university of groningen

An Exploration of On-Shell Recursion Relations

in gauge theory and exceptional scalar field theories

Master's Project Physics

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“Amplitudo potentiae [...] magna abundantia.”

-Marcus Tullius Cicero,
De Inventione 2.166 (around the year 90 B.C.)

The amplitude is a great abundance of power.

Contents

1 Introduction

Quantum mechanical amplitudes are arguably the most interesting objects in physics. An amplitude as a quantity tells us the degree to which two states overlap in the Hilbert space of the quantum mechanical system under consideration. Hence, it is a way of quantifying the degree to which two states are alike. More practically, if a system happens to occupy one of these two states, it almost directly gives us the probability of obtaining the other. As such, amplitudes sit right on the edge between the theoretical machinations that underpin quantum mechanics as a theory and the concrete, measurable evolution of the world we inhabit.

The degree to which it is then the goal of physics to predict the future, to that same degree it is the goal of quantum mechanics to calculate these amplitudes. (Where we ignore for a moment the completely valid goal of discovering what *is* and thus the goal of computing properties of states.) In principle, then, as long as the calculated amplitudes stay the same, the method which produces them can be altered.

As such, one can approach quantum mechanics in a myriad of different ways. This ambiguity necessarily spawns alternative formulations, such as the canonical versus path integral quantizations, and differing 'interpretations' which–though important–tend to verge on the side of philosophy. In addition to all this, the typical Lagrangian is plagued by 'redundancy'. That is, Lagrangians may admit several symmetry transformations which ultimately leave the Lagrangians and therefore the amplitudes invariant. One can conclude that a Lagrangian contains much redundant information which ultimately gets lost at the level of amplitudes.

Furthermore, it is a laborious task to compute the amplitude for a particular process. In quantum field theory, one typically calculates scattering amplitudes as a series of Feynman diagrams. When one calculates amplitudes of processes with increasing numbers of particles or to a higher level of precision, one needs to calculate ever more Feynman diagrams, quickly approaching the tens and hundreds of thousands for relatively small processes. Compare table 1.1.

It would seem that in the typical ways of doing quantum mechanics, there exists a divide between theory and that which the theory aims to achieve, amplitudes. It would then be a very nice surprise if there would exist methods of calculating scattering amplitudes directly without the need of referencing Feynman diagrams or even Lagrangians. As it turns out, this happens to be the precise domain of study of the so-called Scattering Amplitudes Program.

The Scattering Amplitudes Program canonically finds its roots in the discovery of the Parke-Taylor formula in the 1980s [2]. Here, it was discovered that up to $n = 6$, for certain combinations of helicities (the projection of the particle's spin on its momentum), the gluon

n	4	5	6	7	8	9	10
# of diagrams	4	25	220	2485	34300	559405	10525900

Table 1.1: Number of Feynman diagrams contributing to the scattering n-point gluon scattering process [1].

color-ordered scattering amplitude, reduces to

$$A_n[1^+,\ldots,i^-,\ldots,j^-,\ldots,n^+] = \frac{\langle i\,j\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\ldots\langle n\,1\rangle}. \tag{1.1}$$

This notation is explained in great detail in chapter 2. However, even without knowing the notation well, its great simplicity can be admired. Since then, it has been proven that this formula holds for arbitrary n and we will do the same in chapter 4.

Naturally, the simplicity of this formula elicits the thought that there may be some structure to scattering amplitudes not explicit on the level of the theory's Lagrangian. This, especially in contrast to the required amount of Feynman diagrams mentioned before. Since then, many other structures have been discovered in the field of scattering amplitudes. Perhaps most famously, the 'double copy' or BCJ relations.

A 'full' Yang-Mills amplitude scattering amplitude contains contribution from the overall color structure of the SU(N) gauge group and from the momenta and helicities of the external particles. It turns out that a full amplitude can be decomposed into several contibution as follows:

$$A_n(1^{h_1}_{a_1},2^{h_2}_{a_2},\ldots,n^{h_n}_{a_n}) = \sum_{\sigma\in S_n/\mathbb{Z}_n} \mathrm{Tr}\left(T_{\sigma(a_1)}T_{\sigma(a_2)}\ldots T_{\sigma(a_n)}\right) A_n[\sigma(1^{h_1}),\sigma(2^{h_2}),\ldots,\sigma(n^{h_n})].$$

Here, we are summing over all permutations of n labels up to cyclic permutation. T_a is the ath generator of the associated SU(N) Lie group and $A_n[\ldots]$, distinguished by square brackets, is the color ordered amplitude of which equation (1.1) is an example. The labels h_i indicate the helicities of the particles, i.e. whether the particle's spin is alligned with its momentum or points in the opposite direction. (For spin 1/2 particles or massless spin 1 particles, two opposite spin states automatically form a basis for every possible spin state, so it is only necessary to consider two spin states when working generally.)

Subsequently, the traces over the generators can be evaluated and divided into products of structure constants f^{abc}. Upon doing so and grouping together all equivalent structure constant products, one acquires a new decomposition

$$A_n(1^{h_1}_{a_1},2^{h_2}_{a_2},\ldots,n^{h_n}_{a_n}) = \sum_i \frac{c_i n_i}{d_i}. \tag{1.2}$$

Here, c_i are so-called color factors which are products of structure constants,

$$f^{a_1a_2x_1}f^{x_1a_3x_2}f^{x_2a_4x_3}\ldots f^{x_{n-3}a_{n-1}a_n},$$

with each c_i a different ordering of the external labels. The n_i are the kinematic factors, dependent on the particles' helicities and momenta. At 4-point, the color factors unsurprisingly satisfy the Jacobi identity, which, from a Lagrangian perspective, directly follows from the Lie group structure of the gauge group.

$$c_s + c_t + c_u = 0,$$

where each color factor corresponds to a specific channel contributing to the overall amplitude. But seemingly miraculously, the kinematic factors which one would expect to be completely unrelated to any Lie group structure, also satisfy the Jacobi identity,

$$n_s + n_t + n_u = 0,$$

which in this case simply means that they sum to zero.

Hence, one is lead to believe that there may exist some hidden connection between the color and kinematic structures of Yang-Mills. Furthermore, one could be tempted to replace one structure for the other to see what happens. It turns out that if one replaces $c_i \to n_i$ in the expression for the n-point gluon amplitude, one spectacularly gets the n-point graviton amplitude in return:

$$M_n(1^{h_1 h_2}, 2^{h_2 h_2}, \ldots, n^{h_n h_n}) = \sum_i \frac{\tilde{n}_i n_i}{d_i}. \tag{1.3}$$

The duality between (1.2) and (1.3) is called the BCJ relations after Bern, Carrasco and Johansson [3, 4]. The BCJ relations have been proven to hold at tree level for arbitrary n-point and also hold for several examples at loop level where less is understood [5]. Beside this 'double copy' there also exist BCJ relations between several other quantum field theories, where it is a case of mix and match when it comes to their respective kinematic factors and color structures. Beside the BCJ relations, there also exists a different double copy between gauge theory and gravity called the KLT relations (after Kawai, Lewellen and Tye) [6]. This double copy uses and combines amplitudes from a different decomposition of the full gluon amplitude.

However, the focus of this thesis will be on on-shell recursion relations, pioneered by Britto, Cachazo and Feng [7] with later contributions from Witten [8]. In contrast to the double copy, which establishes relationships between amplitudes of different theories, recursion relations establish relationships between amplitudes of the *same* theory. Specifically, they are used to derive higher point scattering amplitudes from lower point amplitudes. Hence, starting from several lower point 'seed amplitudes' it is possible to *recursively* construct the entire tower of higher point scattering amplitudes, given some favorable conditions.

Tree-level on-shell recursion relations rely on the fact that when sums of external particle momenta become on-shell, the amplitude factorizes into lower-point subamplitudes. Let's say we have an n-point amplitude A_n, dependent on a set of momenta $P \equiv \{p_1, p_2, \ldots, p_n\}$, and we sum some subset of these momenta together

$$P_I = \sum_{p_i \in P' \subset P} p_i.$$

Then the statement above means that if we send $P_I^2 \to 0$, we obtain the factorization

$$A_n \longrightarrow A_L \frac{1}{P_I^2} A_R.$$

Here A_R and A_L are lower point amplitudes than A_n.

The trick of on-shell recursion relations is to invert this relationship in order to construct the higher point from the lower point relationships. This is done by introducing a linear shift in the momenta with a complex parameter z. This, when applied to an amplitude, allows one to associate the amplitude's residues in the complex plane with lower point subamplitudes and thereby establish a relationship between these subamplitudes and the amplitude overall. For instance, when $P_I = p_1 + p_2$ goes on shell, the 4-point gluon scattering amplitude, dependent on momenta p_1 through p_4, decomposes into two 3-gluon scattering amplitudes dependent on p_1, p_2 and P_I and p_I, p_3 and p_4.

The problem is that the theory under consideration needs to be sufficiently "special" in order for the recursion relations to work. If the Lagrangian of the theory contains one or more unexpected terms at higher multiplicity, then this introduces behavior into higher point amplitudes which cannot be anticipated at lower point. A more gradual and explicit introduction will follow in later chapters.

It can be quite a task to demonstrate that recursion relations indeed work for a given theory and we will dedicate much attention to this in this thesis. If it turns out that a theory indeed admits recursion relations for tree-amplitudes, we get the following recursion formula:

$$A_n = \frac{1}{2} \sum_{\{I|z_I \text{pole}\}} \hat{A}_{|I|+1}(z_I) \frac{1}{P_I^2} \hat{A}_{n-|I|+1}(z_I). \tag{1.4}$$

Here, one effectively sums over all factorization channels of the amplitude, i.e. over all momentum sums which cause the amplitude to factorize when rendered on-shell.

In this thesis, it will be our goal to explore how recursion relations can be used to derive higher-point interactions in terms of lower-point interactions. Specifically, there is a number of scalar field theories, derived by Li et al. [9], which will be of interest to us. These theories, the gauged non-linear sigma model and DBI-Lovelock, are arrived at by imposing compatibility with the BCJ double copy and certain 'soft limits'. We will ask the question whether recursion relations can be used to derive the higher multiplicity amplitudes of these theories in terms of lower-point amplitudes. If recursion relations work for these theories, it would mean that the requirement of BCJ relations in combination with specific soft limits suffices for fixing *all* tree-level amplitudes in the relevant theories and by extension, for fixing their corresponding Lagrangians.

To build up to this result, we will treat recursion the working of on-shell, tree-level recursion in excruciating detail. In chapter 3, we will discuss the causes of amplitude factorization into subamplitudes together with a general formalism for recursion. In chapter, 4, we will also prove that on-shell recursion works for several theories with a specific focus on Yang-Mills gluon scattering, culminating in a proof of the Parke-Taylor formula (1.1). In appendix A, we will use a similar proof to that used for Yang-Mills to show that recursion also works for graviton scattering. Finally, we will provide an introduction to soft recursion, a type of recursion employing the soft limits of amplitudes. This will lead us to discuss several scalar field theories, among which those derived in [9]. In the next chapter, however, we will start off by discussing the so-called spinor helicity formalism. Apart from being worthwhile in its own right, this formalism features prominently in some of the derivations performed in later chapters. It is the formalism used in equation (1.1). I will introduce the formalism properly at the start of the next section.

Throughout, it has been my goal to justify every statement exactly to the degree required to convince a starting 2nd year master student, largely without deferring to external material. Thus, almost every aspect of an argument that is not treated in the standard course curriculum for particle physics at the University of Groningen is made explicit.

To this end, I have often sacrificed conciseness for explicitness. However, many arguments will still require thought and careful consideration. The thesis merely attempts to make everything explicit and aims to remain self-contained. I hope that these choices have not been made to the detriment of the reading experience, but rather enable any potential future student to continue where I left off, if necessary.

2 The Massless Spinor Helicity Formalism

In this section, we will be interested in producing a formalism which allows for a much simpler expression of scattering amplitudes and BCFW recursion relations, which will be the topic of later chapters. The spinor helicity formalism (originally from [10, 11]) aims to express 4D massless particle states in terms of spinor momentum and helicity eigenstates. That is, it uses massless Dirac spinors, i.e. Weyl spinors, to express states not only for spin-1/2 fermions, but also bosons. This is possible because the direct product of the spinor representation of the Lorentz group with itself contains vector the vector representation as an irrep. Beside the massless spinor helicity formalism, there also exists an extension to the massive case [12], but we will not treat this extension.

The goal of this chapter has largely been to rederive the expressions given by Elvang and Huang [13], but with the $(+,-,-,-)$ metric in concordance with the rest of the thesis. Elvang and Huang's exercises also happen to be a great addition to this chapter. A number of derivations have been directly inspired by this resource, some others by have been adapted from Thomas Bader's lecture notes at the TU München [14].

The most important relations are marked using equation numbering, while those that merely serve to continue the derivation remain unmarked. Naturally, we will be working in 4D in this chapter.

2.1 The Weyl Spinors and Equations

In our pursuit of a formalism which is able to represent the simultaneous momentum and helicity eigenstates of massless fermions, the obvious place to look is at an already known way to represent fermions, i.e. Dirac spinors. For massless particles, $m = 0$, the Dirac equation reduces

$$(i\gamma^\mu \partial_\mu - m)\,\psi = i\gamma^\mu \partial_\mu \psi = 0.$$

Going to momentum space, the momentum eigenstates for momentum $\mathbf{p}$ are given by

$$\psi(x) = u(p)e^{-ipx}, \qquad \psi(x) = v(p)e^{+ipx},$$

for the positive and negative frequency solutions of the Dirac equation respectively. Since we are only considering on-shell massless particles,

$$p^2 = 0, \qquad p^0 = |\mathbf{p}|,$$

the Dirac equation becomes

$$\gamma^\mu p_\mu u(p) = 0, \qquad \gamma^\mu p_\mu v(p) = 0.$$

Considering only $v(p)$ for the moment, we can define the two-component spinors λ_p and $\tilde{\lambda}_p$ such that

$$v(p) \equiv \begin{pmatrix} \lambda_p \\ \tilde{\lambda}_p \end{pmatrix},$$

where $v(p)$ is the solution to the momentum-space massless Dirac equation for the negative frequency solutions or anti-particles.
If we choose the chiral basis for the gamma matrices,

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \qquad \gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{2.1}$$

where $\sigma^\mu = (1, \boldsymbol{\sigma})$ and $\bar{\sigma}^\mu = (1, -\boldsymbol{\sigma})$, then the λ_p and $\tilde{\lambda}_p$ are the so-called Weyl spinors, named after Hermann Weyl. It follows that the massless Dirac equation decouples into two separate equations,

$$\sigma^\mu p_\mu \tilde{\lambda}_p = 0, \qquad \bar{\sigma}^\mu p_\mu \lambda_p = 0. \tag{2.2}$$

These equations are known as the Weyl equation.

In the spinor representation and chiral basis, the helicity operator is given by

$$h \equiv \hat{\mathbf{p}} \cdot \mathbf{S} = -\frac{1}{2}\hat{p}_i \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix},$$

where $\hat{\mathbf{p}}$ is the normalized momentum operator. That is, a state with momentum $\mathbf{p}$ has eigenvalue $\mathbf{p}/|\mathbf{p}|$ for this operator. When acting with the helicity operator on $v(p)$ and using equation (2.2), we get

$$hv(p) = \frac{1}{2|\mathbf{p}|}\begin{pmatrix} \mathbf{p}\cdot\boldsymbol{\sigma}\lambda_p \\ \mathbf{p}\cdot\boldsymbol{\sigma}\tilde{\lambda}_p \end{pmatrix} = \frac{1}{2p^0}\begin{pmatrix} -p^0\lambda_p \\ p^0\tilde{\lambda}_p \end{pmatrix} = \begin{pmatrix} -1/2\lambda_p \\ 1/2\tilde{\lambda}_p \end{pmatrix}.$$

We see that the momentum and helicity operators are simultaneously diagonalizable, where states of the form $(\lambda_p, 0)$ have $h = -1/2$ or 'negative helicity' and states of the form $\left(0, \tilde{\lambda}_p\right)$ have $h = +1/2$ or 'positive helicity'.

It is then natural to invent a symbol for these helicity eigenfunctions. We will write

$$|p] \equiv \begin{pmatrix} 0 \\ \tilde{\lambda}_p \end{pmatrix}, \;\; h = +1/2, \qquad |p\rangle \equiv \begin{pmatrix} \lambda_p \\ 0 \end{pmatrix}, \;\; h = -1/2. \tag{2.3}$$

A similar treatment of the Dirac conjugate of $u(p)$, $\bar{u}(p) \equiv u^\dagger(p)\gamma^0$, reveals that it satisfies

$$\bar{u}(p)\gamma^\mu p_\mu = 0,$$

which, if we define

$$\bar{u}(p) \equiv (\rho_p, \tilde{\rho}_p),$$

results in

$$\tilde{\rho}_p \bar{\sigma}^\mu p_\mu = 0, \qquad \rho_p \sigma^\mu p_\mu = 0. \tag{2.4}$$

This can be rewritten by taking the hermitian conjugate to give

$$\bar{\sigma}^\mu (p_\mu)^* \tilde{\rho}_p^\dagger = 0, \qquad \sigma^\mu (p_\mu)^* \rho_p^\dagger = 0,$$

where we have used the hermiticity of the Pauli matrices. Hence, by comparing to equation (2.2), we can identify,

$$\tilde{\rho}_{p^*} = \lambda_p^\dagger, \qquad \rho_{p^*} = \tilde{\lambda}_p^\dagger, \tag{2.5}$$

Please note that the ρ spinors depend on the complex conjugate of the momentum, the λ do not. We pick the normalisation and relative phase of these spinors such that the identity holds. Also, of course, if p is real then $p^* = p$.

Now acting with the helicity operator h from the right on $\bar{u}(p)$, we can again pick a notation for the helicity eigenstates,

$$[p| \equiv (0, \tilde{\rho}_p)\,, \;\; h = +1/2, \qquad \langle p| \equiv (\rho_p, 0), \;\; h = -1/2. \tag{2.6}$$

Following from equation (2.5), we see that

$$[p^*| = \gamma^0(|p\rangle)^\dagger, \qquad \langle p^*| = \gamma^0(|p])^\dagger. \tag{2.7}$$

The brackets of equation (2.3) and (2.6) are what constitute the spinor helicity formalism. It turns out that we can conveniently express any 4D massless scattering amplitude in terms of these brackets, making it a worthwhile effort to investigate these brackets further. It is thus our goal in the upcoming sections to derive more identities surrounding these brackets, so we can use and interpret them more effectively. Eventually, we will derive enough identities to abandon the notation in terms of λ and ρ in favor of a cleaner and simpler notation in terms of the brackets alone.

We are not interested in the Weyl spinors that are associated with $u(p)$ and $\bar{v}(p)$. This is because we will only look at these brackets inside the expressions for scattering amplitudes. One may recall that $\bar{u}(p)$ and $v(p)$ appear in amplitudes due to the Feynman rules for the external legs of outgoing fermions and anti-fermions respectively. It turns out that, after having calculated an amplitude for a given process, one is allowed to make the substitutions $u(p) \leftrightarrow v(-p)$ and $\bar{v}(p) \leftrightarrow \bar{u}(-p)$ to acquire the amplitude for a new process where one outgoing particle (anti-particle) is now replaced by an incoming anti-particle (particle) and vice versa. This is called crossing symmetry [15]. Hence, by only considering the amplitudes for outgoing particles, one effectively covers the range of all possible interactions.

One should keep in mind that what it means for a particle to be outgoing is that it is a particle that remains after the relevant interaction has taken place and was thus not part of the prepared collection of particles. Hence, a process of only outgoing particles is not physical. Yet, this poses no problems for the calculations that we will be performing.

It is also important to note that the brackets of the spinor helicity formalism are not to be confused with Dirac bras and kets, as they are not the objects that reside in our Hilbert space in a quantum field-theoretical context. They should be thought of as single particle wavefunctions that solve the Dirac equation for a specific momentum and helicity. As such, they simply represent specific functions, which happen to be useful in expressing amplitudes later on. Nevertheless, I will occasionally refer to them as bras and kets (squas and squets) or spinor brackets.

2.2 Defining Spinor Helicity Identities

Here, we will start looking at and proving some identities that hold within the spinor helicity to make the formalism more practical. The identities here also serve to impose a (relative) normalisation on the spinors. Hence, these identities also serve to further define the spinor brackets, as the definitions given in the previous section leave redundancy in the normalisation and phase.

In the next subsection, we will also derive identities; these will directly follow from the Weyl equation and the normalisation imposed in this section.

Following from (2.3) and (2.6), we see that for any four-momenta p and q,

$$\langle p||q] = [p||q\rangle = 0,$$

and,

$$\langle p\,q\rangle \equiv \langle p||q\rangle = \rho_p\lambda_q, \qquad [p\,q] \equiv [p||q] = \tilde{\rho}_p\tilde{\lambda}_q. \tag{2.8}$$

Combinations such as

$$|p\rangle[q|, \quad \text{and,} \quad |p]\langle q|,$$

are 4×4 matrices that take one from the square subspace to the angled subspace or back. There is a nice relationship between σ^μ and $\bar{\sigma}^\mu$ that will allow us to relate some more spinors. Specifically,

$$\mathcal{E}\sigma^\mu\mathcal{E}^T = (\bar{\sigma}^\mu)^*, \text{ where } \mathcal{E} \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Note that $\mathcal{E}$ is simply the two-dimensional Levi-Civita symbol in matrix form. We have $\mathcal{E}^{-1} = \mathcal{E}^T = -\mathcal{E}$. Using this to rewrite the Weyl equation (2.2) into its conjugates (2.4) gives us

$$0 = \sigma^\mu p_\mu\tilde{\lambda}_p = \mathcal{E}\sigma^\mu\mathcal{E}^T\mathcal{E}p_\mu\tilde{\lambda}_p = (\bar{\sigma}^\mu)^*p_\mu\mathcal{E}\tilde{\lambda}_p \Longleftrightarrow 0 = (\mathcal{E}\tilde{\lambda}_p)^\dagger(\bar{\sigma}^\mu)^*(p_\mu)^* = (\mathcal{E}\tilde{\lambda}_p)^T\bar{\sigma}^\mu p_\mu,$$
$$0 = \bar{\sigma}^\mu p_\mu\lambda_p = \mathcal{E}^T\bar{\sigma}^\mu\mathcal{E}\mathcal{E}^T p_\mu\lambda_p = (\sigma^\mu)^*p_\mu\mathcal{E}^T\lambda_p \Longleftrightarrow 0 = (\mathcal{E}^T\lambda_p)^\dagger(\sigma^\mu)^*(p_\mu)^* = (\mathcal{E}^T\lambda_p)^T\sigma^\mu p_\mu.$$

Here, we used the Hermiticity of the Pauli matrices. Again by fixing the relative phase and normalisation, we can conclude

$$\tilde{\rho}_p = (\mathcal{E}\tilde{\lambda}_p)^T, \qquad \rho_p = (\mathcal{E}^T\lambda_p)^T. \tag{2.9}$$

The anti-symmetry of $\mathcal{E}$ gets handed down to the brackets:

$$\langle p\,q\rangle = \rho_p\lambda_p = \lambda_p^T\mathcal{E}\lambda_q = (\lambda_p^T\mathcal{E}\lambda_q)^T = \lambda_q^T\mathcal{E}^T\lambda_p = -\lambda_q^T\mathcal{E}\lambda_p = -\rho_q\lambda_p = -\langle q\,p\rangle.$$

With a near identical calculation for $[p\,q]$, we get

$$\langle p\,q\rangle = -\langle q\,p\rangle, \qquad [p\,q] = -[q\,p]. \tag{2.10}$$

This trivially implies that

$$\langle p\,p\rangle = [p\,p] = 0.$$

We have now fixed the phases and normalizations of the brackets such that given a specific solution for $\tilde{\lambda}_p$, we know the explicit form of Weyl spinors $\tilde{\rho}_p$, λ_{p^*} and ρ_{p^*} through equations (2.5) and (2.9). However given such a solution, it is still possible to shift $\tilde{\lambda}_p \to z_p\tilde{\lambda}_p$, $\tilde{\rho}_p \to z_p\tilde{\rho}_p$, $\lambda_{p^*} \to z_p^*\lambda_{p^*}$ and $\rho_{p^*} \to z_p^*\rho_{p^*}$ to acquire a new solution to the Weyl equation. Furthermore, the conjugate set of spinors ($\tilde{\lambda}_{p^*}$ etc.) remain completely independent and one can shift those with a separate z_{p^*}. Let us say that we have picked a $\tilde{\lambda}_p$, it is then possible to pick a corresponding z_{p^*} such that we acquire a nice property. This we will examine now.

Let us examine the 2×2 matrix $p_\mu\sigma^\mu$. We can compute its determinant to find that $\det(p_\mu\sigma^\mu) = p_\mu p^\mu = 0$. Since this matrix is not the zero-map, it follows that the matrix is of rank 1 implying that its kernel and column space are both one-dimensional. This means that we can write this matrix non-uniquely as the outer product of two non-zero column vectors u and v (not to be confused with the spinors in the previous section), where u is in the column space of our matrix and v inhabits the orthogonal complement of its kernel. That is,

$$p_\mu\sigma^\mu = uv^T.$$

Using the Weyl equation (2.2, 2.4) together with identity (2.9), we find that

$$0 = p_\mu\sigma^\mu\tilde{\lambda}_p = uv^T\tilde{\lambda}_p = u(v^T\mathcal{E}^T\tilde{\rho}_p).$$

Since $u \neq 0$, we find that $v^T\mathcal{E}^T\tilde{\rho}_p = 0$ implying $v \propto \tilde{\rho}_p^T$. We can also act with ρ_p from the left and determine $u \propto \lambda_p$. This implies that

$$p_\mu\sigma^\mu = \alpha_p\lambda_p\tilde{\rho}_p, \tag{2.11}$$

where α_p depends on the normalisation of the relevant spinors. This can be rewritten to obtain

$$(p_\mu\sigma^\mu)^T = p_\mu(\sigma^\mu)^* = \alpha_p\tilde{\rho}_p^T\lambda_p^T = \alpha_p\mathcal{E}\tilde{\lambda}_p\rho_p\mathcal{E}^T \Longrightarrow p_\mu\bar{\sigma}^\mu = \alpha_p\tilde{\lambda}_p\rho_p,$$

with the same α_p. Furthermore, exploiting the hermiticity of σ^μ, we learn that

$$(p_\mu\sigma^\mu)^\dagger = (p_\mu)^*\sigma^\mu = \alpha_p^*\tilde{\rho}_p^\dagger\lambda_p^\dagger = \alpha_p^*\lambda_{p^*}\tilde{\rho}_{p^*}.$$

This is clearly the same equation as (2.11) but for the spinors for momentum p^*. Hence, we can conclude that $\alpha_{p^*} = \alpha_p^*$.

Given all of this information, we can rescale the spinors with z_p and z_{p^*} appropriately, to obtain

$$\begin{aligned} p_\mu\sigma^\mu &= \alpha_p\lambda_p\tilde{\rho}_p, \\ p_\mu\bar{\sigma}^\mu &= \alpha_p\tilde{\lambda}_p\rho_p, \\ (p_\mu)^*\sigma^\mu &= \alpha_p^*\lambda_{p^*}\tilde{\rho}_{p^*}, \\ (p_\mu)^*\bar{\sigma}^\mu &= \alpha_p^*\lambda_{p^*}\tilde{\rho}_{p^*}, \end{aligned} \quad \to \quad \begin{aligned} p_\mu\sigma^\mu &= \alpha_p z_{p^*}^* z_p\lambda'_p\tilde{\rho}'_p &&= \lambda'_p\tilde{\rho}'_p, \\ p_\mu\bar{\sigma}^\mu &= \alpha_p z_{p^*}^* z_p\tilde{\lambda}'_p\rho'_p &&= \tilde{\lambda}'_p\rho'_p, \\ (p_\mu)^*\sigma^\mu &= \alpha_p^*(z_{p^*}^* z_p)^*\lambda'_{p^*}\tilde{\rho}'_{p^*} &&= \lambda'_{p^*}\tilde{\rho}'_{p^*}, \\ (p_\mu)^*\bar{\sigma}^\mu &= \alpha_p^*(z_{p^*}^* z_p)^*\lambda'_{p^*}\tilde{\rho}'_{p^*} &&= \lambda'_{p^*}\tilde{\rho}'_{p^*}, \end{aligned} \tag{2.12}$$

if we pick $z_{p^*} = 1/(z_p\alpha_p)^*$. This can always be done and, in fact, some authors choose these identities as their starting point in developing the spinor helicity formalism [4, 16]. We will always assume that for any p, the spinors have been appropriately shifted such that (2.12) obtains. Note that we can still shift $\lambda_p \to t\lambda_p$ together with $\tilde{\lambda}_p \to t^{-1}\tilde{\lambda}_p$ for some complex t, while still maintaining all previously derived relations. If the momentum associated to the spinor is real, then $z_{p^*} = z_p$ and the only allowed shift will be a pure phase. We will use this fact later in section 2.5.

It follows that for any momentum p,

$$\not{p} \equiv p_\mu \gamma^\mu = |p\rangle[p| + |p]\langle p|. \tag{2.13}$$

2.3 Resultant Spinor Helicity Identities

We continue our task of deriving identities within the spinor helicity formalism. In contrast to the previous section, the identities derived here result directly from the previous sections and do not impose any new conditions on the spinor brackets.

Interestingly, following from equation (2.7), we have the result that

$$(\langle p\, q\rangle)^* = (|q\rangle)^\dagger(\langle p|)^\dagger = \gamma^0[q^*||p^*]\gamma^0 = [q^*\, p^*]. \tag{2.14}$$

Using equation (2.12), we can find

$$\begin{aligned}\langle p\, q\rangle[p\, q] = -\langle p\, q\rangle[q\, p] = -|p]_b\langle p|_a|q\rangle_a[q|_b = -\operatorname{Tr}\{(|p]\langle p|)(|q\rangle[q|)\}\\ = -p_\mu q_\nu \operatorname{Tr}\{\bar{\sigma}^\mu\sigma^\nu\} = -p_\mu q_\nu 2\eta^{\mu\nu} = -2(p\cdot q) = -(p+q)^2.\end{aligned} \tag{2.15}$$

Combining equations (2.14) and (2.15), we can conclude that for real momenta,

$$\langle p\, q\rangle = \sqrt{2p\cdot q}e^{i\phi}, \qquad [p\, q] = -\sqrt{2p\cdot q}e^{-i\phi}, \tag{2.16}$$

where ϕ possibly depends on the (order of the) individual momenta. Hence, if we have an expression given in terms of spinor helicity brackets, we can now make sense of the expression up to phase.

Beside the angled and square brackets we have seen, we are also interested in objects of the form $\langle p|\gamma^\mu|q]$ and other combinations of the spinor brackets and γ^μ. From the definitions of our brackets, it immediately follows that

$$0 = \langle p|\gamma^\mu|q\rangle = [p|\gamma^\mu|q].$$

Considering equation (2.8) and the one above, we see that one can only multiply brackets of the ‘same type’ to acquire a non-zero answer, unless there is a gamma matrix wedged in between. In that case, one can only multiply brackets of the ‘opposite type’.
It also follows that

$$\langle 1|\gamma_\mu|2]\langle 3|\gamma^\mu|4] = 2\langle 1\, 3\rangle[2\, 4]. \tag{2.17}$$

This is the so-called Fierz identity. Here, $|i\rangle \equiv |p_i\rangle$ etc., which is a useful notation if one has multiple momenta $p_1, p_2, \ldots, p_n$. The Fierz identiy follows from the identity $(\sigma_\mu)_{ab}(\sigma^\mu)_{cd} = 2\epsilon_{ac}\epsilon_{bd}$. That is,

$$\langle 1|\gamma_\mu|2]\langle 3|\gamma^\mu|4] = \rho_1\sigma_\mu\tilde{\lambda}_2\rho_3\sigma^\mu\tilde{\lambda}_4 = 2\rho_1\mathcal{E}\rho_3^T\tilde{\lambda}_2^T\mathcal{E}\tilde{\lambda}_4 = 2\langle 1\, 3\rangle[2\, 4].$$

We also have

$$\langle p|\gamma^\mu|q] = [q|\gamma^\mu|p\rangle, \qquad \langle p|\gamma^\mu|q]^* = \langle q^*|\gamma^\mu|p^*]. \tag{2.18}$$

These are easily proved:

$$\langle p|\gamma^\mu|q] = (\langle p|\gamma^\mu|q])^T = \tilde{\lambda}_q^T(\sigma^\mu)^T\rho_p^T = \tilde{\rho}_q\mathcal{E}(\sigma^\mu)^*\mathcal{E}^T\lambda_p = \tilde{\rho}_q\bar{\sigma^\mu}\lambda_p = [q|\gamma^\mu|p\rangle,$$
$$\langle p|\gamma^\mu|q]^* = \langle p|\gamma^\mu|q]^\dagger = \tilde{\lambda}_q^\dagger(\sigma^\mu)^\dagger\rho_p^\dagger = \rho_{q^*}\sigma^\mu\tilde{\lambda}_{p^*} = \langle q^*|\gamma^\mu|p^*].$$

Furthermore,

$$\langle p|\gamma^\mu|p] = \mathrm{Tr}\{\gamma^\mu|p]\langle p|\} = \frac{1}{2}\,\mathrm{Tr}\{\gamma^\mu(|p]\langle p| + |p\rangle[p|)\} = \frac{1}{2}p_\nu\,\mathrm{Tr}\{\gamma^\mu\gamma^\nu\} = 2p^\mu. \tag{2.19}$$

We will often write

$$\langle p|k|q] \equiv \langle p|\not{k}|q], \qquad \langle p|n|q] \equiv \langle p|\not{p}_n|q]. \tag{2.20}$$

Hence, when a number or momentum is wedged between two spinor helicity brackets, we interpret it as the corresponding momentum contracted with a gamma matrix. Hence, it trivially follows from (2.19) that

$$\langle p|q|p] = 2p\cdot q. \tag{2.21}$$

We are almost finished with spinor helicity identities. Continuing, however, it is good to note that any spinor bracket only has two nonzero components. Furthermore, the type of spinor helicity bracket determines which components are nonzero. This means that if we have two linearly independent spinor helicity brackets of the same type (i.e. angled or square), say $|p\rangle$ and $|q\rangle$, we can express any other spinor helicity bracket again of the same type, say $|k\rangle$, as a linear combination of the prior two,

$$|k\rangle = \alpha|p\rangle + \beta|q\rangle.$$

It turns out that our formalism allows us to determine α and β. We simply compute

$$\langle q\,k\rangle = \alpha\langle q\,p\rangle + \beta\langle q\,q\rangle = \alpha\langle q\,p\rangle \Rightarrow \alpha = \frac{\langle q\,k\rangle}{\langle q\,p\rangle}.$$

We get the general identity

$$|p\rangle\langle q\,k\rangle + |k\rangle\langle p\,q\rangle + |q\rangle\langle k\,p\rangle = 0. \tag{2.22}$$

This is called the van Schouten identity. Note the cyclic permutativity.
The final identity to look at only holds in the case we are working with an amplitude consisting of only outgoing particles. When all particles are outgoing, momentum conservation is expressed as $\sum_{i=1}^n p_i^\mu = 0$. This directly implies

$$\sum_{i=1}^n \not{p}_i = \sum_{i=1}^n (|i\rangle[i| + |i]\langle i|) = 0.$$

Furthermore, this holds for the two terms of the summand separately, since they project into two distinct orthogonal subspaces. This can be read off directly from the brackets' definitions. We have

$$\sum_{i=1}^n |i\rangle[i| = 0, \qquad \sum_{i=1}^n |i]\langle i| = 0. \tag{2.23}$$

We now move on to practice with the identities derived.

Example: Calculating Yukawa Diagrams

To demonstrate the power of the spinor helicity formalism and to simultaneously integrate what we have learned in previous sections, let us calculate a couple of tree-level Yukawa processes. Specifically, let us calculate the amplitudes for four outgoing fermions $A(ffff)$ and two outgoing fermions and two outgoing scalars $A(ff\phi\phi)$. We will see that the identities of the previous sections can be a large helping hand in deriving and simplifying amplitude expressions.

The Lagrangian of massless Yukawa theory is given by

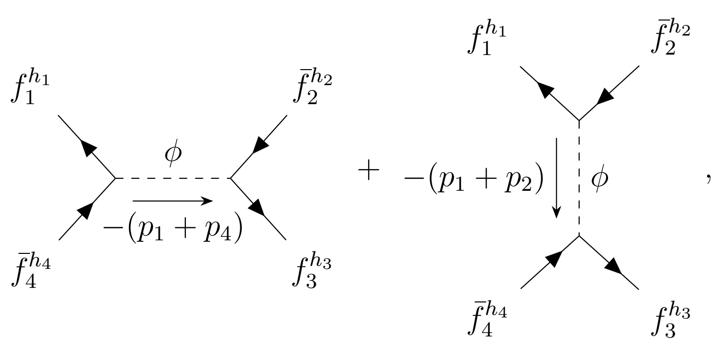
$$\mathcal{L} = i\bar{\psi}\partial\!\!\!/\psi + \frac{1}{2}\partial^{\mu}\phi\partial_{\mu}\phi - g\bar{\psi}\psi\phi.$$

This gives the following Feynman rules:

Diagrammatic feature	Feynman rule
Two fermion, one scalar vertex	$-ig$,
Scalar propagator	i/p^2,
Fermion propagator	$ip\!\!\!//p^2$,
Outgoing scalar	1,
Outgoing fermion	$\bar{u}^s(p)$,
Outgoing anti-fermion	$v^s(p)$.

2.3.1 Four-Fermion Amplitude

At tree level in Yukawa theory, the four-fermion scattering amplitude is given by the following Feynman diagrams,



for arbitrary helicities of the external particles. However, we can see that certain combinations of helicities trivially give rise to a vanishing amplitude. This is because for each diagram, the spinors of particles meeting at the same vertex are multiplied. For example, the first vertex in the first diagram results in a factor of $\bar{u}^{h_1}(p_1)v^{h_4}(p_4)$, which in the spinor helicitiy formalism corresponds to $\langle 1\,4\rangle$, $\langle 1||4]$, $[1||4\rangle$, $[1\,4]$ corresponding to helicities $(h_1, h_4) = (-,-), (-,+), (+,-)$, and $(+,+)$ respectively. Only the first and last of these result in a non-vanishing diagram. We get the non-vanishing amplitudes

$$A_4(f^+\bar{f}^+f^-\bar{f}^-) = (-ig)^2\bar{u}^+(p_1)v^+(p_2)\frac{i}{(p_1+p_2)^2}\bar{u}^-(p_3)v^-(p_4),$$

$$= ig^2 \frac{[1\,2]\langle 3\,4\rangle}{\langle 1\,2\rangle[1\,2]} = ig^2 \frac{\langle 3\,4\rangle}{\langle 1\,2\rangle}.$$

and possibly

$$\begin{aligned}
A_4(f^+\bar{f}^+f^+\bar{f}^+) = (-ig)^2 &\left[\bar{u}^+(p_1)v^+(p_4)\frac{i}{(p_1+p_4)^2}\bar{u}^+(p_3)v^+(p_2)\right.\\
&\left.+\bar{u}^+(p_1)v^+(p_2)\frac{i}{(p_1+p_2)^2}\bar{u}^+(p_3)v^+(p_4)\right],\\
&= ig^2\left[\frac{[1\,4][3\,2]}{\langle 1\,4\rangle[1\,4]} + \frac{[1\,2][3\,4]}{\langle 1\,2\rangle[1\,2]}\right],\\
&= ig^2\left[\frac{[3\,2]\langle 1\,2\rangle + [3\,4]\langle 1\,4\rangle}{\langle 1\,4\rangle\langle 1\,2\rangle}\right].
\end{aligned}$$

Here we have used equation (2.15). We can still rewrite the all-plus amplitude, seeing an opportunity to use momentum conservation (2.23). Using

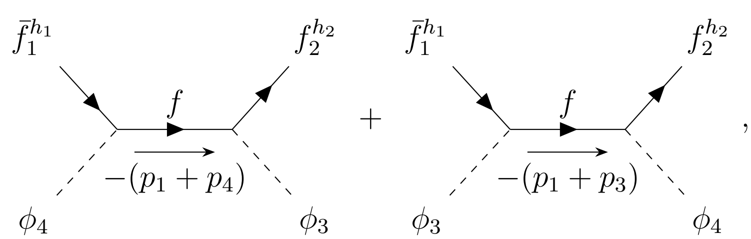
$$[3\,4]\langle 1\,4\rangle = -[3\,4]\langle 4\,1\rangle = -[3|\left(|4]\langle 4|\right)|1\rangle = [3|\left(\sum_{i\neq 4}|i]\langle i|\right)|1\rangle = [3\,2]\langle 2\,1\rangle,$$

we can conclude that $A_4(f^+f^+f^+f^+) = 0$. When the four fermions have identical helicity, the contributions from both diagrams annihilate one another in total destructive interference.

Using the spinor helicity formalism, we could easily see that certain helicity configurations produce vanishing contributions. With typical Dirac spinor notation, this would escape us at first. We would be left with a specific expression in terms of spinors without immediate further insight.

2.3.2 Two-Fermion, Two-Scalar Amplitude

For the two-fermion $A(\bar{f}^{h_1}f^{h_2}\phi\phi)$, two-scalar amplitude, we have the diagrams



which, using the Feynman rules, correspond to

$$= (-ig)^2\bar{u}^{h_2}(p_2)\frac{-i(\not{p}_1+\not{p}_4)}{(p_1+p_4)^2}v^{h_1}(p_1) + (-ig)^2\bar{u}^{h_2}(p_2)\frac{-i(\not{p}_1+\not{p}_3)}{(p_1+p_3)^2}v^{h_1}(p_1).$$

We can simplify this expression using the Weyl equation $\not{p}_1 v^{h_1}(p_1) = 0$ or equivalently using equations (2.10) and (2.13). Then, in the spinor helicity formalism, we can quickly see that the amplitude vanishes for identical helicities. For example,

$$A(\bar{f}^+f^+\phi\phi) = ig^2[2|\left[\frac{|4\rangle[4| + |4]\langle 4|}{\langle 1\,4\rangle[1\,4]} + \frac{|3\rangle[3| + |3]\langle 3|}{\langle 1\,3\rangle[1\,3]}\right]|1] = 0.$$

The external square brackets cancel the angled brackets in the numerators, but this leaves no non-zero term in the entire amplitude.

We only acquire a non-vanishing amplitude for distinct helicities. For instance,

$$\begin{aligned} A(\bar{f}^+ f^- \phi\phi) &= ig^2 \langle 2| \left[\frac{|4\rangle[4| + |4]\langle 4|}{\langle 1\,4\rangle[1\,4]} + \frac{|3\rangle[3| + |3]\langle 3|}{\langle 1\,3\rangle[1\,3]} \right] |1], \\ &= ig^2 \left[\frac{\langle 2\,4\rangle[4\,1]}{\langle 1\,4\rangle[1\,4]} + \frac{\langle 2\,3\rangle[3\,1]}{\langle 1\,3\rangle[1\,3]} \right] = -ig^2 \left[\frac{\langle 2\,4\rangle}{\langle 1\,4\rangle} + \frac{\langle 2\,3\rangle}{\langle 1\,3\rangle} \right]. \end{aligned}$$

2.4 Vectors and Tensors

Up until now, we have developed the spinor helicity formalism in order to describe massless fermions. Perhaps surprisingly, the same formalism can also be used to describe massless vector and tensor particles without introducing new mathematical objects. This opens the formalism up to the calculation of photon, gluon and graviton amplitudes. We will see how we can express the wavefunctions of these particles in the spinor helicity formalism in this section.

2.4.1 Polarization Vectors

Our analysis starts with the observation that we can easily construct objects that transform as vectors (and tensors) from spinors, which is something we have secretly already done. Specifically, we have

$$\langle p|\gamma^\mu|q] \to \langle \Lambda p|\gamma^\mu|\Lambda q] = \langle p|(\Lambda_{1/2}^{-1})\gamma^\mu(\Lambda_{1/2})|q] = \Lambda^\mu_{\ \nu}\langle p|\gamma^\nu|q],$$

where $\Lambda_{1/2}$ is a Lorentz transformation in the spinor representation. The last equality is due to a transformation property of gamma matrices [15, p. 32]. It now simply remains to be shown that these objects solve the relevant equations for polarization vectors.

In the Lorenz gauge, vector potentials satisfy the Klein-Gordon equation per component,

$$(\partial^\mu \partial_\mu + m^2) A^\nu = 0,$$

which in the massless case reduces to the homogeneous Maxwell equations

$$\partial^\mu \partial_\mu A^\nu = 0. \tag{2.24}$$

The Lorenz gauge condition is

$$\partial_\mu A^\mu = 0. \tag{2.25}$$

We can derive equivalent equations for the polarization vectors by Fourier transforming to momentum space

$$p^2 \epsilon^\mu(p) = 0, \quad p_\mu \epsilon^\mu(p) = 0. \tag{2.26}$$

We can see that the homogeneous Maxwell equations kill the polarization vector whenever p becomes off-shell. Hence, these off-shell states are non-physical. The Lorenz gauge condition allows us to solve for the components of $\epsilon^\mu(p)$. We can easily solve (2.26) for a simplified momentum $p^\mu = (1,0,0,1)$. Because (2.26) is Lorentz invariant, we can simply Lorentz transform p to any other momentum and apply to the same transformation to $\epsilon^\mu(p)$ to acquire the polarization vector for the corresponding momentum, i.e.

$$\epsilon^\mu(\Lambda p) = \Lambda^\mu{}_\nu \epsilon^\nu(p).$$

It is not difficult to see that for $p^\mu = (1,0,0,1)$, any linear combination of

$$\epsilon_+^\mu \equiv \frac{-1}{\sqrt{2}}(0,1,i,0), \quad \epsilon_-^\mu \equiv \frac{1}{\sqrt{2}}(0,1,-i,0), \tag{2.27}$$

solves (2.26), normalized such that $\epsilon_\mu^* \epsilon^\mu = 1$. Furthermore any vector proportional to p^μ can be added to such a solution to acquire a new solution. This latter fact reflects the remaining gauge freedom after imposing the Lorenz gauge.

The polarization vectors (2.27) have been chosen such that they are helicity eigenstates. We can show this just as before by introducing the helicity operator in the vector representation of the Lorentz group

$$h \equiv \hat{\mathbf{p}} \cdot \mathbf{S} = S_z = i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We can see that

$$h\epsilon_\pm = \pm \epsilon_\pm, \quad h_\pm = \pm 1.$$

Now, in the spinor helicity formalism, it turns out that we can rewrite

$$\epsilon_+^\mu(p) = -\frac{\langle q|\gamma^\mu|p]}{\sqrt{2}\langle q\,p\rangle}, \quad \epsilon_-^\mu(p) = -\frac{\langle p|\gamma^\mu|q]}{\sqrt{2}[q\,p]}, \tag{2.28}$$

where q is an arbitrary 'reference' four-momentum. As already shown, these objects transform in the desired way under Lorentz transformations. They solve equation (2.26),

$$p_\mu \epsilon_+^\mu(p) = \frac{1}{2}\langle p|\gamma_\mu|p]\epsilon_+^\mu(p) \propto \langle p\,q\rangle[p\,p] = 0,$$

using the Fierz identity (cf. equations 2.10, 2.17, and 2.19). They are normalized identically to (2.27),

$$\epsilon_+^\mu(p)\epsilon_{+\mu}^*(p) = \frac{\langle q|\gamma^\mu|p]\langle p^*|\gamma_\mu|q^*]}{2\langle q\,p\rangle[p^*\,q^*]} = \frac{\langle q\,p^*\rangle[p\,q^*]}{\langle q\,p\rangle[p^*\,q^*]} = 1,$$

for real momenta.

As an exercise, let us analyze the difference between two polarization vectors using distinct reference momenta q and r:

$$\begin{aligned}\epsilon_+^\mu(p;q) - \epsilon_+^\mu(p;r) &= \frac{\langle r|\gamma^\mu|p]}{\sqrt{2}\langle r\,p\rangle} - \frac{\langle q|\gamma^\mu|p]}{\sqrt{2}\langle q\,p\rangle}, \\ &= \frac{1}{\sqrt{2}\langle r\,p\rangle\langle q\,p\rangle}\left(\langle q|\gamma^\mu|p]\langle p\,r\rangle + \langle q\,p\rangle[p|\gamma^\mu|r\rangle\right),\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{2}\langle r\,p\rangle\langle q\,p\rangle}\langle q|\left(\gamma^\mu \not{p} + \not{p}\gamma^\mu\right)|r\rangle, \\
&= \frac{1}{\sqrt{2}\langle r\,p\rangle\langle q\,p\rangle}\langle q\,r\rangle 2\eta^{\mu\nu}p_\nu, \\
&\propto p^\mu.
\end{aligned}
$$

For the last equality we used the defining relation for the gamma matrices $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$. We thus see that picking a different reference momentum is equivalent to applying the transformation $\epsilon^\mu(p) \to \epsilon^\mu(p) + cp^\mu$, i.e. a gauge transformation. We can therefore also expect any amplitude calculated using polarization vectors to be independent from the choice of reference spinor. In order to make sure that the needed cancellation of reference momenta occurs, it is necessary that identical external particles occupy the same gauge and thus use the same reference momentum across Feynman diagrams.

2.4.2 The Polarization (Helicity) of Polarization Vectors

Now, in order to accept the spinor helicity notation for ϵ_+ and ϵ_-, it only remains to be shown that they are indeed helicity eigenstates. However, it turns out that they are not; it also turns out that this is not a problem. To show this, however, requires a somewhat lengthy calculation. Nevertheless, I will present this here, although this part may be skipped if this exact subject is not of much interest to the reader.

We will apply

$$h = \frac{1}{|\vec{p}|}\vec{p}\cdot\vec{S} = -\frac{1}{|\vec{p}|}\frac{\epsilon^{ijk}}{2}p^i J^{jk}$$

to $\epsilon_+(p;q)$. Here, p is the momentum of the polarization vector that h is applied to, $\vec{S}$ is the vector of spin operators in the vector representation and J^{ij} is the generator of Lorentz transformations in the vector representation.
When we apply h to ϵ_+ we get:

$$
\begin{aligned}
h^\mu{}_\nu \epsilon_+^\nu(p) &= -\frac{1}{|\vec{p}|}\frac{\epsilon^{ijk}}{2}p^i\left(J^{jk}\right)^\mu{}_\nu \frac{\langle q|\gamma^\nu|p]}{\sqrt{2}\langle q\,p\rangle}, \\
&= -\frac{1}{|\vec{p}|\sqrt{2}\langle q\,p\rangle}\frac{\epsilon^{ijk}}{2}p^i\langle q|\left[\gamma^\mu, J^{jk}_{1/2}\right]|p], \\
&= -\frac{1}{|\vec{p}|\sqrt{2}\langle q\,p\rangle}p^i\langle q|\left[\gamma^\mu, S^i_{1/2}\right]|p], \\
&= -\frac{1}{|\vec{p}|\sqrt{2}\langle q\,p\rangle}p^i\langle q|\left[\gamma^\mu, S^i_{1/2}\right]|p], \\
&= \frac{-\langle q|\gamma^\mu\left(\frac{1}{|\vec{p}|}\vec{p}\cdot\vec{S}_{1/2}\right)|p]}{\sqrt{2}\langle q\,p\rangle} - \frac{-\langle q|\left(\frac{1}{|\vec{p}|}\vec{p}\cdot\vec{S}_{1/2}\right)\gamma^\mu|p]}{\sqrt{2}\langle q\,p\rangle}.
\end{aligned}
\tag{2.29}
$$

$J^{ij}_{1/2}$ is the generator corresponding to J^{ij} in the spinor representation. Consequently, $\vec{S}_{1/2}$

is the spin operator for spinors. In the second line, we used [15, p. 42],

$$(J^{\rho\sigma})^{\mu}{}_{\nu}\gamma^{\nu} = \left[\gamma^{\mu}, J^{\rho\sigma}_{1/2}\right].$$

The first term in (2.29) is easy to evaluate. We know that the helicity (as a spinor) of $|p]$ is 1/2. Hence,

$$\frac{-\langle q|\gamma^{\mu}\left(\frac{1}{|\vec{p}|}\vec{p}\cdot\vec{S}_{1/2}\right)|p]}{\sqrt{2}\langle q\,p\rangle} = (1/2)\epsilon^{\mu}_{+}(p).$$

The second term in (2.29) requires some more creativity. The helicity of $\langle q|$ is $-1/2$. The helicity operator for this bra would be $\frac{1}{q^0}\vec{q}\cdot\vec{S}_{1/2}$. But in the second term, we are applying $\frac{1}{|\vec{p}|}\vec{p}\cdot\vec{S}_{1/2}$. We can project $\vec{p}$ onto $\vec{q}$ to recover the relevant helicity operator:

$$\langle q|\left(\frac{1}{|\vec{p}|}\vec{p}\cdot\vec{S}_{1/2}\right) = \frac{1}{|\vec{p}|}\langle q|\left[\frac{\vec{p}\cdot\vec{q}}{|\vec{q}|^2}\vec{q} + \left(\vec{p} - \frac{\vec{p}\cdot\vec{q}}{|\vec{q}|^2}\vec{q}\right)\right]\cdot\vec{S}_{1/2}. \tag{2.30}$$

Thus here, in the first term,

$$\langle q|\frac{\vec{p}\cdot\vec{q}}{|\vec{p}||\vec{q}|^2}\vec{q}\cdot\vec{S}_{1/2} = -1/2\left(\frac{\vec{p}\cdot\vec{q}}{|\vec{p}||\vec{q}|}\right)\langle q|.$$

Now, the vector $\left(\vec{p} - \frac{\vec{p}\cdot\vec{q}}{|\vec{q}|^2}\vec{q}\right)$, occurring in the second term of (2.30), is the component of $\vec{p}$ orthogonal to $\vec{q}$. Since $\langle q|$ has spin-down in the $\vec{q}$ direction by construction, $\left(\vec{p} - \frac{\vec{p}\cdot\vec{q}}{|\vec{q}|^2}\vec{q}\right)\cdot\vec{S}_{1/2}$ must be a linear combination of raising and lowering operators for $\langle q|$. The lowering part will annihilate $\langle q|$, the raising part will convert $\langle q|$ into $[q|$. Therefore the second term of (2.29) becomes,

$$\frac{-\langle q|\left(\frac{1}{|\vec{p}|}\vec{p}\cdot\vec{S}_{1/2}\right)\gamma^{\mu}|p]}{\sqrt{2}\langle q\,p\rangle} = -1/2\left(\frac{\vec{p}\cdot\vec{q}}{|\vec{p}||\vec{q}|}\right)\frac{-\langle q|\gamma^{\mu}|p]}{\sqrt{2}\langle q\,p\rangle} + \alpha[q|\gamma^{\mu}|p],$$

for some proportionality constant α. The latter term vanishes, as $[q|\gamma^{\mu}|p] = 0$.

We get

$$\begin{aligned} h^{\mu}{}_{\nu}\epsilon^{\nu}_{+}(p) &= \frac{-\langle q|\gamma^{\mu}\left(\frac{1}{|\vec{p}|}\vec{p}\cdot\vec{S}_{1/2}\right)|p]}{\sqrt{2}\langle q\,p\rangle} - \frac{-\langle q|\left(\frac{1}{|\vec{p}|}\vec{p}\cdot\vec{S}_{1/2}\right)\gamma^{\mu}|p]}{\sqrt{2}\langle q\,p\rangle}, \\ &= 1/2\epsilon^{\mu}_{+}(p) + 1/2\left(\frac{\vec{p}\cdot\vec{q}}{|\vec{p}||\vec{q}|}\right)\epsilon^{\mu}_{+}(p), \\ &= \frac{1}{2}\left(1+\cos\theta\right)\epsilon^{\mu}_{+}(p). \end{aligned}$$

We were expecting an eigenvalue of 1, not an eigenvalue dependent on the angle between $\vec{p}$ and $\vec{q}$. If we wish the eigenvalue to be 1, then we have to require $\vec{p}\cdot\vec{q} = |\vec{p}||\vec{q}|$. However, this

means that $p \cdot q$ (the 4-vector inner product) would vanish. We do not allow this, because then $\langle p\, q\rangle$ is zero and $\epsilon_+^\mu(p)$ contains a division by zero in its definition.
It seems like we are at a loss. But in fact, this is not a problem at all as long as we apply $\epsilon_\pm^\mu$ correctly. As we saw previously, changing q amounts to a gauge transformation of the overall polarization vector. Thus any explicit q dependence will drop out of any physical amplitude. Therefore, in our definition of $\epsilon_\pm$ in the spinor helicity formalism, we can pick a q such that $\epsilon_\pm$ do not actually have helicities ± 1, because when inserted into an amplitude, the expression will simplify as if $\epsilon_\pm$ *do* have the desired helicity. The fact that the polarizations change as a function of q is briefly remarked by [4] as well. It follows that equation (2.28) cannot be used in any gauge dependent expression.

2.4.3 Polarization Tensors

We will briefly remark upon the fact that gravitons can also be expressed in the spinor helicity formalism. For gravitons, we have two distinct polarization tensors $\epsilon_{++}^{\mu\nu}$ and $\epsilon_{--}^{\mu\nu}$. These polarization tensors can be expressed in terms of polarization vectors [17, 13] as follows

$$\epsilon_{\pm\pm}^{\mu\nu}(p) = \epsilon_\pm^\mu(p)\epsilon_\pm^\nu(p). \tag{2.31}$$

Hence, the expression of graviton polarization vectors in terms of spinor brackets immediately follows.
This choice immediately serves to make the graviton tensors symmetric and traceless, which is one way of satisfying the de Donder gauge. Furthermore, contractions with the particle's momentum cause the polarization vector to vanish

$$p_\mu \epsilon_{\pm\pm}^{\mu\nu} = 0.$$

2.5 Little Group Scaling and Bootstrapping Amplitudes

Apart from simplified notation and computation, the spinor helicity formalism also allows one to 'bootstrap' specific three-point amplitudes. That is, simply using various known properties of amplitudes, one can immediately infer what expression an amplitude ought to have without relying on the Lagrangian formalism explicitly. The bootstrap we will see here relies on the 'little group scaling' of spinor brackets, Lorentz invariance of amplitudes and momentum conservation to arrive at the 3-point gluon scattering amplitude. We will largely follow Cheung [4] in this discussion.

We start by analyzing the transformation properties of a spinor helicity bracket under a little group transformation. Given a particular momentum vector, the little group is the unique subgroup of all Poincaré transformations leaving that particular momentum invariant. Massless momenta always come in the shape $(|\vec{p}|, \vec{p})$. Therefore, the little group of such a momentum will, for instance, include those spatial rotations around $\vec{p}$, keeping the length and orientation of $\vec{p}$ fixed. (The full little group of masseless fourmomenta is ISO(2), confer section 2.5 in [18].)

Let us consider a spinor helicity bracket $|p\rangle$ with momentum p^μ. If we act on $|p\rangle$ with a little group transformation in the correct representation, we expect it to transform into a spinor helicity bracket with identical momentum. As we have seen in section 2.2, such a little group transformation need not keep the entire bracket invariant. The spinor helicity brackets still admit an overall phase shift, as long as multiple brackets of the same momentum are shifted simultaneously. The spinor bracket $|p\rangle \to t|p\rangle$ describes the same momentum after being shifted as before given $|p] \to t^{-1}|p]$, where t is a pure phase.

For vector particles, this implies that

$$\epsilon_+^\mu = \frac{\langle q|\gamma^\mu|p]}{\sqrt{2}\langle q\, p\rangle} \to \frac{t^{-1}\langle q|\gamma^\mu|p]}{\sqrt{2}t\langle q\, p\rangle} = t^{-2}\epsilon_+^\mu,$$

and for arbitrary helicity,

$$\epsilon_h^\mu \to t^{-2h}\epsilon_h^\mu. \tag{2.32}$$

Since amplitudes are linear in polarization vectors, one gains the transformation rule

$$A_n(1^{h_1}, 2^{h_2}, \ldots, n^{h_n}) \to \left(\prod_{i=1}^{n} t^{-2h_i}\right) A_n(1^{h_1}, 2^{h_2}, \ldots, n^{h_n}). \tag{2.33}$$

The amplitude need not be fully invariant under Lorentz transformations. A phase change, as occurs here, cancels out in the computation of any physical probability guaranteeing overall Lorentz invariance.

We are then lead to bootstrapping by considering "three particle special kinematics". When we are dealing with a three particle amplitude, momentum conservation guarantees a special relation between the spinor helicity brackets that is particularly useful to us right now. Since $p_1 + p_2 + p_3 = 0$, we have

$$\langle 1\, 2\rangle[1\, 2] = (p_1 + p_2)^2 = p_3^2 = 0,$$

since every external particle is on-shell. So either $\langle 1\, 2\rangle$ or $[1\, 2]$ vanishes. Let us suppose the latter. Then through (2.13) and (2.23),

$$\langle 1\, 2\rangle[2\, 3] = \langle 1|p_2|3] = -\langle 1|(p_1 + p_3)|3] = 0.$$

Hence, it follows that $[2\, 3]$ vanishes. One can continue showing that other bracketes vanish as well.

Generalizing from there, for any three particle amplitude, we automatically obtain

$$[1\, 2] = [2\, 3] = [3\, 1] = 0,$$

or,

$$\langle 1\, 2\rangle = \langle 2\, 3\rangle = \langle 3\, 1\rangle = 0.$$

This is called three particle special kinematics.

We can now perform the bootstrap for the three particle pure Yang-Mills amplitude. Due to three particle special kinematics, we know that the amplitude will either only consist of square brackets or of angled brackets. Let us first consider angled brackets. We make the Ansatz

$$A_3(1^{h_1} 2^{h_2} 3^{h_3}) = \langle 1\, 2\rangle^{n_3} \langle 2\, 3\rangle^{n_1} \langle 3\, 1\rangle^{n_2}.$$

Under a little group transformation of p_1, the left-hand side gains a factor t^{-2h_1}. The right-hand side will gain $t^{(n_2+n_3)}$. In this way, we can generate a system of equations, allowing us to solve for n_1, n_2 and n_3 depending on the helicities of the external particles. We get

$$\begin{cases} -2h_1 = n_2 + n_3, \\ -2h_2 = n_3 + n_1, \\ -2h_3 = n_1 + n_2, \end{cases} \quad \Rightarrow \quad \begin{cases} n_1 = h_1 - h_2 - h_3, \\ n_2 = h_2 - h_3 - h_1, \\ n_3 = h_3 - h_1 - h_2. \end{cases}$$

The mass dimension of the three particle amplitude cannot be negative, because this requires dividing by momenta, which can only be possible if one adds a propagator. Since every contribution at three point requires is contact, we must conclude that

$$0 < n_1 + n_2 + n_3 = -(h_1 + h_2 + h_3).$$

Hence our 'angled bracket only' Ansatz is only compatible with a mostly negative helicity. Naturally, the opposite holds for the 'square' Ansatz. We get the amplitudes

$$A_3(1^-, 2^-, 3^+) = \frac{\langle 1\,2\rangle^3}{\langle 2\,3\rangle\langle 3\,1\rangle}, \qquad A_3(1^+, 2^+, 3^-) = \frac{[1\,2]^3}{[2\,3][3\,1]}. \tag{2.34}$$

This is an example of the three-point Parke-Taylor formula, discussed in the introduction. We will use the amplitudes above to prove the general Parke-Taylor formula recursively in section 4.4.

This is the end of our discussion centered around the spinor helicity formalism. For the remainder of this thesis, we will focus exclusively on on-shell recursion relations. At several points, in certain proofs, the spinor helicity formalism will feature prominently.

“Dent modo Fata recursus!”

-Publius Ovidius Naso,
Heroides 6 (around the year 20 B.C.)

May the Fates just give us recursion!

3 General On-Shell Recursion Relations

In this chapter we will shift our focus to the topic of recursion relations. As stated in the introduction, the goal of recursion is to derive higher-point scattering amplitudes from lower-point scattering amplitudes. That is, for example, if one is given the three-point scattering amplitude of some theory and recursion 'works' for this theory, one can use these three-point amplitudes to generate the expression of the equivalent four-point. The three- and four-point can in turn be used to generate the five-point and so on. Hence, when recursion works, one can construct the entire tower of scattering amplitudes up to an arbitrary amount of particles in, hopefully, a more efficient way than using Feynman diagrams.

There exist several types of recursion. Our scope will be limited to on-shell tree-level recursion relations, meaning that we will only be interested in recursively deriving tree-level scattering amplitudes. Furthermore, all amplitudes we consider will only depend on 'on-shell' momenta. In addition to tree-level, loop-level recursion also exists [13, p. 117].

The main principle that allows recursion to work is that when 'intermediate momenta' go on shell, the scattering amplitudes that depend on these momenta decompose into lower-point 'subamplitudes'. Intermediate momenta are simply sums of external momenta, corresponding to the momenta of intermediate particles or propagators in Feynman diagrams. An example would be $p_1 + p_3 + p_6$ for a six-point amplitude or higher. When this goes on-shell, i.e. $(p_1 + p_3 + p_6)^2 = 0$, any tree-level amplitude which depends on these momenta factorizes into some lower-point amplitudes A_L and A_R,

$$A_n \longrightarrow A_L \frac{1}{(p_1 + p_3 + p_6)^2} A_R.$$

These amplitudes will be tree-level scattering amplitudes as well, describing collisions between the same particle types as the original amplitude.

Recursion employs this relationship between amplitudes of varying particle number to derive higher-point from lower-point. The factorization is, as it were, inverted to give the higher-point using complex analysis. The details will all be treated below.

The precise goal of this chapter will be to derive the recursion formula (1.4) presented in the introduction and to explain how it is used. We will use the first sections to work our way up to the recursion formula. This precise formula will be derived in section 3.4, at first for scalar fields alone. We will then use section 3.5 to generalize the result to other particle types. As mentioned in the introduction, on-shell recursion relations were pioneered by Britto, Cachazo and Feng [7] with later contributions from Witten [8]. However, this chapter will be more general than their seminal contribution. BCFW present a specific momentum shift, whereas this chapter keeps the momentum shift general. (What this means will become clear soon enough.) This is to keep our formalism open to other types of shifts, such as different types of all-line shift [19, 20]. Nevertheless, the BCFW shift will be the main topic of the next chapter, chapter 4. The formalism presented in this chapter is largely taken from Elvang & Huang [13]; subsection 3.1, 3.2 and 3.4 are based on their textbook.

3.1 Definitions and Requirements for General Shifts

We will start by considering an n-point amplitude A_n. We are interested in finding an expression for A_n in terms of $A_{n-1}, A_{n-2}, \dots, A_3$. Consider our amplitude A_n which is a function of n momenta and helicities, corresponding to the external particles:

$$A_n = A_n(p_1^{h_1}, p_2^{h_2}, \dots, p_n^{h_n}).$$

These momenta satisfy momentum conservation, are massless and on-shell,

$$\sum_{i=1}^{n} p_n^\mu = 0, \quad \mu = 0, 1, 2, 3, \qquad\qquad p_i^2 = 0 \quad i = 1, \dots, n.$$

We can now introduce a so-called momentum shift. That is, we introduce a shift to p_i linear in complex variable z such that our momentum becomes a function of this z, $\hat{p}_i(z)$. Specifically,

$$\forall i : \quad p_i^\mu \longrightarrow \hat{p}_i^\mu \equiv p_i^\mu + z r_i^\mu. \tag{3.1}$$

Here, r_i is the "shift vector", which can be different (and generically[1] is) for every i.

The reason for performing this shift may seem somewhat unclear at this point. By shifting these momenta, our ultimate goal is to arrive at an equivalently shifted amplitude, whose poles in the complex plane correspond to factorization channels. It turns out that we will be able to apply Cauchy's residue theorem to this shifted amplitude to relate A_n to its corresponding lower-point subamplitudes. How all of this works, will become clear very soon. Until then, it will be necessary to just follow along, while we make these seemingly arbitrary steps.

We require the shift vectors satisfy certain conditions for our shift to be useful. Namely,

$$\text{(i)} \sum_{i=1}^{n} r_i^\mu = 0, \quad \text{(ii)}\ r_i \cdot r_j = 0\ \forall i, j, \quad \text{(iii)}\ r_i \cdot p_i = 0\ \forall i. \tag{3.2}$$

It trivially follows from condition (ii) that $r_i^2 = 0$. These combined properties guarantee that

$$\text{(i)} \sum_{i=1}^{n} \hat{p}_i^\mu = 0, \qquad \text{(ii)}\ \hat{p}_i^2 = 0.$$

We define

$$P_I^\mu \equiv \sum_{i \in I} p_i^\mu, \qquad R_I \equiv \sum_{i \in I} r_i^\mu.$$

[1] I use the word 'generically' different from 'generally'. If some statement is generally true, it is true for all cases. If something is generically true, it is true for almost all cases. Specifically, some function $f : D \to T$ satisfies some property generically, if it satisfies that property on a large subset of its domain, D. That is, the subset of its domain where it does not satisfy this property is measure-zero or has zero volume in the entire domain.

For each distinct I barring some exceptions, P_I, as a sum of external momenta, represents some 'intermediate momentum' as mentioned above. Hence, these are the quantities, that, when on-shell, cause the amplitude to factorize.

When shifting the intermediate momenta themselves, we acquire

$$\hat{P}_I(z)^2 = P_I^2 + z2P_I \cdot R_I, \tag{3.3}$$

where $\hat{P}_I(z) \equiv P_I + zR_I = \sum_{i\in I} \hat{p}_i(z)$. The R_I^2 term vanishes, because of (3.2.ii). In chapter 5, we will discuss cases where this condition is violated, slightly complicating things.
Equation (3.3) can be rewritten as

$$\hat{P}_I^2 = -\frac{P_I^2}{z_I}(z - z_I), \qquad z_I \equiv -\frac{P_I^2}{2P_I \cdot R_I}. \tag{3.4}$$

We can see on the basis of this equation that the shifted intermediate momentum becomes on-shell exactly when $z = z_I$.
As was our intention, shifting the momenta in z allows us to shift the amplitude A_n as well through analytic continuation. We simply define

$$A_n(p_1, p_2, \dots, p_n) \longrightarrow \hat{A}_n(z) \equiv A_n(\hat{p}_1, \hat{p}_2, \dots, \hat{p}_n). \tag{3.5}$$

The analytic continuation of A_n is well-defined and we recover the original amplitude by $A_n = \hat{A}_n(0)$.
We now have a shifted amplitude which depends on z for which various values of z, (shifted) intermediate momenta to go on-shell. As stated various times now, for these specific values of z, the amplitude factorizes. In the next two sections, we will treat the analytic structure of $\hat{A}_n(z)$ in detail and give a clear demonstration of this much cherised factorization.

3.2 Poles and Residues

For a tree-level scattering process, the only possible poles that an amplitude has are those given by propagators blowing up. This is because propagators are the only objects in Feynman diagrams that scale with some inverse power of momentum, at least for local Lagrangians. The Lagrangian being local means that fields only couple to each other at the same spacetime point, disallowing terms such as

$$\phi(x)\phi(x + a),$$

or expansions thereof. If the Lagrangian is local, we can be assured that vertices only scale with positive powers of momentum.

It turns out that these propagators contain precisely the intermediate momenta P_I. Specifically,

$$\tilde{D}_F(P_I) \propto \frac{1}{P_I^2 - m^2}.$$

Thus now we can see that in the generic massless case $\hat{A}(z)$ is singular when for some sets I, $\hat{P}_I(z)^2 = 0$ or, stated differently, when these external particle momentum sums become on-shell. As we saw before, this occurs exactly at z_I for those propagators containing P_I.

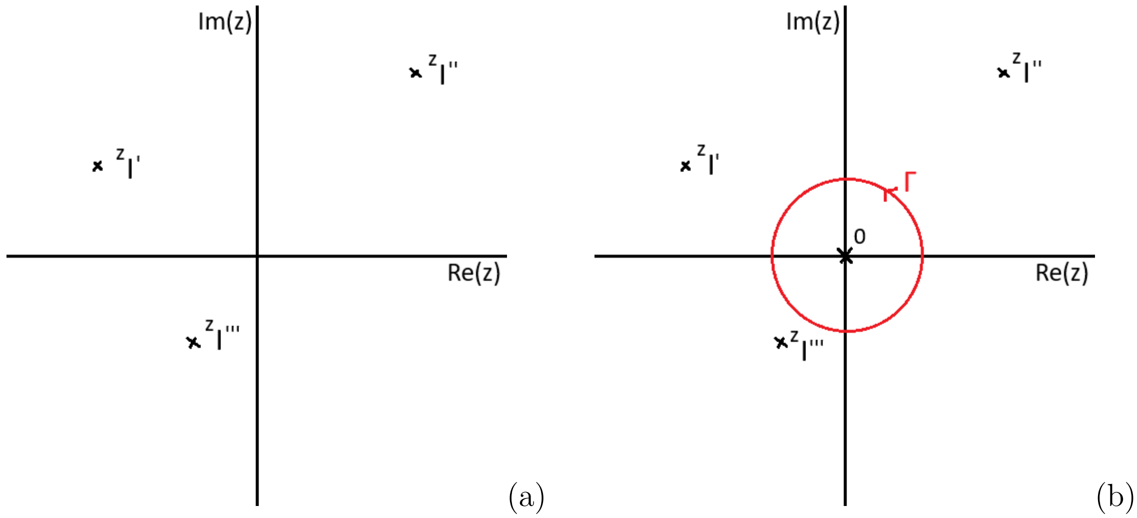


Figure 3.1: (a) The poles of the shifted amplitude $\hat{A}_n(z)$; (b)The contour for integrating $\hat{A}_n(z)/z$.

However, we can only guarantee that $\hat{A}_n(z)$ is singular at z_I if there is an actual propagator with momentum $\propto 1/P_I^2$ for the I under consideration. This complicates the matter. We know for instance that due to momentum conservation, there will not be propagators $\propto 1/P_I^2$ if I contains all indices, $I = \{1, 2, 3, \dots, n\}$. Also, because there only exist vertices with more than two particles, no diagram will contain a propagator $\propto 1/P_I^2$ if I contains only one label (or all but one label). If we want to consider only those I for which there is a singularity, we can exclude these two cases by checking that $R_I^2 \neq 0$ for our particular I.

If we are working in a theory of only one particle type admitting three-particle vertices, the $R_I^2 \neq 0$ condition is sufficient for having a singularity at z_I. For theories of multiple particle types (e.g. QED) or for theories of one particle type but admitting only vertices with more than three particles (e.g. ϕ^4), additional criteria are needed for guaranteeing the presence of a singularity at z_I. These criteria depend on the theory at hand, thus it does not make sense to discuss them here. We can visualize the singularities of the shifted amplitude as in figure 3.1(a). All these singularities are first-order poles in z.

Because the original amplitude A_n is finite-valued for generic external momenta (and thus $\hat{A}_n(z)$ is finite-valued at $z = 0$), we can introduce an additional first order pole at the point $z = 0$ by considering $\hat{A}_n(z)/z$. Subsequently, using Cauchy's residue theorem, we can recover $A_n = \hat{A}_n(0)$ by integrating along a contour Γ around the origin. See figure 3.1(b). We take care to make the contour small enough so it only contains the pole at the origin. We have the expression

$$A_n = \frac{1}{2\pi i} \int_\Gamma \frac{\hat{A}_n(z)}{z} dz. \tag{3.6}$$

In this way, we can recover the original amplitude. However, the residue theorem can be used as well to relate this original amplitude to the residues of all other poles. We see that the counter-clockwise contour around the origin can simultaneously be considered a clockwise contour around all other poles, including a potential pole at $z = \infty$. Summing up the residues at those poles, we get the expression

$$A_n = -\frac{1}{2} \sum_{\{I\}} \text{Res}\left(z_I; \hat{A}_n(z)/z\right) + B_\infty, \tag{3.7}$$

where B_∞ is the contribution from the pole at infinity.

The sum over $\{I\}$ means that we are summing over every possible set $I \subseteq \{1, 2, 3, \ldots, n\}$. We do not have to take care to exclude sets I for which z_I is not a pole, since at those points in the complex plane, we have a vanishing residue anyway. We have to add the factor of 1/2 in front of the sum to prevent double counting. This is because any I produces the same pole as $\{1, 2, 3, \ldots, n\}\backslash I$ (the set of labels with I taken out). It is possible to take away the factor 1/2 one only sums over indexing sets that exclude a specific label, e.g. by summing over all I that do not contain '1'. Something like this is typically done in the literature implicitly.

3.3 Finding the Residue for Scalar Fields

We have now related the unshifted amplitude A_n to the residues of the various poles of the shifted amplitude $\hat{A}_n(z)$. In addition to this, we have argued that these poles occur at exactly those z_I for which intermediate momenta go on-shell and propagators explode. It would be nice if we could perform a more concrete calculation of these residues.

In this section, we will calculate the value of these residues. We will see that amplitudes indeed factorize when intermediate momenta go on-shell and we will relate the residues of the relevant poles to subamplitudes. Here, we will limit ourselves to the case of scalar fields, while in a later section, we will generalize this result.

To calculate the residue of a first order pole at z_I, we can take the following limit

$$\text{Res}\left(z_I; \hat{A}_n(z)/z\right) = \lim_{z \to z_I} (z - z_I) \frac{\hat{A}_n(z)}{z}.$$

It is thus beneficial to find out what happens to $\hat{A}_n(z)$ in the limit of $z \to z_I$.

If there is in fact a pole at z_I, then in this limit $\hat{A}_n(z)$ blows up. If no such pole exists, then the residue simply vanishes. We will assume that we have picked an I with an actual singularity.

Because $\hat{A}_n(z)$ consists of a sum of Feynman diagrams and we have identified that the particular singularity under consideration comes from a propagator $\propto 1/\hat{P}_I^2$, it is precisely those diagrams that contain such a propagator $\propto 1/\hat{P}_I^2$ that become infinitely large in this limit compared to all other diagrams. To improve our understanding of these diagrams in particular, let us consider a specific example.

Consider an arbitrary diagram for ϕ^3 theory contributing to a seven-point amplitude, with a pole at z_I, where $I = \{1, 3, 4\}$. See figure 3.2(a). In order to have a propagator $\propto 1/\hat{P}_I^2$, where $\hat{P}_I = \hat{p}_1 + \hat{p}_3 + \hat{p}_4$, momentum conservation dictates that on one side of the propagator, we encounter precisely the particles with momentum $\hat{p}_1$, $\hat{p}_3$, $\hat{p}_4$. Naturally, on the other side of the propagator, we will then encounter all other momenta, $\hat{p}_2$, $\hat{p}_5$, $\hat{p}_6$, $\hat{p}_7$.

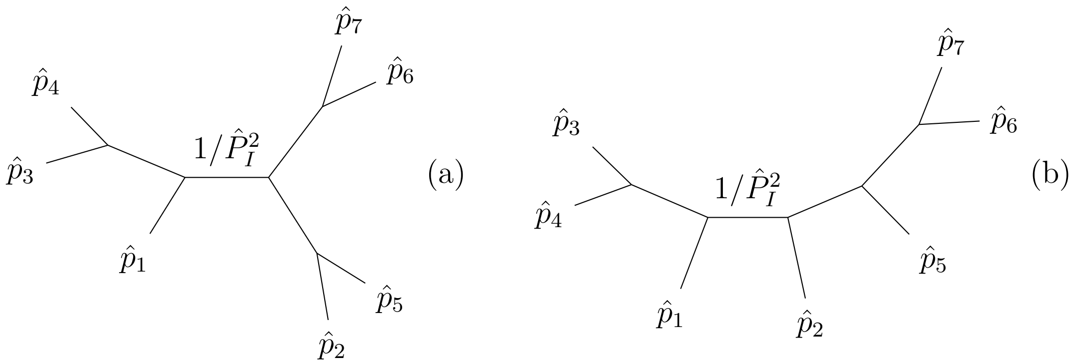
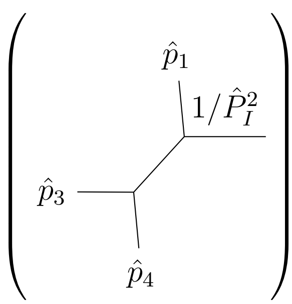


Figure 3.2: Two arbitrary 7-point Feynman diagrams with a pole at z_I for $I = \{1, 3, 4\}$.

Precisely those diagrams with a propagator satisfying this requirement will blow up in the limit $z \to z_I$.

Consider our arbitrary diagram once again. If we vary, for instance, the RHS (right hand side) of the diagram, it is clear that we acquire a new diagram that blows up in the same limit. This is because the external momenta retain their position relative to the propagator after such a variation. Compare the variation on figure 3.2(a) in figure 3.2(b).

We see that all diagrams blowing up in the limit with the exact same LHS (left hand side) can be written as



$$\times \left(V_1 + V_2 + \ldots\right),$$

which is the identical LHS times every possible variation of the RHS. For scalar diagrams, the Feynman rule for external legs is simply 1. This means that all of these variations exactly sum up to the 5-point scalar amplitude for ϕ^3, $A_5(-\hat{P}_I, \hat{p}_2, \hat{p}_5, \hat{p}_6, \hat{p}_7)$. Importantly, if the external leg Feynman rule would have been anything else, the variations $V_1, \ldots$ would not necessarily have summed up to be the amplitude. This follows since we have a propagator connected to these variations and thus these variations would not contain the required external leg to form full-fledged amplitudes. Nevertheless, as we are currently treating scalars, we do not have to worry about this here. We will call the subamplitude that emerges from the RHS, including any potential external legs, $\hat{A}_R$.

To capture all diagrams that blow up in our limit, we can also vary the LHS. This ends up giving a 4-point subamplitude, $\hat{A}_L$. As $z \to z_I$, all diagrams contributing to $\hat{A}(z)$ become negligibly small compared to those that collectively sum up to $\hat{A}_L \hat{A}_R / \hat{P}_I^2$. We can now compute the limit:

$$\operatorname{Res}\left(z_I; \hat{A}_n(z)/z\right) = \lim_{z \to z_I} (z - z_I) \frac{\hat{A}_n(z)}{z},$$

$$
\begin{aligned}
&= \lim_{z\to z_I} \frac{z-z_I}{z}\hat{A}_L(z)\frac{1}{\hat{P}_I^2}\hat{A}_R(z), \\
&= -\lim_{z\to z_I} \frac{z-z_I}{z}\hat{A}_L(z)\frac{z_I}{P_I^2(z-z_I)}\hat{A}_R(z), \\
&= -\hat{A}_L(z_I)\frac{1}{P_I^2}\hat{A}_R(z_I),
\end{aligned}
$$

where we used equation (3.4).

Evidently, the residue at pole z_I is, as predicted, two subamplitudes multiplied together with an additional remnant from the propagator that was taken on-shell. The subamplitudes are shifted themselves with the same shift as the original amplitude to the location of the former pole z_I. From the diagrammatic analysis above, we see that if have an n-point amplitude and we take $\hat{P}_I^2$ on-shell where I has k elements, then the left subamplitude will be $(k+1)$-point, whereas the right subamplitude will be $(n-k+1)$-point. Since indexing sets I with fewer than 2 or more than $n-2$ elements do not contribute any poles, all subamplitudes will always be lower-point than the original n-point amplitude. The propagator included in the residue is unshifted.

3.4 The Recursion Formula

We can now throw the results from the previous sections together to arrive at our desired recursion formula. Combining the previous analysis with equation (3.7) yields us

$$
A_n = \frac{1}{2}\sum_{\{I|z_I\text{pole}\}} \hat{A}_{|I|+1}(z_I)\frac{1}{P_I^2}\hat{A}_{n-|I|+1}(z_I) + B_\infty, \tag{3.8}
$$

This is the same formula as presented in the introduction, i.e. equation (1.4). Here, $|I|$ is the amount of elements in I and $\hat{A}_{|I|+1}(z_I)$ is the $(|I|+1)$-point shifted amplitude. This amplitude depends on the momenta of the particles with labels included in I and an additional external particle with momentum $\hat{P}_I(z_I)$ of the same type as the propagator. $\hat{A}_{n-|I|+1}(z_I)$ depends on the remaining particles and an additional particle with momentum $-\hat{P}_I(z_I)$. It is $(n-|I|+1)$-point. The additional term B_∞ appears from a potential pole that $\hat{A}(z)/z$ has at infinity.

In this case of (3.8), we should take care to not sum over every possible set I, but make sure to only include those I for which z_I is in fact a pole. Whereas in equation (3.7), those sets I for which $\hat{A}_n(z_I)$ was finite-valued were guaranteed to give a vanishing contribution, this is not the case in (3.8).

Equation (3.8) lends itself to a nice interpretation. We see that A_n is given in terms of different amplitudes in addition to a boundary term. We have inferred that only those amplitudes which are lower point than A_n contribute to the sum. If for some reason $B_\infty = 0$, we can see that any n-point amplitude is constructible solely from lower point amplitudes. If $B_\infty = 0$ for any $n \leq N$, we can thus recursively construct A_N from three point amplitudes in our theory. That is, we can construct A_4 from A_3, then A_5 from A_3 and A_4 and build up all the way up to N without ever having to rely on explicit Feynman-diagrammatic calculations. If $B_\infty = 0$ for all n, then we can construct all three level amplitudes of this theory in this fashion. This is the essence of on-shell recursion relations.

It is important to note that B_∞ does not vanish in general. Recursion will not work for the vast majority of conceivable quantum field theories. This makes sense. If one is to calculate a lower-point scattering amplitude using the typical Lagrangian, Feynman-diagrammatic procedure, the higher-order terms of in the Lagrangian will not play any role. These terms do start to play a role when one calculates higher-point scattering amplitudes. It follows that many distinct theories sharing identical lower-point amplitudes, differ in higher-point. It should thus not be possible to derive higher-point amplitudes from lower-point amplitudes in general.

It follows that for recursion to work and for B_∞ to vanish, the theory under consideration needs to be rather special. It needs to be *the* theory whose amplitudes are generated by ignoring B_∞. These are typically theories which satisfy a certain property, allowing one to fix higher-order terms in the Lagrangian based on this property, e.g. gauge symmetry or certain soft limits.

In addition to the fact that a vanishing B_∞ is relatively rare, showing that B_∞ vanishes practically is also hard. One can show that $B_\infty = 0$ by proving that

$$\lim_{z\to\infty} \hat{A}_n(z) = 0. \tag{3.9}$$

If one would already know A_n, then it would be easy to shift all the momenta and acquire $\hat{A}_n(z)$ to subsequently take the limit. However, the whole point of the recursion relations is that we do not know A_n, which we wish to derive. Hence, one needs to use more clever arguments to show that B_∞ vanishes, relying on the special properties of the theory.

In the upcoming chapter, we will show that B_∞ vanishes in the so-called BCFW shift for various amplitudes in various theories, such as Yang-Mills and gravity. We will see that gauge symmetry plays an important role in the argumentation. Afterwards, we will focus on the role of soft limits in allowing recursion to work.

3.5 Generalizing to Other Particle Types: Unitarity

In our derivation of (3.8), we made use of the assumption that we were dealing with scalar fields. We made this assumption in order to justify that the sum of variations of the LHS or RHS $V_1 + V_2 + \ldots$ indeed coverges to a proper tree-level amplitude. We have used the property, particular to scalar fields, that the Feynman rule for external legs is 1, because the variations V_1, V_2 etc. do not contain the external leg Feynman rules for the intermediate particle of the propagator. It is a natural question to ask whether equation (3.8) also holds for other particle types.

In the case of other particle types, there are two relevant changes. First off, as implied above, the external legs change. Specifically, the external legs for some particle are the momentum space wave functions corresponding to that particle type. Secondly, the propagator numerator becomes non-trivial. The propagator numerator for scalar particles is simply 1, but changes for other particle types to a more complicated expression, sometimes dependent on the particle's momentum. If it turns out that these properties exactly cancel each other out when we change particle type, then (3.8) may hold more generally.

For spin-1/2 fermions, this question is relatively easily answered. The external legs for fermions are given by the momentum space Dirac spinors $\bar{u}^s(p), \bar{v}^s(p), u^s(p)$ and $v^s(p)$. With proper normalization, these satisfy the spin sum completion relation

$$\sum_{s=1}^{2} u^s(p)\bar{u}^s(p) = \not{p} + m, \qquad \sum_{s=1}^{2} v^s(p)\bar{v}^s(p) = \not{p} - m.$$

Equation (2.13) from the previous chapter is a special case of this identity. Excitingly, for massless particles the fermion propagator numerator also equals $\not{p}$ up to phase. Hence, equation (3.8) also holds for fermions, as long as we sum over the spin of the exchange particle in the factorization channel in (3.8).

For spin-1 gauge bosons, there is a similar argument. The external legs for fermions are given by the polarization vectors $\epsilon^\mu_\pm(p)$ or $\epsilon^\mu_\pm(p)^*$. Analogously with the fermion case,

$$\sum_{s=1}^{2} \epsilon^\mu_s(p)\epsilon^\nu_s(p)^* = -\eta^{\mu\nu} + \frac{k^\mu p^\nu + k^\nu p^\mu}{p \cdot k}.$$

See, for instance, [15, p. 174]. When contracting this with the subamplitudes, the second term drops away due to the ward identity. Hence, if the gauge boson propagator numerator is $-g^{\mu\nu}$, equation (3.8) also holds for this particle type. Well, we're in luck. In an arbitary ξ Lorenz gauge, the Feynman propagator is given by

$$\tilde{D}^{\mu\nu}_F(p) = \frac{-i}{p^2 + i\epsilon}\left(\eta^{\mu\nu} - (1-\xi)\frac{p^\mu p^\nu}{k^2}\right).$$

See [15, p. 297]. Again, due to the ward identity, the second term drops out in the limit $z \to z_I$ if $p = P_I$. Hence, the propagator numerator is effectively given by $-g^{\mu\nu}$ up to phase. Since each subamplitude is independently gauge invariant, the mere existence of a gauge where the amplitude factorizes properly implies that it does so independently of gauge. We conclude that amplitudes with intermediate gauge bosons also admit on-shell recursion.

For gravitons, we can also consider their sum relation

$$\sum_{s=1}^{2} \epsilon^\mu_s(p)\epsilon^\nu_s(p)\epsilon^\rho_s(p)^*\epsilon^\sigma_s(p)^* = \Pi^{\mu\nu\rho\sigma}(p),$$

where

$$\Pi^{\mu\nu\rho\sigma}(p) = \frac{1}{2}\left[\Pi^{\mu\rho}(p)\Pi^{\nu\sigma}(p) + \Pi^{\mu\sigma}(p)\Pi^{\nu\rho}(p) - \Pi^{\mu\nu}(p)\Pi^{\rho\sigma}(p)\right],$$

$\Pi^{\mu\nu}(p)$ being the spin sum for spin-1 bosons. The propagator numerator will also equal $\Pi^{\mu\nu\rho\sigma}$ [21]. The explicit form of $\Pi^{\mu\nu\rho\sigma}$ in the de Donder gauge in 4D [13] is given by

$$\Pi^{\mu\nu\rho\sigma}(p) = \frac{1}{2}\left[\eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho} - \eta^{\mu\nu}\eta^{\rho\sigma}\right].$$

Admittedly, this striking coincidence for the particles above is no coincidence at all. This fact that the propagator numerator can be substituted by a sum over two combined external

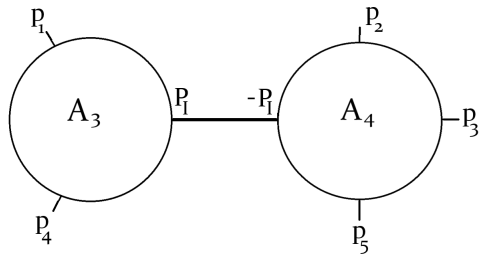


Figure 3.3: A contribution to A_5.

particle states is universal. This is a direct consequence of the optical theorem [15, p. 230], which is a direct consequence of the unitarity of a theory's S matrix,

$$S^{\dagger}S = 1.$$

The S matrix being unitarity has the physical interpretation that if one winds back time after a scattering process has occurred, one retrieves the original state one started with. (Note that the propagator numerator is not always *equal* to the spin sum completion relation. It merely turns out that one can exchange these two inside the expression for an amplitude.)

As stated, the S matrix being unitarity awards one the optical theorem, which states that the imaginary part of any amplitude can be expressed as a sum over amplitudes dependent on every possible intermediate state. Schematically, this can be expressed as:

$$\operatorname{Im} A(a \to b) = \sum_x A^*(b \to x) A(a \to x).$$

The optical theorem in combination with the known pole structure of amplitudes gives us factorization, the latter being due to the locality of the Lagrangian. Hence, amplitude factorization and by extension the recursion formula, (3.8), follow directly from locality and unitarity, as per the standard catchphrase. Since this holds for any quantum field theory that we might be interested in at the moment, we can rest assured that (3.8) holds generally.

3.6 The Recursion Recipe

Here, I will briefly summarize the discussion above with the goal of making it more practical. Let us say that we are interested in determining the n-point amplitude A_n. Let us assume that we already know all amplitudes in the relevant theory A_k with $k < n$. These are the steps to follow in order to derive A_n:

1. Choose a collection of shift vectors $\{r_i\}$ satisfying equation (3.2). Make sure that the residue at infinity of $\hat{A}_n$ vanishes, i.e. $B_\infty = 0$, for this particular shift. Showing this is the most non-trivial.

2. Collect lower point amplitudes together in pairs. For both amplitudes in each pair, select a particle that will function as an intermediate particle. Make sure that the external particles in each pair that are not intermediate particles have the momenta of the external particles of A_n.

That is, if we want to derive $A_5(p_1, p_2, p_3, p_4, p_5)$ collect pairs of A_3 and A_4 amplitudes. A valid pair could be $A_3(p_1, p_4, P_I)$ and $A_4(p_2, p_3, p_5, -P_I)$. The particle with momentum P_I will function as the intermediate particle. In this case $P_I = -(p_1 + p_4) = p_2 + p_3 + p_5$. If we combine these two amplitudes, we will get a contribution to A_5 with P_I taken out. Compare figure 3.3. Make sure to divide the contribution by the intermediate momentum squared.

If the theory involved allows for vertices with multiple particle types, care needs to be taken to make sure that the intermediate particle in either amplitude is of the same type and has identical quantum numbers.

In principle, any pair of amplitudes that 'works' visually as in figure 3.3, is an actual contribution to the greater amplitude. Exchanging external momenta within a subamplitude does not give a new contribution; exchanging external momenta between subamplitudes does.

3. Looking at the current contribution under consideration, determine I for that particular contribution. Shift each momentum using the selected shift from $z = 0$ to $z = z_I$. Then sum each contribution according to equation (3.8).

   It is important to make sure that the subamplitudes are shifted to $\hat{A}_L(z_I)$ and $\hat{A}_R(z_I)$, but the factor $1/P_I^2$ is unshifted.

This is the basic recipe for recursion relations. We will see a couple of examples in the next chapter.

4 BCFW Recursion Relations

In the previous chapter, we saw a relatively general treatment of on-shell tree-level recursion relations. We did not apply recursion to any specific theories and we kept the shift vectors r_i arbitrary (see equation (3.1)).

In this chapter, however, we will discuss a specific type of shift, called the Britto-Cachazo-Feng-Witten (BCFW) shift. This shift keeps all but two momenta unshifted, while shifting two selected momenta in the opposite direction: $r_i = -r_j$ for two specific i and j. We will also treat several examples.

The BCFW recursion relations are the most famous recursion relations, first introduced by BCF, later clarified by Witten and applied to prove of the Parke-Taylor formula [2, 7, 8], which we saw in the introduction. (See equation 1.1.)

The simplicity of this shift makes it relatively easy to work with, compared to certain all-line shifts, where the large number of shifted momenta introduces more computational complexity. In addition to its simplicity, BCFW is also remarkably applicable. BCFW recursion has been shown to work for several important theories, such as Yang-Mills and gravity.

We will start off defining the BCFW shift, both in the spinor helicity formalism and using regular D-vectors. Then, after seeing an example of recursion relations in action, we will move on to prove that BCFW recursion holds for several amplitudes in scalar QED. That means, we will show that the boundary term B_∞ vanishes for certain amplitudes of this theory. This will be a buildup towards showing that BCFW also works in pure Yang-Mills gluon scattering. For both these proofs, we will rely on the work by Nima Arkani-Hamed and Jared Kaplan [22]. After seeing that BCFW works for gluon scattering, the chapter's climax will be to prove the Parke-Taylor formula ourselves, using the spinor helicity formalism in combination with the bootstrapped three-point gluon amplitudes we derived in chapter 2, equation (2.34). We also include a proof that BCFW recursion works for graviton scattering, which has been committed to appendix A.

4.1 Defining the Shift

In the spinor helicity formalism, BCFW recursion relations can be expressed as the shift

$$\begin{aligned} |i] \to |\hat{i}] &\equiv |i] + z|j], \qquad & |j\rangle \to |\hat{j}\rangle &\equiv |j\rangle - z|i\rangle, \\ [i| \to [\hat{i}| &\equiv [i| + z[j|, \qquad & \langle j| \to \langle \hat{j}| &\equiv \langle j| - z\langle i|. \end{aligned} \tag{4.1}$$

Here, $|i]$ and $|j\rangle$ are the positive and negative helicity spinors for the ith and jth particle in the amplitude A_n respectively. All other brackets are left unshifted. This is called an $|i, j\rangle$-shift.

For the sake of defining the BCFW shift, the choice of which momenta to shift and which to keep fixed is essentially arbitrary. However, it turns out that depending on the helicities of the external particles, whether the boundary term B_∞ vanishes can often depend on which particles are picked for the shift. Thus in practice, one needs to be selective in choosing the specific shift.

One can verify that this shift preserves the established identities between the square and angle brackets of the spinor helicity formalism. More pressingly, we can also show that this is a valid shift satisfying (3.2). The shifted momentum for particle i is given by equation (2.19):

$$\hat{p}_i^\mu = \frac{1}{2}\langle\hat{i}|\gamma^\mu|\hat{i}] = \frac{1}{2}\langle i|\gamma^\mu|i] + \frac{z}{2}\langle i|\gamma^\mu|j] = p_i^\mu + \frac{z}{2}\langle i|\gamma^\mu|j].$$

Similarly,

$$\hat{p}_j^\mu = p_j^\mu - \frac{z}{2}\langle i|\gamma^\mu|j].$$

We can thus identify the shift vectors as follows:

$$r_i = q, \qquad r_j = -q, \qquad r_k = 0, \ \ (k \neq i, j) \tag{4.2}$$

where $q^\mu \equiv \langle i|\gamma^\mu|j]/2$. One can use the Fierz identity (2.17) to verify that this satisfies the requirement for shift vectors (3.2). In fact, in arbitrary dimensions, equation (4.2) can be used as the defining feature of the BCFW shift. This, in conjunction with the general requirements of shift vectors, completely fixes q up to normalization. Hence, in 4D equations (4.1) and (4.2) are equivalent.

Interestingly, we can note that for generic momenta we can make a gauge choice such that

$$\epsilon_i^- = \epsilon_j^+ \propto q, \qquad \epsilon_i^+ = \epsilon_j^- \propto q^*,$$

following from the fact that the polarization vectors in the Lorenz gauge and the shift vector have to satisfy the same condition: $p_\mu \epsilon^\mu(p) = p_\mu q^\mu = 0$.

When shifted, this becomes

$$\hat{\epsilon}_i^-(z) = \hat{\epsilon}_j^+(z) \propto q, \qquad \hat{\epsilon}_i^+(z) \propto q - zp_j, \qquad \hat{\epsilon}_j^-(z) \propto q + zp_i. \tag{4.3}$$

We can easily verify this by shifting the spinor helicity expression for the polarization vectors (2.27).

Now that we have a definition for the BCFW shift, it is time we finally treat an example.

Example: Calculating a Scalar QED Amplitude

We will now use the BCFW recursion relations to calculate the scalar QED amplitude $A_4(\phi_1\phi_2^*\gamma_3^+\gamma_4^-)$. That is, the amplitude for two scalars and two photons with opposite helicities.

The three-point amplitudes of this theory are given by

$$A_3(\phi_a\phi_b^*\gamma_c^-) = -i\tilde{e}\frac{\langle a\,c\rangle\langle b\,c\rangle}{\langle a\,b\rangle}, \qquad A_3(\phi_a\phi_b^*\gamma_c^+) = -i\tilde{e}\frac{[a\,c][b\,c]}{[a\,b]}.$$

These can be easily calculated using Feynman diagrams, resulting from the Lagrangian

$$\mathcal{L} = (D_\mu\phi)^*\,D^\mu\phi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{4.4}$$

or they can be bootstrapped using a similar approach to the on used in section 2.5.

We will apply a $|4,3\rangle$-shift: $|4] \to |\hat{4}] = |4] + z|3]$; $|3\rangle \to |\hat{3}\rangle = |3\rangle - z|4\rangle$. It turns out that for this particular shift B_∞ vanishes. For the opposite shift ($|3,4\rangle$) or for shifts of the scalar momenta, the amplitude does not vanish for large z. In this section, we will simply assume that B_∞ vanishes for this particular shift of this amplitude. Later in this chapter, we will prove that $B_\infty = 0$ in this precise case.

Using equation (3.8), we see that

$$A_4(\phi_1\phi_2^*\gamma_3^+\gamma_4^-) = \hat{A}\left(\phi_1\phi^*_{-\hat{P}_{13}}\gamma_{\hat{3}}^+\right)\frac{1}{P_{13}^2}\hat{A}\left(\phi_{\hat{P}_{13}}\phi_2^*\gamma_{\hat{4}}^-\right) + \hat{A}\left(\phi_{-\hat{P}_{23}}\phi_2^*\gamma_{\hat{3}}^+\right)\frac{1}{P_{23}^2}\hat{A}\left(\phi_1\phi^*_{\hat{P}_{23}}\gamma_{\hat{4}}^-\right). \tag{4.5}$$

These are the only two contributions. Importantly, the external particles match on either side of the equation in type and helicity. Furthermore, each subamplitude only contains one particle with shifted momentum. This is because those I which contain both shifted labels do not contribute any poles to the shifted amplitude. Momentum conservation in this case guarantees that the corresponding propagator remains unshifted for any z. Whether a specific I contributes to the amplitude can be verified by confirming that $R_I^2 \neq 0$.

It should be noted that both terms in equation (4.5) come from a different label set I, namely $I = \{1,3\}$ and $I = \{2,3\}$. Consequently, the subamplitudes in these terms are also shifted with different values of z, to $z = z_{13}$ and $z = z_{23}$ respectively. Therefore, when writing out these subamplitudes, a shifted spinor , e.g. $|\hat{3}\rangle$, has a different meaning dependent on the term in which it occurs. Hence, in order to properly evaluate equation (4.5), it is useful to first evaluate each term separately at first and only to combine both expressions into one larger expression once both separate terms have been rewritten in terms of unshifted momenta.

To this end, let us look at the first term. This is given by

$$\hat{A}\left(\phi_1\phi^*_{-\hat{P}_{13}}\gamma_{\hat{3}}^+\right)\frac{1}{P_{13}^2}\hat{A}\left(\phi_{\hat{P}_{13}}\phi_2^*\gamma_{\hat{4}}^-\right) = -\tilde{e}^2\frac{[1\,\hat{3}][-\hat{P}_{13}\,\hat{3}]}{[1\,(-\hat{P}_{13})]}\frac{1}{\langle 1\,3\rangle[1\,3]}\frac{\langle\hat{P}_{13}\,\hat{4}\rangle\langle 2\,\hat{4}\rangle}{\langle\hat{P}_{13}\,2\rangle}.$$

We will now heavily rely on various spinor helicity identities to rework this expression. First, we we will use the identity

$$|-p\rangle = -|p\rangle, \qquad |-p] = -|p], \tag{4.6}$$

derivable from (2.15) and (2.16), to get rid of the minuses inside the spinor brackets. We will use the antisymmetric property (2.10) and the property regarding the contraction with gamma matrices (2.13) to rewrite the expression:

$$\frac{[1\,\hat{3}][-\hat{P}_{13}\,\hat{3}]}{[1\,(-\hat{P}_{13})]}\frac{1}{\langle 1\,3\rangle[1\,3]}\frac{\langle\hat{P}_{13}\,\hat{4}\rangle\langle 2\,\hat{4}\rangle}{\langle\hat{P}_{13}\,2\rangle} = -\frac{[1\,\hat{3}][\hat{3}|p_1+\hat{p}_3|\hat{4}\rangle\langle 2\,\hat{4}\rangle}{\langle 1\,3\rangle[1\,3][1|p_1+\hat{p}_3|2\rangle}.$$

Here we are using the 'slashless' notation (2.20). We can use the Weyl equation to kill some momenta wedged between spinor brackets. We can then rewrite the expression purely in terms of spinor brackets once again.

$$-\frac{[1\,\hat{3}][\hat{3}|p_1+\hat{p}_3|\hat{4}\rangle\langle 2\,\hat{4}\rangle}{\langle 1\,3\rangle[1\,3][1|p_1+\hat{p}_3|2\rangle} = -\frac{[1\,\hat{3}][\hat{3}|p_1|\hat{4}\rangle\langle 2\,\hat{4}\rangle}{\langle 1\,3\rangle[1\,3][1|\hat{p}_3|2\rangle} = -\frac{[1\,\hat{3}][1\,\hat{3}]\langle 1\,\hat{4}\rangle\langle 2\,\hat{4}\rangle}{\langle 1\,3\rangle[1\,3][1\,\hat{3}]\langle 2\,\hat{3}\rangle}.$$

Since $|\hat{3}] = |3]$ and $|\hat{4}\rangle = |4\rangle$, we can simplify this expression.

$$\hat{A}\left(\phi_1\phi^*_{-\hat{P}_{13}}\gamma^+_{\hat{3}}\right)\frac{1}{P^2_{13}}\hat{A}\left(\phi_{\hat{P}_{13}}\phi^*_2\gamma^-_{\hat{4}}\right) = \tilde{e}^2\frac{\langle 1\,4\rangle\langle 2\,4\rangle}{\langle 1\,3\rangle\langle 2\,\hat{3}\rangle}.$$

It is now our task to write this in terms of unshifted momentum only. Firstly, note that z_{13} has the exact value such that $\hat{P}_{13}$ becomes on-shell. Hence,

$$0 = \hat{P}^2_{13} = -\langle 1\,\hat{3}\rangle[1\,\hat{3}] = -\langle 1\,\hat{3}\rangle[1\,3]. \tag{4.7}$$

Since $[1\,3]$ is nonzero for generic momenta, this leads us to conclude that $\langle 1\,\hat{3}\rangle = 0$. Of course, this identity does not hold in general. It only works for shifts where $\hat{P}_{13}$ becomes on shell.

Secondly, if we analyse the unrelated bracket $\langle 3\,\hat{3}\rangle$, using the antisymmetric property, we deduce that

$$\langle 3\,\hat{3}\rangle = \langle 3|(|3\rangle - z_{13}|4\rangle) = -z_{13}\langle 3\,4\rangle.$$

Using these two identities together with the Van Schouten identity (2.22), we get

$$\hat{A}\left(\phi_1\phi^*_{-\hat{P}_{13}}\gamma^+_{\hat{3}}\right)\frac{1}{P^2_{13}}\hat{A}\left(\phi_{\hat{P}_{13}}\phi^*_2\gamma^-_{\hat{4}}\right) = -\tilde{e}^2\frac{\langle 1\,4\rangle\langle 2\,4\rangle}{\langle 1\,\hat{3}\rangle\langle 3\,2\rangle + \langle 1\,2\rangle\langle \hat{3}\,3\rangle} = -\frac{\tilde{e}^2}{z_{13}}\frac{\langle 1\,4\rangle\langle 2\,4\rangle}{\langle 1\,2\rangle\langle 3\,4\rangle}.$$

One can easily calculate the value of z_{13} using equation (3.4). For this shift $z_{13} = \langle 1\,3\rangle/\langle 1\,4\rangle$, hence

$$\hat{A}\left(\phi_1\phi^*_{-\hat{P}_{13}}\gamma^+_{\hat{3}}\right)\frac{1}{P^2_{13}}\hat{A}\left(\phi_{\hat{P}_{13}}\phi^*_2\gamma^-_{\hat{4}}\right) = -\tilde{e}^2\frac{\langle 1\,4\rangle^2\langle 2\,4\rangle}{\langle 1\,2\rangle\langle 1\,3\rangle\langle 3\,4\rangle}.$$

With near identical reasoning, we can determine that the second term of equation (4.5) is

$$\hat{A}\left(\phi_{-\hat{P}_{23}}\phi^*_2\gamma^+_{\hat{3}}\right)\frac{1}{P^2_{23}}\hat{A}\left(\phi_1\phi^*_{\hat{P}_{23}}\gamma^-_{\hat{4}}\right) = \tilde{e}^2\frac{\langle 1\,4\rangle\langle 2\,4\rangle^2}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle}.$$

Combining these two expressions gives

$$A_4(\phi_1\phi^*_2\gamma^+_3\gamma^-_4) = -\tilde{e}^2\frac{\langle 1\,4\rangle\langle 2\,4\rangle}{\langle 1\,3\rangle\langle 2\,3\rangle}\left(\frac{\langle 1\,4\rangle\langle 2\,3\rangle - \langle 1\,3\rangle\langle 2\,4\rangle}{\langle 1\,2\rangle\langle 3\,4\rangle}\right) = \tilde{e}^2\frac{\langle 1\,4\rangle\langle 2\,4\rangle}{\langle 1\,3\rangle\langle 2\,3\rangle},$$

where the Van Schouten identity has been used once again in the last step.

And there we have it. Acquiring a valid expression for the amplitude was in essence effortless, as we already obtained a valid expression in equation (4.5). Reworking this expression into an elegant form completely given in terms of unshifted momenta did require some work. In contrast, the calculation of this amplitude using traditional methods requires calculating three Feynman diagrams, two exchange diagrams and one contact diagram. It is overall difficult to say which method was simpler in this case. Of course, for higher-point amplitudes, where the amount of Feynman diagrams explodes, this method wins out rather quickly.

4.2 Proving BCFW Recursion for Scalar QED

In the previous example, we calculated a 4-point scalar QED scattering amplitude using recursion relations. In the process of calculating this amplitude, we simply assumed that the residue at infinity B_∞ vanishes. It thus remains to be shown that in the specific example above B_∞ indeed vanishes before we can fully trust the result.

In this section, we will show that for a wide range of scalar QED amplitudes, it is indeed true that there is a vanishing residue at infinity, allowing one to perform recursion. We will largely restrict our focus to the type of diagram with two shifted scalars and n unshifted photons. As is typical, we will demonstrate that that scalar QED amplitudes of the type we are considering vanish in the high-z limit,

$$\hat{A}(z) \to 0, \text{ as } z \to \infty,$$

guaranteeing a vanishing B_∞. Despite the fact that this differs from the scenario of two shifted photons in the example above, we can still use this result for arguing that B_∞ vanishes in the previous example. We will briefly touch upon this.

However, the main motivation for treating scalar QED amplitudes in this section with two shifted scalars and n unshifted photons is to build up towards showing that B_∞ vanishes for gluon scattering amplitudes in Yang-Mills in the next section. The proof we will see in the next section for Yang-Mills is comparatively technical compared to rest of the thesis. Hence, it serves us well to treat an easier, yet analogous proof in an earlier section. In either section, we will follow Arkani-Hamed et al. [22]. The concepts introduced in this section will return in the next.

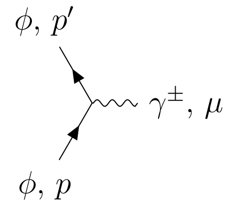
After showing that BCFW recursion works for Yang-Mills in the next section, we will arrive at this chapter's climax: proving the Parke-Taylor formula.

We start by considering the scalar QED Lagrangian once more

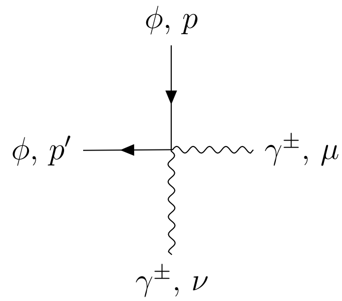
$$\mathcal{L} = (D_\mu \phi)^* D^\mu \phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu},$$

with $D_\mu = \partial_\mu - ieA_\mu$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. We will analyze the various components of Feynman diagrams in this theory and determine their z-dependence separately. We will then be able to make conclusions about the overall z-dependence of various amplitudes.

The Lagrangian provides us with the following vertices:



$$\propto -ie(p-p')^{\mu},$$



$$\propto 2ie^2 g^{\mu\nu}. \tag{4.8}$$

We can see that since the second of these vertices does not depend on the momenta of its particles, this vertex is always independent of z and thus scales with $\propto z^0$.

The first of these vertices, as it scales linearly with scalar momentum, can have a potential z-dependence, with a term $\propto z^1$. This of course if and only if some of the contributions to $\hat{p}$ or $\hat{p}'$ are shifted, with the added condition that these shifts do not cancel each other. Otherwise, the vertex will have ordinary $\propto z^0$ dependence.

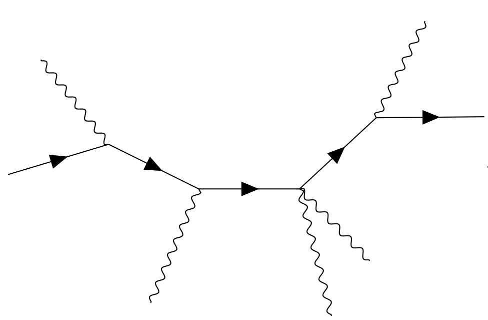
The propagators of the theory are all proportional to

$$\frac{1}{\hat{p}^2} \propto \frac{1}{p^2(z-z_I)} \propto z^{-1},$$

if the momentum of the intermediate particle is shifted, as appears from the discussion in sections 3.1 and 3.2.

Now, let us consider the amplitude with n external photons and two external scalars. We will call this amplitude $M_{2;n}$. If we choose our two shifted particles to be the two external scalars, then all external leg contributions to the Feynman diagram will scale with z^0, as the scalar external leg is independent of particle momentum and the photon polarization vectors are unshifted.

Because of the two vertex types available in this theory, we can already conclude that every Feynman diagram contributing to this amplitude will have the shape



. (4.9)

That is, we have one continuous scalar line from one shifted particle to the other with photon branches. There is no gauge boson self-interaction in this theory. This precludes us from having any intermediate photons at tree level for a two-scalar amplitude.

We can now make an estimate of the overall z-dependence of such diagrams. First of all, we know that for any scalar propagator in this type of diagram, we will gain an overall z^{-1}-dependence in the limit. We also know that for each three point vertex, we will gain an overall z^1-dependence.

Thus the worst contribution to the amplitude $M_{2;n}$ will be a diagram only featuring three-point vertices. One can see that in that case, one will have an overall z-dependence of

$$M_{2;n} \sim z^{n-(n-1)} = z^1 \to \infty.$$

It seems as if the amplitude blows up in the limit and thus that scalar QED amplitudes of this type are not on-shell constructible.

However, this argumentation has been a little too quick. We have assumed that there are no cancellations of the z-dependence inside or between the diagrams. It is actually possible to pick a specific gauge, where it becomes visible that much of this z-dependence actually cancels. This gauge choice is the so-called lightcone gauge.

4.2.1 The Lightcone Gauge for Gauge Bosons

The lightcone gauge for some 4-vector q is a specific Lorenz gauge choice such that

$$q_\mu A^\mu = 0,$$

where A^μ is the gauge boson field. If for q we pick our shift vector from equation (4.2), the lightcone gauge gives us the property that

$$\hat{P}_I^\mu \epsilon_\mu = (P_I^\mu + zq^\mu)\,\epsilon_\mu = P_I^\mu \epsilon_\mu, \tag{4.10}$$

if P_I contains the positively shifted external momentum p_i.

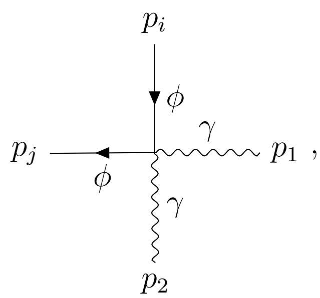
Typically, we can use the lightcone gauge to manifestly eliminate all z-dependence from vertices. For the case of diagrams of the same type as diagram (4.9), all positive z-dependence comes from three-particle vertices (4.8). At every such vertex, a contraction occurs similar to equation (4.10). Hence, by picking the lightcone gauge for the gauge boson field, we can eliminate all such z-dependence from the vertices. Hence, this demonstrates that the actual z-dependence of scalar QED amplitudes is markedly better than stated in the previous section. By choosing the lightcone gauge, this improved z-dependence can be made manifest. Since amplitudes are gauge invariant, amplitudes in other gauges feature identically improved z-dependence.

To make things explicit, in the case of scalar QED the 'worst offender' in terms of z-dependence contributing to a typical $M_{2;n}$ amplitude will be the diagram featuring only four-particle vertices, contrary to what seemed to be the case before we discussed the lightcone gauge. This is because this is the diagram featuring the smallest amount of propagators. After all, every propagator adds a factor of $\propto 1/z$, whereas the amount of vertices does not alter the overall z-dependence. Such a diagram gains a propagator for every two additional external gauge bosons.

As the z-dependence of an amplitude in the limit is dominated by the z-dependence of the least favorable diagram, this gives us a maximum z-dependence (for $n \geq 2$) of

$$M_{2;n} \sim \begin{cases} z^{1-n/2} & n \text{ is even,} \\ z^{1-(n+1)/2} & n \text{ is odd.} \end{cases} \tag{4.11}$$

We observe that $M_{2;n} \to 0$ for every n, except for the $n = 2$ case. This is due to the contact diagram



which lacks propagators. We thus see that the BCFW recursion relations work for constructing amplitudes $M_{2;n}$ for $n > 2$.

This result seems problematic, given that the example calculation we performed above featured an $M_{2;2}$ amplitude. However, whereas in this section, we are discussing shifts of the two external scalar momenta, in the example, we shifted two photon momenta. It turns out that the shifting of the two photons confers slightly improved z-dependence in our case. This is due to the shifted polarization vectors. In combination with the Ward identity, it is possible to show that this results in an additional factor of $1/z$, making also the $M_{2;2}$ vanish in the large-z limit. This topic will be left for what it is now. It is more appropriate to discuss the Ward identity in the next section, as this will be very relevant for Yang-Mills amplitudes.

We have now argued that $M_{2;n}$ is on-shell constructible using a BCFW shift for $n > 2$. Even though this conclusion is correct, the argument has left out an important detail. We assumed that we could pick the lightcone gauge freely, but it turns out there are some technicalities to consider in doing so. This will be important later, so we must discuss this now.

Let A^μ be some vector field in some Lorenz gauge. We acquire the lightcone gauge by imposing

$$A^\mu \to A'^\mu = A^\mu + \partial^\mu \Lambda(x),$$

such that

$$q_\mu A'^\mu = 0.$$

To evaluate the feasibility of the existence of such a Λ, we can go to momentum space by Fourier-transforming the above condition. We get

$$q_\mu \epsilon'^\mu(k) = q_\mu \epsilon^\mu(k) + q_\mu k^\mu \tilde{\Lambda}(k) = 0, \qquad \forall k.$$

This poses a problem, since for any q, there is definitely some collection of ks such that $q \cdot k = 0$, eliminating the second term in the equation above. These k lie on the lightcone originating from q in the vector space of all four-momenta. (See figure 4.1.) For these k, it is impossible to transform $\epsilon(k)$ to the q-lightcone gauge unless $\epsilon(k)$ was already in the lightcone gauge to begin with.

This, however, is not a problem for our purposes in most cases. We do not need $q_\mu \epsilon^\mu(k)$ to vanish for every k, but only for those specific k which would result in the cancellation of z-dependence from vertices. These k are specifically the momenta of gauge bosons meeting a particle with a shifted momentum at a vertex. Which k these are is determined by external particle momentum.

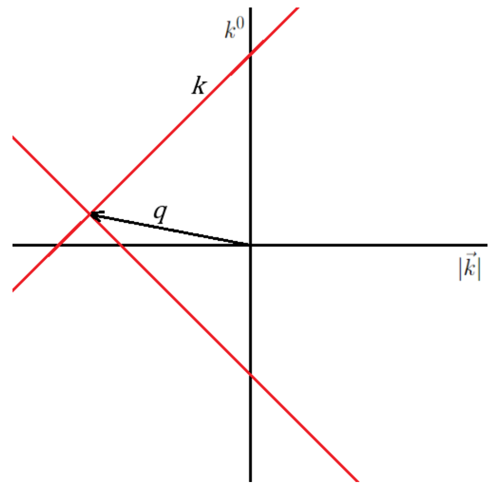
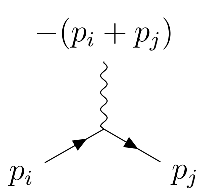


Figure 4.1: The lightcone of momenta k orthogonal to momentum q, i.e. $k \cdot q = 0$.

Luckily, the lightcone on which k is orthogonal to q is in fact very small in comparison to the entire space of possible momenta k, as can be seen in the figure. In fact, these k form a measure zero subset of the total possible space of allowed momenta. Importantly, every k for which $k \cdot q = 0$ borders a region where $k \cdot q \neq 0$.

Since every problematic k borders a region of non-problematic k, we can typically construct the amplitude for this untroublesome region by recursion to subsequently take the limit to this problematic k. It seems as if this would require us to first construct the amplitude for every external momentum configuration not resulting in any problematic k. One would then have to take the limit of the external momenta such that we enter the difficult region and get the expression of the amplitude there too. In practice, however, it simply means that this entire concern can be ignored. By continuity arguments, one can show that B_∞ vanishes on this lightcone as well, allowing for undisturbed recursion.

After this initial concern, it seems like we are back on solid ground when it comes to our previously derived result. However, we have now glossed over another fact. For some specific amplitudes, it is never possible to enter the non-problematic region. This occurs when there is a Feynman diagram contributing to the amplitude, where momentum conservation dictates that a gauge boson always has a momentum orthogonal to q. A specific example of such a Feynman diagram would be



,

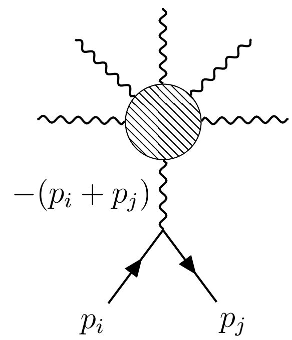
where, by construction, p_i and p_j are orthogonal to q, thus $-(p_i + p_j)$ is as well. (Compare equation 3.2.iii.)

If we would perform the BCFW shift for this diagram, the vertex in this diagram would have z-dependence which we could not explicitly eliminate with a lightcone gauge transformation. For scalar QED, this is the only diagram of this type. Since this diagram contributes to the $M_{2;1}$ amplitude, we conclude that

$$M_{2;1}(z) \to \infty,$$

as $z \to \infty$ for scalar QED. But since we are never interested in derived a three-point amplitude through recursion, this is of little concern to us.

In general, we have the following type of diagram posing a similar problem:



(4.12)

That is, a diagram featuring the two shifted particles coupled directly to a gauge boson. This gauge boson, in case it is self-interacting, may in turn be coupled to any number of external particles.

These diagrams also contain a vertex whose z-dependence cannot be eliminated by choosing the lightcone gauge. For scalar QED, the only diagram of this type is the one seen above. For other theories, however, there can often be more diagrams of such a problematic nature. When discussing those theories, we will have to pay special attention to these diagrams, as they will be leading the the theory's z-dependence.

We will now move on to prove that BCFW recursion works for general Yang-Mills amplitudes. There, such diagrams feature extensively, which is something we will have to address in the proof.

4.3 Proving BCFW Recursion for Yang-Mills

In this subsection, we will prove that one can use the BCFW recursion relations to construct any pure Yang-Mills amplitude beyond three-point. This is in contrast to the scalar QED case, where we only looked at two-scalar amplitudes. Our arguments will be valid for full and partial amplitudes. Afterwards, we will spend a section to use this result in proving the famous Parke-Taylor formula.

Our method for proving the validity of BCFW for Yang-Mills will again rely on a consideration of the lightcone gauge, but also the so-called background field method [23, p. 249]. That is, since in the large z limit of the amplitude, we can consider the two shifted particles to have a very large momentum, much larger than any other particles involved in the amplitude. This allows us to make the substitution $A_\mu \to A_\mu + a_\mu$, where A_μ is the original, low energy, background field and a_μ the boosted field. How this is exactly applied will become apparent soon.

We will use all of this to calculate the z-dependence of $\hat{M}_n^{\mu\nu}(z)$, which is the shifted n-point amplitude, as of yet to be contracted with the polarization vectors of the two external particles. Focusing on this amplitude instead of the complete amplitude at first will allow us to determine the z-dependence in a more organized fashion.

As before, we will start by stating the Lagrangian. We will then perform some manipulations to cast the Lagrangian in a form that is convenient for our purposes, following [22].

The Yang-Mills Lagrangian is given by

$$\mathcal{L} = -\frac{1}{2}\,\mathrm{Tr}\left[F^{\mu\nu}F_{\mu\nu}\right],$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig\left[A_\mu, A_\nu\right]$. After performing the shift $A_\mu \to A_\mu + a_\mu$, we get

$$F_{\mu\nu} \to F_{\mu\nu} + D_{[\mu}\,a_{\nu]} - ig a_{[\mu}\,A_{\nu]} - ig\left[a_\mu, a_\nu\right].$$

The Lagrangian will then consist of many terms. However, we will specifically be interested in those terms which contain a^μ twice, i.e. quadratic in a^μ. Those are

$$\mathcal{L}_{|a^2} = -\frac{1}{2}\,\mathrm{Tr}\left(D_{[\mu}\,a_{\nu]}D^{[\mu}\,a^{\nu]} - 2ig\left[a_\mu, a_\nu\right]F^{\mu\nu}\right).$$

Here, the covariant derivative is given by the traditional $D_\mu = \partial_\mu - igA_\mu$. We are interested specifically in the terms containing a^μ, because these terms alone will provide the z-dependent vertices. After all, a^μ is the shifted field. Furthermore, we are never interested in terms with a higher power in a^μ, since momentum conservation dictates that two (and only two) shifted fields meet at any given vertex.

We add a gauge fixing term for a^μ,

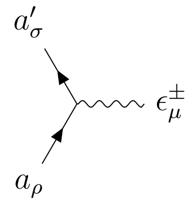
$$-\frac{1}{2}\left(D_\mu a^\mu\right)^2.$$

Now our Lagrangian of interest becomes

$$\mathcal{L}_{|a^2} = -\frac{1}{2}\,\mathrm{Tr}\left(\eta^{\rho\sigma}D_\mu a_\rho D^\mu a_\sigma - 2ig\left[a_\rho, a_\sigma\right]F^{\rho\sigma}\right).$$

Here, the indices have been chosen to make the scaling with $\eta^{\rho\sigma}$ of the first term more explicit. When performing the BCFW shift, we will always have two shifted external particles. Each diagram contributing to the overall amplitude will then have a single unique path of shifted internal lines from shifted external particle to shifted external particle. The typical amplitude will thus again have the same shape as figure (4.9), where the straight lines correspond to the shifted field, but now with possible gluon self-interactions.

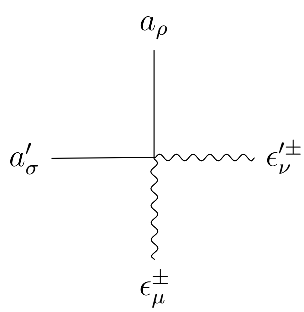
The first term in the Lagrangian gives the kinetic term for a^μ, but also vertices that scale with z, due to the derivative acting on the shifted field. Since this first term scales with $\eta^{\rho\sigma}$, the three particle vertex gives us



$$\propto \eta^{\rho\sigma}\,(p^\mu + zq^\mu).$$

In a diagram, $\eta^{\rho\sigma}$ is contracted with external or internal shifted particle lines and the momenta are contracted with the unshifted gluon field.

Both the first and second term of the Lagrangian produce a four particle vertex of the type



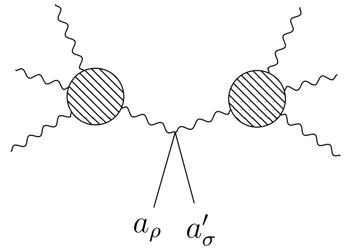
$$\propto \eta^{\rho\sigma} + A^{\rho\sigma}.$$

Here, $A^{\rho\sigma}$ is a tensor already containing the contribution from the unshifted gluons, to be contracted with the shifted gluons within an actual diagram. $A^{\rho\sigma}$ is anti-symmetric, inheriting its anti-symmetry from the field strength tensor $F^{\rho\sigma}$ in the second term.

Since the z-dependence of any diagram is contained in terms where q^μ contracts with gluon fields, we can once again apply the lightcone gauge. Note that since we have gluon self-interactions, not every z-dependent vertex is directly contracted with an external gluon, but sometimes with gluon propagators. However, as discussed in section 3.5, the propagator numerators can be substituted with polarization vectors. Hence, choosing the lightcone gauge is effective in eliminating the z-dependence in these vertices as well.

Thus everything discussed in section 4.2.1 for scalar QED holds here too. The only truly z-dependent diagram with a diverging limit is of the type of (4.12).

Furthermore, every shifted propagator scales with $\propto 1/z$. Thus we can conclude that any diagram with at least one shifted propagator scales with $\propto 1/z$ or better. Hence, the only diagrams which have a term scaling with z^0 are again the type of diagram of (4.12) above and a similar diagram with a four particle vertex:



We get an overall z-dependence of

$$\hat{M}_n^{\rho\sigma} \propto (cz+1)\,\eta^{\rho\sigma} + A^{\rho\sigma} + \frac{1}{z}B^{\rho\sigma} + \dots,$$

where c is a proportionality constant and B an unknown tensor.

The task at hand is to contract this incomplete amplitude above with the polarization vectors of the shifted particles. For this, we will use the Ward identity for gluons, which is

$$p_\mu A^\mu = 0,$$

if $\epsilon_\mu(p)A^\mu$ is the complete amplitude. Similarly, it will hold that

$$\hat{p}_{i\rho}\hat{M}_n^{\rho\sigma}\hat{\epsilon}_{j\sigma} = 0,$$

where we are using the shifted quantities. This holds identically for exchanged i and j. Since $\hat{p}_i = p_i + zq$ and $\hat{p}_j = p_j - zq$, the Ward identity gives us

$$q_\rho \hat{M}^{\rho\sigma}\hat{\epsilon}_{j\sigma} = -\frac{1}{z}p_{i\rho}\hat{M}^{\rho\sigma}\hat{\epsilon}_{j\sigma}, \qquad q_\rho \hat{M}^{\rho\sigma}\hat{\epsilon}_{i\sigma} = +\frac{1}{z}p_{j\rho}\hat{M}^{\rho\sigma}\hat{\epsilon}_{i\sigma}. \tag{4.13}$$

Hence, contracting with the shift vector q results in an additional factor $1/z$ of overall z-dependence.

Using this together with the identities in equation (4.3), we can easily determine the overall z-dependence based on the helicities of the external particles. For example,

$$\begin{aligned}
\hat{M}_n^{-+} &= \hat{\epsilon}_{i\rho}^{-}\hat{M}^{\rho\sigma}\hat{\epsilon}_{j\sigma}^{+}, \\
&= q_\rho \hat{M}^{\rho\sigma} q_\sigma, \\
&\propto q_\rho \left[(cz+1)\,\eta^{\rho\sigma} + A^{\rho\sigma} + \frac{1}{z}B^{\rho\sigma} + \dots\right] q_\sigma, \\
&= \frac{1}{z} q_\rho B^{\rho\sigma} q_\sigma + \dots,
\end{aligned}$$

relying on the on-shell property of q and the anti-symmetry of A. Or,

$$\begin{aligned}
\hat{M}_n^{--} &= \hat{\epsilon}_{i\rho}^{-}\hat{M}^{\rho\sigma}\hat{\epsilon}_{j\sigma}^{-}, \\
&= q_\rho \hat{M}^{\rho\sigma}\epsilon_{j\sigma}^{+}, \\
&= -\frac{1}{z}p_{i\rho}\hat{M}^{\rho\sigma}\epsilon_{j\sigma}^{+}, \\
&= -\frac{1}{z}p_{i\rho}\hat{M}^{\rho\sigma}(q_\sigma + zp_{i\sigma}), \\
&\propto \frac{1}{z}p_{i\rho}A^{\rho\sigma}q_\sigma + \dots.
\end{aligned}$$

Here, both equations (4.13) and (4.3) have been used. Subsequently, the on-shellness of p_i,the anti-symmetry of A and the orthogonality of q and p_i.

$\epsilon_i \backslash \epsilon_j$	$-$	$+$
$-$	$1/z$	$1/z$
$+$	z^3	$1/z$

Table 4.1: The z-dependence of pure Yang-Mills amplitudes depending on shifted particle amplitude.

For scalar QED, the ward identity holds as well. So when we shift the photons instead of the scalars in the two-scalar two-photon case of example 4.1, we gain an additional factor of $1/z$ compared to (4.11). This additional factor makes the amplitude vanish in the large z limit, proving that our original recursion works.

Doing similar calculations for every polarization combination in Yang-Mills gets us the result shown in table 4.1. We see thus that depending on the polarizations of the shifted particles, the gluon amplitude is either on-shell constructible or it is not. The only non-constructible case is the $\hat{M}_n^{+-}$ amplitude. However, this can be made constructible simply by deciding to shift p_i and p_j oppositely. That is, by shifting p_i as if it were p_j ($p_i \to p_i - zq$) and p_j as if it were p_i ($p_j \to p_j + zq$), effectively exchanging labels i and j. In this way, we can see that every tree-level pure gluon amplitude is on-shell constructible.

4.4 Proving the Parke-Taylor Formula

The Parke-Taylor formula is a very simple formula, giving the expression for maximally helicity violating, color-ordered Yang-Mills tree amplitudes.

An n-point Yang-Mills amplitude is maximally helicity violation (MHV) when $n-2$ of its external gluons have a helicity opposite to the remaining 2 gluons. This is *maximally* helicity violating, because any greater uniformity in helicities reduces the amplitude to zero. That is,

$$A_n(p_1^{\pm}, p_2^{\pm}, \ldots, p_n^{\pm}) = A_n(p_1^{\mp}, p_2^{\pm}, \ldots, p_n^{\pm}) = 0,$$

for $n \geq 4$.

The Parke-Taylor formula then gives the color-ordered MHV tree amplitude:

$$A_n[p_1^+, \ldots, p_i^-, \ldots, p_j^-, \ldots, p_n^+] = \frac{\langle i\, j\rangle^4}{\langle 1\, 2\rangle\langle 2\, 3\rangle \ldots \langle n\, 1\rangle}, \tag{4.14}$$

and for flipped helicities,

$$A_n[p_1^-, \ldots, p_i^+, \ldots, p_j^+, \ldots, p_n^-] = \frac{[i\, j]^4}{[1\, 2][2\, 3] \ldots [n\, 1]}. \tag{4.15}$$

In this section, we will prove the Parke-Taylor formula using BCFW recursion, now that we know that this method holds for Yang-Mills amplitudes. We will do so inductively, limiting to the mostly minus case.

We inadvertedly showed that the Parke-Taylor formula holds for the 3-point case in section 2.5, by deriving (2.34). The complete result is given here.

$$A_3[p_1^-, p_2^-, p_3^+] = \frac{\langle 1\, 2\rangle^4}{\langle 1\, 2\rangle\langle 2\, 3\rangle\langle 3\, 1\rangle}, \qquad A_3[p_1^+, p_2^+, p_3^-] = \frac{[1\, 2]^4}{[1\, 2][2\, 3][3\, 1]}.$$

Hence, we need to show that if Parke-Taylor formula holds up to $(n-1)$-point, it will also hold for n point. In this, we will follow the derivation laid out in Elvang & Huang [13].

We will use a $|-,-\rangle = |1,2\rangle$ BCFW shift, applied to the recursion formula (3.8). We will be able to ignore the factor 1/2 in the recursion formula by only summing over indexing sets that contain momentum label '1'.

It is important to note that we want to derive a color-ordered amplitude. The only Feynman diagrams that contribute to such an amplitude have a specific ordering of their external legs. For instance, the color-ordered amplitude $A_n[1,2,3,4]$ will only consist of Feynman diagrams where the external momenta p_1, p_2 etc. are arranged in a clockwise fashion. This fact means that color-ordered tree amplitudes decompose solely into color-ordered amplitudes when intermediate momenta go on-shell. Hence, these are the only contribution in our recursion formula.

In this case, we get

$$A_n = \sum_{h=\pm}\sum_{k=4}^{n} \hat{A}_{3+(n-k)}[k^+,(k+1)^+\dots,n^+,\hat{1}^-,\hat{P}_I^h]\frac{1}{P_I^2}\hat{A}_{k-1}[-\hat{P}_I^{-h},\hat{2}^-,3^+,\dots,(k-1)^+].$$

Here, $P_I = p_2 + p_3 + \ldots + p_{k-1}$. We sum over the helicity of the internal line, because both helicities contribute diagrams to the overall amplitude. The mediating gluon has an opposite helicities in the two amplitudes (h and $-h$). One needs to flip the momentum in one of the two amplitudes to make sure that the particle is outgoing in both amplitudes. Since spin stays the same, the helicity is then flipped as well.

Interestingly, since the mediating particle can only have negative helicity in one subamplitude at the same time, always one of the two amplitudes will be 'more than maximally helicity violating' at the same time. This is true, except when one of the two subamplitudes is three-point, where the MHV amplitude has only one distinct particle. Hence, we are only left with the contributions:

$$\begin{aligned}A_n[1^-,2^-,3^+,4^+,\dots,n^+] = {}&\hat{A}_3[n^+,\hat{1}^-,\hat{P}_{n1}^+]\frac{1}{P_{n1}^2}\hat{A}_{n-1}[-\hat{P}_{n1}^-,\hat{2}^-,3^+,\dots,(n-1)^+]\\ &+\hat{A}_{n-1}[4^+,5^+,\dots,\hat{1}^-,\hat{P}_{23}^-]\frac{1}{P_{23}^2}\hat{A}_3[-\hat{P}_{23}^+,\hat{2}^-,3^+].\end{aligned}$$

We will treat these two terms separately. Starting with the first term,

$$\begin{aligned}&\hat{A}_3[n^+,\hat{1}^-,\hat{P}_{n1}^+]\frac{1}{P_{n1}^2}\hat{A}_{n-1}[-\hat{P}_{n1}^-,\hat{2}^-,3^+,\dots,(n-1)^+]\\
&\qquad= \frac{[n\,\hat{P}_{n1}]^4}{[n\,\hat{1}][\hat{1}\,\hat{P}_{n1}][\hat{P}_{n1}\,n]}\frac{1}{P_{n1}^2}\frac{\langle -\hat{P}_{n1}\,\hat{2}\rangle^4}{\langle -\hat{P}_{n1}\,\hat{2}\rangle\langle\hat{2}\,3\rangle\dots\langle (n-1)\,(-\hat{P}_{n1})\rangle},\\
&\qquad= -\frac{[n\,\hat{P}_{n1}]^3}{[n\,\hat{1}][\hat{1}\,\hat{P}_{n1}]}\frac{1}{P_{n1}^2}\frac{\langle \hat{P}_{n1}\,\hat{2}\rangle^3}{\langle\hat{2}\,3\rangle\dots\langle (n-1)\,\hat{P}_{n1}\rangle},\\
&\qquad= \frac{([n|\hat{1}+n|\hat{2}\rangle)^3}{[n\,\hat{1}]\langle n\,1\rangle[n\,1]\langle\hat{2}\,3\rangle\dots\langle n-1|\hat{1}+n|\hat{1}]},\\
&\qquad= \frac{[n\,\hat{1}]^3\langle\hat{1}\,\hat{2}\rangle^3}{[n\,\hat{1}]\langle n\,1\rangle[n\,1]\langle\hat{2}\,3\rangle\dots\langle (n-1)\,n\rangle[n\,\hat{1}]},\end{aligned}$$

$$= \frac{[n\,\hat{1}]}{[n\,1]} \frac{\langle \hat{1}\,\hat{2}\rangle^4}{\langle \hat{1}\,\hat{2}\rangle\langle \hat{2}\,3\rangle \ldots \langle n\,1\rangle}.$$

Here, we used equation (2.10) and (2.13). We can already recognize the outline of the Parke-Taylor formula. Now, because of the BCFW shift, we know that $|\hat{1}\rangle = |1\rangle$. This means that

$$\langle \hat{1}\,\hat{2}\rangle = \langle 1\,\hat{2}\rangle = \langle 1|(|2\rangle - z_{n1}|1\rangle) = \langle 1\,2\rangle.$$

Hence, we get

$$\frac{[n\,\hat{1}]}{[n\,1]} \frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\langle \hat{2}\,3\rangle \ldots \langle n\,1\rangle}.$$

Finally, since the shifted internal particle goes on-shell, we have

$$0 = \hat{P}_{n1}^2 = \langle n\,\hat{1}\rangle[n\,\hat{1}] = \langle n\,1\rangle[n\,\hat{1}] \Rightarrow [n\,\hat{1}] = 0,$$

in the same vein as in equation (4.7). This naturally yet surprisingly means that the first term entirely vanishes,

$$\hat{A}_3[n^+, \hat{1}^-, \hat{P}_{n1}^+] \frac{1}{P_{n1}^2} \hat{A}_{n-1}[-\hat{P}_{n1}^-, \hat{2}^-, 3^+, \ldots, (n-1)^+] = 0.$$

It turns out, however, that our efforts were not for nothing. Using a near identical calculation for the second term, we get

$$\hat{A}_{n-1}[4^+, 5^+, \ldots, \hat{1}^-, \hat{P}_{23}^-] \frac{1}{P_{23}^2} \hat{A}_3[-\hat{P}_{23}^+, \hat{2}^-, 3^+] = \frac{\langle \hat{2}\,3\rangle}{\langle 2\,3\rangle} \frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\langle \hat{2}\,3\rangle \ldots \langle n\,1\rangle}.$$

Since, in this term, the shift is such that $\hat{P}_{23}^2$ vanishes, it can be shown here that $\langle \hat{2}\,3\rangle$ disappears. But now, since this same bracket appears in the denominator, the zeroes cancel. (Dividing by zero is not problematic here, as the final value of these terms is formally arrived at by taking a limit in the residue calculation.)

We are left with

$$A_n[1^-, 2^-, 3^+, 4^+, \ldots, n^+] = \frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle \ldots \langle n\,1\rangle}.$$

Hence, we have shown that given the validity of the (mostly positive helicity) Parke-Taylor formula up to multiplicity $n-1$, it follows that it also holds for n. Since the Parke-Taylor formula holds for $n = 3$, it must therefore hold for arbitrary n. The mostly negative helicity case follows from a similar line of argumentation. Hence, we have shown that the Parke-Taylor formula holds.

We have now completed the chapter on BCFW recursion. For the interested reader, a section discussing a proof that BCFW recursion works for gravity has been added to the appendix (appendix A). This proof follows from the same ideas as the ones for scalar QED and Yang-Mills, but with slightly increased complexity.

In the main text, we now move on to discuss our main topic of interest, which is soft recursion for scalar effective field theories. This chapter will feature a detailed exposition of this topic.

“Cui permittit necessitas sua,
circumspiciat exitum mollem.”

-Lucius Annaeus Seneca Minor,
Epistulae Morales 70 (between 62 and 65 A.D.)

To whom it is permitted by necessity, let him consider a soft method.

5 Soft Recursion

Our research question is whether recursion relations can be used to fix the higher multiplicity amplitudes in terms of lower-point amplitudes for the theories derived in [9]. These are a type of theory called scalar effective field theories. It turns out that BCFW recursion does not work for scalar effective field theories and that we will have to consider a new type of recursion: soft recursion. Soft recursion is a type of recursion that employs an amplitude's soft limit to acquire an improved z-dependence compared to BCFW. Similarly to BCFW, it employs the factorization of amplitudes into subamplitudes in the limit where intermediate particles go on-shell.

Hence, in this chapter, after seeing a brief introduction to scalar effective field theories and their incompatibility with BCFW, we will provide a reasonably in-depth discussion of soft recursion. Here, the particular shift and recursion method stray slightly from the principles discussed in chapter 3. Therefore it will be necessary to expand upon the formalism of chapter 3 before we can put this new form of recursion to the test. We will distinguish between two types of soft recursion, i.e. graded and non-graded soft recursion. The former is relevant for theories with 'variable power counting', the latter for theories with 'fixed power counting'. What this distinction entails, will be explained later. After discussing soft recursion in a more general fashion, we will pay specific attention to the theories derived by Li et al. in [9], called Gauged NLSM and DBI-Lovelock. We will be interested in discussing the feasibility of various (soft) recursion methods working for these theories. Afterwards, we will conclude the thesis.

5.1 Scalar EFTs and BCFW

A scalar effective field theory is an effective field theory (EFT) describing the interactions between scalars. These theories serve to capture the low energy behaviour of more complicated theories by removing less relevant degrees of freedom at the low energy scale, typically involving some form of spontaneous symmetry breaking.

Some famous examples of scalar EFTs include the Dirac-Bord-Infeld model (DBI), so-called non-linear sigma models (NLSM) and various galileon models. These theories have applicability in a large range of areas within physics. Some examples include: the Goldstone boson for chiral symmetry breaking in QCD describing pion scattering is a NLSM [24], DBI has been used to model inflation [25] and the galileon has been proposed as a modifcation to gravity [26, 27]. Interestingly, it has been shown that DBI and the special galileon (a galileon with specific, special properties) satisfy an analogue to GR's equivalence principle [28].

It turns out that the theories mentioned above all share an interesting property. That is, when you scale any one of the external momenta of these theories' amplitudes with some constant sent to zero, i.e. you take the soft limit, the amplitude vanishes. That is,

$$\lim_{c \to 0} A_n(p_1, \ldots, cp_i, \ldots, p_n) = 0.$$

This property is called the Adler zero [29] or a vanishing soft limit. Some other theories have non-vanishing soft limits, such as gauge theory and gravity, with different, yet interesting scaling behaviour[17, 18].

In the case of an Adler zero, we can speak of a theory's soft degree σ, expressing the speed at which the amplitude goes to zero in the soft limit. A theory is said to have soft degree σ if the amplitude scales with c^σ when c is small. That is,

$$A_n \sim c^\sigma, \text{ if } c \text{ small.}$$

Alternatively, this is often written as $A_n \sim p_i^\sigma$.[2] The theories above all have a specific soft degree σ, valid for every amplitude in the theory.

Given the wide range of use cases for these theories, it would be useful to have a way of performing recursion on these theories. However, if try to shift these amplitudes with a BCFW shift, we run into trouble. The Lagrangians of these theories contain many high-derivative operators, creating a large positive z-dependence in the amplitudes in a BCFW shift, giving us a persistent boundary term and obstructing BCFW recursion. Furthermore, there is no gauge symmetry enforcing cancellations in this z-dependence. BCFW does not work.

If recursion is to work for scalar EFTs, the absence of BCFW recursion necessitates an alternative recursion method. Such an alternative recursion method is precisely what has been found by Cheung et al. [20]. This method exploits the soft behaviour of these scalar EFTs discussed above. It turns out that the presence of a consistent soft degree for all amplitudes in a theory functions to constrain the higher-point amplitudes in a similar way that gauge freedom constrains the amplitudes in the theories discussed earlier.

We will develop this method of recursion in this chapter. In the next section, we will provide the necessary generalizations of chapter 3 in order to treat this type of recursion.

5.2 Generalized Contour Integral

One of the insights leading into soft recursion is the realization that one can construct alternative contour integrals beside the one given in equation (3.6). That is, in chapter 3 we saw that if we integrate $\hat{A}_n(z)/z$ along a contour tightly encircling the origin, we recover the original unshifted amplitude A_n. This relied on the fact that $\hat{A}_n$ equals the original amplitude at the origin,

$$\hat{A}_n(0) = A_n,$$

and that in at least some neighbourhood containing the origin, $\hat{A}_n$ is analytic. We then concluded that we could derive A_n from the residues of $\hat{A}_n$'s poles situated at finite z, if $\hat{A}_n(z) \to 0$ as $z \to \infty$. These particular residues could be calculated in terms of lower-point subamplitudes.

The novelty here is that there exist alternatives to the integrand of the previously stated contour integral, which are equally valid and–in some cases–more useful than $\hat{A}_n(z)/z$. We will label these alternatives by $f(z)/z$. Naturally, a valid f should satisfy the same conditions as $\hat{A}_n$ listed above. That is, if we can find some f such that

[2] Technically, a better definition would be the maximal σ such that for any A_n, $\lim_{p_i \to 0} [A_n(p_1, \dots, p_n)/p_i^\sigma] < \infty$.

- $f(0) = A_n$,
- The residues of $f(z)$ can be given in terms of subamplitudes, lower-point than A_n,
- $f_n(z) \to 0$ as $z \to \infty$, especially when $\hat{A}_n(z)$ blows up,

then f is a good alternative to $\hat{A}_n$. That is, it can be substituted in the contour integral to give the (improved) recursion relations for the original amplitude A_n.

Given these properties, in analogy with equation (3.6), we have

$$A_n = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z}dz = \sum \text{Some expression containing lower point amplitudes.}$$

Of course, finding such an f is the actual difficult part. In practice, $f(z)$ will be given by some expression containing shifted amplitudes. We will now see how we can construct an adequate function f for theories with soft theorems.

In the next section, we will apply this to derive a new way of doing recursion using vanishing soft limits, which will be advantageous compared to BCFW in the case of scalar EFTs.

5.3 Soft Recursion

The advantage that soft limits provide in the realm of recursion, is that the amplitudes of a theory with soft limits feature zeroes at known locations and of known degree. Indeed, these locations are where individual external momenta go to zero; the zeroes are of degree σ, the soft degree of the theory. Here, we will discuss the exact method by which these zeroes are exploited.

In the case of soft recursion, we perform a shift which is distinct from BCFW. We shift

$$p_i \to \hat{p}_i(z) \equiv p_i(1 - za_i) \tag{5.1}$$

and accordingly shift

$$A_n \to \hat{A}_n(z) \equiv A_n(\hat{p}_1(z), \dots, \hat{p}_n(z)),$$

in analogy to equations (3.1) and (3.5) respectively. After performing this shift, sending $z \to 1/a_i$ is equivalent to sending $p_i \to 0$. Hence, given that we are working within a theory with soft degree σ, we expect that

$$\hat{A}_n(z) \sim (1 - za_i)^\sigma,$$

as $z \to 1/a_i$. Subsequently, a nice choice for $f(z)$ presents itself

$$f(z) \equiv \frac{\hat{A}_n(z)}{\prod_{i=1}^n (1 - a_i z)^\sigma}. \tag{5.2}$$

This f nicely satisfies all three conditions mentioned above. Clearly, $f(0) = A_n$ since $\hat{A}_n(0) = A_n$. Furthermore, f has the same poles as $\hat{A}_n$. Even though we divide by a polynomial, this does not introduce any new poles, since they are exactly cancelled by the soft zeroes of $\hat{A}_n$. Consequently, f also has residues given in terms of subamplitudes, inherited for $\hat{A}_n$ as well. And crucially, f has improved z-dependence compared to $\hat{A}_n$ since we are dividing by a polynomial in z. Particularly,

$$f(z) \sim z^{m-n\sigma},$$

if $\hat{A}_n(z) \sim z^m$ for large z.

If $m - n\sigma < 0$, then $f(z)$ will not feature any pole at infinity and every residue of $f(z)$ will be expressible in terms of subamplitudes. We conclude that if BCFW or some other recursion method fails, this method of recursion might still yield results.

There are, however, a few caveats to take into account when performing this shift. First of all, the shift vectors $a_i p_i$ do not satisfy all three shift conditions outlined in equation (3.2). They particularly do not satisfy condition (iii), as for a generic collection of momenta, $p_i \cdot p_j \neq 0$. Therefore, intermediate momenta squared contain an extra term

$$\hat{P}_I^2(z) = P_I^2 + z2P_I \cdot R_I + z^2 R_I^2,$$

in contrast to equation (3.3). This entails that there are twice as many points in the complex plane where our shifted amplitude factorizes, because $\hat{P}_I^2(z)$, being a second degree polynomial, now has two zeroes instead of only one. We will label these two zeroes z_I^+ and z_I^-. Note that due to the factorization of polynomials, we can write

$$\hat{P}_I^2(z) = R_I^2(z - z_I^+)(z - z_I^-).$$

As far as the other conditions are concerned, (3.2.ii), the requirement that $p_i \cdot r_i = p_i \cdot a_i p_i = 0$, is automatically satisfied by the on-shell $\{p_i\}$. However, the first condition (3.2.i),

$$\sum_{i=1}^{n} a_i p_i = 0,$$

is not satisfied automatically and puts a constraint on the constants $\{a_i\}$.

In addition, there are two other conditions that $\{a_i\}$ have to satisfy. All a_i need to be nonzero, since otherwise the additional coveted z-dependence in the denominator of f is immediately cancelled. Furthermore, all a_i need to be distinct. That is because if we take the limit of $z \to 1/a_i$, we want to be sure that we are taking the soft limit of only one of the n external momenta at the same time. (If for instance $a_i = a_j$ where $i \neq j$, then $z \to 1/a_i$ takes both $\hat{p}_i(z) \to 0$ and $\hat{p}_j(z) \to 0$ simultaneously.) In this simultaneous soft limit, we can no longer be guaranteed that the pole in the denominator of $f(z)$ is exactly cancelled by its zero in the numerator.

So in summary, given a set of momenta $\{p_i\}$, we need to pick our $\{a_i\}$ such that

$$\text{(i)} \sum_{i=1}^{n} a_i p_i = 0, \qquad \text{(ii)}\ a_i \neq 0\ \ \forall i, \qquad \text{(iii)}\ i \neq j \Rightarrow a_i \neq a_j\ \ \forall i, j.$$

This is only possible for a generic set of momenta if

$$n > d + 1,$$

where d is the spacetime dimension and n the amount of particles in the interaction. Hence, with this type of recursion, depending on the dimension, there will be a certain amount of particles above which this recursion method will start working.

This requirement comes from the fact that requirement (i) can be reformulated in terms of the matrix equation

$$\begin{pmatrix} p_1 & p_2 & \dots p_n \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = 0.$$

Due to momentum conservation, we know that the nth vector can always be given as a linear combination of the first $(n-1)$ vectors. Also, since the momentum vectors are d-dimensional, at most d of these vectors will be linearly independent. For a generic set of momenta, then, the rank of the matrix $\begin{pmatrix} p_1 & p_2 & \dots p_n \end{pmatrix}$ will be $n-1$ if $n \leq d+1$ and d if $n > d+1$. From the rank-nullity theorem, we get that the dimension of the null-space (nullity) of this matrix is n minus the matrix' rank. Hence, if $n \leq d+1$, we have a nullity of only one, while if $n > d+1$, we get nullities greater than one.

Since, due to momentum conservation, we know that $(1, 1, 1, \dots, 1)$ is a solution to the equation, we also know that if we have a nullity of only one, all solutions will be spanned by this vector. Hence, all of those solutions will have identical $a_1 = a_2 = \dots = a_n$ in violation of requirement (iii). It follows that we need $n > d+1$ to satisfy requirement (i) and (iii) simultaneously.

Furthermore, for a generic set of momenta with $n > d+1$, one expects that the solutions to requirement (i) will typically have coefficients $a_1 \neq a_2 \neq \dots \neq a_n$. This follows since for a generic set of momenta, there will be no relationship between any individual momenta conspiring to make individual coefficients equal. As before, we only care about recursion for generic sets of momenta, For recursion of an amplitude to work, it is sufficient we show that recursion works almost everywhere in the domain of the desired amplitude.

If we have a nullity of greater than one, it is also easy to satisfy requirement (ii) simply by tweaking the solution.

Luckily, most of the scalar EFTs described in section 5.1 do not admit a non-vanishing three- or five-point interaction. (The most general galileon admits 5-point.) In addition, the theories in which we are interested by Li et al. in [9] lack three-point and five-point interactions as well. Hence, in so far as there is a vanishing boundary term at infinity, one can use these theories' four-point interactions as seed amplitudes to generate the complete tree-level structures in four dimensions, i.e. for six-point and beyond.

Hence, if we have $m < n\sigma$ and $n > d+1$, we can find a shift $p_i \to p_i(1 - za_i)$, correspondingly $A_n \to \hat{A}_n(z)$, such that for f given in equation (5.2),

$$A_n = \int_\Gamma dz \frac{f(z)}{z} = -\frac{1}{2} \sum_{\{I\}} \sum_{s=\pm} \mathrm{Res}\{f(z)/z; z_I^s\}.$$

Computing the residue, we get

$$
\begin{aligned}
\mathrm{Res}\big\{f(z)/z; z_I^+\big\} &= \lim_{z\to z_I^+}(z-z_I^+)\frac{f(z)}{z}, \\
&= \lim_{z\to z_I^+}\frac{z-z_I^+}{z}\frac{\hat{A}_L(z)\hat{A}_R(z)}{\hat{P}_I^2(z)\prod_{i=1}^n(1-a_iz)^\sigma}, \\
&= \lim_{z\to z_I^+}\frac{z-z_I^+}{zR_I^2(z-z_I^+)(z-z_I^-)}\frac{\hat{A}_L(z)\hat{A}_R(z)}{\prod_{i=1}^n(1-a_iz)^\sigma}, \\
&= \lim_{z\to z_I^+}\frac{z_I^+z_I^-}{zP_I^2(z-z_I^-)}\frac{\hat{A}_L(z)\hat{A}_R(z)}{\prod_{i=1}^n(1-a_iz)^\sigma}, \\
&= -\frac{1}{P_I^2(1-z_I^+/z_I^-)}\frac{\hat{A}_L(z_I^+)\hat{A}_R(z_I^+)}{\prod_{i=1}^n(1-a_iz_I^+)^\sigma}.
\end{aligned}
$$

We can then, analogously to equation (3.8), derive the corresponding recursion formula:

$$
A_n = \frac{1}{2}\sum_{\{I\}}\frac{1}{P_I^2(1-z_I^+/z_I^-)}\frac{\hat{A}_L(z_I^+)\hat{A}_L(z_I^+)}{\prod_{i=1}^n(1-a_iz_I^+)^\sigma} + (z_I^+ \leftrightarrow z_I^-). \tag{5.3}
$$

For a discussion regarding the factor 1/2, please consult page 31 (and by extension page 29).

Example: Deriving the NLSM Six-Point Amplitude

To demonstrate this form of recursion relations, an example is in order. Here, we will derive the six-point flavor-ordered non-linear sigma model amplitude. This amplitude is the easiest to derive for the scalar EFTs mentioned above, since the flavor-ordering reduces the amount of contributing factorization channels, decreasing the amount of residues we have to calculate. This works similarly to the color-ordered amplitudes we calculated for Yang-Mills.

The six-point NLSM amplitude is has a large enough multiplicity to satisfy $n > d+1$ in $4D$. Furthermore, all NLSM amplitudes satisfy $m < n\sigma$. We will hear more about this latter condition below. NLSM has $\sigma = 1$. We will use this fact below.

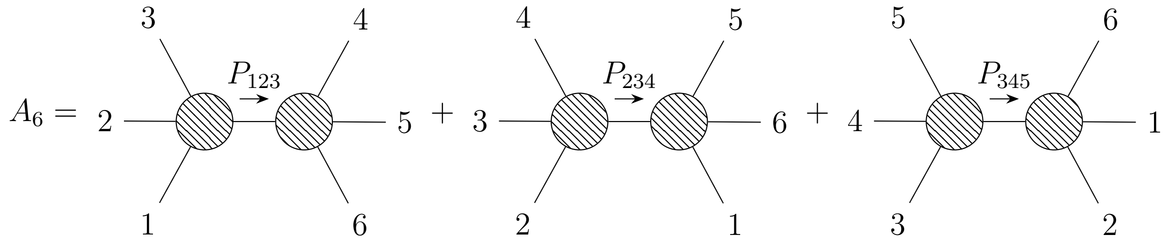
Superficially looking at equation (5.3) gives the impression that soft recursion is a more complicated computational process than BCFW. This impression is not wrong. In our effort to calculate the six-point amplitude, we will use Cauchy's residue theorem three times, following [20]. Our particular computational method will allow us to avoid having to pick concrete $\{a_i\}$, being eliminate the amplitude's dependence on these constants.

We start off by considering the four-point flavor-ordered NLSM amplitude,

$$
A_4[1,2,3,4] = s_{12} + s_{23}, \tag{5.4}
$$

which is the only amplitude we will need to construct the six-point. Here, the mandelstams are defined as $s_{ij} \equiv (p_i + p_j)^2$.

The six-point amplitude will have three factorization channels,



We will label these contributions $A^{(123)}$, $A^{(234)}$ and $A^{(345)}$ respectively for obvious reasons.

We will first only focus on $A^{(123)}$ and then generate the complete result using cyclic permutability. The contribution $A^{(123)}$ is given by

$$A^{(123)} = \frac{1}{P_{123}^2(1 - z_{123}^+/z_{123}^-)} \frac{(\hat{s}_{12}(z_{123}^+) + \hat{s}_{23}(z_{123}^+))(\hat{s}_{45}(z_{123}^+) + \hat{s}_{56}(z_{123}^+))}{\prod_{i=1}^{6}(1 - a_i z_{123}^+)} + (z_{123}^+ \leftrightarrow z_{123}^-),$$

where we have combined equation (5.3) and (5.4). Note that we have picked $\sigma = 1$ in the denominator.

Instead of moving forward from this result, we will retrace a couple of our steps in the derivation of equation (5.3) to reintroduce a more explicit residue dependent calculation. Recall that the above contribution equals

$$\begin{aligned} A^{(123)} &= -\sum_{z_p = z_{123}^\pm} \lim_{z \to z_p} \frac{z - z_p}{z} \frac{(\hat{s}_{12}(z) + \hat{s}_{23}(z))(\hat{s}_{45}(z) + \hat{s}_{56}(z))}{\hat{P}_{123}^2(z) \prod_{i=1}^{6}(1 - a_i z)}, \\ &= -\sum_{z_p = z_{123}^\pm} \mathrm{Res}\left[\frac{(\hat{s}_{12}(z) + \hat{s}_{23}(z))(\hat{s}_{45}(z) + \hat{s}_{56}(z))}{z\hat{P}_{123}^2(z) \prod_{i=1}^{6}(1 - a_i z)}; z_p \right]. \end{aligned}$$

In this expression, we are summing over two of the residues of the function which is contained within the brackets. These two residues directly stem from the poles introduced by the factor of $1/\hat{P}_{123}^2(z)$. Apart from these two poles, we have seven other poles in the function under consideration, one pole at $z = 0$ and one pole at each $z = 1/a_i$. Using Cauchy's residue theorem again, we can equate the residues above to the residues of these seven other poles. We then receive

$$A^{(123)} = \frac{(s_{12} + s_{23})(s_{45} + s_{56})}{P_{123}^2} + \sum_{j=1}^{6} \mathrm{Res}\left[\frac{(\hat{s}_{12}(z) + \hat{s}_{23}(z))(\hat{s}_{45}(z) + \hat{s}_{56}(z))}{z\hat{P}_{123}^2(z) \prod_{i=1}^{6}(1 - a_i z)}; z = 1/a_j \right]. \tag{5.5}$$

The first term comes from the pole at $z = 0$. This gives us a very nice result in terms of unshifted amplitudes. The second term, however, will require some more work to interpret. Let us explicitly calculate a couple of residues in this sum to see what happens. First, it is useful to derive an expression for the shifted Mandelstam variables:

$$\hat{s}_{ij}(z) = 2\hat{p}_i(z) \cdot \hat{p}_j(z) = 2p_i \cdot p_j(1 - a_i z)(1 - a_j z) = s_{ij}(1 - a_i z)(1 - a_j z).$$

We see that as z appraoches $1/a_i$ or $1/a_j$, s_{ij} goes to zero. It is also useful to expand the momentum $\hat{P}^2_{123}(z)$.

$$\begin{aligned}\hat{P}^2_{123} &= (\hat{p}_1 + \hat{p}_2 + \hat{p}_3)^2, \\ &= 2\hat{p}_1 \cdot \hat{p}_2 + 2\hat{p}_1 \cdot \hat{p}_3 + 2\hat{p}_2 \cdot \hat{p}_3, \\ &= \hat{s}_{12} + \hat{s}_{13} + \hat{s}_{23},\end{aligned}$$

where we have omitted explicit z-dependence. If we then, for instance, calculate the residue for $j = 1$, we get

$$\mathrm{Res}_{j=1}[\ldots] = \lim_{z \to 1/a_1} \frac{z - 1/a_1}{z} \frac{(\hat{s}_{12}(z) + \hat{s}_{23}(z))(\hat{s}_{45}(z) + \hat{s}_{56}(z))}{(\hat{s}_{12}(z) + \hat{s}_{13}(z) + \hat{s}_{23}(z)) \prod_{i=1}^{6}(1 - a_i z)}.$$

Here, the fraction $(z-1/a_1)/z$ will cancel the zero in the denominator. But more interestingly, every mandelstam dependent on momentum $\hat{p}_1$ will vanish. This will cause $\hat{s}_{12}$ to disappear from the numerator and denominator and $\hat{s}_1 3$ from the denominator alone. The two $\hat{s}_{23}$ will subsequently cancel. We can rewrite this limit as an identical limit of a different function:

$$\mathrm{Res}_{j=1}[\ldots] = \lim_{z \to 1/a_1} \frac{z - 1/a_1}{z} \frac{\hat{s}_{45}(z) + \hat{s}_{56}(z)}{\prod_{i=1}^{6}(1 - a_i z)}.$$

Continuing this logic, for $j = 2$, the entire contribution will be killed, for $j = 3$ the we get a similar result as for $j = 1$ but then with the limit going to $1/a_3$. For $j = 4, 5, 6$ we get very similar results to $j = 1, 2, 3$, since we can use momentum conservation to rewrite $P^2_{123} = s_{45} + s_{56} + s_{64}$. For example, we get

$$\mathrm{Res}_{j=6}[\ldots] = \lim_{z \to 1/a_6} \frac{z - 1/a_6}{z} \frac{\hat{s}_{12}(z) + \hat{s}_{23}(z)}{\prod_{i=1}^{6}(1 - a_i z)}.$$

It is not obvious how we could get a nice result from this independent of $\{a_i\}$. But it turns out there is a way. Recall that this is the result for only one of the three factorization channels. We can get the other factorization channels simply by cyclically permuting this result $1 \to 2 \to \ldots \to 6 \to 1$. We see that the $A^{(234)}$ channel also gives a $\mathrm{Res}[\ldots]_{j=1}$ contribution, which we obtain by applying the permutation once to the $j = 6$ result for the $A^{(123)}$ channel. Since the $j = 5$ contribution vanishes for $A^{(123)}$, $A^{(345)}$ will not contribute to $j = 1$. We can write the complete $j = 1$ contribution as

$$\begin{aligned}&\lim_{z \to 1/a_1} \frac{z - 1/a_1}{z} \frac{\hat{s}_{23}(z) + \hat{s}_{34}(z) + \hat{s}_{45}(z) + \hat{s}_{56}(z)}{\prod_{i=1}^{6}(1 - a_i z)} \\ &\qquad = \lim_{z \to 1/a_1} \frac{z - 1/a_1}{z} \frac{\hat{s}_{12}(z) + \hat{s}_{23}(z) + \hat{s}_{34}(z) + \hat{s}_{45}(z) + \hat{s}_{56}(z) + \hat{s}_{61}(z)}{\prod_{i=1}^{6}(1 - a_i z)}, \\ &\qquad = \mathrm{Res}\left[\frac{\hat{s}_{12}(z) + \hat{s}_{23}(z) + \hat{s}_{34}(z) + \hat{s}_{45}(z) + \hat{s}_{56}(z) + \hat{s}_{61}(z)}{z \prod_{i=1}^{6}(1 - a_i z)}; z = 1/a_1\right].\end{aligned}$$

This is the result for $j = 1$ resulting from the second term of equation (5.5) but then for all three factorization channels. We could simply add in the additional mandelstams since they vanish in the limit. We then rewrote this result as a residue.

If we repeat this for other values of j, we get a similar result. Adding this all up, we get

$$\sum_{j=1}^{6} \text{Res}\left[\frac{\hat{s}_{12}(z) + \hat{s}_{23}(z) + \ldots + \hat{s}_{61}(z)}{z\prod_{i=1}^{6}(1 - a_i z)}; z = 1/a_j\right].$$

However, this is nothing more than a sum over residues. Specifically, this is a sum over the residue of every pole except for the pole at the origin. Hence, we can use Cauchy's residue theorem once more to get

$$\sum_{j=1}^{6} \text{Res}\left[\frac{\hat{s}_{12}(z) + \hat{s}_{23}(z) + \ldots + \hat{s}_{61}(z)}{z\prod_{i=1}^{6}(1 - a_i z)}; z = 1/a_j\right] = -(s_{12} + s_{23} + \ldots + s_{61}).$$

Now also adding the contributions from the first term of equation (5.5), we get the complete flavor-ordered amplitude

$$A_6 = \frac{(s_{12} + s_{23})(s_{45} + s_{56})}{P_{123}^2} + \frac{(s_{23} + s_{34})(s_{56} + s_{61})}{P_{234}^2} + \frac{(s_{34} + s_{45})(s_{61} + s_{12})}{P_{345}^2} - (s_{12} + s_{23} + s_{34} + s_{45} + s_{56} + s_{61}).$$

This is the same as the result from Feynman diagrams. The first three terms are from different exchange diagram while the second line is from a contract contribution.

5.4 Power Counting Parameter

It turns out that it is possible to classify a subset of scalar effective field theories in terms of the soft degree, σ, which we have already seen, and a new parameter, ρ, called the power countering parameter. The power counting parameter expresses the ratio between the mass dimension of Feynman diagrams (or amplitudes) and the number of external fields. Directly related to this, it expresses the amount of derivatives per field present in the terms of a Lagrangian of a scalar field theory. A more comprehensive definition follows below.

In this section, we will focus on this power counting parameter. We do this for several reasons. First off, it turns out that it is possible to formulate a sufficient condition for soft on-shell constructibility in terms of the power counting parameter and the soft degree of a scalar field theory. This condition, $\rho \leq \sigma$, directly follows from $m < n\sigma$, but is independent of the amplitude-specific parameter n. Rather this condition is specific to theories. Secondly, as mentioned before, it is possible to classify a subset of scalar effective field theories using ρ and σ. It turns out that for various combinations of these two parameters, there only exists one possible scalar EFT. Hence, the subject at hand will allow us to briefly talk about this classification. Thirdly and most importantly, the theories we are particularly concerned with in Li et al. are of a specific type of theory called variable-ρ or variable power counting parameter theories. Hence, in order to understand the behaviour of these theories when it comes to recursion, we need to discuss power counting. We will largely base this discussion on [30].

Let us consider the Lagrangian of some field theory with an arbitrary amount of terms

$$\mathcal{L} = \sum_V g_V \mathcal{L}_V.$$

Here, each $g_V \mathcal{L}_V$ is a unique term with coupling constant g_V, responsible for a unique vertex type in this theory's resultant Feynman diagram. For each term, it is possible to define a so-called power counting parameter ρ_V, which in the case of scalar fields is given by

$$\rho_V \equiv \frac{D_V - 2}{N_V - 2}.$$

N_V is the power of scalar fields featured in $\mathcal{L}_V$ and D_V is the mass dimension of the vertex, equivalent to the amount of derivatives acting on the fields in $\mathcal{L}_V$ or momenta in its corresponding vertex.

The amount of derivatives per field in each term of the Lagrangian is of interest to us from a recursion perspective, since each derivative is responsible for a momentum in the corresponding vertex rule in Feynman diagrams. If we wish to find the behaviour of an amplitude in the high z limit, then finding how the amplitude depends on momentum is crucial. Thus, in this way, we are naturally led to the power counting parameter.

Given the power counting parameter(s) of the Lagrangian terms of a theory, we want to translate this information to the z-dependence of individual diagrams and amplitudes. It turns out that this can be done very naturally by extending the notion of the power counting parameter to individual diagrams. For each diagram Γ, we define

$$\rho_\Gamma \equiv \frac{D_\Gamma - 2}{N_\Gamma - 2},$$

where D_Γ is the mass dimension of a single Feynman diagram and N_Γ the amount of its external legs.

As we will see, if a theory only features terms for one particular ρ_V, i.e. there is some ρ such that for all V, $\rho_V = \rho$, then every diagram Γ will also have the same value: for all Γ $\rho_\Gamma = \rho$.

Indeed, some theories only feature diagrams with one particular ρ_Γ, these are called single-ρ theories. Other theories feature diagrams with variable ρ_Γ, called multi-ρ theories. These are also called graded and non-graded or interpolating versus non-interpolating. The scalar EFTs mentioned above are all single-ρ theories. However, we will treat the most general case here.

We now wish to perform a general derivation of ρ_Γ by considering mass dimension of all objects that collectively shape Feynman diagram Γ. We will thus consider each vertex V entering into the Feynman diagram, for which we have already seen D_V and N_V above. In addition, we will consider the mass dimension due to propagators entering into the diagram. In particular, we have the relation

$$D_\Gamma = -2I_\Gamma + 4L_\Gamma + \sum_V D_V,$$

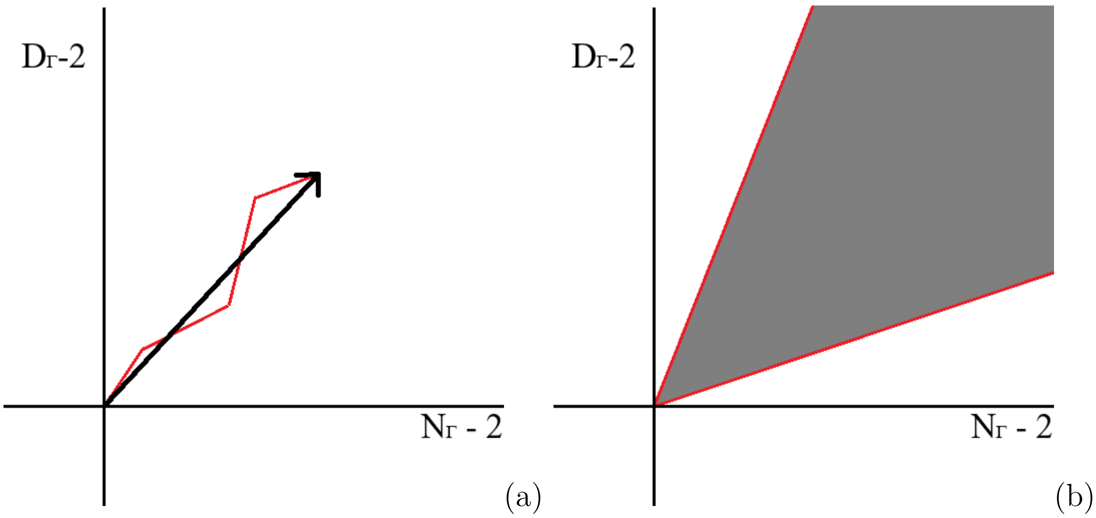


Figure 5.1: Diagrams representing the power counting parameters of Feynman diagrams. (a) The slope of this vector represents the power counting parameter of a single tree-level diagram. One can see that the vector consists of the vectors corresponding to the various vertices the diagram contains. (b) The gray region (wedge) represents the power counting parameters accessible to a single theory. The red lines indicate the diagrams with the maximal or minimal power counting parameter inside a theory.

$$N_\Gamma = -2I_\Gamma + \sum_V N_V.$$

Each time, we are summing over each vertex the amount of times that it enters into Γ. Here, I_Γ is the amount of internal lines in the diagram, L_Γ the amount of loops and, as before, D_V and N_V are the mass dimension and fields per vertex respectively. Every loop increases the mass dimension by four, because of the loop integration measure d^4p. Every internal line replaces two external particles by connecting two vertices and decreases the mass dimension by two because propagators scale with p^{-2}.

We combine this with

$$L_\Gamma = I_\Gamma - V_\Gamma + 1,$$

where V_Γ is the amount of vertices in the diagram. (At tree level, every vertex is connected by exactly one internal line; hence, adding more internal lines adds loops.) Then we arrive at

$$D_\Gamma - 2 = \sum_V (D_V - 2) + 2L_\Gamma,$$
$$N_\Gamma - 2 = \sum_V (N_V - 2) - 2L_\Gamma.$$

This shows that in the case of scalar theories, the theory is single-ρ at tree-level if and only if its vertices are also single-ρ. This means that if there exists a ρ such that for all V, $\rho_V = \rho$, then and only then it holds that $\rho = \rho_\Gamma$ for any Γ. If the ratio between $D_V - 2$ and $N_V - 2$ is constant, then so will the ratio between $D_\Gamma - 2$ and $N_\Gamma - 2$ be.

For the general case, we can define a vector for each vertex

$$\vec{v}_V = \begin{pmatrix} D_V - 2 \\ N_V - 2 \end{pmatrix},$$

which we can then represent in a 2-dimensional diagram. The slope of this vector is identical to ρ_V. Then in the same diagram, using vector addition, we can represent each Feynman diagram using the vector

$$\sum_V \vec{v}_V + \begin{pmatrix} 2L_\Gamma \\ -2L_\Gamma \end{pmatrix},$$

the slope of which equals the power counting parameter of the diagram, compare figure 5.1(a).

In the case of a multi-ρ theory, it is possible to define the minimal ρ and maximal ρ of a theory

$$\rho_{\min} \equiv \min_\Gamma \rho_\Gamma, \qquad \rho_{\max} \equiv \max_\Gamma \rho_\Gamma.$$

Those diagrams with $\rho_{\max}$ or $\rho_{\min}$ lie on the upper or lower red line in figure 5.1(b) respectively.

To get a diagram with maximal or minimal ρ, one needs to construct said diagram only using vertices that have the greatest and smallest ρ_V respectively. Diagrams that lie within the gray area of figure 5.1(b) potentially contain vertices with mixed ρ_V. For those diagrams with $\rho_{\min}$ or $\rho_{\max}$, then, one can separate out those vertices and, by extension, those terms in the Lagrangian responsible for those those diagrams. By doing so, one acquires two separate theories, so-called 'subtheories' of our multi-ρ theory, given by

$$\mathcal{L}_{\min} = \sum_{V, \rho_V = \rho_{\min}} g_V \mathcal{L}_V, \qquad \mathcal{L}_{\max} = \sum_{V, \rho_V = \rho_{\max}} g_V \mathcal{L}_V.$$

It is said that an multi-ρ theory 'interpolates' between its subtheories.

We deduce that the diagrams that lie on the red lines separately sum up to form amplitudes of these two subtheories. In general, we can define $A_n^{(\rho)}$ to be the sum of all diagrams with $\rho_\Gamma = \rho$. Typically, these are only proper amplitudes if $\rho = \rho_{\min}$ or $\rho_{\max}$. We have

$$A_n = \sum_\rho A_n^{(\rho)}.$$

It is not uncommon for these different contributions to have distinct soft limits. We can define $\sigma^{(\rho)}$ as the softness of contribution $A_n^{(\rho)}$. Then $\sigma^{(\rho_{\max})}$ and $\sigma^{(\rho_{\min})}$ are the soft degrees of the two subtheories. We can also define $\sigma_{\max}$ and $\sigma_{\min}$ as the maximal and minimal $\sigma^{(\rho)}$ respectively. We typically have that $\sigma_{\max} = \sigma^{(\rho_{\max})}$ and $\sigma_{\min} = \sigma^{(\rho_{\min})}$, so we will not make this distinction below and opt for simpler notation.

We will now use what we have seen here to increase our understanding of soft recursion.

5.5 Recursion Condition with Power Counting Parameter

We are now in a position to derive a condition for recursion which does not depend on the individual amplitude under consideration, but on conditions dependent on the theory as a whole, replacing $m < n\sigma$. This will allow us to quickly gauge whether a scalar EFT admits recursion or not.

We start by assuming that the mass dimension of the worst behaving diagram dictates the scaling of the shifted amplitude at large z. That is, if $\hat{A}_n(z) \sim z^m$ as $z \to \infty$, then we assume that $m = \max_\Gamma D_\Gamma$. This is plausible, because the mass dimension gives the total power of momenta present in the diagram, which all scale with z. This assumption is, of course, only valid if there are no cancellations that occur inside the amplitude, eliminating some z-dependence. This is not always the case. The arguments used in chapter 4 were used to show that such cancellation *does* occur for Yang-Mills and GR. Nevertheless, it only matters for our argument if this cancellation occurs in propagator denominators, since any other cancellation results in improved scaling behaviour. This does not happen for generic $\{a_i\}$ as shown by Cheung et al. [20].

Then in the case of a single-ρ theory with power counting parameter ρ, we get

$$m < n\sigma \Leftrightarrow \rho - 1 = \frac{m-n}{n-2} < \frac{n\sigma - n}{n-2} = (\sigma - 1)\frac{1}{1-2/n}.$$

Since $1/(1-2/n) > 1$ for $n \geq 3$ (which holds for any amplitude we might consider), the above condition is implied by

$$\rho \leq \sigma, \tag{5.6}$$

if $(\rho, \sigma) \neq (1, 1)$. This is the condition for recursion we were looking for, first found in [20].

Equation (5.6) turns out to be the defining condition for so-called ‘exceptional’ scalar field theories. The term ‘exceptional’ is justified by the fact that this condition implies a higher soft limit than what you would expect based on the power counting parameter.

As we saw before, a higher power counting parameter of a theory is directly related to the strength of the momentum dependence of the amplitudes of that theory. If we have a higher power counting parameter, we can expect the amplitude to scale with momentum more strongly. It turns out that any theory with a specific ρ has $\sigma \geq \rho - 1$. Hence, by adding more momenta to the vertices, one is guaranteed to get a soft limit eventually. Theories that saturate this inequality are called trivial. It turns out that it is never possible for σ to exceed ρ by more than one.

In order to have a σ better than the lower bound and thus satisfy (5.6), some non-trivial cancellation needs to occur between diagrams, only possible in certain ‘exceptional circumstances’. That is, when the theory satisfies some kind of symmetry.

It is then not very surprising that this condition is special enough to fix the higher-point amplitudes in terms of the lower-point amplitudes through recursion. It turns out that some combinations of (ρ, σ) are sufficient to bootstrap the lower point amplitudes of some exceptional scalar field theories [31] for which all higher point amplitudes are subsequently fixed. Notably, DBI and the special galileon are unique for their values of (ρ, σ). Furthermore, a complete classifcation of all exceptional scalar field theories exists in terms of four parameters, of which σ and ρ are two [32]. A table listing the exceptional scalar EFTs mentioned in this thesis is given in table 5.1.

$\rho\backslash\sigma$	1	2	3	∞
0	NLSM	Not Possible	Not Possible	Free
1	Multiple Theories	DBI	Not Possible	
2	Trivial	Galileon	Special Galileon	

Table 5.1: Some exceptional scalar field theories organized according to the power counting parameter (ρ) and soft degree (σ) [31, 32].

If we want to derive the equivalent condition for a multi-ρ theory, we simply compare the 'worst' power counting parameter contribution to the 'worst' soft degree of the overall amplitude

$$\rho_{\max} \leq \sigma_{\min}.$$

The overall amplitude scales with $p^{\sigma_{\min}}$ as $p \to 0$, since the contribution to the amplitude scaling in this way is the slowest to diminish. Hence, $\sigma_{\min} = \sigma$, effectively. The contribution with $\rho_{\max}$ increases the fastest in the limit $z \to \infty$, hence this is the part that needs to be suppressed by the denominator of equation (5.2). Recall that in any case, we still need $n > d+1$ in order to perform the desired shift. Hence, this recursion method will only start working from 6-point onward in a 4D context.

In the case of a multi-ρ theory, there is a modification we can make to the function f of the contour integral to derive a new recursion condition which holds in slightly more situations than $\rho_{\max} \leq \sigma_{\min}$. This will also be relevant for discussing the theories in Li et al. We will discuss this in the next section.

5.6 Split Graded Soft Recursion

Here, we will discuss the interesting case if a multi-ρ theory fails to satisfy $\rho_{\max} \leq \sigma_{\min}$. In this case, there is a certain edge case in which it is still possible to perform recursion. This will, however, require a different $f(z)$ and recursion formula, alternative to eqs. (5.2) and (5.3). In short, the method we will present here will be to split the amplitude into parts with different mass dimensions $A^{(\rho)}$ which behave differently in the high z limit. We will then recursively construct these various parts independently and recombine them to form a complete amplitude. We will derive the recursion formula presented by Kampf et al. in [30].

The interesting case appears when a multi-ρ theory is not constructible in the way presented above, but whose maximal-ρ subtheory, $\mathcal{L}_{\max}$, is constructible in its own right. If the rest of the theory is well-behaved enough, this can be sufficient for recursion.

Consider the following situation, where we have an independently constructible maximal-ρ subtheory with amplitude $A_n^{(\rho_{\max})}$ contributing to the overall amplitude A_n. Given that this subtheory is constructible in accordance with the description above, we have

$$\rho_{\max} \leq \sigma_{\max}.$$

Naturally, this means that $A_n^{(\rho_{\max})} \sim p^{\sigma_{\max}}$ for small p and that

$$\frac{\hat{A}_n^{(\rho_{\max})}(z)}{\prod_{i=1}^{n}(1 - za_i)^{\sigma_{\max}}} \to 0$$

as $z \to \infty$ as per the discussion above.

We can then construct

$$f(z) = \frac{\hat{A}_n(z) - \hat{A}_n^{(\rho_{\max})}(z)}{\prod_{i=1}^n (1 - za_i)^{\sigma_{\min}}} + \frac{\hat{A}_n^{(\rho_{\max})}(z)}{\prod_{i=1}^n (1 - za_i)^{\sigma_{\max}}}. \tag{5.7}$$

This is possibly a good alternative to equation (5.2), because it can potentially satisfy the conditions outlined in section 5.2. Clearly, putting $z = 0$ recovers A_n. The numerators consist solely of terms which constitute amplitudes in their own right, thus we can be guaranteed that its residues consist of subamplitudes. (This would not be the case if we singled out contributions to the amplitude $A^{(\rho)}$ with ρ different from $\rho_{\max}$ or $\rho_{\min}$.) Furthermore, because of the assumed constructibility of $\mathcal{L}_{\max}$, we can be sure that the second term vanishes in the large z limit. The only question that remains is what requirements suffice to make the first term vanish as well.

Since $\hat{A}_n(z) = \sum_\rho \hat{A}_n^{(\rho)}(z)$, we can see that the numerator of the first in equation (5.7) term equals the full amplitude, stripped of its highest ρ contribution. The next highest ρ contribution will thus be the worst behaving part of the numerator of the first term in the high z limit. Let us call this $\rho_{\max-1}$. Then, clearly, one needs

$$\rho_{\max-1} \leq \sigma_{\min}$$

to guarantee the vanishing of the first term.

The reason why we cannot divide by anything with a greater z-dependence than $(1 - za_i)^{\sigma_{\min}}$ is because the numerator as a whole will have a softness of $\sigma_{\min}$. Hence, by dividing by any factor more favorable at large z will inevitably introduce additional poles into $f(z)$ not situated at points where the amplitude factorizes.

We can thus summarize this recursion method as follows. Given that we are attempting to derive an n-point amplitude with $n > d + 1$, we perform the same shift as before: $p_i \to p_i(1 - za_i)$. Consequently, if

$$\rho_{\max} \leq \sigma_{\max}, \qquad \rho_{\max-1} \leq \sigma_{\min},$$

then $f(z)$ in equation (5.7) provides a useful function for recursion.[3]

Now, in a calculation similar to the one at page 60, one can calculate the residues of $f(z)$ in equation (5.7). This results in a recursion formula

$$\begin{aligned} A_n = \sum_{\{I\}} \frac{1}{P_I^2(1 - z_I^+/z_I^-)} \Bigg[& \frac{\hat{A}_L(z_I^+)\hat{A}_R(z_I^+) - \hat{A}_L^{(\rho_{\max})}(z_I^+)\hat{A}_R^{(\rho_{\max})}(z_I^+)}{\prod_{i=1}^n \left(1 - a_i z_I^+\right)^{\sigma_{\min}}} \\ & + \frac{\hat{A}_L^{(\rho_{\max})}(z_I^+)\hat{A}_R^{(\rho_{\max})}(z_I^+)}{\prod_{i=1}^n \left(1 - a_i z_I^+\right)^{\sigma_{\max}}} \Bigg] + (z_I^+ \leftrightarrow z_I^-), \end{aligned} \tag{5.8}$$

in analogy with equation (5.3). Here, $\hat{A}_{L,R}^{(\rho_{\max})}$ are amplitudes of the maximal-ρ subtheory. Thus, one needs to find these amplitudes independently, working with this subtheory and

[3]Note that the condition mentioned here differs from the one cited by Kampf et al. [30]. The condition here is the most general out of the two.

the theory as a whole independently. In order to apply recursion for multiple iterations, one will have to use the previously discussed soft recursion method to derive the higher point amplitudes of this subtheory, which will then have to fed into this recursion formula.

Kampf et al. use this recursion method on a theory called extended DBI, with $\mathcal{L}_{\text{min}}$ being NLSM and $\mathcal{L}_{\text{max}}$ DBI [30]. For this combined theory, $(\rho_{\text{max}}, \sigma_{\text{min}}) = (1, 1)$, thus disallowing the more basic recursion method above. Obviously, this recursion method is less convenient than this formerly discussed soft recursion method. Hence, whenever $\rho_{\text{max}} \leq \sigma_{\text{min}}$ and $(\rho_{\text{max}}, \sigma_{\text{min}}) \neq (1, 1)$, that recursion method is to be preferred.

5.7 DBI-Lovelock and Gauged NLSM

We are now finally in a position to talk about the two theories derived by Li, Roest and Ter Veldhuis [9], which was our main objective from the start. We are specifically interested in answering the question whether these two theories are on-shell constructible in light of the different theoretical frameworks we have been able to observe in the previous sections.

In [9], the authors bootstrap two different variable-ρ theories on the basis of BCJ compatibility. As mentioned in the introduction, there exist several theories that admit the so-called BCJ double copy, where one can mix and match different 'BCJ numerators' to rework scattering amplitudes of one theory into scattering amplitudes of another, cf. equations (1.2) and (1.3).

These BCJ numerators satisfy specific transformational properties under exchange of particle label. As such, depending on particle multiplicities, these numerators transform according to specific representations of the permutation group S_n. The authors of [9] then use representation theory to find all possible combinations of mandelstam variables that satisfy the transformational properties of BCJ numerators. These will exhaust the list of possible kinematic BCJ numerators for scalar field theories, since for scalars, only mandelstams appear in the numerators. These numerators are subsequently used to construct amplitudes of several scalar theories, which are further refined by imposing specific soft limits on these theories.

It will be our goal to perform an analysis regarding whether the two theories derived in the theory are on-shell constructible or not.

The two theories presented in the paper are called Gauged NLSM and DBI-Lovelock. The former is a single-copy theory, meaning that its amplitudes are constructed using a color and kinematic factor akin to Yang-Mills in equation (1.2). This theory features pions coupled to gluons, with NLSM for its $\mathcal{L}_{\text{max}}$ together with subleading terms.

The latter, DBI-Lovelock, is a theory solely consisting of scalar fields. It is a double copy of a gauged and ungauged NLSM, i.e. it has a kinematic numerator from both theories. This theory interpolates between DBI and the special galileon, satisfying $(\rho_{\text{min}}, \sigma_{\text{min}}) = (1, 2)$ and $(\rho_{\text{max}}, \sigma_{\text{max}}) = (2, 3)$ respectively.

We will now attempt to analyze for both of these theories their potential for on-shell constructibility.

For DBI-Lovelock, it is easiest to evaluate its prospects for recursion. This theory clearly satisfies

$$\rho_{\max} \leq \sigma_{\min},$$

meaning that recursion formula (5.3) works for this theory. The four-point seed amplitude should be sufficient to construct the entire theory.

For Gauged NLSM, however, the result is less obvious. Unfortunately, due to lack of time, it is not possible anymore to do a very thorough, let alone conclusive analysis. Nevertheless, there are some ideas to be shared, which could be used in the future when looking at this theory.

The Gauged NLSM interpolates between a $(\rho_{\max}, \sigma_{\max}) = (0, 1)$ $\mathcal{L}_{\max}$ and $\sigma_{\min} = 0$ $\mathcal{L}_{\min}$ for which $\rho_{\min}$ is difficult to identify according to the prescription established in section 5.4. This is because the gauged NLSM features gluon exchange, whereas section 5.4 was only valid for scalar fields.

Nevertheless, it may very well be possible to apply the same derivative counting method to this theory as it is for scalar fields, because for gluons the momentum dependence can be eliminated from propagators by picking the right Lorenz gauge. This would indicate that also for NLSM amplitudes, momentum dependence only comes from derivatives. This would mean that when looking at pure pion (scalar) scattering, the momentum dependence of the relevant amplitude could be determined by treating gluons effectively as if they were scalars, i.e. looking at the amount of derivatives acting on the gluons alone to determine z-dependence. In this case,

$$\rho_{\max} \leq \sigma_{\min},$$

would indeed be satisfied. This would mean that recursion according to (5.3) would indeed work. This is all somewhat hypothetical, since without any concrete calculations, it is difficult to say if some aspect is being overlooked.

In this scenario, there are some oddities to take into account. First off, because $\sigma = \sigma_{\min} = 0$, the denominator in equation (5.3) would be reduced to 1, thus making it so that one would not gain any improved z-dependence due to the soft-limits. This is not necessarily a problem. If the analysis above is correct, it would mean that the complete pion scattering amplitude A_n should vanish for high z on its own after an all-line shift of the type we have been considering, i.e. equation (5.1).

Secondly, when practically calculating an amplitude using (5.3) for the Gauged NLSM, one has to take into account that since these amplitudes feature gluon exchange, there will also be gluon factorization channels that will contribute residues. This means that in order to calculate an all-pion scattering amplitude using recursion, one would also need to know lower-point 1-gluon $(n-1)$-pion amplitudes, unless there is a good reason to argue that these contributions vanish. If these gluon-pion scattering amplitudes do not vanish, one would also need to know a large amount of gluon-pion scattering amplitudes. If one want to derive these through recursion, one would need even amplitudes with even more gluons. This would be true for any recursion method for the gauged NLSM.

If one could show that there is a vanishing boundary term for scattering amplitudes with both external scalars and gluons (or that these amplitudes vanish), then that would be sufficient in our scenario to prove that recursion works. This is because then lower-point amplitude entering into the calculation of some higher-point scattering amplitude can be derived recursively itself except for the very initial seed amplitudes. In this sense, it can be shown that recursion already works to fix the higher-point scattering amplitudes without even having to calculate them. Nevertheless, if one would actually want to calculate all tree-level pion scattering amplitudes, one would still be burdened with increased computational complexity if these mixed amplitudes do not vanish.

If the z-dependence of gauged NLSM amplitudes are worse than assumed above, there still exist some avenues to probe before giving up on recursion entirely. Following Kampf et al. [30], one can find alternative $f(z)$ in the contour integral to find new recursion formulas, specifically by breaking up the amplitude in different parts with different soft limits and z-dependence.

One could try to see if the amplitude, with the pure pion section split off, has markedly better z-dependence than the amplitude as a whole. That is, if

$$\hat{A}_n(z) - \hat{A}_n^{(\rho_{\max})}(z) \to 0,$$

as $z \to \infty$, then

$$f(z) = \left[\hat{A}_n(z) - \hat{A}_n^{(\rho_{\max})}(z)\right] + \frac{\hat{A}_n^{(\rho_{\max})}(z)}{\prod_{i=1}^n (1 - za_i)}, \tag{5.9}$$

would be a good integrand for the contour integral with vanishing boundary term. Note that this is nothing other than (5.7) filled in.[4]

Unfortunately, it will still require significant effort to determine for each of these suggested methods whether they would bear fruit. This is something that must be left for a future project.

[4]Alternatively, one could even apply different shifts to different parts of $f(z)$. One could, for instance, apply a BCFW shift to the $\hat{A}_n(z) - \hat{A}_n^{(\rho_{\max})}(z)$ part of (5.9), while applying the soft all-line shift to the second term. Or one could even cut up the amplitude further, also subtracting the $\rho_{\min}$ contribution from the complete amplitude, $f(z) = \left[\hat{A}_n(z) - \hat{A}_n^{(\rho_{\max})}(z) - \hat{A}_n^{(\rho_{\min})}(z)\right] + \frac{\hat{A}_n^{(\rho_{\max})}(z)}{\prod_{i=1}^n (1-za_i)} + \hat{A}_n^{(\rho_{\min})}(z)$. Here, one could again choose to apply a different shift for each term in $f(z)$. But there is currently no good reason to assume that these methods would be effective.

6 Conclusion

In this thesis, we set out to explore the application of recursion relations to derive higher-point tree-level scattering amplitudes from lower-point interactions in several different field theories. We were specifically interested in applying the theoretical apparatus of various external sources to the study of scalar effective field theories.

We started off by discussing the spinor helicity formalism, which was applied several times in chapter 4. We then moved on to discuss on-shell tree-level recursion in a rather general fashion. We subsequently dedicated a chapter to BCFW recursion, culminating in a proof of the Parke-Taylor formula for Yang-Mills. Finally, we spent significant time discussing scalar effective field theories, concluding the thesis with a discussion of the two interpolating theories found in Li et al. [9].

Our research question was whether recursion relations can be used to derive the higher multiplicity amplitudes of the gauged NLSM and DBI-Lovelock theories, derived in [9], from lower-point amplitudes.

For DBI-Lovelock, we were able to answer this question relatively easily, where the theory plainly satisfies the constructibility condition, equation (5.6), meaning that recursion formula (5.3) can indeed be applied to generate all higher-point tree-level scattering amplitudes of this theory. Unfortunately, due to time constraints, we were unable to provide an example calculation for this theory. However, an example calculation of a different theory using the same recursion formula was shown on page 60.

For the gauged NLSM, we were not able to give an unequivocal result. This theory proves much more difficult, as in addition to pure scalar interactions, this theory also admits gluon exchange, making it difficult to apply the previously treated frameworks by Cheung et al. [20] and Kampf et al. [30]. Nevertheless, we have alluded to the possibility that the original framework of Cheung et al. might still be applicable to this theory, given some caveats. This, however, is very speculative and can hardly be called a result.

Here, several paths are suggested for future research. In order to determine whether the gauged NLSM is on-shell constructible, several suggestions have already been made in section 5.7. One can perform a Feynman-diagrammatic analysis for this theory, akin to what we have seen several times in chapter 4, in an attempt to resolve the z-dependence of the theory under several different shifts. Furthermore, one could attempt to perform an example calculation, using various recursion methods, in an attempt to derive some already known amplitude to check if an effective recursion method can be found. We suggest the same be done for DBI-Lovelock.

This research has been an initial step in understanding the whether the theories featured in Li et al., and specifically the gauged NLSM, are on-shell constructible or not. This thesis has clarified and elaborated upon several different results within the area of on-shell recursion relations, thereby making these results more accessible. In this way, it has become easier in a potential future project to devote more concentrated attention to (the) individual theories under consideration.

A Proving BCFW Recursion for Gravity

In this appendix, we will show that BCFW recursion also holds for (perturbative) graviton amplitudes following a similar line of reasoning to that of Yang-Mills amplitudes. That is, we will also appeal to the lightcone gauge and use Lorentz symmetry to analyze the z-dependence of our one lightcone gauge resistent diagram.

However, contrasted with the gluon case, the increased complexity of the graviton Lagrangian results in a need for more involved argumentation. It is then useful to recapitulate some basic concepts within GR that will feature in the subsequent argumentation. This is how we will start this subsection.

A.1 Focused Recapitulation of GR

This section will contain a summary of various relevant concepts within in GR. For a more in-depth look, one can consult Carroll [33].

In general relativity, spacetime is a four-dimensional smooth manifold M with a smooth Lorentzian metric g, defining an inner product on each tangent space of the manifold. That is, at each point $p \in M$, the metric is a map

$$g : T_p \times T_p \to \mathbb{R},$$

such that

$$V \cdot W \equiv g(V, W) = g_{\mu\nu} V^\mu V^\nu,$$

where $g_{\mu\nu}$ are the metric's components. That is, at each point p, there exists a tangent space, containing vectors which can be thought of as being tangent to the manifold at point p.

Furthermore, the manifold is endowed with a torsion-free, metric compatible connection, $\Gamma^\nu_{\mu\lambda}$. The connection is defined such that for vectors,

$$\nabla_\mu = \partial_\mu + \Gamma^\nu_{\mu\lambda}$$

is a proper covariant derivative, meaning that components $\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda$ transform as tensor components upon coordinate transformation,

$$\nabla_\mu V^\nu \to \nabla_{\mu'} V^{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \nabla_\mu V^\nu.$$

When applied to a tensor with multiple indices, the connection must be used multiple times

$$\nabla_\mu T_\nu{}^{\rho\sigma} = \partial_\mu T_\nu{}^{\rho\sigma} - \Gamma^\lambda_{\mu\nu} T_\lambda{}^{\rho\sigma} + \Gamma^\rho_{\mu\lambda} T_\nu{}^{\lambda\sigma} + \Gamma^\sigma_{\mu\lambda} T_\nu{}^{\rho\lambda},$$

with the sign dependent on whether the connection is contracted with a lower or upper index. General Relativity requires that the connection be symmetric in its lower indices, $\Gamma^\nu_{[\mu,\lambda]} = 0$ (torsion-free), and be defined such that $\nabla_\rho\, g_{\mu\nu} = 0$ (metric compatible). These conditions uniquely fix the connection in terms of the metric, such that in any coordinate system

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\rho} \left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu} \right).$$

A connection satisfying these properties is called a Christoffel connection with coefficients called Christoffel symbols.

Given a connection, it is then possible to construct the Riemann tensor, whose components are given by

$$R^{\rho}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma},$$

expressing the curvature of the manifold at any specific point. Specifically,

$$\delta V^\rho = R^{\rho}_{\sigma\mu\nu} V^\sigma A^\mu B^\nu$$

expresses how much the ρth component of V changes when it is parallel transported in an infinitesimal loop, first along A, then along B, then backwards along A and finally backwards along B. The Riemann tensor vanishes inside any flat region of the manifold.

Subsequently, the dynamics of spacetime in vacuum are governed by the Einstein-Hilbert action,

$$S_H = \int d^4x \sqrt{-g} R. \tag{A.1}$$

Here, g is the determinant of the metric tensor and R is the so-called Ricci scalar, given by $R = g^{\mu\nu} R_{\mu\nu}$, $R_{\mu\nu} = R^{\rho}_{\mu\rho\nu}$. $R_{\mu\nu}$ is called the Ricci tensor and $R^{\rho}_{\mu\sigma\nu}$ is our familiar Riemann tensor. The square root of the determinant of the metric tensor functions as an inverse Jacobian under coordinate transformations, allowing for an expression of S_H independent from choice of coordinates.

It is very easy to couple matter, governed by its own Lagrangian $\mathcal{L}_M$, to gravity. This is done in the following way:

$$S_M = \int d^4x \sqrt{-g} \mathcal{L}_M.$$

One can then recover the complete Einstein equation from the action

$$S = S_H + S_M,$$

through application of the Euler-Lagrange equations. As we are interested in amplitudes containing gravitons alone, we will solely focus on equation A.1.

In addition to the points above, we will also make use of vielbeine. Since the tangent space at a particular spacetime point p, T_pM, is a vector space, one can make various choices with regards to the basis in which one would like to express vectors. Vectors (or tensors) in the tangent space are often expressed in terms of the coordinate basis, where basis vector $\hat{e}_\mu$ points in the direction of increasing coordinate x^μ. The Greek index indicates that we are working with a coordinate basis, the hat that the basis vectors are normalized.

When one takes the inner product between coordinate basis unit vectors, one recovers the local metric components in terms of the coordinate basis:

$$\langle \hat{e}_\mu, \hat{e}_\nu \rangle = g_{\mu\nu}.$$

However, it is also possible to introduce a basis transformation at each separate spacetime point such that these new basis vectors recover the Minkowski metric. That is, at point p,

$$\hat{e}_\mu \to e(p)^{\mu}{}_{a} \hat{e}_\mu \equiv \hat{e}_a,$$

where $\hat{e}_\mu, \hat{e}_a \in T_pM$, such that

$$\langle \hat{e}_a, \hat{e}_b \rangle = \eta_{ab}.$$

One can orthonormalize the coordinate basis like this at each spacetime point separately, creating a basis for the entire tangent bundle of the manifold. The set of new orthonormal basis vectors is called a vielbein or tetrad. Oftentimes these terms are also used to refer to the basis transformation itself $e(p)^\mu{}_a$ and its inverse $e(p)_\mu{}^a$. We will drop the explicit mention of individual spacetime point dependence and simply write $e^\mu{}_a$ and $e_\mu{}^a$. A tensor whose components are expressed with respect to a vielbein are labeled with Latin indices, while those components carrying Greek indices indicate one is working with a coordinate basis.

One thing to consider when working with vielbeine is the fact that covariant derivatives acting on components expressed in a viebein require a different type of connection than hitherto seen,

$$\nabla_\mu X^a = \partial_\mu X^a + \omega_\mu{}^a{}_b X^b.$$

This is the so-called spin connection, $\omega_\mu{}^a{}_b$. One can derive how the spin connection depends on the Christoffel connection by applying the covariant derivative to an arbitrary vector field X and later expressing the resulting tensor both in the coordinate basis and with respect to the vielbein and equating the two.

$$(\nabla_\mu X^\nu)\hat{e}^\mu \otimes \hat{e}_\nu = \nabla X = (\nabla_\mu X^a)\hat{e}^\mu \otimes \hat{e}_b.$$

One finds that

$$\omega_\mu{}^a{}_b = e_\nu{}^a e^\lambda{}_b \Gamma^\nu_{\mu\lambda} - e^\lambda{}_b \partial_\mu e_\lambda{}^a.$$

This neatly results in

$$\nabla_\mu e_\nu{}^a = 0.$$

Interestingly, vielbeine themselves are subject to a type of gauge freedom. Since Lorentz transformations preserve the Minkowiski metric, one can simply perform a Lorentz transformation on the vielbein basis vectors to acquire a different vielbein, also reproducing the Minkowski metric.

$$e_\mu{}^a \to e_\mu{}^b \Lambda^a{}_b,$$

where Λ can depend on spacetime point. This results in the following transformation of the spin connection:

$$\omega_\mu{}^a{}_b \to \Lambda^a{}_{a'} \Lambda^b{}_{b'} \omega_\mu{}^{a'}{}_{b'} - \Lambda^c{}_b \partial_\mu \Lambda^a{}_c.$$

This sums up what we need to know in order to understand the argument that follows.

A.2 The Argument

As before, we are interested in the z-dependence of the shifted tree-level amplitude $\hat{M}_n$, this only containing external gravitons. The argument will proceed along the same lines as before. We will once again consider the background field method, treating the shifted particles as high energy particles in a soft background. We will also apply the lightcone gauge to eliminate the z-dependence of the vast majority of diagrams contributing to $\hat{M}_n$. We will then deduce the overall z-dependence by scrutinizing the diagrams that are resistant to the lightcone gauge.

We start with action

$$S = \int d^4x \sqrt{-g} R,$$

which we perturb by introducing the shift $g^{\mu\nu} \to g^{\mu\nu} + h^{\mu\nu}$, resulting in [23, 34]

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{4} g^{\mu\nu} \, \nabla_\mu h_\alpha{}^\beta \, \nabla_\nu h_\beta{}^\alpha - \frac{1}{8} g^{\mu\nu} \, \nabla_\mu h_\alpha{}^\alpha \, \nabla_\nu h_\beta{}^\beta - h_{\alpha\beta} h_{\mu\nu} R^{\beta\mu\alpha\nu} \right].$$

Here we have kept only the terms that are quadratic in $h^{\mu\nu}$. Note that we are not introducing $h^{\mu\nu}$ as a perturbation away from a specific metric. Rather, $h^{\mu\nu}$ represents the shifted particle in an arbitrary background, since we want to keep our treatment as general as possible.

As a trick, we now introduce a scalar field ϕ to our current theory in an additional term. We introduce this ϕ only for the purposes of performing a field redefinition and casting our Lagrangian in an agreeable form. We acquire

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{4} g^{\mu\mu} \, \nabla_\mu h_\alpha{}^\beta \, \nabla_\nu h_\beta{}^\alpha - \frac{1}{8} g^{\mu\nu} \, \nabla_\mu h_\alpha{}^\alpha \, \nabla_\nu h_\beta{}^\beta - h_{\alpha\beta} h_{\mu\nu} R^{\beta\mu\alpha\nu} + \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right].$$

We now perform the field redefinition

$$h_{\mu\nu} \to h_{\mu\nu} + g_{\mu\nu} \sqrt{\frac{2}{D-2}} \phi, \qquad \phi \to \frac{1}{2} g^{\mu\nu} h_{\mu\nu} + \sqrt{\frac{D-2}{2}} \phi,$$

giving

$$\mathcal{L} = \sqrt{-g} \left[\frac{1}{4} g^{\mu\nu} g^{\alpha\rho} g^{\beta\sigma} \nabla_\mu h_{\alpha\beta} \nabla_\nu h_{\rho\sigma} - \frac{1}{2} h_{\alpha\beta} h_{\mu\nu} R^{\beta\mu\alpha\nu} + \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right].$$

Here, D is the amount of spacetime dimensions, which in our treatment equals 4.

Now that we have performed the field redefinition, we can cast off the last dilaton term, since it does not play a role at tree-level pure graviton scattering. We rewrite the tensors using vielbein $e_\mu{}^a$,

$$\begin{aligned} \mathcal{L} &= \sqrt{-g} \left[\frac{1}{4} g^{\mu\nu} g^{\alpha\rho} g^{\beta\sigma} \nabla_\mu \left(e_\alpha{}^a e_\beta{}^{\tilde{a}} h_{a\tilde{a}} \right) \nabla_\nu \left(e_\rho{}^b e_\sigma{}^{\tilde{b}} h_{b\tilde{b}} \right) - \frac{1}{2} e_\alpha{}^a e_\beta{}^{\tilde{a}} e_\mu{}^b e_\nu{}^{\tilde{b}} h_{a\tilde{a}} h_{b\tilde{b}} e^\beta{}_c e^\mu{}_d e^\alpha{}_{\tilde{c}} e^\nu{}_{\tilde{d}} R^{cd\tilde{c}\tilde{d}} \right], \\ &= \sqrt{-g} \left[\frac{1}{4} g^{\mu\nu} \eta^{ab} \eta^{\tilde{a}\tilde{b}} \nabla_\mu h_{a\tilde{a}} \nabla_\nu h_{b\tilde{b}} - \frac{1}{2} h_{a\tilde{a}} h_{b\tilde{b}} R^{ab\tilde{a}\tilde{b}} \right]. \end{aligned}$$

Here, we used the product rule in conjunction with the fact that $\nabla_\mu e_\nu{}^a = 0$. Furthermore, we used that $g^{\mu\nu} e_\mu{}^a e_\nu{}^b = \eta^{ab}$ and $e^\mu{}_b e_\mu{}^a = \delta^a_b$. The tildes on the indices are used to elucidate the fact that the left and right indices on $h_{a\tilde{a}}$ don't mix: right indices are only contracted with right indices etc. This will be a fact useful to us soon.

It is now our task to write out the Lagrangian using $\nabla_\mu h_{a\tilde{a}} = \partial_\mu h_{a\tilde{a}} - \omega_\mu{}^r{}_a h_{r\tilde{a}} - \omega_\mu{}^{\tilde{r}}{}_{\tilde{a}} h_{a\tilde{r}}$. This will allow us to make those vertices manifest scaling with z or z^2 (those containing derivatives) and those that scale with z^0. We get

$$\begin{aligned}\mathcal{L} =&\frac{1}{4}g^{\mu\nu}\eta^{ab}\eta^{\tilde{a}\tilde{b}}\left(\partial_\mu h_{a\tilde{a}}\partial_\nu h_{b\tilde{b}} - 2\omega_\mu{}^r{}_a h_{r\tilde{a}}\partial_\nu h_{b\tilde{b}} - 2\omega_\mu{}^{\tilde{r}}{}_{\tilde{a}} h_{a\tilde{r}}\partial_\nu h_{b\tilde{b}}\right)\\ &+\frac{1}{2}h_{r\tilde{a}}h_{a\tilde{r}}R^{ra\tilde{a}\tilde{r}} + \frac{1}{4}g^{\mu\nu}\left[\omega_\mu{}^{ra}\omega_\nu{}^{\tilde{r}\tilde{a}}h_{r\tilde{a}}h_{a\tilde{r}} + \eta^{\tilde{a}\tilde{b}}\omega_\mu{}^{ra}\omega_\nu{}^s{}_a h_{r\tilde{a}}h_{s\tilde{b}} + \eta^{ab}\omega_\mu{}^{\tilde{r}\tilde{a}}\omega_\nu{}^{\tilde{s}}{}_{\tilde{a}}h_{a\tilde{r}}h_{b\tilde{b}}\right].\end{aligned} \tag{A.2}$$

The first term provides vertices scaling with z^2, the next two terms with z and the entire last line gives terms with no z-dependence.

We can once again eliminate the z-dependence of all vertices except for the vertex appearing in the familiar diagram A.3 below. The method for and problems with picking the lightcone gauge here are nearly identical to those discussed for spin-1 bosons in section 4.2.1, slightly modified for the graviton discussion.

First of all, interestingly enough, we can pick the lightcone gauge for the spin connection using a local Lorentz transformation

$$q^\mu\omega_\mu{}^{ab} = 0.$$

For the actual gravitons, within the de Donder gauge, we have the remaining freedom to perform a gauge transformation of the form

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu,$$

for any arbitrary vector field ξ. If we thus wish to impose the lightcone gauge, we must pick ξ such that in momentum space

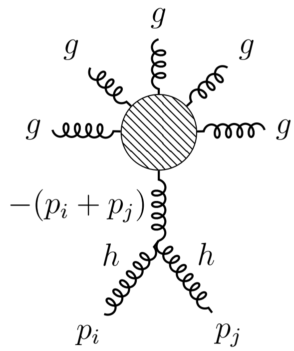
$$\forall k: \quad q_\mu\left(\epsilon'_\pm\right)^{\mu\nu}(k) = q_\mu\epsilon_\pm^{\mu\nu}(k) + q_\mu k^\mu\tilde{\xi}^\nu(k) + k^\nu q_\mu\tilde{\xi}^\mu(k) = 0.$$

For some of these k, however, $q\cdot k$ vanishes, eliminating the second term in the expression. As discussed previously, these k lie on the lightcone emanating from q in the space of all possible four-momenta. (Compare figure 4.1.) For these particular k, the condition above reduces to

$$q_\mu\epsilon^{\mu\nu}(k) = q_\mu\epsilon_\pm^\mu(k)\epsilon_\pm^\nu(k) = -k^\nu q_\mu\tilde{\xi}^\mu(k).$$

Here we used that the graviton polarization tensor can be written as the tensor product of two polarization vectors, $\epsilon_\pm^{\mu\nu} = \epsilon_\pm^\mu\epsilon_\pm^\nu$. Since $q_\mu\tilde{\xi}^\mu(k)$ and $q_\mu\epsilon_\pm^\mu(k)$ are scalars, the only possible way to satisfy the condition above is if $\epsilon_\pm^\nu$ is proportional to k^ν. However, this analysis is specific to those k, for which $k\cdot q = 0$, implying that also $q\cdot\epsilon_\pm(k) = 0$, in turn requiring that $q_\mu\epsilon_\pm^{\mu\nu}(k)$ vanish. We can thus see that for k such that $k\cdot q = 0$, it is only possible to pick the lightcone gauge if $\epsilon_\pm^{\mu\nu}(k)$ was in the lightcone gauge to begin with! That is not particularly useful. We are left with the conclusion that we can only pick the lightcone gauge for gravitons off of the lightcone emanating from q, identically to the situation with vector bosons.

We once again fail to eliminate the z-dependence in those vertices where shifted momenta couple to graviton lines carrying momenta orthogonal to q. As discussed previously, this is only relevant for those diagrams where this condition does not depend on external particle momenta. Hence, indeed, in practice this means that we get z-dependence only from diagrams of type 4.12, which is displayed in figure A.3 for gravitons. It is now our task to determine the z-dependence of diagrams of this type in light of the Lagrangian in equation A.2.



(A.3)

In this diagram, the only z-dependent vertex is the one connected to the shifted particles.

Reading off from the first three terms in the Lagrangian in equation A.2, the z-dependent contribution to the amplitude can be written as

$$\hat{M}^{a\tilde{a}b\tilde{b}} = cz^2\eta^{ab}\eta^{\tilde{a}\tilde{b}} + z\left(\eta^{ab}\tilde{A}^{\tilde{a}\tilde{b}} + \eta^{\tilde{a}\tilde{b}}A^{ab}\right) + \mathcal{O}\left(z^0\right). \tag{A.4}$$

Here, the amplitude is still uncontracted with the external, shifted particles. We will perform this contraction at the end of this argument, just as in the gluon case.

We can see from the Lagrangian that the z^2 term must be proportional to two factors of η. The z terms come from the next two terms in the Lagrangian, proportional to a factor η and a spin connection with raised indices $\omega_\mu{}^{ab}$, resulting in a similar factor η and a tensor A^{ab}. Metric compatibility, however, is equivalent to the anti-symmetry of $\omega_\mu{}^{ab}$ in its vielbein indices. (This can be verified by setting $\nabla_\mu g^{\rho\sigma}$ to zero and rewriting the expression with respect to a vielbein.) Hence, the tensors A^{ab} and $\tilde{A}^{\tilde{a}\tilde{b}}$ are anti-symmetric.

Let us now consider the $\mathcal{O}(z^0)$ contributions. We know that all order z^0 contribution must come from the same diagram (A.3), because all other diagrams will at the very least contain one shifted propagator, scaling $\propto 1/z$, with all vertex z-dependence eliminated due to the lightcone gauge. Hence, we simply continue reading off from the Lagrangian. We get

$$M^{a\tilde{a}b\tilde{b}}_{\mathcal{O}(z^0)} = A^{ab\tilde{a}\tilde{b}} + \eta^{ab}\tilde{B}^{\tilde{a}\tilde{b}} + B^{ab}\eta^{\tilde{a}\tilde{b}}.$$

The first term comes from the first two terms in the second line of the Lagrangian in equation A.2. The tensor $A^{ab\tilde{a}\tilde{b}}$ is anti-symmetric in its first two (a, b) and last two indices $(\tilde{a}, \tilde{b})$, inheriting the symmetry properties of the Riemann curvature tensor $R^{ab\tilde{a}\tilde{b}}$. Furthermore, the second term in the second line of the Lagrangian has the same symmetry properties, as it contains two spin connections with raised indices, which are separately anti-symmetric. Hence, both contributions of these terms can be unified in a single tensor, $A^{ab\tilde{a}\tilde{b}}$.

The last two terms in the Lagrnagian are proportional to the Minkowski metric, resulting in the last to terms in the expression above. There are no clear symmetry properties of the tensors B^{ab} and $\tilde{B}^{\tilde{a}\tilde{b}}$.

$\epsilon_i \backslash \epsilon_j$	$-$	$+$
$-$	$1/z$	$1/z$
$+$	z^3	$1/z$

Table A.1: The z-dependence of pure Yang-Mills amplitudes depending on shifted particle amplitude.

We get the full z-dependence of the uncontracted amplitude

$$\hat{M}^{a\tilde{a}b\tilde{b}} = cz^2\eta^{ab}\eta^{\tilde{a}\tilde{b}} + z\left(\eta^{ab}\tilde{A}^{\tilde{a}\tilde{b}} + \eta^{\tilde{a}\tilde{b}}A^{ab}\right) + A^{ab\tilde{a}\tilde{b}} + \eta^{ab}\tilde{B}^{\tilde{a}\tilde{b}} + B^{ab}\eta^{\tilde{a}\tilde{b}} + \mathcal{O}(1/z). \tag{A.5}$$

Once more, to get the z-dependence of the full amplitude, we contract with the polarization tensors of the external, shifted particles. We use the Ward identity for gravitons, which is

$$p_\mu A^{\mu\nu} = 0,$$

if $\epsilon_{\mu\nu}(p)A^{\mu\nu}$ is the complete amplitude. Similarly, it will hold that

$$\hat{p}_{ia}M^{a\tilde{a}b\tilde{b}}\hat{\epsilon}^{\pm}_{jb\tilde{b}} = 0,$$

where we are using the shifted quantities. Just as with gluons (eq. 4.13), we get

$$q_a\hat{M}^{a\tilde{a}b\tilde{b}}\hat{\epsilon}_{jb\tilde{b}} = -\frac{1}{z}p_{ia}\hat{M}^{a\tilde{a}b\tilde{b}}\hat{\epsilon}_{jb\tilde{b}}, \qquad q_a\hat{M}^{a\tilde{a}b\tilde{b}}\hat{\epsilon}_{ib\tilde{b}} = +\frac{1}{z}p_{ja}\hat{M}^{a\tilde{a}b\tilde{b}}\hat{\epsilon}_{ib\tilde{b}}. \tag{A.6}$$

Adapting the identities for shifted polarization vectors to gravitons (eq. 4.3), we have

$$\begin{gathered}\hat{\epsilon}^{-}_{ia\tilde{a}}(z) = \hat{\epsilon}^{+}_{ja\tilde{a}}(z) \propto q_a q_{\tilde{a}}, \\ \hat{\epsilon}^{+}_{ia\tilde{a}}(z) \propto (q_a - zp_{ja})(q_{\tilde{a}} - zp_{j\tilde{a}}), \qquad \hat{\epsilon}^{-}_{ja\tilde{a}}(z) \propto (q_a + zp_{ia})(q_{\tilde{a}} + zp_{i\tilde{a}}).\end{gathered} \tag{A.7}$$

Combining A.5 and A.7, we can infer, for instance,

$$\begin{aligned} M^{--} &= \epsilon^{-}_{ia\tilde{a}}M^{a\tilde{a}b\tilde{b}}\epsilon^{-}_{jb\tilde{b}}, \\ &= q_a q_{\tilde{a}} q_b q_{\tilde{b}} M^{a\tilde{a}b\tilde{b}}, \\ &\propto 1/z. \end{aligned}$$

We see that M^{--} scales as $1/z$ in the limit of $z \to \infty$, since every term in equation A.5, containing a factor of η, gets cancelled because $q^2 = 0$.

Also utilizing equation A.6 and using the orthogonality of q with p_i and p_j, one can derive the full table of z-dependence, in table A.1.

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Explanations of Latin Quotes

The quote on page 3 has–of course–been taken out of context. In the relevant passage of *De Inventione*, a handbook for orators, Cicero describes features of a person that are to be considered 'useful yet honorable'. One of these features he calls 'amplitude' ('amplitudo' in Latin) which in its foremost sense means 'size' or 'magnitude'. For the purposes of his work, Cicero defines amplitude as a great abundance of power, majesty and resources of some sort. This is what the quote thus originally means, however, out of context, the translation given is also completely valid.

The quote on page 26 occurs in a collection of poems called *Epistulae Heroidum* ("Letters by (the) Heroines") by Roman poet Ovid. This particular poem (*Heroides* 6) presents itself as a letter from Hypsipyle to Jason, both mythological figures. The quote "Dent modo fata recursus." should properly be translated as "May the fates just give a way back." "Recursus" literally means "a running back" and here figuratively stands for an escape from the current situation to a former one. Of course, our word "recursion" is etymologically closely related to "recursus" and this connection also makes sense in terms of meaning. Recursion is after all a running back to the start of some procedure in order to perform the procedure once more.

The quote on page 54 is contained within a collection of letters written by famous philosopher Seneca nearing the end of his lifetime. According to the standards of our modern culture, the true context of this quote can be considered rather dark. I feel that explaining the context does not fit the tone of the rest of the thesis.

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