters studying the inflationary universe.

3 Shortcomings of the Big Bang

While the Big Bang model provides a comprehensive framework for understanding the evolution of the universe, several significant shortcomings arise if we assume the universe has always been dominated by some form of matter with $w \geq 0$ [2]. These assumptions lead to unusual initial conditions that current physics struggles to explain. Typically, physics predicts the evolution of a given initial state rather than providing a theory for the initial state itself. In cosmology, the peculiar initial state required for the Big Bang model poses several puzzles if normal matter has always dominated.

3.1 Horizon Problem

Despite the universe being extremely small in its early stages, rapid expansion prevented causal contact across the entire universe. The horizon problem in cosmology arises due to the the fact that CMB radiation has a uniform temperature across the entire sky. This is puzzling because, according to the Big Bang model, regions of the universe separated by large distances should not have been able to exchange information or energy due to the finite speed of light and the finite age of the universe. This lack of causal connection suggests that these regions should not have had the same temperature, yet observations show otherwise.

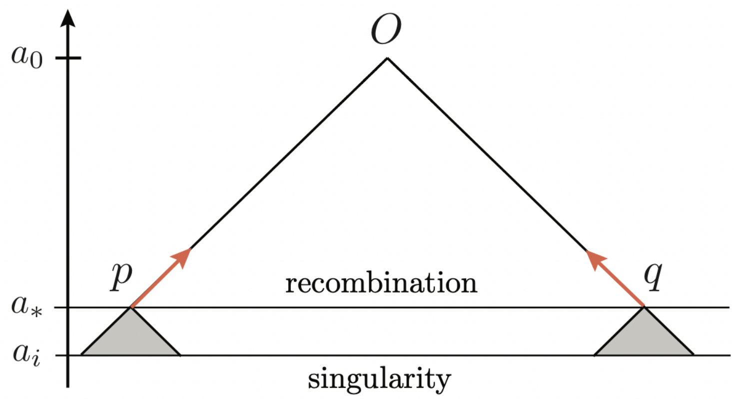

Figure 4: *The events we observe are within the past light cone and the intersection of this light cone with "recombination" marks the "surface of last scattering," shown by using the opposite points on the sky, labeled p and q. Picture from [9]*

Photons follow null geodesics, which for a radial route are $dr = dt/a(t)$ and $ds^2 = 0$. Because space is uniform, we may assume $r_0 = 0$ without losing generality [10]. Similar to great circles that pass through a two-sphere's poles are lines of constant longitude (θ), meaning $d\theta = d\phi = 0$, geodesics traveling through $r = 0$ are characterized by constant θ and ϕ. Space is isotropic, which suggests that the direction (θ_0, ϕ_0) is not important. Consequently, at time $t = 0$, a light signal coming from the coordinates (r_H, θ_0, ϕ_0) will arrive at $r_0 = 0$ at a time t specified by:

$$R_p(t) = a(t) \int_0^t \frac{dt'}{a(t')} = a(t) \int_0^a \frac{d(\ln a')}{a' H(a')} \propto H^{-1} \tag{3.1}$$

This is the the particle horizon, the greatest length of time a light beam can travel from $t = 0$ to t, marking the boundaries of a causal area. Particles separated by a distance greater than the

Hubble radius H^{-1} could never have been in causal contact. This Hubble horizon, expressed in comoving coordinates, is the maximum distance over which causal interactions can occur. Thus, the comoving Hubble radius $(aH)^{-1}$ determines the particle horizon R_p. As the universe expands, physical lengths stretch, and scales inside the horizon today were outside the horizon in the past.

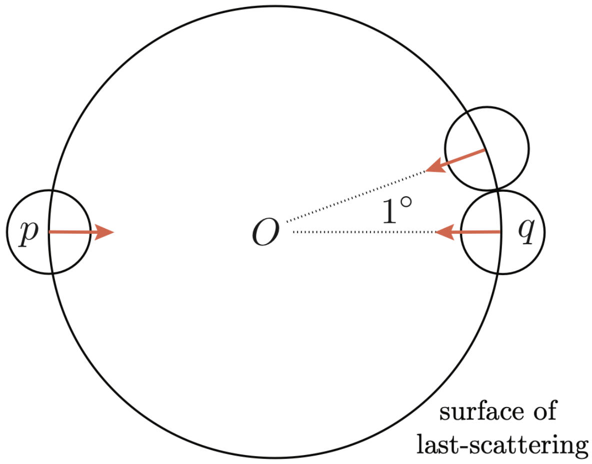

Figure 5: *Points p and q on this surface separated by more than 1° have never interacted causally. Their past light cones do not overlap before the singularity. Picture from [9].*

Imagine two CMB photons that were released during the last scattering and that are currently being seen in the sky at an angle of separation. These photons have almost the same temperature even though they originate from causally separate places. The horizon problem is defined by this disparity. The particle horizon at last scattering is calculated as:

$$R_p(t_{ls}) = H_{ls}^{-1} \approx R_p(t_0)\left(\frac{T_0}{T_{ls}}\right)^{3/2}$$

where T_0 and T_{ls} are the temperatures of the universe now and at last scattering, respectively. This results in a significant number of regions within our current horizon being causally disconnected at that time. Yet, observations of CMB indicate that these regions maintain remarkably consistent temperatures. How can such precise uniformity occur without any information exchange to enable comparison?

3.2 Flatness Problem

Another significant issue in cosmology is the flatness problem. This problem emerges from the dynamics described by the Friedmann equation 2.5 where after rearranging terms, we obtained in the previous section:

$$(\Omega - 1) = \frac{k}{a^2 H^2}$$

or equivalently,

$$|\Omega - 1| = \frac{|k|}{a^2 H^2}$$

In various cosmological models, it is found that the factor a^2H^2 in the denominator scales with time in a manner that reveals the evolution of the universe. Specifically, in models that have a radiation or matter-dominated era, we see that:

$$a^2H^2 \propto t^{-\alpha}$$

with $\alpha > 0$. This scaling behavior implies that:

$$|\Omega - 1| \propto t^{\alpha}$$

Any small deviation from flatness ($|\Omega - 1| = 0$) is amplified with time. This poses a significant problem because our current observations indicate that the universe is approximately flat, with $|\Omega - 1| < 0.005$ [1]. The value of $|\Omega - 1|$ in the early universe has to be incredibly small—tens of orders of magnitude smaller—for this to be the case today. This implies that the universe must have been extremely flat as it came into existence.

The flatness problem can be better understood by considering specific epochs in the universe's history. During the radiation-dominated (RD) era ($a \propto t^{1/2}$), and the matter-dominated (MD) era ($a \propto t^{2/3}$), the product a^2H^2 scales as $a^2H^2 \propto$ constant and $a^2H^2 \propto t^{-1}$, respectively. Thus, during these epochs:

$$|\Omega - 1| \propto t \quad \text{(MD)}$$

$$|\Omega - 1| \propto \text{constant} \quad \text{(RD)}$$

We can trace this back to previous eras since the universe is now seen to be almost flat with $|\Omega_0 - 1| < 0.005$. Based on these estimations, the universe has to have started off extremely near to being flat, suggesting an implausibly precise fine-tuning of initial conditions.

3.3 Solving the Problems: Conditions

The particle horizon, which is again the maximum comoving distance at which light signals can be received by an observer at time t, is given in 3.1. Early contributions dominate the integral if $(aH)^{-1}$ is considerable in the past. This implies that rather than current temporal quantities like the Hubble scale, the true size of the particle horizon is dictated by circumstances in the early universe. The horizon problem arises in conventional cosmology because late times dominate the integral. To address these issues, we require a mechanism that can trigger an accelerated expansion phase in the early universe. An era in which the comoving Hubble radius $(aH)^{-1}$ diminishes with time is a crucial requirement. For this, the universe must experience an accelerated expansion:

$$\frac{d}{dt}\left(\frac{1}{aH}\right) < 0 \implies \ddot{a} > 0$$

This implies that the physical wavelengths of perturbations become longer than the Hubble radius H^{-1}, effectively solving the horizon problem by allowing regions of the universe to come into causal contact. The condition for accelerated expansion can also be expressed in terms of the equation of state parameter w. From the continuity equation 2.7, we have:

$$\ddot{a} = -\frac{a}{2}(\rho + 3p) = -a\rho(1 + 3w)$$

For $\ddot{a} > 0$, it is required that:

$$w < -\frac{1}{3}$$

Thus, inflation necessitates a phase where the universe is dominated by an energy form with $w < -1/3$ if $\rho > 0$. This ensures that the comoving Hubble radius decreases, allowing the universe to expand in such a way that previously causally disconnected regions come into contact.

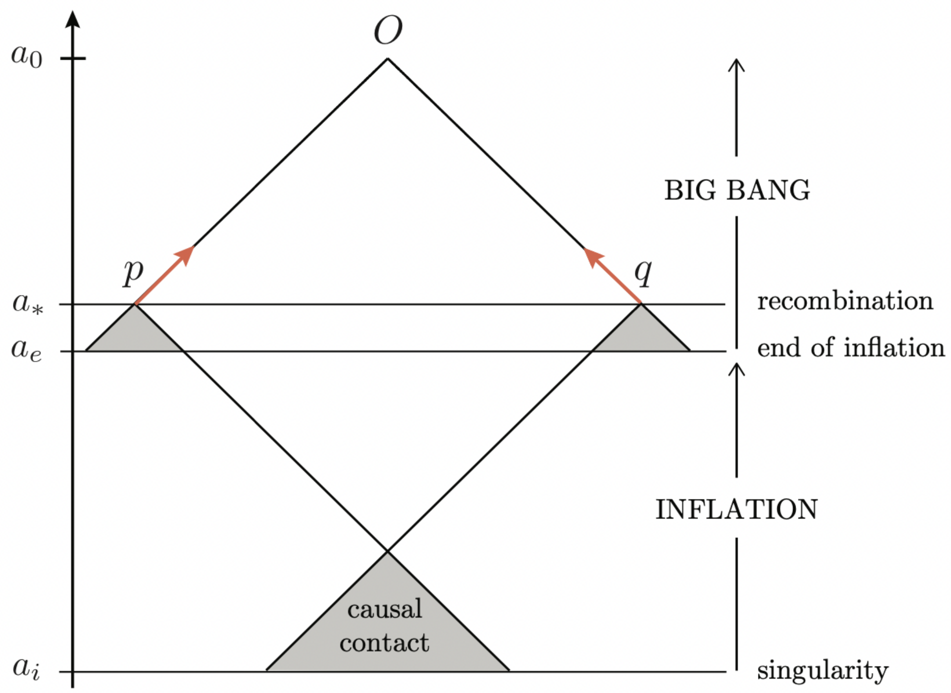

Figure 6: *The inflationary solution to horizon problem. Instead of having a singularity, we have the end of inflation where conformal time on the vertical axis is pushed back to the negative values. As a result, all points in the CMB share overlapping past light cones, indicating they originated from a causally connected region of space. Figure from [9].*

4 The Theory of Inflation

Inflation is proposed to be a period in the early universe dominated by an energy form with $w \approx -1$, leading to a nearly constant Hubble parameter H. This can be achieved through the dynamics of a scalar field, known as the inflaton, which drives the exponential expansion. Since neither radiation- or matter-dominated satisfy such a condition, it therefore needs to be the cosmological constant driving the expansion. This is called the de Sitter universe, in which the universe consists simply of the cosmological constant. The equation of state 2.8 in this instance is as follows:

$$\rho_\Lambda = \frac{\Lambda}{8\pi G} = -P_\Lambda$$

and using the Friedmann equations 2.5 and 2.6, we find:

$$H^2 = \frac{\Lambda}{3}$$

In a standard cosmology with $w > -1/3$, the scale factor $a(t)$ behaves as 2.10, leading to a singularity at $t \to 0$. However, with $w < -1/3$, the singularity is pushed back in conformal time, extending t to negative values. During de Sitter, H is approximately constant, and thus during inflation we have:

$$a(\tau) = -\frac{1}{H\tau}$$

where $-\infty < \tau \leq 0$. This makes the horizon much larger than H^{-1}. Observations of Big Bang Nucleosynthesis indicate that the universe was radiation-dominated around $t \approx 1-100$ seconds [11]. This suggests that inflation occurred much earlier, providing the initial conditions for the hot Big Bang phase. Combining the Friedmann equation 2.5 and the acceleration equation 2.6, we can derive the expansion rate of the universe during different epochs as:

$$a(t) \propto \begin{cases} t^{\frac{2}{3(1+w)}} & \text{if } w \neq -1 \\ e^{H(t-t_i)} & \text{if } w = -1 \end{cases}$$

or more comprehensively:

$$a(t) \propto \begin{cases} t^{\frac{2}{3(1+w)}} & \text{for } t < t_i \\ e^{H(t-t_i)} & \text{for } t_i < t < t_f \\ e^N t^{\frac{2}{3(1+w)}} & \text{for } t > t_f \end{cases}$$

where we consider t_i is the start time and t_f is the end time of inflation, and the ratio is given by

$$\frac{a(t_f)}{a(t_i)} \sim e^{H(t_f - t_i)} = e^N$$

where N is the number of e-folds, telling us the number of times the universe has expanded by a factor of e.

- **Solving the horizon problem:** Let's find the horizon distance during the radiation-dominated early universe. The scale factor is $a(t) \propto t^{1/2}$ for the RD era, and thus:

  $$R_{\text{hor}}(t_i) = a(t_i) \int_0^{t_i} \frac{dt'}{a(t'/t_i)^{1/2}} = 2t_i$$

  Assuming $a(t_i)$ and t_i are constant, the horizon distance at the end of inflation becomes:

  $$R_{\text{hor}}(t_f) = a(t_i)e^N \left(\int_0^{t_i} \frac{dt'}{a(t')(t'/t_i)^{1/2}} + \int_{t_i}^{t_f} \frac{dt'}{a(t')e^{H(t'-t_i)}} \right)$$

  Evaluating the second integral and assuming e^N is large (with $N \approx 60$ which is enought to solve both the horizon and flatness problems):

$$\int_{t_i}^{t_f} \frac{dt'}{a(t')e^{H(t'-t_i)}} \approx \frac{1-e^{-N}}{a(t_i)H_i} \approx \frac{1}{a(t_i)H_i}$$

So, the horizon distance at the end of inflation is:

$$d_{\text{hor}} = e^N \left(2t_i + \frac{1}{H_i}\right)$$

This indicates that the horizon grew from roughly 10^{-27}m at $t_i = 10^{-36}$s to about 15m, with rough estimates $N \approx 60$ and $\frac{1}{H_i} \approx t_i$. This effectively solves the horizon problem by allowing points that were initially close and causally connected to expand to sizes much larger than the observable universe [1].

- **Solving the flatness problem:** The flatness problem is addressed by considering the behavior of the curvature term $|1-\Omega|$ during inflation. Assuming a constant Hubble parameter H during inflation, the curvature term evolves as:

$$|1-\Omega| \propto \frac{1}{a(t)^2}$$

During exponential expansion, the scale factor $a(t) \propto e^{Ht}$, leading to:

$$|1-\Omega| \propto \frac{1}{e^{2N}}$$

Given that $N \approx 60$ is sufficient to solve the flatness problem, this mechanism ensures that any initial curvature rapidly diminishes, driving Ω towards 1 and making the universe appear flat [9].

- **Monopole problem:** Originally, inflation was suggested as a solution to the horizon and flatness issues [2]. It also turns out to be a solution to the monopole problem, though.The tremendous energy in the early universe permitted the unification of the fundamental forces. The weak and electromagnetic forces combine at around 1 TeV, while the strong and electroweak forces combine at about 10^{12} TeV. Symmetry breaking resulted from the decoupling of these forces when the universe cooled, creating topological defects like magnetic monopoles [1]. Magnetic monopoles are predicted to have high energy densities, roughly 10^{94} TeV m^{-3}. If these monopoles were abundant, they would dominate the universe's energy density shortly after the Big Bang. For instance, if their energy density, ρ_{mp}, were to decrease as the universe expands, it would follow the relation $\rho_{mp} \propto a^{-3}$, where a is the scale factor of the universe. This rate is slower than the radiation background, $\rho_r \propto a^{-4}$, leading to monopoles becoming the dominant energy component early in the universe's history, conflicting with current observations. However, observations place a stringent upper bound on the monopole density parameter, Ω_{mp}, such that $\Omega_{mp} < 5 \times 10^{-16}$. This discrepancy indicates that a mechanism must have suppressed monopole density. Inflation offers a solution to this problem. During inflation,

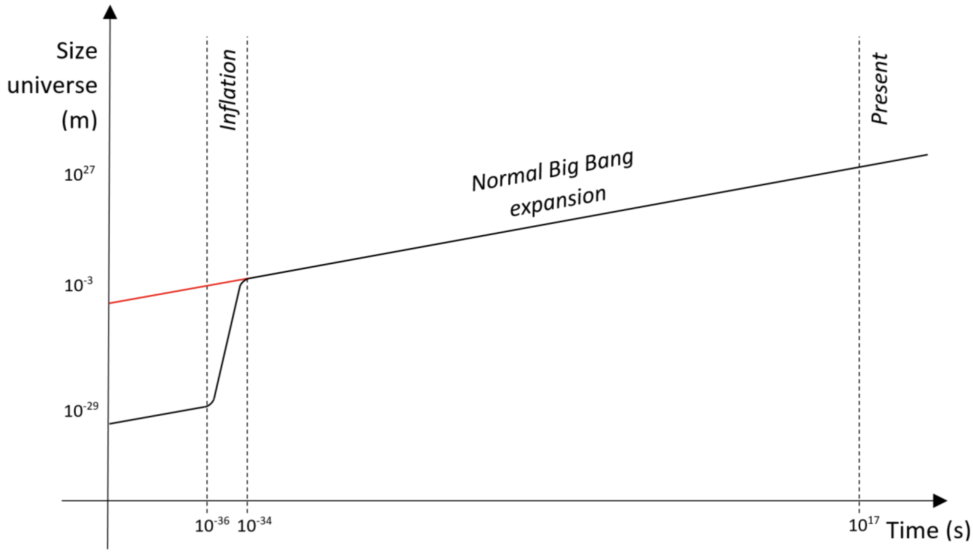

Figure 7: *Expansion of the universe with (black line) and without (red line) an epoch of inflationary expansion. All scales are approximate. Based on similar pictures from NASA and [9].*

the universe undergoes exponential expansion, $a(t) \propto e^{Ht}$. This rapid expansion dilutes the monopole density. If inflation occurs after the monopole-forming phase transition, the number density of monopoles, n_{mp}, would be diluted according to:

$$n_{mp} \propto e^{-3Ht}$$

As a result, the monopole density becomes negligible:

$$\Omega_{mp} \propto e^{-3Ht} \to 0$$

This drastic reduction ensures that monopoles do not dominate the universe's energy density, aligning with observational limits. Thus, inflation effectively resolves the monopole problem by diluting any initial abundance of monopoles to an undetectable level, supporting the consistency of the inflationary model with current cosmological observations.

4.1 Inflaton

To achieve inflation, we require a component with negative pressure, specifically with an equation of state $w < -1/3$. A scalar field, known as the *inflaton*, can drive the early universe's accelerated expansion under certain assumptions. While vector or tensor fields could also be candidates, they present challenges due to their preference for specific directions in space. Scalar fields, on the other hand, are the simplest and most natural choice, as they are predictive and align well with observations [11].

As a basic model for inflation, we examine the dynamics of the inflaton $\phi(t, x)$. This field can vary with both time t and spatial position x. Each value of the field corresponds to a potential energy density $V(\phi)$, as illustrated in Figure 8. When the field changes over time, it also possesses kinetic energy density. The evolution of the FRW background is driven by the energy density of the scalar field if it is the dominant field in the universe. Our goal is to determine the conditions under which this dominance results in accelerated expansion. For this scalar field, the Lagrangian density is:

$$\mathcal{L} = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi), \tag{4.1}$$

where the potential corresponding to ϕ is denoted by $V(\phi)$ and usually includes a quadratic term that represents the mass. We can decompose the field into two parts:

$$\phi(t, x) = \phi_0(t) + \delta\phi(t, x),$$

The vacuum expectation value $\phi_0(t)$ represents the classical solution in a FRW universe, whereas the perturbation is represented by $\delta\phi(t, x)$. Because of isotropy and homogeneity, $\phi_0(t)$ is merely time-dependent. Since perturbations are considered to be minor, the background field governs the field's dynamics primarily. This scalar field has the action:

$$S = \int d^4x\sqrt{-g}\mathcal{L} = \int d^4x\sqrt{-g}\left(-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right)$$

where the Lagrangian density, denoted by $\mathcal{L} = \int dx_i\mathcal{L}$ and, for the FRW metric, $\sqrt{-g} = a^3$, is given. In this case, the scalar field's kinetic and potential terms are included in the action. Because the potential's shape determines how the universe evolves throughout the inflationary phase, it is essential to the inflation process. Although it had its own problems, Alan Guth's original inflation model was revolutionary. Among the first to tackle these issues was Andrei Linde, who presented a novel model based on a scalar field that was gradually rolling down a potential hill. The idea of slow-roll inflation originated with this gradual rolling of the scalar field. We shall calculate the slow-roll parameters and associated observables in the next subsections.

4.2 Slow-roll Inflation

For the scalar field, we get the following relations for the stress-energy tensor:

$$T_{\mu\nu} = \frac{\delta\mathcal{L}}{\delta(\partial^\mu\phi)}\partial_\nu\phi - g_{\mu\nu}\mathcal{L} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\left(\partial_\rho\phi\partial^\rho\phi - V(\phi)\right)$$

For the scalar field energy density (T_{00}) and pressure (T_{ii}), we have:

$$T_{00} = \rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi)$$

$$T_{ii} = p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi)$$

Then the equation of state 2.8 becomes:

$$\omega_\phi = \frac{p_\phi}{\rho_\phi} = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)} \approx -1$$

when the potential energy dominates the kinetic energy, such that:

$$\dot{\phi}^2 \ll V(\phi) \tag{4.2}$$

We derive the following expression using the Friedmann and acceleration equations, knowing that $H = \dot{a}/a$:

$$\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} = -4\pi G(\rho + p) - \frac{k}{a^2} \tag{4.3}$$

Assuming no curvature $k = 0$, and substituting T_{00} and T_{ii}, we obtain the following relation:

$$3H^2 = 8\pi G\rho = 8\pi G(\frac{1}{2}\dot{\phi}^2 + V(\phi)) \tag{4.4}$$

$$\dot{H} = -4\pi G(\rho + p) = -4\pi G\dot{\phi}^2 \tag{4.5}$$

From the continuity equation 2.7, we can also deduce:

$$\ddot{\phi} + \frac{dV(\phi)}{d\phi} = -3H\dot{\phi}$$

A friction term $3H\dot{\phi}$ is included in this equation of motion corresponding to the Lagrangian density 4.1, allowing the field to roll down the potential [9]. The field rolls down slowly under the condition $\dot{\phi}^2 \ll V(\phi)$, and equation 4.4 shows that the universe is expanding at a constant rate, meaning that $\dot{H}$ in equation 4.5 is small. Rewriting equation 4.3, we obtain:

$$\dot{H} + H^2 = \frac{8\pi G}{3}(\dot{\phi}^2 - V(\phi))$$

This indicates that $p < \frac{\rho}{3}$ is necessary for inflation to occur. This requirement is the same as what we said earlier: $\dot{\phi}^2 \ll V(\phi)$. Furthermore, we obtain from this that we may assert $|\ddot{\phi}| \ll V_\phi$, where a derivative with regard to ϕ is indicated by the subscript ϕ. From 4.3, we can now define a useful parameter ϵ_H:

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 = H^2\left(1 + \frac{\dot{H}}{H^2}\right) = H^2(1 - \epsilon_H)$$

where

$$\epsilon_H = -\frac{\dot{H}}{H^2}$$

Using the slow-roll approximations and the potential as an expression for this parameter, we write:

$$H^2 = \frac{8\pi G}{3}V(\phi) \tag{4.6}$$

$$3H\dot{\phi} = V_\phi \tag{4.7}$$

Using equation 4.5, we derive:

$$\dot{\phi}^2 = \frac{V_\phi^2}{3H^2}$$

$$\dot{H} = -4\pi G\frac{V_\phi^2}{9H^2}$$

$$\frac{\dot{H}}{H^2} = -\frac{4\pi G V_\phi^2}{9H^4} = -\frac{V_\phi^2}{16\pi G V^2}$$

Thus, the first slow-roll parameter, ϵ_V, is given by:

$$\epsilon_V = \frac{1}{16\pi G}\left(\frac{V_\phi}{V}\right)^2$$

This parameter ϵ must lie between 0 and 1 for inflation to occur, and ideally, $\epsilon \ll 1$ to satisfy slow-roll conditions. Additionally, we set $8\pi G = 1/M_{pl}^2 = 1$. With H remaining nearly constant during slow-roll, the universe undergoes a near-exponential expansion. The scale factor a grows as $a \propto e^{Hdt} = e^N$, where N represents the number of e-folds of inflation which was introduced in the previous sections. We define:

$$dN = -d\ln a = -Hdt$$

$$N = \int_{a_i}^{a_f} d\ln a = \int_{t_i}^{t_f} H(t)dt$$

By reversing the integration restrictions, the negative sign before N is eliminated, resulting in $N = 0$ at the conclusion of inflation (integrating backwards in time). Rewriting Hdt:

$$Hdt = H\frac{dt}{d\phi}d\phi = \frac{H}{\dot{\phi}}d\phi = -\frac{3H^2 d\phi}{V_\phi} = \frac{V}{V_\phi}d\phi$$

Hence,

$$N = -\int_{\phi_i}^{\phi_f} \frac{V}{V_\phi}d\phi$$

We used equations 4.6 and 4.7 to rewrite the first relation. Next, we define a second slow-roll parameter, η. Assuming the scalar field acceleration is small ($|\ddot{\phi}| \ll |3H\dot{\phi}| \approx |V_\phi|$), we find:

$$\frac{d}{dt}(3H\dot{\phi}) \approx \frac{d}{dt}(-V_\phi)$$

$$3H\ddot{\phi} + 3H^2\dot{\phi} \approx -V_{\phi\phi}\dot{\phi}$$

$$3\dot{\phi}\frac{\dot{H}}{H^2} \approx \frac{V_{\phi\phi}\dot{\phi}}{H^2} - \frac{3\dot{H}}{H^2}$$

Given $|\dot{\phi}| \ll |3H\dot{\phi}|$, the left side of the above equation is small. Knowing the right side term, ϵ_H, is also small, the term $V_{\phi\phi}/H^2$ must be small too. Next, we define η_H and η_V as follows:

$$\eta_H = \frac{\ddot{\phi}}{H\dot{\phi}} = \left(\frac{1}{2}\frac{\dot{H}}{H^2}\right)$$

$$\eta_V = \frac{V_{\phi\phi}}{V}$$

Therefore, in order to have a successful inflation phase, we have two conditions: $\epsilon \ll 1$ and $|\eta| \ll 1$. Although it is possible to define higher order parameters, we are only considering these for this thesis. Moreover, when neglecting those higher order parameters, we have the relations: $\epsilon_V \approx \epsilon_H$ and $\eta_V = \epsilon_H + \eta_H$.

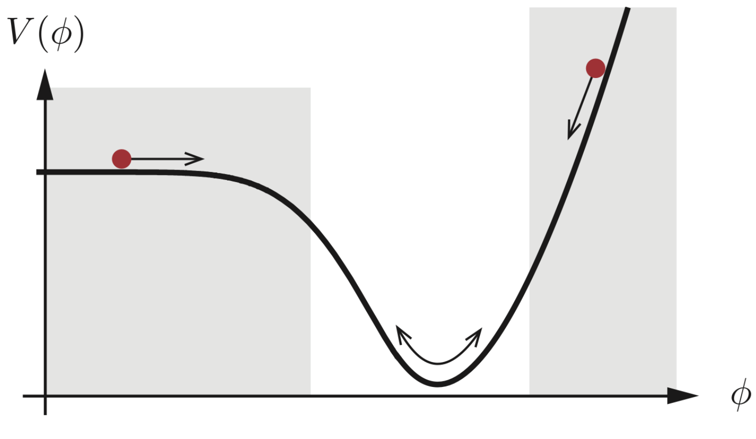

Figure 8: *Example of a slow-roll potential. Inflation occurs in the grey areas of the figure. Figure from [9].*

5 Cosmological Perturbations

We looked at how the early universe was almost uniform due to the slow-roll model of inflation in the previous section. But the structures we observe now, such galaxy clusters and galaxies, started off as tiny "seed" perturbations that evolved over time. These initial density inhomogeneities expanded as a result of gravitational instabilities once the universe became dominated by matter, generating the structures we see today [10].Recent observations of the Cosmic Microwave Background (CMB) have revealed these initial conditions for structure formation as temperature fluctuations in the background radiation. Since microphysical processes are unable to produce temperature anisotropies on sizes larger than 1°, inflationary inhomogeneities are the cause of temperature anisotropies on these larger scales.

Quantum fluctuations during the inflationary era are the most likely origin of these perturbations. While inflation was initially proposed to solve the horizon, flatness, and entropy problems, its ability to produce gravitational wave and density perturbation spectra is its most important characteristic. Due to the stretching of space during inflation, these perturbations range from cosmic sizes to incredibly tiny scales. When inflation stops, the Hubble radius grows more quickly than the scale factor, which leads to fluctuations reentering the Hubble radius during MD or RD eras. Before reheating, fluctuations that left at around 60 e-foldings re-enter at physical wavelengths that can be seen in cosmic data. These spectra, measurable through various methods, including microwave background anisotropy analysis, provide a distinctive signature of inflation.

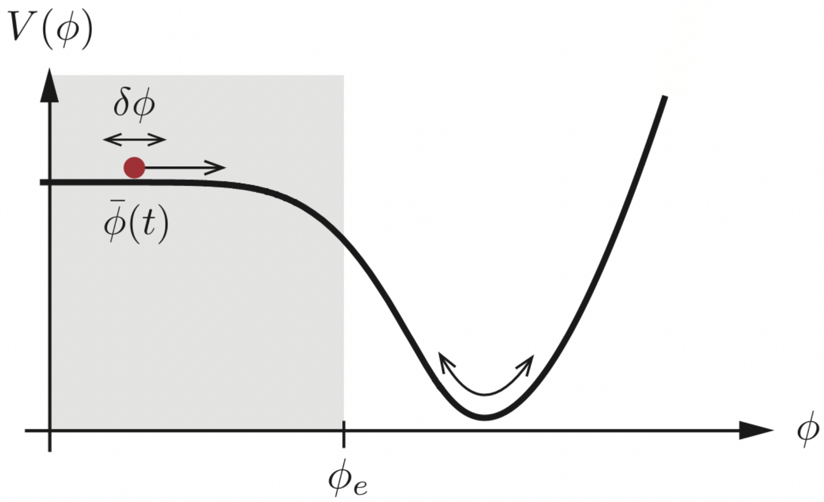

Figure 9: *The classical background evolution $\bar{\phi}(t)$ is perturbed by quantum fluctuations of a sample field $\delta\phi(x,t)$ so that areas with negative fluctuations $\delta\phi$ remain potential-dominated longer than those with positive $\delta\phi$. diverse regions of the the universe go through somewhat diverse evolutionary processes. Figure from [9]*

We will now investigate the process by which quantum fluctuations are produced during inflation. We will begin by looking at the most straightforward issue: understanding how perturbations change over time and calculating their power spectrum by analyzing the quantum fluctuations of a generic scalar field during inflation. The analysis in this chapter will be based on [1], [9], [10] and [12].

5.1 Quantum Fluctuations

The behaviour of these fluctuations can be analyzed in two regimes. First, let us define a physical length scale:

$$\lambda = \frac{2\pi a}{k}$$

where k is a comoving wavenumber for the corresponding λ. We say that λ is within the Hubble radius if $\lambda < H^{-1}$, and corresponds to the sub-Hubble scales. Whereas, λ is outside the Hubble radius if $\lambda > H^{-1}$. Then we have the following relations:

$$\frac{k}{aH} \ll 1 \qquad \Rightarrow \qquad \lambda \gg H^{-1} \tag{5.1}$$

$$\frac{k}{aH} \gg 1 \qquad \Rightarrow \qquad \lambda \ll H^{-1} \tag{5.2}$$

Notice that, from the particle horizon 3.1, we have:

$$\frac{\lambda}{R_H} = \lambda H \sim \frac{aH}{k}$$

These fluctuations get "frozen" in place as inflation stretches them to sizes greater than the horizon ($\lambda > H^{-1}$). The wavelength of these fluctuations increases with the scale factor a, whereas the amplitude stays approximately constant on superhorizon scales. As a result, a classical field $\delta\phi$ is created, which persists when averaged throughout a macroscopic time interval. The inflaton's quantum fluctuations create perturbations in the metric, which then have an impact on the inflaton.

We take a different scalar field than the inflaton ϕ to perform the analysis. The back response on the metric is negligible because of its fluctuations, which have little effect on the overall energy density. We can look at the perturbed Klein-Gordon equation for this scalar in a given spacetime background in order to determine the quantum fluctuations created during inflation. Please take note that we will be using Mathematica to do calculations and extract solutions in the upcoming subsections. The Appendix contains the associated code for every section.

5.1.1 Massless Scalar Field in de Sitter

Consider the quantum fluctuations of a generic massless scalar field χ during a de Sitter stage. We start by splitting the field into two parts in a homogenous background:

$$\chi(\mathbf{x}, t) = \chi(t) + \delta\chi(\mathbf{x}, t)$$

where $\chi(\mathbf{x}, t)$ is the expectation value of the field on the initial isotropic and homogeneous state, and $\delta\chi(\mathbf{x}, t)$ represents its fluctuation. We obtain the following after applying the Fourier transform to the fields χ in Fourier modes:

$$\delta\chi(\mathbf{x}, t) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \delta\chi_{\mathbf{k}}(t),$$

with $\mathbf{k}$ and $\mathbf{x}$ are the comoving momenta and distance, respectively. We can then write the equation for the fluctuations as:

$$\delta\ddot{\chi}_{\mathbf{k}} + 3H\delta\dot{\chi}_{\mathbf{k}} + \frac{k^2}{a^2}\delta\chi_{\mathbf{k}} = 0 \tag{5.3}$$

- For the sub-Hubble scales 5.1, the friction term $3H\delta\chi_{\mathbf{k}}$ is negligible, and the equation simplifies to:

$$\delta\ddot{\chi}_{\mathbf{k}} + \left(\frac{k^2}{a^2}\right)\delta\chi_{\mathbf{k}} = 0$$

  which is the equation of motion of a harmonic oscillator. Because the scaling factor a rises exponentially, the frequency term k^2/a^2 depends on time. The fluctuations exhibit oscillations on sub-Hubble scales.

- For the super-Hubble scales ($\lambda \gg H^{-1}$), where $k \ll aH$, the term k^2/a^2 is negligible, and the equation reduces to:

$$\delta\ddot{\chi}_{\mathbf{k}} + 3H\delta\dot{\chi}_{\mathbf{k}} = 0 \tag{5.4}$$

  This indicates that on super-Hubble scales, the fluctuations remain constant.

This means that a specific fluctuation with an initial wavelength $\lambda \sim a/k$ that lies within the Hubble radius. This fluctuation oscillates until its wavelength approaches the scale of the horizon. Once the wavelength exceeds the Hubble radius, the fluctuation stops oscillating and becomes fixed. To study the evolution of the fluctuation more quantitatively, we introduce the following variable:

$$\delta\sigma_{\mathbf{k}} = a\delta\chi_{\mathbf{k}}$$

and work in conformal time $d\tau = dt/a$. The conformal factor for a pure de Sitter expansion with the scale factor $a \sim e^{Ht}$ is as follows:

$$a(\tau) = -\frac{1}{H\tau} \quad (\tau < 0).$$

We find that 5.4 becomes:

$$\delta\sigma''_{\mathbf{k}} + \left(k^2 - \frac{a''}{a}\right)\delta\sigma_{\mathbf{k}} = 0 \tag{5.5}$$

where $-a''/a = -2/\tau^2$ is the negative mass term.

- For sub-Hubble scales ($k^2 \gg a''/a$), 5.5 reduces to:

$$\delta\sigma''_{\mathbf{k}} + k^2\delta\sigma_{\mathbf{k}} = 0,$$

  whose solution is a plane wave:

$$\delta\sigma_{\mathbf{k}} = \frac{1}{\sqrt{2k}}e^{-ik\tau}.$$

  This demonstrates that fluctuations with wavelengths inside the horizon oscillate just like they would in a flat space-time configuration.

- For super-Hubble scales ($k^2 \ll a''/a$), 5.5 reduces to:

$$\delta\sigma''_{\mathbf{k}} - \frac{a''}{a}\delta\sigma_{\mathbf{k}} = 0,$$

which is satisfied by:

$$\delta\sigma_{\mathbf{k}} = B(k)\frac{1}{a} \quad (k \ll aH),$$

where B(k) is the integration constant. By matching the solutions at $k = aH$ (i.e. $-k\tau = 1$, we may calculate the constant $B(k)$:

$$|B(k)| = \frac{1}{a\sqrt{2k}} = \frac{H}{\sqrt{2k^3}}.$$

Going back to the initial variable $\delta\chi_{\mathbf{k}}$, we see that on super-Hubble scales, the quantum fluctuation of the χ field is roughly constant:

$$|\delta\chi_{\mathbf{k}}| \approx \frac{H}{\sqrt{2k^3}}.$$

In fact, 5.5 has an exact solution that gives the behavior shown by qualitative arguments in the two extreme regimes $k \ll aH$ and $k \gg aH$,

$$\delta\sigma_{\mathbf{k}} = \frac{1}{\sqrt{2k}}e^{-ik\tau}\left(1 - \frac{i}{k\tau}\right)$$

5.1.2 Massive Scalar Field in de Sitter

We disregarded the mass squared component m_χ^2 in the previous section. Let's now look at the answer in the case that this phrase exists. After accounting for the mass term, 5.5 is as follows:

$$\delta\sigma''_{\mathbf{k}} + \left(k^2 + M^2(\tau)\right)\delta\sigma_{\mathbf{k}} = 0 \tag{5.6}$$

such that,

$$M^2(\tau) = m_\chi^2 a^2(\tau) - \frac{2}{\tau^2} = \left(\frac{m_\chi^2}{H^2} - 2\right)\frac{1}{\tau^2}.$$

Rewriting 5.6 in the form:

$$\delta\sigma''_{\mathbf{k}} + \left(k^2 - \frac{1}{\tau^2}\left(\nu_\chi^2 - \frac{1}{4}\right)\right)\delta\sigma_{\mathbf{k}} = 0, \tag{5.7}$$

where

$$\nu_\chi^2 = \frac{9}{4} - \frac{m_\chi^2}{H^2}.$$

The Bessel equation of order ν_χ is represented by equation 5.7. The Hankel functions of the first and second kinds are the combination of two separate solutions that make up the general solution. This can be expressed as:

$$\delta\sigma_{\mathbf{k}} = \sqrt{-\tau}\left[c_1(k)H^{(1)}_{\nu_\chi}(-k\tau) + c_2(k)H^{(2)}_{\nu_\chi}(-k\tau)\right]$$

Here, $H^{(1)}_{\nu_\chi}$ and $H^{(2)}_{\nu_\chi}$ are the Hankel functions of the first and second kind, respectively, and c_1 and c_2 are arbitrary functions of momentum that can be found by applying boundary conditions. It is reasonable to require that $\delta\sigma_{\mathbf{k}}(\tau) \to e^{-ik\tau}$ for $-k\tau \gg 1$ in the setting of cosmic perturbations. This is consistent with adopting the Bunch-Davies vacuum. The Hankel functions' asymptotic behavior is given by:

$$H^{(1)}_{\nu}(x \to 0) \sim \sqrt{\frac{2}{\pi x}} e^{i(x-\frac{\pi}{2}\nu-\frac{\pi}{4})} + e^{i(x-\frac{\pi}{2}\nu+\frac{\pi}{4})}\frac{\nu^2-\frac{1}{4}}{x\sqrt{2\pi x}},$$

$$\begin{aligned} H^{(1)}_{\nu}(x \to -\infty) \sim & -i\frac{2^{\nu}\Gamma(\nu)}{\pi}x^{-\nu} - i\frac{2^{\nu-2}\Gamma(\nu)}{\pi(\nu-1)}x^{2-\nu} \\ & + \left(-i\frac{\cos(\pi\nu)\Gamma(-\nu)}{2^{\nu}\pi} + \frac{1}{2^{\nu}\Gamma(\nu+1)}\right)x^{\nu}, \end{aligned}$$

and the first and second orders are related by $H^{(2)}_{\nu} = H^{(1)*}_{\nu}$. Using the previous equations, one can find that $c_2(k) = 0$ and:

$$c_1(k) = \frac{\sqrt{\pi}}{2}e^{i\left(\nu_\chi+\frac{1}{2}\right)\frac{\pi}{2}}.$$

The exact solution then becomes:

$$\delta\sigma_{\mathbf{k}} = \frac{\sqrt{\pi}}{2}e^{i\left(\nu_\chi+\frac{1}{2}\right)\frac{\pi}{2}}\sqrt{-\tau}H^{(1)}_{\nu_\chi}(-k\tau). \tag{5.8}$$

On super-Hubble scales, the fluctuation 5.8 becomes:

$$\delta\sigma_{\mathbf{k}} = e^{i\left(\nu_\chi-\frac{1}{2}\right)\frac{\pi}{2}}\left(\frac{2^{\nu_\chi-\frac{3}{2}}\Gamma(\nu_\chi)}{\Gamma(3/2)}\right)\frac{1}{\sqrt{2k}}(-k\tau)^{\frac{1}{2}-\nu_\chi}.$$

Returning to the original variable $\delta\chi_{\mathbf{k}}$, on super-horizon scales, the field perturbation's modulus can be rewritten as follows:

$$|\delta\chi_{\mathbf{k}}| \approx \sqrt{\frac{H^2}{2k^3}\left(\frac{k}{aH}\right)^{\frac{3}{2}-\nu_\chi}}.$$

Since $\nu \approx \frac{3}{2}$, A perturbation with wavenumber k has an amplitude that is proportional to its expansion rate H, which remains nearly constant during inflation. Defining the parameter $\eta_\chi = m_\chi^2/3H^2 \ll 1$ Comparable to the inflaton field's slow roll parameters η and ϵ, we find:

$$\frac{3}{2} - \nu_\chi \approx \eta_\chi.$$

5.2 The Power Spectrum

Let's now define the power spectrum, a useful tool to describe the characteristics of perturbations. In Fourier space, the generic quantity $g(\mathbf{x}, t)$, can be extended as follows:

$$g(\mathbf{x}, t) = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} g_{\mathbf{k}}(t),$$

the power spectrum is defined as:

$$\langle 0|g_{\mathbf{k}_1} g_{\mathbf{k}_2}|0\rangle \equiv (2\pi)^3 \delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2)|g_{\mathbf{k}}|^2,$$

where $|0\rangle$ denotes the system's vacuum quantum state. This brings up the following relationship:

$$\langle 0|g^2(\mathbf{x}, t)|0\rangle = \int \frac{d^3k}{(2\pi)^3} g_{\mathbf{k}} g_{-\mathbf{k}} = \int \frac{d^3k}{(2\pi)^3} |g_{\mathbf{k}}|^2 \equiv \int \frac{dk}{k} \mathcal{P}_g(k),$$

This describes the following as the power spectrum of the field $g(\mathbf{x}, t)$ perturbations:

$$\mathcal{P}_g(k) = \frac{k^3}{2\pi^2} |g_{\mathbf{k}}|^2.$$

It is crucial to remember that the fluctuations of a scale-invariant spectrum (also known as white noise) are not constant across all scales k. Specifically, because the size of fluctuations needs to be connected to a certain physical length, the power spectrum has a k^{-3} scaling. The Hubble radius, which changes throughout time, provides this scale. The following is the power spectrum of the scalar field χ fluctuations:

$$\mathcal{P}_{\delta\chi}(k) \equiv \frac{k^3}{2\pi^2} |\delta\chi_{\mathbf{k}}|^2 = \left(\frac{H}{2\pi}\right)^2 \left(\frac{k}{aH}\right)^{3-2\nu_\chi}$$

Next, we establish the fluctuations' spectral index, $n_{\delta\chi}$ as:

$$n_{\delta\chi} - 1 = \frac{d \ln \mathcal{P}_{\delta\phi}}{d \ln k} = 3 - 2\nu_\chi = 2\eta_\chi - 2\epsilon \tag{5.9}$$

The spectral index n_s is a measure of divergence from scale invariance. If $n_s = 1$ (or $n_s - 1 = 0$, the scalar power spectrum is scale-invariant, which means its power is independent of the size of the angular scale ℓ. Any spectrum with $n_s \neq 1$ is considered tilted. For example, when $n_s < 1$, the spectrum is characterized as a red spectrum, showing higher power at large scales (smaller ℓ) than in a scale-invariant universe. In contrast, if $n_s > 1$, it is considered a blue spectrum, indicating less power at small scales.

5.3 Observables

As we can see from 5.9, for single-field inflation, $n_s \approx 1$ since $\nu \approx \frac{3}{2}$, yielding a power spectrum that is almost scale-invariant, with deviations from scale invariance on the order of the slow-roll parameters. According to recent measurements made by the Planck spacecraft, $n_s = 0.9645 \pm 0.0049$ has been found [6]. Remarkably, there is a red tilt in the primordial power spectrum ($n_s - 1 < 0$). This agreement between theoretical predictions and empirical

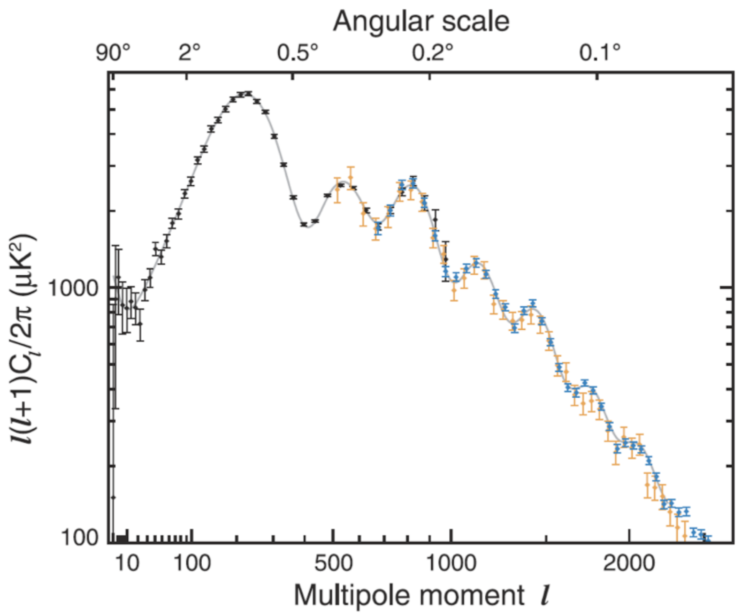

Figure 10: *Power spectrum of fluctuations in the CMB. The temperature variations are represented on the vertical axis. The image is from [13], which aggregates data from many CMB experiments, including ACT [14], SPT [15], and WMAP.*

evidence is one of the biggest achievements of single-field slow-roll inflation. It is important to note that, except from its flatness, no particular assumptions were made regarding the shape of the potential. Thus, for all single-field models satisfying the slow-roll requirements, an almost scale-invariant power spectrum is a generic prediction.

The peaks of the spectrum as seen in Figure 10 correspond to different stages of these acoustic oscillations. The first peak represents regions that had just undergone partial compression at recombination, while subsequent peaks correspond to regions that had completed various stages of oscillation. This observed oscillatory behavior in the power spectrum, plotted against angular scale, reflects the early universe's physical processes. The CMB power spectrum depends on the matter content of the the universe in addition to the perturbation spectrum. This enables us to derive important parameters of the primordial density variations. The CMB is nevertheless an effective instrument for learning about the early universe even given the existence of possible sources of mistake, such as parameter degeneracies, cosmic noise on small sizes, and cosmic variation on large scales.

6 Conclusion

In this thesis, we conducted a detailed mathematical analysis of cosmic inflation. We began with an introduction to the Big Bang theory and standard cosmology, exploring the Friedmann-Robertson-Walker (FRW) metric and Friedmann equations. While the Big Bang theory explains many aspects of the universe, it has significant shortcomings, such as the horizon, monopole, and flatness problems. To address these issues, we introduced inflation, a rapid exponential expansion of the early universe. We examined the slow-roll inflationary model and its parameters, which simplify the dynamics of inflation. We then analyzed quantum fluctuations of scalar fields, computed the power spectrum, and derived the spectral index. This theoretical framework allowed us to compare our predictions with observational constraints, such as that from the Wilkinson Microwave Anisotropy Probe (WMAP) and the Cosmic Microwave Background (CMB).

The predictions of single field slow roll inflation present a strong agreement with observational data, supporting the success of inflation The nearly scale-invariant power spectrum and the slight red tilt predicted by the theory match observations, providing strong evidence for inflation. In conclusion, inflation effectively addresses key problems in the Big Bang theory and provides a robust framework for understanding the early universe's quantum fluctuations. The agreement between the derived spectral index and observational data shows us that the inflationary paradigm works quite well at solving a number of puzzles in cosmology and is robust against current observations.

Future research could further our understanding of inflation by investigating alternative inflationary models such as multifield inflation or k-inflation. Additionally, studying the reheating process after inflation and looking for non-Gaussian features in the CMB could reveal more about the early universe. Detecting primordial gravitational waves would also be significant. Finally, high-precision observations from upcoming space missions could refine our understanding of inflation and its implications for cosmology. By pursuing these areas of research, we can build on the successes of the inflation and enhance our understanding of the universe's origins and fundamental physics.

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Appendix

.1 Christoffel Symbols

We write the FRW metric as:

$$ds^2 = -dt^2 + a^2(t)\tilde{g}_{ij}dx^i dx^j$$

From now on, all objects with a tilde will refer to three-dimensional quantities calculated with the metric $\tilde{g}_{ij}$. One can then calculate the Christoffel symbols in terms of $a(t)$ and $\tilde{\Gamma}^i_{jk}$. Recalling from the GR course that the Christoffel symbols are:

$$\Gamma^\mu_{\nu\lambda} = \frac{1}{2}g^{\mu\rho}\left(\frac{\partial g_{\rho\nu}}{\partial x^\lambda} + \frac{\partial g_{\rho\lambda}}{\partial x^\nu} - \frac{\partial g_{\nu\lambda}}{\partial x^\rho}\right),$$

we may compute the non-vanishing components:

$$\begin{aligned}
\Gamma^i_{jk} &= \tilde{\Gamma}^i_{jk}, \\
\Gamma^i_{j0} &= \frac{\dot{a}}{a}\delta^i_j, \\
\Gamma^0_{ij} &= \frac{\dot{a}}{a}g_{ij} = a\dot{a}\tilde{g}_{ij}.
\end{aligned}$$

The relevant components of the Riemann tensor for the FRW metric are:

$$R^\lambda_{\mu\nu\sigma} = \partial_\nu\Gamma^\lambda_{\mu\sigma} - \partial_\sigma\Gamma^\lambda_{\mu\nu} + \Gamma^\lambda_{\nu\rho}\Gamma^\rho_{\mu\sigma} - \Gamma^\lambda_{\sigma\rho}\Gamma^\rho_{\mu\nu},$$

$$\begin{aligned}
R^i_{0j0} &= -\frac{\ddot{a}}{a}\delta^i_j, \\
R^0_{i0j} &= a\ddot{a}\tilde{g}_{ij}, \\
R^i_{jkl} &= \tilde{R}^i_{jkl} + 2\dot{a}^2\tilde{g}^i_{[k}\delta^i_{l]}.
\end{aligned}$$

Now we can use $\tilde{R}_{ij} = 2k\tilde{g}_{ij}$ (as a consequence of the maximal symmetry of $\tilde{g}_{ij}$) to calculate $R_{\mu\nu}$. The nonzero components are:

$$\begin{aligned}
R_{00} &= -3\frac{\ddot{a}}{a}, \\
R_{ij} &= \left(a\ddot{a} + 2\dot{a}^2 + 2k\right)\tilde{g}_{ij}, \\
&= \left(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right)g_{ij}.
\end{aligned}$$

The Ricci scalar is:

$$R = \frac{6}{a^2}\left(a\ddot{a} + \dot{a}^2 + k\right),$$

and the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ has the components:

$$\begin{aligned}
G_{00} &= 3\left(\frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right), \\
G_{0i} &= 0, \\
G_{ij} &= -\left(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right)g_{ij}.
\end{aligned}$$

.2 Derivation of the specific Christoffel symbols

The FRW metric in comoving coordinates (t, r, θ, ϕ) is:

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right).$$

For simplicity, let's consider the flat case $K = 0$:

$$ds^2 = -dt^2 + a^2(t)(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2).$$

The Christoffel symbols are given by:

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right).$$

For the FRW metric, the relevant components are $g_{00} = -1$ and the fact that $\partial_0 g_{00} = 0$. Therefore:

$$\Gamma^0_{00} = \frac{1}{2} g^{00}(\partial_0 g_{00} + \partial_0 g_{00} - \partial_0 g_{00}) = 0.$$

Actually, for Γ^0_{00}, we need to use the fact that the dt term does not change directly. For the spatial components, $g_{ij} = a^2(t)\delta_{ij}$. Therefore:

$$\Gamma^0_{ij} = \frac{1}{2} g^{00}(\partial_i g_{0j} + \partial_j g_{0i} - \partial_0 g_{ij}).$$

Since $g_{0j} = g_{0i} = 0$, this simplifies to:

$$\Gamma^0_{ij} = -\frac{1}{2} g^{00}\partial_0 g_{ij}.$$

Now, $g^{00} = -1$ and $g_{ij} = a^2(t)\delta_{ij}$, thus:

$$\partial_0 g_{ij} = 2a(t)\dot{a}(t)\delta_{ij}.$$

Substituting this, we get:

$$\Gamma^0_{ij} = -\frac{1}{2}(-1)2a\dot{a}\delta_{ij} = a\dot{a}\delta_{ij}.$$

Since $H = \frac{\dot{a}}{a}$:

$$\Gamma^0_{ij} = Ha^2\delta_{ij} = Hg_{ij}.$$

For $\Gamma^\mu_{0\mu}$, we sum over all μ:

$$\Gamma^\mu_{0\mu} = \Gamma^0_{00} + \Gamma^i_{0i}.$$

We already have $\Gamma^0_{00} = 0$. For Γ^i_{0i}, where i represents spatial indices:

$$\Gamma^i_{0i} = \frac{1}{2} g^{i\sigma}(\partial_0 g_{i\sigma} + \partial_i g_{0\sigma} - \partial_\sigma g_{0i}).$$

Since $g_{0\sigma} = 0$, this simplifies to:

$$\Gamma^i_{0i} = \frac{1}{2} g^{ii}\partial_0 g_{ii}.$$

Now, $g^{ii} = \frac{1}{a^2(t)}$ and $g_{ii} = a^2(t)\delta_{ii}$:

$$\partial_0 g_{ii} = 2a(t)\dot{a}(t)\delta_{ii}.$$

Thus:

$$\Gamma^i_{0i} = \frac{1}{2}\frac{1}{a^2(t)} 2a(t)\dot{a}(t)\delta_{ii} = \frac{\dot{a}(t)}{a(t)}\delta_{ii} = H.$$

Summing over the three spatial dimensions (since i can be 1, 2, or 3):

$$\Gamma^\mu_{0\mu} = 3H.$$

.3 Mathematica code for the quantum fluctuations of a generic scalar field $v(t)$

.3.1 Massless case

For the sub-Horizon scale: (.1)

(.2)

$$\text{DSolve}\left[\frac{\partial^2 v(t)}{\partial t^2} + k^2 v(t) = 0, v(t), t\right]$$

(.3)

$$\{v(t) \to c_1 \cos(kt) + c_2 \sin(kt)\}$$

(.4)

For the super-Horizon scale: (.5)

(.6)

$$\text{DSolve}\left[\frac{\partial^2 v(t)}{\partial t^2} - \frac{2v(t)}{t^2} = 0, v(t), t\right]$$

(.7)

$$\left\{v(t) \to c_2 t^2 + \frac{c_1}{t}\right\}$$

(.8)

The exact solution of 5.5: (.9)

(.10)

$$\text{DSolve}\left[\frac{\partial^2 v(t)}{\partial t^2} + \left(k^2 - \frac{2}{t^2}\right) v(t) = 0, v(t), t\right]$$

(.11)

$$\left\{v(t) \to \sqrt{\frac{2}{\pi}} c_1 \left(\frac{\sin(kt)}{t} - \cos(kt)\right) + \sqrt{\frac{2}{\pi}} c_2 \left(-\sin(kt) - \frac{\cos(kt)}{t}\right)\right\}$$

(.12)

$$\left(t\sqrt{\frac{2}{\pi}} c_1 \left(\frac{\sin(kt)}{t} - \cos(kt)\right) + \sqrt{\frac{2}{\pi}} c_2 \left(-\sin(kt) - \frac{\cos(kt)}{t}\right)\right)(\infty)$$

(.13)

$$c_1\text{Interval}\left[\left\{-\sqrt{\frac{2}{\pi}}, \sqrt{\frac{2}{\pi}}\right\}\right] + c_2\text{Interval}\left[\left\{-\sqrt{\frac{2}{\pi}}, \sqrt{\frac{2}{\pi}}\right\}\right] \quad (.14)$$

.3.2 Massive case

$$\text{DSolve}\left[\frac{\partial^2 v(t)}{\partial t\, \partial t} + v(t)\left(\frac{\frac{m^2}{H^2} - 2}{t^2} + k^2\right) = 0, v(t), t\right]$$

$$\left\{\left\{v(t) \to c_1 \sqrt{t} J_{\frac{1}{2}\sqrt{9 - \frac{4m^2}{H^2}}}(kt) + c_2 \sqrt{t} Y_{\frac{1}{2}\sqrt{9 - \frac{4m^2}{H^2}}}(kt)\right\}\right\}$$

$$\text{DSolve}\left[\frac{\partial^2 v(t)}{\partial t^2} + v(t)\left(\frac{\frac{m^2}{H^2}-2}{t^2}+k^2\right)=0, v(t), t\right]$$

$$\left\{\left\{v(t) \to \frac{1}{2}\sqrt{\pi}\sqrt{t}e^{\frac{1}{2}i\pi\left(\sqrt{9-\frac{4m^2}{H^2}}+\frac{1}{2}\right)}Y_{\frac{1}{2}\sqrt{9-\frac{4m^2}{H^2}}}(kt)\right\}\right\}$$

$$\text{Series}\left[\text{Expand}\left[v(t) = \frac{1}{2}\sqrt{\pi}\sqrt{t}e^{\frac{1}{2}i\pi\left(\sqrt{9-\frac{4m^2}{H^2}}+\frac{1}{2}\right)}Y_{\frac{1}{2}\sqrt{9-\frac{4m^2}{H^2}}}(\text{kt})\right], \{\text{kt}, \infty, 1\}\right]$$

$$v(t) = \sin\left(\text{kt} + \frac{1}{4}\left(-\left(\sqrt{9-\frac{4m^2}{H^2}}+1\right)\right)\pi + O\left(\left(\frac{1}{\text{kt}}\right)^2\right)\right)$$

$$\left(\frac{e^{\frac{1}{2}\left(\sqrt{9-\frac{4m^2}{H^2}}+\frac{1}{2}\right)i\pi}\sqrt{t}\sqrt{\frac{1}{\text{kt}}}}{\sqrt{2}} + O\left(\left(\frac{1}{\text{kt}}\right)^{3/2}\right)\right)$$

$$+\cos\left(\text{kt} + \frac{1}{4}\left(-\left(\sqrt{9-\frac{4m^2}{H^2}}+1\right)\right)\pi + O\left(\left(\frac{1}{\text{kt}}\right)^2\right)\right)$$

$$\left(\frac{\left(e^{\frac{1}{2}\left(\sqrt{9-\frac{4m^2}{H^2}}+\frac{1}{2}\right)i\pi}\left(8-\frac{4m^2}{H^2}\right)\right)\sqrt{t}\left(\frac{1}{\text{kt}}\right)^{3/2}}{8\sqrt{2}} + O\left(\left(\frac{1}{\text{kt}}\right)^2\right)\right)$$

Here, J and Y represent the Henkel functions.