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Wave propagation in variations of the Lorenz-96 model

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Abstract

The Lorenz 96 (L96) model, developed by Edward Lorenz, has been broadly researched and applied to many areas such as oceanography and climate modelling. In previous studies, travelling and stationary waves were observed in the model, for suitable forcing parameter values, as they represent stable periodic orbits born after supercritical Hopf bifurcations. The system can be generalized by modifying the indices of the advection term in the differential equation function. In this paper, suitable modifications to the original L96 system are studied, in order to derive variations of the travelling and stationary waves. The variations of the model and the L96 model are related by a linear change of coordinates, which is presented in the form of propositions. Additionally, an analysis of eigenvalues of the Jacobian matrix evaluated at the equilibrium solution was preformed for the lowest possible dimension $n = 4$ in order to identify a Hopf bifurcation, which produces a stable periodic orbit seen as a travelling wave. The periodic orbit is approximated in order to examine spatial and temporal properties of the travelling waves in the variations of the model. When analysing stationary waves, the dimension $n = 6$ was taken. A pitchfork bifurcation of the original equilibrium produces two new equilibrium solutions, whose further supercritical Hopf bifurcations give rise to two coexisting stable periodic orbits, which can be interpreted as stationary waves. Spatial and temporal properties of the waves are examined such as wave number and period. All waves are represented using a Hovmöller diagram.

Contents

1 Introduction

The Lorenz-96 (L96) model, developed by Edward Lorenz [1], is one of the most widely used models for studying predictability and weather modelling. Lorenz initially developed the model to study error growth, however due to it's simplicity compared to the other models, it is still heavily used and allows us to explore certain mathematical properties. Given a positive integer n, the L96 model is described by the following differential equations:

$$\frac{d}{dt}x_i = x_{i-1}(x_{i+1} - x_{i-2}) - x_i + F,$$

where $x_i = x_{i+n}$ for all times t and all integers i. The parameter F is usually called the *forcing* parameter. The points $x_0, \ldots, x_{n-1}$ can be seen as n equidistantly distributed points, circulating the globe latitudinally as their indices are integers modulo n. The model also has physical interpretations as we can view the term $x_{i-1}(x_{i+1} - x_{i-2})$ as advection of some atmospheric property (such as temperature or pressure), while the term $-x_i$ represents dissipation or cooling. The parameter F is called the forcing parameter precisely because it can resemble some external forcing applied to the system.

When studying the L96 model, we can examine bifurcations occurring in the system. The type of the bifurcation depends on the parameter F and the dimension of the system n. When the forcing parameter reaches the first positive critical point, the process undergoes a supercritical Hopf or a double-Hopf bifurcation. This bifurcation produces a periodic attractor which represents a traveling wave. For $F < 0$ and an even integer n, the first bifurcation is a pitchfork bifurcation, which gives birth to two new equilibrium solutions, whose further Hopf bifurcations induce stable periodic attractors. These attractors represent stationary wave propagation. Usually, the spatial wave number and the period of the wave are studied as seen in [2].

Since Lorenz introduces the famous L96 model, many variations and adaptations have been studied. One of them is the generalized Lorenz-96 model (see [3] and [4]) and it is defined as follows:

Definition 1. *Let α, β and γ be three integers and n a positive integer. The system:*

$$\frac{d}{dt}x_i(t) = x_{i+\alpha}(x_{i+\beta} - x_{i+\gamma}) - x_i + F,$$

where the indices of x_i are taken modulo n. The system above is usually identified by the symbol $\mathcal{L}_{\alpha,\beta,\gamma}(n)$ and the integers α, β and γ are called system parameters.

This way, L96 model with the dimension n is given by $\mathcal{L}_{-1,1,-2}(n)$. Notice that if $\beta = \gamma$ we get the system

$$\frac{d}{dt}x_i = -x_i + F\ ,$$

which is a simple system with the solution given by:

$$x_i(t) = x_i(0)e^{-t} - e^{-t} + F,$$

where $x_i(0)$ is the initial state of the system. Notice that as $t \to \infty$, we have that $e^{-t} \to 0$, so the solution of the system will converge to the value of the parameter F. For this reason, only the systems $\mathcal{L}_{\alpha,\beta,\gamma}(n)$ with $\beta \neq \gamma$, are considered.

The aim is to examine the spatial and temporal properties of waves occurring in some of the $\mathcal{L}_{\alpha,\beta,\gamma}(n)$ models and explore the relation between them and the original L96 model. These models have been studied in the papers [3] and [4]. In particular, the goal is to show some previous results for the system described by $\mathcal{L}_{\alpha,\beta,\gamma}(n)$ and examine the waves in systems with suitable system parameters α, β and γ. The representation of the waves is done using the Hovmöller diagram. Usually, on the x-axis we plot coordinates, while the y-axis represents time. The points on the diagram are then coloured depending on the value of the data point at the given time.

2 Prerequisites

In the field of dynamical systems, we explore behaviour of systems that evolve with time. In the remainder of this section, we will explore the dynamics of an n-dimensional, non-linear systems that are given by a first order differential equation, with one parameter. Firstly, we introduce the notion of dynamical systems and their flow through a formal definition, which was taken from [5].

Definition 2. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a map and $\frac{dx}{dt} = f(x)$ a differential equation. Then the dynamical system associated with the differential equation is a map $\phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ with the following properties:*

- *$\phi(0, x) = x$, for all $x \in \mathbb{R}^n$*
- *$\phi(t + s, x) = \phi(t, x) \circ \phi(s, x)$, for all $x \in \mathbb{R}^n$*

We refer to the function ϕ as flow and we write $\phi_t(x) = \phi(t, x)$.

Given that the flow is a continuously differentiable function, the dynamical system will be smooth. Moreover, in this case, the solution to the system will be unique for a given initial state.

One of the most important concepts is *an equilibrium* solution of the system, which is a time-invariant solution. Namely, consider any system of differential equations as follows:

$$\frac{dx}{dt} = f(x, \mu),$$

where $x \in \mathbb{R}^n$, μ is a parameter with respect to which we will observe bifurcations occurring in the system and $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ a (differentiable) map. An *equilibrium* solution with respect to the parameter value μ_0 is a vector $x_0 \in \mathbb{R}^n$ if and only if $f(x_0, \mu_0) = 0$.

In $\mathcal{L}_{\alpha,\beta,\gamma}(n)$ systems, the vector $x^0 = (F, F, \ldots, F)$ is always an equilibrium solution as we have $x^0_{i+\alpha}(x^0_{i+\beta} - x^0_{i+\gamma}) - x^0_i + F = 0$, for all $i \in \{0, \ldots, n-1\}$. Depending on the behavior of trajectories with initial conditions near the equilibrium, we can classify the equilibrium as stable or unstable. When it is stable we say that it is of physical significance as trajectories of initial points near the equilibrium stay near the equilibrium. The formal definition below, as well as the others in this section are taken from [5].

Definition 3. *Suppose that $x^0 \in \mathbb{R}^n$ is an equilibrium solution of the system:*

$$\frac{dx}{dt} = f(x).$$

*Then x^0 is called a **stable equilibrium** if for every neighbourhood $\mathcal{O}$ of x^0 there exists a neighbourhood $\mathcal{O}_1$ of x^0 in $\mathcal{O}$ such that for every solution $X(t)$ with $X(0) \in \mathcal{O}_1$ we have that $X(t) \in \mathcal{O}$, for all $t > 0$. The equilibrium is called **unstable** if it is not stable.*

When examining dynamical systems, bifurcations are an indicator of a change in equilibrium solutions. There can be many different bifurcations occurring in the system, but for the purpose of this paper, we are mainly interested in the Hopf and pitchfork bifurcations. We give formal definitions for both below.

Definition 4. *Consider a dynamical system*

$$\frac{dx}{dt} = f(x, \mu) \ ,$$

*where $x \in \mathbb{R}^n$, μ a parameter and $f(x^0, \mu_0) = 0$. The system undergoes a **Hopf bifurcation** at μ_0 if the Jacobian matrix of $f(x, \mu)$ at (x^0, μ_0) has purely imaginary eigenvalues.*

Hopf bifurcation indicates a new periodic solution that arises from the equilibrium. This periodic orbit can be stable or unstable, which is defined in the same way as the stability of equilibrium. When the period orbit is stable, preceded by a stable equilibrium, we say that the Hopf bifurcation is *supercritical.* In the examples below, we illustrate the difference between supercritical and subcritical Hopf bifurcation in the 2-dimensional systems.

Example 1. *Consider the system*

$$\begin{aligned} x' &= ax - y - x(x^2 + y^2) \\ y' &= x + ay - y(x^2 + y^2). \end{aligned}$$

An equilibrium solution is given by $x^0 = y^0 = 0$. By evaluating the Jacobian of the system at the equilibrium, we get the matrix:

$$\begin{pmatrix} a & -1 \\ 1 & a \end{pmatrix},$$

which has two eigenvalues: $\lambda_{0,1} = a \pm i$. This indicates a Hopf bifurcation at $a = 0$. In order to observe the behaviour of this planar system better, we switch to polar coordinates, which gives us the following:

$$r' = ar - r^3$$
$$\theta' = 1.$$

Notice that the equilibrium (x^0, y^0) is the only one because $\theta' \neq 0$. For $a < 0$ we have that r' is negative for all $r > 0$. This means that the origin is a stable equilibrium for $a < 0$. Now if $a > 0$, we have that $r' = 0$ for $r = \sqrt{a}$. This means that the circle in xy-plane, with radius $\sqrt{a}$ for $a > 0$, is a periodic solution to the system with period 2π.

Additionally, notice that for $0 < r < \sqrt{a}$ we have $r' > 0$ and if $r > \sqrt{a}$, then $r' < 0$. We can interpret this as follows: for points in the plane with radius less than $\sqrt{a}$, the radius will increase over time, but if the point has a radius bigger than $\sqrt{a}$, it will decrease over time. This means that all points will tend to the limit cycle given by $x^2 + y^2 = a$. This can be seen in Figure 1.

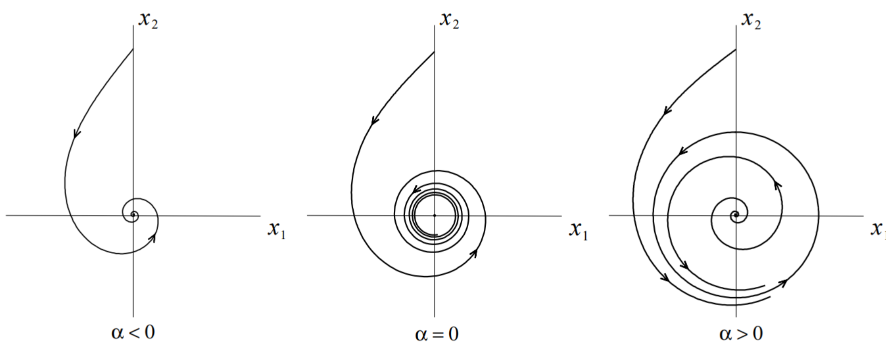

Figure 1: A phase portrait representing a stable limit cycle born at parameter value $a = 0$ from the equilibrium at origin that loses its stability. The picture is taken from [6] and represents a supercritical Hopf bifurcation.

Example 2. *Now, consider the system*

$$x' = ax - y + x(x^2 + y^2)$$
$$y' = x + ay + y(x^2 + y^2).$$

Note that the system is the same as in the previous example, except for the opposite signs in front of the cubic terms. Again, the equilibrium is $x^0 = y^0 = 0$ and the Jacobian matrix evaluated at the equilibrium is given by:

$$\begin{pmatrix} a & -1 \\ 1 & a \end{pmatrix},$$

which has two eigenvalues $\lambda_{0,1} = a \pm i$. The equilibrium undergoes a Hopf bifurcation at $a = 0$ as it's Jacobian has two purely imaginary eigenvalues at this parameter value. We can write the system in the polar form by changing to complex form $z = x + iy$:

$$z' = (a + i)z + z|z|^2,$$

and by taking $z = re^{i\theta}$, we get:

$$\begin{aligned} r' &= ar + r^3 \\ \theta' &= 1. \end{aligned}$$

This implies that the only equilibrium solution is indeed the origin as $\theta' \neq 0$. Similarly as in the previous example, the limit cycle is given by $r = \sqrt{-a}$ for $a < 0$ as $r' = 0$ for this value of r. This gives a circle in the xy-plane with radius $\sqrt{-a}$ and period 2π.

In the case when $a < 0$ we have that $r' < 0$ for $0 < r < \sqrt{-a}$ indicating that the points with the radius less than $\sqrt{-a}$ in the plane will converge to the origin. On the other hand, when $r > \sqrt{-a}$, we have that $r' > 0$, indicating that the points outside of the circle enclosed by the limit cycle diverge further away from the cycle. This can be observed in the Figure 2.

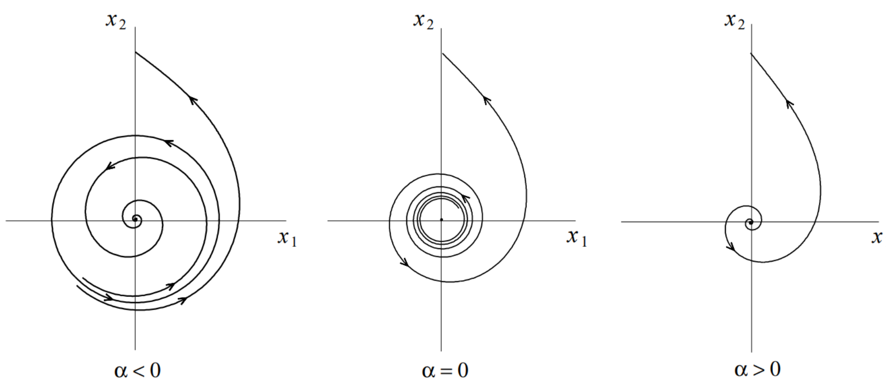

Figure 2: A phase portrait representing an unstable limit cycle disappearing at the parameter value $a = 0$, from where the equilibrium changes its stability from stable to unstable. The picture is taken from [6] and represents the 2-dimensional subcritical Hopf bifurcation.

In the following we define a pitchfork bifurcation.

Definition 5. *Consider the same system with x^0 and μ_0 as in the previous definition. We say that the system goes under* ***pitchfork bifurcation*** *at μ_0 if x^0 is the only equilibrium for $\mu < \mu_0$ ($\mu > \mu_0$), with two more equilibrium arise for $\mu > \mu_0$ ($\mu < \mu_0$).*

Now assume that x^0 is an equilibrium, $f(x^0, \mu_0) = 0$ and the imaginary eigenvalues of the Jacobian evaluated at x^0 are given by $\pm\omega i$. In the paper [7], for some small ϵ with $\epsilon = \sqrt{|\mu - \mu_0|}$, the periodic orbit born at the Hopf bifurcation can be approximated by:

$$x(t) = x^0 + \frac{\epsilon}{\|u+iv\|} Re((u+iv)e^{i\omega t}) + O(\epsilon^2), \tag{1}$$

where $u + iv$ is the eigenvector corresponding to the eigenvalue $\pm\omega i$. As mentioned before, we are mainly interested in supercritical Hopf bifurcations because then, the arising periodic orbit is stable. We say that stable orbits are of physical significance because then, the points in its neighbourhood converge to the orbit.

3 General results for $\mathcal{L}_{\alpha,\beta,\gamma}(n)$ systems

In this section we present some general properties for the family of systems $\{\mathcal{L}_{\alpha,\beta,\gamma}(n) | \alpha, \beta, \gamma \in \mathbb{Z}/n\mathbb{Z}\}$. Every $\mathcal{L}_{\alpha,\beta,\gamma}(n)$ system has an equilibrium solution at $x^0 = (F, F, \ldots, F)$. In order to examine stability at the equilibrium, the Jacobian matrix of the system is observed. The system $\mathcal{L}_{\alpha,\beta,\gamma}(n)$ can be written as:

$$\begin{pmatrix} \dot{x_0} \\ \dot{x_1} \\ \vdots \\ \dot{x}_{n-1} \end{pmatrix} = \begin{pmatrix} f_0(x;\alpha,\beta,\gamma) \\ f_1(x;\alpha,\beta,\gamma) \\ \vdots \\ f_{n-1}(x;\alpha,\beta,\gamma) \end{pmatrix} = f(x;\alpha,\beta,\gamma) \ ,$$

where $x = (x_0, x_1, \ldots, x_{n-1})^\top$, $f_i(x;\alpha,\beta,\gamma) = x_{i+\alpha}(x_{i+\beta} - x_{i+\gamma}) - x_i + F$, for $i = 0, 1, \ldots n-1$ and indices of x_i's are taken modulo n. Note that the Jacobian matrix of the system is given by

$$\mathcal{J}_A(x) = \begin{pmatrix} \frac{\partial f_0}{\partial x_0} & \frac{\partial f_0}{\partial x_1} & \cdots & \frac{\partial f_0}{\partial x_{n-1}} \\ \frac{\partial f_1}{\partial x_0} & \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_{n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_{n-1}}{\partial x_0} & \frac{\partial f_{n-1}}{\partial x_1} & \cdots & \frac{\partial f_{n-1}}{\partial x_{n-1}} \end{pmatrix} .$$

By observing the function $f_0(x;\alpha,\beta,\gamma)$, we can derive the first row of the Jacobian matrix $\mathcal{J}_A(x)$. Indeed, it is given by $c = (c_0, c_1, \ldots, c_{n-1})$, where:

$$c_i = \begin{cases} -1 & \text{if } i = 0, \\ x_\beta - x_\gamma & \text{if } i = \alpha, \\ x_\alpha & \text{if } i = \beta, \\ -x_\alpha & \text{if } i = \gamma, \\ 0 & \text{otherwise.} \end{cases}$$

Since the formula of the function f_1 is obtained by increasing all indices in the formula of f_0 by one, the second row of the Jacobian matrix is given by shifting all of the entries of the first row to the right. Namely, if the first row of the matrix $\mathcal{J}_A(x)$ is $c^0 = (c_0, c_1, \ldots, c_{n-1})$, the second row is given by $c^1 = (c_{n-1}, c_0, c_1, \ldots, c_{n-2})$.

This rule can be applied further, meaning that the entire matrix $\mathcal{J}_A(x)$ is generated by cyclic right shifts of the first row. These types of matrices are called *circulant matrices* and we know how to find their eigenvalues and the corresponding eigenvectors. In the following theorem, eigenvalues and eigenvectors of the Jacobian matrix of the system $\mathcal{L}_{\alpha,\beta,\gamma}(n)$ are given in terms of n, α, β and γ.

Theorem 3.1. *The eigenvalues of the Jacobian matrix of $\mathcal{L}_{\alpha,\beta,\gamma}$ at $x^0 = (F, F, ..., F)$ are given by:*

$$\lambda_j = -1 + F(\eta(j, n; \beta, \gamma) + i\mu(j, n; \beta, \gamma)),\ j = 0, 1, \ldots, n-1,$$

where

$$\begin{aligned}\eta(j, n; \beta, \gamma) &= \cos\left(\frac{2\pi j\beta}{n}\right) - \cos\left(\frac{2\pi j\gamma}{n}\right)\\ \mu(j, n; \beta, \gamma) &= \sin\left(\frac{2\pi j\gamma}{n}\right) - \sin\left(\frac{2\pi j\beta}{n}\right).\end{aligned}$$

Moreover, the eigenvectors v_j corresponding to the eigenvalues λ_j as above, are given by

$$v_j = \frac{1}{\sqrt{n}}(1, \rho_j, \ldots, \rho_j^{n-1})^\top,$$

where $\rho_j = e^{\frac{-2\pi ij}{n}}$ are the n-th roots of unity.

Proof. The Jacobian matrix $\mathcal{J}_A(x^0)$ is a circulant matrix. The article [8] states that if the first row of a circulant matrix is $c = (c_0, c_1, \ldots, c_{n-1})$, the eigenvalues and the corresponding eigenvectors are given by

$$\begin{aligned}\lambda_j &= \sum_{k=0}^{n-1} c_k \rho_j^k\ ,\\ v_j &= \frac{1}{\sqrt{n}}(1, \rho_j, \ldots, \rho_j^{n-1})^\top\ ,\end{aligned}$$

where ρ_j are the n-th roots of unity. Note that the first row of the Jacobian matrix, in this case, is $c^0 = (c_0, c_1, \ldots, c_{n-1})$, where $c_0 = -1$, $c_\alpha = x_\beta - x_\gamma$, $c_\beta = x_\alpha$ and $c_\gamma = -x_\alpha$. Since the Jacobian matrix is evaluated at $x^0 = (F, F, \ldots, F)$, we have that $c_\alpha = 0$, $c_\beta = F$ and $c_\gamma = -F$, with the remaining entries zeros. This means that we can write:

$$
\begin{aligned}
\lambda_j &= \sum_{k=0}^{n-1} c_k \rho_j^k = c_0\rho_j^0 + c_\beta \rho_j^\beta + c_\gamma \rho_j^\gamma \\
&= -1 + Fe^{\frac{-2\pi i j\beta}{n}} - Fe^{\frac{-2\pi i j\gamma}{n}} \\
&= -1 + F\left(\cos\left(\frac{-2\pi j\beta}{n}\right) + i\sin\left(\frac{-2\pi j\beta}{n}\right)\right) - F\left(\cos\left(\frac{-2\pi j\gamma}{n}\right) + i\sin\left(\frac{-2\pi j\gamma}{n}\right)\right) \\
&= -1 + F\left(\cos\left(\frac{2\pi j\beta}{n}\right) - \cos\left(\frac{2\pi j\gamma}{n}\right)\right) + iF\left(\sin\left(\frac{2\pi j\gamma}{n}\right) - \sin\left(\frac{2\pi j\beta}{n}\right)\right),
\end{aligned}
$$

which completes the proof.

□

Notice that for $F = 0$ we have that $\lambda_j = -1$ for all $j \in \{0, \ldots, n-1\}$, which means that the equilibrium solution x^0 is stable at $F = 0$.

4 Travelling waves in $\mathcal{L}_{\alpha,\beta,\gamma}(n)$ systems for $n = 4$

In the Lorenz 96 model, travelling waves can be observed near a supercritical Hopf bifurcation. In this sub-section, we will dive into the case where the dimension is $n = 4$ and explore in which systems in the family $\{\mathcal{L}_{\alpha,\beta,\gamma}(4) | \alpha, \beta, \gamma \in \mathbb{Z}/4\mathbb{Z}\}$ do the travelling waves occur. To do that, we will identify the systems in which the Hopf bifurcation gives birth to a periodic solution from a stable equilibrium x^0, as such occurrence indicates a strong possibility of existence of a stable periodic orbit.

In the following proposition, different sets of eigenvalues are given for the dimension $n = 4$, depending on the choices for β and γ, since eigenvalues and eigenvectors do not depend on the choice of α. Note that in the table below, all possible cases are discussed for the dimension $n = 4$.

Proposition 4.1. *Let $\mathcal{L}_{\alpha,\beta,\gamma}(4)$ be a system and $x^0 = (F, F, F, F)$ the equilibrium solution. Then the eigenvalues λ_j (for $j = 0, 1, 2, 3$) of the Jacobian at x^0, depending on the choice of β and γ, are given in the table below.*

(β,γ)	λ_0	λ_1	λ_2	λ_3
$(0,1)$	-1	$-1+F+iF$	$-1+2F$	$-1+F-iF$
$(0,2)$	-1	$-1+2F$	-1	$-1+2F$
$(0,3)$	-1	$-1+F-iF$	$-1+2F$	$-1+F+iF$
$(1,0)$	-1	$-1-F-iF$	$-1-2F$	$-1-F+iF$
$(1,2)$	-1	$-1+F-iF$	$-1-2F$	$-1+F+iF$
$(1,3)$	-1	$-1-2iF$	-1	$-1+2iF$
$(2,0)$	-1	$-1-2F$	-1	$-1-2F$
$(2,1)$	-1	$-1-F+iF$	$-1+2F$	$-1-F-iF$
$(2,3)$	-1	$-1-F-iF$	$-1+2F$	$-1-F+iF$
$(3,0)$	-1	$-1-F+iF$	$-1-2F$	$-1-F-iF$
$(3,1)$	-1	$-1+2iF$	-1	$-1-2iF$
$(3,2)$	-1	$-1+F+iF$	$-1-2F$	$-1+F-iF$

Proof. Taking $n = 4$ and using the expression for eigenvalues of the Jacobian of the $\mathcal{L}_{\alpha,\beta,\gamma}(4)$ system, evaluated at $x^0 = (F, F, F, F)$, we get:

$$\begin{aligned}\lambda_j &= -1 + F\left(\cos\left(\frac{2\pi j\beta}{4}\right) - \cos\left(\frac{2\pi j\gamma}{4}\right)\right) + iF\left(\sin\left(\frac{2\pi j\gamma}{4}\right) - \sin\left(\frac{2\pi j\beta}{4}\right)\right)\\ &= -1 + 2F\sin\left(\frac{\pi j(\beta+\gamma)}{4}\right)\sin\left(\frac{\pi j(\gamma-\beta)}{4}\right) + 2iF\cos\left(\frac{\pi j(\beta+\gamma)}{4}\right)\sin\left(\frac{\pi j(\gamma-\beta)}{4}\right).\end{aligned}$$

From here we can conclude that $\lambda_0 = -1$ for any choice of β and γ, since $\sin(0) = 0$. Additionally, if $\beta - \gamma$ is an even number, we have that $\lambda_2 = -1$, as $\sin(\pi j) = 0$ for any integer j. This means that for $(\alpha, \beta) = (0, 2), (2, 0), (1, 3)$ or $(3, 1)$, $\lambda_2 = -1$.

The rest of the values in the table are found by a simple substitution of sine and cosine function values into the expressions. We show the computations of λ_1,λ_2 and λ_3 for $(\beta, \gamma) = (0, 1)$ system parameters:

$$\lambda_j = -1 + 2F\sin^2(\frac{\pi j}{4}) + iF\sin\left(\frac{\pi j}{2}\right),$$

which allows us to find:

$$\begin{aligned}\lambda_1 &= -1 + 2F(\frac{1}{\sqrt{2}})^2 + iF\cdot 1 = -1 + F + iF\\ \lambda_2 &= -1 + 2F\cdot 1 + iF\cdot 0 = -1 + 2F\\ \lambda_3 &= -1 + 2F(-\frac{1}{\sqrt{2}})^2 + iF\cdot(-1) = -1 + F - iF\end{aligned}$$

$\square$

The theorem above allows us to observe when does a Hopf bifurcations occur for $n = 4$. The stability of equilibrium solution is indicated by the sign of real parts of eigenvalues. If all of the values are negative, the equilibrium is stable, otherwise it is unstable.

Notice that the Hopf bifurcation occurs in all 4-dimensional systems, except when $(\beta, \gamma) = (0, 2)$, $(1, 3)$, $(2, 0)$ or $(3, 1)$. This is because when $(\beta, \gamma) = (0, 2)$ or $(2, 0)$, all eigenvalues are real numbers, and in the cases when $(\beta, \gamma) = (1, 3)$ or $(3, 1)$, all eigenvalues have a constant real part equal to -1, so they can never cross the imaginary axis.

For the systems $(\beta, \gamma) = (0, 1)$ and $(0, 3)$, the Hopf bifurcation occurs at $F = 1$ since for this parameter value we have $Re(\lambda_1) = Re(\lambda_3) = 0$ and $Im(\lambda_{1,3}) = \pm i$. However, in both cases, x^0 is unstable at $F = 1$, as the equilibrium x^0 loses stability at $F = \frac{1}{2}$. This indicates a possible supercritical pitchfork bifurcation at $F = \frac{1}{2}$, preceding a Hopf bifurcation, which would mean that a supercritical Hopf bifurcation doesn't occur.

When $(\beta, \gamma) = (1, 0)$ or $(3, 0)$, the Hopf bifurcation occurs at the parameter value $F = -1$. It can easily be verified that $Re(\lambda_1) = Re(\lambda_3) = 0$ in both cases, while the imaginary parts are non-zero. However, same as for $(\beta, \gamma) = (0, 1)$ and $(0, 3)$, a possible pitchfork bifurcation at $F = -\frac{1}{2}$ implies that the equilibrium loses stability at this value. Moreover, as x^0 is stable for $-\frac{1}{2} < F < 0$, the pitchfork bifurcation would be supercritical, meaning that the Hopf bifurcation is unlikely to occur.

This leaves only four pairs of (β, γ) values for which the system could have a supercritical Hopf bifurcation at the equilibrium solution $x^0 = (F, F, F, F)$. These are: $(\beta, \gamma) = (1, 2)$, $(2, 1)$, $(2, 3)$ and $(3, 2)$. For $(\beta, \gamma) = (1, 2)$ and $(3, 2)$, the system goes under Hopf bifurcation at $F = 1$, while for $(\beta, \gamma) = (2, 1)$ and $(2, 3)$ the Hopf bifurcation occurs at $F = -1$. The table below gives an overview for the four pairs of parameters (β, γ) for which the Hopf bifurcation occurs, their eigenvalues corresponding to the Jacobian of the system at the equilibrium and value of the parameter for which the Hopf bifurcation occurs.

(β, γ)	λ_0	λ_1	λ_2	λ_3	Hopf bifurcation at:
$(1, 2)$	-1	$-1 + F - iF$	$-1 - 2F$	$-1 + F + iF$	$F = 1$
$(2, 1)$	-1	$-1 - F + iF$	$-1 + 2F$	$-1 - F - iF$	$F = -1$
$(2, 3)$	-1	$-1 - F - iF$	$-1 + 2F$	$-1 - F + iF$	$F = -1$
$(3, 2)$	-1	$-1 + F + iF$	$-1 - 2F$	$-1 + F - iF$	$F = 1$

In order to formally prove that the Hopf bifurcation is supercritical in all four cases, non-degeneracy conditions (given in [6]) would need to be satisfied. This would require significant amount of computations and work, which is why it was not proven in this paper.

For all systems, the first eigenvalue crosses the imaginary axis with $\lambda_1 = \pm i$, which means that $\omega = \pm 1$. We find the corresponding eigenvector to λ_1 as described in the theorem:

$$v_1 = \frac{1}{\sqrt{n}}(1, \rho_1, \rho_1^2, \rho_1^3)^\top = \frac{1}{\sqrt{4}}(1, -i, -1, i)^\top.$$

Notice that if $v_1 = (x_0, x_1, x_2, x_3)$, then we can write $x_j = \frac{1}{\sqrt{4}} e^{-\frac{\pi i j}{2}}$. This means that we can express the periodic orbit as:

$$\begin{aligned} x_j(t) &= x_j^0 + \frac{1}{\sqrt{4}} \epsilon \cdot Re\left[e^{-\frac{\pi i j}{2} + i\omega t} \right] + \mathcal{O}(\epsilon^2) \\ &= x_j^0 + \frac{1}{\sqrt{4}} \epsilon \cdot Re\left[e^{i(\omega t - \frac{\pi j}{2})} \right] + \mathcal{O}(\epsilon^2) \\ &= F + \frac{1}{\sqrt{4}} \epsilon \cos\left(\frac{\pi j}{2} - \omega t \right) + \mathcal{O}(\epsilon^2), \end{aligned}$$

where $x^0 = (F, F, F, F)^\top$ is the equilibrium solution of the system and $\epsilon = |F - F_0|$ (F_0 is the parameter value for which the supercritical Hopf bifurcation occurs). From here we find that the period of the travelling wave described by the equation above is $T = \frac{2\pi}{\omega} = 2\pi$ and the wave number is 1. Period of the wave is determined by the $-\omega t$ term in the cosine function and the wave number is indicated by $\frac{\pi j}{2}$ in the same function. As j takes $0, 1, 2$ and 3 as values, $\frac{\pi j}{2}$ ranges from 0 to 2π, which is exactly one length of a period of the cosine function, meaning that the wave number is 1.

This can be seen in the diagrams below as there is one wave in the x-axis direction (confirming that the wave number is one). Period of the wave can be observed in the y-axis direction. It is important to see that in the first two diagrams wee can observe westward propagation of the waves, while in the other two, eastward propagation appears.

Note that we chose a specific α for each pair (β, γ) in order to plot the Hovmöller diagram. It can be shown that for each (non-ordered) pair, we can take precisely two value for α in order to ensure appearance of the waves. This is due to the *boundedness of orbits*, as for certain parameters (α, β, γ), the total energy of the system is not bounded. As we are interested in systems with bounded orbits, some restrictions on choices for the system parameters need to be introduced. In [3], it was shown that if $A_n = 0$, where:

$$A_n = \max_{x \in \mathbb{S}^n} \sum_{i=0}^{n-1} x_i x_{i+\alpha} (x_{i+\beta} - x_{i+\gamma}),$$

then the orbits for these system parameters are bounded. It can be shown that for all four choices of (α, β, γ) that were taken, we have $A_n = 0$.

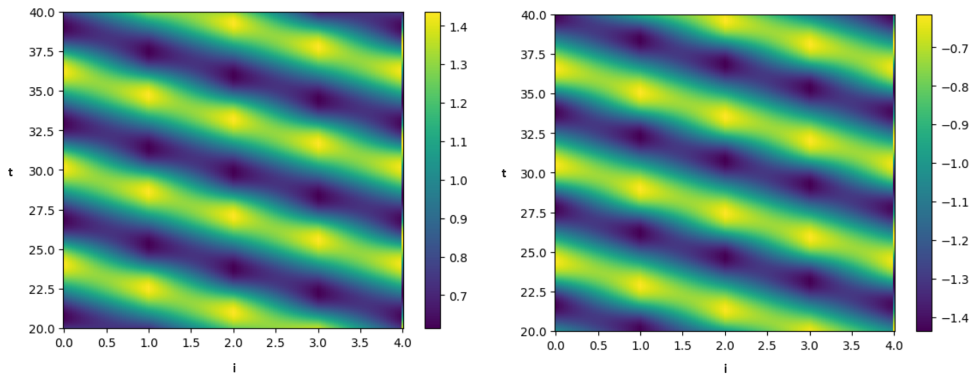

Figure 3: A Hovmöller diagram where the y-axis represents time and $x_j(t)$ is represented by a colour. Linear interpolation was done in order to obtain the values between $x_j(t)$ and $x_{j+1}(t)$ for visualization purposes. The picture on the left represents the L96 model and parameter value $F = 1.1$ was taken, while the diagram on the right corresponds to the system $\mathcal{L}_{-1,-2,1}(4)$ and parameter value $F = -1.1$. The plot was done using python and the code can be found in SectionA.

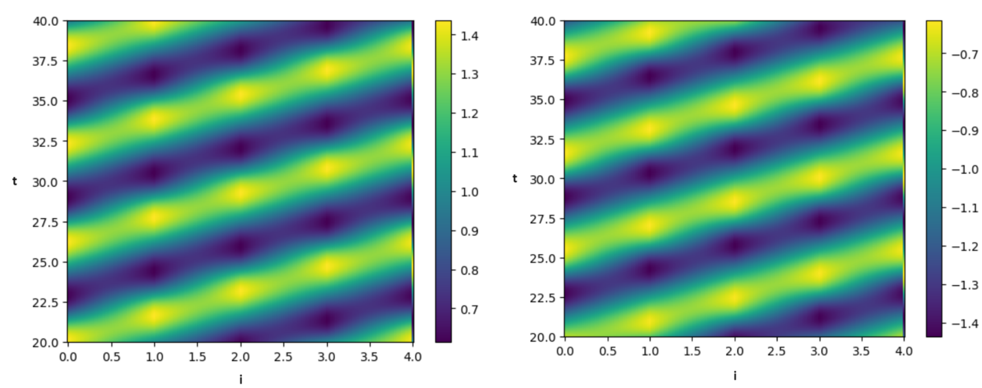

Figure 4: The diagram on the left represents the system $\mathcal{L}_{1,-1,2}(4)$ with parameter value $F = 1.1$, while the diagram on the right corresponds to the system $\mathcal{L}_{1,2,-1}(4)$ with $F = -1.1$.

5 Systems $\mathcal{L}_{-1,-2,1}(n)$, $\mathcal{L}_{1,-1,2}(n)$ and $\mathcal{L}_{1,2,-1}(n)$

In this section, we will use the following notation: $f(x,F) = f(x;-1,1,-2)$, $g(x,F) = f(x;1,-1,2)$ and $h(x,F) = f(x;-1,-2,1)$ where F is omitted if such specification is not needed in the given context. We show that the systems given by $\mathcal{L}_{-1,-2,1}(n)$, $\mathcal{L}_{1,-1,2}(n)$ and $\mathcal{L}_{1,2,-1}(n)$ are related to the original L96 model by simple transformations.

Proposition 5.1. *Let the solution to the system $\mathcal{L}_{-1,1,-2}(n)$ with the initial condition $x_{in} = (x^0_{in}, \ldots, x^{n-1}_{in})$ be given by $\phi_t(x_{in})$. Then the solution to the system $\mathcal{L}_{1,-1,2}(n)$, with the initial condition $y_{in} = (y^0_{in}, \ldots, y^{n-1}_{in})$, is given by $\phi^1_t(y_{in}) = T\phi_t(Ty_{in})$, where T is a linear transformation, whose matrix representation is given below.*

$$T = \begin{pmatrix} 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \end{pmatrix}$$

Proof. Let $y = Tx$ be a linear transformation of vector x as described in the proposition. We show that $g_i(x) = f_{n-i-1}(y)$ using the fact that $x_i = y_{n-i-1}$ and that indices of x_i's and y_i's are taken modulo n:

$$\begin{aligned} g_i(x) &= x_{i+1}(x_{i-1} - x_{i+2}) - x_i + F \\ &= y_{n-i-2}(y_{n-i} - y_{n-i-3}) - y_{n-i-1} + F \\ &= y_{(n-i-1)-1}(y_{(n-i-1)+1} - y_{(n-i-1)-2}) - y_{n-i-1} + F \\ &= f_{n-i-1}(y) \\ &= f_{n-i-1}(Tx). \end{aligned}$$

This implies that:

$$g(x) = Tf(Tx) \ ,$$

for any vector x. Note that $T^2 = I$, so we can also write the above as $f(x) = Tg(Tx)$. From here we can conclude that the system $\mathcal{L}_{1,-1,2}(n)$ is the same as $\mathcal{L}_{-1,1,-2}(n)$ under the linear transformation T. This means that if the solution to the L96 model with the initial condition $x_{in} = (x_{in,0}, \ldots, x_{in,n-1})^\top$ is given by $x = (x_0, \ldots, x_{n-1})^\top$, then the solution to the system $\mathcal{L}_{1,-1,2}(n)$ with the initial condition $y_{in} = (x_{in,n-1}, \ldots, x_{in,0})^\top$ is given by $y = (x_{n-1}, \ldots, x_0)^\top = Tx$.

□

Since $\mathcal{L}_{1,-1,2}(n)$ system is equivalent to the L96 model and is related to it by a linear transformation T, we expect the waves observed in the L96 model to appear in this system as well, but as a mirrored image. This explains eastward wave propagation seen in Figure 4 that arises when looking at travelling waves in the dimension $n = 4$. Moreover, all bifurcations observed in the L96 system will also be present in the $\mathcal{L}_{1,-1,2}(n)$ system, at the same parameter values. In [9], bifurcations in L96 model for a positive forcing parameter F are explored in more detail. Additionally, as a consequence of the proposition, we also expect stationary waves in the system $\mathcal{L}_{1,-1,2}(n)$ to behave in the same way as observed in [2] while studying the L96 model.

In the same way as in the previous proposition, the dynamical systems $\mathcal{L}_{-1,-2,1}(n)$ and $\mathcal{L}_{1,2,-1}(n)$ are related via the same linear transformation T.

Proposition 5.2. *Let the solution to the system $\mathcal{L}_{-1,-2,1}(n)$ with the initial condition $x_{in} = (x_{in}^0, \ldots, x_{in}^{n-1})$ be given by $\phi_t(x_{in})$. Then the solution to the system $\mathcal{L}_{1,2,-1}(n)$, with the initial condition $y_{in} = (y_{in}^0, \ldots, y_{in}^{n-1})$, is given by $\phi_t^1(y_{in}) = T\phi_t(Ty_{in})$, where T is the same as in Proposition 5.1.*

We can show that $f(x; 1, 2, -1) = Tf(Tx; -1, -2, 1)$, meaning that the solution to the system $\mathcal{L}_{1,2,-1}(n)$ can be obtained by a linear transformation of the solution to the $\mathcal{L}_{-1,-2,1}(n)$ system and vice versa. We omit the proof as it is identical to the one in the previous proposition. As a consequence, the same wave characteristics can be observed in $\mathcal{L}_{-1,-2,1}(n)$ and $\mathcal{L}_{1,2,-1}(n)$ models.

Now, we show the relation between $\mathcal{L}_{-1,-2,1}(n)$ and the original L96 model in the proposition below.

Proposition 5.3. *Let the solution to the system $\mathcal{L}_{-1,1,-2}(n)$ with the initial condition $x_{in} = (x_{in}^0, \ldots, x_{in}^{n-1})$ be given by $\phi_t(x_{in}; F)$, where F is the forcing parameter. Then the solution to the system $\mathcal{L}_{-1,-2,1}(n)$, with the initial condition $y_{in} = (y_{in}^0, \ldots, y_{in}^{n-1})$, is given by $\phi_t^1(y_{in}; F) = -\phi_t(-y_{in}; -F)$.*

Proof. we can show that $h(x, F) = -f(-x, -F)$ for all $x \in \mathbb{R}^n$ and $F \in \mathbb{R}$:

$$\begin{aligned} -f_i(-x, -F) &= -\left(-x_{i-1}(x_{i-2} - x_{i+1}) + x_i - F\right) \\ &= x_{i-1}(x_{i-2} - x_{i+1}) - x_i + F \\ &= h_i(x, F), \end{aligned}$$

for all $i = 0, \ldots, n-1$. This means that the dynamical system given by the $\mathcal{L}_{-1,-2,1}(n)$ model can be obtained by a simple transformation of the dynamical system of the L96 model. Concretely, we can now write the following:

$$\phi_t^1(x_{in}; F) = -\phi_t(-x_{in}; -F) \ ,$$

where $\phi_t(x; F)$ is the flow of the L96 model with forcing parameter F and the initial condition x. Similarly, $\phi_t^1(x)$ is the flow of the $\mathcal{L}_{-1,-2,1}(n)$ model. This completes the proof. $\square$

As a consequence, the waves observed in the L96 model at parameter value F will also appear in the $\mathcal{L}_{-1,-2,1}(n)$ system at $-F$, but with the values of the solution vectors of opposite signs. Notice that by using the Proposition 5.3 and 5.2, we can represent the solution to the $\mathcal{L}_{1,2,-1}(n)$ system as a combination of the two transformations of the solution to the L96 model. We show this in a corollary.

Corollary 5.3.1. *Let the solution to the system $\mathcal{L}_{-1,1,-2}(n)$ with the initial condition $x_{in} = (x_{in}^0, \ldots, x_{in}^{n-1})$ be given by $\phi_t(x_{in}; F)$, where F is the forcing parameter. Then the solution to the system $\mathcal{L}_{1,2,-1}(n)$, with the initial condition $y_{in} = (y_{in}^0, \ldots, y_{in}^{n-1})$, is given by $\phi_t^1(y_{in}; F) = -T\phi_t(-Ty_{in}; -F)$, where T is the same as in Proposition 5.1.*

Proof. According to Proposition 5.2, we can write:

$$\phi_t^1(y_{in}; F) = T\phi_t^2(Ty_{in}; F) \ ,$$

where $\phi_t^2(x; F)$ is the flow of the system $\mathcal{L}_{-1,-2,1}(n)$ with initial condition x. Now, by using Proposition 5.3, we get:

$$\begin{aligned}\phi_t^1(y_{in}; F) &= T\phi_t^2(Ty_{in}; F) \\ &= -T\phi_t(-Ty_{in}; -F) \ ,\end{aligned}$$

so the proof is complete.

□

6 Stationary waves in systems $\mathcal{L}_{-1,-2,1}(6)$,$\mathcal{L}_{1,-1,2}(6)$ and $\mathcal{L}_{1,2,-1}(6)$

In this section the dimension $n = 6$ is taken and stationary waves in $\mathcal{L}_{-1,-2,1}(6)$,$\mathcal{L}_{1,-1,2}(6)$ and $\mathcal{L}_{1,2,-1}(6)$ systems will be examined. In the paper [2] which explored $\mathcal{L}_{-1,1,-2}(6)$ system, a pitchfork bifurcation at $F = -0.5$ leads to two new equilibrium solutions, whose further Hopf bifurcation induces two co-existing stable periodic orbits, which can be interpreted as stationary waves. As it was shown in the previous sections, the systems that we want to discuss in this section are closely related to the original L96, so we expect similar results when exploring Hopf bifurcation of the non-trivial equilibrium solutions.
First, let us consider the eigenvalues of the Jacobian matrix of each system, evaluated at the equilibrium solution $x^0 = (F, F, F, F, F, F)$. The values of all 6 eigenvalues for each system are listed in the table below.

$\mathcal{L}_{\alpha,\beta,\gamma}(6)$	λ_0	λ_1	λ_2	λ_3	λ_4	λ_5
$\mathcal{L}_{-1,-2,1}(6)$	-1	$-1 - F + iF\sqrt{3}$	-1	$-1 + 2F$	-1	$-1 - F - iF\sqrt{3}$
$\mathcal{L}_{1,-1,2}(6)$	-1	$-1 + F + iF\sqrt{3}$	-1	$-1 - 2F$	-1	$-1 + F - iF\sqrt{3}$
$\mathcal{L}_{1,2,-1}(6)$	-1	$-1 - F - iF\sqrt{3}$	-1	$-1 + 2F$	-1	$-1 - F + iF\sqrt{3}$

By observing the table, notice that the eigenvalue λ_3 changes sign at $F = \frac{1}{2}$ for the systems $\mathcal{L}_{-1,-2,1}(6)$ and $\mathcal{L}_{1,2,-1}(6)$, and at $F = -\frac{1}{2}$ for the system $\mathcal{L}_{1,-1,2}(6)$. This indicates a bifurcation occurring at these parameter values. Since the equilibrium x^0 exists for all values of $F \in \mathbb{R}$, a saddle-node bifurcation can be ruled out, meaning that a possible pitchfork bifurcation occurs, leading to two new equilibrium solutions. In the following, each system will be discussed separately in order two examine new equilibrium solutions arising and their further Hopf bifurcations.

6.1 Stationary waves in $\mathcal{L}_{-1,-2,1}(6)$

Consider the equilibrium solution given by $x^1 = (a, b, a, b, a, b)$, with $a, b \in \mathbb{R}$. Then it has to satisfy $f_j(x^1; -1, -2, 1) = 0$ for all $j = 0, \dots, 5$. This yields two equations:

$$\begin{aligned} b(a-b) - a + F &= 0 \\ a(b-a) - b + F &= 0. \end{aligned}$$

By solving the system above, we get:

$$a = \frac{1+\sqrt{2F-1}}{2} \text{ and } b = \frac{1-\sqrt{2F-1}}{2}.$$

Note that $x^2 = (b, a, b, a, b, a)$ is also an equilibrium solution since $f_j(x^2; -1, -2, 1) = 0$ for all $j = 0, \dots, 5$. Since a and b are real number when $F \geq \frac{1}{2}$, there must be a pitchfork bifurcation occurring at $F = \frac{1}{2}$, leading to two new equilibrium solutions that exist for all $F \geq \frac{1}{2}$. The Jacobian matrix of the system $\mathcal{L}_{-1,-2,1}(6)$ evaluated at x^1 is given by:

$$\mathcal{J}_A(x^1) = \begin{pmatrix} -1 & -b & 0 & 0 & b & a-b \\ b-a & -1 & -a & 0 & 0 & a \\ b & a-b & -1 & -b & 0 & 0 \\ 0 & a & b-a & -1 & -a & 0 \\ 0 & 0 & b & a-b & -1 & -b \\ -a & 0 & 0 & a & b-a & -1 \end{pmatrix},$$

which is not a circulant matrix. By doing some numerical experimentation, we find the characteristic polynomial of this system at $F = 3.5$ is:

$$p^1(\lambda) = (\lambda^2 + 3)(\lambda^2 + \lambda + 12)(\lambda^2 + 5\lambda + 13),$$

indicating an eigenvalue crossing of the imaginary axis at this parameter value, as $\lambda = \pm i\sqrt{3}$ are roots of the characteristic polynomial. Notice that all of the other eigenvalues have a negative real part, indicating that the equilibrium x^1 is stable for $F \in (\frac{1}{2}, \frac{7}{2})$, but looses it's stability at $F = \frac{7}{2}$. In order to find the eigenvector v_0 corresponding to the eigenvalue $\lambda_0 = i\sqrt{3}$, we solve:

$$\begin{pmatrix} -1-i\sqrt{3} & -b & 0 & 0 & b & a-b \\ b-a & -1-i\sqrt{3} & -a & 0 & 0 & a \\ b & a-b & -1-i\sqrt{3} & -b & 0 & 0 \\ 0 & a & b-a & -1-i\sqrt{3} & -a & 0 \\ 0 & 0 & b & a-b & -1-i\sqrt{3} & -b \\ -a & 0 & 0 & a & b-a & -1-i\sqrt{3} \end{pmatrix} \begin{pmatrix} v_{0,0} \\ v_{0,1} \\ v_{0,2} \\ v_{0,3} \\ v_{0,4} \\ v_{0,5} \end{pmatrix} = 0.$$

Using Mathematica [10], we find:

$$\begin{pmatrix} v_{0,0} \\ v_{0,1} \\ v_{0,2} \\ v_{0,3} \\ v_{0,4} \\ v_{0,5} \end{pmatrix} = \begin{pmatrix} \frac{9102+273i\sqrt{2}+246i\sqrt{3}+3367\sqrt{6}}{2(-6i+\sqrt{2})(12i+\sqrt{3})(22+18i\sqrt{2}+11i\sqrt{3}+12\sqrt{6})}x \\ -\frac{4374-1209i\sqrt{2}-438i\sqrt{3}+2239\sqrt{6}}{2(-6i+\sqrt{2})(12i+\sqrt{3})(22+18i\sqrt{2}+11i\sqrt{3}+12\sqrt{6})}x \\ -\frac{5(600-837i\sqrt{2}-540i\sqrt{3}+310\sqrt{6})}{(-6i+\sqrt{2})(12i+\sqrt{3})(-1+\sqrt{6})(22+18i\sqrt{2}+11i\sqrt{3}+12\sqrt{6})}x \\ \frac{(-13-18i\sqrt{2}-31i\sqrt{3}+2\sqrt{6})(78+45i\sqrt{2}+18i\sqrt{3}+65\sqrt{6})}{4(-6i+\sqrt{2})(12i+\sqrt{3})(22+18i\sqrt{2}+11i\sqrt{3}+12\sqrt{6})}x \\ -\frac{78+45i\sqrt{2}+18i\sqrt{3}+65\sqrt{6}}{2(-6i+\sqrt{2})(12i+\sqrt{3})}x \\ x \end{pmatrix},$$

for any $x \in \mathbb{C}$. By dividing the entries of the eigenvector, we can conclude the following:

$$\frac{v_{0,2}}{v_{0,0}} = -\frac{1}{2} + i\frac{\sqrt{3}}{2} = e^{\frac{2\pi i}{3}} \ , \ \frac{v_{0,4}}{v_{0,0}} = -\frac{1}{2} - i\frac{\sqrt{3}}{2} = e^{-\frac{2\pi i}{3}}$$
$$\frac{v_{0,3}}{v_{0,1}} = -\frac{1}{2} + i\frac{\sqrt{3}}{2} = e^{\frac{2\pi i}{3}} \ , \ \frac{v_{0,5}}{v_{0,1}} = -\frac{1}{2} - i\frac{\sqrt{3}}{2} = e^{-\frac{2\pi i}{3}}.$$

This allows us to write the eigenvector v_0 as:

$$\begin{pmatrix} v_{0,0} \\ v_{0,1} \\ v_{0,2} \\ v_{0,3} \\ v_{0,4} \\ v_{0,5} \end{pmatrix} = \begin{pmatrix} v_{0,0} \\ v_{0,1} \\ v_{0,0}e^{\frac{2\pi i}{3}} \\ v_{0,1}e^{\frac{2\pi i}{3}} \\ v_{0,0}e^{-\frac{2\pi i}{3}} \\ v_{0,1}e^{-\frac{2\pi i}{3}} \end{pmatrix}.$$

Using the approximation of the periodic orbit given by 1, we can write:

$$x_0(t) = \frac{1+\sqrt{6}}{2} + \frac{\epsilon}{\|v_0\|} Re\left(v_{0,0} \cdot e^{i\sqrt{3}t}\right)$$
$$x_1(t) = \frac{1-\sqrt{6}}{2} + \frac{\epsilon}{\|v_0\|} Re\left(v_{0,1} \cdot e^{i\sqrt{3}t}\right)$$
$$x_2(t) = \frac{1+\sqrt{6}}{2} + \frac{\epsilon}{\|v_0\|} Re\left(v_{0,0} \cdot e^{\frac{2\pi i}{3}+i\sqrt{3}t}\right)$$
$$x_3(t) = \frac{1-\sqrt{6}}{2} + \frac{\epsilon}{\|v_0\|} Re\left(v_{0,1} \cdot e^{\frac{2\pi i}{3}+i\sqrt{3}t}\right)$$
$$x_4(t) = \frac{1+\sqrt{6}}{2} + \frac{\epsilon}{\|v_0\|} Re\left(v_{0,0} \cdot e^{-\frac{2\pi i}{3}+i\sqrt{3}t}\right)$$
$$x_5(t) = \frac{1-\sqrt{6}}{2} + \frac{\epsilon}{\|v_0\|} Re\left(v_{0,1} \cdot e^{-\frac{2\pi i}{3}+i\sqrt{3}t}\right).$$

From here we can observe that $x_i(t)$ is always positive for $i \in \{0, 2, 4\}$ and $x_i(t)$ is always negative for $i \in \{1, 3, 5\}$. This indicates the appearance of stationary waves, which can be seen in Figure 5. The period of the wave is $T = \frac{2\pi}{\sqrt{3}}$ as it can be derived from the periodic orbit approximation. The period is given by the period of the cosine function(which is 2π) divided by the the term in front of t(which is $\sqrt{3}$). We find that the wave number is 3, which is determined by the fact that 3 data points are always positive, while the other 3 always negative. This can also be seen on the plot.

In the same way, we observe the other equilibrium solution x^2 that arises at $F = \frac{1}{2}$. Namely, the Jacobian matrix at x^2 is the same as the Jacobian at x^1, but with a and b interchanged, so it's expression is omitted. Similarly, the characteristic polynomial at $F = 3.5$ is $p^2(\lambda) = (\lambda^2 + 3)(\lambda^2 + \lambda + 12)(\lambda^2 + 5\lambda + 13)$, indicating that the eigenvalues given by $\lambda_{0,1} = \pm i\sqrt{3}$ cross the imaginary axis at this parameter value. Consequently, the Hopf bifurcation of x^2 equilibrium at $F_0 = 3.5$ gives birth to a periodic orbit for $F > F_0$. Notice that again, x^2 loses it's stability at F_0. In order to derive an approximation of this orbit, the eigenvector of the Jacobian at x^2 is needed. Again, using Mathematica, we derive an eigenvector v_0 corresponding to λ_0 as :

$$\begin{pmatrix} v_{0,0} \\ v_{0,1} \\ v_{0,2} \\ v_{0,3} \\ v_{0,4} \\ v_{0,5} \end{pmatrix} = \begin{pmatrix} \frac{9102-273i\sqrt{2}+246i\sqrt{3}-3367\sqrt{6}}{2(6i+\sqrt{2})(12i+\sqrt{3})(-22+18i\sqrt{2}-11i\sqrt{3}+12\sqrt{6})}x \\ -\frac{4374+1209i\sqrt{2}-438i\sqrt{3}-2239\sqrt{6}}{2(6i+\sqrt{2})(12i+\sqrt{3})(-22+18i\sqrt{2}-11i\sqrt{3}+12\sqrt{6})}x \\ -\frac{5(-600-837i\sqrt{2}+540i\sqrt{3}+310\sqrt{6})}{(6i+\sqrt{2})(12i+\sqrt{3})(1+\sqrt{6})(-22+18i\sqrt{2}-11i\sqrt{3}+12\sqrt{6})}x \\ \frac{(13-18i\sqrt{2}+31i\sqrt{3}+2\sqrt{6})(-78+45i\sqrt{2}-18i\sqrt{3}+65\sqrt{6})}{4(6i+\sqrt{2})(12i+\sqrt{3})(-22+18i\sqrt{2}-11i\sqrt{3}+12\sqrt{6})}x \\ -\frac{-78+45i\sqrt{2}-18i\sqrt{3}+65\sqrt{6}}{2(6i+\sqrt{2})(12i+\sqrt{3})}x \\ x \end{pmatrix},$$

where x is any complex number. Notice that the this eigenvector is very similar to the one we obtained for the x^1 equilibrium solution case. Moreover, by dividing the entries of the eigenvector, we can conclude that v_0 is of the form:

$$v_0 = \begin{pmatrix} v_{0,0} \\ v_{0,1} \\ v_{0,0}e^{\frac{2\pi i}{3}} \\ v_{0,1}e^{\frac{2\pi i}{3}} \\ v_{0,0}e^{-\frac{2\pi i}{3}} \\ v_{0,1}e^{-\frac{2\pi i}{3}} \end{pmatrix}.$$

This gives us the same expression for the periodic orbit as the one obtained from x^1 equilibrium, but the values $1+\sqrt{6}$ and $1-\sqrt{6}$ interchange places. As such, this periodic orbit $x_i(t)$ is always positive for $j \in \{1, 3, 5\}$ and negative for $j \in \{0, 2, 4\}$, which is the opposite from the stationary wave obtained from x^1. These two stable periodic orbits arising from the two equilibrium solutions x^1 and x^2 represent stationary waves, which co-exist and depending on the initial state of the system, the solution will converge to one or the other orbit. This can be seen in Figure 5 where two different initial states were taken, leading to the convergence to two different stationary waves.

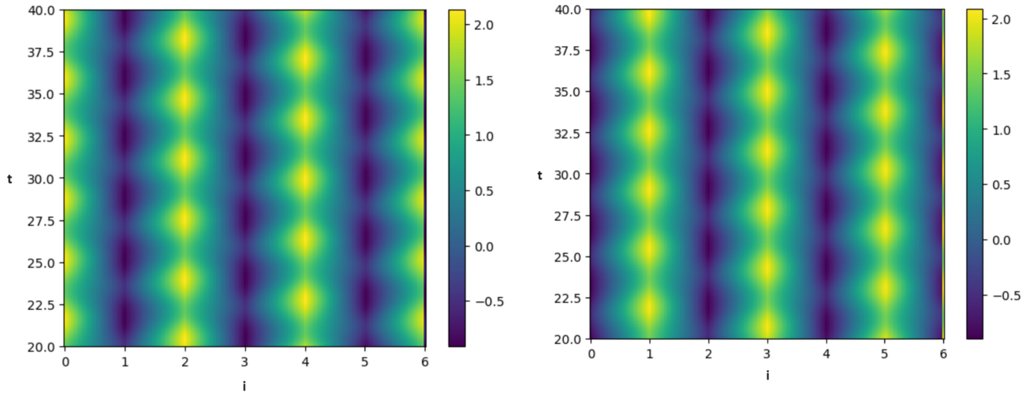

Figure 5: Hovmöller diagram which represents stationary waves in $\mathcal{L}_{-1,-2,1}(6)$ model for the parameter value $F = 3.6$. The picture on the left corresponds to the periodic orbit born from x^1 equilibrium solution, while the picture on the right represents the periodic orbit born at x^2.

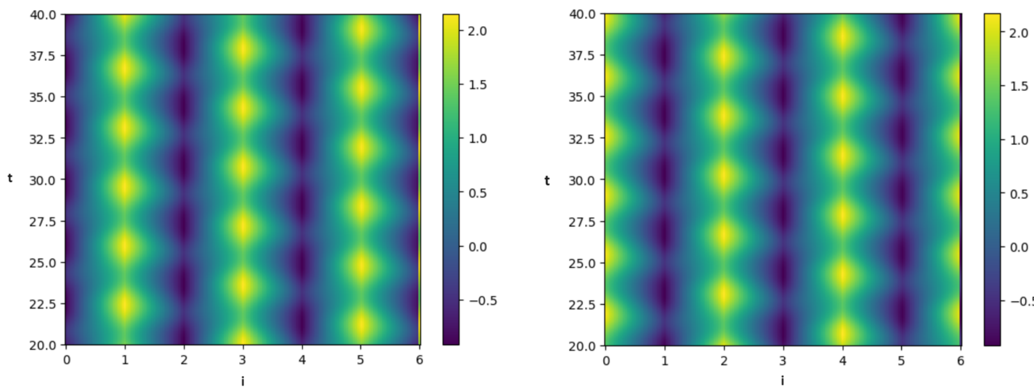

Figure 6: Hovmöller plot representing stationary waves in $\mathcal{L}_{1,2,-1}(6)$ model at parameter value $F = 3.6$.

6.2 Stationary waves in $\mathcal{L}_{1,2,-1}(6)$

As it was shown in section 5, the solution to the system $\mathcal{L}_{1,2,-1}(n)$ can be obtained by "flipping" the solution vector to the $\mathcal{L}_{-1,-2,1}(n)$. This means that the stable periodic orbits found in the system $\mathcal{L}_{-1,-2,1}(6)$ will also appear in the $\mathcal{L}_{-1,-2,1}(6)$ system, but as a mirrored image. This can be better observed for travelling waves as stationary waves are fairly symmetrical, differing only by a fraction of the period. The Hovmöller plot of stationary waves in this system are represented in Figure 6.

The period and the wave number will be the same as observed in the previous subsection. It is also visible from the plot that the wave number is 3. The period is given by $T = \frac{2\pi}{\sqrt{3}}$.

6.3 Stationary waves in $\mathcal{L}_{1,-1,2}(6)$

Again, consider the equilibrium solution of the system given by $x^1 = (a, b, a, b, a, b)$, for some real numbers a and b. It has to be that $f_j(x^1; 1, -1, 2) = 0$ for all $j \in \{0, 1, \ldots, 5\}$. This gives us two equations:

$$b(b-a) - a + F = 0$$
$$a(a-b) - b + F = 0.$$

By solving the system of equations above, we get the values:

$$a = \frac{-1 + \sqrt{-1-2F}}{2} \text{ and } b = \frac{-1 - \sqrt{-1-2F}}{2}.$$

Notice that if x^1 is an equilibrium, then so is $x^2 = (b, a, b, a, b, a)$. For $F = -\frac{1}{2}$, $a = b = -\frac{1}{2}$, so the two new equilibrium points are "born" at parameter value $F = -\frac{1}{2}$, implying that the pitchfork bifurcation occurs at this parameter value. The Jacobian matrix of system evaluated at x^1 is given by:

$$\mathcal{J}_A(x^1) = \begin{pmatrix} -1 & b-a & -b & 0 & 0 & b \\ a & -1 & a-b & -a & 0 & 0 \\ 0 & b & -1 & b-a & -b & 0 \\ 0 & 0 & a & -1 & a-b & -a \\ -b & 0 & 0 & b & -1 & b-a \\ a-b & -a & 0 & 0 & a & -1 \end{pmatrix},$$

which is not a circulant matrix. We can observe that the equilibrium x^1 now undergoes a Hopf bifurcation at $F = -\frac{7}{2}$, since for this parameter value we have $a = \frac{-1+\sqrt{6}}{2}$ and $b = \frac{-1-\sqrt{6}}{2}$, so using a characteristic polynomial calculator, we find:

$$p^1(\lambda) = (\lambda^2 + 3)(\lambda^2 + \lambda + 12)(\lambda^2 + 5\lambda + 13)$$

as a characteristic polynomial of the Jacobian $\mathcal{J}_A(x^1)$ at $F = -3.5$. From here we see that the eigenvalues which cross the imaginary axis at $F = -3.5$ are given by $\lambda_{0,1} = \pm i\sqrt{3}$, while the other four eigenvalues have a negative real part, which implies stability of equlibrium x^1. Furthermore, by finding a solution to:

$$\begin{pmatrix} -1-i\sqrt{3} & b-a & -b & 0 & 0 & b \\ a & -1-i\sqrt{3} & a-b & -a & 0 & 0 \\ 0 & b & -1-i\sqrt{3} & b-a & -b & 0 \\ 0 & 0 & a & -1-i\sqrt{3} & a-b & -a \\ -b & 0 & 0 & b & -1-i\sqrt{3} & b-a \\ a-b & -a & 0 & 0 & a & -1-i\sqrt{3} \end{pmatrix} \begin{pmatrix} v_{0,0} \\ v_{0,1} \\ v_{0,2} \\ v_{0,3} \\ v_{0,4} \\ v_{0,5} \end{pmatrix} = 0,$$

we get the eigenvector $v_0 = (v_{0,0}, \ldots, v_{0,5})^\top$ that corresponds to the eigenvalue $\lambda_0 = i\sqrt{3}$. Using Mathematica software [10], we find that:

$$
\begin{pmatrix} v_{0,0} \\ v_{0,1} \\ v_{0,2} \\ v_{0,3} \\ v_{0,4} \\ v_{0,5} \end{pmatrix} = \begin{pmatrix} \frac{2(-1221+1080i\sqrt{2}-33i\sqrt{3}+286\sqrt{6})}{(6i+\sqrt{2})(12i+\sqrt{3})(-20-21i\sqrt{2}-24i\sqrt{3}+7\sqrt{6})}x \\ \frac{2(696-249i\sqrt{2}-1038i\sqrt{3}+359\sqrt{6})}{(6i+\sqrt{2})(12i+\sqrt{3})(-20-21i\sqrt{2}-24i\sqrt{3}+7\sqrt{6})}x \\ \frac{2(1821+2850i\sqrt{2}-2565i\sqrt{3}+164\sqrt{6})}{(6i+\sqrt{2})(12i+\sqrt{3})(\sqrt{6}-1)(-20-21i\sqrt{2}-24i\sqrt{3}+7\sqrt{6})}x \\ -\frac{2(1905+414i\sqrt{2}-171i\sqrt{3}+304\sqrt{6})}{(6i+\sqrt{2})(12i+\sqrt{3})(-20-21i\sqrt{2}-24i\sqrt{3}+7\sqrt{6})}x \\ -\frac{66+75i\sqrt{2}-30i\sqrt{3}-55\sqrt{6}}{2(6i+\sqrt{2})(12i+\sqrt{3})}x \\ x \end{pmatrix},
$$

for any $x \in \mathbb{C}$. From here we find that:

$$
\frac{v_{0,2}}{v_{0,0}} = -\frac{1}{2} - i\frac{\sqrt{3}}{2} = e^{-\frac{2\pi i}{3}}, \; \frac{v_{0,4}}{v_{0,0}} = -\frac{1}{2} + i\frac{\sqrt{3}}{2} = e^{\frac{2\pi i}{3}}, \text{ and}
$$
$$
\frac{v_{0,3}}{v_{0,1}} = -\frac{1}{2} - i\frac{\sqrt{3}}{2} = e^{-\frac{2\pi i}{3}}, \; \frac{v_{0,5}}{v_{0,1}} = -\frac{1}{2} + i\frac{\sqrt{3}}{2} = e^{\frac{2\pi i}{3}}.
$$

This means that the eigenvector is of the form:

$$
\begin{pmatrix} v_{0,0} \\ v_{0,1} \\ v_{0,2} \\ v_{0,3} \\ v_{0,4} \\ v_{0,5} \end{pmatrix} = \begin{pmatrix} v_{0,0} \\ v_{0,1} \\ v_{0,0}e^{-\frac{2\pi i}{3}} \\ v_{0,1}e^{-\frac{2\pi i}{3}} \\ v_{0,0}e^{\frac{2\pi i}{3}} \\ v_{0,1}e^{\frac{2\pi i}{3}} \end{pmatrix}.
$$

Now, according to 1, the periodic orbit born at the Hopf bifurcation can be written as follows:

$$
\begin{aligned}
x_0(t) &= \frac{-1+\sqrt{6}}{2} + \frac{\epsilon}{\|v_0\|} Re\left(v_{0,0} \cdot e^{i\sqrt{3}t}\right) \\
x_1(t) &= \frac{-1-\sqrt{6}}{2} + \frac{\epsilon}{\|v_0\|} Re\left(v_{0,1} \cdot e^{i\sqrt{3}t}\right) \\
x_2(t) &= \frac{-1+\sqrt{6}}{2} + \frac{\epsilon}{\|v_0\|} Re\left(v_{0,0} \cdot e^{-\frac{2\pi i}{3}+i\sqrt{3}t}\right) \\
x_3(t) &= \frac{-1-\sqrt{6}}{2} + \frac{\epsilon}{\|v_0\|} Re\left(v_{0,1} \cdot e^{-\frac{2\pi i}{3}+i\sqrt{3}t}\right) \\
x_4(t) &= \frac{-1+\sqrt{6}}{2} + \frac{\epsilon}{\|v_0\|} Re\left(v_{0,0} \cdot e^{\frac{2\pi i}{3}+i\sqrt{3}t}\right) \\
x_5(t) &= \frac{-1-\sqrt{6}}{2} + \frac{\epsilon}{\|v_0\|} Re\left(v_{0,1} \cdot e^{\frac{2\pi i}{3}+i\sqrt{3}t}\right).
\end{aligned}
$$

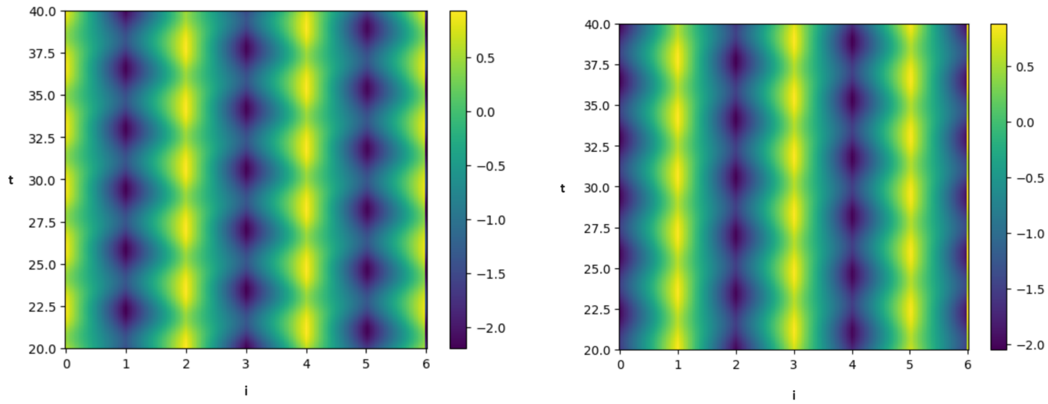

Figure 7: Hovmöller diagram which represents stationary waves in $\mathcal{L}_{1,-1,2}(6)$ model for the parameter value $F = -3.6$. Again, the left picture represents periodic orbit at x^1, while the other represents the periodic orbit born after Hopf-bifurcation of x^2.

From here we can conclude that the periodic orbit will take the form of stationary waves as $x_j(t)$ is always positive for $j \in \{0, 2, 4\}$ and always negative for $j \in \{1, 3, 5\}$(for a sufficiently small ϵ). Notice that the period of the waves is given by $T = \frac{2\pi}{\sqrt{3}}$ and the spatial wave number is 3.

Same as for the system $\mathcal{L}_{-1,-2,1}(6)$, we now consider the other new equilibrium solution given by $x^2 = (b, a, b, a, b, a)$. Since the computations are repetitive, we will omit that part. The approximated periodic orbit born at the equilibrium x^2 is the same as for the equilibrium x^1, but with $-1 + \sqrt{6}$ and $-1 - \sqrt{6}$ interchanged. Again, we find two stable periodic orbits, which represent stationary waves that co-exist. For this reason, depending on the initial state of the system, the solution will converge to one of the two stable orbits. This can be seen in Figure 7.

7 Conclusion

The variations of the L96 model described by $\mathcal{L}_{\alpha,\beta,\gamma}(n)$ can have widely different dynamics for different choices of system parameters α, β and γ. In this paper, the lowest possible dimension 4 was chosen, in order to explore all possible Hopf bifurcations of the $x^0 = (F, \ldots, F)$ equilibrium in $\mathcal{L}_{\alpha,\beta,\gamma}(4)$ systems. By doing so, we concluded that there are only for pairs of values for (β, γ) for which the system definitely undergoes a supercritical Hopf bifurcation. These are $(\beta, \gamma) = (1, 2)$, $(2, 1)$, $(2, 3)$ and $(3, 2)$.

For suitable values for α, we found travelling waves with very similar properties to the ones found in the L96 model. To be precise, we found travelling in systems $\mathcal{L}_{-1,-2,1}(4)$, $\mathcal{L}_{1,-1,2}(4)$ and $\mathcal{L}_{1,2,-1}(4)$, which can be observed in the Hovmöller diagram. The period of all waves we found is 2π and the wave number is 1. This was obtained by approximating the period orbit born at the Hopf bifurcation. The waves in systems $\mathcal{L}_{1,-1,2}(4)$ and $\mathcal{L}_{1,2,-1}(4)$ exhibit eastward propagation, while in the original L96 and $\mathcal{L}_{-1,-2,1}(4)$ the waves propagate in the "west" direction.

When observing the systems $\mathcal{L}_{-1,-2,1}(n)$, $\mathcal{L}_{1,-1,2}(n)$ and $\mathcal{L}_{1,2,-1}(n)$, we found that they can be transformed into the original L96 model by simple linear transformations. Namely, the solution to the $\mathcal{L}_{-1,-2,1}(n; F)$ system can be obtained as a negative solution to the negative initial condition and negative parameter value of the L96 solution. The solution to the system $\mathcal{L}_{1,-1,2}(n)$ can be obtained by "flipping" the solution vector of the flipped initial condition of L96 model. Similarly, the solution to the system $\mathcal{L}_{1,2,-1}(n)$ can be obtained by applying both transformations described above to the solution of the L96 system.

As stationary waves were found in the L96 model, we discussed stationary waves in $\mathcal{L}_{-1,-2,1}(6)$, $\mathcal{L}_{1,-1,2}(6)$ and $\mathcal{L}_{1,2,-1}(6)$. The same properties of stationary waves were expected, as the Section 5 shows clear relation between them and the famous L96 system. Indeed, by following the two equilibria after the pitchfork bifurcation, we found two coexisting stable periodic orbits. As expected, the wave number in this scenario is 3, and the period is given by $T = \frac{2\pi}{\sqrt{3}}$. The two stable periodic orbits were represented by a Hovmöller diagram for all three variations of the L96 model.

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A Python code

```
import numpy as np
from scipy.integrate import odeint
import matplotlib.pyplot as plt
import math

N = 6 #dimension of the system
F = -3.6 #focring parameter
x0 = [1,-0.7,0.1,0.5,0.1,-1] #initial state
t = np.linspace(20.0, 40.0, 2000) #times


def GL96(x,t):
  alpha = -1 #system parameter alpha
  beta = 1 #system parameter beta
  gamma = -2 #system parameter gamma
  d = np.zeros(N)
  for i in range(N):
    d[i] = (x[(i + beta) % N] - x[(i + gamma) % N]) * x[(i + alpha) % N] - x[i] + F
  return d

def matrix(x_values, t, x, N): #preforms linear interpolation in order to get values in
  time = len(t)
  num_points = len(x_values)
  step = N/num_points
  M = np.zeros([time,num_points])
  for i in range(time):
    for j in range(num_points):
      if x_values[j] == math.trunc(x_values[j]):
        M[i][j] = x[i][int(j*step)]
      else:
        M[i][j] = x[i][math.trunc(j*step)]+(j*step-math.trunc(j*step))*
        (x[i][(math.trunc(j*step)+1)%N]-x[i][math.trunc(j*step)])
  return M

x_values = np.linspace(0,N,240)
x = odeint(GL96, x0, t)
x_new = odeint(GL96,x[1999],t)
Z = matrix(x_values, t, x_new, N)
plt.pcolor(x_values, t, Z)
plt.colorbar()
plt.show()
```