university of groningen / faculty of science and engineering

Phase Space on a Noncommutative de Sitter Spacetime

Author:
João Paulo ESPÍRITO SANTO
(s4738748)

Supervisor:
prof. Daniël BOER

Bachelor's Thesis
To fulfill the requirements for the degree of
Bachelor of Science in Physics
at the University of Groningen

July 14, 2024

Abstract

In this thesis, an exploration of formulations of phase space in a non-commutative de Sitter spacetime is undergone. Mappings between classical phase space and quantum mechanics are discussed. Methods of defining a phase space in a Lorentz-violating noncommutative de Sitter space are considered and its spacetime symmetries are discussed.

An overview of classical phase space and its QM formulation is given through Weyl quantization and the Groenewold-Moyal bracket. Different bracket structures are explored for the case of the Poisson and Groenewold-Moyal bracket. It's demonstrated that the Jacobi identity of both the Groenewold-Moyal bracket and the Dirac commutator in a noncommutative Minkowski spacetime requires a constant self-commutator of the 4-position operators $[\hat{q}^{\mu}, \hat{q}^{\nu}]$.

A short introduction to de Sitter spacetime, its group and algebra is given. It's concluded that a noncommutative de Sitter spacetime lacks both complete Lorentz symmetry and translational invariance.

It is speculated that, in contrast to the case of a noncommutative Minkowski spacetime, the de Sitter momentum operator and its commutation relations lead to the Jacobi Identity demanding a non-constant self-commutator of 4-position operators.

It is finally concluded that to define a phase space on a noncommutative de Sitter spacetime, its symmetry group and algebra must first be explicitly defined.

Acknowledgements

The author would like to thank his supervisor, Daniël Boer, for his patience, curiosity and guidance. His insistence in questioning often overlooked issues motivated the research undergone in this thesis. The author would also like to express his extreme gratitude towards his parents for providing for his education and life. Thanks are also extended to Tom, Bastiaan and Leah for their open ears and attentive responses. Finally, he would like to thank Maya for her support and kindness throughout the entire research process.

Contents

1 Introduction **6**

1.1 Motivations 7

1.1.1 Noncommutative Spacetime 7

1.1.2 de Sitter Space 8

1.2 Research Questions 9

2 Noncommutative Spacetime **10**

2.1 Physical Interpretation of $\theta^{\mu\nu}$ 10

2.2 Lorentz Symmetry Violation 11

2.3 Hermiticity of $\theta^{\mu\nu}$ 11

3 Poisson Structures **13**

3.1 Canonical Poisson Bracket in Phase Space 14

3.2 Noncommutative Poisson Structure 15

3.3 Affine Poisson Structure 16

3.4 The Jacobi identity 16

4 Groenewold-Moyal Bracket **18**

4.1 Dirac's Canonical Quantization 18

4.2 Weyl Quantization 19

4.3 Bracket Mapping 20

4.4 Noncommutative Groenewold-Moyal Bracket 20

4.5 Affine Groenewold-Moyal Bracket 21

4.6 Jacobi Identity Limitation 22

5 de Sitter Space **23**

5.1 de Sitter Group 24

5.2 Inönü-Wigner Contraction 25

5.3 Classical Phase Space in a Noncommutative de Sitter Spacetime 25

6 de Sitter Phase Space **27**

6.1 de Sitter Jacobi Identity 27

6.2 Q-Position 28

6.3 Non-constant Canonical Commutator 29

7 θ Group **32**

7.1 θ's Rotational Symmetries 32

7.2 θ's Boost Symmetries 33

7.3 Noncommutative de Sitter group 34

8 Conclusion **36**

8.1 Further Research 38

1 Introduction

When attempting to describe nature, physicists tend to take a modelling approach. We select a system in nature, we observe its behaviour and we come up with a (mathematical) model of that behaviour. Then, if we can use that model to make accurate predictions of the selected system's behaviour, we gain confidence in the belief that the model we created is representative of that system. It is through the systematic modelling of different systems' behaviours that we can "connect the dots" and peek into the underlying fundamental behaviour of nature to form theories. For us to describe the behaviour of a system, it is often helpful to rely on a well-defined and accepted formalism. We seek a formalism that allows for the compatibility of different models and allows us to identify the general properties of all systems.

The formalism that will be used throughout this thesis is that of phase space. Phase space is defined as the space of all states that a dynamic (changing) system can be in. We can describe points (states) in a phase space using coordinates and typically, a mechanical system's phase space is described in terms of position q and momentum p coordinates. For example, consider a mass hanging from a spring and oscillating vertically.

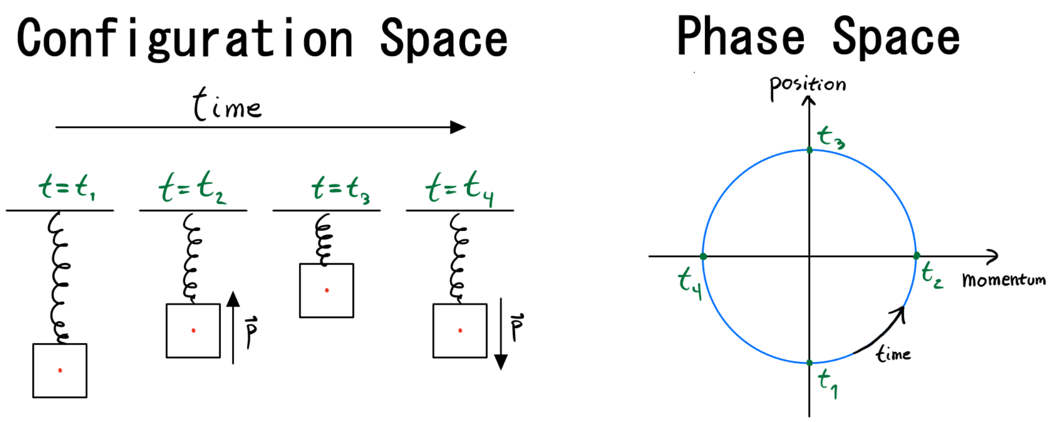

Figure 1: *Diagram of a mass hanging from a spring oscillating vertically and its corresponding graph in phase space.*

We can define functions in phase space that represent certain properties of the system that we want to describe. Through tools available in Hamiltonian mechanics, we can obtain equations of motion that allow us to make predictions of how a system's states and its behaviour evolve with time.

One way of checking that a model is reasonable is by verifying that there are no contradictions in combining it with other accepted models. In this endeavour, it is important to find ways to combine different accepted theories in a coherent manner. One of the biggest issues in this aspect, for the past century, has been unifying our theory of gravity and quantum mechanics. This will relate to the motivations outlined in the next section.

In this thesis, formulations of phase space on a noncommutative spacetime and on a Sitter spacetime will be explored. The goal is to understand how to define a phase space of systems in a noncommutative de Sitter spacetime. Furthermore, it is relevant to **guarantee that this formulation is properly defined in quantum mechanics and reduces correctly in the classical limit** ($\hbar \to 0$).

To begin, the motivations for this research are outlined in the next section along with a series of research questions that will be explored. In section 2, there is an introduction to noncommutative spacetime and some of its properties are discussed. There is an explanation of Poisson structures and their application in Phase space in section 3. In section 4, a prescription for mapping Classical phase space to quantum mechanics and vice versa is introduced. In sections 5 and 6, de Sitter space is described and how a phase space formalism could be defined on it. Finally, in section 7, the effect of a noncommutative spacetime on the symmetries of spacetime will be discussed.

1.1 Motivations

1.1.1 Noncommutative Spacetime

Noncommutative spacetime is used in certain approaches towards combining gravity with quantum mechanics and quantum field theory. It defines a minimum uncertainty in position measurements. In doing so, it serves as a way of implementing, into quantum mechanics, the experimental and theoretical limits of making arbitrarily small length measurements. Different thought experiments and motivations for these limits are discussed at length in Hossenfelder's review of the topic [21]. One of the more easily digestible arguments made involves the Hoop Conjecture, which states:

"If an amount of energy ω is compacted at any time into a region whose circumference in every direction is $R \leq 4\pi G\omega$, then the region will eventually develop into a black hole."

In which G is Newton's gravitational constant. While, experimentally unproven, it has been shown that the conjecture holds in analytic and numerical models very well [13][22]. This conjecture is relevant to our topic due to the inverse proportionality between the length scale of a measurement and the energy required to make that measurement. In general, matter on the smallest scales is probed through scattering experiments in which higher energies of particles directly relate to shorter wavelengths through the de Broglie relation $\lambda = \frac{h}{p}$.

So, one can expect that there exists some limit in which the energy required for a precision scattering experiment is so high and concentrated that the Hoop conjecture comes into play and the system that is being measured is destroyed by the formation of a black hole. Not only that but, in theory, no measurements can be made past the event horizon of a black hole. This argument is explored in thorough detail in Mead [26] and shortly in Hossenfelder [21]. The important takeaway of the argument is that our theory of gravity, by not allowing arbitrarily high concentrations of energy, along with the inverse relation between energy and measurement scale, leads to an emergent limit in the precision of position measurements. This is a feature that could be included explicitly in a theory of quantum gravity and one of the ways that can be done is through defining a noncommutative spacetime.

Noncommutative spacetime is a theory that has been explored for over 90 years since Heisenberg first sent a letter to his student mentioning that he was trying to understand how to define a minimal length through setting the commutator of 4-position operators be non-zero $[\hat{q}^{\mu}, \hat{q}^{\nu}] \neq 0$ [19]. While initially disregarded, the idea has expanded into a mathematical subject of its own named noncommutative geometry [9] and physical theories such as noncommutative field theories [11], amongst others. Our focus will be on its most basic form, in which the commutator of 4-position operators defines a constant background field $\hat{\theta}^{\mu\nu}$. Through the uncertainty principle, this field defines minimum areas of uncertainty in position measurement, effectively defining a maximum resolution in position space.

1.1.2 de Sitter Space

Typically, when one is discussing Special Relativity or General Relativity, it is assumed that the spacetime in the absence of mass or energy is a Minkowski spacetime with the metric $\eta^{\mu\nu} = \text{diag}(-1,1,1,1)$. This metric is a solution to the Einstein Field Equations for the sourceless case (no mass or energy). The Einstein Field Equations allow one to solve for the geometry of spacetime given a certain mass-energy distribution. This geometry and its effect on the motion of matter describes the theory of gravity we know as General Relativity. Many of the effects predicted by General Relativity have been proven [32], which along with its beauty and applications, has led to it being the most widely accepted theory of gravity.

The sourceless Einstein Field equations (sEFE) have the following covariant form [28]:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \tag{1}$$

in which $R_{\mu\nu}$ and R are the Ricci curvature tensor and the scalar curvature, respectively. $g_{\mu\nu}$ is the metric of the spacetime. The most important component with respect to this motivation is the cosmological constant Λ. Einstein originally included the constant in 1917 to counteract the effect of gravity such that the solution was that of a static universe [14]. Once the expansion of the Universe was discovered, the idea was disregarded. However, it was later revived when, in 1998, it was found that the expansion of the universe was accelerating, indicating a positive cosmological constant ($\Lambda > 0$) [16] [29].

Minkowski spacetime is the solution of the sEFE for a vanishing cosmological constant ($\Lambda = 0$). However, when it is a positive constant ($\Lambda > 0$), the (maximally symmetric) solution is instead de Sitter spacetime [1]. de Sitter spacetime has a few fundamental differences from

Minkowski spacetime that have to be taken into account in our formulation of phase space and so will be a part of our exploration. Furthermore, "momentum" in de Sitter space has a noncommutative aspect that, much like our noncommutative spacetime, should be included in our phase space coordinates. The details of this are discussed in sections 5 and 6.

1.2 Research Questions

So, given the motivations that have been laid out. We will look to explore ways of including noncommutative spacetime and de Sitter spacetime in a formulation of phase space. Therefore, this thesis will attempt to tackle the following research questions:

1. Can noncommutative position and noncommutative momentum coordinates be defined consistently in phase space?
2. Can the commutator of 4-position operators be non-constant in phase space?
3. Does noncommutative spacetime violate Lorentz symmetry? If so, what proper subgroup remains?

2 Noncommutative Spacetime

2.1 Physical Interpretation of $\theta^{\mu\nu}$

As mentioned in the introduction, we define spacetime as noncommutative when, in quantum mechanics, we define the position vector operators to not commute with themselves. We may refer to this as the position vector operator having a non-zero self-commutator. For the type of noncommutative spacetime we will be using, the commutator between two 4-position operators $\hat{q}^{\mu} = (\hat{q}^0, \hat{q}^1, \hat{q}^2, \hat{q}^3)$ has the following form [21]:

$$[\hat{q}^{\mu}, \hat{q}^{\nu}] \equiv \hat{q}^{\mu}\hat{q}^{\nu} - \hat{q}^{\nu}\hat{q}^{\mu} = i\hbar\hat{\theta}^{\mu\nu} \tag{2}$$

$\hat{q}^{\mu} \longrightarrow$ Hermitian operators associated with each spacetime coordinate
$\hat{\theta}^{\mu\nu} \longrightarrow$ Real-valued, hermitian, anti-symmetric rank-2 tensor operator of dimension position/momentum.

When two operators don't commute, they cannot be simultaneously diagonalizable. In other words, no quantum state can simultaneously be an eigenstate of both operators. This means that when a state is an eigenstate of one observable operator $\hat{O}_1$, it must be a superposition of states with respect to the other operator $\hat{O}_2$. From this, arises an uncertainty principle between the quantities represented by those operators. The relation between the uncertainty principle and the commutator of those operators is generally defined by [17]:

$$\Delta O_1 \Delta O_2 \geq |\frac{1}{2i}\langle[\hat{O}_1, \hat{O}_2]\rangle| \tag{3}$$

ΔO is the uncertainty in a measurement of O. $\langle\hat{O}\rangle$ denotes the expectation value of $\hat{O}$. For the case of the commutator between 4-position operators as defined in equation 2, we obtain the uncertainty principle:

$$\Delta q^{\mu} \Delta q^{\nu} \geq \frac{\hbar}{2}|\langle\hat{\theta}^{\mu\nu}\rangle| \tag{4}$$

In which Δq^{μ} is the uncertainty in the measurement of the μth component of a quantum state's spacetime 4-position. From this inequality, one can interpret that $\frac{\hbar}{2}|\langle\hat{\theta}^{\mu\nu}\rangle|$ defines a minimum area of uncertainty in every $\mu\nu$-plane in spacetime that a measurement can have.

Given that the effects of this principle have yet to be observed experimentally, we expect that its area of uncertainty is of the Planck scale [21]. Due to this, there are currently no known measurements of the non-zero entries of $\theta^{\mu\nu}$, thus they remain free parameters. The upper bound of any spatial component of θ, currently seems to be [7]:

$\hbar\hat{\theta} \leq 10^{-40}\text{m}^2$,

or in natural units ($\hbar = 1$, $c = 1$),

$\hat{\theta} \leq 10^{-8}\text{GeV}^{-2}$.

2.2 Lorentz Symmetry Violation

The Principle of Relativity states the following:
"**The Laws of Physics remain the same for all observers moving with respect to one another within inertial frames.**"

In other words, experimental results should be independent of lab orientation or boost velocities. This principle's claim is a result of the Lorentz symmetry of Minkowski spacetime. This means that Minkowski spacetime is invariant under rotations and boosts. Typically, on most scales, one would want to include Lorentz symmetry when constructing a physical theory. One of the main reasons for this is that a violation of Lorentz symmetry has never been detected. Upper bounds on different parameters representing violations of Lorentz symmetry have been set experimentally, leading to very high constraints on the possible scale of a violation [25].

However, noncommutative spacetime seems to have an intrinsic inconsistency with this symmetry. $\theta^{\mu\nu}$ defines a minimal observable length scale. Its values must be defined in some frame, which will be a preferred frame. Its components would then transform and mix under boosts and rotations. If a system's behaviour is dependent on $\theta^{\mu\nu}$, it would violate the Principle of Relativity by behaving differently in different inertial frames.

A measurement of this variation between frames would allow one to distinguish (by isolated experiment) which frame of reference they are in and so the Principle of Relativity no longer holds. By $\theta^{\mu\nu}$ defining a preferred frame, it violates Lorentz symmetry.

There are theories of a Lorentz invariant noncommutative spacetime that take varying approaches [6][4]. But, that will not be the focus of this thesis. We will remain with this type of noncommutative spacetime with a focus on its inclusion in a de Sitter phase space. Nevertheless, in section 7 we will consider how different forms of $\theta^{\mu\nu}$ might affect its violation of Lorentz symmetry and under what proper Lorentz subgroups it remains invariant.

2.3 Hermiticity of $\theta^{\mu\nu}$

Let's verify the hermiticity of $\hat{\theta}^{\mu\nu}$ to verify that equation 2 is consistent in that regard. To do so, we have to observe how both sides of the equation behave under hermitian conjugation:
We assume that $([\hat{q}^{\mu},\hat{q}^{\nu}])^{\dagger} = (i\hbar\hat{\theta}^{\mu\nu})^{\dagger}$
The hermitian conjugate of the left-hand side can be easily evaluated by taking into account that the operator of an observable such as position must be hermitian $(\hat{q}^{\mu})^{\dagger} = \hat{q}^{\mu}$ and the hermitian conjugate follows the product rule $(A\cdot B)^{\dagger} = B^{\dagger}\cdot A^{\dagger}$. So we find that:

$$([\hat{q}^{\mu},\hat{q}^{\nu}])^{\dagger} = -[\hat{q}^{\mu},\hat{q}^{\nu}] \tag{5}$$

The commutator of hermitian operators is anti-hermitian. So for consistency, the right-hand side must also be anti-hermitian $(i\hbar\hat{\theta}^{\mu\nu})^\dagger = -i\hbar\hat{\theta}^{\mu\nu}$

$$(i\hbar\hat{\theta}^{\mu\nu})^\dagger = -i(\hbar\hat{\theta}^{\mu\nu})^\dagger = -i\hbar\hat{\theta}^{\mu\nu} \tag{6}$$

Hence, we can conclude that $(\hat{\theta}^{\mu\nu})^\dagger = \hat{\theta}^{\mu\nu}$ is hermitian.

This can be surprising if one assumes that $\hat{\theta}^{\mu\nu}$ follows a matrix algebra where $(A^{\mu\nu})^\dagger = ((A^{\mu\nu})^T)^* = A^{\nu\mu}$ and so when A is an anti-symmetric, real-valued matrix we would expect that $(A^{\mu\nu})^\dagger = -A^{\mu\nu}$ is anti-hermitian. This is not the case for $\hat{\theta}^{\mu\nu}$.

3 Poisson Structures

To include the commutator result of a noncommutative spacetime in both a quantum and classical formulation of phase space, we need to understand bracket structures. More specifically, we need to understand what conditions brackets must satisfy and how we can modify their structure based on the results we want. From these conditions, we will extract further conditions on what type of noncommutative spacetime can be defined consistently within the phase space formulation.

Furthermore, we need to understand the purpose that bracket structures serve in phase space formulations of classical mechanics. In classical mechanics, the Poisson bracket is used in evaluating the behaviour of classical systems defined in phase space. Poisson brackets and their structure are defined in the following way:

Definition: A **Poisson bracket** is a binary map on the space $C^\infty(M)$ of smooth functions on a smooth manifold. This (Poisson) manifold is denoted $(M, \{\cdot,\cdot\})$ [10]. The mapping of the Poisson bracket is defined as follows:

$$\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$$

Let $f, g, h \in C^\infty(M)$. The conditions this bracket must satisfy are:
$\mathbb{R}$-bilinearity,

$$\{af + bg, h\} = a\{f,h\} + b\{g,h\} \qquad a, b \in \mathbb{R} \tag{7}$$

skew-symmetty,

$$\{f,g\} = -\{g,f\} \tag{8}$$

and the Jacobi Identity.

$$\{f,\{g,h\}\} + \{h,\{f,g\}\} + \{g,\{h,f\}\} = 0 \tag{9}$$

By satisfying the above conditions, $\{\cdot,\cdot\}$ is a Lie bracket. To be a Poisson bracket it must also satisfy the Leibniz Rule:

$$\{f, gh\} = g\{f,h\} + \{f,g\}h \tag{10}$$

If we consider a Poisson Structure on an n-dimensional smooth Manifold M with local coordinates $\{x^i\}_{i=1}^n$, we can describe a Poisson Bracket of two functions f and g on M through its structure functions π^{ij} in the following form:

$$\{f(x), g(x)\} = \pi^{ij} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j}, \qquad i, j = 1, 2, ..., n \tag{11}$$

The structure functions must be smooth on M and anti-symmetric under $i \leftrightarrow j$ exchange to satisfy the bracket's skew-symmetry.

3.1 Canonical Poisson Bracket in Phase Space

For example, the canonical Poisson bracket on $\mathbb{R}^2$ phase space, in which $(x^1, x^2) = (q, p)$ and q is canonical position and p is canonical momentum, has the (constant) structure functions:

$$\pi^{ij} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{12}$$

Hence, if we sum over the structure functions, the Poisson bracket takes the explicit form:

$$\{f(q,p), g(q,p)\} = \left(\frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q}\right) \tag{13}$$

It can more generally be defined on $\mathbb{R}^{2n}$ phase space of coordinates $(x^1, x^2, ..., x^n | x^{n+1}, ..., x^{2n}) = (q^1, q^2, ..., q^n | p^1, ..., p^n)$ with the following structure function tensor:

$$\pi^{ij} = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \tag{14}$$

with $\mathbb{1}$ being the nth-dimensional identity. With this form, our higher dimension Poisson bracket takes the form:

$$\{f(\vec{q},\vec{p}), g(\vec{q},\vec{p})\} = \sum_{i=1}^{n}\left(\frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p^i} - \frac{\partial f}{\partial p^i}\frac{\partial g}{\partial q^i}\right) \tag{15}$$

With the bracket explicitly defined, we can discuss its purpose in classical phase space mechanics.

The canonical Poisson bracket is used in Hamiltonian Mechanics as a binary operator that evaluates the time evolution of functions on phase space. This can be seen in the Liouville equation:

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\vec{q},\vec{p},t) = \{f, H\} + \frac{\partial f}{\partial t} \tag{16}$$

in which H is the Hamiltonian of the system as a function of the canonical phase space coordinates. The derivation of this is simple, one can apply the chain rule to the left-hand side:

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\vec{q},\vec{p},t) = \frac{\partial f}{\partial q^i}\frac{\mathrm{d}q^i}{\mathrm{d}t} + \frac{\partial f}{\partial p^j}\frac{\mathrm{d}p^j}{\mathrm{d}t} + \frac{\partial f}{\partial t} \tag{17}$$

Utilize Hamilton's equations of motion:

$$\frac{\partial q^i}{\partial t} = \frac{\partial H}{\partial p^i} \qquad \frac{\partial p^i}{\partial t} = -\frac{\partial H}{\partial q^i} \tag{18}$$

Then given that $\frac{\mathrm{d}p^i}{\mathrm{d}t} = \frac{\partial p^i}{\partial t}$ and the same for q^i, by applying substitution we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\vec{q},\vec{p},t) = \frac{\partial f}{\partial q^i}\frac{\partial H}{\partial p^i} - \frac{\partial f}{\partial p^i}\frac{\partial H}{\partial q^i} + \frac{\partial f}{\partial t} = \{f, H\} + \frac{\partial f}{\partial t} \tag{19}$$

Which is the Liouville equation. Hence, if one is looking to see how certain properties of a system evolve over time, a function representative of the property can be defined on phase space. The Poisson bracket of that function with the system's Hamiltonian will then equate to that function's implicit time dependence.

For the specific case in which the function has no explicit time dependence ($\frac{\partial f}{\partial t} = 0$), then the equation simplifies to:

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\vec{q},\vec{p}) = \{f,H\} \tag{20}$$

For this case, when $\{f,H\} = 0$, the property represented by $f(\vec{q},\vec{p})$ is known as a constant of motion as its value is constant in time and so is conserved in that system. By the anti-symmetry condition of the Poisson bracket, H itself is a constant of motion $\{H,H\} = 0$. This is nothing more than conservation of energy, given that the system is closed ($\frac{\partial H}{\partial t} = 0$).

3.2 Noncommutative Poisson Structure

We can include our noncommutative spacetime result in the classical Poisson bracket by modifying the structure function. A good way to go about this is to recognize that the structure function of a bracket is always returned by the bracket of local coordinates [10]:

$$\pi^{ij} = \{x^i, x^j\} \tag{21}$$

Hence, if we want to describe a phase space of 4-position q^μ and 4-momentum p^ν with an analogy to the noncommutative result, we can define:

$$\{q^\mu, q^\nu\}_{\mathrm{NC}} = \theta^{\mu\nu} \tag{22}$$

This leads to the overall structure function having the following form:

$$\pi^{ij} = \begin{pmatrix} \theta & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \qquad i,j = 1,2,..,8 \tag{23}$$

If we write out our noncommutative Poisson bracket in terms of 4-position and 4-momentum:

$$\{f,g\}_{\mathrm{NC}} = \lambda\theta^{\mu\nu}\frac{\partial f}{\partial q^\mu}\frac{\partial g}{\partial q^\nu} + \eta^{\mu\nu}\left(\frac{\partial f}{\partial q^\mu}\frac{\partial g}{\partial p^\nu} - \frac{\partial f}{\partial p^\mu}\frac{\partial g}{\partial q^\nu}\right) \qquad \mu,\nu = 0,1,2,3 \tag{24}$$

In which $\eta^{\mu\nu} = diag(-1,+1,+1,+1)$ is the Minkowski metric and λ is a dimensionful constant that maintains the same dimensions between terms. This results in the following structural relations:

$$\{q^\mu, q^\nu\}_{\mathrm{NC}} = \lambda\theta^{\mu\nu} \qquad \{q^\mu, p^\nu\}_{\mathrm{NC}} = \eta^{\mu\nu} \qquad \{p^\mu, p^\nu\}_{\mathrm{NC}} = 0 \tag{25}$$

We can define $\theta^{\mu\nu}$ as being a function of local coordinates. If the function is linear, this leads to what is called an affine Poisson structure ([24], pg. 21).

3.3 Affine Poisson Structure

So, if we are looking to construct a Poisson Structure with a $\lambda\theta^{\mu\nu} = \{q^\mu, q^\nu\}$ linear in q. We simply define our structure functions to be such:

$$\{q^\mu, q^\nu\}_{\text{Affine}} = \lambda(a_\gamma^{\mu\nu} q^\gamma + b^{\mu\nu}) \tag{26}$$

The bracket can be written in terms of 4-position and 4-momentum.

$$\{f,g\}_{\text{Affine}} = \lambda(a_\gamma^{\mu\nu} q^\gamma + b^{\mu\nu}) \frac{\partial f}{\partial q^\mu}\frac{\partial g}{\partial q^\nu} + \eta^{\mu\nu}\left(\frac{\partial f}{\partial q^\mu}\frac{\partial g}{\partial p^\nu} - \frac{\partial f}{\partial p^\mu}\frac{\partial g}{\partial q^\nu}\right) \qquad \mu,\nu = 0,1,2,3 \tag{27}$$

$a_\gamma^{\mu\nu}$ and $b^{\mu\nu}$ must continue to maintain anti-symmetry in $\mu \leftrightarrow \nu$ exchange such that the skew-symmetry condition is satisfied (eq. 8).

3.4 The Jacobi identity

The Jacobi identity as displayed in equation 9 is a condition that every Lie bracket must satisfy. A Poisson bracket is, by definition, a Lie bracket [10]. Furthermore, the Dirac commutator used in quantum mechanics is also a Lie bracket and so, must also satisfy the Jacobi identity. We will be using this identity as both a sanity check when making major modifications to bracket structures and as a base to argue consequential conditions on the brackets applicable to the phase space of a system in a noncommutative de Sitter spacetime.

We need to recognize an important distinction between the Jacobi identity of a Poisson bracket and the Jacobi identity of the commutators of generators of a symmetry. Due to the definition of the Poisson bracket and the Dirac commutator, both always satisfy the Jacobi identity, as long as the functions used in it belong to the algebra on which the bracket is defined. If we find that the Jacobi identity isn't satisfied, we must consider what needs to be changed. What do we need to sacrifice and remove from our formulation of phase space? The functions used in the identity or the structure of the bracket itself?

The symmetries, defined by the space-time we use, lead to Lie algebras with generators and a bracket that must also satisfy the Jacobi identity (see section 5). If we wish to include them as operators in quantum mechanics, then they will also need to satisfy the Jacobi identity between all operators. If we agree that the generators of the symmetry of the spacetime being used are fundamental elements that must be included, we should not conclude that the Jacobi identity can enforce conditions on them. Instead, we should modify other aspects that can guarantee the satisfaction of the identity, such as the bracket structure.

With this line of reasoning, in section 4.6, we will see that the Jacobi identity of quantum operators results in a limitation of the types of noncommutative Minkowski spacetime that can be defined. We will also later argue, in section 6.3, how this condition might also limit the types of noncommutative de Sitter spacetime. These limitations are not absolute, but, something to take into account when looking to define a formulation of phase space on a noncommutative de Sitter spacetime.

4 Groenewold-Moyal Bracket

According to the Ehrenfest theorem, the time-evolution of a quantum operator $\hat{f}$ of functions in Hilbert space and its expectation value $\langle \hat{f} \rangle$ is dictated by [17]:

$$\frac{d\langle \hat{f} \rangle}{dt} = \frac{i}{\hbar}\langle [\hat{H}, \hat{f}] \rangle + \left\langle \frac{\partial \hat{f}}{\partial t} \right\rangle \tag{28}$$

This looks like a Quantum Mechanical version of the Liouville equation. It is clear that the Dirac commutator $[a,b] = ab - ba$ serves a similar purpose in quantum mechanics as the canonical Poisson bracket serves in classical phase space (equation 16). So, if we want to define a map between classical and quantum mechanics such that we can describe quantum and classical equivalents of a system, a reasonable goal would be to map the Poisson bracket to the Dirac commutator:

$$\{f, g\} \to \frac{1}{i\hbar}[\hat{f}, \hat{g}] \tag{29}$$

4.1 Dirac's Canonical Quantization

The first attempt at this was Dirac's proposition in 1930 [24]. He proposed simply replacing the position q^i and momentum p^i coordinates with their respective operators. Let's denote this quantization by $\hat{Q}()$.

$$\hat{Q}(q^i) = \hat{q}^i \qquad \hat{Q}(p^i) = \hat{p}^i \tag{30}$$

When acting on a wavefunction ϕ in Hilbert space, they have the following explicit form:

$$\hat{q}^i \phi = q^i \phi \qquad \hat{p}^i \phi = -i\hbar \frac{\partial}{\partial q^i} \phi \tag{31}$$

Furthermore, Dirac's canonical quantization also maps the Poisson bracket in the following way:

$$\hat{Q}(\{f, g) = \frac{1}{i\hbar}[\hat{f}, \hat{g}] \tag{32}$$

Under this quantization, the Liouville equation (16) maps to equation 28 describing the time evolution of operators. So, this quantization serves the purpose of allowing for a phase space formulation of quantum mechanics. However, it has two major problems.

One is that it is not well-defined for terms containing products of q and p. This is due to an issue of ordering. In classical phase space, the coordinate variables are commutative ($[q^i, p_j] = 0$), while their corresponding operators are not ($[\hat{q}^i, \hat{p}_j] = i\hbar\delta^i_j$). So, the quantization of the term $q \cdot p$ can map to multiple orderings of their operators ($\hat{q} \cdot \hat{p}$, $\hat{p} \cdot \hat{q}$) which are not equivalent. Dirac's quantization is hence not well-defined for the quantization of certain functions on phase space.

Furthermore, according to the Groenewold-van Hove Theorem, there can be no consistent quantization of the Poisson algebra of polynomials on phase space [24]. Groenewold realized that any quantization of the Poisson bracket was only precise up to first-order powers of $\hbar$ [24]:

$$\hat{Q}(\{f,g\}) = [\hat{f},\hat{g}] + \mathcal{O}(\hbar^2) \tag{33}$$

Hence, if one aims to describe a phase space formulation of Quantum Mechanics, one must define a quantization and bracket that allow for a consistent 1-1 mapping.

This was achieved through the Weyl-Wigner Mappings and the Groenewold-Moyal Bracket which allowed for a consistent quantization of a classical bracket to the Dirac commutator of operators in quantum mechanics [24].

4.2 Weyl Quantization

Weyl quantization is a general prescription that describes how to map functions of (q,p) in classical phase space to their equivalent operator as a function of $(\hat{q},\hat{p})$ in Hilbert Space. This is done through a Weyl Map W ([31], Chapter 6 F. Lizzi, pg. 103) here defined in $\mathbb{R}^2$ phase space $(x^1,x^2) = (q,p)$:

$$\hat{W}(f)(\hat{q},\hat{p}) = \frac{1}{2\pi}\int \mathrm{d}\zeta \mathrm{d}\eta \mathrm{d}q \mathrm{d}p f(q,p) e^{\frac{i}{\hbar}(\zeta(\hat{p}-p)+\eta(\hat{q}-q))} \tag{34}$$

which can be seen as two consecutive Fourier transforms $(q,p) \to (\eta,\zeta) \to (\hat{q},\hat{p})$. The inverse map is called the Wigner Map and can be formulated as such:

$$W^{-1}(\hat{F})(q,p) = \int \frac{\mathrm{d}\eta \mathrm{d}\zeta}{(2\pi)^2\hbar} e^{\frac{i}{\hbar}(\zeta(\hat{p}-p)+\eta(\hat{q}-q))} Tr\hat{F} \tag{35}$$

We can define the $\star$-product (Moyal Product) as the following:

$$f \star g = W^{-1}(\hat{W}(f)\hat{W}(g)) \tag{36}$$

This product is associative but noncommutative and so can reproduce quantum mechanical commutation relations with functions of classical phase space. This definition can be expanded into different forms, one of which is the differential form [3] in local coordinates of our phase space $\{x^i\}_{i=1}^8$:

$$(f \star g)(x) = \exp\left(\frac{i\hbar}{2}\frac{\partial}{\partial x^j}\pi^{jk}\frac{\partial}{\partial y^k}\right) f(x)g(y)\Big|_{y=x} \qquad j,k = 1,2,...,8 \tag{37}$$

We can expand the exponent into a power series in $\hbar$ in which the second term is proportional to the Poisson Bracket:

$$f \star g = fg + i\hbar\{f,g\} + \mathcal{O}(\hbar^2) \tag{38}$$

4.3 Bracket Mapping

Now that the Weyl-Wigner mappings and the $\star$-product have been defined, we can define the Groenewold-Moyal bracket. It is defined as follows:

$$\{f,g\}_{GM} = \frac{1}{i\hbar}(f \star g - g \star f) \tag{39}$$

Recalling the definition of the star-product in equation 36 we see that by construction, one can map this bracket to the quantum Dirac commutator through a Weyl Map $\hat{W}()$:

$$\hat{W}(\{f,g\}_{GM}) = \frac{1}{i\hbar}[\hat{W}(f), \hat{W}(g)] \tag{40}$$

It is this compatibility between the Weyl Map and the Groenewold-Moyal bracket that allows for a 1-1 mapping of the Groenewold-Moyal classical bracket to the commutator of quantum operators (Groenewold, 1946 [18]). With this, one can have a phase space formulation of Quantum Mechanics.

We can represent the bracket more explicitly through the following expression:

$$\{f,g\}_{GM} = \frac{2}{\hbar}\sin\left(\frac{\hbar}{2}\pi^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial y^j}\right)f(x)g(y)\Big|_{y=x} \qquad i,j = 1,2,...,8 \tag{41}$$

π^{ij} is the structure function of this bracket. The standard GM bracket for a commutative phase space has the same structure function as the canonical Poisson bracket (eq. 14). This standard bracket can be written in terms of 4-momentum and 4-position:

$$\{f,g\}_{GM}(q,p) = \frac{2}{\hbar}\sin\left(\frac{\hbar}{2}\eta^{\mu\nu}\left(\frac{\partial}{\partial q^\mu}\frac{\partial}{\partial p'^\nu} - \frac{\partial}{\partial p^\nu}\frac{\partial}{\partial q'^\mu}\right)\right)f(q,p)g(q',p')\Big|_{(q',p')=(q,p)} \tag{42}$$

in which $\eta^{\mu\nu} = diag(-1,1,1,1)$ is the Minkowski metric.

The major difference between this bracket and the canonical Poisson bracket is a deformation in powers of $\hbar$. This allows for one to obtain the Poisson bracket from the GM bracket in the limit of $\hbar \to 0$:

$$\{f,g\}_{GM} = \{f,g\}_{Poisson} + \mathrm{O}(\hbar) \tag{43}$$

A summary of the different brackets discussed until now and their maps is displayed in the diagram in Figure 2.

4.4 Noncommutative Groenewold-Moyal Bracket

Much like the Poisson bracket, if we want to include a noncommutative bracket result, we simply need to modify the structure function. It turns out that despite the higher complexity

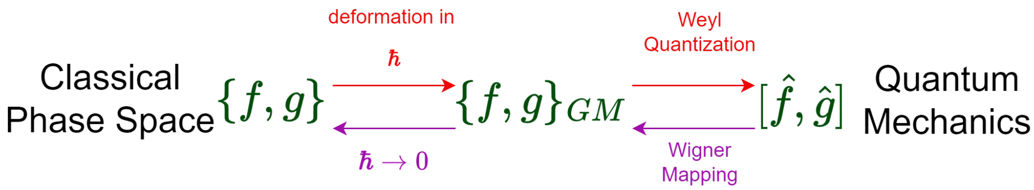

Figure 2: *Summarizing diagram of mappings between phase space of classical and quantum mechanics.*

of the Groenewold-Moyal bracket, we can continue to define the structure function (in local coordinates) as:

$$\{x^i,x^j\}_{GM} = \{x^i,x^j\}_{Poisson} + \mathrm{O}(\hbar) = \{x^i,x^j\}_{Poisson} = \pi^{ij}(x) \qquad i,j = 1,2,...,8 \tag{44}$$

The higher-order terms vanish due to the higher-order derivatives of x vanishing. So, to include the noncommutative result we can modify the structure function to have the same form as the noncommutative Poisson structure shown in equation 23.

We can again write the bracket in terms of 4-position and 4-momentum (q^μ, p^ν). However, much like in equation 24, the inclusion of $\frac{\partial}{\partial q^\mu}\frac{\partial}{\partial q'^\nu}$ and $\frac{\partial}{\partial q^\mu}\frac{\partial}{\partial p'^\nu}$ terms in the same sum will require splitting up our structure function into different tensors and including dimensionful constants λ, τ to take into account dimensional coherence.

$$\begin{aligned}\{f,g\}_{GM}(q,p) = \frac{2}{\hbar}\sin\Bigg(\frac{\hbar}{2}\Bigg(\lambda\theta^{\mu\nu}\frac{\partial}{\partial q^\mu}\frac{\partial}{\partial q'^\nu} + \\ \tau\eta^{\mu\nu}\left(\frac{\partial}{\partial q^\mu}\frac{\partial}{\partial p'^\nu} - \frac{\partial}{\partial p^\nu}\frac{\partial}{\partial q'^\mu}\right)\Bigg)\Bigg) f(q,p)g(q',p')\Big|_{(q',p')=(q,p)}\end{aligned} \tag{45}$$

So, in terms of 4-position and 4-momentum, this noncommutative bracket structure gives the result:

$$\{q^\mu,q^\nu\}_{\text{NCGM}} = \lambda\theta^{\mu\nu} \quad \{q^\mu,p^\nu\}_{\text{NCGM}} = \tau\eta^{\mu\nu} \quad \{p^\mu,p^\nu\}_{\text{NCGM}} = 0 \tag{46}$$

4.5 Affine Groenewold-Moyal Bracket

We are free to define the structure functions to be linear in an analogy to the Affine Poisson Structure. For simplicity, let's define this formulation in local coordinates (recall section 3.1):

$$\pi^{ij}(x) = a^{ij}_k x^k + b^{ij} \tag{47}$$

$$\{f,g\}_{AGM} = \frac{2}{\hbar}\sin\left(\frac{\hbar}{2}\pi^{ij}(z)\frac{\partial}{\partial x^i}\frac{\partial}{\partial y^j}\right) f(x)g(y)\Big|_{y=z=x} \qquad i,j,k = 1,2,...,n \tag{48}$$

If we re-formulate this in terms of canonical 4-momentum and 4-position, we can define a linear $\theta^{\mu\nu}$ for this bracket (similarly to section 3.3):

$$\{q^\mu,q^\nu\}_{AGM} = \{q^\mu,q^\nu\}_{\text{Affine}} + \mathrm{O}(\hbar) = \theta^{\mu\nu} = c^{\mu\nu}_\gamma q^\gamma + d^{\mu\nu}_\gamma p^\gamma + k^{\mu\nu} \tag{49}$$

Once again, the higher order terms in the expansion only contain higher order derivatives and so they vanish since $\partial\partial x = 0$.

However, in any modification of the bracket structure, it's important to verify that our bracket remains a Poisson bracket or even a Lie bracket. Other than checking that linearity, skew-symmetry and the Leibniz Rule still hold, it's important we verify the Jacobi Identity as a test.

4.6 Jacobi Identity Limitation

When we apply the Jacobi Identity to the Groenewold-Moyal bracket with the structure functions defined in eq. 46 with respect to q^μ, q^ν and p^σ and let $\{\cdot,\cdot\}_{GM} = \{\cdot,\cdot\}$ for simplicity:

$$J(q^\mu, q^\nu, p^\sigma) \equiv \{q^\mu, \{q^\nu, p^\sigma\}\} + \{p^\sigma, \{q^\mu, q^\nu\}\} + \{q^\nu, \{p^\sigma, q^\mu\}\} = 0 \tag{50}$$

The first and third term vanish due to:

$$\{q^\mu, \{q^\nu, p^\sigma\}\} = \{q^\mu, \tau\eta^{\nu\sigma}\} = 0 = \{q^\nu, \{p^\sigma, q^\mu\}\} \tag{51}$$

By substituting $\{q^\mu, q^\nu\} = \lambda\theta^{\mu\nu}$, we obtain the equation:

$$\{p^\sigma, \theta^{\mu\nu}\} = 0 \tag{52}$$

Given that this is the Groenewold-Moyal bracket, we can apply a Weyl Transformation to both sides and obtain the equation (with the hat on a symbol (e.g. Ô) denoting an operator):

$$[\hat{p}^\sigma, \hat{\theta}^{\mu\nu}] = 0 \tag{53}$$

Given that the momentum operator $\hat{p}^\sigma = -i\hbar\partial^\sigma$, if we include a test function $f(\hat{q})$ we obtain:

$$\partial^\sigma(\hat{\theta}^{\mu\nu} f) - \hat{\theta}^{\mu\nu}\partial^\sigma(f) = \hat{\theta}^{\mu\nu}\partial^\sigma(f) + f\partial^\sigma(\hat{\theta}^{\mu\nu}) - \hat{\theta}^{\mu\nu}\partial^\sigma(f) = 0 \tag{54}$$

this then cancels to:

$$\partial^\sigma(\hat{\theta}^{\mu\nu}) = 0 \tag{55}$$

So, the Jacobi Identity directly requires that the quantized structure function of $\{q,q\}$ be constant with respect to position. If we were to define the structure function of the Groenwold-Moyal Bracket $\theta^{\mu\nu}$ as any function dependent on position, assuming that the Weyl mapping continues to apply, the Jacobi identity would not be satisfied and our modified bracket would no longer satisfy the conditions to be a Lie bracket, nor a Poisson bracket. This is a serious limitation that will henceforth be referred to as the "Jacobi limitation".

To drive this point further, we obtain the same equation (55) if we evaluate the Jacobi identity of the commutators of quantum operators ($J(\hat{q}^\mu, \hat{q}^\nu, \hat{p}^\sigma) = 0$). So, regardless of the applicability of the GM bracket and the Weyl-Wigner mappings, this limitation remains for the following structure:

$$[\hat{q}^\mu, \hat{q}^\nu] = i\hbar\hat{\theta}^{\mu\nu} \qquad [\hat{q}^\mu, \hat{p}^\nu] = i\hbar\eta^{\mu\nu} \qquad [\hat{p}^\mu, \hat{p}^\nu] = 0 \tag{56}$$

What modifications we can make to avoid this result will be discussed in section 6.

5 de Sitter Space

Similarly to the noncommutative position defined in a noncommutative spacetime. A de Sitter spacetime has a noncommutative momentum. This originates from its symmetry group and the respective algebra of infinitesimal generators. However, before we can approach the noncommutative aspects of that algebra, let's revise what de Sitter space is.

To describe the structure and form of spacetime in a universe, we want to first describe the spacetime in the simplest case, that of an empty universe, without mass or energy. To do this, one solves the Einstein field equations without a source term (a.k.a. sourceless) to find the spacetime metric that describes the geometry of that spacetime solution.

de Sitter spacetime is a solution to the sourceless Einstein equation in the case of a non-vanishing cosmological constant Λ [1]. For a vanishing Λ, the solution to the sourceless Einstein field equations is special relativity's Minkowski spacetime. However, given the current observational evidence for a non-vanishing cosmological constant [29], there are reasons to believe we exist in a universe with a spacetime whose sourceless condition is that of the de Sitter spacetime. One might also call our universe an asymptotically de Sitter universe, meaning that as the universe expands and becomes less matter-dominant, our spacetime asymptotically approaches de Sitter spacetime.

A constant non-vanishing cosmological constant in the sourceless limit leads to a space with a constant curvature. There are two viable solutions: one with a positive curvature (de Sitter with $\varepsilon = +1$), and one with a negative curvature (anti-de Sitter with $\varepsilon = -1$). Both have the following 5-D metric:

$$\eta_{AB}\xi^A\xi^B = -(\xi^0)^2 + (\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2 + \varepsilon(\xi^4)^2 = \varepsilon R^2 \tag{57}$$

in which R is the pseudo-radius of the de Sitter Space and $\xi^A = (\xi^0, \xi^1, \xi^2, \xi^3, \xi^4)$ are the Cartesian coordinates of the space. We will be focusing on the solution with positive curvature $(\varepsilon = +1)$.

Defined by this metric equation, one can model the de Sitter space as a 4-D one-sheeted hyperboloid submanifold embedded in a (4+1)-D Minkowski space [27]. A 2-D example of a hyperboloid is shown in Figure 3. This hyperboloid can be further interpreted as a spatial 3-sphere of radius R with a time dimension in which it is hyperbolic. Due to this geometry, the de Sitter Space of diagonal metric $\eta_{AB} = \mathrm{diag}(-1,1,1,1,1)$ has the symmetry of the pseudo-orthogonal group SO(4,1) [15], henceforth referred to as the de Sitter group.

We can decompose this metric into a combination of the Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$ and the 4th component $\varepsilon = \eta_{44}$:

$$\eta_{ab}\xi^a\xi^b + \varepsilon(\xi^4)^2 = \varepsilon R^2. \qquad a,b = 0,1,2,3 \tag{58}$$

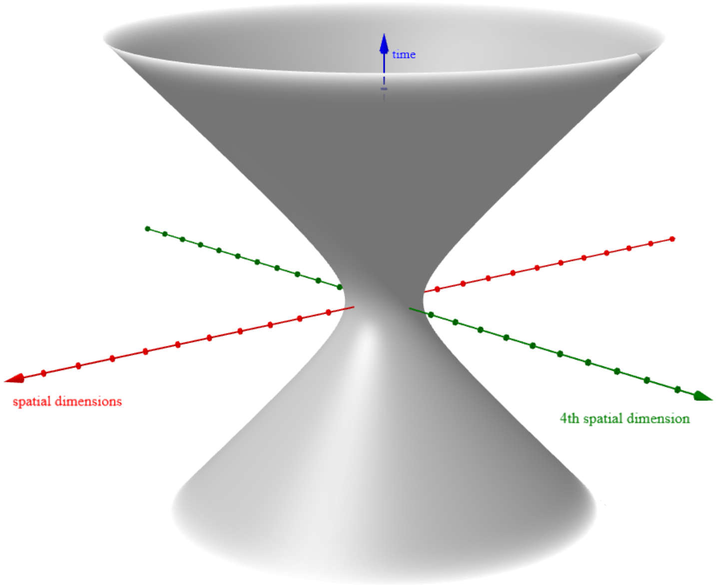

Figure 3: *2-D Hyperboloid in 3-D space rendered using GeoGebra. Three spatial dimensions are contracted into one horizontal axis, leaving the other horizontal axis as the fourth spatial dimension and the vertical axis as the time dimension.*

We can apply a stereographic mapping from the curved surface of the hyperboloid embedded in 5-D space to a 4-D Minkowski plane. The coordinates on this mapped plane q^μ are obtained through the following formula [2]:

$$q^\mu \equiv \frac{1}{2}(1-\frac{\xi^4}{R})\delta^\mu_\nu \xi^\nu \tag{59}$$

The first 4 components of the de Sitter metric $\eta_{\mu\nu}$ map to the metric on the Minkowski plane $g_{\mu\nu}$ with a conformal factor n:

$$g_{\mu\nu} \equiv n^2(q)\eta_{\mu\nu} \qquad n(q) = \frac{1}{1+\varepsilon\frac{\sigma^2}{4R^2}} \tag{60}$$

in which $\sigma^2 = \eta_{\alpha\beta}q^\beta q^\alpha$. For further details on this mapping, see [2] [12].

5.1 de Sitter Group

While, in a Minkowski spacetime, isometry transformations are determined by the Poincaré group. In a de Sitter spacetime, they are instead determined by the de Sitter group. This group follows a certain representative algebra (de Sitter algebra). The generators of infinitesimal de Sitter transformations in terms of our stereographic coordinates are the following [2]:

$$J_{ab} = \delta^\mu_a \delta^\nu_b \left(g_{\rho\mu}q^\rho P_\nu - g_{\rho\nu}q^\rho P_\mu\right) \tag{61}$$

in which

$$P_\mu = -i\partial_\mu \tag{62}$$

are the generators of Poincaré translations and

$$L_{\mu\nu} = g_{\mu\rho} q^{\rho} P_{\nu} - g_{\nu\rho} q^{\rho} P_{\mu} \tag{63}$$

are the generators of the Lorentz group. The 4th component (5th dimension) of the de Sitter transformations is:

$$J_{a4} = R\delta_{a}^{\mu}\left(\varepsilon(P_{\mu} + \frac{\varepsilon}{4R^2}K_{\mu})\right) \tag{64}$$

in which

$$\Pi_{\mu} = \varepsilon(P_{\mu} + \frac{\varepsilon}{4R^2}K_{\mu}) \tag{65}$$

are the generators of de Sitter Translations with,

$$K_{\mu} = (2g_{\mu\lambda}q^{\lambda}q^{\rho} - \sigma^2\delta_{\mu}^{\rho})P_{\rho} \tag{66}$$

as the generator of special conformal transformations. We can for now interpret Π_{μ} as the de Sitter momentum operator and it has the following self-commutation relation:

$$[\Pi_{\mu}, \Pi_{\nu}] = -\varepsilon i R^{-2} L_{\mu\lambda} \tag{67}$$

This non-zero self-commutator is another noncommutative aspect that must be included in a phase space formulation of de Sitter space.

5.2 Inönü-Wigner Contraction

If one takes the limit of the pseudo-radius $R \to \infty$, de Sitter space and the de Sitter group both contract to Minkowski space and the Poincaré group, respectively. This limit is called the contraction limit. We find, mathematically, that:

$$\lim_{R\to\infty} \Pi_{\mu} = P_{\mu} \qquad \lim_{R\to\infty} L_{\mu\nu} = L_{\mu\nu} \tag{68}$$

The generator of de Sitter translations contracts to that of Poincaré translations. Furthermore, the metric of the stereographic coordinates on the hyperboloid itself reduces to the Minkowski metric:

$$\lim_{R\to\infty} g_{\mu\nu} = \eta_{\mu\nu} \tag{69}$$

Effectively showing that this limit contracts the de Sitter geometry to the Minkowski geometry for a vanishing Λ.

5.3 Classical Phase Space in a Noncommutative de Sitter Spacetime

Let's define a bracket that describes a formulation of phase space in which we have both noncommutative aspects from de Sitter Space and the noncommutative geometry described by the Groenewold-Moyal Bracket.

We can formulate a Poisson bracket with the desired structure and then apply a deformation to have a modified Groenewold-Moyal bracket. Then, if we can map this bracket to a theory of quantum mechanics and avoid the Jacobi limitation (section 4.6), we can obtain a quantum noncommutative de Sitter phase space.

A Poisson bracket with the combined non-zero structure function results of the non-commutative spacetime ($\{q^\mu, q^\nu\} = \theta^{\mu\nu} \neq 0$), de Sitter Space ($\{p^\mu, p^\nu\} = \Omega^{\mu\nu} \neq 0$) and classical phase space ($\{q^\mu, p_\nu\} = \delta^\mu_\nu =$ Kronecker delta) has the following form:

$$\{f,g\}_{NC} = \lambda\theta^{\mu\nu}\frac{\partial f}{\partial q^\mu}\frac{\partial g}{\partial q^\nu} + \kappa\Omega^{\mu\nu}\frac{\partial f}{\partial p^\mu}\frac{\partial g}{\partial p^\nu} + \tau g^{\mu\nu}\left(\frac{\partial f}{\partial q^\mu}\frac{\partial g}{\partial p^\nu} - \frac{\partial f}{\partial p^\nu}\frac{\partial g}{\partial q^\mu}\right) \tag{70}$$

This bracket is defined on an algebra of functions in 4-position and 4-momentum phase space with the same coordinates as described in section 3.1. $\theta^{\mu\nu}$ and $\Omega^{\mu\nu}$ are real and anti-symmetric. The constants λ, κ, τ are to maintain consistency of dimension of the terms being added.

One can rewrite this bracket in terms of local coordinates $\{x^i\}_{i=1}^8$ with the following form and structure function tensor:

$$\{f(x), g(x)\}_{NC} = \pi^{ij}\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^j} \qquad \pi^{ij} = \begin{pmatrix} \theta & \mathbb{1} \\ -\mathbb{1} & \Omega \end{pmatrix} \tag{71}$$

In which every block is a 4×4 matrix. We can see that the structure function has the same form as the canonical Poisson bracket except we've replaced the diagonal block elements with anti-symmetric tensors. Given that they are anti-symmetric, the tensor as a whole remains anti-symmetric and with a 0 diagonal.

Now, if we wish to define non-constant structure functions, we can, as long as they satisfy the Jacobi identity and so maintain a coherent Poisson Structure. One option is to define:

$$\theta^{\mu\nu} = \Omega^{\mu\nu} = L^{\mu\nu} = q^\mu p^\nu - q^\nu p^\mu \tag{72}$$

in which $L^{\mu\nu}$ is the generator of Lorentz transformations. However as we saw in section 4.6, it is not possible to have a Groenewold-Moyal bracket with a non-constant $\{q, q\}$ structure function. How we can try to overcome this will be discussed in the next section. It seems that given the limitation, we must define θ as a constant. What form of θ is most suitable and its symmetries is discussed in section 7.

6 de Sitter Phase Space

It seems that the issue of the Jacobi limitation essentially resides in the algebra of quantum operators as we only ran into the limitation when we mapped the GM bracket to the Dirac commutator. So, let's attempt to define a noncommutative quantum algebra that avoids that limitation. We can then consider what classical phase space formulation can be defined with the results obtained.

We will be using the momentum operator defined as $\hat{p}^\mu = -i\hbar\partial^\mu$. With that, the de Sitter momentum operator will simply be defined as:

$$\hat{\Pi}_\mu = \varepsilon(\hat{p}_\mu + \frac{\varepsilon}{4R^2}(2g_{\mu\lambda}q^\lambda q^\rho - \sigma^2\delta^\rho_\mu)\hat{p}_\rho) \tag{73}$$

Its self-commutator will then gain a factor of $\hbar$:

$$[\hat{\Pi}_\mu, \hat{\Pi}_\nu] = -\varepsilon i\hbar R^{-2}\hat{L}_{\mu\lambda} \tag{74}$$

6.1 de Sitter Jacobi Identity

Before attempting to make the position self-commutator non-zero ($[\hat{q}^\mu, \hat{q}^\nu] \neq 0$), we first have to check that the position operator and de Sitter momentum operator can satisfy the Jacobi identity. To do this, while allowing $[\hat{q}^\mu, \hat{q}^\nu] = 0$ for simplicity, let's verify that the Jacobi Identity holds for:

$$J(\hat{q}^\mu, \hat{\Pi}^\nu, \hat{\Pi}^\sigma) = [\hat{q}^\mu, [\hat{\Pi}^\nu, \hat{\Pi}^\sigma]] + [\hat{\Pi}^\sigma, [\hat{q}^\mu, \hat{\Pi}^\nu]] + [\hat{\Pi}^\nu, [\hat{\Pi}^\sigma, \hat{q}^\mu]] = 0 \tag{75}$$

Using the relation $[\hat{\Pi}^\nu, \hat{\Pi}^\sigma] = -\varepsilon i\hbar R^{-2}L_{\nu\sigma}$ we find that:

$$-\varepsilon i\hbar R^{-2}[\hat{q}^\mu, L^{\nu\sigma}] = [\hat{\Pi}^\sigma, [\hat{q}^\mu, \hat{\Pi}^\nu]] - [\hat{\Pi}^\nu, [\hat{q}^\mu, \hat{\Pi}^\sigma]] \tag{76}$$

Is this true? Both sides certainly demonstrate the same skew-symmetry in $\nu \leftrightarrow \sigma$ exchange. If we simplify and evaluate both sides, we find that:

$$-\varepsilon i\hbar R^{-2}[\hat{q}^\mu, L^{\nu\sigma}] = \varepsilon^3\hbar^2 R^{-2}(\hat{q}^\nu g^{\mu\sigma} - \hat{q}^\sigma g^{\mu\nu}) \tag{77}$$

$$[\hat{q}^\mu, L^{\nu\sigma}] = \varepsilon^2 i\hbar(\hat{q}^\nu g^{\mu\sigma} - \hat{q}^\sigma g^{\mu\nu}) \tag{78}$$

In which $\varepsilon^2 = (\pm 1)^2 = +1$. This is the correct result! Hence, the identity is satisfied for our de Sitter algebra with the following commutation relations:

$$[\hat{q}^\mu, \hat{q}^\nu] = 0 \qquad [\hat{q}^\mu, P^\nu] = i\hbar g^{\mu\nu} \qquad [\hat{\Pi}^\mu, \hat{\Pi}^\nu] = -\varepsilon i\hbar R^{-2}L^{\mu\nu} \tag{79}$$

It seems that with this definition of momentum, the Jacobi limitation (section 4.6) for the case of $J(\hat{q}^\mu, p^\nu, p^\sigma) = 0$ has been overcome in a sense. However, how are we to do the same with position? We cannot set $[\hat{q}, \hat{q}]$ to be non-constant as this would directly contradict the limitation once again.

A possible way of avoiding this would be to define a modified Dirac commutator or Groenewold-Moyal bracket that can satisfy the Jacobi identity with a non-constant $\theta^{\mu\nu}$. However, how this could be defined correctly is beyond the expertise of this thesis. For now, let's consider doing the following:

Re-define canonical position and momentum such that the Jacobi limitation does not apply in any case.

6.2 Q-Position

In transitioning from the Poincaré group to the de Sitter group the momentum operator changes to a noncommutative momentum due to the change in the generators of translations.

$$\hat{p}_\mu \to \hat{\Pi}_\mu \qquad [\hat{p}_\mu, \hat{p}_\nu] = 0 \to [\hat{\Pi}_\mu, \hat{\Pi}_\nu] = -\varepsilon i\hbar R^{-2} L_{\mu\nu} \tag{80}$$

As shown in the previous chapter, this allowed the JI limitation to be avoided.

Following the same reasoning, we can re-define canonical position itself in a similar manner:

$$\hat{q}^\mu \to \hat{Q}^\mu \qquad [\hat{q}^\mu, \hat{q}^\nu] = 0 \to [\hat{Q}^\mu, \hat{Q}^\nu] = \lambda \hat{L}^{\mu\nu} \tag{81}$$

In which λ would maintain dimensional consistency (units of position/momentum).

In this way, given that the generator of de Sitter Translations Π is a function of q, we can avoid modifying its commutation relations by maintaining that $[\hat{q}, \hat{q}] = 0$. If Q can be defined in a consistent and motivated manner such that we avoid the Jacobi limitation and we define Q as the canonical position, we will have established a noncommutative space of Q and Π.

If we follow the same form of $\hat{\Pi}$ for our definition of $\hat{Q}$, we look for something of the form:

$$\hat{Q}^\mu = \hat{q}^\mu + a\hat{F}^\mu \tag{82}$$

In which the non-commutativity is contained in the $a\hat{F}^\mu$ term and $[\hat{q}^\mu, \hat{q}^\nu] = 0$. a must have units of (position/units of F).

If we want the commutator result $[\hat{Q}^\mu, \hat{Q}^\nu] = \lambda \hat{L}^{\mu\nu}$, we must look for an $\hat{F}^\mu$ that satisfies the following commutation relation:

$$[\hat{Q}^\mu, \hat{Q}^\nu] = a([\hat{q}^\mu, \hat{F}^\nu] - [\hat{q}^\nu, \hat{F}^\mu]) + a^2[\hat{F}^\mu, \hat{F}^\nu] = \lambda \hat{L}^{\mu\nu} \tag{83}$$

A simple demand would be to find an $\hat{F}^\mu$ such that:

- $[\hat{q}^\mu, \hat{F}^\nu]$ is symmetric in $\mu \leftrightarrow \nu$ exchange
- $[\hat{F}^\mu, \hat{F}^\nu] = \frac{\lambda}{a^2} \hat{L}^{\mu\nu}$

These conditions would allow the relation to be satisfied.

It just so happens that $\hat{F}^\mu = \hat{\Pi}^\mu$ satisfies these conditions. Let's explicitly show this. If we let $\hat{Q}^\mu = \hat{q}^\mu + a\hat{\Pi}^\mu$, we then find that:

$$[\hat{q}^\mu, \hat{\Pi}^\nu] = (\varepsilon - \frac{\varepsilon^2}{4R^2}\sigma^2)g^{\mu\nu} + \frac{\varepsilon^2}{R^2}\hat{q}^\mu \hat{q}^\nu \tag{84}$$

Which is symmetric in index exchange (given that $[\hat{q}^\mu, \hat{q}^\nu] = 0$). By definition, the second condition is also satisfied $[\hat{\Pi}^\mu, \hat{\Pi}^\nu] = \frac{\lambda}{a^2}\hat{L}^{\mu\nu}$ as long as:

$$\frac{\lambda}{a^2} = -\frac{\varepsilon i\hbar}{R^2} \tag{85}$$

A simple way of satisfying this, given that both constants have the same units, is to let $\lambda = a$ and define:

$$\lambda = -\varepsilon \frac{R^2}{i\hbar} \tag{86}$$

This then results in:

$$\hat{Q}^\mu = \hat{q}^\mu + i\varepsilon \frac{R^2}{\hbar}\hat{\Pi}^\mu \qquad [\hat{Q}^\mu, \hat{Q}^\nu] = i\varepsilon \frac{R^2}{\hbar}\hat{L}^{\mu\nu} \tag{87}$$

We have defined a noncommutative 4-position operaator that gives the non-constant commutator we were searching for. This definition, by construction, should avoid the JI limitation given that we've verified that $\hat{\Pi}$ and $\hat{q}$ satisfy the Jacobi identity (6.1). There is, however, **a major problem in this definition**.

Contrary to the requirements set out in sections 1 and 5.2, it does not reduce to a viable position coordinate in the classical limit ($\hbar \to 0$) nor in the Inönü-Wigner Contraction limit ($R \to \infty$). For both cases:

$$\lim_{\hbar \to 0} \hat{Q}^\mu = \infty \qquad \lim_{R \to \infty} \hat{Q}^\mu = \infty \tag{88}$$

So not only does it not reduce correctly, it diverges to infinity. This is unacceptable. If we want a classical formulation of noncommutative spacetime, we need it to reduce properly in the classical limit.

With no other simple options in the de Sitter algebra to make $[\hat{Q}^\mu, \hat{Q}^\nu]$ equal to, we've seemingly exhausted our search for a re-definition of position in this way. However, there is one more way we could deal with the Jacobi limitation.

6.3 Non-constant Canonical Commutator

One of the conditions for the Jacobi limitation (discussed in section 4.6) to apply is that the canonical commutator $[\hat{q}^\mu, \hat{p}^\nu]$ is a constant. This property arises from the generators of the Poincaré group since $\hat{p}^\nu = -i\hbar\partial^\nu$ is defined as the generator of translations on a Minkowski spacetime. With this definition, we obtain the canonical commutator result $[\hat{q}^\mu, \hat{p}^\nu] = i\hbar\eta^{\mu\nu}$ which is a constant.

If we take another look at the same Jacobi Identity but, this time, of commutators of quantum operators:

$$J(\hat{q}^\mu, \hat{q}^\nu, \hat{p}^\sigma) = 0 \Leftrightarrow [\hat{p}^\sigma, i\hbar\hat{\theta}^{\mu\nu}] = [\hat{q}^\nu, [\hat{q}^\mu, \hat{p}^\sigma]] - [\hat{q}^\mu, [\hat{q}^\nu, \hat{p}^\sigma]] \tag{89}$$

If we evaluate the LHS of the equation using $\hat{p}^\sigma = -i\hbar\partial^\sigma$

$$\hbar^2 \partial^\sigma(\hat{\theta}^{\mu\nu}) = [\hat{q}^\nu, [\hat{q}^\mu, \hat{p}^\sigma]] - [\hat{q}^\mu, [\hat{q}^\nu, \hat{p}^\sigma]] \tag{90}$$

With this, we see that the RHS would not necessarily vanish if $[\hat{q}^\mu, \hat{p}^\sigma]$ is not constant. This would establish a differential equation of $\hat{\theta}^{\mu\nu}$ and possibly allow for non-constant solutions.

However, that is not a result that could exist in a Minkowski spacetime due to its Poincaré symmetry and its generator of translations, assuming that the momentum operator must be defined the same as the generator of translations of the spacetime being used. In a de Sitter spacetime, however, the canonical commutator is not so trivially constant ($[\hat{q}^\mu, \hat{\Pi}^\nu]$ in eq. 84). If one were to write the Jacobi Identity $J(\hat{q}^\mu, \hat{q}^\nu, \hat{\Pi}^\sigma) = 0$ for a noncommutative spacetime $[\hat{q}^\mu, \hat{q}^\nu] = i\hbar\hat{\theta}^{\mu\nu}$, it would simplify to this equation:

$$\hbar^2 \partial^\sigma(\hat{\theta}^{\mu\nu}) = [\hat{q}^\nu, [\hat{q}^\mu, \hat{\Pi}^\sigma]] - [\hat{q}^\mu, [\hat{q}^\nu, \hat{\Pi}^\sigma]] - \frac{1}{4R^2}[\hat{K}^\mu, \hat{\theta}^{\mu\nu}] \tag{91}$$

if we assume that Π^μ maintains the same definition and form for a noncommutative spacetime. We then see that $\hat{\theta}^{\mu\nu}$'s position-dependence depends on the RHS of equation 91. The extra $\frac{1}{4R^2}[\hat{K}^\mu, \hat{\theta}^{\mu\nu}]$ term arises from the second term of the definition of $\hat{\Pi}^\mu$ as defined in equation 74.

From this expression, we can conclude that **a noncommutative de Sitter spacetime may require a non-constant commutator of position operators** $\theta^{\mu\nu}$ through the Jacobi identity. This would be in line with the loss of (Poincaré) translational invariance of the de Sitter spacetime mentioned in section 5, as a position-dependent $\hat{\theta}^{\mu\nu}$ would also violate the same symmetry. However, we cannot be sure of this conclusion until we define the symmetry group of a noncommutative de Sitter spacetime. Why this is the case will be discussed in subsection 7.3.

Furthermore, it is important to keep in mind that $J(\hat{q}^\mu, \hat{q}^\nu, \hat{p}^\sigma) = 0$ continues to demand that $\hat{\theta}$ has no position-dependence, while $J(\hat{q}^\mu, \hat{q}^\nu, \hat{\Pi}^\sigma) = 0$ might demand a non-constant $\hat{\theta}$. If the latter is definitely the case, then both Jacobi identities cannot be simultaneously satisfied. This is not an issue if we accept that in a de Sitter spacetime, since $\hat{p}^\sigma$ no longer serves as a generator of translations, it does not need to be the momentum coordinate in phase space nor the momentum operator and so it does not need to satisfy the Jacobi identity in either context. We can require that $J(\hat{q}^\mu, \hat{q}^\nu, \hat{\Pi}^\sigma) = 0$ must be satisfied in a de Sitter spacetime but not in a Minkowski spacetime and vice versa for $J(\hat{q}^\mu, \hat{q}^\nu, \hat{p}^\sigma) = 0$.

Nevertheless, once we define the symmetry group and algebra of a noncommutative de Sitter spacetime, we can explicitly evaluate the Jacobi identity $J(q^\mu, q^\nu, \Pi^\sigma) = 0$ to find the conditions on the position-dependence of $\theta^{\mu\nu}$. So, let's then discuss what are the symmetries of a noncommutative spacetime.

7 θ Group

As mentioned in section 2.2, the frame-dependence of $\hat{\theta}^{\mu\nu}$ leads to a violation of Lorentz symmetry. In each frame it defines a particular direction in space $\hat{\theta}^{i} \approx \varepsilon_{ijk}\hat{\theta}^{jk}$ [20].

It is common practice to set the time components to vanish $\hat{\theta}^{0i} = 0$ to avoid loss of unitarity (S matrix being unitary) and causality in a noncommutative QFT [5]. However, given that any violation of these properties would be on the Planck scale, there is nothing to say these should hold. Furthermore, these difficulties can be avoided by perturbative approaches [30]. Nevertheless, this condition is important to take into account when defining the form of $\hat{\theta}^{\mu\nu}$.

We can understand in what ways the symmetry is violated by understanding under what proper Lorentz subgroup $\hat{\theta}^{\mu\nu}$ remains invariant, essentially representing the symmetry of $\hat{\theta}^{\mu\nu}$. The result of this will be dependent on its form. We will need to specify a specific form to be able to reach any conclusions.

$\hat{\theta}^{\mu\nu}$'s violation of Lorentz symmetry has no effect on the translational invariance of spacetime as long as the commutator and its uncertainty principle are independent of 4-position. If it were to be position-dependent it would define an origin in spacetime and so lose translational symmetry as mentioned at the end of section 6.3. However, regardless of it being constant, it violates rotational and boost symmetry. So, we might be looking at a subgroup of Poincaré that represents what is called a Very Special Relativity (VSR).

Definition: **Very Special Relativity** is a scheme whose spacetime symmetries consist of translations and a proper Lorentz subgroup. Physically meaning that the translation symmetry of spacetime remains while its boost or rotation symmetries are less than that of the Lorentz group [8].

7.1 θ's Rotational Symmetries

The group of rotations in space SO(3)'s defining vector representation is an irreducible representation. Then, according to Schur's lemmas, a tensor that is invariant under any rotation must be proportional to the identity ($\hat{\theta}^{ij} \propto \mathbb{1}$) [23]. This condition requires that all of the spatial block $\hat{\theta}^{ij}$'s non-diagonal elements be zero. For $\hat{\theta}$ to satisfy this condition, along with its anti-symmetry $\hat{\theta}^{\mu\nu} = -\hat{\theta}^{\nu\mu}$ (which requires diagonal elements to be zero), then all of its spatial components must be zero.

$$\hat{\theta}^{xy} = \hat{\theta}^{xz} = \hat{\theta}^{yz} = ... = \hat{\theta}^{ij} = 0 \tag{92}$$

However, this very condition would mean that a minimal length would no longer be defined. It was this feature of noncommutative spacetime that served as motivation to investigate it (section 1.1) in the first place. So, it seems that if we want to maintain that feature, **we must discard complete rotational symmetry** of a noncommutative spacetime.

Nevertheless, for certain forms of the spatial block $\hat{\theta}$, there could be a subgroup of rotations that leave it invariant. Firstly, recall that a member of SO(3)'s matrix representation R must satisfy the identity:

$$RR^T = \mathbb{1} \tag{93}$$

A spatial rotation $R(\phi,\vec{n})$ of angle ϕ around axis $\vec{n}$ acts on $\hat{\theta}$ in the following way:

$$R(\phi,\vec{n})^{\alpha}_{\ \mu}\hat{\theta}^{\mu\nu}R^T(\phi,\vec{n})^{\beta}_{\ \nu} = \tilde{\hat{\theta}}^{\alpha\beta} \tag{94}$$

$\tilde{\hat{\theta}}$ is the transformed tensor. Let's consider if a certain form of $\hat{\theta}$ can be invariant under any rotation around a specific axis. For convenience, the z-axis. A spatial counterclockwise rotation in spacetime of angle ϕ around the positive z-axis has the following vector representation:

$$R(\phi,\vec{z}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi & 0 \\ 0 & \sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{95}$$

$\vec{z}$ is the z-direction unit vector. If we are looking for a form of $\hat{\theta}$ that is invariant under this rotation, then it must satisfy the equation:

$$R(\phi,\vec{z})^{\alpha}_{\ \mu}\hat{\theta}^{\mu\nu}R^T(\phi,\vec{z})^{\beta}_{\ \nu} = \hat{\theta}^{\alpha\beta} \tag{96}$$

The solution to this equation turns out to be:

$$\hat{\theta} = \begin{pmatrix} 0 & 0 & 0 & \hat{\theta}^{tz} \\ 0 & 0 & \hat{\theta}^{xy} & 0 \\ 0 & -\hat{\theta}^{xy} & 0 & 0 \\ -\hat{\theta}^{tz} & 0 & 0 & 0 \end{pmatrix} \tag{97}$$

So, we have a noncommutative spacetime that is invariant under rotations about a specific direction. Given that this form is not boost-invariant, $\hat{\theta}$'s rotational invariance only holds in a preferred frame.

In the frame in which this form is defined, there is only a single plane in which there is an uncertainty principle arising from the commutator, the xy-plane. In measurements in the xz- and yz-planes there would be no uncertainty principle arising from the noncommutative spacetime. Whether this fails to align with our motivation of defining a minimal length scale, it is hard to say.

With regard to the rest of the Lorentz group, let's consider if there is any boost-invariant form of $\hat{\theta}$ that is non-trivial ($\hat{\theta}^{ij} \neq 0$).

7.2 θ's Boost Symmetries

Given that the Lorentz group is SO(3,1), any member of the group's matrix representation Λ must satisfy the identity:

$$\Lambda\Lambda^T = \mathbb{1} \tag{98}$$

$\Lambda(\vec{\beta})$ acts on a 4-vector as a boost of velocity $c\vec{\beta} = \vec{v}$. If we want to find a boost $\vec{\beta}$ and a form of $\hat{\theta}^{\mu\nu}$ such that $\hat{\theta}$ is left invariant under such a boost, we must solve the following equation:

$$\Lambda(\vec{\beta})^{\alpha}_{\ \mu}\hat{\theta}^{\mu\nu}(\Lambda(\vec{\beta})^{T})_{\beta}^{\ \nu} = \hat{\theta}^{\alpha\beta} \tag{99}$$

With some re-arranging,

$$\Lambda\hat{\theta}\Lambda^{T}\Lambda = \hat{\theta}\Lambda \Leftrightarrow \Lambda\hat{\theta} - \hat{\theta}\Lambda = 0 \tag{100}$$

this is equivalent to:

$$[\Lambda, \hat{\theta}] = 0 \tag{101}$$

So, to find what boosts $\hat{\theta}^{\mu\nu}$ would be invariant under, one could solve equation 99 or the commutator equation 101. Finding the solutions to these equations is beyond the scope of this thesis. However, it is a possible approach recommended to the interested reader, although likely not the easiest.

7.3 Noncommutative de Sitter group

So, we've established that a noncommutative spacetime of this form leads to a loss of Lorentz symmetry. If we were to define a noncommutative de Sitter spacetime, its symmetry group would be smaller than the de Sitter group due to the loss of Lorentz symmetry as a result of the non-commutativity of spacetime.

As explained in section 5, the de Sitter group has 10 generators: the 6 generators of Lorentz transformations $L^{\mu\nu}$ and the 4 generators of de Sitter translations Π^{μ}. It is a fair assumption that the loss of Lorentz symmetry of a noncommutative de Sitter spacetime would lead to fewer generators of boosts and rotations, if any, in its Lie algebra. With this, the algebra of a noncommutative de Sitter group must be different from that of its commutative counterpart.

More importantly, the generator of translations in the algebra of a noncommutative de Sitter group Π_{NC} should have a different self-commutator. This is because, according to the closure condition, a commutator of generators of a Lie algebra must also belong to that algebra. The commutator is a Lie bracket and Lie brackets must satisfy the closure condition. The closure condition requires that any Lie bracket of two elements of an algebra must also be a part of that algebra. We then expect that the self-commutator of the generator of translations in the noncommutative de Sitter algebra must not be the same as the one of the commutative de Sitter algebra.

$$[\Pi^{\mu}_{\text{NC}}, \Pi^{\nu}_{\text{NC}}] \neq -\varepsilon i\hbar R^{-2} L^{\mu\nu} \tag{102}$$

We expect this because many of the group elements generated by the Lorentz generators $L^{\mu\nu}$ would not be a part of the noncommutative de Sitter group and so the generator itself would not be a part of the algebra. Essentially, the closure condition of a Lie bracket and the loss of Lorentz symmetry in a noncommutative de Sitter spacetime would require the self-commutator of Π^{μ}_{NC} to be different from the self-commutator of Π^{μ}. Given this, we cannot expect to use the self-commutator result from the de Sitter algebra in verifying the satisfaction of the Jacobi identity in a noncommutative de Sitter algebra.

So, as we have not managed to define what is the Lie algebra of a noncommutative de Sitter group, this thesis has reached its limit of exploration of this topic. In the next section, the conclusions we've reached are stated. What next steps need to be taken to further our understanding of a noncommutative de Sitter spacetime and its phase space will also be discussed.

8 Conclusion

Defining a self-consistent phase space formulation of a noncommutative de Sitter space is not a trivial task.

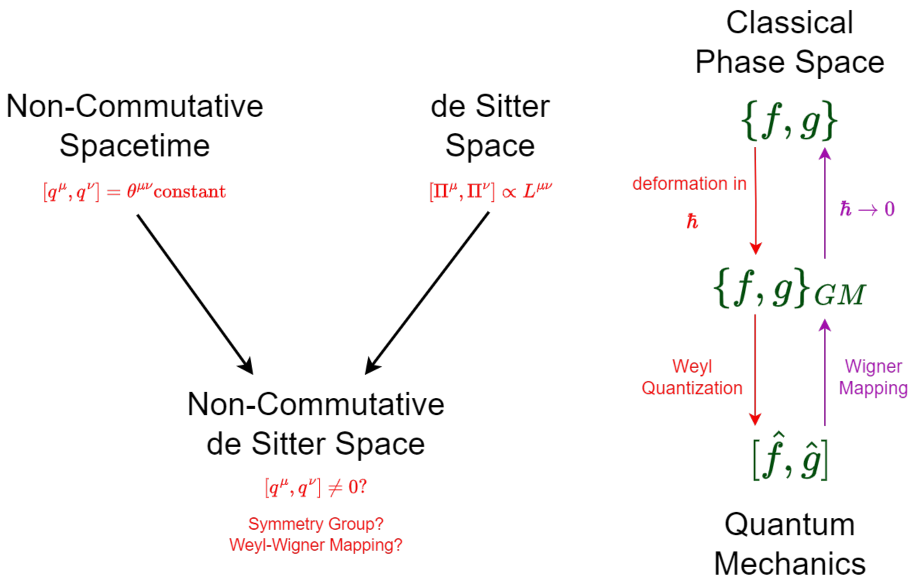

Figure 4: *On the left-hand side is a diagram summarizing the research goal and some unanswered questions. On the right-hand side is a diagram summarizing the bracket mappings between classical and quantum mechanics.*

Let's analyze the progress we made in answering the research questions set out in section 1.2.

Question 1: **Can noncommutative position and noncommutative momentum coordinates be defined consistently in phase space?**

We have seen that it is certainly possible to define a phase space with noncommutative position coordinates (section 3.2) for a constant $\{q^\mu, q^\nu\} = \theta$ and that it can be mapped to quantum mechanics through the Groenewold-Moyal bracket and the Weyl-Wigner mappings (section 4.4).

We have also seen that we can include the noncommutative de Sitter momentum in a quantum phase space (section 6.1) with commutative position. However, we did not verify if this could be done together with noncommutative position coordinates. As discussed in sections 2.2 and 7, this type of noncommutative spacetime violates Lorentz symmetry and so the symmetry group of a non-commutative de Sitter spacetime would be even smaller than that of a commutative de Sitter spacetime. If $\hat{\theta}$ is constant, translational symmetry should remain the same and so the generator of translations Π should also remain the same, but its commutator would be different as argued in section 7.3. So, to evaluate the Jacobi Identity, one might need to first define the Lie algebra of a noncommutative de Sitter spacetime.

This task is beyond the scope of this thesis but is a necessary step in better understanding the properties of a noncommutative de Sitter spacetime.

However, the expectation remains: A phase space formulation on a noncommutative spacetime de Sitter spacetime **must exist**, given the motivations laid out in section 1.1. Nevertheless, its possibility was **not verified** in this thesis.

Question 2: **Can the commutator of 4-position operators be non-constant in phase space?**

This question was quite easily answered for the case of a phase space in Minkowski space (in section 4.6). Given that classical phase space relies on a Poisson algebra and quantum phase space relies on a Lie algebra, in both cases, it is required that the brackets defined satisfy the Jacobi identity. It was found that the Jacobi identity $J(q, p, p) = 0$ directly requires that $\hat{\theta}^{\mu\nu}$ has no position-dependence. There is also no reason to believe it has any momentum-dependence.

So, for the case of the GM bracket and the Dirac commutator bracket, assuming a constant canonical commutator $[\hat{q}, \hat{p}]$, we have reason to believe that $\hat{\theta}^{\mu\nu}$, the self-commutator of the 4-position operator and the structure function of the position-position bracket, **must be constant with respect to position** in a phase space in Minkowski space.

However, it was also found that this need not be the case in a de Sitter spacetime. In such a spacetime, the canonical commutator $[\hat{q}^{\mu}, \hat{\Pi}^{\nu}]$ is not constant. This changes the situation we had in the JI limitation in section 4.6. It turns out that **a noncommutative de Sitter spacetime could require a non-constant commutator of 4-position operators** $\theta^{\hat{\mu}\nu}$ given that the Jacobi identity is satisfied in section 6.3.

We have then, that in a Minkowski spacetime, $\hat{\theta}$ must be constant. However, in a de Sitter spacetime, $\hat{\theta}$ must not be constant given the Jacobi identities discussed in sections 4.6 and 6.3. Furthermore, $\hat{\theta}$'s position dependence for this case must vanish under the Inönü-Wigner contraction limit ($R \to \infty$).

Question 3: **Does noncommutative spacetime violate Lorentz symmetry? If so, what proper subgroup remains?**

We have concluded that a noncommutative spacetime has reduced boost and rotational symmetries and so does not have a full Lorentz symmetry. In section 7, we also found that a noncommutative spacetime that defines a minimum length cannot have complete rotational invariance. So, complete rotational symmetry is disregarded. With regard to the boost symmetries, a possible method for verifying them is laid out but the calculation is not undergone in this thesis. Furthermore, a constant $\hat{\theta}^{\mu\nu}$ does not result in any loss of translational invariance of spacetime.

The main interest of this thesis was to define a phase space in a noncommutative de Sitter spacetime. In section 6.3, we concluded that a noncommutative de Sitter spacetime might require that $\hat{\theta}^{\mu\nu}$ be non-constant if the Jacobi identity $J(\hat{q}^{\mu}, \hat{q}^{\nu}, \hat{\Pi}^{\sigma}) = 0$ is to be satisfied, resulting in a $\hat{\theta}^{\mu\nu}$ that is not translationally invariant. What dependencies $\hat{\theta}^{\mu\nu}$ would be required to have from the Jacobi identity are defined in equation 91. However, they cannot be explicitly calculated without first defining an algebra of the symmetry group of a noncommutative de Sitter spacetime. Finally, this leads us to what further research can be done to continue the exploration undergone in this thesis.

8.1 Further Research

Regardless of the accuracy of the conclusions stated in this thesis, **the next step in better understanding noncommutative de Sitter spacetime is to define its symmetry group and algebra**. Once this is defined, one can attempt to formulate a phase space with its algebra. The Weyl-Wigner mappings would likely no longer apply due to the change in phase space coordinates ($p \to \Pi$) and would need to be modified.

Other curiosities related to this exploration would be:

- What physical quantities are conserved locally and globally in a universe whose spacetime symmetry is that of a noncommutative de Sitter group?
- What are the effects of a noncommutative de Sitter spacetime on the time-evolution of important physical systems?
- What experiments can be designed to test for physical features unique to a QFT in a noncommutative de Sitter spacetime?

We then conclude this thesis and what further research is suggested.

References

[1] R. Aldrovandi, J. P. B. Almeida, and J. G. Pereira, "De sitter special relativity," *Classical and Quantum Gravity*, vol. 24, no. 6, pp. 1385–1404, Mar. 2007, ISSN: 1361-6382. DOI: 10.1088/0264-9381/24/6/002. [Online]. Available: http://dx.doi.org/10.1088/0264-9381/24/6/002.

[2] R. Aldrovandi and J. G. Pereira, *A second poincare' group*, 1998. arXiv: gr-qc/9809061 [gr-qc].

[3] P. Aschieri, C. Blohmann, M. Dimitrijević, F. Meyer, P. Schupp, and J. Wess, "A gravity theory on noncommutative spaces," *Classical and Quantum Gravity*, vol. 22, no. 17, pp. 3511–3532, Aug. 2005, ISSN: 1361-6382. DOI: 10.1088/0264-9381/22/17/011. [Online]. Available: http://dx.doi.org/10.1088/0264-9381/22/17/011.

[4] X. Calmet, "Space-time symmetries of noncommutative spaces," *Physical Review D*, vol. 71, no. 8, Apr. 2005, ISSN: 1550-2368. DOI: 10.1103/physrevd.71.085012. [Online]. Available: https://arxiv.org/abs/hep-th/0411147.

[5] M. Chaichian, A. Demichev, P. Prešnajder, and A. Tureanu, "Space-time noncommutativity, discreteness of time and unitarity," *The European Physical Journal C*, vol. 20, no. 4, pp. 767–772, May 2001, ISSN: 1434-6052. DOI: 10.1007/s100520100664. [Online]. Available: http://dx.doi.org/10.1007/s100520100664.

[6] M. Chaichian, P. Kulish, K. Nishijima, and A. Tureanu, "On a lorentz-invariant interpretation of noncommutative space–time and its implications on noncommutative qft," *Physics Letters B*, vol. 604, no. 1–2, pp. 98–102, Dec. 2004, ISSN: 0370-2693. DOI: 10.1016/j.physletb.2004.10.045. [Online]. Available: http://dx.doi.org/10.1016/j.physletb.2004.10.045.

[7] M. Chaichian, M. M. Sheikh-Jabbari, and A. Tureanu, "Hydrogen atom spectrum and the lamb shift in noncommutative qed," *Physical Review Letters*, vol. 86, no. 13, pp. 2716–2719, Mar. 2001, ISSN: 1079-7114. DOI: 10.1103/physrevlett.86.2716. [Online]. Available: http://dx.doi.org/10.1103/PhysRevLett.86.2716.

[8] A. G. Cohen and S. L. Glashow, "Very special relativity," *Physical Review Letters*, vol. 97, no. 2, Jul. 2006, ISSN: 1079-7114. DOI: 10.1103/physrevlett.97.021601. [Online]. Available: http://dx.doi.org/10.1103/PhysRevLett.97.021601.

[9] A. Connes, *Noncommutative Geometry*. Elsevier Science, 1994, ISBN: 9780121858605. [Online]. Available: https://www.google.nl/books/edition/Noncommutative_Geometry/ZnGUGTa99Q0C?hl=en&gbpv=0.

[10] M. Crainic, R. Fernandes, and I. Mărcuţ, *Lectures on Poisson Geometry*, ser. Graduate Studies in Mathematics. American Mathematical Society, 2021, ISBN: 9781470466671. [Online]. Available: https://bookstore.ams.org/gsm-217/.

[11] M. Douglas and N. Nekrasov, "Noncommutative field theory," *Reviews of Modern Physics*, vol. 73, no. 4, pp. 977–1029, Nov. 2001, ISSN: 1539-0756. DOI: 10.1103/

revmodphys.73.977. [Online]. Available: http://dx.doi.org/10.1103/RevModPhys.73.977.

[12] R. Duff, "Conservation of energy in de sitter space," 2021. [Online]. Available: https://fse.studenttheses.ub.rug.nl/25147/8/bPHYS_2021_DuffR.pdf.

[13] D. M. Eardley and S. B. Giddings, "Classical black hole production in high-energy collisions," *Physical Review D*, vol. 66, no. 4, Aug. 2002, ISSN: 1089-4918. DOI: 10.1103/physrevd.66.044011. [Online]. Available: http://dx.doi.org/10.1103/PhysRevD.66.044011.

[14] A. Einstein, "Kosmologische betrachtungen zur allgemeinen relativitätstheorie," *Annalen der Physik*, Aug. 1917. [Online]. Available: https://echo.mpiwg-berlin.mpg.de/ECHOdocuView?url=/permanent/echo/einstein/sitzungsberichte/S250UZ0K/index.meta&pn=10.

[15] M. Enayati, J.-P. Gazeau, H. Pejhan, and A. Wang, "The de sitter (ds) group and its representations," in Springer International Publishing, 2024, ch. 8, ISBN: 978-3-031-56551-9. DOI: 10.1007/978-3-031-56552-6. [Online]. Available: https://link.springer.com/10.1007/978-3-031-56552-6.

[16] G. Goldhaber and D. B. Cline, "The acceleration of the expansion of the universe: A brief early history of the supernova cosmology project (scp)," in *AIP Conference Proceedings*, AIP, 2009. DOI: 10.1063/1.3232196. [Online]. Available: http://dx.doi.org/10.1063/1.3232196.

[17] D. J. Griffiths, "Introduction to quantum mechanics," in Cambridge University Press, 2017, ch. 3.5.

[18] H. Groenewold, "On the principles of elementary quantum mechanics," *Physica*, vol. 12, no. 7, pp. 405–460, Oct. 1946. [Online]. Available: https://www.rug.nl/research/vsi/events/groenewold/groenewold-article.pdf.

[19] A. Hermann, K. Meyenn, W. Pauli, and V. Weisskopff, *Scientific correspondence with Bohr, Einstein, Heisenberg a. o*, ser. Sources in the History of Mathematics and Physical Sciences Series. Springer New York, 1985, ISBN: 9783540788010. [Online]. Available: https://books.google.nl/books?id=uYwN-D6hgE0C.

[20] I. HINCHLIFFE, N. KERSTING, and Y. L. MA, "Review of the phenomenology of noncommutative geometry," *International Journal of Modern Physics A*, vol. 19, no. 02, pp. 179–204, Jan. 2004, ISSN: 1793-656X. DOI: 10.1142/s0217751x04017094. [Online]. Available: http://dx.doi.org/10.1142/S0217751X04017094.

[21] S. Hossenfelder, "Minimal length scale scenarios for quantum gravity," *Living Reviews in Relativity*, vol. 16, p. 2, 1 Dec. 2013, ISSN: 2367-3613. DOI: 10.12942/lrr-2013-2. [Online]. Available: http://link.springer.com/10.12942/lrr-2013-2.

[22] S. D. Hsu, "Quantum production of black holes," *Physics Letters B*, vol. 555, no. 1–2, pp. 92–98, Feb. 2003, ISSN: 0370-2693. DOI: 10.1016/s0370-2693(03)00012-1. [Online]. Available: http://dx.doi.org/10.1016/S0370-2693(03)00012-1.

[23] H. Jones, "Groups, representations and physics," in CRC Press, 2020, ch. 4.1, ISBN: 9781420050295. [Online]. Available: https://books.google.nl/books?id=OvNp8Df5XukC.

[24] W. A. Kok, "The relation between feynman diagrams and kontsevich graphs in non-commutative star products," Rijksuniversiteit Groningen, 2021. [Online]. Available: https://fse.studenttheses.ub.rug.nl/id/eprint/25793.

[25] S. LIBERATI, T. A. JACOBSON, and D. MATTINGLY, "High energy constraints on lorentz symmetry violations," in *CPT and Lorentz Symmetry*, WORLD SCIENTIFIC, Feb. 2002. DOI: 10.1142/9789812778123_0036. [Online]. Available: http://dx.doi.org/10.1142/9789812778123_0036.

[26] C. A. Mead, "Possible connection between gravitation and fundamental length," *Phys. Rev.*, vol. 135, B849–B862, 3B Aug. 1964. DOI: 10.1103/PhysRev.135.B849. [Online]. Available: https://link.aps.org/doi/10.1103/PhysRev.135.B849.

[27] J. Mejía, "A noncommutative de sitter space," M.S. thesis, Universidade de los Andes, Bogotá, Colombia, 2018. [Online]. Available: https://repositorio.uniandes.edu.co/server/api/core/bitstreams/0862aafb-9d22-4368-9e6a-c0ceea0a4928/content.

[28] S. H. Øyvind Grøn, "Einstein's general theory of relativity: With modern applications in cosmology," in Springer Science, 2007, ch. 8.1, ISBN: 9780387692005. [Online]. Available: https://link.springer.com/book/10.1007/978-0-387-69200-5.

[29] A. G. Riess, A. V. Filippenko, P. Challis, *et al.*, "Observational Evidence from Supernovae for an Accelerating Universe and a Cosmological Constant," vol. 116, no. 3, pp. 1009–1038, Sep. 1998. DOI: 10.1086/300499. arXiv: astro-ph/9805201 [astro-ph].

[30] A. Saha, "Upper bound of the time-space non-commutative parameter from gravitational quantum well experiment," *Journal of Physics: Conference Series*, vol. 484, no. 1, p. 012 071, Mar. 2014. DOI: 10.1088/1742-6596/484/1/012071. [Online]. Available: https://dx.doi.org/10.1088/1742-6596/484/1/012071.

[31] J. Wess and P. Aschieri, *Noncommutative Spacetimes: Symmetries in Noncommutative Geometry and Field Theory*. Springer, 2009, vol. LNP 774. [Online]. Available: https://link.springer.com/book/10.1007/978-3-540-89793-4.

[32] C. M. Will, "The confrontation between general relativity and experiment," *Living Reviews in Relativity*, vol. 17, 1 Dec. 2014. DOI: 10.12942/lrr-2014-4. [Online]. Available: https://doi.org/10.12942/lrr-2014-4.