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Comparing frameworks for modelling non-linear electrical circuits

Bachelor's Project Mathematics

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Abstract

Huge amounts of power are being transmitted constantly through various electrical circuits on all scales. In order to minimise transmission losses, we want to control this power flow. Control is achieved through mathematical models. We cover two models, one based on energy, the other based on power. We investigate the implications of the presence of non-linear elements. We consider stability results based on different constraints. Finally, we establish a duality relation between the models and compare their applicability to real-life situations.

Contents

1 Introduction

This paper deals with two modelling frameworks for non-linear RLC circuits. An RLC circuit is a kind of electrical network. Often, a network is compared to a graph, with the edges representing elements and the vertices representing their interconnection points. An important feature of a circuit is that its graph is connected; it is a closed loop, which means that electric current is able to flow. Elements of an RLC circuit are resistors, inductors and capacitors. These elements are defined, and the relations between them explained, in Section 2.1. Electrical circuits appear everywhere; from tiny microchips in computers and mobile phones to huge power grids stretching over hundreds of kilometres. It is important that, when current travels through these circuits and power is transmitted, losses are minimised. This is achieved through the field of *control*. A mathematical model for a circuit is created. The model describes the circuit in ideal terms. Then, controllers are designed to favourably change the behaviour of the system. Two examples of such models are the port-Hamiltonian model and the Brayton-Moser model. The port-Hamiltonian model has been most notably treated by van der Schaft ([17], [16], [15] and many more), who was also the first to propose it in 1992, together with Maschke [12]. The Brayton-Moser model has been around since the 1960s. It was developed by Moser while he was researching stability of systems with tunnel diodes [13], and then he generalised it a few years later with Brayton [1], [2]. MacFarlane also did considerable research on the topic, exploring an approach related to invariant integrals [11]. Both models are derived and explained on a simple example of an RLC circuit in Sections 2.2 and 2.3. Using these models, we can also investigate *stability* properties. Stability is desirable because it allows us to predict the system's behaviour. Stability is generally determined with Lyapunov functions and the functions appearing in these two models happen to be candidate Lyapunov functions. Stability results are covered in Section 3. One of three theorems by Brayton and Moser is proven, with the condition of linear resistance. Jeltsema [6] later proved the theorem that was missing from Brayton and Moser's original paper. Jeltsema's work was crucial to the writing of this thesis as he has, together with Scherpen, done a lot of research on the relations between the two models [7], [6], [8]. In [7], they define a dual relation connecting the two models, which is explored in Section 4.

2 Modelling frameworks

2.1 Electrical circuits

In this section we lay out the physical quantities pertaining to electrical circuits and the laws dictating the relations between these quantities. We also introduce standard circuit elements [5].
For the purpose of this text we apply the modelling frameworks to an RLC circuit. The physical quantities relevant to this kind of electrical circuit are electric current, voltage, electric charge and magnetic flux.
Electric current quantifies the amount of charged particles passing through a surface over time. It is denoted by i. Voltage corresponds to the work required for an electric charge to move from one place to another. It is denoted by v. Electric charge denotes the state of a particle where it is affected by an electromagnetic field. Charge is denoted by q. Finally, magnetic flux quantifies the magnetic field passing through a surface and is denoted by φ.

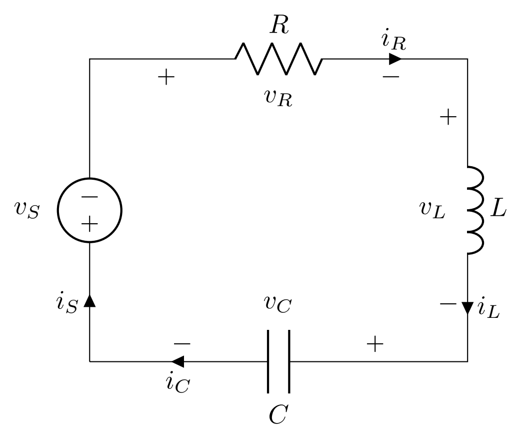

Figure 1: RLC circuit.

The simplest version of an RLC circuit consists of a source, a resistor (R), an inductor (L), and a capacitor (C) in a series configuration, see Figure 1.

Definition 2.1 (Source)**.** A *voltage source* is an element that maintains a prescribed voltage in the circuit. A *current source* maintains a prescribed current in the circuit.

An example of a source is a battery or a generator. The source used in our example is a voltage source and its voltage is denoted by v_S. Note that the direction of the voltage of the source is opposite to the rest of the circuit; this is the conventional way to express that the source delivers power to the rest of the circuit, so the power (voltage times current) of the source is negative.

Definition 2.2 (Resistor)**.** A *resistor* is an element described by a relation between its voltage and current:

$$R(v_R, i_R) = 0. \tag{2.1.1}$$

This relation is called the *characteristic* of the resistor. In case the characteristic is a linear function, (2.1.1) can be written as

$$v_R(t) = -Ri_R(t). \tag{2.1.2}$$

This equation is known as *Ohm's law*. In this case, the resistor is said to be linear.

As the name suggests, a resistor "resists" the flow of electricity by limiting the current passing through it. In (2.1.2), R denotes the *resistance*, a constant inherent to the resistor which represents the scale at which this occurs. The bigger the resistance, the more the current is reduced passing through the resistor.

Definition 2.3 (Capacitor)**.** A *capacitor* is an element described by a relation between its charge and voltage:

$$C(q_C, v_C) = 0.$$

In case the charge can be expressed as a function of the voltage ($q_C = \hat{q}_C(v_C)$), the capacitor is called *voltage-controlled*. If the voltage is a function of the charge ($v_C = \hat{v}_C(q_C)$), it is *charge-controlled*.

A capacitor is able to store electric charge and release it at a later point in time.
The current passing through the capacitor is given by the following relation:

$$i(t) = \frac{dq}{dt}. \tag{2.1.3}$$

Definition 2.4 (Inductor)**.** An *inductor* is an element described by a relation between its magnetic flux and current:

$$L(\varphi_L, i_L) = 0.$$

If the flux can be expressed as a function of the current ($\varphi_L = \hat{\varphi}_L(i_L)$), the inductor is called *current-controlled*. If the current is a function of the flux ($i_L = \hat{i_L}(\varphi_L)$), it is *flux-controlled*.

An inductor is also able to store energy in the magnetic field it creates.
The voltage across the inductor is given by

$$v(t) = \frac{d\varphi}{dt}. \tag{2.1.4}$$

This relation is known as *Faraday's induction law*.

There are two more important laws related to electrical circuits. Both were formulated by the German physicist Gustav Kirchhoff in the mid-19th century. *Kirchhoff's current law* states that the sum of currents entering a node[1] is the same as the sum of currents exiting the node. In our RLC circuit, this is equivalent to saying the current passing through each element is the same, that is:

$$i_S = i_R = i_C = i_L. \tag{2.1.5}$$

Kirchhoff's second law is the *voltage law*, which states the sum of all voltages around a loop[2] is equal to zero. Since our RLC circuit is indeed a loop, we have:

$$-v_S + v_R + v_C + v_L = 0. \tag{2.1.6}$$

The minus sign in front of the source voltage represents the opposite reference direction of the source.

In the following sections, we use these relations to arrive at different mathematical models for the RLC circuit.

2.2 The Port-Hamiltonian approach

One way of modelling physical systems is by viewing them through a port-Hamiltonian lens. This model was introduced by Maschke and van der Schaft in 1992 [12] and it combines *port-based modelling* with the *Hamiltonian* equations used in mechanics. *Port-based modelling* was introduced by Henry Paynter in the 1960s [14]. The idea is to view a physical system as an interconnection of elements through so-called ports.

[1] A circuit can be regarded as consisting of branches and nodes. The branches contain the circuit elements while the nodes are the connections between them. If looking at Figure 1, the nodes are the vertices of the rectangle representing the circuit.

[2] A loop is a closed path of branches, that is, its start node is the same as its end node

For a mechanical system, the *Hamiltonian equations* are given as follows:

$$\dot{q} = \frac{\partial H}{\partial p}(q, p)$$

$$\dot{p} = -\frac{\partial H}{\partial q}(q, p) + F.$$

Here, q is the vector of displacements, p is the vector of momenta, H is the Hamiltonian equation representing the total energy stored in the system, and F is the input of external forces. Combining the port-based model and the idea of the Hamiltonian, we can create a more general model which can be used to describe different kinds of physical systems [17]. We separate the elements of the system into three categories: energy-storing, energy-dissipating and energy-routing. Energy-storing elements are those which are able to store energy. In mechanical systems, it can be a mass at a height storing gravitational potential energy, while in an electrical circuit it might be a capacitor. Energy-dissipating elements reduce the energy of the system by releasing it into the environment. Examples are dampers and resistors. Energy-routing elements are for example gyrators and transformers. We model the system by grouping elements in this way and observing the relations between the groups, which are expressed through *effort* (e) and *flow* (f) variables. These variables come in pairs (e, f) which represent the ports between the groups of elements and their product is equal to power. We can visualise this representation as in Figure 2: We denote the group of energy routing elements by $\mathcal{D}$. This stands for *Dirac structure*; this is a notion from geometry which we use to describe the interconnection structure together with the set of interconnection and constraint equations. These equations describe relations between the efforts, and relations between the flows. For an electrical circuit these equations are precisely Kirchhoff's laws. The group of energy-storing elements, which we denote by $\mathcal{S}$, is connected to $\mathcal{D}$ by (e_S, f_S). The energy-dissipating elements, denoted $\mathcal{R}$ for "resistive", are similarly connected to $\mathcal{D}$ by (e_R, f_R). The external port is connected to $\mathcal{D}$ by (e_P, f_P).

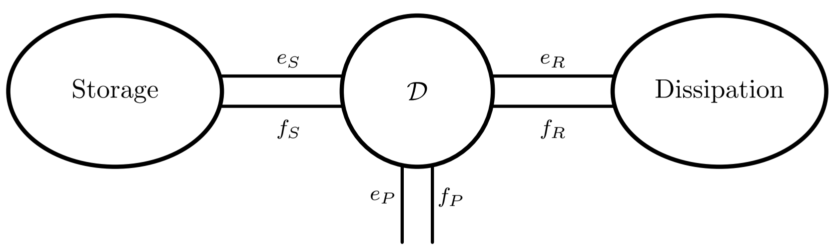

Figure 2: Representation of a port-Hamiltonian system.

The Dirac structure is crucial to this way of representing a physical system. In an electrical circuit, it can physically be imagined as the wiring connecting the elements. Mathematically, it can be viewed as a generalisation of Kirchhoff's laws. Formally, it is defined as follows [17]:

Definition 2.5. Consider a finite-dimensional[3] linear space $\mathcal{F}$ (the space of flows, usually $\mathbb{R}^n$) with dual space $\mathcal{E} = \mathcal{F}^*$. A subspace $\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$ is a *Dirac structure* if it satisfies the following conditions:

1. The duality product $< e \mid f >$ (for $\mathcal{F} = \mathbb{R}^n$, $< e \mid f >= e^T f$) is equal to zero for all $(f, e) \in \mathcal{D}$.

2. $\dim D = \dim F$.

[3]The Dirac structure, and thus the port-Hamiltonian formulation can be generalised to infinite-dimensional systems using Stokes' theorem, see [17].

The first condition tells us total power flowing into the Dirac structure is equal to zero. This implies power conservation in the system which is an important feature of the interconnection structure.
We define the energy-storing port. The energy stored in the system is given by the Hamiltonian, a function $H : \mathcal{X} \to \mathbb{R}$, where $\mathcal{X}$ is the state space of the energy-storing elements. The vector of flow variables f_S is given by the derivative of the vector of state variables:

$$f_S = -\dot{x}.$$

The vector of effort variables is some function of the state:

$$e_S = F(x).$$

We can use this to find the Hamiltonian. The Hamiltonian is an energy function, and the time derivative of energy is power, which is the product of effort and flow[4]:

$$\frac{dH}{dt} = \left\langle \frac{\partial H}{\partial x}, \dot{x} \right\rangle = -\langle e_S, f_S \rangle = \langle e_S, \dot{x} \rangle.$$

We conclude:

$$\frac{\partial H}{\partial x} = e_S = F(x).$$

We find the Hamiltonian:

$$H(x) = \int F(x) dx.$$

The effort and flow of the resistive port are given by a relation $D(e_R, f_R)$ such that $\langle e_R, f_R \rangle \leqslant 0$, indicating energy dissipation. When the relation is linear, we can express it by

$$f_R = -De_R, \tag{2.2.1}$$

where D is a positive semidefinite symmetric matrix.
Given that the resistive structure is linear and the relation $\langle e_S, f_S \rangle + \langle e_R, f_R \rangle + \langle e_P, f_P \rangle = 0$, we can represent the combination of the Dirac structure and the resistive elements as:

$$\begin{bmatrix} f_S \\ f_P \end{bmatrix} = \left(\begin{bmatrix} -J & -B \\ B^T & 0 \end{bmatrix} + \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} e_S \\ e_P \end{bmatrix}. \tag{2.2.2}$$

Using the fact that $f_S = -\dot{x}$, $e_S = \frac{\partial H}{\partial x}$, and setting $u = e_S$ (input) and $y = f_S$ (output) we can write the *port-Hamiltonian equations*:

$$\dot{x} = (J - D)\frac{\partial H}{\partial x}(x) + Bu \tag{2.2.3}$$

$$y = B^T \frac{\partial H}{\partial x}(x). \tag{2.2.4}$$

The matrix J is skew-symmetric ($J^T = -J$) and represents the interconnection structure, and its graph $\{(f, e) \in \mathcal{F} \times \mathcal{E} \mid f = Je\}$ is a Dirac structure. Namely, we have

$$\begin{aligned} \langle e, f \rangle &= \langle e, Je \rangle \\ &= e^T Je \\ &= (e^T Je)^T \\ &= e^T J^T e \\ &= -e^T Je \\ &= 0. \end{aligned}$$

The matrix D is as in (2.2.1). The matrix B represents the interconnection of the external port to the rest of the system.

[4]The minus sign in the second equality represents that $\frac{\partial^T H}{\partial x}\dot{x}$ is the power flowing into the storage port, while $e_S^T f_S$ is the power flowing into the Dirac structure so they must have opposite sign.

2.2.1 Example

Let us apply this approach to the RLC circuit of Figure 1. First we establish the storage port. The energy-storing elements of the circuit are the capacitor and inductor, as they are indeed able to store energy. From (2.1.3) and (2.1.4) we can conclude the state variable of the capacitor is charge, while the state variable of the inductor is magnetic flux:

$$x = \begin{bmatrix} q \\ \varphi \end{bmatrix}.$$

From the same equations we also find that the flow variable of the capacitor is current and the flow variable of the inductor is voltage: $f_S = \begin{bmatrix} f_C \\ f_L \end{bmatrix} = \begin{bmatrix} i_C \\ v_L \end{bmatrix}$. Finally, we find the effort variables using the fact that the effort is a function of the state. The effort variables for the capacitor and inductor are then voltage and current respectively: $e_S = \begin{bmatrix} e_C \\ e_L \end{bmatrix} = \begin{bmatrix} v_C \\ i_L \end{bmatrix}$.
We find the Hamiltonian of the system as the sum of the Hamiltonians of the capacitor and inductor:

$$H(q, \varphi) = H_C(q) + H_L(\varphi) = \int^{q_C} \hat{v}_C(q)\, dq + \int^{\varphi_L} \hat{i}_L(\varphi)\, d\varphi. \tag{2.2.5}$$

We express the effort variables through the Hamiltonian:

$$e_S = \begin{bmatrix} v_C \\ i_L \end{bmatrix} = \begin{bmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial \varphi} \end{bmatrix}.$$

The resistive port consists only of the resistor. The effort variable is the current, $e_R = i_R$, and the flow variable is $f_R = v_R$. Our resistor is linear which means the relation between effort and flow is given by (2.1.2): $f_R = -Ri_R$.
The external port consists of the source. The effort, i.e., the input, is the source voltage $e_P = u = v_S$ and the flow (output) is the source current $f_P = y = i_S$.
We want to find an expression for $\dot{x}$, that is, $\dot{q}$ and $\dot{\varphi}$, in terms of $\frac{\partial H}{\partial x}(x)$ and u.
We take (2.1.3) and use the equality from Kirchhoff's current law (2.1.5):

$$\dot{q} = i_C = i_L = \frac{\partial H}{\partial \varphi}(\varphi) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q}(q) \\ \frac{\partial H}{\partial \varphi}(\varphi) \end{bmatrix}. \tag{2.2.6}$$

Similarly, we take (2.1.4) and use Kirchhoff's voltage law (2.1.6) and Ohm's law (2.1.2):

$$\begin{aligned} \dot{\varphi} = v_L &= v_S - v_R - v_C \\ &= -v_C - Ri_R + v_S \\ &= -V_C - Ri_L + v_S \\ &= \begin{bmatrix} -1 & -R \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q}(q) \\ \frac{\partial H}{\partial \varphi}(\varphi) \end{bmatrix} + v_S. \end{aligned} \tag{2.2.7}$$

In the second to last step we also use Kirchhoff's current law again.
We combine (2.2.6) and (2.2.7) to obtain

$$\dot{x} = \begin{bmatrix} \dot{q} \\ \dot{\varphi} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -R \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q}(q) \\ \frac{\partial H}{\partial \varphi}(\varphi) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v_S. \tag{2.2.8}$$

To complete the port-Hamiltonian formulation, we also include the output equation. In this case the output is the current through the source, which we can find using Kirchhoff's current law:

$$i_S = i_L = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q}(q) \\ \frac{\partial H}{\partial \varphi}(\varphi) \end{bmatrix}. \tag{2.2.9}$$

Indeed, the equations (2.2.8) and (2.2.9) match the requirements from (2.2.3) and (2.2.4), with $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ skew-symmetric and $D = \begin{bmatrix} 0 & 0 \\ 0 & R \end{bmatrix}$ positive semidefinite as R, the resistance of the resistor, is a non-negative quantity.

2.3 The Brayton-Moser equations

In the 1960s, Robert Brayton and Jürgen Moser developed another way to characterise nonlinear networks [1]. Their so-called mixed-potential function is similar to the Hamiltonian in the sense that it is a single function describing the behaviour of the system. However, there are some differences: while the Hamiltonian is given in terms of energy, the mixed-potential function is given in terms of power. Furthermore, while the Hamiltonian does not include resistive elements, which are included by expanding to the port-Hamiltonian formulation, the mixed-potential inherently includes resistive elements. The mixed potential function is a function $\mathscr{P}(i_L, v_C)$ satisfying the following relations:

$$v_L = -\frac{\partial \mathscr{P}}{\partial i_L} \tag{2.3.1}$$

$$i_C = \frac{\partial \mathscr{P}}{\partial v_C}. \tag{2.3.2}$$

Using this, we can represent the system as follows:

$$\begin{bmatrix} -L & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} \frac{di_L}{dt} \\ \frac{dv_C}{dt} \end{bmatrix} = \begin{bmatrix} \frac{\partial \mathscr{P}}{\partial i_L} \\ \frac{\partial \mathscr{P}}{\partial v_C} \end{bmatrix}, \tag{2.3.3}$$

where $L = -\frac{d\hat{\varphi}}{di_L}$ (known as the incremental inductance matrix), $C = -\frac{d\hat{q}}{dv_C}$ (known as the incremental capacitance matrix)[5]. We can verify this corresponds to (2.3.1) and (2.3.2):

$$-\left\langle \frac{d\hat{\varphi}}{di_L}, \frac{di_L}{dt} \right\rangle = -\frac{d\hat{\varphi}}{dt} = -v_L = \frac{\partial \mathscr{P}}{\partial i_L}$$

$$-\left\langle \frac{d\hat{q}}{dv_C}, \frac{dv_C}{dt} \right\rangle = \frac{d\hat{q}}{dt} = i_C = \frac{\partial \mathscr{P}}{\partial v_C}.$$

The existence of the mixed-potential function was proven by Brayton and Moser for the class of complete RLC networks, that is, networks that satisfy the following conditions [6], [4]:

1. There are no cutsets[6] formed exclusively by inductors and/or current sources. There are no loops formed exclusively by capacitors and/or voltage sources.

2. Each current-controlled (but not voltage-controlled) resistor is in series with an inductor and each voltage-controlled (but not current-controlled) resistor is in parallel with a capacitor.

3. Each remaining resistor has a bijective characteristic relation, i.e., current-voltage relation.

An algorithm for the construction of the mixed-potential was given for the class of topologically complete networks. These networks, in addition to being complete, it needs to satisfy the following condition [6]: all current-controlled resistors are in series with the inductors and all voltage-controlled resistors are in parallel with the capacitors. Sources are also considered resistive elements. A topologically complete circuit is visualised in Figure 3.

[5] It makes sense to define the matrices L and C without the minus signs. However, Brayton and Moser use an opposite sign convention for Faraday's law and the corresponding law for current ($v = -\dot{\varphi}, i = -\dot{q}$) to what is used in van der Schaft's texts on port-Hamiltonian systems ($v = \dot{\varphi}, i = \dot{q}$). In order to maintain consistency for equations (2.1.3) and (2.1.4), the minus signs are inserted here instead.

[6] The notion of a cutset is borrowed from graph theory: it is a set of edges which, if removed from the graph, would create a disconnected graph.

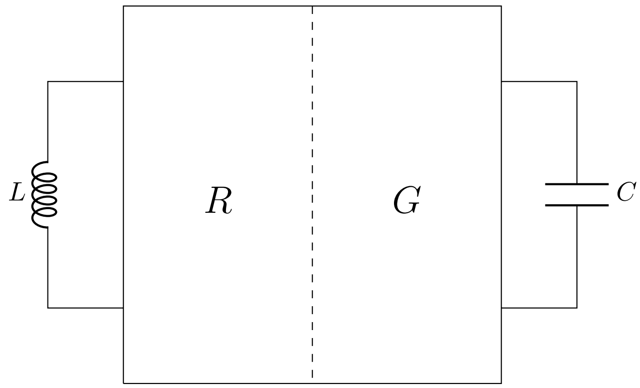

Figure 3: Rough representation of a topologically complete circuit. Note that L and C represent all the inductors and all the capacitors in the circuit respectively. R represents the current-controlled resistive elements while G represents the voltage-controlled resistive elements, also known as conductors.

Then, the network is described by the following equations. which can be obtained using Kirchhoff's laws:

$$v_L = -Dv_C - D\hat{v}_R(D_R^T i_L) \tag{2.3.4}$$

$$i_C = D^T i_L - D_G^T \hat{i}_G(D_G v_C), \tag{2.3.5}$$

where $D \in \mathbb{R}^{n_L \times n_C}$ represents the interconnection of the inductors and capacitors, $D_R \in \mathbb{R}^{n_R \times n_L}$ represents the interconnection of the current-controlled resistive elements with the inductors and $D_R^T i_L = i_R$. The matrix $D_G \in \mathbb{R}^{n_C \times n_G}$ represents the interconnection of the voltage-controlled resistive elements with the capacitors and $D_G v_C = v_G$. When the network has this form, it can be represented by the mixed-potential function as follows:

$$\mathscr{P}(i_L, v_C) = \mathscr{G}_R(i_L) - \mathscr{J}_G(v_C) + \langle i_L, Dv_c \rangle. \tag{2.3.6}$$

The quantity $\mathscr{G}_R$ is called the content or current potential of the current-controlled resistive elements and is given by the following equation:

$$\mathscr{G}_R(i_L) = \int^{i_L} (D_R \hat{v}_R(D_R^T i))^T di.$$

Similarly, $\mathscr{J}_G$ is called the co-content or voltage potential of the voltage-controlled resistive elements and is given by:

$$\mathscr{J}_G(v_C) = \int^{v_C} (D_G^T \hat{i}_G(D_G v))^T dv.$$

An illustration of the notions of content and co-content is given in Figure 4.

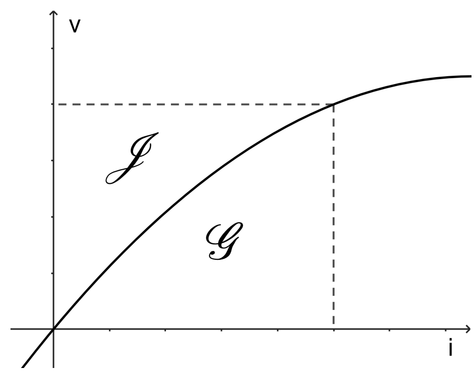

Figure 4: A resistive element is described by a curve in the iv plane. The content $\mathscr{G}$ is then simply the area under the curve, while the co-content $\mathscr{J}$ is the area above the curve.

Finally, $\langle i_L, Dv_c \rangle$ represents the power delivered from the capacitors to the inductors.

2.3.1 Example

Before applying (2.3.6) to find the mixed-potential function for our simple RLC circuit, we verify that it is a topologically complete circuit. For this purpose, we consider the linear resistor as a current-controlled resistor, and the voltage source as a current-controlled resistive element too. Indeed, conditions 1-3 for completeness are then easily verified. Furthermore, both current-controlled resistive elements are in series with the inductor so the circuit is topologically complete. Let us represent the circuit as in (2.3.4) and (2.3.5). We note since there are no voltage-controlled resistive elements in the circuit, the co-content element of (2.3.5) is equal to zero. So, to compute the mixed-potential, we only need to find $D_R \hat{v}_R(D_R^T i_L)$ and D. We find D, D_R as follows: since we have one inductor and one capacitor, D is a scalar. It relates v_L and v_C. Since both of these voltages have the same direction, we have $D = 1$. Furthermore, D_R is a 2×1 matrix. As the voltage of the source and the voltage of the resistor have opposite directions, we have $D_R = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. There is one more thing to clarify: the function $\hat{v}_R(D_R^T i_L)$ gives us the voltages of the resistor and source. The resistor voltage is a linear function of i_R, which is equal to i_L, so $\hat{v}_R(i_L) = -Ri_L$. The source voltage is not a function of the current, but it can be written simply as $\hat{v}_S(i_L) = v_S$. With all this, we have:

$$\begin{aligned} D_R \hat{v}_R(D_R^T i_L) &= \begin{bmatrix} 1 & -1 \end{bmatrix} \hat{v_R} \begin{bmatrix} 1 \\ -1 \end{bmatrix} i_L \\ &= \begin{bmatrix} 1 & -1 \end{bmatrix} \hat{v_R} \begin{bmatrix} i_L \\ -i_L \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} -Ri_L \\ v_S \end{bmatrix} \\ &= -Ri_L - v_S. \end{aligned}$$

We plug this into (2.3.5) to find the content $\mathscr{G}(i_L)$:

$$\begin{aligned} \mathscr{G}_R(i_L) &= \int^{i_L} (D_R \hat{v}_R(D_R^T i))^T di \\ &= \int^{i_L} -Ri - v_S di \\ &= \left[-R\frac{i^2}{2} - v_S i \right]^{i_L} \\ &= -R\frac{i_L^2}{2} - i_L v_S. \end{aligned}$$

The power $\langle i_L, Dv_c \rangle$ is simply equal to $i_L v_C$ since $D = 1$, so we have the mixed-potential function:

$$\mathscr{P}(i_L, v_C) = -R\frac{i_L^2}{2} - i_L v_S + i_L v_C. \tag{2.3.7}$$

We differentiate (2.3.7) with respect to i_L and v_C respectively to see if it matches (2.3.1) and (2.3.2):

$$\frac{\partial \mathscr{P}}{\partial i_L} = -Ri_L - v_S + v_C = -v_L.$$

The second equality follows from Kirchhoff's voltage law.

$$\frac{\partial \mathscr{P}}{\partial v_C} = i_L = i_C.$$

The second equality follows from Kirchhoff's current law.
So, $\mathscr{P}(i_L, v_C) = R\frac{i_L^2}{2} - i_L v_S + i_L v_C$ is indeed the mixed-potential function representing our circuit. Then, we can also represent the circuit as in (2.3.3):

$$\begin{bmatrix} -L & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} \frac{di_L}{dt} \\ \frac{dv_C}{dt} \end{bmatrix} = \begin{bmatrix} -Ri_L - v_S + v_C \\ i_L \end{bmatrix}.$$

3 Stability

When studying a certain physical system, the goal is often to determine stability conditions, that is, under which conditions the system's trajectories stay near one another or tend towards an equilibrium state. This is usually accomplished using the theory developed by Lyapunov in the late 19th century [10]. A Lyapunov function is defined as follows [9]:

Definition 3.1. Consider a system defined by $\dot{x} = f(x)$ with an equilibrium at the origin. A function $V(x)$ is called a *Lyapunov function* if it satisfies the following properties:

1. The function and its first derivatives are continuous on an open set Ω around the origin.
2. The function is equal to zero at the origin and positive elsewhere on Ω.
3. The derivative $\dot{V} = \frac{dV}{dt} = \nabla V \cdot \dot{x}$ is non-positive on Ω.

When a Lyapunov function exists, the origin is *stable*. If the derivative $\dot{V}$ is strictly negative on Ω, the origin is *asymptotically stable*. Furthermore, if the Lyapunov function is *radially unbounded*, that is, when $V(x) \to \infty$ as $|x| \to \infty$, the origin is *globally asymptotically stable*, that is, all trajectories in the state space tend to the equilibrium.
It turns out that the Hamiltonian and the mixed-potential function can be used to find Lyapunov functions for the RLC system. In fact, Moser first formulated a version of the mixed-potential function while trying to define stability for circuits with tunnel diodes [13], and the papers where the mixed-potential was officially introduced [1], [2], which he published with Brayton a few years later, also contain stability theorems.

3.1 The Hamiltonian as a Lyapunov function

We consider the port-Hamiltonian system

$$\dot{x} = (J - D)\frac{\partial H}{\partial x}. \tag{3.1.1}$$

We omit the input as, when determining stability, we generally look at zero-input systems.

Lemma 3.1. *For the system in (3.1.1), assuming*[7] *the Hamiltonian $H(x)$ is continuously differentiable on a set $\Omega \subset \mathcal{X}$, equal to 0 at the origin and positive elsewhere on Ω, the Hamiltonian $H(x)$ can be used as a Lyapunov function to show stability.*

Proof. We can assume that the Hamiltonian is indeed continuously differentiable. Furthermore, as it represents the energy of the system, it is clear that it is zero when the state variables are zero and positive otherwise. It is straightforward to verify that $\dot{H} \leqslant 0$. This follows from the power-conserving nature of the Dirac structure and the dissipative nature of the resistive structure:

$$\begin{aligned}
\frac{dH}{dt} &= \left\langle \frac{\partial H}{\partial x}, \frac{dx}{dt} \right\rangle \\
&= \left\langle \frac{\partial H}{\partial x}, (J-D)\frac{\partial H}{dx} \right\rangle \\
&= \left\langle \frac{\partial H}{\partial x}, J\frac{\partial H}{\partial x} \right\rangle - \left\langle \frac{\partial H}{\partial x}, D\frac{\partial H}{\partial x} \right\rangle \\
&= 0 - \left\langle \frac{\partial H}{\partial x}, D\frac{\partial H}{\partial x} \right\rangle \qquad (3.1.2)\\
&\leqslant 0. \qquad (3.1.3)
\end{aligned}$$

The equality (3.1.2) follows from the skew-symmetry of J, while the inequality (3.1.3) follows from the fact that D is positive semidefinite. □

3.2 Finding a Lyapunov function from the mixed-potential function [1]

In the paper introducing the mixed-potential function, Brayton and Moser also prove several stability theorems. The theorem and proof presented here concerns a class of systems with linear resistors and nonlinear inductors and capacitors, which matches our RLC circuit. With some additional requirements, global asymptotic stability is proven for these systems.

Before stating and proving the theorem, we do a preliminary analysis. For the purposes of determining stability, Brayton and Moser write the equation (2.3.3) as follows:

$$-J\dot{x} = \frac{\partial \mathscr{P}}{\partial x}(x), \qquad (3.2.1)$$

where $J = \begin{bmatrix} -L & 0 \\ 0 & C \end{bmatrix}$, $x = \begin{bmatrix} i_L \\ v_C \end{bmatrix}$ and $\mathscr{P}$ is the mixed-potential function as before. We take $\mathscr{P}$ as a candidate Lyapunov function. We differentiate $\mathscr{P}$ to see whether it requires the Lyapunov conditions. This gives

$$\frac{d\mathscr{P}}{dt} = \left\langle \frac{\partial \mathscr{P}}{\partial x}(x), \dot{x} \right\rangle = \left\langle \dot{x}, \frac{\partial \mathscr{P}}{\partial x}(x) \right\rangle = -\langle \dot{x}, J\dot{x} \rangle.$$

This is non-positive when J is positive semidefinite, but the matrices L and C are taken to be positive definite, which means J is indefinite. So, we search for a transformation $(J, \mathscr{P}) \to (J^*, \mathscr{P}^*)$ such that (3.2.1) still holds:

$$-J^*\dot{x} = \frac{\partial \mathscr{P}^*}{\partial x}(x). \qquad (3.2.2)$$

We plug in $\dot{x} = -J^{-1}\frac{\partial \mathscr{P}}{\partial x}(x)$ (J^{-1} exists because L and C are assumed to be invertible):

$$J^* J^{-1}\frac{\partial \mathscr{P}}{\partial x}(x) = \frac{\partial \mathscr{P}^*}{\partial x}(x).$$

We want $(J^*, \mathscr{P}^*)$ to satisfy the Lyapunov conditions for global asymptotic stability.

[7] In general, when dealing with energy functions, we can assume the first two conditions of Definition 3.1 hold. The same applies for the mixed-potential function.

Lemma 3.2. *If we set*

$$J^* = \left(\lambda I + \frac{\partial^2 \mathscr{P}}{\partial x^2}(x) M\right) J, \quad \mathscr{P}^* = \lambda \mathscr{P} + \frac{1}{2}\left\langle \frac{\partial \mathscr{P}}{\partial x}(x), M \frac{\partial \mathscr{P}}{\partial x}(x) \right\rangle, \tag{3.2.3}$$

where M is any constant symmetric matrix, (3.2.2) holds.

Proof. Note that if $(J_1,\ \mathscr{P}_1)$ and $(J_2,\ \mathscr{P}_2)$ satisfy (3.2.2), so does $(J_1 + J_2, \mathscr{P}_1 + \mathscr{P}_2)$. This follows from the sum rule. Clearly $(\lambda J,\ \lambda \mathscr{P})$ satisfies (3.2.2), it is simply (3.2.1) multiplied by λ on both sides. So, it remains to show $\left(\frac{\partial^2 \mathscr{P}}{\partial x^2}(x) M J,\ \frac{1}{2}\left\langle \frac{\partial \mathscr{P}}{\partial x}(x), M \frac{\partial \mathscr{P}}{\partial x}(x)\right\rangle\right)$ satisfies (3.2.2). For simplicity, denote $\left(\frac{\partial^2 \mathscr{P}}{\partial x^2}(x) M J,\ \frac{1}{2}\left\langle \frac{\partial \mathscr{P}}{\partial x}(x), M \frac{\partial \mathscr{P}}{\partial x}(x)\right\rangle\right) = (J_2, \mathscr{P}_2)$. Consider $\frac{\partial \mathscr{P}_2}{\partial x}(x)$:

$$\begin{aligned} \frac{\partial \mathscr{P}_2}{\partial x}(x) &= \frac{1}{2}\left(\left\langle \frac{\partial^2 \mathscr{P}}{\partial x^2}(x), M \frac{\partial \mathscr{P}}{\partial x}(x)\right\rangle + \left\langle \frac{\partial \mathscr{P}}{\partial x}(x), M \frac{\partial^2 \mathscr{P}}{\partial x^2}(x)\right\rangle\right) \\ &= \frac{1}{2} \cdot 2 \left\langle \frac{\partial^2 \mathscr{P}}{\partial x^2}(x), M \frac{\partial \mathscr{P}}{\partial x}(x)\right\rangle \\ &= \left\langle \frac{\partial^2 \mathscr{P}}{\partial x^2}(x), M \frac{\partial \mathscr{P}}{\partial x}(x)\right\rangle. \end{aligned}$$

Then,

$$\begin{aligned} J_2 J^{-1} \frac{\partial \mathscr{P}}{\partial x}(x) &= \frac{\partial^2 \mathscr{P}}{\partial x^2} M J J^{-1} \frac{\partial \mathscr{P}}{\partial x}(x) \\ &= \frac{\partial^2 \mathscr{P}}{\partial x^2} M \frac{\partial \mathscr{P}}{\partial x}(x) \\ &= \frac{\partial \mathscr{P}_2}{\partial x}(x) \end{aligned}$$

as required. □

With this, we can introduce the theorem [1], [6][8].

Theorem 3.1. *If $\mathscr{G}(i_L)$ has the form $\mathscr{G}(i_L) = -\frac{1}{2}\langle i_L, R i_L\rangle$ (where $R = \frac{\partial^2 \mathscr{G}}{\partial i_L^2}(i_L)$ is a symmetric positive definite constant matrix), $-\mathscr{J}_G(v_C) + |D v_C| \to \infty$ as $|v_C| \to \infty$ and there exists $\delta > 0$ such that*

$$\left\| L^{\frac{1}{2}}(i_L) R^{-1} D C^{-\frac{1}{2}}(v_C) \right\| \leqslant 1 - \delta \tag{3.2.4}$$

where $\|\cdot\|$ is the operator norm, for all $(i_L, v_C) \in \mathbb{I}_L \times \mathbb{V}_C$ (the space of inductor currents and capacitor voltages), then all trajectories of (3.2.1) tend to the set of equilibrium points as $t \to \infty$.

Proof. As in (3.2.3), we take

$$M = \begin{bmatrix} 2R^{-1} & 0 \\ 0 & 0 \end{bmatrix}, \quad \lambda = 1.$$

Then,

$$\begin{aligned} J^* &= J + \frac{\partial^2 \mathscr{P}}{\partial x^2}(x) M J \\ &= J + \begin{bmatrix} -R & D \\ D^T & -\frac{d^2 \mathscr{J}}{d v_C^2} \end{bmatrix} \begin{bmatrix} 2R^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -L & 0 \\ 0 & C \end{bmatrix} \\ &= \begin{bmatrix} -L & 0 \\ 0 & C \end{bmatrix} + \begin{bmatrix} 2L & 0 \\ -2D^T R^{-1} L & 0 \end{bmatrix} \\ &= \begin{bmatrix} L & 0 \\ -2D^T R^{-1} L & C \end{bmatrix}. \end{aligned}$$

[8] Both the theorem and the proof are adapted from the combination of [1] and [6] with some added clarifications.

We check $\langle \dot{x}, J^* \dot{x} \rangle$:

$$\begin{aligned}\langle \dot{x}, J^* \dot{x} \rangle &= \begin{bmatrix} \frac{di^T}{dt} & \frac{dv^T}{dt} \end{bmatrix} \begin{bmatrix} L & 0 \\ -2D^T R^{-1} L & C \end{bmatrix} \begin{bmatrix} \frac{di}{dt} \\ \frac{dv}{dt} \end{bmatrix} \\ &= \left\langle \frac{di}{dt}, L\frac{di}{dt} \right\rangle - 2\left\langle \frac{dv}{dt}, D^T R^{-1} L \frac{di}{dt} \right\rangle + \left\langle \frac{dv}{dt}, C\frac{dv}{dt} \right\rangle. \end{aligned} \tag{3.2.5}$$

If we set $y = L^{\frac{1}{2}} \frac{di}{dt}$, $z = C^{\frac{1}{2}} \frac{dv}{dt}$ and $K = L^{\frac{1}{2}} R^{-1} D C^{-\frac{1}{2}}$ as in (3.2.4), we can rewrite (3.2.5) as:

$$\langle \dot{x}, J^* \dot{x} \rangle = \langle y, y \rangle - 2\langle z, Ky \rangle + \langle z, z \rangle = \langle y - Kz, y - Kz \rangle + \langle z, z \rangle - \langle Kz, Kz \rangle.$$

Using the fact that $\|K\| \leqslant 1 - \delta$ from (3.2.4):

$$\begin{aligned}\langle \dot{x}, J^* \dot{x} \rangle &= |y - Kz|^2 + |z|^2 - |Kz|^2 \\ &\geqslant |y - Kz|^2 + |z|^2 - \|K\|^2 |z|^2 && (3.2.6) \\ &= |y - Kz|^2 + |z|^2 (1 - \|K\|^2) \\ &\geqslant |y - Kz|^2 + |z|^2 (1 - \|K\|) && (3.2.7) \\ &\geqslant |y - Kz|^2 + \delta |z|^2 && (3.2.8) \\ &\geqslant 0. && (3.2.9)\end{aligned}$$

The inequality (3.2.6) follows from the property $|Kx| \leqslant \|K\| |x|$ of the operator norm. The inequality (3.2.7) follows from the fact that $0 \leqslant \|K\| \leqslant 1 - \delta < 1$ so $\|K\|^2 \leqslant \|K\|$. Finally, (3.2.8) follows from (3.2.4) and (3.2.9) from the fact that $\delta > 0$ and a vector norm is always greater than or equal to zero. This shows J^* is positive semidefinite, which means the system is stable. In fact, we only have $\frac{d\mathscr{P}^*}{dt} = \langle \dot{x}, J^* \dot{x} \rangle = 0$ when $|y - Kz| = |Kz| = 0$ which happens precisely when $\frac{di}{dt} = \frac{dv}{dt} = 0$, that is, at the equilibria of the system. With Lasalle's invariance principle we can conclude asymptotic stability. Namely, if we have a set $E = \{(i, v) \in \Omega : \frac{d\mathscr{P}^*}{dt} = 0\}$, and its largest invariant[9] subset, M, then all trajectories on Ω tend to M. But since in this case E is the set of equilibria, which stay at the same point forever, we have $M = E$, so solutions in Ω tend to the equilibria. If we show radial unboundedness, this means $\Omega = \mathbb{I}_L \times \mathbb{V}_C$, that is, the entire state space, so the system is globally asymptotically stable. To show this, we show $\mathscr{P}^* \to \infty$ as $|x| \to \infty$.
We set $\alpha = \frac{\partial \mathscr{P}}{\partial i_L} = -Ri_L + Dv$. Then, we can write $\mathscr{P}$ as $\mathscr{P}(\alpha, v_C)$:

$$\mathscr{P}(\alpha, v_C) = -\frac{1}{2}\langle \alpha, R^{-1}\alpha \rangle + U(v_C). \tag{3.2.10}$$

where $U(v_C) = -\mathscr{J}(v_C) + \frac{1}{2}\langle Dv_C, R^{-1} Dv_C \rangle$. Indeed:

$$\begin{aligned}\mathscr{P}(\alpha, v_C) &= -\frac{1}{2}\langle \alpha, R^{-1}\alpha \rangle - \mathscr{J}(v_C) + \frac{1}{2}\langle Dv_C, R^{-1} Dv_C \rangle \\ &= -\frac{1}{2}\langle -Ri_L + Dv_C, R^{-1}(-Ri_L + Dv_C) \rangle - \mathscr{J}(v_C) + \frac{1}{2}\langle Dv_C, R^{-1} Dv_C \rangle \\ &= -\frac{1}{2}\langle -Ri_L + Dv_C, -i_L + R^{-1} Dv_C) \rangle - \mathscr{J}(v_C) + \frac{1}{2}\langle Dv_C, R^{-1} Dv_C \rangle \\ &= -\frac{1}{2}\left(\langle -Ri_L, -i_L \rangle + \langle Dv_C, -i_L \rangle + \langle -Ri_L, R^{-1} Dv_C \rangle + \langle Dv_C, R^{-1} Dv_C \rangle \right) - \mathscr{J}(v_C) + \frac{1}{2}\langle Dv_C, R^{-1} Dv_C \rangle \\ &= -\frac{1}{2}\left(\langle i_L, Ri_L \rangle - 2\langle i_L, Dv_C \rangle \right) - \mathscr{J}(v_C) \\ &= -\frac{1}{2}\langle i_L, Ri_L \rangle - \mathscr{J}(v_C) + \langle i_L, Dv_C \rangle \\ &= \mathscr{P}(i_L, v_C).\end{aligned}$$

[9] An invariant set corresponding to a system $\dot{x} = f(x)$ is a set such that trajectories of solutions that are in the set at $t = 0$ remain in the set for all $t \geqslant 0$.

Next, we note that $\mathscr{P}^*$ can be written in terms of $\mathscr{P}$ and α:

$$\begin{aligned}
\mathscr{P}^*(i_L, v_C) &= \mathscr{P} + \frac{1}{2}\left\langle \frac{\partial \mathscr{P}}{\partial x}(x), M\frac{\partial \mathscr{P}}{\partial x}(x) \right\rangle \\
&= \mathscr{P} + \frac{1}{2}\begin{bmatrix} \frac{\partial^T \mathscr{P}}{\partial i_L} & \frac{\partial^T \mathscr{P}}{\partial v_C} \end{bmatrix}\begin{bmatrix} 2R^{-1} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \frac{\partial \mathscr{P}}{\partial i_L} \\ \frac{\partial \mathscr{P}}{\partial v_C} \end{bmatrix} \\
&= \mathscr{P} + \left\langle \frac{\partial \mathscr{P}}{\partial i_L}, R^{-1}\frac{\partial \mathscr{P}}{\partial i_L} \right\rangle \\
&= \mathscr{P} + \langle \alpha, R^{-1}\alpha \rangle.
\end{aligned}$$

Plugging in (3.2.10), we get:

$$\mathscr{P}^*(\alpha, v_C) = \frac{1}{2}\langle \alpha, R^{-1}\alpha \rangle + U(v_c).$$

Since by assumption $-\mathscr{J}(v_C) + |Dv_C| \to \infty$ as $|v_C| \to \infty$, we also have that $U(v_C) \to \infty$ as $|v_C| \to \infty$. It remains to show that $|\alpha| \to \infty$ as $|i_L| \to \infty$, or equivalently that $|\alpha| + |v_C| \to \infty$ as $|i_L| + |v_C| \to \infty$. Denote the map which sends (i_L, v_C) to (α, v_C) by S:

$$S = \begin{bmatrix} -R & D \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} \alpha \\ v_C \end{bmatrix} = Sx.$$

The matrix $S^T S$ is positive semidefinite ($\langle Sx, Sx \rangle \geqslant 0$) and S is non-singular so $S^T S$ is positive definite; it only has positive eigenvalues. We have that $\langle Sx, Sx \rangle \geqslant \lambda_{\min}\langle x, x \rangle$ where $\lambda_{\min} > 0$ is the minimum eigenvalue of $S^T S$. With this we have

$$|\alpha|^2 + |v_C|^2 \geqslant \lambda_{\min}\left(|i_L|^2 + |v_C|^2\right),$$

which means as $|i_L| + |v_C| \to \infty$, also $|\alpha| + |v_C| \to \infty$ and therefore $\mathscr{P}^* \to \infty$. As $\mathscr{P}^*$ is radially unbounded, the system is globally asymptotically stable, that is, all trajectories tend to the set of equilibria as $t \to \infty$. □

4 Discussion

4.1 Duality

It is apparent that there are some similarities between the port-Hamiltonian equations (2.2.3), (2.2.4) and the Brayton-Moser equations (2.3.3). In a way, these models are *dual*. The port-Hamiltonian approach requires the capacitors to be charge-controlled and the inductors to be flux-controlled, while in the Brayton-Moser model the capacitors are voltage-controlled and the inductors are current-controlled. In an LC-circuit (only inductors and capacitors), in case we have bijective mappings from charge to voltage for the capacitors and from flux to current for the inductors, a relation between the two models can be expressed [7]. We can consider q_C, φ_L to be *energy* variables. Then their duals v_C, i_L are *co-energy* variables. Similarly as in (2.2.5), we can express the total co-energy of the system with the *co-Hamiltonian*:

$$H^*(v_C, i_L) = \int^{v_C} \hat{q}_C(v)\, dv + \int^{i_L} \hat{\varphi}_L(i)\, di.$$

Then, we can write the Brayton-Moser equations as follows:

$$\frac{d}{dt}\frac{\partial H^*}{\partial x}(x) = \Phi\frac{\partial \mathscr{P}}{\partial x}(x),$$

where Φ is a diagonal matrix with a number of copies of the identity matrix corresponding to the amount of capacitors followed by a number of copies of the identity matrix times -1 corresponding

to the amount of inductors.
An important element in the relation of the two models is the Dirac structure. Since it represents the interconnection structure and Kirchhoff's laws, it makes sense that it is inherent to the system regardless of the model. In fact, if we set $J = \begin{bmatrix} 0 & \psi \\ -\psi^T & 0 \end{bmatrix}$, then ψ is precisely the interconnection matrix D from (2.3.6) and the port-Hamiltonian equations can be obtained from the Brayton-Moser equations by multiplying with the Dirac structure and integrating with respect to time. As mentioned, the Hamiltonian and mixed-potential (co-Hamiltonian) are given in terms of energy and co-energy respectively. The duality between these two quantities can be defined with a Legendre transformation.

4.2 Practical use

In terms of real-life applications, both models have their advantages and disadvantages. When designing feedback controllers, we rely on measuring the state or output from the system. In general, it is easier to measure current and voltage than flux and charge. So if we are measuring the output, the port-Hamiltonian approach may be preferable, while if we measure the state, the Brayton-Moser model is more applicable. Furthermore, as flux and charge are the "natural" quantities of the inductor and capacitor respectively, the port-Hamiltonian model may seem like a more obvious choice. However, when it comes to circuits with resistive elements that are defined by a relation between current and voltage, this reasoning falls short as the resistive elements do not fit so nicely into the model. As shown, when there are bijective flux-current and charge-voltage relations, the models can be interchanged. So, it is possible, for example, to create a controller using one model and translate it to the other model.

5 Conclusion

This thesis gives an overview of two ways to model non-linear electrical circuits. The port-Hamiltonian model regards the system as an interconnection of elements, with the "language" of the model being energy, expressed through the Hamiltonian. The Brayton-Moser model takes a similar approach, with the mixed-potential function given in terms of power. These functions are also useful because they can be used to show stability. For the port-Hamiltonian example, stability is easy to show. However, the entire formulation given by (2.2.2) relies on the resistive structure being linear. Therefore, in case of non-linear resistors this does not apply. On the other hand, Brayton and Moser proved three stability theorems in their paper on the mixed-potential function [1]: each theorem required a certain class of elements to be linear. The first theorem required linear resistors, the second, linear conductors, and the third theorem allowed nonlinear resistive elements but required the inductors and capacitors to be linear. A fourth theorem, where all elements are allowed to be non-linear, was proven by Jeltsema over 40 years later [6]. So, the stability results in the Brayton-Moser model are somewhat stronger.
When there are bijective relations between the pairs of dual variables, namely charge and voltage, and flux and current, the two models can be related. The Dirac structure, representing Kirchhoff's laws and the interconnection structure, plays an important role in this relation. This interchangeability also proves useful for practical applications, as there are instances where one model might be more appropriate than the other, but perhaps in the other model it is easier to create a controller which can then be translated into the first model.
With the port-Hamiltonian model only being developed in the last two decades, there is still a lot to be explored. And of course, while the Brayton-Moser model is well-established as it has been around for 60 years, there are always many interesting new approaches or applications. In particular, in terms of these two models there is not much research about the so-called memristor. This theoretical element was proposed by Chua [3] in the 1970s as the element to fulfill the remaining relation between the four main quantities of electrical circuits: the resistor relates current and voltage, the inductor relates current and flux and the capacitor relates charge and voltage. Chua proposed

the memristor, an element relating voltage and flux. An ideal memristor has not yet been developed, but memristive systems (circuits with memristive properties) exist. The implications of the existence of memristors are very exciting for the field of computing science. Therefore it could be interesting to look into how memristors or memristive systems would fit into the port-Hamiltonian and Brayton-Moser models.

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