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Revealing unique scattering amplitudes from hidden zeros in Tr(ϕ^3) and NLSM Theories

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Abstract

This study investigates unique scattering amplitudes arising from hidden zeros of colored-ordered scalars in $\mathrm{Tr}(\phi^3)$ and Non-Linear Sigma Model (NLSM) theories. Exploiting the combinatorial and geometric properties of the scalar's theoretical amplitudes, the general prediction is made and further simplified using the hidden zeros. Detailed analysis was done of various n-point interactions, including 4-point, 5-point, 6-point, and 7-point interactions within $\mathrm{Tr}(\phi^3)$ theory and 6-point and 8-point interactions within NLSM, to validate the simplification and significance of these hidden zeros. The results reveal unique and simplified amplitude equations, demonstrating the effectiveness of hidden zeros and locality constraints in yielding symmetrical and minimal sets of terms. These insights contribute to the broader understanding of scattering amplitudes and suggest potential applications in more complex field theories.

1 Introduction

In the realm of particle physics, scattering amplitudes are of vital significance, as they are crucial for understanding the behavior of quanta at the most fundamental levels of nature. These processes have widespread applications in both theoretical and experimental studies. In collider physics, scattering amplitudes are essential for high-precision calculations that reduce theoretical uncertainties and match the experimental precision needed to explore new physics at facilities like the Large Hadron Collider (LHC). In gravitational wave physics, scattering amplitudes, combined with effective field theory methods, improve the modeling of gravitational wave sources, such as binary black hole mergers, providing accurate predictions for observatories like LIGO. Additionally, their importance extends into theoretical physics concepts, such as gravity as a Double Copy of Gauge Theory, String Scattering Amplitudes, the S-Matrix function space, and much more.[1]

Nevertheless, scattering amplitudes are still being studied to simplify theoretical frameworks and understand interactions. Recently, in the area of quantum chromodynamics, for colored scalar fields, Arkani-Hamed and collaborators published a groundbreaking paper uncovering special amplitude properties[2]. Starting from the simplest $\text{Tr}(\phi^3)$ theory, they discovered a hidden pattern of zeros in partial amplitudes in special loci defined by the momentum of the interactions [2]. This theory can be then extended to more complex models as the Non-Linear Sigma Model with pions, or non-supersymmetric gluons in any dimension.

Given the extensive nature of the field, only a few theories, will be examined for different interactions. Starting with the simplest interactions in $\text{Tr}(\phi^3)$ theory where three fields interact at once. Additionally, the more complex Non-Linear Sigma Model (NLSM) will be explored, which describes the dynamics of a pion scalar field constrained to lie on a curved target manifold. By applying the discovery made by Arkani-Hamed, the study aims to identify a unique partial amplitude that results from imposing the assumed hidden zeros on a general amplitude. This approach validates the authenticity and usefulness of these hidden zeros, demonstrating their role in simplifying and accurately predicting scattering processes.

The paper is structured to first provide a theoretical foundation, detailing the concept of scattering amplitudes by initially studying normal scalars and then exploring how the imposition of color-order theory changes the description. Once color-ordered scalars are comprehended, this understanding is applied to theories such as $\text{Tr}(\phi^3)$ and NLSM, which are described in detail for various types of interactions. In these theories, hidden zeros can be revealed, as stated by Arkani-Hamed. Building on this theoretical groundwork, the methods for identifying unique scattering amplitudes are then applied using a Mathematica code.

Finally, the presentation of results highlights the unique and simplified amplitude equations derived from the imposed conditions of hidden zeros and locality constraints. These findings demonstrate the effectiveness of this approach in yielding symmetrical and minimal sets of terms in the scattering amplitudes. Finally, the discussion suggests avenues for further research, including the exploration of additional constraints and the application of this methodology to more complex field theories.

2 Theory

2.1 Scattering amplitudes

In particle interactions, the probability of scattering between particles is described by their scattering amplitude. This amplitude summarizes all potential interaction processes between the incoming and outgoing particles and so can be described by the following formula of an amplitude between initial i and final state f:

$$\mathcal{A} = \langle f|S|i\rangle = \lim_{t_\pm \to \pm\infty} \langle f|U(t_+, t_-)|i\rangle \tag{1}$$

where S is the unitary operator known as the S-matrix relating the initial and final state. It is assumed that both states are eigenstates of the free theory [3]. $U(t_+, t_-)$ represents the unitary time evolution operator. This latter is described by Dyson's formula defined by the interaction Hamiltonian of the scattering process. To compute such amplitude, Wick's theorem needs to be utilized to account for the Hamiltonian characteristics. Performing such a calculation is tedious and lengthy work, so another simpler approach is to use Feynman diagrams.

To start using these, calculations must begin with the Lagrangian $\mathcal{L}(\phi)$ in scalar field theory describing the interaction. The particles involved and their dynamics are defined by their scalar field ϕ, which forms the basis of this Lagrangian. This latter can be written as the difference between the kinetic and potential terms $\mathcal{L} = T(\phi) - V(\phi)$, later expanded to the following [3]:

$$\mathcal{L} = \mathcal{L}_{kinetic} + \mathcal{L}_{mass} + \mathcal{L}_{interaction} \tag{2}$$

The kinetic term remains invariant, however, the potential is divided into the mass and interaction term. This latter one can be represented in the Feynman diagram, giving a graphical representation to visualize and calculate the interactions between particles.

In this diagram, the lines symbolize the particles involved while the vertices represent the interaction points. These former lines can be classified into two types: internal and external. External lines describe incoming and outgoing particles, normally accompanied by an arrow to indicate the direction of movement with respect to time. Internal lines connect vertices within the diagram and represent virtual particles that mediate the interaction but do not appear in the initial or final states of the amplitude. These are the ones that encode the conservation of energy and momentum. In the diagram, these two types are represented by different lines, generally, solid lines represent fermions (external) such as electrons, quarks, or neutrinos, and wavy or dotted lines represent bosons (internal)[3]. For this study both will be represented by solid lines and later indicated and explained.

Feynman diagrams are governed by a specific set of rules that must be adhered to. The most fundamental rule pertains to momentum conservation. Both internal and external lines in the diagram must satisfy the condition of four-momentum $p^\mu p_\mu = m^2$, which is also commonly written as $p^2 = m^2$ in contracted form. This implies that each internal line must be assigned a designated momentum p_i. Additionally, at each vertex in the diagram, a corresponding factor is introduced [4]:

$$(-ig)(2\pi^4)\delta^4(\sum_i k_i) \tag{3}$$

where g is the coupling constant at each vertex, and the sum represents the sum of momenta flowing into the vertex. This way of analyzing the momentum assumes all momenta are incoming.
The amplitude then has another term corresponding to the propagators of the internal lines with a factor to be multiplied of:

$$\int \frac{d^4p}{2\pi^4} \frac{i}{p^2 - m^2 + i\epsilon} \tag{4}$$

where the ϵ dictates the way the contour must be deformed in the complex plane to avoid the poles, leading to the correct causal ordering of events [5]. The total amplitude will therefore be the multiplication of the both above terms. As seen, both factors mostly contribute redundant constant terms, therefore for simplicity, the amplitude can be taken to be proportional to the reciprocal of the four-momenta squared. In many scattering amplitudes, the same combination of momenta terms appears repeatedly in the denominator. These can be collected and redefined using the Mandelstam variables for simplicity. In a general interaction between four incoming particles p_1, p_2, p_3 and p_4 as seen in Figure 1:

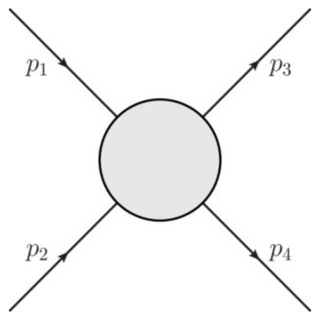

Figure 1: *Generic 4-point interaction between four scalar fields [6]*

the Madelstam variables are expressed as follows[6]:

$$s = (p_1 + p_2)^2 = 2p_1p_2$$
$$t = (p_1 + p_3)^2 = 2p_1p_3$$
$$u = (p_1 + p_4)^2 = 2p_1p_4$$

Therefore from these, the corresponding amplitude can be written to be proportional to a set of these variables.

2.1.1 Cubic interaction example

To understand scattering amplitudes using Mandelstam variables, one can examine a more specific case of Figure 1, such as a 4-point interaction in the ϕ^3 theory. This theory involves interactions where only three fields can interact at a single vertex. Thus, for a 4-particle interaction, there are three possible configurations:

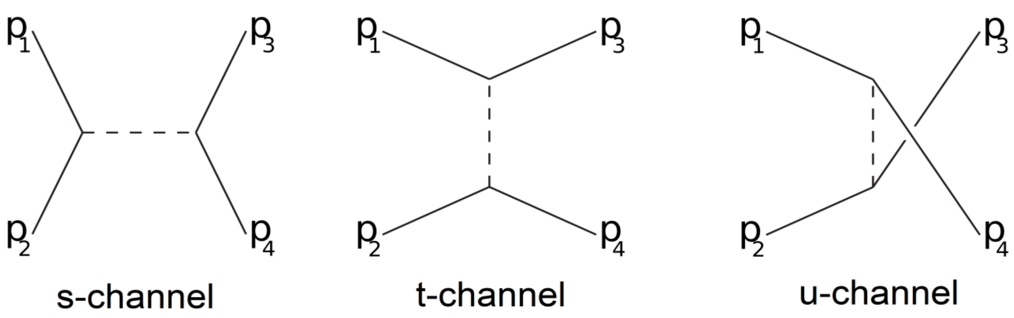

Figure 2: *Feynman diagrams for the three possible interactions between 4 scalar fields labeled p1, p2, p3, and p4. Solid lines represent the particles and dotted the mediator. [7]*

As illustrated by the Feynman diagrams in Figure 2, the s-channel, t-channel, and u-channel all contribute to the 4-point amplitude in ϕ^3 theory. Each of these channels represents a different way in which the particles can interact: The s-channel involves two particles combining to form an intermediate state, which then decays into two final particles. In the t-channel, one particle is exchanged between two other particles, leading to a scattering event. The u-channel is similar to the t-channel but involves different particles in the exchange and final states. For massless particles, the amplitude for such an interaction can be approximated by the following equation using all contributions:

$$\mathcal{A}_4^{\phi^3} \approx \frac{1}{s} + \frac{1}{t} + \frac{1}{u} \tag{5}$$

Where s, t, u are the different channels that satisfy the relation s + t + u = 0 for massless particles.

2.2 Colored-ordered theory

After establishing the foundational principles of scattering amplitudes, the study delves deeper into the interactions of scalar fields by introducing a new property: color ordering. In quantum field theory, and particularly in the context of Quantum Chromodynamics (QCD), particles such as quarks and gluons carry a property called "color charge". The interactions involving these are described by scattering amplitudes that account for both the kinematic properties and the color flow of the particles involved. At the tree level (most general case with no loops), the full amplitude is therefore described as the sum of all possible color flows with its corresponding kinematics factor [8]:

$$\mathcal{A} = \sum_{permutations} Tr[A \cdot B \cdot C ...] A_p \tag{6}$$

where the trace $Tr[A \cdot B \cdot C...]$ represents the color structure and the A_p represents the kinematics also called the partial amplitude. The color structure is defined by a set of letters, the number of which corresponds to the number of interacting fields. These letters rearrange in different orders to show the various color flows. For this study, the color-ordered partial amplitude studied will be the alphabetic simplest case. By assuming a fixed color order, this study focuses exclusively on the kinematics of the partial amplitude while accounting for the conditions imposed by color.

The Feynman rules and Lagrangian now apply solely to the kinematic partial amplitude term and experience slight modifications accordingly. The Feynman diagrams are now simplified to include only planar diagrams, which are diagrams that can be drawn on a plane without any crossing lines. This change significantly simplifies the calculations by reducing the number of diagrams that need to be considered. Propagators and vertices remain the same as in the full theory, but the summation over permutations is restricted to those respecting the specific color ordering.

In order to deal with these colored ordered fields at high-point interactions, momenta must be again redefined into a variable of the following form to make it easier to work with [9]:

$$X_{i,j} = (p_i + p_{i+1} + ... + p_{j-1})^2 \tag{7}$$

2.2.1 Cubic interaction example with color ordering

The same example previously discussed can now examined for a color-ordered scattering amplitude, showing the reduced amplitude. The interaction will now be for a 4-point interaction in the Tr(ϕ^3) theory. In line with the previously defined ϕ^3 theory, only three fields can interact at a single vertex, however, the number of possible Feynman diagrams is now reduced to two due to the planar diagrams condition for color ordering:

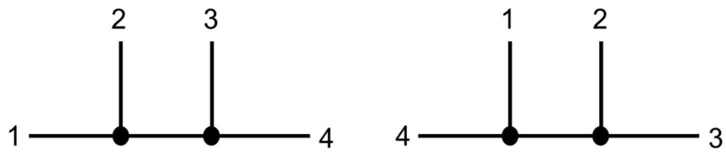

Figure 3: *Two possible configurations of the Feynman diagrams for a 4-point interaction for Tr(ϕ^3) theory. Lines represent the particles labeled by the numbers, the points are the vertices, and in between vertices lie the mediator*

As seen in Figure 3, there are no intersecting lines, unlike the u-channel in Figure 2, due to the color-ordered property. Instead, the two possible configurations for the four-particle interactions are visible, with indices in ascending order due to the cyclic symmetry defined by the assumed simple alphabetical color order. Both diagrams depict the same interaction but illustrate two different possible configurations that can be transformed into each other by permuting the indices of the fields. From these diagrams, one can analyze the momentum from the single propagator present as the line between the two vertices. Using the redefined momentum variable $X_{i,j}$ from equation 7, two possible $X_{i,j}$ are extracted from each diagram respectively from left to right:

$$(X_{1,3}, X_{2,4}) \tag{8}$$

And so its amplitude taking both contributions will take the form of [2]:

$$\mathcal{A}_4^{Tr(\phi^3)}(X) \approx \frac{1}{X_{1,3}} + \frac{1}{X_{1,4}} \tag{9}$$

Once this concept is understood, this type of colored order scattering amplitudes for scalar fields are further studied for different theories.

2.3 Tr(ϕ^3) theory

The first theory to discuss is the previously briefly mentioned Tr(ϕ^3) theory, describing interactions between cubic vertices scalar fields ϕ. The fields unlike for ϕ^3 theory, do not describe single scalar fields, but a matrix-valued field transforming under a gauge group like SU(N), meaning it symmetrically transforms as an N×N unitary matrix of determinant 1. The trace will then allow to sum over all indices in the matrix accounting for internal symmetry where the color-ordering lies.

Its defining Lagrangian takes the form of[2]:

$$\mathcal{L} = \frac{1}{2}Tr(\partial_\mu\phi\partial^\mu\phi) - \frac{g}{3!}Tr(\phi^3) \tag{10}$$

Its first term represents the kinetics of the scalar field ϕ, described by the partial derivatives ∂_μ denoting field changes in space and time. The second term corresponds to the interactions of the field: $\frac{g}{3!}Tr(\phi^3)$ captures the cubic interactions characteristic proper of the $Tr(\phi^3)$, where g is the coupling constant that describes the strength interaction at each vertex proper of $Tr(\phi^3)$. The factorial 3! ensures the correct accounting for the indistinguishability of the fields involved.

The amplitude, derived from the Lagrangian mentioned above, can be expressed using the general Feynman rules. Specifically, the amplitude is proportional to the reciprocal of the squared momenta, as indicated by the propagators. By redefining the momenta in terms of $X_{i,j}$, the amplitude can be formulated as follows:

$$\mathcal{A}_p(X) \approx \sum \frac{1}{X_{i,j}X_{i',j'}X_{i'',j''}...X_{i^\beta,j^\beta}} \approx \sum \frac{1}{X^\beta} \tag{11}$$

where the sum is over all possible fractions of combinations of $X_{i,j}$ in the interaction coming from the propagators. This can be simplified for notation as the expression seen to the right where β defines the number of propagators present.

2.3.1 5-point theory

As done in the example for cubic interaction at 4-point described in section 2.2.1, the theory can be extended to a more complex 5-point interaction, meaning 5 fields are scattering, where the amplitude should take the form of [2]:

$$\mathcal{A}_5^{Tr(\phi^3)} = \frac{1}{X_{1,3}X_{1,4}} + \frac{1}{X_{2,4}X_{2,5}} + \frac{1}{X_{1,3}X_{3,5}} + \frac{1}{X_{1,4}X_{2,4}} + \frac{1}{X_{2,5}X_{3,5}} \tag{12}$$

One can then analyze a 5-point interaction with the following Feynman diagram:

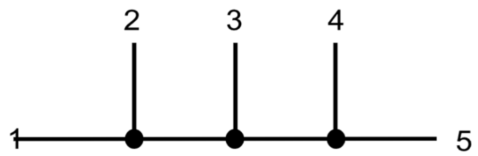

Figure 4: *Simplest Feynman diagram for a 5-point interaction in Tr(ϕ^3). Numbered lines represent particles, the points are vertices, and in between vertices lies the mediator.*

For simplicity the simplest Feynman diagram is seen above, however, there exist five different permutations of the indices therefore obtaining five possible $X_{i,j}$:

$$(X_{1,4}, X_{1,3}, X_{1,5}, X_{2,4}, X_{2,5}, X_{3,5}) \tag{13}$$

2.3.2 6-point theory

The above process is then repeated for a 6-point, for which the amplitude takes the following form [2]:

$$\mathcal{A}_5^{Tr(\phi^3)} = (\frac{1}{X_{1,3}X_{1,4}X_{1,5}} + \frac{1}{X_{1,3}X_{3,6}X_{4,6}} + cyclic) + \frac{1}{X_{1,3}X_{3,5}X_{1,5}} + \frac{1}{X_{2,4}X_{4,6}X_{2,6}} \tag{14}$$

Studying its interaction, its corresponding Feynman diagram is shown below:

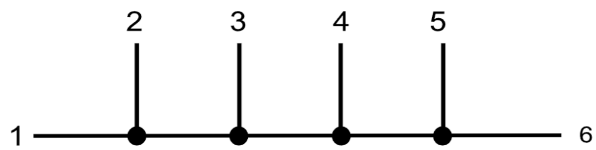

Figure 5: *Simplest Feynman diagram for a 6-point interaction in Tr(ϕ^3). Numbered lines represent particles, the points are vertices, and in between vertices lies the mediator.*

In Figure 5 one can see there are four vertices and therefore three propagators extracted from one possible configuration. To account again for the possible configurations the fields are permuted and from the possible cyclic permutations the following possible $X_{i,j}$ are extracted:

$$(X_{1,3}, X_{1,4}, X_{1,5}, X_{2,4}, X_{2,5}, X_{2,6}, X_{3,5}, X_{3,6}, X_{4,6}) \tag{15}$$

2.3.3 7point theory

The amplitude for a 7-point interaction can be again obtained analytically through the Lagrangian. The corresponding Feynman diagram of the interaction is shown below:

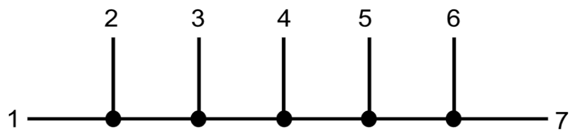

Figure 6: *Simplest Feynman diagram for a 7-point interaction in Tr(ϕ^3). Numbered lines represent particles, the points are vertices, and in between vertices lies the mediator.*

As seen from Figure 6 above there are five vertices and therefore four propagators to take into account. The possible cyclic permutations of such a diagram give the following $X_{i,j}$:

$$(X_{1,3}, X_{1,4}, X_{1,5}, X_{1,6}, X_{2,4}, X_{2,5}, X_{2,6}, X_{2,7}, X_{3,5}, X_{3,6}, X_{3,7}, X_{4,6}, X_{4,7}, X_{5,7}) \tag{16}$$

2.4 NLSM

These scalar-colored ordered fields can be extended to another more complex theory; The Non-Linear Sigma Model. The NLSM describes how scalar fields map points from Euclidean space-time to a more complex manifold with curved Riemann metric. This theory allows for the study of fields that live of complex geometric which represent physical systems better than the linear models [10]. The fields studied for this research are for pion particles which arise from spontaneous breaking of chiral symmetry. Its Lagrangian takes the following form[11]:

$$\mathcal{L} = \frac{f_\pi^2}{4} Tr(\partial_\mu U^\dagger \partial^\mu U) \tag{17}$$

where f_π^2 is the pion decay constant. The following partial derivatives show the change in time of the unitary matrix U and its hermitian conjugate $U^\dagger$ described by Chiral perturbation theory. This matrix takes the complex form of an exponential, expanded through power series to take an expression describing the interaction terms as a series of quadratic derivatives. Its complete expansion is not the primary focus; rather, the key takeaway is the quadratic derivatives that arise from such, imposing on the NLSM interactions an even point condition. Fields can only appear as even powers, making n-point interactions occur only for even "n" orders, and even "n" interaction vertices.

The amplitude equation for the NLSM theory will then vary, accounting for its intricate properties and so the Feynman rules describing such function are no longer defined as like for the Tr(ϕ^3) theory. The propagators of the interaction remain unchanged, as described by Equation 4, while the vertex undergoes significant alteration. The interaction vertices in the NLSM arise from the expansion of the non-linear field U(x) in the Lagrangian, resulting in a more complex structure due to the non-linear nature of these interactions. Describing these intricate vertex factors exceeds the scope of this research. However, both the vertex and propagator contributions simplify to yield an amplitude proportional to the squared momenta $\mathcal{A}(X) \propto p^2$. This form of amplitude, incorporating contributions from both vertex and propagator terms, can be expressed using the variable $X_{i,j}$ [1] as follows:

$$\mathcal{A}_p(X) \approx X_{i,j}X_{i',j'}X_{i'',j''}... \tag{18}$$

2.4.1 6-point NLSM

This theory can then be applied to the different n-point interactions. Starting again with the theoretical amplitude of the interaction [2]:

$$\mathcal{A}_6^{NLSM} = (\frac{(X_{1,3}+X_{2,4})(X_{1,5}+X_{4,6})}{X_{1,4}} - X_{1,3} - X_{2,4} + (cyclic, i \rightarrow i+1) \tag{19}$$

Furthermore, from the Lagrangian for NLSM one can deduce the types of interactions allowed in such theory. As only even orders of fields interact, only vertices with even fields interacting can be seen in the Feynman diagram, starting from the lowest 4 field interaction with shape as seen in the top of Figure 7. The maximum even interaction vertex is for this case a point for the 6 fields interacting shown in the bottom of Figure 7.

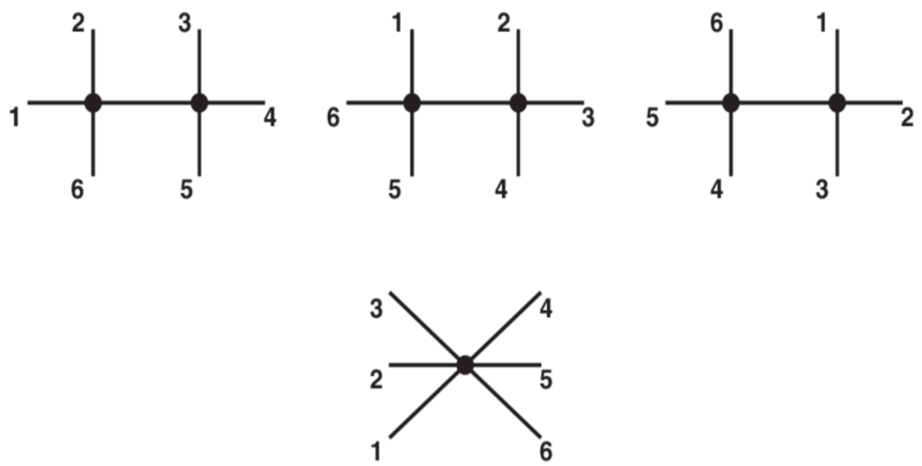

Figure 7: *Two different Feynman diagrams possible for a 6-point interaction with NLSM theory. At the top are the three possible permutations of the 4-fields per vertex diagram. Below stands the diagram of all 6 fields interacting at one vertex.*

At $\text{Tr}(\phi^3)$ all possible $X_{i,j}$ found from the permutations of Feynman diagrams corresponded to the same variables found from the combinations of momenta from equation 7. However in this case, as seen from Figure 7, the three possible $X_{i,j}$ from the Feynman diagrams are the ones seen $(X_{1,4}, X_{2,5}, X_{3,6})$, however all possible $X_{i,j}$ from all possible interactions are shown below to be:

$$(X_{1,3}, X_{1,4}, X_{1,5}, X_{1,6}, X_{2,4}, X_{2,5}, X_{2,6}, X_{3,5}, X_{3,6}, X_{4,6}) \tag{20}$$

[1] $X_{i,j}$ should be re-defined again for NLSM as the kinematic variables often involve momenta directly due to the nature of the interaction vertices. However, for the sake of simplicity, they will be defined the same as for $\text{Tr}(\phi^3)$ like the paper from Arkani-Hamed states too[2]

2.4.2 8-point NLSM

For 8-point interactions, the amplitude can be found through its Lagrangian. From such, there can be three possible ways of interacting between the 8 fields. These are shown in the Feynman diagram shown in Figure 8. The top left shows six and four field vertices, the top right shows three four field vertices, and finally the bottom shows the 8 fields interacting at once.

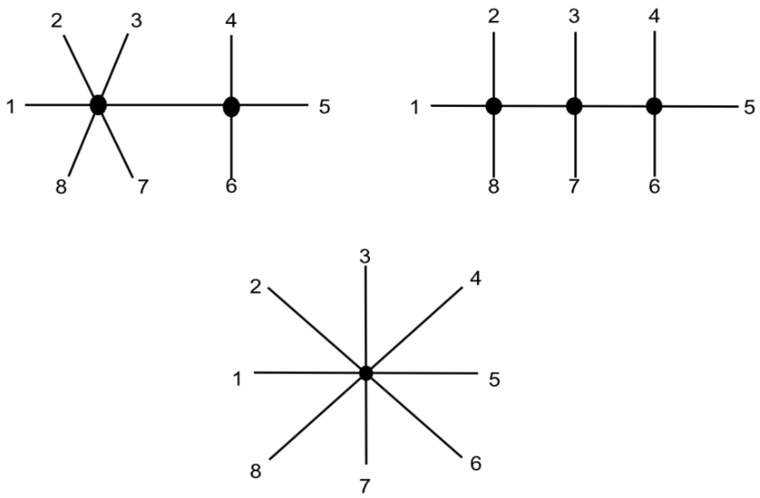

Figure 8: *All three possible Feynman diagrams for an 8-point interaction with NLSM theory.* Top right has three 4-fields per vertex contribution, top left has one 6-field vertex together with a 4-field vertex. Below is the simplest of all eight fields interacting in single vertex.

Again the possible $X_{i,j}$ from the Feynman diagram is shown as pairs of $((X_{1,4}, X_{5,8}), (X_{1,6}, X_{2,5}), (X_{3,8}, X_{4,7}))$. However all possible $X_{i,j}$ as deduced from the interactions found from equation 7 are as follows:

$$(X_{1,4}, X_{1,5}, X_{1,6}, X_{1,7}, X_{2,4}, X_{2,5}, X_{2,6}, X_{2,7}, X_{2,8}, X_{3,5}, X_{3,6}, X_{3,7}, X_{3,8}, X_{4,6}, X_{4,7}, X_{4,8}, X_{5,7}, X_{5,8}, X_{6,8}) \tag{21}$$

2.5 Arkani-Hamed's hidden zer0s

Colored-ordered scattering amplitudes for scalar fields can be described for both theories of $\text{Tr}(\phi^3)$ and NLSM. Their diverse n-point interactions have different amplitudes describing it, which have many properties yet to discover and study.

In 2023 Arkani-Hamed and collaborators published a paper titled "Hidden zeros for particle/string amplitudes and the unity of colored scalars, pions, and gluons" unveiling properties of these scattering amplitudes never thought of before. By using colored scalar theories the paper discovers a hidden pattern of zeros when a specific set of Mandelstam variables are set to zero. This was firstly observed in the tree level $\text{Tr}(\phi^3)$ and was quickly extended to amplitudes for pions and gluons too [2].

To understand Arkani-Hamed's work one must start from the amplitude. As explained above, it can be redefined using the variable $X_{i,j}$ shown in equations 11 for $\text{Tr}(\phi^3)$ and 18 for NLSM, however, this is the simplest form of it, its most general pattern involves a fraction of the form $A = \frac{N(X)}{D(X)}$. The denominator $(D(X))$ represents the combinatorial property from the structure and relationships between terms in the amplitude. This is proportional to the propagators of the interaction, visualized through its Feynman diagram. The denominator will be proportional to the possible $X_{i,j}$ obtained from the propagators shown in such diagram. To do so, in the Feynman diagram, external lines with their momenta are labeled with the natural numbers together with the internal lines. The sum of the momenta of the particles connected by each propagator all squared as shown in equation 7 gives the primary $X_{i,j}$ variables the denominator includes.

The numerator ($N(X)$) represents the algebraic geometric properties of the amplitude as described by the polynomials that form it. This emphasizes the geometric structure and symmetries underlying particle interactions. The polynomials are proportional to all possible $X_{i,j}$, found from the permutations of particle indices and formation of all possible combinations of momenta. These can be visualized from the $X_{i,j}$ that appear in the kinematic mesh later explained.
Based on this numerator, numerous constraints can be applied to the amplitude, leading to significant simplifications [12]. Firstly cyclicity is imposed meaning that the scattering amplitude remains invariant under cyclic permutations of its external particles. This property reflects the physical fact that the scattering process is symmetric under the interchange of particles, as long as the order is preserved cyclically. Additionally, the amplitude must satisfy the physicality constraint, which is composed of locality and unitarity. Locality refers to the principle that interactions occur at specific, discrete points in spacetime. In terms of scattering amplitudes in the defining Feynman diagram of interaction, the locality represents the interaction at a single point. This property dictates the type of singularities (poles) that appear in scattering amplitudes. Specifically, in tree-level diagrams, the poles correspond to the propagators of the theory. It also ensures that the singularities of the amplitude are simple poles rather than higher-order or overlapping poles. Higher-order or overlapping poles would indicate non-local interactions, which are not physical in a local QFT. Unitarity is equally important; it is a fundamental principle in quantum field theory that ensures the conservation of probability in scattering processes. It reflects the requirement that the sum of probabilities of all possible outcomes of a scattering process equals one [12].

Finally, the most used property of the numerator for this research that constrains the amplitude massively are Arkani-Hamed's hidden zeros. This is a property of the amplitude discovered, which makes a partial amplitude vanish at certain loci in space defined by their momentum. From a $X_{i,j}$ basis one can define all the other kinematic variables using the following $c_{i,j} = -2p_ip_j$, which is related to $X_{i,j}$ by the following:

$$c_{ij} = -s_{ij} = X_{i,j} + X_{i+1,j+1} - X_{i,j+1} - X_{i+1,j} \tag{22}$$

All of this kinematic data can then be represented visually using a kinematic mesh. This is not only useful to organize the momentum invariants but to also understand certain features of the amplitudes such as the hidden zeros. To build such mesh, equation 7 is used. Each $X_{i,j}$ is the vertex of a 45° rotated square, and the whole square is the named $c_{i,j}$. From a vertex point up right there is $X_{i,j+1}$, and to the left $X_{i+1,j}$ seen in the example for a 6-point interaction in Figure 9 to the left. Placing all these squares together creates the grid seen in Figure 9 in the middle, where the boundaries that touch the vertical lines represent $X_{i,i+1} = {p_i}^2$ points, equal to zero. The mesh extends infinitely long by Mobius symmetry where $X_{i,j} = X_{j,i}$ and $c_{i,j} = c_{j,i}$. From such mesh, one can observe the causal diamonds that lie inside it defining later the hidden zeros. These are regions defined by four boundary points: X_B bottom point, X_R right side point, X_T top point, and X_L left side point. To form this start at a boundary point defined X_L and continue the line until the $X_{i,j}$ before the next boundary point is hit, named X_T. From there bounce down to the boundary point called X_R. There bounce back to again the $X_{i,j}$ before the next boundary point named X_B, to bounce back to the initial X_L. That will create the causal diamond shown the the right of Figure 9. In this case, the resulting region is referred to as a "skinny rectangle" due to its elongated form. At higher-order kinematic meshes, there are also regions known as "squares." These squares resemble skinny rectangles but include an additional diamond structure extending from X_B to X_L, transforming them into nm areas, where n represents the length and m represents the height. As seen to the right of Figure 9, $X_B + X_T - X_L - X_R = \sum_{c_{i,j} \in \Diamond} c_{i,j}$, and so meaning that the skinny rectangles or squares formed from a kinematic mesh are the sum of their enclosing $c_{i,j}$. Setting these to zero simultaneously would give the hidden zeros of the partial amplitudes. This just gives one possibility of simultaneous $c_{i,j}$ that gives a vanishing partial amplitude, however, this can be repeated all over the kinematic mesh until all possible combinations of $c_{i,j}$ are found, giving many different ways to vanish the partial amplitude.

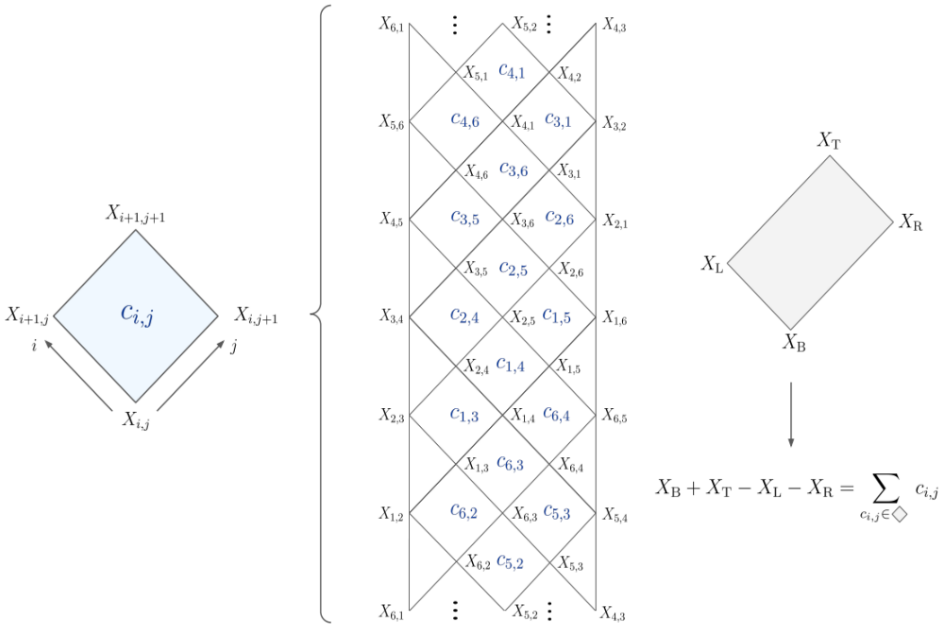

Figure 9: *Kinematic mesh in the center, showing all possible $X_{i,j}$ and $c_{i,j}$. To the left an explanation of the formation of one of the squares and its labeling. To the right is the explanation of the formation of the causal diamond.*

These hidden zeros can be found in both $\text{Tr}(\phi^3)$ and NLSM theories at different n-point interactions leading to different kinematic meshes.

2.5.1 4-point

Starting at the simplest case for a 4-point interaction. The kinematic mesh can be built from the possible $X_{i,j}$ that the Lagrangian allows for. For four points, correspond to the ones stated in equation 8. From such variables, there would be the same indices $s_{i,j}$ made up of a combination of these, that will later be of use for the zeros:

$$s_{1,3} = s_{2,4} = X_{1,3} + X_{2,4}$$

From such variables, one can construct the kinematic mesh shown below:

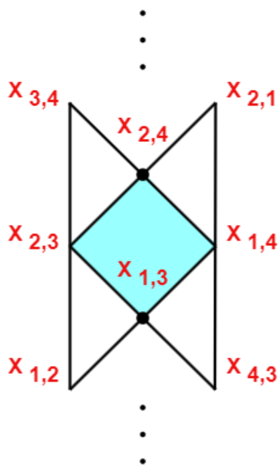

Figure 10: *Kinematic mesh for a 4-point interaction with a single skinny rectangle shaded in blue*

Using this mesh, one can deduce the hidden zeros by applying Arkani-Hamed's method. There exists only one skinny rectangle obtained from such kinematic mesh seen colored in blue in Figure 10:

$$s_{1,3} = s_{2,4}$$

Concluding, for a 4-point $\mathrm{Tr}(\phi^3)$ interaction, this single distinct configuration, when applied with their corresponding $s_{i,j}$ conditions, results in a hidden zero. This analysis reveals where the interaction amplitude vanishes under specific circumstances.

2.5.2 5-point

This can then be repeated for a 5-point interaction with possible $X_{i,j}$ from equation 13, which then have their corresponding $s_{i,j}$:

$$\begin{aligned} s_{1,3} &= X_{1,3} + X_{2,4} - X_{1,4}; \\ s_{1,4} &= X_{1,4} + X_{2,5} - X_{2,4}; \\ s_{2,4} &= X_{2,4} + X_{3,5} - X_{2,5}; \\ s_{2,5} &= X_{2,5} + X_{1,3} - X_{3,5}; \\ s_{3,5} &= X_{3,5} + X_{1,4} - X_{1,3}; \end{aligned} \tag{23}$$

Using the above $X_{i,j}$ and $s_{i,j}$ possible, the following kinematic mesh is plotted in Figure 11:

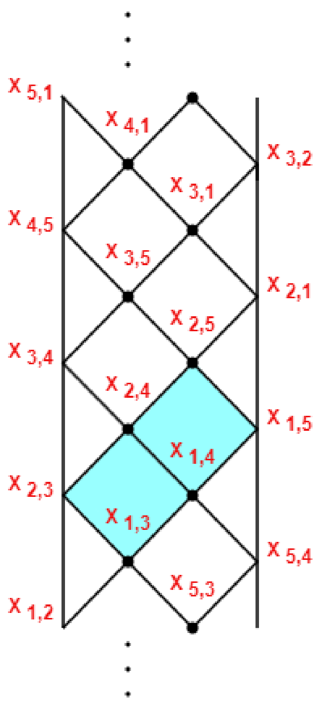

Figure 11: *Kinematic mesh for a 5-point interaction with a single skinny rectangle shaded in blue*

The hidden zeros can be seen through the five possible configurations of skinny rectangles, each formed by two causal diamonds:

$$\begin{gathered} s_{1,3}, s_{1,4} \\ s_{2,4}, s_{2,5} \\ s_{1,3}, s_{3,5} \\ s_{1,4}, s_{2,4} \\ s_{2,5}, s_{3,5} \end{gathered}$$

These pairs applied independently will result in Arkani-Hamed's hidden zeros.

2.5.3 6-point

The possible $X_{i,j}$ for 6-point interactions are as shown in equation 15, with the corresponding $s_{i,j}$ made of the combinations of the above are as follows:

$$\begin{aligned} s13 &= X_{1,3} + X_{2,4} - X_{1,4}; \\ s14 &= X_{2,5} + X_{1,4} - X_{1,5} - X_{2,4}; \\ s15 &= X_{1,5} + X_{2,6} - X_{2,5}; \\ s24 &= X_{3,5} + X_{2,4} - X_{2,5}; \\ s25 &= X_{2,5} + X_{3,6} - X_{3,5} - X_{2,6}; \\ s26 &= X_{1,3} + X_{2,6} - X_{3,6}; \\ s35 &= X_{3,5} + X_{4,6} - X_{3,6}; \\ s36 &= X_{1,4} + X_{3,6} - X_{1,3} - X_{4,6}; \\ s46 &= X_{1,5} + X_{4,6} - X_{1,4}; \end{aligned} \tag{24}$$

From these, the following kinematic mesh is obtained:

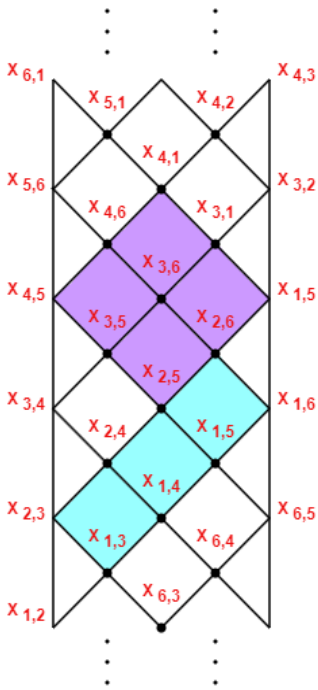

Figure 12: *Kinematic mesh for a 6-point interaction with a single skinny rectangle shaded in blue, and a square shaded in purple*

Using this mesh, the hidden zeros are again deduced. In this case, there exist two types of causal diamonds shown in Figure 12 above shaded. The one shaded in blue shows the familiar skinny rectangle with 6 possible combinations of 3 diamonds of length. The purple-shaded region shows the so-called squares of 2x2 diamonds with 3 possible combinations. These are shown below:

$$\begin{aligned}&Rectangles:\\&s_{1,3}, s_{1,4}, s_{1,5}\\&s_{2,4}, s_{2,5}, s_{2,6}\\&s_{3,5}, s_{3,6}, s_{1,3}\\&s_{4,6}, s_{1,4}, s_{2,4}\\&s_{1,5}, s_{2,5}, s_{3,5}\\&s_{2,6}, s_{3,6}, s_{4,6}\end{aligned}$$

$$\begin{aligned}&Squares:\\&s_{1,3}, s_{3,6}, s_{4,6}, s_{1,4}\\&s_{2,4}, s_{1,5}, s_{1,5}, s_{2,5}\\&s_{3,5}, s_{2,5}, s_{2,6}, s_{3,6}\end{aligned}$$

2.5.4 7-point

For a 7-point interaction with possible $X_{i,j}$ from equation 16, which then have their corresponding $s_{i,j}$: The corresponding $s_{i,j}$ taken:

$$\begin{aligned}s_{1,3} &= X_{1,3} + X_{2,4} - X_{1,4}\\s_{1,4} &= X_{2,5} + X_{1,4} - X_{1,5} - X_{2,4}\\s_{1,5} &= X_{1,5} + X_{2,6} - X_{1,6} - X_{2,5}\\s_{1,6} &= X_{1,6} + X_{2,7} - X_{2,6}\\s_{2,4} &= X_{3,5} + X_{2,4} - X_{2,5}\\s_{2,5} &= X_{2,5} + X_{3,6} - X_{3,5} - X_{2,6}\\s_{2,6} & X_{2,6} + X_{3,7} - X_{3,6} - X_{2,7}\\s_{2,7} &= X_{2,7} + X_{1,3} - X_{3,7}\\s_{3,5} &= X_{3,5} + X_{4,6} - X_{3,6}\\s_{3,6} &= X_{4,7} + X_{3,6} - X_{3,7} - X_{4,6}\\s_{3,7} &= X_{3,7} + X_{4,1} - X_{4,7} - X_{1,3}\\s_{4,6} &= X_{4,6} + X_{5,7} - X_{4,7}\\s_{4,7} &= X_{4,7} + X_{1,5} - X_{5,7} - X_{1,4}\\s_{5,7} &= X_{5,7} + X_{1,6} - X_{1,5}\end{aligned} \tag{25}$$

The corresponding kinematic mesh is seen below:

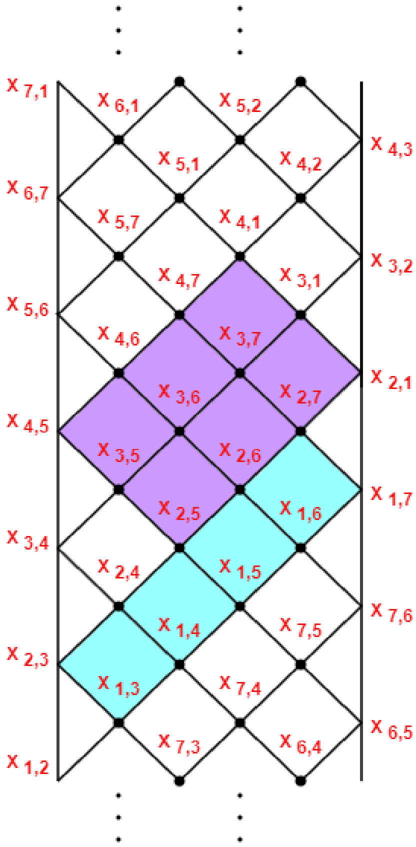

Figure 13: *Kinematic mesh for a 7-point interaction with a single skinny rectangle shaded in blue, and a square shaded in purple*

Using the mesh the hidden zeros are again deduced from the 7 possible skinny rectangles of 4 diamonds in length, with 5 squares of 3x2 shown below:

$Rectangles:$	$Squares:$
$s_{1,3}, s_{1,4}, s_{1,5}, s_{1,6}$	$s_{1,3}, s_{1,4}, s_{1,5}, s_{7,3}, s_{7,4}, s_{7,5}$
$s_{2,4}, s_{2,5}, s_{2,6}, s_{2,7}$	$s_{2,4}, s_{2,5}, s_{2,6}, s_{1,4}, s_{1,5}, s_{1,6}$
$s_{3,5}, s_{3,6}, s_{3,7}, s_{1,3}$	$s_{3,5}, s_{3,6}, s_{3,7}, s_{2,5}, s_{2,6}, s_{2,7}$
$s_{4,6}, s_{4,7}, s_{1,4}, s_{2,4}$	$s_{4,6}, s_{4,7}, s_{4,1}, s_{3,6}, s_{3,7}, s_{3,1}$
$s_{5,7}, s_{1,5}, s_{2,5}, s_{3,5}$	$s_{5,7}, s_{5,1}, s_{5,2}, s_{4,7}, s_{4,1}, s_{4,2}$
$s_{1,6}, s_{2,6}, s_{3,6}, s_{4,6}$	$s_{5,1}, s_{5,2}, s_{5,3}, s_{6,1}, s_{6,2}, s_{6,3}$
$s_{2,7}, s_{3,7}, s_{4,7}, s_{5,7}$	$s_{6,2}, s_{6,3}, s_{6,4}, s_{7,2}, s_{7,3}, s_{7,4}$

2.5.5 8-point

Finally for a 8-point interaction with possible $X_{i,j}$ from equation 33, which then have their corresponding $s_{i,j}$:

$$
\begin{aligned}
s_{1,3} &= X_{1,3} + X_{2,4} - X_{1,4} \\
s_{1,4} &= X_{2,5} + X_{1,4} - X_{1,5} - X_{2,4} \\
s_{1,5} &= X_{1,5} + X_{2,6} - X_{2,5} - X_{1,6} \\
s_{1,6} &= X_{1,6} + X_{2,7} - X_{2,6} - X_{1,7} \\
s_{1,7} &= X_{1,7} + X_{2,8} - X_{2,7} \\
s_{2,4} &= X_{2,4} + X_{3,5} - X_{2,5} \\
s_{2,5} &= X_{2,5} + X_{3,6} - X_{3,5} - X_{2,6} \\
s_{2,6} &= X_{2,6} + X_{3,7} - X_{3,6} - X_{2,7} \\
s_{2,7} &= X_{2,7} + X_{3,8} - X_{3,7} - X_{2,8} \\
s_{2,8} &= X_{2,8} + X_{1,3} - X_{3,8} \\
s_{3,5} &= X_{3,5} + X_{4,6} - X_{3,6} \\
s_{3,6} &= X_{3,6} + X_{4,7} - X_{3,7} - X_{4,6} \\
s_{3,7} &= X_{3,7} + X_{4,8} - X_{4,7} - X_{3,8} \\
s_{3,8} &= X_{3,8} + X_{1,4} - X_{4,8} - X_{1,3} \\
s_{4,6} &= X_{4,6} + X_{5,7} - X_{4,7} \\
s_{4,7} &= X_{4,7} + X_{5,8} - X_{5,7} - X_{4,8} \\
s_{4,8} &= X_{4,8} + X_{1,5} - X_{5,8} - X_{1,4} \\
s_{5,7} &= X_{5,7} + X_{1,6} - X_{1,5} \\
s_{5,8} &= X_{5,8} + X_{1,6} - X_{6,8} - X_{1,5} \\
s_{6,8} &= X_{6,8} + X_{1,7} - X_{1,6}
\end{aligned}
$$

From the above, the kinematic mesh is built:

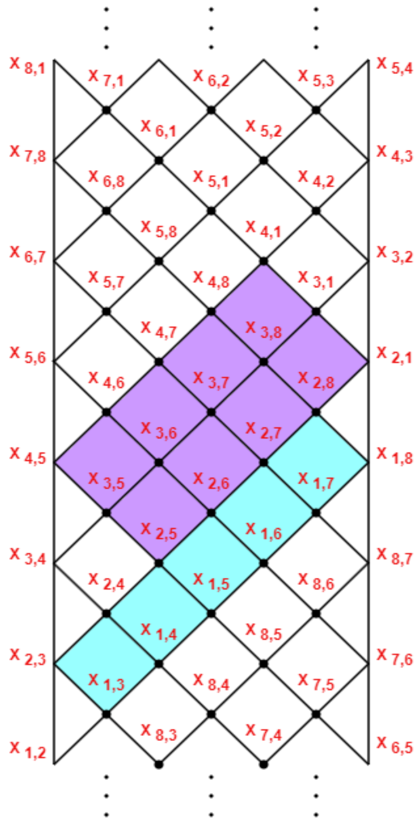

Figure 14: *Kinematic mesh for a 6-point interaction with a single skinny rectangle shaded in blue, and a square shaded in purple*

From such a mesh there can be deduced eight possible skinny rectangles with 5 diamonds of length and 8 squares of 4x2 :

Rectangles :	*Squares* :
$s_{1,3}, s_{1,4}, s_{1,5}, s_{1,6}, s_{1,7}$	$s_{1,4}, s_{1,5}, s_{1,6}, s_{1,7}, s_{2,4}, s_{2,5}, s_{2,6}, s_{2,7}$
$s_{2,4}, s_{2,5}, s_{2,6}, s_{2,7}, s_{2,8}$	$s_{2,5}, s_{2,6}, s_{2,7}, s_{2,8}, s_{3,5}, s_{3,6}, s_{3,7}, s_{3,8}$
$s_{3,5}, s_{3,6}, s_{3,7}, s_{3,8}, s_{1,3}$	$s_{3,6}, s_{3,7}, s_{3,8}, s_{3,1}, s_{4,6}, s_{4,7}, s_{4,8}, s_{4,1}$
$s_{4,6}, s_{4,7}, s_{4,8}, s_{1,4}, s_{2,4}$	$s_{4,7}, s_{4,8}, s_{4,1}, s_{4,2}, s_{5,7}, s_{5,8}, s_{5,1}, s_{5,2}$
$s_{5,7}, s_{5,8}, s_{1,5}, s_{2,5}, s_{3,5}$	$s_{5,8}, s_{5,1}, s_{5,2}, s_{5,3}, s_{6,8}, s_{6,1}, s_{6,2}, s_{6,3}$
$s_{6,8}, s_{1,6}, s_{2,6}, s_{3,6}, s_{4,6}$	$s_{6,1}, s_{6,2}, s_{6,3}, s_{6,4}, s_{7,1}, s_{7,2}, s_{7,3}, s_{7,4}$
$s_{1,7}, s_{2,7}, s_{3,7}, s_{4,7}, s_{5,7}$	$s_{7,2}, s_{7,3}, s_{7,4}, s_{7,5}, s_{8,2}, s_{8,3}, s_{8,4}, s_{8,5}$
$s_{2,8}, s_{3,8}, s_{4,8}, s_{5,8}, s_{6,8}$	$s_{8,3}, s_{8,4}, s_{8,5}, s_{8,6}, s_{1,3}, s_{1,4}, s_{1,5}, s_{1,6}$

3 Methods

Once the hidden zeros of the partial amplitude equation are determined, one can proceed to analyze how to reverse engineer the process and compute a unique partial amplitude assuming these zeros as valid. Starting from the theoretical amplitude proposed in the preceding theory A(X), a general partial amplitude can be inferred. By imposing conditions that incorporate these hidden zeros, a unique partial amplitude can then be derived. This derivation involves the use of a Mathematica code to simulate the amplitude, which is presented directly in the Appendix 6.2.

3.1 Tr(ϕ^3)

As previously mentioned, the amplitude in terms of $X_{i,j}$ follows as $\mathcal{A} = \frac{N(X)}{D(X)}$. The algebraic-geometric numerator is represented by all the possible polynomial combinations formed from the possible $X_{i,j}$ available. However, for Tr(ϕ^3) combinations with $X_{i,j}$ of orders higher than 1 are not possible as these variables do not appear in the final expected partial amplitude and so are not needed. Therefore the numerator is the monomial combinations of possible $X_{i,j}$, each weighted by a constant term that defines its contribution. The denominator on the other hand is the combinatorial multiplication of possible $X_{i,j}$ defined by the propagators. Together, these elements construct the generalized amplitude of interaction, serving as a foundational assumption in the Mathematica code.

Once the ansatz is created the hidden zeros and locality conditions are imposed. The hidden zeros will be used by setting the pairs of $s_{i,j}$ previously defined by skinny rectangles and squares for each case to zero and solving the equation. This will impose some conditions on the constants multiplying the combinations of $X_{i,j}$ in the numerator.

Following, numerator algebraic property constraints, the locality plays a crucial condition on the amplitude ensuring that the singularities of the amplitude are simple poles rather than higher order. To do so, one must eliminate possible overlapping poles of order bigger than the number of propagators (n) as defined by equation 18. This can be applied to the Mathematica code using the reminder theorem. This method in polynomial algebra states that the remainder of the division of a polynomial f(x) by a linear divisor (x-a) is equal to f(a) [13]. This can be extended to multiple variables and higher-order terms to adhere to the amplitude constraints. The process involves creating a table of residues for the number of $X_{i,j}$ variables allowed and then solving such equation by equating it to zero. By calculating these residues, higher-order poles are isolated and eliminated. It's crucial to note that an additional condition is imposed to ensure that the code recognizes $X_{i,j}$ as equivalent to $X_{j,i}$.

After both conditions are applied the results are fully simplified and a unique amplitude should show.

3.2 NLSM

For the Non-Linear Sigma Model procedure takes the same form as for Tr(ϕ^3) with some small changes. As the amplitude desired is different, the numerator N(X) will take a different form. Higher order variables are possible, hence there could be an $X_{i,j}^2$ or $X_{i,j}^3$ depending on the n-point representation. Therefore the possible numerator is a function of all monomial combinations between the possible $X_{i,j}$, added to the possible higher-order possibilities, each being multiplied by a constant to restrict their weights. As explained in the theory above, for 6-point only three $X_{i,j}$ will appear in the denominator separately, and for 8-point, they are three pairs of variables. The denominator, as like for Tr(ϕ^3) theory is the combinational multiplication of the possible variables, which in the case of this model is reduced. From Figure 7 and 8 one can observe different Feynman representations with different propagators that should be included in the calculations. However, the most general case used is the one with the most propagators, and that will already encapsulate the rest of them.

Once the ansatz is created only the hidden zeros condition needs to be imposed, as there can not exist non-physical poles due to the nature of the amplitude of the NLSM. The hidden zeros are applied as before. The same condition of $X_{i,j} = X_{j,i}$ is also used. Moreover, in order to simplify the amplitude further one can use the property of the reduced possible $X_{i,j}$ that the NLSM states. As seen, there is a pattern between the possible variables which is $X_{i,j} \to X_{i+1,j+1}$ which is imposed in the Mathematica code too.

After such conditions are applied the result is fully simplified to show the unique amplitude.

4 Results

By applying the methods detailed in the previous section, unique amplitude equations are derived for interactions at various points. The findings illustrate how the imposed conditions of hidden zeros and locality constraints lead to simplified and symmetrical sets of terms in the amplitudes. Each interaction point, from the simpler 4-point interactions to the more complex 7-point interactions, is examined, showcasing the effectiveness of the methodology in isolating the specific configurations where the scattering amplitudes vanish

The uniqueness of the outcome can be established from the clear and structured answer that simplifies to a minimal and symmetric set of terms. This is seen by one or a set of the weighing constants all multiplying a combination of $X_{i,j}$ variables.

4.1 Tr(ϕ^3) for 4-point

For a 4-point interaction of Tr(ϕ^3), as it deals with such little variables, there is no need for a Mathematica code and can be all done analytically.
The partial amplitude for such interaction contains only the two variables shown in equation 8, and so the ansatz is as follows:

$$\frac{c_1 X_{2,4} X_{1,3} + c_2 X_{1,3}^2 + c_3 X_{2,4}^2}{X_{1,3} X_{2,4}} \tag{26}$$

As seen there is a monomial of the combination of both variables, together with two second-order variables. As explained before this should not be allowed. The numerator should only contain monomials of combinations of the possible $X_{i,j}$, if not their variables won't cancel and display the amplitude needed. However, for very simple cases of amplitudes, these polynomials are needed, if not the constants will all just cancel, and the conditions imposed will just show $c_i \to 0$.
From such ansatz, one must now impose the conditions to simplify it. Setting the single causal diamond found on the kinematic mesh, it can be observed that for the expression to equal zero $X_{1,3} = -X_{2,4}$. Imposing non-physical pole conditions the obtained equation is of the form of:

$$c_1 \left(\frac{X_{1,3} + X_{2,4}}{X_{1,3} X_{2,4}}\right) \tag{27}$$

This has shown to be a unique solution due to its simplicity and structure.

4.2 Tr(ϕ^3) for 5-point

The following code results for 5 and 6 points are taken from a previous bachelor thesis done by Marc Coll Puig. [14]

A 5-point interaction as shown in Figure 4 contains three vertices and so two propagators. Due to this the amplitude should ultimately have the form of $\sum \frac{1}{X^2}$. There are 5 possible $X_{i,j}$ as discovered from the Feynman diagrams that will exist in the denominator, therefore to obtain the amplitude form just mentioned, the numerator must have combinations of order three. These combinations are again too simple and so in addition to adding the monomials, the polynomials for each $X_{i,j}$ are to be included too. Therefore the numerator ansatz shows 12 monomials and 23 polynomials:

$$\frac{1}{X_{1,3}X_{1,4}X_{2,4}X_{2,5}X_{3,5}}(c_{35}X_{1,3}^3 + c_{34}X_{1,3}^2X_{1,4} + c_{30}X_{1,3}X_{1,4}^2 + c_{20}X_{1,4}^3 + c_{33}X_{1,3}^2X_{2,4} + c_{29}X_{1,3}X_{1,4}X_{2,4}+$$

$$c_{19}X_{1,4}^2X_{2,4} + c_{26}X_{1,3}X_{2,4}^2 + c_{16}X_{1,4}X_{2,4}^2 + c_{10}X_{2,4}^3 + c_{32}X_{1,3}^2X_{2,5} + c_{28}X_{1,3}X_{1,4}X_{2,5} + c_{18}X_{1,4}^2X_{2,5}+$$

$$c_{31}X_{1,3}^2X_{3,5} + c_{27}X_{1,3}X_{1,4}X_{3,5} + c_{17} + X_{1,4}^2X_{3,5} + c_{24}X_{1,3}X_{2,4}X_{3,5} + c_{14}X_{1,4}X_{2,4}X_{3,5} + c_8 + X_{2,4}^2X_{3,5}+$$

$$c_{22}X_{1,3}X_{2,5}X_{3,5} + c_{12}X_{1,4}X_{2,5}X_{3,5} + c_6X_{2,4}X_{2,5}X_{3,5} + c_3 + X_{2,5}^2X_{3,5} + c_{21}X_{1,3}X_{3,5} + X_{3,5}^2 + c_{11}X_{1,4}X_{3,5}+$$

$$c_5X_{2,4}X_{3,5} + c_2X_{2,5}X_{3,5} + c_1X_{3,5}^2) \tag{28}$$

Once the ansatz is set, conditions to simplify it are applied. Five skinny rectangles are applied together with locality avoiding non-physical poles. To account for these non-physical poles the residue theorem explained before is utilized. As the final amplitude form is of X of order 2 in the denominator, three residues must be computed, as shown in the code in the Appendix 6.2. The resulting partial amplitude is in the form of:

$$c_{29}(\frac{X_{1,3}X_{1,4}X_{2,4} + X_{1,4}X_{2,4}X_{2,5} + X_{1,3}X_{1,4}X_{3,5} + X_{1,3}X_{2,5}X_{3,5} + X_{2,4}X_{2,5}X_{3,5}}{X_{1,3}X_{1,4}X_{2,4}X_{2,5}X_{3,5}}) \tag{29}$$

4.3 Tr(ϕ^3) for 6-point

For a 6-point interaction, as shown in the Feynman diagram, there are three propagators, making the final partial amplitude the form of $\sum \frac{1}{X^3}$. As there are 6 possible $X_{i,j}$ variables obtained from the propagators and shown in the denominator, the numerator should have combinations of variables of order 9. In this case, the amplitude is complex enough to be sufficient and simple enough with only monomial contributions in the numerator. This will also reduce the computation time needed as dealing with 84 monomials is already a complex task for the Mathematica code and takes too long.

$$\frac{1}{X_{1,3}X_{1,4}X_{1,5}X_{2,4}X_{2,5}X_{2,6}X_{3,5}X_{3,6}X_{4,6}}c_1X_{1,3}X_{1,4}X_{1,5}X_{2,4}X_{2,5}X_{2,6} + c_2X_{1,3}X_{1,4}X_{1,5}X_{2,4}X_{2,5}X_{3,5} + ...$$
$$(81).... + c_{83}X_{1,5}X_{2,5}X_{2,6}X_{3,5}X_{3,6}X_{4,6} + c_{84}X_{2,4}X_{2,5}X_{2,6}X_{3,5}X_{3,6}X_{4,6}) \tag{30}$$

The conditions of hidden zeros can then be enforced. As previously discussed, for the 6-point interaction, certain configurations derived from the kinematic mesh, namely skinny rectangles and squares, lead to the vanishing of the amplitude. Remarkably, it has been demonstrated that using only skinny rectangles or a combination of skinny rectangles and squares yields the same result. However, relying solely on squares is insufficient due to a phenomenon known as factorization, also identified by Arkani-Hamed et al. [2](more to be explained here). For computational efficiency, we consider only skinny rectangles in our approach. These conditions, coupled with the exclusion of non-physical poles via four residues, lead to the following derived result:

$$c_{84}\Big(\frac{1}{X_{1,3}X_{1,4}X_{1,5}}+\frac{1}{X_{1,4}X_{1,5}X_{2,4}}+\frac{1}{X_{1,5}X_{2,4}X_{2,5}}+\frac{1}{X_{2,4}X_{2,5}X_{2,6}}+\frac{1}{X_{1,3}X_{1,5}X_{3,5}}+\frac{1}{X_{1,5}X_{2,5}X_{3,5}}+ \frac{1}{X_{2,5}X_{2,6}X_{3,5}}+\frac{1}{X_{1,3}X_{3,5}X_{3,6}}+\frac{1}{X_{2,6}X_{3,5}X_{3,6}}+\frac{1}{X_{1,3}X_{1,4}X_{4,6}}+\frac{1}{X_{1,4}X_{2,4}X_{4,6}}+\frac{1}{X_{2,4}X_{2,6}X_{4,6}}+ \frac{1}{X_{1,3}X_{3,6}X_{4,6}}+\frac{1}{X_{2,6}X_{3,6}X_{4,6}}\Big) \tag{31}$$

4.4 Tr(ϕ^3) for 7-point

At 7-point, there are four propagators present in the interaction, and therefore the amplitude is in the form of $\frac{1}{X^4}$. As there are 14 possible $X_{i,j}$ imposed from the propagators, the numerator should have combinations of tenth-order monomials. This gives a total of 1001 sets of monomials, expressed below:

$$\frac{c_1X_{1,3}X_{1,4}X_{1,5}X_{1,6}X_{2,4}X_{2,5}X_{2,6}X_{2,7}X_{3,5}X_{3,6}+\cdots(999)\cdots+c_{1001}X_{2,4}X_{2,5}X_{2,6}X_{2,7}X_{3,5}X_{3,6}X_{4,6}X_{4,7}X_{5,7}}{X_{1,3}X_{1,4}X_{1,5}X_{1,6}X_{2,4}X_{2,5}X_{2,6}X_{2,7}X_{3,5}X_{3,6}X_{4,6}X_{4,7}X_{5,7}} \tag{32}$$

These conditions are then imposed. While both skinny rectangles and squares exist, only skinny rectangles are used, which suffice and provide a satisfactory outcome with a simpler computational approach. Incorporating the 7 skinny rectangles alongside handling non-physical poles with 5 residues delivers the desired result. Due to the substantial number of monomials and variables involved, it was anticipated that the computation might take too long. To address this, the code was submitted to a cluster utilizing the supercomputer resources provided by the University of Groningen. We thank the Center for Information Technology of the University of Groningen for their support and for providing access to the Hábrók high-performance computing cluster. The obtained result is presented below:

Supplementary Equations Document

seen to be all proportional to the coefficient c_{1001}, therefore showing its uniqueness.

4.5 NLSM for 6-point

For the Non-Linear Sigma Model as explained before can be represented through different Feynman diagrams depending on the amount of pions interacting per vertex, always in even numbers. Only the diagram with the most propagators is used, as the rest are already included in such. For 6-point interactions, there is only one propagator and so takes the form of $\frac{1}{X}$. However its amplitude is of the form of X, therefore the numerator must be or second order accommodate for this. As previously mentioned, both monomial and polynomial contributions must be incorporated into the numerator for completeness. Interestingly, it has been confirmed through computational analysis that using monomials alone yields a satisfactory and unique result. This approach simplifies the computation significantly. There are 36 monomials for each of the there possible $X_{i,j}$, therefore 108 variables to compute, together with the 9 extra polynomials not to be included. The ansatz is shown below:

$$\begin{aligned} &a_1X_{1,3} + a_2X_{1,4} + a_3X_{1,5} + ...(33) \\ +&\frac{1}{X_{1,4}}(b_1X_{1,3}X_{1,4} + b_2X_{1,3}X_{1,5} + b_3X_{1,4}X_{1,5} + ...(33)) \\ +&\frac{1}{X_{2,5}}(c_1X_{1,3}X_{1,4} + c_2X_{1,3}X_{1,5} + c_3X_{1,4}X_{1,5} + ...(33)) \\ +&\frac{1}{X_{3,6}}(d_1X_{1,3}X_{1,4} + d_2X_{1,3}X_{1,5} + d_3X_{1,4}X_{1,5} + ...(33)) \end{aligned} \tag{33}$$

To this, the conditions can be applied. The hidden zeros can be firstly applied with the same six skinny rectangles as for $\mathrm{Tr}(\phi^3)$ theory. Non-physical poles are not required for this theory, and so the final result is shown below:

$$\begin{aligned} &-\frac{1}{X_{1,4}X_{2,5}X_{3,6}}(a_9 + c_{15} + f_{30} + k_{36}) \\ &(X_{2,4}X_{2,5}X_{3,6}\left(X_{1,5} + X_{4,6}\right) + X_{1,3}X_{2,5} \\ &(X_{3,6}\left(X_{1,5} + X_{4,6}\right) + X_{1,4}\left(X_{3,5} - X_{3,6} + X_{4,6}\right)) + \\ &X_{1,4}\left(\left(X_{1,5} + X_{2,6}\right)\left(X_{2,4} + X_{3,5}\right)X_{3,6} + X_{2,5}\left(-X_{3,6}\right.\right. \\ &\left.\left.\left(X_{1,5} + X_{2,4} + X_{3,5} + X_{4,6}\right) + X_{2,6}\left(X_{3,5} - X_{3,6} + X_{4,6}\right)\right)\right)) \end{aligned} \tag{34}$$

4.6 NLSM for 8-point

The same procedure applies to 8-point interactions. In this scenario, there are two propagators, resulting in a denominator of the form $\frac{1}{X^2}$. However, the amplitude must again simplify to X necessitating a denominator of third order. Only including monomials for simplicity, there exist 1140 different monomials for each set of $X_{i,j}$ pairs obtained from the propagators, dealing in total with 3420 variables as shown below:

$$\begin{aligned} &\frac{1}{X_{1,4}X_{5,8}}(a_{36}X_{1,7}X_{2,1}X_{2,4} + \cdots 1138 \cdots + a_{1140}X_{5,7}X_{5,8}X_{6,8}) + \\ &\frac{1}{X_{1,6}X_{2,5}}(b_{36}X_{1,7}X_{2,1}X_{2,4} \cdots 1138 \cdots + b_{1140}X_{3,7}X_{4,8}X_{6,8}) + \\ &\frac{1}{X_{3,8}X_{4,7}}(c_{36}X_{1,7}X_{2,1}X_{2,4} \cdots 1138 \cdots + c_{1140}X_{5,7}X_{5,8}X_{6,8}) \end{aligned} \tag{35}$$

To this 8 skinny rectangles are applied and condition the result. Again as seen that the variables computed is so large, the supercomputer was used to perform such calculation. Unfortunately, the code was submitted to the cluster, but the results didn't arrive on time. Therefore, the outcome will be analyzed in future calculations.

All the simplifications above are not needed for low-order interactions such as 4 and 5-point interactions, however, they served as good proof that the simplifications work to aid the more complex ones that have long computation times.

The results above demonstrate that starting from a general prediction of partial amplitude and assuming Arkani-Hamed's hidden zeros hold true, a unique amplitude can be derived. This methodology applies not only to the basic case of $\mathrm{Tr}(\phi^3)$ but also extends to higher-order interactions and more complex models such as the Non-Linear Sigma Model (NLSM). This might also be as it is theorized that the Tr(ϕ^3) theory secretly contains Non-linear Sigma Model (NLSM) amplitudes to all loop orders. The NLSM amplitudes are obtained from Tr(ϕ^3) amplitudes by a unique shift of kinematic variables [15].

4.7 Further research

This topic presents numerous avenues for further exploration. As previously discussed, the Non-Linear Sigma Model (NLSM) includes an additional constraint known as the Adler zero, which can be integrated into the computational framework to validate hidden zeros accuracy. Correctly applying the hidden zeros should inherently satisfy the Adler zero, given its more specific nature.

Furthermore, a deeper investigation into the necessity of using only skinny rectangles for the computation of hidden zeros is required. This arises from an intriguing property discovered by Arkani-Hamed and collaborators, known as factorization. Analyzing this property reveals why squares in the kinematic mesh are supplementary rather than essential for the calculations.

Lastly, the methodology for identifying hidden zeros can be extended to more complex theories, starting with Yang-Mills theory, another color-ordered theory that involves gluons. Its theoretical approach is however much more complex as it deals with vector fields associated with gauge symmetry groups instead of scalar fields. Additionally, the approach can be expanded to include flavor ordering, touching upon the field of double zeros and the double copy. This expansion includes theories such as Scaffolded General Relativity (GR), Multi-DBI, and even triple zeros in the context of the Special Galileon[16][4].

5 Conclusion

The study presented in this paper highlights the significant role of hidden zeros in the computation of scattering amplitudes within Tr(ϕ^3) and Non-Linear Sigma Model (NLSM) theories. We have shown that by assuming and imposing the conditions of hidden zeros on general amplitude frameworks, it is possible to derive unique partial amplitudes that simplify the traditionally complex calculations associated with particle interactions.
The analysis of different n-point interactions for both Tr(ϕ^3) and NLSM theories demonstrates that these hidden zeros not only exist but also provide a practical means of achieving more accurate predictions of scattering processes. The results affirm the theoretical predictions made by Arkani-Hamed and extend their applicability, showing that these zeros can indeed lead to simplified and symmetrical sets of terms in the scattering amplitudes.
The study revealed unexpected findings that need further analysis in the future. The condition of skinny rectangles was sufficient to simplify the amplitude, likely due to factorization, and only monomial configurations on the general amplitude prediction were required.
The research highlights the importance of hidden zeros as a valuable tool in quantum field theory, offering a new perspective on the mathematical structures underlying particle interactions. The findings suggest that further exploration of hidden zeros in other field theories, such as Yang-Mills theory and double copy theories, could provide additional insights and advancements in the field.

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6 Appendix

6.1 Extra information on NLSM

The NLSM is built upon the concept of spontaneous symmetry breaking in a Lie group G. When a symmetry group G×G is spontaneously broken down to its diagonal subgroup G, massless Goldstone bosons emerge. In this case, the pion meson is studied. The NLSM describes how scalar fields map points from Euclidean space-time to a more complex manifold with curved Riemann metric. This theory allows for the study of fields that live of complex geometric which represent physical systems better than the linear models. The fields studied for this research are for pion particles which arise from spontaneous breaking of chiral symmetry. Its Lagrangian takes the form shown in equation 17. Where f_π^2 is the pion decay constant. The following partial derivatives show the change in time of the unitary matrix U and its hermitian conjugate $U^\dagger$ described by Chiral perturbation theory. These matrices parametrize the space of $SU(N)_{N_f}$, N_f denoting the number of light quarks involved. The U field can be parametrized in order to simplify it as an exponential:

$$U = exp(i\frac{\phi}{F}) \tag{36}$$

Where $\phi = \phi^a t^a$ is the matrix of the pion fields and t^a are the generators of the Lie group G. This exponential can then be further expanded as a power series.[11]

The NLSM already has its own vanishing amplitude loci due to a phenomenon called the Adler zero. This describes a property where scattering amplitudes vanish as the field momentum tends to zero. These are not studied further in this research. This is because Arkani-Hamed's hidden zeros give a more general vanishing loci theory, and so if these hidden zeros hold, the Adler zero must too(since setting $s_{i,j}$ to zero implies p_i is zero). [19]

6.2 Mathematica code for each interaction

5-point Tr(Φ^3)

Set of possible $X_{i,j}$:

In[93]:=

```
Xset5 = {X1,3, X1,4, X2,4, X2,5, X3,5};
```

$$\text{Xset5} = \{X_{1,3}, X_{1,4}, X_{2,4}, X_{2,5}, X_{3,5}\};$$

Possible combinations:

In[94]:=

$$\text{cubmono} = \{X_{3,5}^3, X_{2,5} X_{3,5}^2, X_{2,5}^2 X_{3,5}, X_{2,5}^3, X_{2,4} X_{3,5}^2, X_{2,4} X_{2,5} X_{3,5}, X_{2,4} X_{2,5}^2, X_{2,4}^2 X_{3,5}, X_{2,4}^2 X_{2,5}, X_{2,4}^3, X_{1,4} X_{3,5}^2, X_{1,4} X_{2,5} X_{3,5}, X_{1,4} X_{2,5}^2, X_{1,4} X_{2,4} X_{3,5}, X_{1,4} X_{2,4} X_{2,5}, X_{1,4} X_{2,4}^2, X_{1,4}^2 X_{3,5}, X_{1,4}^2 X_{2,5}, X_{1,4}^2 X_{2,4}, X_{1,4}^3, X_{1,3} X_{3,5}^2, X_{1,3} X_{2,5} X_{3,5}, X_{1,3} X_{2,5}^2, X_{1,3} X_{2,4} X_{3,5}, X_{1,3} X_{2,4} X_{2,5}, X_{1,3} X_{2,4}^2, X_{1,3} X_{1,4} X_{3,5}, X_{1,3} X_{1,4} X_{2,5}, X_{1,3} X_{1,4} X_{2,4}, X_{1,3} X_{1,4}^2, X_{1,3}^2 X_{3,5}, X_{1,3}^2 X_{2,5}, X_{1,3}^2 X_{2,4}, X_{1,3}^2 X_{1,4}, X_{1,3}^3\};$$

Ansatz made from the combination of monomials and polynomials with a constant c_i at ech term to describe the weight:

In[95]:=

$$\text{ansatz} = \frac{\text{Sum}[c_i\,\text{cubmono}[\![i]\!], \{i, 1, \text{Length@cubmono}\}]}{\text{Times @@ Xset5}}$$

Out[95]=

$$\frac{1}{X_{1,3} X_{1,4} X_{2,4} X_{2,5} X_{3,5}} \left(c_{35} X_{1,3}^3 + c_{34} X_{1,3}^2 X_{1,4} + c_{30} X_{1,3} X_{1,4}^2 + c_{20} X_{1,4}^3 + c_{33} X_{1,3}^2 X_{2,4} + c_{29} X_{1,3} X_{1,4} X_{2,4} + c_{19} X_{1,4}^2 X_{2,4} + c_{26} X_{1,3} X_{2,4}^2 + c_{16} X_{1,4} X_{2,4}^2 + c_{10} X_{2,4}^3 + c_{32} X_{1,3}^2 X_{2,5} + c_{28} X_{1,3} X_{1,4} X_{2,5} + c_{18} X_{1,4}^2 X_{2,5} + c_{25} X_{1,3} X_{2,4} X_{2,5} + c_{15} X_{1,4} X_{2,4} X_{2,5} + c_9 X_{2,4}^2 X_{2,5} + c_{23} X_{1,3} X_{2,5}^2 + c_{13} X_{1,4} X_{2,5}^2 + c_7 X_{2,4} X_{2,5}^2 + c_4 X_{2,5}^3 + c_{31} X_{1,3}^2 X_{3,5} + c_{27} X_{1,3} X_{1,4} X_{3,5} + c_{17} X_{1,4}^2 X_{3,5} + c_{24} X_{1,3} X_{2,4} X_{3,5} + c_{14} X_{1,4} X_{2,4} X_{3,5} + c_8 X_{2,4}^2 X_{3,5} + c_{22} X_{1,3} X_{2,5} X_{3,5} + c_{12} X_{1,4} X_{2,5} X_{3,5} + c_6 X_{2,4} X_{2,5} X_{3,5} + c_3 X_{2,5}^2 X_{3,5} + c_{21} X_{1,3} X_{3,5}^2 + c_{11} X_{1,4} X_{3,5}^2 + c_5 X_{2,4} X_{3,5}^2 + c_2 X_{2,5} X_{3,5}^2 + c_1 X_{3,5}^3\right)$$

Describe the skinny rectangles available and show how these condition the $X_{i,j}$'s:

In[140]:=

$$s13 = X_{1,3} + X_{2,4} - X_{1,4};$$
$$s14 = X_{2,5} + X_{1,4} - X_{2,4};$$
$$s24 = X_{3,5} + X_{2,4} - X_{2,5};$$
$$s25 = X_{2,5} + X_{1,3} - X_{3,5};$$
$$s35 = X_{3,5} + X_{1,4} - X_{1,3};$$

```
locus0 = Solve[0 == {s13, s14}][[1]];
locus1 = Solve[0 == {s24, s25}][[1]];
locus2 = Solve[0 == {s13, s35}][[1]];
locus3 = Solve[0 == {s14, s24}][[1]];
locus4 = Solve[0 == {s25, s35}][[1]];
Totloc = {locus0, locus1, locus2, locus3, locus4}
```

Out[150]=

$$\{\{X_{2,4} \to -X_{1,3} + X_{1,4}, X_{2,5} \to -X_{1,3}\}, \{X_{2,4} \to -X_{1,3}, X_{3,5} \to X_{1,3} + X_{2,5}\}, \{X_{2,4} \to -X_{1,3} + X_{1,4}, X_{3,5} \to X_{1,3} - X_{1,4}\}, \{X_{2,5} \to -X_{1,4} + X_{2,4}, X_{3,5} \to -X_{1,4}\}, \{X_{2,5} \to -X_{1,4}, X_{3,5} \to X_{1,3} - X_{1,4}\}\}$$

Apply the above conditions on the ansatz to reduce the c_i terms:

In[151]:=

```
Loc = SolveAlways[0 == ansatz /. Totloc, Xset5][[1]]
```

Out[151]=

$\{c_1 \to 0,\ c_2 \to c_{24} - c_{34},\ c_3 \to -c_{24} + c_{34},\ c_4 \to 0,\ c_{10} \to 0,\ c_7 \to -c_{24} + c_{28} - 2\,c_{33},$
$c_9 \to c_{24} - c_{28} + 2\,c_{33},\ c_5 \to -c_{24} - c_{33},\ c_{11} \to c_{24} + c_{33},\ c_{13} \to -c_{28} + 2\,c_{33} + c_{34},$
$c_{12} \to -c_{24} + c_{28} - 3\,c_{33} - c_{34},\ c_8 \to -c_{24} - c_{33} + c_{34},\ c_{16} \to -c_{28} + c_{33} + c_{34},$
$c_6 \to 3\,c_{24} + c_{29} + 3\,c_{33} - 3\,c_{34},\ c_{15} \to 3\,c_{28} + c_{29} - 3\,c_{33} - 3\,c_{34},\ c_{14} \to c_{24} - c_{28} + 2\,c_{33} - c_{34},$
$c_{17} \to c_{28} - c_{33},\ c_{18} \to -c_{28} + c_{33},\ c_{19} \to c_{28} - c_{33} - c_{34},\ c_{20} \to 0,\ c_{21} \to -c_{33} - c_{34},$
$c_{23} \to c_{28} - 2\,c_{33} - c_{34},\ c_{22} \to c_{29} + 3\,c_{33},\ c_{26} \to c_{24} + c_{33} - c_{34},\ c_{25} \to -c_{24} - c_{28} + c_{33} + 2\,c_{34},$
$c_{27} \to c_{29} - 3\,c_{34},\ c_{30} \to -c_{34},\ c_{31} \to c_{33} + c_{34},\ c_{32} \to -c_{33},\ c_{35} \to 0\}$

Use the above reduction to simplify the ansatz:

In[152]:=

```
ans2 = ansatz /. Loc
```

Out[152]=

$$\frac{1}{X_{1,3}\,X_{1,4}\,X_{2,4}\,X_{2,5}\,X_{3,5}}\Big(c_{34}\,X_{1,3}^2\,X_{1,4} - c_{34}\,X_{1,3}\,X_{1,4}^2 + c_{33}\,X_{1,3}^2\,X_{2,4} + c_{29}\,X_{1,3}\,X_{1,4}\,X_{2,4} + (c_{28} - c_{33} - c_{34})\,X_{1,4}^2\,X_{2,4} +$$
$$(c_{24} + c_{33} - c_{34})\,X_{1,3}\,X_{2,4}^2 + (-c_{28} + c_{33} + c_{34})\,X_{1,4}\,X_{2,4}^2 - c_{33}\,X_{1,3}^2\,X_{2,5} +$$
$$c_{28}\,X_{1,3}\,X_{1,4}\,X_{2,5} + (-c_{28} + c_{33})\,X_{1,4}^2\,X_{2,5} + (-c_{24} - c_{28} + c_{33} + 2\,c_{34})\,X_{1,3}\,X_{2,4}\,X_{2,5} +$$
$$(3\,c_{28} + c_{29} - 3\,c_{33} - 3\,c_{34})\,X_{1,4}\,X_{2,4}\,X_{2,5} + (c_{24} - c_{28} + 2\,c_{33})\,X_{2,4}^2\,X_{2,5} +$$
$$(c_{28} - 2\,c_{33} - c_{34})\,X_{1,3}\,X_{2,5}^2 + (-c_{28} + 2\,c_{33} + c_{34})\,X_{1,4}\,X_{2,5}^2 + (-c_{24} + c_{28} - 2\,c_{33})\,X_{2,4}\,X_{2,5}^2 +$$
$$(c_{33} + c_{34})\,X_{1,3}^2\,X_{3,5} + (c_{29} - 3\,c_{34})\,X_{1,3}\,X_{1,4}\,X_{3,5} + (c_{28} - c_{33})\,X_{1,4}^2\,X_{3,5} +$$
$$c_{24}\,X_{1,3}\,X_{2,4}\,X_{3,5} + (c_{24} - c_{28} + 2\,c_{33} - c_{34})\,X_{1,4}\,X_{2,4}\,X_{3,5} + (-c_{24} - c_{33} + c_{34})\,X_{2,4}^2\,X_{3,5} +$$
$$(c_{29} + 3\,c_{33})\,X_{1,3}\,X_{2,5}\,X_{3,5} + (-c_{24} + c_{28} - 3\,c_{33} - c_{34})\,X_{1,4}\,X_{2,5}\,X_{3,5} +$$
$$(3\,c_{24} + c_{29} + 3\,c_{33} - 3\,c_{34})\,X_{2,4}\,X_{2,5}\,X_{3,5} + (-c_{24} + c_{34})\,X_{2,5}^2\,X_{3,5} + (-c_{33} - c_{34})\,X_{1,3}\,X_{3,5}^2 +$$
$$(c_{24} + c_{33})\,X_{1,4}\,X_{3,5}^2 + (-c_{24} - c_{33})\,X_{2,4}\,X_{3,5}^2 + (c_{24} - c_{34})\,X_{2,5}\,X_{3,5}^2\Big)$$

Apply the Non-physical poles Condition:

In[153]:=

```
trip = Subsets[Xset5, {4}]
trires[X1_, X2_, X3_] := Residue[Residue[Residue[ans2, {X1, 0}], {X2, 0}], {X3, 0}];
poletab = Table[trires[trip[[i, 1]], trip[[i, 2]], trip[[i, 3]]], {i, 1, Length@trip}];
molpole = SolveAlways[poletab == 0, Xset5]
```

Out[153]=

$\{\{X_{1,3}, X_{1,4}, X_{2,4}, X_{2,5}\}, \{X_{1,3}, X_{1,4}, X_{2,4}, X_{3,5}\},$
$\{X_{1,3}, X_{1,4}, X_{2,5}, X_{3,5}\}, \{X_{1,3}, X_{2,4}, X_{2,5}, X_{3,5}\}, \{X_{1,4}, X_{2,4}, X_{2,5}, X_{3,5}\}\}$

Out[156]=

$\{\{c_{24} \to 0,\ c_{33} \to 0,\ c_{28} \to 0,\ c_{34} \to 0\}\}$

Result:

In[157]:=

```
res = ans2 /. molpole
```

Out[157]=

$\{(c_{29}\,X_{1,3}\,X_{1,4}\,X_{2,4} + c_{29}\,X_{1,4}\,X_{2,4}\,X_{2,5} + c_{29}\,X_{1,3}\,X_{1,4}\,X_{3,5} + c_{29}\,X_{1,3}\,X_{2,5}\,X_{3,5} + c_{29}\,X_{2,4}\,X_{2,5}\,X_{3,5})\ /$
$(X_{1,3}\,X_{1,4}\,X_{2,4}\,X_{2,5}\,X_{3,5})\}$

6-point Tr(Φ^3)

Set of possible $X_{i,j}$:

In[ ]:=
```
Xset9 = {X_{1,3}, X_{1,4}, X_{1,5}, X_{2,4}, X_{2,5}, X_{2,6}, X_{3,5}, X_{3,6}, X_{4,6}}
```

Out[ ]=

$\{X_{1,3}, X_{1,4}, X_{1,5}, X_{2,4}, X_{2,5}, X_{2,6}, X_{3,5}, X_{3,6}, X_{4,6}\}$

Possible combinations:

In[ ]:=
```
sixterm = Subsets[Xset9, {6}];
Length[sixterm]
```

Out[ ]=

84

Ansatz made from the combination of monomials with a constant c_i at ech term to describe the weight:

In[ ]:=
```
ansatz = Sum[c_i Times @@ sixterm[[i]], {i, 1, Length@sixterm}] / (Times @@ Xset9);
```

Describe the skinny rectangles and squares available and show how these condition the $X_{i,j}$'s:

In[ ]:=
```
s13 = X_{1,3} + X_{2,4} - X_{1,4} ;
s14 = X_{2,5} + X_{1,4} - X_{1,5} - X_{2,4} ;
s15 = X_{1,5} + X_{2,6} - X_{2,5} ;
s24 = X_{3,5} + X_{2,4} - X_{2,5} ;
s25 = X_{2,5} + X_{3,6} - X_{3,5} - X_{2,6} ;
s26 = X_{1,3} + X_{2,6} - X_{3,6} ;
s35 = X_{3,5} + X_{4,6} - X_{3,6} ;
s36 = X_{1,4} + X_{3,6} - X_{1,3} - X_{4,6} ;
s46 = X_{1,5} + X_{4,6} - X_{1,4} ;
```

In[ ]:=
```
(*s_1,3=s_1,4=s_1,5=0*)
locus0 =  Solve[0=={s13,s14,s15}][[1]];
(*s_2,4=s_2,5=s_2,6=0*)
locus1 = Solve[0=={s24,s25,s26}][[1]];
(*s_3,5=s_3,6=s_1,3=0*)
locus2 = Solve[0=={s35,s36,s13}][[1]];
(*s_4,6=s_1,4=s_2,4=0*)
locus3 = Solve[0=={s46,s14,s24}][[1]];
(*s_1,5=s_2,5=s_3,5=0*)
locus4 = Solve[0=={s15,s25,s35}][[1]];
(*s_2,6=s_3,6=s_4,6=0*)
locus5 = Solve[0=={s26,s36,s46}][[1]];
(*s_1,3=s_3,6=s_4,6=s,_1,4=0*)
locus6 = Solve[0=={s13,s36,s46,s14}][[1]];
(*s_2,4=s_1,4=s_1,5=s_2,50*)
locus7 = Solve[0=={s24,s14,s15,s25}][[1]];
(*s_3,5=s_2,5=s_2,6=s_3,6=0*)
locus8 = Solve[0=={s35,s25,s26,s36}][[1]];

Totlocu = {locus0,locus1,locus2,locus3,locus4,locus5,locus6,locus7,locus8};
```

Apply the above conditions on the ansatz to reduce the c_i terms:

In[ ]:=
```
Locs = SolveAlways[0 == ansatz /. Totlocu, Xset9][[1]]
```

Out[ ]=

$\{c_1 \to 0, c_8 \to 0, c_9 \to 0, c_{15} \to 0, c_{19} \to 0, c_{21} \to 0, c_{22} \to 0, c_{23} \to 0, c_{24} \to 0, c_{25} \to 0, c_{26} \to 0, c_{27} \to 0, c_{28} \to 0, c_3 \to 0, c_4 \to c_{84}, c_6 \to 0, c_7 \to 0, c_{29} \to c_{84}, c_{10} \to c_{84}, c_2 \to c_{84}, c_5 \to 0, c_{30} \to 0, c_{31} \to 0, c_{32} \to 0, c_{12} \to 0, c_{13} \to 0, c_{33} \to 0, c_{16} \to 0, c_{11} \to 0, c_{34} \to 0, c_{17} \to 0, c_{20} \to c_{84}, c_{35} \to c_{84}, c_{14} \to c_{84}, c_{18} \to 0, c_{36} \to 0, c_{37} \to 0, c_{38} \to 0, c_{39} \to 0, c_{40} \to 0, c_{41} \to 0, c_{42} \to 0, c_{43} \to 0, c_{44} \to 0, c_{45} \to 0, c_{46} \to c_{84}, c_{47} \to 0, c_{48} \to 0, c_{49} \to 0, c_{50} \to 0, c_{51} \to 0, c_{52} \to 0, c_{53} \to 0, c_{54} \to 0, c_{55} \to 0, c_{56} \to c_{84}, c_{57} \to c_{84}, c_{58} \to 0, c_{59} \to c_{84}, c_{60} \to 0, c_{61} \to 0, c_{62} \to 0, c_{63} \to 0, c_{64} \to 0, c_{65} \to 0, c_{66} \to 0, c_{67} \to 0, c_{68} \to 0, c_{69} \to 0, c_{70} \to 0, c_{71} \to 0, c_{72} \to 0, c_{73} \to 0, c_{74} \to c_{84}, c_{75} \to 0, c_{76} \to 0, c_{77} \to 0, c_{78} \to c_{84}, c_{79} \to 0, c_{80} \to 0, c_{81} \to 0, c_{82} \to 0, c_{83} \to 0\}$

Use the above reduction to simplify the ansatz:

In[ ]:=
```
anss = ansatz /. Locs
```

Out[ ]=

$$(c_{84} X_{1,3} X_{1,4} X_{1,5} X_{2,4} X_{2,5} X_{3,5} + c_{84} X_{1,4} X_{1,5} X_{2,4} X_{2,5} X_{2,6} X_{3,5} + c_{84} X_{1,3} X_{1,4} X_{1,5} X_{2,5} X_{3,5} X_{3,6} + c_{84} X_{1,3} X_{1,5} X_{2,5} X_{2,6} X_{3,5} X_{3,6} + c_{84} X_{1,5} X_{2,4} X_{2,5} X_{2,6} X_{3,5} X_{3,6} + c_{84} X_{1,3} X_{1,4} X_{1,5} X_{2,4} X_{2,5} X_{4,6} + c_{84} X_{1,4} X_{1,5} X_{2,4} X_{2,5} X_{2,6} X_{4,6} + c_{84} X_{1,3} X_{1,4} X_{1,5} X_{2,4} X_{3,6} X_{4,6} + c_{84} X_{1,3} X_{1,4} X_{2,4} X_{2,6} X_{3,6} X_{4,6} + c_{84} X_{1,4} X_{2,4} X_{2,5} X_{2,6} X_{3,6} X_{4,6} + c_{84} X_{1,3} X_{1,4} X_{1,5} X_{3,5} X_{3,6} X_{4,6} + c_{84} X_{1,3} X_{1,4} X_{2,6} X_{3,5} X_{3,6} X_{4,6} + c_{84} X_{1,3} X_{2,5} X_{2,6} X_{3,5} X_{3,6} X_{4,6} + c_{84} X_{2,4} X_{2,5} X_{2,6} X_{3,5} X_{3,6} X_{4,6}) / (X_{1,3} X_{1,4} X_{1,5} X_{2,4} X_{2,5} X_{2,6} X_{3,5} X_{3,6} X_{4,6})$$

Apply the Non-physical poles Condition:

In[ ]:=
```
triress[X1_, X2_, X3_, X4_] :=
  Residue[Residue[Residue[Residue[anss, {X1, 0}], {X2, 0}], {X3, 0}], {X4, 0}];
poletabi =
  Table[triress[sixt[[i, 1]], sixt[[i, 2]], sixt[[i, 3]], sixt[[i, 4]]], {i, 1, Length@sixt}];
olpole = SolveAlways[poletabi == 0, Xset9][[1]]
```

Out[ ]=

{}

Result:

In[ ]:=
```
res = anss /. olpole // FullSimplify // Expand
```

Out[ ]=

$$\frac{c_{84}}{X_{1,3} X_{1,4} X_{1,5}} + \frac{c_{84}}{X_{1,4} X_{1,5} X_{2,4}} + \frac{c_{84}}{X_{1,5} X_{2,4} X_{2,5}} + \frac{c_{84}}{X_{2,4} X_{2,5} X_{2,6}} + \frac{c_{84}}{X_{1,3} X_{1,5} X_{3,5}} + \frac{c_{84}}{X_{1,5} X_{2,5} X_{3,5}} + \frac{c_{84}}{X_{2,5} X_{2,6} X_{3,5}} + \frac{c_{84}}{X_{1,3} X_{3,5} X_{3,6}} + \frac{c_{84}}{X_{2,6} X_{3,5} X_{3,6}} + \frac{c_{84}}{X_{1,3} X_{1,4} X_{4,6}} + \frac{c_{84}}{X_{1,4} X_{2,4} X_{4,6}} + \frac{c_{84}}{X_{2,4} X_{2,6} X_{4,6}} + \frac{c_{84}}{X_{1,3} X_{3,6} X_{4,6}} + \frac{c_{84}}{X_{2,6} X_{3,6} X_{4,6}}$$

7-point $\mathrm{Tr}(\Phi^3)$

Set of possible $X_{i,j}$:

```
In[ ]:= Xset7 = {X1,3, X1,4, X1,5, X1,6, X2,4, X2,5, X2,6, X2,7, X3,5, X3,6, X3,7, X4,6, X4,7, X5,7};
        Length[Xset7]
Out[ ]=
        14
```

Possible combinations:

```
In[ ]:= seventerm = Subsets[Xset7, {10}]
        Length[seventerm]
Out[ ]=
        {{X1,3, X1,4, X1,5, X1,6, X2,4, X2,5, X2,6, X2,7, X3,5, X3,6},
         {X1,3, X1,4, X1,5, X1,6, X2,4, X2,5, X2,6, X2,7, X3,5, X3,7},
         {X1,3, X1,4, X1,5, X1,6, X2,4, X2,5, X2,6, X2,7, X3,5, X4,6},
         {X1,3, X1,4, X1,5, X1,6, X2,4, X2,5, X2,6, X2,7, X3,5, X4,7},
         … 993 …, {X1,6, X2,4, X2,5, X2,7, X3,5, X3,6, X3,7, X4,6, X4,7, X5,7},
         {X1,6, X2,4, X2,6, X2,7, X3,5, X3,6, X3,7, X4,6, X4,7, X5,7},
         {X1,6, X2,5, X2,6, X2,7, X3,5, X3,6, X3,7, X4,6, X4,7, X5,7},
         {X2,4, X2,5, X2,6, X2,7, X3,5, X3,6, X3,7, X4,6, X4,7, X5,7}}
        Full expression not available (original memory size: 1.1 MB)
Out[ ]=
        1001
```

Ansatz made from the combination of monomials with a constant c_i at ech term to describe the weight:

```
In[ ]:= ansatz = Sum[ci Times @@ seventerm[[i]], {i, 1, Length@seventerm}] / Times @@ Xset7
Out[ ]=
        (c1 X1,3 X1,4 X1,5 X1,6 X2,4 X2,5 X2,6 X2,7 X3,5 X3,6 + c2 X1,3 X1,4 X1,5 X1,6 X2,4 X2,5 X2,6 X2,7 X3,5 X3,7 +
           … 998 … + c1001 X2,4 X2,5 X2,6 X2,7 X3,5 X3,6 X3,7 X4,6 X4,7 X5,7) /
         (X1,3 X1,4 X1,5 X1,6 X2,4 X2,5 X2,6 X2,7 X3,5 X3,6 X3,7 X4,6 X4,7 X5,7)
        Full expression not available (original memory size: 1.2 MB)
```

Extra condition to ensure $X_{j,i} = X_{i,j}$:

```
In[ ]:= replX = Table[Table[Xj,i → Xi,j, {i, 1, j - 1}], {j, 1, 7}] // Flatten;
```

Describe the skinny rectangles available and show how these condition the $X_{i,j}$'s:

In[ ]:=
```
s13 = X1,3 + X2,4 - X1,4 ;
s14 = X2,5 + X1,4 - X1,5 - X2,4 ;
s15 = X1,5 + X2,6 - X1,6 - X2,5 ;
s16 = X1,6 + X2,7 - X2,6;
s24 = X3,5 + X2,4 - X2,5 ;
s25 = X2,5 + X3,6 - X3,5 - X2,6 ;
s26 = X2,6 + X3,7 - X3,6 - X2,7;
s27 = X2,7 + X1,3 - X3,7;
s35 = X3,5 + X4,6 - X3,6 ;
s36 = X4,7 + X3,6 - X3,7 - X4,6 ;
s37 = X3,7 + X4,1 - X4,7 - X1,3;
s46 = X4,6 + X5,7 - X4,7 ;
s47 = X4,7 + X1,5 - X5,7 - X1,4;
s57 = X5,7 + X1,6 - X1,5;
locus0 = Solve[0 == {s13, s14, s15, s16}][[1]];
locus1 = Solve[0 == {s24, s25, s26, s27}][[1]];
locus2 = Solve[0 == {s35, s36, s37, s13}][[1]];
locus3 = Solve[0 == {s46, s47, s14, s24}][[1]];
locus4 = Solve[0 == {s57, s15, s25, s35}][[1]];
locus5 = Solve[0 == {s16, s26, s36, s46}][[1]];
locus6 = Solve[0 == {s27, s37, s47, s57}][[1]];
Totloc = {locus0, locus1, locus2, locus3, locus4 , locus5, locus6} /. replX;
```

Apply the above conditions on the ansatz to reduce the c_i terms:

In[ ]:=
```
Loc = SolveAlways[0 == ansatz /. Totloc, Xset7][[1]]
```

ReplaceAll: {Totloc} is neither a list of replacement rules nor a valid dispatch table, and so cannot be used for replacing.

SolveAlways: ! (0 == ansatz /. Totloc) is not a well-formed equation.

Out[ ]=
```
0 == ansatz /. Totloc
```

In[ ]:=
```
ans2 = ansatz /. Loc
```

ReplaceAll: {{}[[1]]} is neither a list of replacement rules nor a valid dispatch table, and so cannot be used for replacing.

Out[ ]=

$$\left(c_1 X_{1,3} X_{1,4} X_{1,5} X_{1,6} X_{2,4} X_{2,5} X_{2,6} X_{2,7} X_{3,5} X_{3,6} + c_2 X_{1,3} X_{1,4} X_{1,5} X_{1,6} X_{2,4} X_{2,5} X_{2,6} X_{2,7} X_{3,5} X_{3,7} + \ldots 997 \ldots + c_{1000} X_{1,6} X_{2,5} X_{2,6} X_{2,7} X_{3,5} X_{3,6} X_{3,7} X_{4,6} X_{4,7} X_{5,7} + c_{1001} X_{2,4} X_{2,5} X_{2,6} X_{2,7} X_{3,5} X_{3,6} X_{3,7} X_{4,6} X_{4,7} X_{5,7}\right) / \left(X_{1,3} X_{1,4} X_{1,5} X_{1,6} X_{2,4} X_{2,5} X_{2,6} X_{2,7} X_{3,5} X_{3,6} X_{3,7} X_{4,6} X_{4,7} X_{5,7}\right) /. \{\}[[1]]$$

Full expression not available (original memory size: 1.2 MB)

Set of possible monomials of order 5, as the denominator has to be of maximum order four:

In[ ]:=
```
trip = Subsets[Xset7, {5}]
```
Out[ ]=

{{$X_{1,3}$, $X_{1,4}$, $X_{1,5}$, $X_{1,6}$, $X_{2,4}$}, {$X_{1,3}$, $X_{1,4}$, $X_{1,5}$, $X_{1,6}$, $X_{2,5}$},
{$X_{1,3}$, $X_{1,4}$, $X_{1,5}$, $X_{1,6}$, $X_{2,6}$}, {$X_{1,3}$, $X_{1,4}$, $X_{1,5}$, $X_{1,6}$, $X_{2,7}$},
{$X_{1,3}$, $X_{1,4}$, $X_{1,5}$, $X_{1,6}$, $X_{3,5}$}, {$X_{1,3}$, $X_{1,4}$, $X_{1,5}$, $X_{1,6}$, $X_{3,6}$}, {$X_{1,3}$, $X_{1,4}$, $X_{1,5}$, $X_{1,6}$, $X_{3,7}$},
{$X_{1,3}$, $X_{1,4}$, $X_{1,5}$, $X_{1,6}$, $X_{4,6}$}, {$X_{1,3}$, $X_{1,4}$, $X_{1,5}$, $X_{1,6}$, $X_{4,7}$}, ⋯ 1984 ⋯,
{$X_{2,7}$, $X_{3,6}$, $X_{3,7}$, $X_{4,7}$, $X_{5,7}$}, {$X_{2,7}$, $X_{3,6}$, $X_{4,6}$, $X_{4,7}$, $X_{5,7}$}, {$X_{2,7}$, $X_{3,7}$, $X_{4,6}$, $X_{4,7}$, $X_{5,7}$},
{$X_{3,5}$, $X_{3,6}$, $X_{3,7}$, $X_{4,6}$, $X_{4,7}$}, {$X_{3,5}$, $X_{3,6}$, $X_{3,7}$, $X_{4,6}$, $X_{5,7}$}, {$X_{3,5}$, $X_{3,6}$, $X_{3,7}$, $X_{4,7}$, $X_{5,7}$},
{$X_{3,5}$, $X_{3,6}$, $X_{4,6}$, $X_{4,7}$, $X_{5,7}$}, {$X_{3,5}$, $X_{3,7}$, $X_{4,6}$, $X_{4,7}$, $X_{5,7}$}, {$X_{3,6}$, $X_{3,7}$, $X_{4,6}$, $X_{4,7}$, $X_{5,7}$}}

Full expression not available (original memory size: 1.1 MB)

Apply Non-physical poles:

In[ ]:=
```
trires[X1_, X2_, X3_, X4_, X5_ ] := Residue[Residue[
    Residue[Residue[Residue[ans2, {X1, 0}], {X2, 0}], {X3, 0}], {X4, 0}], {X5, 0}];
poletab = Table[trires[trip[[i, 1]], trip[[i, 2]],
    trip[[i, 3]], trip[[i, 4]], trip[[i, 5]]], {i, 1, Length@trip}];
molpole = SolveAlways[poletab == 0, Xset7]
```
Out[ ]=

{{}}

6-point NLSM

Set of possible $X_{i,j}$:

In[ ]:=
```
Xset9 = {X1,3, X1,4, X1,5, X2,4, X2,5, X2,6, X3,5, X3,6, X4,6}
```

Out[ ]=

$\{X_{1,3}, X_{1,4}, X_{1,5}, X_{2,4}, X_{2,5}, X_{2,6}, X_{3,5}, X_{3,6}, X_{4,6}\}$

Extra set to help the code:

In[ ]:=
```
Sett = Subsets[Xset9, {1}]
```

Out[ ]=

$\{\{X_{1,3}\}, \{X_{1,4}\}, \{X_{1,5}\}, \{X_{2,4}\}, \{X_{2,5}\}, \{X_{2,6}\}, \{X_{3,5}\}, \{X_{3,6}\}, \{X_{4,6}\}\}$

Possible combinations:

In[ ]:=
```
sixterm = Subsets[Xset9, {2}]
Length[sixterm]
```

Out[ ]=

$\{\{X_{1,3}, X_{1,4}\}, \{X_{1,3}, X_{1,5}\}, \{X_{1,3}, X_{2,4}\}, \{X_{1,3}, X_{2,5}\}, \{X_{1,3}, X_{2,6}\}, \{X_{1,3}, X_{3,5}\},$
$\{X_{1,3}, X_{3,6}\}, \{X_{1,3}, X_{4,6}\}, \{X_{1,4}, X_{1,5}\}, \{X_{1,4}, X_{2,4}\}, \{X_{1,4}, X_{2,5}\}, \{X_{1,4}, X_{2,6}\},$
$\{X_{1,4}, X_{3,5}\}, \{X_{1,4}, X_{3,6}\}, \{X_{1,4}, X_{4,6}\}, \{X_{1,5}, X_{2,4}\}, \{X_{1,5}, X_{2,5}\}, \{X_{1,5}, X_{2,6}\},$
$\{X_{1,5}, X_{3,5}\}, \{X_{1,5}, X_{3,6}\}, \{X_{1,5}, X_{4,6}\}, \{X_{2,4}, X_{2,5}\}, \{X_{2,4}, X_{2,6}\}, \{X_{2,4}, X_{3,5}\},$
$\{X_{2,4}, X_{3,6}\}, \{X_{2,4}, X_{4,6}\}, \{X_{2,5}, X_{2,6}\}, \{X_{2,5}, X_{3,5}\}, \{X_{2,5}, X_{3,6}\}, \{X_{2,5}, X_{4,6}\},$
$\{X_{2,6}, X_{3,5}\}, \{X_{2,6}, X_{3,6}\}, \{X_{2,6}, X_{4,6}\}, \{X_{3,5}, X_{3,6}\}, \{X_{3,5}, X_{4,6}\}, \{X_{3,6}, X_{4,6}\}\}$

Out[ ]=

36

Extra condition to ensure $X_{j,i} = X_{i,j}$ and that only possible $X_{i,j}$ are taken into account:

In[ ]:=
```
replX = Table[Table[Xj,i → Xi,j, {i, 1, j - 1}], {j, 1, 6}] // Flatten
replX1 = Table[Table[Xi,j → XMod[i,6]+1,Mod[j,6]+1, {i, 1, j - 1}], {j, 1, 6}] // Flatten
```

Out[ ]=

$\{X_{2,1} \to X_{1,2}, X_{3,1} \to X_{1,3}, X_{3,2} \to X_{2,3}, X_{4,1} \to X_{1,4}, X_{4,2} \to X_{2,4}, X_{4,3} \to X_{3,4}, X_{5,1} \to X_{1,5}, X_{5,2} \to X_{2,5},$
$X_{5,3} \to X_{3,5}, X_{5,4} \to X_{4,5}, X_{6,1} \to X_{1,6}, X_{6,2} \to X_{2,6}, X_{6,3} \to X_{3,6}, X_{6,4} \to X_{4,6}, X_{6,5} \to X_{5,6}\}$

Out[ ]=

$\{X_{1,2} \to X_{2,3}, X_{1,3} \to X_{2,4}, X_{2,3} \to X_{3,4}, X_{1,4} \to X_{2,5}, X_{2,4} \to X_{3,5}, X_{3,4} \to X_{4,5}, X_{1,5} \to X_{2,6}, X_{2,5} \to X_{3,6},$
$X_{3,5} \to X_{4,6}, X_{4,5} \to X_{5,6}, X_{1,6} \to X_{2,1}, X_{2,6} \to X_{3,1}, X_{3,6} \to X_{4,1}, X_{4,6} \to X_{5,1}, X_{5,6} \to X_{6,1}\}$

Ansatz made from the combination of monomials in the numerator with constants a_i, b_i, c_i at ech term to describe the weight and the three possible $X_{i,j}$ in the denominator :

In[ ]:=
```
ansatz1 =
 Sum[Subscript[a, i] Times @@ sixterm[[i]], {i, 1, Length@sixterm}] * 1 / Sett[[2]] +
  Sum[Subscript[b, i] Times @@ sixterm[[i]], {i, 1, Length@sixterm}] * 1 / Sett[[5]] +
  Sum[Subscript[c, i] Times @@ sixterm[[i]], {i, 1, Length@sixterm}] * 1 / Sett[[8]]
```

Out[ ]=

$$\Big\{\frac{1}{X_{1,4}}(a_1 X_{1,3} X_{1,4} + a_2 X_{1,3} X_{1,5} + a_9 X_{1,4} X_{1,5} + a_3 X_{1,3} X_{2,4} + a_{10} X_{1,4} X_{2,4} + a_{16} X_{1,5} X_{2,4} + a_4 X_{1,3} X_{2,5} + a_{11} X_{1,4} X_{2,5} + a_{17} X_{1,5} X_{2,5} + a_{22} X_{2,4} X_{2,5} + a_5 X_{1,3} X_{2,6} + a_{12} X_{1,4} X_{2,6} + a_{18} X_{1,5} X_{2,6} + a_{23} X_{2,4} X_{2,6} + a_{27} X_{2,5} X_{2,6} + a_6 X_{1,3} X_{3,5} + a_{13} X_{1,4} X_{3,5} + a_{19} X_{1,5} X_{3,5} + a_{24} X_{2,4} X_{3,5} + a_{28} X_{2,5} X_{3,5} + a_{31} X_{2,6} X_{3,5} + a_7 X_{1,3} X_{3,6} + a_{14} X_{1,4} X_{3,6} + a_{20} X_{1,5} X_{3,6} + a_{25} X_{2,4} X_{3,6} + a_{29} X_{2,5} X_{3,6} + a_{32} X_{2,6} X_{3,6} + a_{34} X_{3,5} X_{3,6} + a_8 X_{1,3} X_{4,6} + a_{15} X_{1,4} X_{4,6} + a_{21} X_{1,5} X_{4,6} + a_{26} X_{2,4} X_{4,6} + a_{30} X_{2,5} X_{4,6} + a_{33} X_{2,6} X_{4,6} + a_{35} X_{3,5} X_{4,6} + a_{36} X_{3,6} X_{4,6}) +$$

$$\frac{1}{X_{2,5}}(b_1 X_{1,3} X_{1,4} + b_2 X_{1,3} X_{1,5} + b_9 X_{1,4} X_{1,5} + b_3 X_{1,3} X_{2,4} + b_{10} X_{1,4} X_{2,4} + b_{16} X_{1,5} X_{2,4} + b_4 X_{1,3} X_{2,5} + b_{11} X_{1,4} X_{2,5} + b_{17} X_{1,5} X_{2,5} + b_{22} X_{2,4} X_{2,5} + b_5 X_{1,3} X_{2,6} + b_{12} X_{1,4} X_{2,6} + b_{18} X_{1,5} X_{2,6} + b_{23} X_{2,4} X_{2,6} + b_{27} X_{2,5} X_{2,6} + b_6 X_{1,3} X_{3,5} + b_{13} X_{1,4} X_{3,5} + b_{19} X_{1,5} X_{3,5} + b_{24} X_{2,4} X_{3,5} + b_{28} X_{2,5} X_{3,5} + b_{31} X_{2,6} X_{3,5} + b_7 X_{1,3} X_{3,6} + b_{14} X_{1,4} X_{3,6} + b_{20} X_{1,5} X_{3,6} + b_{25} X_{2,4} X_{3,6} + b_{29} X_{2,5} X_{3,6} + b_{32} X_{2,6} X_{3,6} + b_{34} X_{3,5} X_{3,6} + b_8 X_{1,3} X_{4,6} + b_{15} X_{1,4} X_{4,6} + b_{21} X_{1,5} X_{4,6} + b_{26} X_{2,4} X_{4,6} + b_{30} X_{2,5} X_{4,6} + b_{33} X_{2,6} X_{4,6} + b_{35} X_{3,5} X_{4,6} + b_{36} X_{3,6} X_{4,6}) +$$

$$\frac{1}{X_{3,6}}(c_1 X_{1,3} X_{1,4} + c_2 X_{1,3} X_{1,5} + c_9 X_{1,4} X_{1,5} + c_3 X_{1,3} X_{2,4} + c_{10} X_{1,4} X_{2,4} + c_{16} X_{1,5} X_{2,4} + c_4 X_{1,3} X_{2,5} + c_{11} X_{1,4} X_{2,5} + c_{17} X_{1,5} X_{2,5} + c_{22} X_{2,4} X_{2,5} + c_5 X_{1,3} X_{2,6} + c_{12} X_{1,4} X_{2,6} + c_{18} X_{1,5} X_{2,6} + c_{23} X_{2,4} X_{2,6} + c_{27} X_{2,5} X_{2,6} + c_6 X_{1,3} X_{3,5} + c_{13} X_{1,4} X_{3,5} + c_{19} X_{1,5} X_{3,5} + c_{24} X_{2,4} X_{3,5} + c_{28} X_{2,5} X_{3,5} + c_{31} X_{2,6} X_{3,5} + c_7 X_{1,3} X_{3,6} + c_{14} X_{1,4} X_{3,6} + c_{20} X_{1,5} X_{3,6} + c_{25} X_{2,4} X_{3,6} + c_{29} X_{2,5} X_{3,6} + c_{32} X_{2,6} X_{3,6} + c_{34} X_{3,5} X_{3,6} + c_8 X_{1,3} X_{4,6} + c_{15} X_{1,4} X_{4,6} + c_{21} X_{1,5} X_{4,6} + c_{26} X_{2,4} X_{4,6} + c_{30} X_{2,5} X_{4,6} + c_{33} X_{2,6} X_{4,6} + c_{35} X_{3,5} X_{4,6} + c_{36} X_{3,6} X_{4,6})\Big\}$$

Describe the skinny rectangles available and show how these condition the $X_{i,j}$'s:

In[ ]:=
```
s13 = X_{1,3} + X_{2,4} - X_{1,4} ;
s14 = X_{2,5} + X_{1,4} - X_{1,5} - X_{2,4} ;
s15 = X_{1,5} + X_{2,6} - X_{2,5} ;
s24 = X_{3,5} + X_{2,4} - X_{2,5} ;
s25 = X_{2,5} + X_{3,6} - X_{3,5} - X_{2,6} ;
s26 = X_{1,3} + X_{2,6} - X_{3,6} ;
s35 = X_{3,5} + X_{4,6} - X_{3,6} ;
s36 = X_{1,4} + X_{3,6} - X_{1,3} - X_{4,6} ;
s46 = X_{1,5} + X_{4,6} - X_{1,4} ;
(*s_1,3=s_1,4=s_1,5=0*)
locus0 = Solve[0 == {s13, s14, s15}][[1]];
(*s_2,4=s_2,5=s_2,6=0*)
locus1 = Solve[0 == {s24, s25, s26}][[1]];
(*s_3,5=s_3,6=s_1,3=0*)
locus2 = Solve[0 == {s35, s36, s13}][[1]];
(*s_4,6=s_1,4=s_2,4=0*)
locus3 = Solve[0 == {s46, s14, s24}][[1]];
(*s_1,5=s_2,5=s_3,5=0*)
locus4 = Solve[0 == {s15, s25, s35}][[1]];
(*s_2,6=s_3,6=s_4,6=0*)
locus5 = Solve[0 == {s26, s36, s46}][[1]];
(*s_1,3=s_3,6=s_4,6=s,_1,4=0*)
locus6 = Solve[0 == {s13, s36, s46, s14}][[1]];
(*s_2,4=s_1,4=s_1,5=s_2,50*)
locus7 = Solve[0 == {s24, s14, s15, s25}][[1]];
(*s_3,5=s_2,5=s_2,6=s_3,6=0*)
locus8 = Solve[0 == {s35, s25, s26, s36}][[1]];

Totlocu = {locus0, locus1, locus2, locus3, locus4, locus5, locus6, locus7, locus8}
```

Out[ ]=

$$
\begin{aligned}
\{&\{X_{2,4} \to -X_{1,3} + X_{1,4},\ X_{2,5} \to -X_{1,3} + X_{1,5},\ X_{2,6} \to -X_{1,3}\},\\
&\{X_{2,4} \to -X_{1,3},\ X_{3,5} \to X_{1,3} + X_{2,5},\ X_{3,6} \to X_{1,3} + X_{2,6}\},\\
&\{X_{2,4} \to -X_{1,3} + X_{1,4},\ X_{3,5} \to X_{1,3} - X_{1,4},\ X_{4,6} \to -X_{1,3} + X_{1,4} + X_{3,6}\},\\
&\{X_{2,5} \to -X_{1,4} + X_{1,5} + X_{2,4},\ X_{3,5} \to -X_{1,4} + X_{1,5},\ X_{4,6} \to X_{1,4} - X_{1,5}\},\\
&\{X_{2,6} \to -X_{1,5} + X_{2,5},\ X_{3,6} \to -X_{1,5} + X_{3,5},\ X_{4,6} \to -X_{1,5}\},\\
&\{X_{2,6} \to -X_{1,5},\ X_{3,6} \to X_{1,3} - X_{1,5},\ X_{4,6} \to X_{1,4} - X_{1,5}\},\\
&\{X_{2,4} \to -X_{1,3} + X_{1,4},\ X_{2,5} \to -X_{1,3} + X_{1,5},\ X_{3,6} \to X_{1,3} - X_{1,5},\ X_{4,6} \to X_{1,4} - X_{1,5}\},\\
&\{X_{2,5} \to -X_{1,4} + X_{1,5} + X_{2,4},\ X_{2,6} \to -X_{1,4} + X_{2,4},\ X_{3,5} \to -X_{1,4} + X_{1,5},\ X_{3,6} \to -X_{1,4}\},\\
&\{X_{2,5} \to -X_{1,4},\ X_{3,5} \to X_{1,3} - X_{1,4},\ X_{3,6} \to X_{1,3} + X_{2,6},\ X_{4,6} \to X_{1,4} + X_{2,6}\}\}
\end{aligned}
$$

Apply the above conditions on the ansatz to reduce the constant terms:

In[ ]:=
```
Locs = SolveAlways[0 == Flatten[{ansatz1 /. Totlocu, ansatz1 - (ansatz1)}], Xset9][[1]]
```

Out[ ]=

$\{a_3 \to 0, a_4 \to 0, a_5 \to 0, a_6 \to 0, a_7 \to 0, a_2 \to -a_{15} - b_{30} - c_{36}, a_8 \to -a_{15} - b_{30} - c_{36}, a_{17} \to 0, a_{18} \to 0, a_{19} \to 0, a_{20} \to 0, a_{16} \to -a_{15} - b_{30} - c_{36}, a_{21} \to 0, a_{22} \to 0, a_{23} \to 0, a_{24} \to 0, a_{25} \to 0, a_{26} \to -a_{15} - b_{30} - c_{36}, a_{27} \to 0, a_{28} \to 0, a_{29} \to 0, a_{30} \to 0, a_{31} \to 0, a_{32} \to 0, a_{33} \to 0, a_{34} \to 0, a_{35} \to 0, a_{36} \to 0, b_1 \to 0, b_3 \to 0, b_5 \to 0, b_7 \to 0, b_6 \to 0, b_2 \to 0, b_8 \to 0, b_9 \to 0, b_{10} \to 0, b_{13} \to 0, b_{14} \to 0, b_{12} \to 0, b_{15} \to 0, b_{16} \to -a_{15} - b_{30} - c_{36}, b_{18} \to 0, b_{19} \to -a_{15} - b_{30} - c_{36}, b_{20} \to 0, b_{21} \to 0, b_{24} \to 0, b_{25} \to 0, b_{26} \to 0, b_{23} \to -a_{15} - b_{30} - c_{36}, a_{14} \to -b_{29}, b_{32} \to 0, b_{31} \to -a_{15} - b_{30} - c_{36}, b_{33} \to 0, b_{34} \to 0, b_{35} \to 0, b_{36} \to 0, c_1 \to 0, c_3 \to 0, c_4 \to 0, c_5 \to 0, c_2 \to 0, c_6 \to -a_{15} - b_{30} - c_{36}, a_1 \to a_{15} - b_4 + b_{30} - c_7 + c_{36}, c_8 \to -a_{15} - b_{30} - c_{36}, c_9 \to 0, c_{10} \to 0, c_{11} \to 0, c_{12} \to 0, c_{13} \to 0, b_{11} \to -c_{14}, c_{15} \to 0, c_{16} \to 0, c_{17} \to 0, c_{18} \to 0, c_{19} \to 0, a_9 \to a_{15} - b_{17} + b_{30} - c_{20} + c_{36}, c_{21} \to 0, c_{22} \to 0, c_{23} \to 0, c_{24} \to 0, a_{10} \to a_{15} - b_{22} + b_{30} - c_{25} + c_{36}, c_{26} \to 0, c_{27} \to 0, c_{28} \to 0, a_{11} \to -c_{29}, c_{30} \to 0, c_{31} \to -a_{15} - b_{30} - c_{36}, a_{12} \to a_{15} - b_{27} + b_{30} - c_{32} + c_{36}, c_{33} \to -a_{15} - b_{30} - c_{36}, a_{13} \to a_{15} - b_{28} + b_{30} - c_{34} + c_{36}, c_{35} \to 0\}$

Result:

In[ ]:=
```
anss = ansatz1 /. Locs // FullSimplify
```

Out[ ]=

$$\Big\{-\frac{1}{X_{1,4} X_{2,5} X_{3,6}} (a_{15} + b_{30} + c_{36}) \big(X_{2,4} X_{2,5} X_{3,6} (X_{1,5} + X_{4,6}) + X_{1,3} X_{2,5} (X_{3,6} (X_{1,5} + X_{4,6}) + X_{1,4} (X_{3,5} - X_{3,6} + X_{4,6})) + X_{1,4} ((X_{1,5} + X_{2,6}) (X_{2,4} + X_{3,5}) X_{3,6} + X_{2,5} (-X_{3,6} (X_{1,5} + X_{2,4} + X_{3,5} + X_{4,6}) + X_{2,6} (X_{3,5} - X_{3,6} + X_{4,6})))\big)\Big\}$$

8-point NLSM

Set of possible $X_{i,j}$:

```
In[∘]:= Xset8 = {X1,3, X1,4, X1,5, X1,6, X1,7, X2,4, X2,5, X2,6,
   X2,7, X2,8, X3,5, X3,6, X3,7, X3,8, X4,6, X4,7, X4,8, X5,7, X5,8, X6,8}
```

Out[∘]=

$\{X_{1,3}, X_{1,4}, X_{1,5}, X_{1,6}, X_{1,7}, X_{2,4}, X_{2,5}, X_{2,6}, X_{2,7}, X_{2,8}, X_{3,5}, X_{3,6}, X_{3,7}, X_{3,8}, X_{4,6}, X_{4,7}, X_{4,8}, X_{5,7}, X_{5,8}, X_{6,8}\}$

Extra set to help the code:

```
In[∘]:= Sett = Subsets[Xset8, {1}]
Length[Sett]
```

Out[∘]=

$\{\{X_{1,3}\}, \{X_{1,4}\}, \{X_{1,5}\}, \{X_{1,6}\}, \{X_{1,7}\}, \{X_{2,4}\}, \{X_{2,5}\}, \{X_{2,6}\}, \{X_{2,7}\}, \{X_{2,8}\}, \{X_{3,5}\}, \{X_{3,6}\}, \{X_{3,7}\}, \{X_{3,8}\}, \{X_{4,6}\}, \{X_{4,7}\}, \{X_{4,8}\}, \{X_{5,7}\}, \{X_{5,8}\}, \{X_{6,8}\}\}$

Out[∘]=

20

Possible combinations:

```
In[∘]:= sixterm1 = Subsets[Xset8, {3}]
Length[sixterm1]
```

Out[∘]=

$\{\{X_{1,3}, X_{1,4}, X_{1,5}\}, \{X_{1,3}, X_{1,4}, X_{1,6}\}, \{X_{1,3}, X_{1,4}, X_{1,7}\}, \{X_{1,3}, X_{1,4}, X_{2,4}\}, \{X_{1,3}, X_{1,4}, X_{2,5}\}, \{X_{1,3}, X_{1,4}, X_{2,6}\}, \{X_{1,3}, X_{1,4}, X_{2,7}\}, \{X_{1,3}, X_{1,4}, X_{2,8}\}, \{X_{1,3}, X_{1,4}, X_{3,5}\}, \{X_{1,3}, X_{1,4}, X_{3,6}\}, \{X_{1,3}, X_{1,4}, X_{3,7}\}, \{X_{1,3}, X_{1,4}, X_{3,8}\},$ ⋯ 1117 ⋯ $, \{X_{4,6}, X_{5,8}, X_{6,8}\}, \{X_{4,7}, X_{4,8}, X_{5,7}\}, \{X_{4,7}, X_{4,8}, X_{5,8}\}, \{X_{4,7}, X_{4,8}, X_{6,8}\}, \{X_{4,7}, X_{5,7}, X_{5,8}\}, \{X_{4,7}, X_{5,7}, X_{6,8}\}, \{X_{4,7}, X_{5,8}, X_{6,8}\}, \{X_{4,8}, X_{5,7}, X_{5,8}\}, \{X_{4,8}, X_{5,7}, X_{6,8}\}, \{X_{4,8}, X_{5,8}, X_{6,8}\}, \{X_{5,7}, X_{5,8}, X_{6,8}\}\}$

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Out[∘]=

1140

Extra condition to ensure $X_{j,i} = X_{i,j}$ and that only possible $X_{i,j}$ are taken into account:

```
In[∘]:= replX = Table[Table[Xj,i → Xi,j, {i, 1, j - 1}], {j, 1, 6}] // Flatten
replX1 = Table[Table[Xi,j → XMod[i,6]+1,Mod[j,6]+1, {i, 1, j - 1}], {j, 1, 6}] // Flatten
```

Out[∘]=

$\{X_{2,1} \to X_{1,2}, X_{3,1} \to X_{1,3}, X_{3,2} \to X_{2,3}, X_{4,1} \to X_{1,4}, X_{4,2} \to X_{2,4}, X_{4,3} \to X_{3,4}, X_{5,1} \to X_{1,5}, X_{5,2} \to X_{2,5}, X_{5,3} \to X_{3,5}, X_{5,4} \to X_{4,5}, X_{6,1} \to X_{1,6}, X_{6,2} \to X_{2,6}, X_{6,3} \to X_{3,6}, X_{6,4} \to X_{4,6}, X_{6,5} \to X_{5,6}\}$

Out[∘]=

$\{X_{1,2} \to X_{2,3}, X_{1,3} \to X_{2,4}, X_{2,3} \to X_{3,4}, X_{1,4} \to X_{2,5}, X_{2,4} \to X_{3,5}, X_{3,4} \to X_{4,5}, X_{1,5} \to X_{2,6}, X_{2,5} \to X_{3,6}, X_{3,5} \to X_{4,6}, X_{4,5} \to X_{5,6}, X_{1,6} \to X_{2,1}, X_{2,6} \to X_{3,1}, X_{3,6} \to X_{4,1}, X_{4,6} \to X_{5,1}, X_{5,6} \to X_{6,1}\}$

In[ ]:=
```
sixterm = sixterm1 /. replX1
```
Out[ ]=

$\{\{X_{2,4}, X_{2,5}, X_{2,6}\}, \{X_{2,4}, X_{2,5}, X_{2,1}\}, \{X_{2,4}, X_{2,5}, X_{1,7}\}, \{X_{2,4}, X_{2,5}, X_{3,5}\}, \{X_{2,4}, X_{2,5}, X_{3,6}\}, \{X_{2,4}, X_{2,5}, X_{3,1}\}, \{X_{2,4}, X_{2,5}, X_{2,7}\}, \{X_{2,4}, X_{2,5}, X_{2,8}\}, \{X_{2,4}, X_{2,5}, X_{4,6}\}, \{X_{2,4}, X_{2,5}, X_{4,1}\}, \{X_{2,4}, X_{2,5}, X_{3,7}\}, \{X_{2,4}, X_{2,5}, X_{3,8}\}, \ldots 1117 \ldots, \{X_{5,1}, X_{5,8}, X_{6,8}\}, \{X_{4,7}, X_{4,8}, X_{5,7}\}, \{X_{4,7}, X_{4,8}, X_{5,8}\}, \{X_{4,7}, X_{4,8}, X_{6,8}\}, \{X_{4,7}, X_{5,7}, X_{5,8}\}, \{X_{4,7}, X_{5,7}, X_{6,8}\}, \{X_{4,7}, X_{5,8}, X_{6,8}\}, \{X_{4,8}, X_{5,7}, X_{5,8}\}, \{X_{4,8}, X_{5,7}, X_{6,8}\}, \{X_{4,8}, X_{5,8}, X_{6,8}\}, \{X_{5,7}, X_{5,8}, X_{6,8}\}\}$

Full expression not available (original memory size: 0.4 MB)

Ansatz made from the combination of monomials in the numerator with constants a_i, b_i, c_i, d_i at ech term to describe the weight and the four possible order two $X_{i,j}$ in the denominator :

In[ ]:=
```
ansatz1 = Sum[Subscript[a, i] Times @@ sixterm[[i]], {i, 1, Length@sixterm}] *
    1 / (Sett[[2]] * Sett[[19]]) +
  Sum[Subscript[b, i] Times @@ sixterm[[i]], {i, 1, Length@sixterm}] *
    1 / (Sett[[4]] * Sett[[7]]) + Sum[Subscript[c, i] Times @@ sixterm[[i]],
     {i, 1, Length@sixterm}] * 1 / (Sett[[9]] * Sett[[12]]) +
  Sum[Subscript[d, i] Times @@ sixterm[[i]], {i, 1, Length@sixterm}] *
    1 / (Sett[[14]] * Sett[[16]])
```
Out[ ]=

$$\left\{\frac{1}{X_{1,4} X_{5,8}} \left(a_{36} X_{1,7} X_{2,1} X_{2,4} + a_{189} X_{1,7} X_{2,1} X_{2,5} + a_3 X_{1,7} X_{2,4} X_{2,5} + a_2 X_{2,1} X_{2,4} X_{2,5} + \ldots 1132 \ldots + a_{1136} X_{4,7} X_{5,8} X_{6,8} + a_{1139} X_{4,8} X_{5,8} X_{6,8} + a_{1130} X_{5,1} X_{5,8} X_{6,8} + a_{1140} X_{5,7} X_{5,8} X_{6,8}\right) + \frac{b_{36} X_{1,7} X_{2,1} X_{2,4} + \ldots 1138 \ldots + b_{1140} \ldots 2 \ldots X_{6\ldots 1 \ldots} \ldots 1 \ldots}{X_{1,6} X_{2,5}} + \frac{\ldots 1 \ldots}{X_{\ldots 1 \ldots} \ldots 1 \ldots} + \frac{d_{36} X_{1,7} X_{2,1} X_{2,4} + d_{189} X_{1,7} X_{2,1} X_{2,5} + \ldots 1137 \ldots + d_{1140} X_{5,7} X_{5,8} X_{6,8}}{X_{3,8} X_{4,7}}\right\}$$

Full expression not available (original memory size: 2 MB)

Describe the skinny rectangles available and show how these condition the $X_{i,j}$'s:

In[ ]:=
```
s13 = X1,3 + X2,4 - X1,4 ;
s14 = X2,5 + X1,4 - X1,5 - X2,4 ;
s15 = X1,5 + X2,6 - X2,5 - X1,6;
s16 = X1,6 + X2,7 - X2,6 - X1,7;
s17 = X1,7 + X2,8 - X2,7 ;
s24 = X2,4 + X3,5 - X2,5 ;
s25 = X2,5 + X3,6 - X3,5 - X2,6 ;
s26 = X2,6 + X3,7 - X3,6 - X2,7;
s27 = X2,7 + X3,8 - X3,7 - X2,8;
s28 = X2,8 + X1,3 - X3,8 ;
s35 = X3,5 + X4,6 - X3,6 ;
s36 = X3,6 + X4,7 - X3,7 - X4,6 ;
s37 = X3,7 + X4,8 - X4,7 - X3,8;
s38 = X3,8 + X1,4 - X4,8 - X1,3;
s46 = X4,6 + X5,7 - X4,7 ;
s47 = X4,7 + X5,8 - X5,7 - X4,8;
s48 = X4,8 + X1,5 - X5,8 - X1,4;
s57 = X5,7 + X1,6 - X1,5;
s58 = X5,8 + X1,6 - X6,8 - X1,5;
s68 = X6,8 + X1,7 - X1,6;
locus0 = Solve[0 == {s13, s14, s15, s16, s17}][[1]];
locus1 = Solve[0 == {s24, s25, s26, s27, s28 }][[1]];
locus2 = Solve[0 == {s35, s36, s37, s38, s13}][[1]];
locus3 = Solve[0 == {s46, s47, s48, s14, s24}][[1]];
locus4 = Solve[0 == {s57, s58, s15, s25, s35}][[1]];
locus5 = Solve[0 == {s68, s16, s26, s36, s46}][[1]];
locus6 = Solve[0 == {s17, s27, s37, s47, s57}][[1]];
locus7 = Solve[0 == {s28, s38, s48, s58, s68}][[1]];
Totlocu = {locus0, locus1, locus2, locus3, locus4, locus5, locus6, locus7}
```

Out[ ]=

$\{\{X_{2,4} \to -X_{1,3} + X_{1,4},\ X_{2,5} \to -X_{1,3} + X_{1,5},\ X_{2,6} \to -X_{1,3} + X_{1,6},\ X_{2,7} \to -X_{1,3} + X_{1,7},\ X_{2,8} \to -X_{1,3}\},$
$\{X_{2,4} \to -X_{1,3},\ X_{3,5} \to X_{1,3} + X_{2,5},\ X_{3,6} \to X_{1,3} + X_{2,6},\ X_{3,7} \to X_{1,3} + X_{2,7},\ X_{3,8} \to X_{1,3} + X_{2,8}\},$
$\{X_{2,4} \to -X_{1,3} + X_{1,4},\ X_{3,5} \to X_{1,3} - X_{1,4},\ X_{4,6} \to -X_{1,3} + X_{1,4} + X_{3,6},\ X_{4,7} \to -X_{1,3} + X_{1,4} + X_{3,7},$
$X_{4,8} \to -X_{1,3} + X_{1,4} + X_{3,8}\},\ \{X_{2,5} \to -X_{1,4} + X_{1,5} + X_{2,4},\ X_{3,5} \to -X_{1,4} + X_{1,5},$
$X_{4,6} \to X_{1,4} - X_{1,5},\ X_{5,7} \to -X_{1,4} + X_{1,5} + X_{4,7},\ X_{5,8} \to -X_{1,4} + X_{1,5} + X_{4,8}\},$
$\{X_{2,6} \to -X_{1,5} + X_{1,6} + X_{2,5},\ X_{3,6} \to -X_{1,5} + X_{1,6} + X_{3,5},\ X_{4,6} \to -X_{1,5} + X_{1,6},\ X_{5,7} \to X_{1,5} - X_{1,6},$
$X_{6,8} \to -X_{1,5} + X_{1,6} + X_{5,8}\},\ \{X_{2,7} \to -X_{1,6} + X_{1,7} + X_{2,6},\ X_{3,7} \to -X_{1,6} + X_{1,7} + X_{3,6},$
$X_{4,7} \to -X_{1,6} + X_{1,7} + X_{4,6},\ X_{5,7} \to -X_{1,6} + X_{1,7},\ X_{6,8} \to X_{1,6} - X_{1,7}\},\ \{X_{2,8} \to -X_{1,7} + X_{2,7},$
$X_{3,8} \to -X_{1,7} + X_{3,7},\ X_{4,8} \to -X_{1,7} + X_{4,7},\ X_{5,7} \to X_{1,5} - X_{1,6},\ X_{5,8} \to X_{1,5} - X_{1,6} - X_{1,7}\},$
$\{X_{2,8} \to -X_{1,7},\ X_{3,8} \to X_{1,3} - X_{1,7},\ X_{4,8} \to X_{1,4} - X_{1,7},\ X_{5,8} \to X_{1,5} - X_{1,7},\ X_{6,8} \to X_{1,6} - X_{1,7}\}\}$

Apply the above conditions on the ansatz to reduce the constant terms:

In[ ]:=
```
Locs = SolveAlways[
   0 == Flatten[{ansatz1 /. Totlocu, ansatz1 - ( ansatz1 /. replX1 /. replX) }], Xset9][[1]]
```

Out[ ]=

{}

```
In[ ]:= anss = ansatz1 /. Locs // FullSimplify
```

6.2.1 Proof only skinny rectangles needed from code

The following codes prove that skinny rectangles are sufficient in the calculation of the unique amplitude. The code used is the same except for the locus applied to simplify the general function. As seen the result is the same as before, proving squares are not necessary. This proof is only done for 6-point for both $\mathrm{Tr}(\phi^3)$ and NLSM and assumed to be true for higher-point interactions, as its explanation (factorization) applies for any n-point interaction.

6-point Tr(Φ^3) with no squares

Set of possible $X_{i,j}$:

In[ ]:=
```
Xset9 = {X1,3, X1,4, X1,5, X2,4, X2,5, X2,6, X3,5, X3,6, X4,6}
```

Out[ ]=

$\{X_{1,3}, X_{1,4}, X_{1,5}, X_{2,4}, X_{2,5}, X_{2,6}, X_{3,5}, X_{3,6}, X_{4,6}\}$

Possible combinations:

In[ ]:=
```
sixterm = Subsets[Xset9, {6}];
Length[sixterm]
```

Out[ ]=

84

Ansatz made from the combination of monomials with a constant c_i at ech term to describe the weight:

In[ ]:=
```
ansatz = Sum[ci Times @@ sixterm[[i]], {i, 1, Length@sixterm}] / (Times @@ Xset9);
```

Describe the skinny rectangles and squares available and show how these condition the $X_{i,j}$'s:

In[ ]:=
```
s13 = X1,3 + X2,4 - X1,4 ;
s14 = X2,5 + X1,4 - X1,5 - X2,4 ;
s15 = X1,5 + X2,6 - X2,5 ;
s24 = X3,5 + X2,4 - X2,5 ;
s25 = X2,5 + X3,6 - X3,5 - X2,6 ;
s26 = X1,3 + X2,6 - X3,6 ;
s35 = X3,5 + X4,6 - X3,6 ;
s36 = X1,4 + X3,6 - X1,3 - X4,6 ;
s46 = X1,5 + X4,6 - X1,4 ;
```

In[ ]:=
```
(*s_1,3=s_1,4=s_1,5=0*)
locus0 =  Solve[0=={s13,s14,s15}][[1]];
(*s_2,4=s_2,5=s_2,6=0*)
locus1 = Solve[0=={s24,s25,s26}][[1]];
(*s_3,5=s_3,6=s_1,3=0*)
locus2 = Solve[0=={s35,s36,s13}][[1]];
(*s_4,6=s_1,4=s_2,4=0*)
locus3 = Solve[0=={s46,s14,s24}][[1]];
(*s_1,5=s_2,5=s_3,5=0*)
locus4 = Solve[0=={s15,s25,s35}][[1]];
(*s_2,6=s_3,6=s_4,6=0*)
locus5 = Solve[0=={s26,s36,s46}][[1]];
(*s_1,3=s_3,6=s_4,6=s,_1,4=0*)
locus6 = Solve[0=={s13,s36,s46,s14}][[1]];
(*s_2,4=s_1,4=s_1,5=s_2,50*)
locus7 = Solve[0=={s24,s14,s15,s25}][[1]];
(*s_3,5=s_2,5=s_2,6=s_3,6=0*)
locus8 = Solve[0=={s35,s25,s26,s36}][[1]];

Totlocu = {locus0,locus1,locus2,locus3,locus4,locus5};
```

Apply the above conditions on the ansatz to reduce the c_i terms:

In[ ]:=
```
Locs = SolveAlways[0 == ansatz /. Totlocu, Xset9][[1]]
```

Out[ ]=

$\{c_1 \to 0, c_8 \to 0, c_9 \to 0, c_{15} \to 0, c_{19} \to 0, c_{21} \to 0, c_{22} \to 0, c_{23} \to 0, c_{24} \to 0, c_{25} \to 0, c_{26} \to 0, c_{27} \to 0, c_{28} \to 0, c_3 \to 0, c_4 \to c_{84}, c_6 \to 0, c_7 \to 0, c_{29} \to c_{84}, c_{10} \to c_{84}, c_2 \to c_{84}, c_5 \to 0, c_{30} \to 0, c_{31} \to 0, c_{32} \to 0, c_{12} \to 0, c_{13} \to 0, c_{33} \to 0, c_{16} \to 0, c_{11} \to 0, c_{34} \to 0, c_{17} \to 0, c_{20} \to c_{84}, c_{35} \to c_{84}, c_{14} \to c_{84}, c_{18} \to 0, c_{36} \to 0, c_{37} \to 0, c_{38} \to 0, c_{39} \to 0, c_{40} \to 0, c_{42} \to 0, c_{43} \to 0, c_{44} \to 0, c_{45} \to 0, c_{47} \to 0, c_{48} \to 0, c_{49} \to 0, c_{50} \to 0, c_{51} \to 0, c_{52} \to 0, c_{53} \to 0, c_{41} \to 0, c_{54} \to 0, c_{55} \to 0, c_{46} \to c_{84}, c_{56} \to c_{84}, c_{57} \to c_{84}, c_{58} \to 0, c_{59} \to c_{84}, c_{60} \to 0, c_{61} \to 0, c_{62} \to 0, c_{63} \to 0, c_{64} \to 0, c_{65} \to 0, c_{66} \to 0, c_{67} \to 0, c_{68} \to 0, c_{69} \to 0, c_{70} \to 0, c_{71} \to 0, c_{72} \to 0, c_{73} \to 0, c_{74} \to c_{84}, c_{75} \to 0, c_{76} \to 0, c_{77} \to 0, c_{78} \to c_{84}, c_{79} \to 0, c_{80} \to 0, c_{81} \to 0, c_{82} \to 0, c_{83} \to 0\}$

Use the above reduction to simplify the ansatz:

In[ ]:=
```
anss = ansatz /. Locs
```

Out[ ]=

$$\frac{1}{X_{1,3} X_{1,4} X_{1,5} X_{2,4} X_{2,5} X_{2,6} X_{3,5} X_{3,6} X_{4,6}} (c_{84} X_{1,3} X_{1,4} X_{1,5} X_{2,4} X_{2,5} X_{3,5} + c_{84} X_{1,4} X_{1,5} X_{2,4} X_{2,5} X_{2,6} X_{3,5} + c_{84} X_{1,3} X_{1,4} X_{1,5} X_{2,5} X_{3,5} X_{3,6} + c_{84} X_{1,3} X_{1,5} X_{2,5} X_{2,6} X_{3,5} X_{3,6} + c_{84} X_{1,5} X_{2,4} X_{2,5} X_{2,6} X_{3,5} X_{3,6} + c_{84} X_{1,3} X_{1,4} X_{1,5} X_{2,4} X_{2,5} X_{4,6} + c_{84} X_{1,4} X_{1,5} X_{2,4} X_{2,5} X_{2,6} X_{4,6} + c_{84} X_{1,3} X_{1,4} X_{1,5} X_{2,4} X_{3,6} X_{4,6} + c_{84} X_{1,3} X_{1,4} X_{2,4} X_{2,6} X_{3,6} X_{4,6} + c_{84} X_{1,4} X_{2,4} X_{2,5} X_{2,6} X_{3,6} X_{4,6} + c_{84} X_{1,3} X_{1,4} X_{1,5} X_{3,5} X_{3,6} X_{4,6} + c_{84} X_{1,3} X_{1,4} X_{2,6} X_{3,5} X_{3,6} X_{4,6} + c_{84} X_{1,3} X_{2,5} X_{2,6} X_{3,5} X_{3,6} X_{4,6} + c_{84} X_{2,4} X_{2,5} X_{2,6} X_{3,5} X_{3,6} X_{4,6})$$

Apply the Non-physical poles Condition:

In[ ]:=
```
triress[X1_, X2_, X3_, X4_] :=
  Residue[Residue[Residue[Residue[anss, {X1, 0}], {X2, 0}], {X3, 0}], {X4, 0}];
poletabi =
  Table[triress[sixt⟦i, 1⟧, sixt⟦i, 2⟧, sixt⟦i, 3⟧, sixt⟦i, 4⟧], {i, 1, Length@sixt}];
olpole = SolveAlways[poletabi == 0, Xset9]⟦1⟧
```
Out[ ]=

{}

Result:

In[ ]:=
```
res = anss /. olpole // FullSimplify // Expand
```
Out[ ]=

$$\frac{c_{84}}{X_{1,3} X_{1,4} X_{1,5}} + \frac{c_{84}}{X_{1,4} X_{1,5} X_{2,4}} + \frac{c_{84}}{X_{1,5} X_{2,4} X_{2,5}} + \frac{c_{84}}{X_{2,4} X_{2,5} X_{2,6}} + \frac{c_{84}}{X_{1,3} X_{1,5} X_{3,5}} + \frac{c_{84}}{X_{1,5} X_{2,5} X_{3,5}} + \frac{c_{84}}{X_{2,5} X_{2,6} X_{3,5}} + \frac{c_{84}}{X_{1,3} X_{3,5} X_{3,6}} + \frac{c_{84}}{X_{2,6} X_{3,5} X_{3,6}} + \frac{c_{84}}{X_{1,3} X_{1,4} X_{4,6}} + \frac{c_{84}}{X_{1,4} X_{2,4} X_{4,6}} + \frac{c_{84}}{X_{2,4} X_{2,6} X_{4,6}} + \frac{c_{84}}{X_{1,3} X_{3,6} X_{4,6}} + \frac{c_{84}}{X_{2,6} X_{3,6} X_{4,6}}$$

6-point NLSM with no squares

Set of possible $X_{i,j}$:

```
In[ ]:= Xset9 = {X1,3, X1,4, X1,5, X2,4, X2,5, X2,6, X3,5, X3,6, X4,6}
```

Out[ ]=

$\{X_{1,3}, X_{1,4}, X_{1,5}, X_{2,4}, X_{2,5}, X_{2,6}, X_{3,5}, X_{3,6}, X_{4,6}\}$

Extra set to help the code:

```
In[ ]:= Sett = Subsets[Xset9, {1}]
```

Out[ ]=

$\{\{X_{1,3}\}, \{X_{1,4}\}, \{X_{1,5}\}, \{X_{2,4}\}, \{X_{2,5}\}, \{X_{2,6}\}, \{X_{3,5}\}, \{X_{3,6}\}, \{X_{4,6}\}\}$

Possible combinations:

```
In[ ]:= sixterm = Subsets[Xset9, {2}]
Length[sixterm]
```

Out[ ]=

$\{\{X_{1,3}, X_{1,4}\}, \{X_{1,3}, X_{1,5}\}, \{X_{1,3}, X_{2,4}\}, \{X_{1,3}, X_{2,5}\}, \{X_{1,3}, X_{2,6}\}, \{X_{1,3}, X_{3,5}\},$
$\{X_{1,3}, X_{3,6}\}, \{X_{1,3}, X_{4,6}\}, \{X_{1,4}, X_{1,5}\}, \{X_{1,4}, X_{2,4}\}, \{X_{1,4}, X_{2,5}\}, \{X_{1,4}, X_{2,6}\},$
$\{X_{1,4}, X_{3,5}\}, \{X_{1,4}, X_{3,6}\}, \{X_{1,4}, X_{4,6}\}, \{X_{1,5}, X_{2,4}\}, \{X_{1,5}, X_{2,5}\}, \{X_{1,5}, X_{2,6}\},$
$\{X_{1,5}, X_{3,5}\}, \{X_{1,5}, X_{3,6}\}, \{X_{1,5}, X_{4,6}\}, \{X_{2,4}, X_{2,5}\}, \{X_{2,4}, X_{2,6}\}, \{X_{2,4}, X_{3,5}\},$
$\{X_{2,4}, X_{3,6}\}, \{X_{2,4}, X_{4,6}\}, \{X_{2,5}, X_{2,6}\}, \{X_{2,5}, X_{3,5}\}, \{X_{2,5}, X_{3,6}\}, \{X_{2,5}, X_{4,6}\},$
$\{X_{2,6}, X_{3,5}\}, \{X_{2,6}, X_{3,6}\}, \{X_{2,6}, X_{4,6}\}, \{X_{3,5}, X_{3,6}\}, \{X_{3,5}, X_{4,6}\}, \{X_{3,6}, X_{4,6}\}\}$

Out[ ]=

36

Extra condition to ensure $X_{j,i} = X_{i,j}$ and that only possible $X_{i,j}$ are taken into account:

```
In[ ]:= replX = Table[Table[Xj,i → Xi,j, {i, 1, j - 1}], {j, 1, 6}] // Flatten
replX1 = Table[Table[Xi,j → XMod[i,6]+1,Mod[j,6]+1, {i, 1, j - 1}], {j, 1, 6}] // Flatten
```

Out[ ]=

$\{X_{2,1} \to X_{1,2}, X_{3,1} \to X_{1,3}, X_{3,2} \to X_{2,3}, X_{4,1} \to X_{1,4}, X_{4,2} \to X_{2,4}, X_{4,3} \to X_{3,4}, X_{5,1} \to X_{1,5}, X_{5,2} \to X_{2,5},$
$X_{5,3} \to X_{3,5}, X_{5,4} \to X_{4,5}, X_{6,1} \to X_{1,6}, X_{6,2} \to X_{2,6}, X_{6,3} \to X_{3,6}, X_{6,4} \to X_{4,6}, X_{6,5} \to X_{5,6}\}$

Out[ ]=

$\{X_{1,2} \to X_{2,3}, X_{1,3} \to X_{2,4}, X_{2,3} \to X_{3,4}, X_{1,4} \to X_{2,5}, X_{2,4} \to X_{3,5}, X_{3,4} \to X_{4,5}, X_{1,5} \to X_{2,6}, X_{2,5} \to X_{3,6},$
$X_{3,5} \to X_{4,6}, X_{4,5} \to X_{5,6}, X_{1,6} \to X_{2,1}, X_{2,6} \to X_{3,1}, X_{3,6} \to X_{4,1}, X_{4,6} \to X_{5,1}, X_{5,6} \to X_{6,1}\}$

Ansatz made from the combination of monomials in the numerator with constants a_i, b_i, c_i at ech term to describe the weight and the three possible $X_{i,j}$ in the denominator :

In[ ]:=
```
ansatz1 =
 Sum[Subscript[a, i] Times @@ sixterm[[i]], {i, 1, Length@sixterm}] * 1 / Sett[[2]] +
  Sum[Subscript[b, i] Times @@ sixterm[[i]], {i, 1, Length@sixterm}] * 1 / Sett[[5]] +
  Sum[Subscript[c, i] Times @@ sixterm[[i]], {i, 1, Length@sixterm}] * 1 / Sett[[8]]
```

Out[ ]=

$$\Big\{\frac{1}{X_{1,4}}(a_1 X_{1,3} X_{1,4} + a_2 X_{1,3} X_{1,5} + a_9 X_{1,4} X_{1,5} + a_3 X_{1,3} X_{2,4} + a_{10} X_{1,4} X_{2,4} + a_{16} X_{1,5} X_{2,4} + a_4 X_{1,3} X_{2,5} + a_{11} X_{1,4} X_{2,5} + a_{17} X_{1,5} X_{2,5} + a_{22} X_{2,4} X_{2,5} + a_5 X_{1,3} X_{2,6} + a_{12} X_{1,4} X_{2,6} + a_{18} X_{1,5} X_{2,6} + a_{23} X_{2,4} X_{2,6} + a_{27} X_{2,5} X_{2,6} + a_6 X_{1,3} X_{3,5} + a_{13} X_{1,4} X_{3,5} + a_{19} X_{1,5} X_{3,5} + a_{24} X_{2,4} X_{3,5} + a_{28} X_{2,5} X_{3,5} + a_{31} X_{2,6} X_{3,5} + a_7 X_{1,3} X_{3,6} + a_{14} X_{1,4} X_{3,6} + a_{20} X_{1,5} X_{3,6} + a_{25} X_{2,4} X_{3,6} + a_{29} X_{2,5} X_{3,6} + a_{32} X_{2,6} X_{3,6} + a_{34} X_{3,5} X_{3,6} + a_8 X_{1,3} X_{4,6} + a_{15} X_{1,4} X_{4,6} + a_{21} X_{1,5} X_{4,6} + a_{26} X_{2,4} X_{4,6} + a_{30} X_{2,5} X_{4,6} + a_{33} X_{2,6} X_{4,6} + a_{35} X_{3,5} X_{4,6} + a_{36} X_{3,6} X_{4,6}) +$$

$$\frac{1}{X_{2,5}}(b_1 X_{1,3} X_{1,4} + b_2 X_{1,3} X_{1,5} + b_9 X_{1,4} X_{1,5} + b_3 X_{1,3} X_{2,4} + b_{10} X_{1,4} X_{2,4} + b_{16} X_{1,5} X_{2,4} + b_4 X_{1,3} X_{2,5} + b_{11} X_{1,4} X_{2,5} + b_{17} X_{1,5} X_{2,5} + b_{22} X_{2,4} X_{2,5} + b_5 X_{1,3} X_{2,6} + b_{12} X_{1,4} X_{2,6} + b_{18} X_{1,5} X_{2,6} + b_{23} X_{2,4} X_{2,6} + b_{27} X_{2,5} X_{2,6} + b_6 X_{1,3} X_{3,5} + b_{13} X_{1,4} X_{3,5} + b_{19} X_{1,5} X_{3,5} + b_{24} X_{2,4} X_{3,5} + b_{28} X_{2,5} X_{3,5} + b_{31} X_{2,6} X_{3,5} + b_7 X_{1,3} X_{3,6} + b_{14} X_{1,4} X_{3,6} + b_{20} X_{1,5} X_{3,6} + b_{25} X_{2,4} X_{3,6} + b_{29} X_{2,5} X_{3,6} + b_{32} X_{2,6} X_{3,6} + b_{34} X_{3,5} X_{3,6} + b_8 X_{1,3} X_{4,6} + b_{15} X_{1,4} X_{4,6} + b_{21} X_{1,5} X_{4,6} + b_{26} X_{2,4} X_{4,6} + b_{30} X_{2,5} X_{4,6} + b_{33} X_{2,6} X_{4,6} + b_{35} X_{3,5} X_{4,6} + b_{36} X_{3,6} X_{4,6}) +$$

$$\frac{1}{X_{3,6}}(c_1 X_{1,3} X_{1,4} + c_2 X_{1,3} X_{1,5} + c_9 X_{1,4} X_{1,5} + c_3 X_{1,3} X_{2,4} + c_{10} X_{1,4} X_{2,4} + c_{16} X_{1,5} X_{2,4} + c_4 X_{1,3} X_{2,5} + c_{11} X_{1,4} X_{2,5} + c_{17} X_{1,5} X_{2,5} + c_{22} X_{2,4} X_{2,5} + c_5 X_{1,3} X_{2,6} + c_{12} X_{1,4} X_{2,6} + c_{18} X_{1,5} X_{2,6} + c_{23} X_{2,4} X_{2,6} + c_{27} X_{2,5} X_{2,6} + c_6 X_{1,3} X_{3,5} + c_{13} X_{1,4} X_{3,5} + c_{19} X_{1,5} X_{3,5} + c_{24} X_{2,4} X_{3,5} + c_{28} X_{2,5} X_{3,5} + c_{31} X_{2,6} X_{3,5} + c_7 X_{1,3} X_{3,6} + c_{14} X_{1,4} X_{3,6} + c_{20} X_{1,5} X_{3,6} + c_{25} X_{2,4} X_{3,6} + c_{29} X_{2,5} X_{3,6} + c_{32} X_{2,6} X_{3,6} + c_{34} X_{3,5} X_{3,6} + c_8 X_{1,3} X_{4,6} + c_{15} X_{1,4} X_{4,6} + c_{21} X_{1,5} X_{4,6} + c_{26} X_{2,4} X_{4,6} + c_{30} X_{2,5} X_{4,6} + c_{33} X_{2,6} X_{4,6} + c_{35} X_{3,5} X_{4,6} + c_{36} X_{3,6} X_{4,6})\Big\}$$

Describe the skinny rectangles available and show how these condition the $X_{i,j}$'s:

In[ ]:=

```
s13 = X1,3 + X2,4 - X1,4 ;
s14 = X2,5 + X1,4 - X1,5 - X2,4 ;
s15 = X1,5 + X2,6 - X2,5 ;
s24 = X3,5 + X2,4 - X2,5 ;
s25 = X2,5 + X3,6 - X3,5 - X2,6 ;
s26 = X1,3 + X2,6 - X3,6 ;
s35 = X3,5 + X4,6 - X3,6 ;
s36 = X1,4 + X3,6 - X1,3 - X4,6 ;
s46 = X1,5 + X4,6 - X1,4 ;
(*s_1,3=s_1,4=s_1,5=0*)
locus0 = Solve[0 == {s13, s14, s15}][[1]];
(*s_2,4=s_2,5=s_2,6=0*)
locus1 = Solve[0 == {s24, s25, s26}][[1]];
(*s_3,5=s_3,6=s_1,3=0*)
locus2 = Solve[0 == {s35, s36, s13}][[1]];
(*s_4,6=s_1,4=s_2,4=0*)
locus3 = Solve[0 == {s46, s14, s24}][[1]];
(*s_1,5=s_2,5=s_3,5=0*)
locus4 = Solve[0 == {s15, s25, s35}][[1]];
(*s_2,6=s_3,6=s_4,6=0*)
locus5 = Solve[0 == {s26, s36, s46}][[1]];
(*s_1,3=s_3,6=s_4,6=s,_1,4=0*)
locus6 = Solve[0 == {s13, s36, s46, s14}][[1]];
(*s_2,4=s_1,4=s_1,5=s_2,50*)
locus7 = Solve[0 == {s24, s14, s15, s25}][[1]];
(*s_3,5=s_2,5=s_2,6=s_3,6=0*)
locus8 = Solve[0 == {s35, s25, s26, s36}][[1]];

Totlocu = {locus0, locus1, locus2, locus3, locus4, locus5, locus6}
```

Out[ ]=

$$\{\{X_{2,4} \to -X_{1,3} + X_{1,4},\ X_{2,5} \to -X_{1,3} + X_{1,5},\ X_{2,6} \to -X_{1,3}\},$$
$$\{X_{2,4} \to -X_{1,3},\ X_{3,5} \to X_{1,3} + X_{2,5},\ X_{3,6} \to X_{1,3} + X_{2,6}\},$$
$$\{X_{2,4} \to -X_{1,3} + X_{1,4},\ X_{3,5} \to X_{1,3} - X_{1,4},\ X_{4,6} \to -X_{1,3} + X_{1,4} + X_{3,6}\},$$
$$\{X_{2,5} \to -X_{1,4} + X_{1,5} + X_{2,4},\ X_{3,5} \to -X_{1,4} + X_{1,5},\ X_{4,6} \to X_{1,4} - X_{1,5}\},$$
$$\{X_{2,6} \to -X_{1,5} + X_{2,5},\ X_{3,6} \to -X_{1,5} + X_{3,5},\ X_{4,6} \to -X_{1,5}\},$$
$$\{X_{2,6} \to -X_{1,5},\ X_{3,6} \to X_{1,3} - X_{1,5},\ X_{4,6} \to X_{1,4} - X_{1,5}\},$$
$$\{X_{2,4} \to -X_{1,3} + X_{1,4},\ X_{2,5} \to -X_{1,3} + X_{1,5},\ X_{3,6} \to X_{1,3} - X_{1,5},\ X_{4,6} \to X_{1,4} - X_{1,5}\}\}$$

Apply the above conditions on the ansatz to reduce the constant terms:

In[ ]:=
```
Locs = SolveAlways[0 == Flatten[{ansatz1 /. Totlocu, ansatz1 - (ansatz1)}], Xset9][[1]]
```

Out[ ]=

$$\{a_3 \to 0,\ a_4 \to 0,\ a_5 \to 0,\ a_6 \to 0,\ a_7 \to 0,\ a_2 \to -a_{15} - b_{30} - c_{36},\ a_8 \to -a_{15} - b_{30} - c_{36},\ a_{17} \to 0,$$
$$a_{18} \to 0,\ a_{19} \to 0,\ a_{20} \to 0,\ a_{16} \to -a_{15} - b_{30} - c_{36},\ a_{21} \to 0,\ a_{22} \to 0,\ a_{23} \to 0,\ a_{24} \to 0,$$
$$a_{25} \to 0,\ a_{26} \to -a_{15} - b_{30} - c_{36},\ a_{27} \to 0,\ a_{28} \to 0,\ a_{29} \to 0,\ a_{30} \to 0,\ a_{31} \to 0,\ a_{32} \to 0,$$
$$a_{33} \to 0,\ a_{34} \to 0,\ a_{35} \to 0,\ a_{36} \to 0,\ b_1 \to 0,\ b_3 \to 0,\ b_5 \to 0,\ b_7 \to 0,\ b_6 \to 0,\ b_2 \to 0,\ b_8 \to 0,$$
$$b_9 \to 0,\ b_{10} \to 0,\ b_{13} \to 0,\ b_{14} \to 0,\ b_{12} \to 0,\ b_{15} \to 0,\ b_{18} \to 0,\ b_{20} \to 0,\ b_{16} \to -a_{15} - b_{30} - c_{36},$$
$$b_{19} \to -a_{15} - b_{30} - c_{36},\ b_{21} \to 0,\ b_{24} \to 0,\ b_{25} \to 0,\ b_{26} \to 0,\ b_{23} \to -a_{15} - b_{30} - c_{36},\ a_{14} \to -b_{29},$$
$$b_{32} \to 0,\ b_{31} \to -a_{15} - b_{30} - c_{36},\ b_{33} \to 0,\ b_{34} \to 0,\ b_{35} \to 0,\ b_{36} \to 0,\ c_1 \to 0,\ c_3 \to 0,\ c_4 \to 0,$$
$$c_5 \to 0,\ a_1 \to a_{15} - b_4 + b_{30} - c_7 + c_{36},\ c_6 \to -a_{15} - b_{30} - c_{36},\ c_2 \to 0,\ c_8 \to -a_{15} - b_{30} - c_{36},$$
$$c_9 \to 0,\ c_{10} \to 0,\ c_{11} \to 0,\ c_{12} \to 0,\ c_{13} \to 0,\ b_{11} \to -c_{14},\ c_{15} \to 0,\ c_{17} \to 0,\ c_{18} \to 0,$$
$$c_{19} \to 0,\ a_9 \to a_{15} - b_{17} + b_{30} - c_{20} + c_{36},\ c_{16} \to 0,\ c_{21} \to 0,\ c_{22} \to 0,\ c_{23} \to 0,\ c_{24} \to 0,$$
$$a_{10} \to a_{15} - b_{22} + b_{30} - c_{25} + c_{36},\ c_{26} \to 0,\ c_{27} \to 0,\ c_{28} \to 0,\ a_{11} \to -c_{29},\ c_{30} \to 0,\ c_{31} \to -a_{15} - b_{30} - c_{36},$$
$$a_{12} \to a_{15} - b_{27} + b_{30} - c_{32} + c_{36},\ c_{33} \to -a_{15} - b_{30} - c_{36},\ a_{13} \to a_{15} - b_{28} + b_{30} - c_{34} + c_{36},\ c_{35} \to 0\}$$

Result:

In[ ]:=
```
anss = ansatz1 /. Locs // FullSimplify
```

Out[ ]=

$$\Big\{-\frac{1}{X_{1,4} X_{2,5} X_{3,6}} (a_{15} + b_{30} + c_{36}) \big(X_{2,4} X_{2,5} X_{3,6} (X_{1,5} + X_{4,6}) +$$
$$X_{1,3} X_{2,5} (X_{3,6} (X_{1,5} + X_{4,6}) + X_{1,4} (X_{3,5} - X_{3,6} + X_{4,6})) + X_{1,4} ((X_{1,5} + X_{2,6}) (X_{2,4} + X_{3,5}) X_{3,6} +$$
$$X_{2,5} (-X_{3,6} (X_{1,5} + X_{2,4} + X_{3,5} + X_{4,6}) + X_{2,6} (X_{3,5} - X_{3,6} + X_{4,6})))\big)\Big\}$$