University of Groningen

Master Research Project

On Duality and Unity: Constructing Hybrid Soft Behaviour with the Double Copy and Transmutation Operators

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Abstract

Scattering amplitudes play a critical role in understanding fundamental interactions in quantum field theory. This thesis delves into the unifying relations for these amplitudes and applies the BCJ bootstrap method to construct kinematic numerators that respect the colour-kinematics duality. We began with a detailed analysis of Yang-Mills (gluon) and pion (NLSM) amplitudes, investigating their duality and behaviour under specific kinematic conditions. We then applied the BCJ bootstrap method to derive amplitudes for the gauged nonlinear sigma model, observing a hybrid soft degree where some amplitudes conform to $\sigma - 0$ while others exhibit pion contact terms with $\sigma = 1$.

In this work we have developed of a generalised transmutation (GT) operator, which allowed us to generate and analyze mixed amplitudes of scalars and pions interacting through gluons. The GT operator was validated by computing 6-point amplitudes and tracking Mandelstam variables, revealing correct results for Yang-Mills-Scalar (YMS) and Nonlinear Sigma Model (NLSM) amplitudes, and uncovering unexpected terms like the 4-point ϕ^4 amplitude.

We found that while the GT operator reconstructs some interactions observed in the BCJ bootstrap method, there are contributions to the $\sigma = 0$ amplitude that are not generated by the BCJ bootstrap. We therefore conclude that the BCJ bootstrap does not uniquely determine amplitudes based solely on the soft degree $\sigma = 0$.

Future work could examine how the GT operator acts on higher-derivative corrections in Yang-Mills theory in order to investigate their compatibility with CK-duality. We also suggest the possibility of applying GT to other theoretical frameworks, which could land on mixed theories such as Born-Infeld photons interacting with Maxwell theory photons through graviton exchange.

Contents

Acknowledgements

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Introduction

The study of quantum field theory has been an active area of research since the 1920s. One of the first fields to be successfully quantised was the electromagnetic field [1], which led to the development of the theory of quantum electrodynamics (QED). This theory was able to explain the existence of relativistic particles, namely photons, which appear as quantisations of the electromagnetic field. Such an explanation was not possible in the ordinary theory of quantum mechanics at the time.

With quantum field theory, a new method of calculating the *scattering matrix*, or S-matrix, of particle interactions was formulated. The S-matrix relates the initial and final state of particle processes, allowing for the calculation of interaction probabilities. The elements of this matrix are known as *scattering amplitudes*, which can be pertubatively computed using the famous Feynman diagram approach [2]. These diagrams are a diagrammatic representation of the scattering process, where each line and vertex can be associated to a mathematical expression. These associations are called the 'Feynman rules', and their specific form is based on the theory on is studying, as the Feynman rules can be derived from the interaction Lagrangian $\mathcal{L}_{int}$.

There are two types of theories that are fundamental to our understanding of particle interactions. The first is gauge theory, which is a core part of the Standard Model of particle physics. In this model, three of the four fundamental forces of the universe are described as interacting particles exchanging gauge bosons, which are referred to as the 'force carriers'. There are three kinds of gauge bosons, the first being the photon that mediates the electromagnetic interactions. The second are the W and Z bosons, which are responsible for mediating the weak interaction. Finally, there are gluons, which mediate the strong interaction between quarks that are described in quantum chromodynamics (QCD).

The different interactions above arise from different symmetry groups under which the gauge field transforms. The standard model interactions are defined by the gauge symmetry $U(1) \times SU(2) \times SU(3)$[3], where each indiviual group is responsible for the electromagnetic, weak and strong interaction respectively. Associated to each symmetry group is a number of colours or flavours that dictate the interactions of the theory through the structure constants of the group f^{abc}.

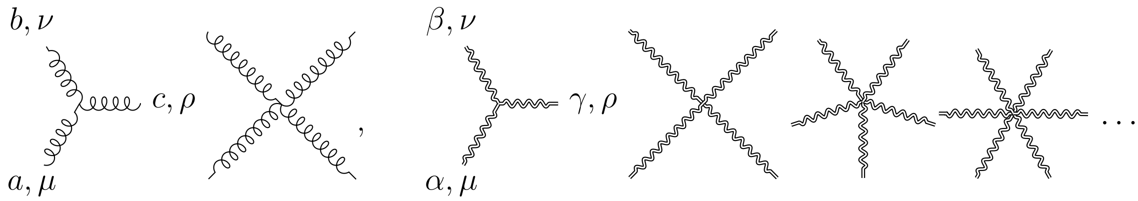


Figure 1: Diagrams of the types of self-interactions of gluons (curly lines) and gravitons (double wavy lines). The self interaction of gravitons arise at every other in the pertubative expansion of the theory, while gluons are restricted to cubic and quartic self-interactions.

The other fundamental theory is that of gravity, which describes the interactions of the fourth fundamental force of the universe. Gravity is not incorporated in the standard model of particle physics due to the fact that it is not straightforward to correctly quantise the gravitational field. The best theory of gravity that we do have is general relativity, but a quantisation of this theory leads to divergences of the S-matrix for high energies (the so-called UV divergences)[4]. Nevertheless, it is possible to formulate these theories of quantum gravity [5] to study the interactions that arise from it in the hopes of ironing out these UV divergences [6]. Moreover, the amplitudes of the particles that arise from the quantisation of the gravitational field are called gravitons. It is also possible to use graviton scattering amplitudes to aid in the calculations of inspiraling binary black holes and neutron stars that produce gravitation waves [7, 8].

Gauge theory and gravity are fundamentally different theories. The first describe the interactions of elementary particles, while the second shapes the curvature of spacetime and dictates the evolution of the large scale structure of the universe. At a mathematical level these QFTs give rise to very different types of interactions. It can be shown that (non-Abelian) gauge theory allows for cubic and quartic self-interactions of gluons through the Yang-Mills Lagrangian

$$\mathcal{L}_{\text{YM}} = -\frac{1}{4} F^a_{\mu\nu} F^{a\,\mu\nu}, \qquad \text{with} \qquad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu. \tag{1}$$

Gravitons on the other hand, arise from a pertubative expansion of the Einstein-Hilbert action

$$\mathcal{L} = -\frac{1}{16\pi G}\sqrt{-g}R, \qquad \text{with} \qquad g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{2}$$

where the amplitudes are typically built from an expansion in $h_{\mu\nu}$ [9]. Due to this expansion, we find that there are self-interaction at every order. The self interactions of gluons and gravitons are displayed in Figure 1.

While in theory it is possible to calculate the scattering amplitudes of gravitons from their Lagrangian, it is in practice a computational nightmare. Not only do we have to consider all the possible self-interactions, but already for each cubic graviton vertex we have to account for 171 terms [10, 11]. Regardless of this, a great number of simplifications occurs, which immensely reduces the expression for the resulting amplitude. This behaviour hints at hidden symmetries at the level of the amplitude that are not visible in the Lagrangian of the theory. These types of features that are hidden at the level of the Lagrangian, are the main motivation of the modern S-matrix programme, where scattering amplitudes are built from first principles such as symmetries, factorisation, and infrared behaviour.

There is a particularly interesting hidden connection in the amplitudes of gauge theory and gravity. The story starts in gauge theory, where the tree-level amplitudes $\mathcal{A}_m^{\text{tree}}$ can be formatted as sums over distinct cubic diagrams

$$\mathcal{A}_m^{\text{tree}} \sim \sum_{i \in \text{cubic}} \frac{c_i n_i}{D_i}, \tag{3}$$

where each diagram is assigned a colour factor c_i, a kinematic numerator n_i and a denominator D_i. It was presented by Zvi Bern, John Joseph Carrasco and Henrik Johansson [12] that a general property of gauge theory amplitudes is that the kinematic numerators satisfy the same algebraic identity as the colour factors

$$c_i + c_j + c_k = 0, \qquad\qquad n_i + n_j + n_k = 0, \tag{4}$$

which is known as the *colour kinematics* (CK) *duality*. The CK-duality allows for a replacement of colour factors by another copy of kinematic numerators

$$\mathcal{A}_m^{\text{tree}} \sim \sum_{i \in \text{cubic}} \frac{c_i n_i}{D_i} \xrightarrow{c_i \to \tilde{n}_i} \sum_{i \in \text{cubic}} \frac{\tilde{n}_i n_i}{D_i} \sim \mathcal{M}_m^{\text{tree}}. \tag{5}$$

The objects $\mathcal{M}_4^{\text{tree}}$ that are constructed through this replacement turn out to be the tree-level graviton amplitudes. This procedure is known as the *double copy*, which lends its name to the fact that we are creating a copy of the kinematic numerators, and doubling it in the expression for the amplitude [13].The relation between the two theories is often referred to as gravity being the 'square' of Yang-Mills, summarised as

$$\text{Gravity} = (\text{Yang-Mills})^2. \tag{6}$$

Remarkably, CK-duality is not limited to gauge and gravity theories; it extends to a whole web of theories, ranging from effective scalar field theories to supersymmetric field theories.[14, 15].

The fact that there exist numerators that satisfy the same algebra as the colour factors is quite remarkable. The search for a *kinematic algebra*, analogous to the Lie algebra of the colour factors, is an active area of research [16, 17]. Equally important is the effort to understand this duality at the level of the Lagrangian [18].

Given a class of field theory, it is not immediately clear that such numerators even exist. If they do exist, finding expressions for the numerators n_i that satisfy CK-duality is a nontrivial task, as standard methods like Feynman rules generally do not automatically produce such numerators. A recent publication by Li, Roest and Ter Veldhuis [19] approaches the construction of numerators of scalar field theory amplitudes from a group theory perspective, categorising the representations of the symmetric group S_n that satisfy the CK-duality conditions and relating them to the representations that the kinematics live in.

This classification allows for the construction of 'kinematic building blocks', out of which numerators at any order in Mandelstam invariants can be bootstrapped. From these numerators it is possible to construct amplitudes of the *gauged nonlinear sigma-model* (gNLSM) that describes pions interacting with gluons. We will refer to this bootstrapping of numerators and amplitudes from the BCJ classifications as the 'BCJ bootstrap'. The full theory that is constructed obeys to a *soft degree* of $\sigma = 0$, which indicates that the amplitude scales as $A \sim p^0$ in infrared (IR) regime, where p is tuned to 0. However, there is a subsector of amplitudes that can be constructed from this theory that obey a soft degree of $\sigma = 1$.

This *hybrid soft behaviour* is interesting, as typically theories have a singular soft degree. In the case of the NLSM and other exceptional effective field theories, it is possible to uniquely define the theory based on the soft degree of the amplitudes. The soft degree is then strongly related to the symmetries of these theories. The uniqueness of a theory through its symmetries and soft theorems makes theories with a hybrid soft degree particularly intriguing to study [20].

The idea of a relationship between gauge theory and gravity amplitudes first emerged from string theory through the Kawai-Lewellen-Tye (KLT) relations [21]. These relations show how closed string (graviton) amplitudes can be expressed in terms of products of open string (gauge theory) amplitudes.

A different approach of relating the amplitudes within the double copy framework was proposed by Cheung, Shen and Wen [22]. These *Unifying Relations* are a set of differential operators, referred to as *transmutation operators* $\mathcal{T}$ that act on the tree-level amplitudes of one theory (A_a) to produce (partial) amplitudes of a different theory (A_b)

$$\mathcal{T} \cdot A_a \to A_b. \tag{7}$$

This is a top-down approach that reduces the spin of the particles in the amplitudes to construct (mixed) amplitudes of a variety of theories. Of particular interest to this work are the amplitudes of scalar fields coupled to pion fields through gauge interactions, which can be constructed from a transmutation of Yang-Mills gluon amplitudes.

The aim of this thesis will be to use the framework of unifying relations to construct a new type of transmutation operator that lands on the amplitudes of the gNLSM constructed using the BCJ bootstrap. Through this construction, we aim to answer several questions on transmutations of amplitudes and the amplitudes of the BCJ bootstrap:

1. How can we construct a transmutation operator that generates amplitudes for comparison with those derived from the BCJ bootstrap?

2. What types of amplitudes can be derived from a generalised transmutation operator?

3. How can generalised transmutation operators inform us about the uniqueness of the theory developed via the BCJ bootstrap?

To answer these questions, we first need a solid understanding of the amplitudes we will be studying. We begin by analysing Yang-Mills amplitudes in section 1.1, where we will explain the details of the colour factors and how they can be used to decompose amplitudes. After the discussion on Yang-Mills amplitudes we turn our attention to pion amplitudes, described by the nonlinear sigma-model section 1.2. We conclude chapter 1 with a discussion on two topics from the modern amplitudes programme, where we will see that both of the amplitudes of Yang-Mills theory and the NLSM have interesting properties in certain kinematic regimes.

Following this, we will take a closer look at the BCJ double copy in section 1.5. Here we explore the objects that can be constructed using the CK-duality, such as graviton amplitudes and amplitudes of the so-called 'biadjoint scalar theory'. Furthermore, we will discuss how higher-derivative operators can be added to the Lagrangians of gluons and pions and what this means for the duality between colour and kinematics.

After we have discussed the details of scattering amplitudes and the double copy, we address the challenge of constructing duality-satisfying numerators in chapter 2. In this section, we elaborate on the approach of Li, Roest and Ter Veldhuis to construct amplitudes that contain the hybrid soft degree using the BCJ bootstrap.

Keeping in mind our goal of constructing these amplitudes using transmutation operators, we dedicate section chapter 3 to discuss the details of the unifying relations of Cheung and collaborators. We will see how we can construct operators that transmute gluons into pions scalars coupled to gluon, which will be generalised in chapter 4 with the aim of landing on the same amplitudes of the BCJ bootstrap. Finally, chapter 5 will be dedicated to comparing the amplitudes of these two methods and addressing the above research questions.

Chapter 1

Scattering Amplitudes

We have seen that gluon amplitudes play a fundamental role in the double copy formalism. The amplitudes are made up of both elements that are essential in the formalism: colour and kinematics. In the first section of this chapter, we will discuss the theory behind gluon amplitudes, which is known as Yang-Mills theory. In this discussion, we will take a closer look at the colour factors and algebra that dictate the theory and its amplitudes.

After discussing gluons, we will turn our attention to the theory and amplitudes of a different type of particle: pions. This theory, part of a class of particle models known as Nonlinear Sigma Models (NLSM), is also governed by symmetry groups. However, instead of focusing on the colour structure as we did with gluons, we will examine the flavour structure of pions. We will explore how these amplitudes exhibit interesting properties, such as a flavour-kinematics decomposition and exceptional behaviour in the infrared (IR) regime.

The properties of scattering amplitudes in the IR regime are then further studied in the following section on *soft theorems.*. We find that by tuning the momenta of our external legs to vanish, we can not only recover general properties of physics such as conservation of charge and the equivalence principle, but also build up higher-point amplitudes from lower point amplitudes and heavily constrain the amplitudes of pions.

Throughout this entire work, when talking about scattering amplitudes, we will refer to tree-level amplitudes of massless, on-shell particles (obeying the EOM of the respective field with $p_\mu p^\mu = 0$), unless stated otherwise. Many of the properties of amplitudes that we derive do not easily generalise to loop-level amplitudes, and have sometimes only been proven to hold at low loop orderS.

1.1 Gluons - Yang-Mills Theory

1.1.1 Gauge Theories

When studying physics, the first field theory we are introduced to is that of electrodynamics. After we have familiarised ourselves with Maxwell's equations, we encounter the relativistic formulation of this theory that describes a four-potential A_μ whose components relate to the electric and magnetic field. We discover that different four-potentials can give rise to the same electromagnetic fields, which introduces us to the concept of gauge-freedom. The four-potential gives rise to the formulation of the electromagnetic tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{1.1}$$

The Lagrangian for Maxwell's equations (in the absence of any sources) is then given by

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \tag{1.2}$$

In a quantum field theory description, we can quantise the field A_μ after choosing a specific gauge [2]. The quantisation gives rise to a photon with two polarisation states, as required. The gauge invariance of electrodynamics is one example of a gauge symmetry.

The generalisation of such field theories is called Yang-Mills theory [23]. These theories describes massless, spin-1 vector fields A_μ in D dimensions that transform according to the adjoint representation of a gauge goup G. The above example of such a gauge group is $U(1)$ which gives rise to the dynamics of the photon field. The group $U(1)$ is an Abelian group, but we will be investigating non-Abelian gauge groups, such as $SU(N)$. Examples of interactions that arise from non-Abelian groups are the electroweak interactions which are described by the gauge group $SU(2) \times U(1)$ and QCD, which is described by $SU(3)$. This last example describes the theory of gluons that we are interested in, but we will keep our discussion generalised to the group $SU(N)$. The Yang-Mills Lagrangian is denoted as

$$\mathcal{L}_{\text{YM}} = -\frac{1}{4} F^a_{\mu\nu} F^{a\,\mu\nu}, \qquad \text{with} \qquad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu, \tag{1.3}$$

where g denotes the coupling strength, the index $a = 1, 2, \ldots, \dim G$ denotes the colour index of the field A_μ and we use notation where summation over repeated indices is implied. The structure constants f^{abc} are derived from the defining Lie algebra of $SU(N)$

$$\left[t^a, t^b\right] = i f^{abc} t^c, \tag{1.4}$$

where t^a denote the generators of the Lie group.

The Lagrangian (1.3) is invariant under gauge transformations of the fields [24]

$$A^a_\mu \to A^a_\mu + \frac{1}{g} \partial_\mu \theta^a - f^{abc} \theta^b A^c_\mu. \tag{1.5}$$

To construct amplitudes from this theory we first have to remove the gauge redundancy through the process of gauge-fixing[1]. We will choose to work in the *Feynman- 't Hooft gauge* $\xi = 1$. The scattering amplitude, sometimes referred to as the S-matrix, of the theory will also be invariant under gauge transformations.

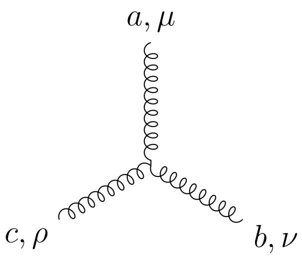
As a consequence of this gauge invariance, for external on-shell particles the polarisation $e_\mu(k)$ of some gluon with momentum k can be decoupled from the amplitude. A gauge transformation $e_\mu(k) \to e_\mu(k) + k_\mu$ then leads to the vanishing of the amplitude

$$\mathcal{A}(k) = e_\mu(k)\mathcal{A}^\mu(k) \xrightarrow{e_\mu(k)\to e_\mu(k)+k_\mu} k_\mu \mathcal{A}^\mu(k) = 0. \tag{1.6}$$

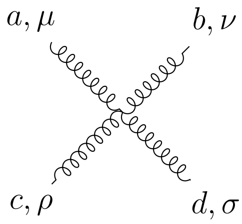
This vanishing of the amplitude is referred to as the Ward identity [25].

1.1.2 Interactions and Amplitudes

Now that the Lagrangian of Yang-Mills theory has been properly defined, we look towards creating the amplitudes that arise from it. The type of interactions that arise from the Lagrangian are cubic and quartic interactions. The Feynman rules of these interactions are given by [2]



$$= g f^{abc} \left[g^{\mu\nu}(k_1 - k_2)^\rho + g^{\nu\rho}(k_2 - k_3)^\mu + g^{\rho\mu}(k_3 - k_1^\nu) \right], \tag{1.7}$$



$$= -ig^2 \begin{Bmatrix} f^{abe} f^{cde} \left(g^{\mu\rho} g^{\nu\rho} - g^{\mu\sigma} g^{\nu\rho} \right) \\ + f^{ace} f^{bde} \left(g^{\mu\nu} g^{\rho\sigma} - g^{\mu\sigma} g^{\nu\rho} \right) \\ + f^{ade} f^{bce} \left(g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma} \right) \end{Bmatrix}. \tag{1.8}$$

With these Feynman rules, we can construct an m-point amplitude by considering the different possible topologies that can be constructed from the 3- and 4-point diagrams. For reasons that will be explained shortly, we will specifically be looking to create diagrams out of the 3-point vertex, which we call *cubic* diagrams.

At tree level, there are $(2m-5)!!$ possible cubic distinct diagrams. This number is found by considering that we can construct higher-point diagrams by attaching a new external

[1] A thorough method of gauge-fixing is the Faddeev-Popov method. The treatment requires introducing new fields to the Lagrangian, whose excitations are called *Faddeev-Popov ghosts*. Luckily (or by design of the universe) the contribution of these ghosts exactly cancel other unwanted/unphysical contributions.[24]

leg to a lower-point diagram. Given an n-point diagram, there are $(2n-3)$ edges onto which we can attach a new leg. Therefore the number of cubic diagrams at a given order is given by

$$1 \times 3 \times 5 \times \cdots \times (2n-3) = (2m-5)!!. \tag{1.9}$$

An example that we will consider shortly is the $m = 4$-point amplitude, which will have 3 distinct cubic diagrams, constructed out of the three possibilities of attaching a fourth leg to a 3-point diagram. The three distinct diagrams are shown in Figure 1.1.

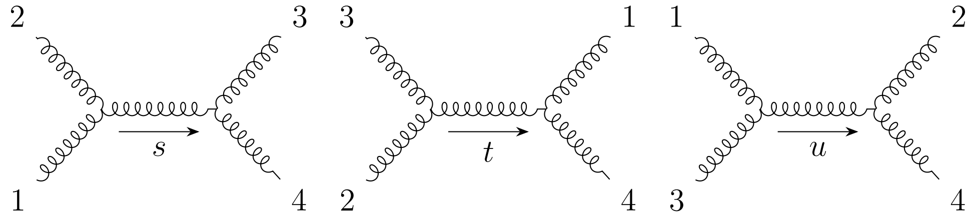


Figure 1.1: Feynman diagrams that depict the three possible 4-point cubic diagrams. The momenta of the propagators are indicated by the Mandelstam invariants s, t and u.

The term 'distinct' means that each diagram has a unique propagator contribution D_i. Moreover, each diagram has its own colour factor c_i made out of contractions of structure constants and kinematic numerator n_i that contains information about the momenta p_i and the polarisations e_i of the external legs.

After assigning a colour factor, kinematic factor, and propagator to each distinct diagram, the total amplitude can then be organised as a summation over all diagrams given by

$$\mathcal{A}_m^{\text{tree}} = -ig^{m-2} \sum_{i=1}^{(2m-5)!!} \frac{c_i n_i}{D_i}, \tag{1.10}$$

where the sum runs over the possible cubic diagrams. A useful example is the 4-point gluon amplitude. There are two topologies that contribute to the amplitude, namely the s, t and u exchange diagrams and a 4-point contact diagram. The full tree-level 4-point amplitude can be written as

$$i\mathcal{A}_4^{\text{tree}} = g^2 \left(\frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u} \right), \tag{1.11}$$

where the Mandelstam variables indicate the exchange channels depicted in Figure 1.1 and which are defined as $s = (p_1+p_2)^2$, $t = (p_2+p_3)^2$ and $u = (p_1+p_3)^2$. You may wonder what happened to the 4 point contact diagram of equation (1.8). Such diagrams are reformatted in terms of cubic vertices by multiplying and dividing the numerator of the contact diagram by one of the propagators, i.e. $\frac{s}{s}$. This already implies that the formulation of the kinematic numerators is not unique, as the contact vertex can be absorbed into each of the n_i.

We will return to the example of the 4-point gluon amplitude in our discussion of the BCJ double copy in subsection 1.5.1, but first we will need to deepen our understanding of the details of the colour algebra and structuring of the theory.

1.1.3 Colour factors and algebra

When we are discussing the duality between colour and kinematics, the colour factors c_i and their associated algebra are fundamental. To fully understand their significance, let's delve into some of the essential details and key concepts that will be referenced throughout this work.

We first start with the statement that for gauge theories the fields transform according to representations of a gauge-group G. Inherent to this Lie group is the (Lie) algebra of it's generators t^a and the structure constants f^{abc}. The structure constants are defined through the Lie product, which for us physicist is the commutator of the matrices of the generators

$$\left[t^a, t^b\right] = i f^{abc} t^c, \tag{1.12}$$

where the summation over the index c is implied. The elements of the Lie algebra satisfy a Jacobi identity

$$\left[t^a, \left[t^b, t^c\right]\right] + \left[t^b, \left[t^c, t^a\right]\right] + \left[t^c, \left[t^a, t^b\right]\right] = 0, \tag{1.13}$$

which can be rewritten in terms of structure constants as

$$f^{ade} f^{bcd} + f^{bde} f^{cad} + f^{cde} f^{abd} = 0, \tag{1.14}$$

$$(t^a)_i^{\ k} (t^b)_k^{\ j} - (t^b)_i^{\ k} (t^a)_k^{\ j} = i f^{abc} (t^c)_i^{\ j}. \tag{1.15}$$

It should be noted that the generators can be normalised in different ways. The above follows the normalisations of $\mathrm{Tr}\left[t^a t^b\right] = \delta^{ab}/2$, but when calculating amplitudes this leads to cumbersome factors of 2 that can be avoided when renormalising the generators and structure constants as

$$T^a \equiv \sqrt{2} t^a. \qquad \tilde{f}^{abc} \equiv i\sqrt{2} f^{abc}. \tag{1.16}$$

The identities of equation (1.14) are then redefined as

$$\tilde{f}^{ade} \tilde{f}^{bcd} + \tilde{f}^{bde} \tilde{f}^{cad} + \tilde{f}^{cde} f^{abd} = 0, \tag{1.17}$$

$$(T^a)_i^{\ k} (T^b)_k^{\ j} - (T^b)_i^{\ k} (T^a)_k^{\ j} = \tilde{f}^{abc} (T^c)_i^{\ j}. \tag{1.18}$$

Note that the summation over the index k equates to taking the trace of these products of matrices, $\mathrm{Tr}\left(T^a T^b\right)$.

When we construct the amplitude, each diagram has an associated colour factor c_i which is constructed out of contractions of representations of structure constants f^{abc} that are associated to the vertices of the diagrams. In the next section, we will construct the colour factors explicitly and see how these allow us to remove some of the redundancy in the number of diagrams that we have to sum over in order to construct amplitudes.

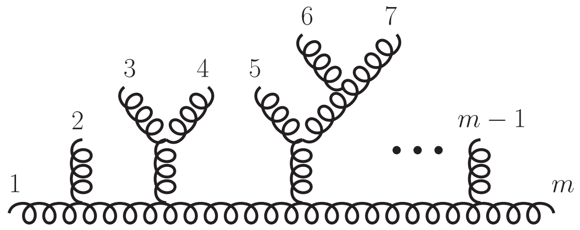


Figure 1.2: Feynman diagram of m-point gluon amplitude. The colour factor can be deduced by following a path from leg 1 through leg m. [15]

1.1.4 Colour-Ordered Partial Amplitudes

The number of diagrams at a given multiplicity can be reduced by employing relations between diagrams such as the Jacobi identity. At m-point, the $(2m-5)!!$ diagrams each have their own colour factor c_i, but there are $(2m-5)!! - (m-2)!$ independent Jacobi relations of the form (1.17).

Besides the Jacobi identities, through the definition of the algebra $\left[t^a, t^b\right] = if^{abc}t^c$ we see that the interchanging of a and b will result in a sign flip for the structure constant. Take for example the four point diagram g_s that indicates the s-channel exchange. This specific ordering of the external legs leads to a colour factor of $c(g_s) = \tilde{f}^{a_1a_2b}\tilde{f}^{ba_3a_4}$. Flipping legs 1 and 2 still results in an s-channel diagram, g_s', but with a new colour factor $c(g_s') = \tilde{f}^{a_2a_1b}\tilde{f}^{ba_3a_4}$. We therefore realise that there is an anti-symmetry under the exchange of these two legs $c(g_s') = -c(g_s)$.

For an adjoint-only theory, any colour factor can be written as a product of adjoint generator matrices $(\tilde{f}^a)_{bc} \equiv \tilde{f}^{bac}$. By following a path from leg 1 to leg m of a given diagram and denoting the corresponding (commutators of) $\tilde{f}^a$'s of the vertices we encounter along the path, we can denote the colour factor of any diagram. For example, the diagram in Figure 1.2 results in the colour factor

$$\left(\tilde{f}^{a_2}\left[\tilde{f}^{a_3}, \tilde{f}^{a_4}\right]\left[\tilde{f}^{a_5}\left[\tilde{f}^{a_6}, \tilde{f}^{a_7}\right]\right]\cdots\tilde{f}^{a_{m-1}}\right)_{a_1a_m}, \tag{1.19}$$

where we have used the Lie-algebra identity $\tilde{f}^{abc}\tilde{f}^c = [\tilde{f}^a, \tilde{f}^b]$. The colour factor of (1.19) can be generalised to any permutation σ of external legs, giving

$$c_i = \sum_{\sigma\in S_{m-2}} b_{i\sigma}\left(\tilde{f}^{a_{\sigma(2)}}\tilde{f}^{a_{\sigma(3)}}\tilde{f}^{a_{\sigma(4)}}\cdots\tilde{f}^{a_{\sigma(m-1)}}\right)_{a_1a_m}, \tag{1.20}$$

where the prefactor $b_{i\sigma} \in \{0, \pm1\}$ depends on the specific permutation and colour factor. There are $(m-2)!$ permutations of σ, as indicated by the permutation group S_{m-2}.

Returning to the summation over colour factors to obtain the total amplitude in (1.10), we find that this transforms into a sum over $(m-2)!$ ordered partial amplitudes

$$A_m^{\text{tree}}(1, \sigma(2), \sigma(3), \ldots, \sigma(m-1), m), \tag{1.21}$$

where leg 1 and m are fixed. These define a basis called the Kleiss-Kuijf (KK) basis. The summation over partial amplitudes and their corresponding colour trace is then given by

$$\mathcal{A}_m^{\text{tree}} = g^{m-2} \sum_{\sigma \in S_{m-2}} A_m^{\text{tree}}(1, \sigma(2), \sigma(3), \ldots, \sigma(m-1), m) \left(\tilde{f}^{a_{\sigma(2)}} \tilde{f}^{a_{\sigma(3)}} \cdots \tilde{f}^{a_{\sigma(m-1)}} \right)_{a_1 a_m}, \tag{1.22}$$

and is referred to as the Del Duca-Dixon-Maltoni (DDM) colour decomposition [26].

The partial tree amplitudes $A_m^{\text{tree}}(1, 2, \ldots, m)$ in Yang-Mills theory have several useful properties. The following properties will be of relevance to our discussion of gluon amplitudes:

1. **Gauge invariance:** Individually each partial amplitude is invariant under gauge transformations.

2. **Functions of kinematic variables e_i and p_i only**. Specifically, they are functions of the Lorentz invariant dot products of e_i and p_i; $(e_i \cdot e_j)$, $(e_i \cdot p_j)$, and $(p_i \cdot p_j)$. This will become a key aspect of the *Unifying Relations* between different theories discussed in chapter 3.

3. **Cyclicity:** partial amplitudes are invariant under cyclic permutations.

$$A_m^{\text{tree}}(1, 2, \ldots, m) = A_m^{\text{tree}}(2, \ldots, m, 1). \tag{1.23}$$

4. **Reflectivity:** partial amplitudes exhibit a sign flip under reversal of the ordering

$$A_m^{\text{tree}}(m, \ldots, 2, 1) = (-1)^m A_m^{\text{tree}}(1, 2, \ldots, m). \tag{1.24}$$

5. A **Photon-decoupling identity**[2] is satisfied by the partial amplitudes.

$$\sum_{\sigma \in cyclic} A_m^{\text{tree}}(1, \sigma(2), \ldots, \sigma(m)) = 0, \tag{1.25}$$

where we sum over cyclic permutations of all-but-one external leg.

6. **Obey (fundamental) BCJ relations** which take the form

$$\sum_{i=2}^{m-1} p_1 \cdot (p_2 + \ldots + p_i) A_m^{\text{tree}}(2, \ldots, i, 1, i+1, \ldots, m) = 0. \tag{1.26}$$

For example, at four point such a relation is $tA_4(1324) - sA_4(1234) = 0$.

These properties describe linear relations between partial amplitudes, allowing us to reduce the summation over all possible diagrams to a summation over the independent partial amplitudes. After considering all permutations of the above BCJ relation, there are only $(m-3)!$ independent partial tree amplitudes. The position of the three consecutive legs can be fixed in the cyclic ordering; for example $A_m^{\text{tree}}(1, \sigma(2), \ldots, \sigma(m))$ can be chosen as the independent BCJ basis.

[2]This can be seen by replacing one generator in the trace decomposition of the next section by a $U(1)$ generator $T_{U(1)} = \mathbb{I}$. Gluons do not couple directly to photons because photons have no colour charge. Explicitely $\tilde{f}^{abU(1)} = \text{Tr}\left(\left[T^a, T^b\right] 1\right) = 0$, implying that the amplitude vanishes for one photon.

1.1.5 Trace basis decomposition

By relating the structure constants to traces of the generators,

$$\tilde{f}^{abc} \equiv i\sqrt{2} f^{abc} = \mathrm{Tr}\left(\left[T^a, T^b\right] T^c\right) = \mathrm{Tr}\left(T^a T^b T^c\right) - \mathrm{Tr}\left(T^b T^a T^c\right). \tag{1.27}$$

We can decompose the product of $\tilde{f}$'s that is present in the decomposition of the total amplitude into partial amplitudes as a summation over permutations of trace structures.

Let's continue with one of our four point colour factors, $c(g_s) = \tilde{f}^{a_1 a_2 b} \tilde{f}^{b a_3 a_4}$. We can use the $SU(N)$ Fierz completeness relations,

$$\left(T^a\right)_i^{\ j} \left(T^a\right)_k^{\ l} = \delta_i^l \delta_k^j - \frac{1}{N} \delta_i^j \delta_k^l, \tag{1.28}$$

to derive the trace decomposition of the colour factor. We can perform this calculation explicitly for the four-point colour factor, but for higher multiplicities it is convenient to employ an algorithm for the decomposition. For the colour factor $c(g_s)$ we find

$$\begin{aligned}
\tilde{f}^{a_1 a_2 b} \tilde{f}^{b a_3 a_4} &= \mathrm{Tr}\left[\left(T^{a_1} T^{a_2} - T^{a_2} T^{a_1}\right) T^b\right] \times \mathrm{Tr}\left[T^b \left(T^{a_3} T^{a_4} - T^{a_4} T^{a_3}\right)\right] \\
&= \left(T^{a_1}_{ij} T^{a_2}_{jk} - T^{a_2}_{ij} T^{a_1}_{jk}\right) T^b_{ki} \times T^b_{lm} \left(T^{a_3}_{mn} T^{a_4}_{nl} - T^{a_4}_{mn} T^{a_3}_{nl}\right) \\
&= \left(T^{a_1}_{ij} T^{a_2}_{jk} - T^{a_2}_{ij} T^{a_1}_{jk}\right) \left(\delta_{km}\delta_{li} - \frac{1}{N}\delta_{ki}\delta_{lm}\right) \left(T^{a_3}_{mn} T^{a_4}_{nl} - T^{a_4}_{mn} T^{a_3}_{nl}\right) \\
&= \left(T^{a_1}_{ij} T^{a_2}_{jk} T^{a_3}_{kn} T^{a_4}_{ni} - T^{a_1}_{ij} T^{a_2}_{jk} T^{a_4}_{kn} T^{a_3}_{ni} - T^{a_2}_{ij} T^{a_1}_{jk} T^{a_3}_{kn} T^{a_4}_{ni} + T^{a_2}_{ij} T^{a_1}_{jk} T^{a_4}_{kn} T^{a_3}_{ni}\right) \\
&= \mathrm{Tr}\left(T^{a_1} T^{a_2} T^{a_3} T^{a_4}\right) - \mathrm{Tr}\left(T^{a_1} T^{a_2} T^{a_4} T^{a_3}\right) \\
&\quad - \mathrm{Tr}\left(T^{a_1} T^{a_4} T^{a_3} T^{a_2}\right) + \mathrm{Tr}\left(T^{a_1} T^{a_3} T^{a_4} T^{a_2}\right).
\end{aligned} \tag{1.29}$$

Note that implicit summation over repeated indices is implied. The terms proportional to $1/N$ cancel out. Each trace factor occurs uniquely in every permutation, and will belong to the partial amplitude of its permutation. In other words, the partial tree amplitude $A_4^{\mathrm{tree}}(1,2,3,4)$ is the kinematic coefficient of $\mathrm{Tr}\left(T^{a_1} T^{a_2} T^{a_3} T^{a_4}\right)$ and so forth for other permutations. The total amplitude is then denoted as

$$\mathcal{A}_m^{\mathrm{tree}} = g^2 \left(A_4(1,2,3,4)\, \mathrm{Tr}\left(T^{a_1} T^{a_2} T^{a_3} T^{a_4}\right) + \mathrm{perm}(2,3,4)\right). \tag{1.30}$$

Combining this decomposition with (1.11) we find the following expressions for the partial amplitudes in terms of the kinematic numerators

$$iA_4^{\mathrm{tree}}(1,2,3,4) = \frac{n_s}{s} - \frac{n_t}{t}, \tag{1.31}$$

$$iA_4^{\mathrm{tree}}(1,3,2,4) = \frac{n_t}{t} - \frac{n_u}{u}, \tag{1.32}$$

$$iA_4^{\mathrm{tree}}(1,2,4,3) = \frac{n_u}{u} - \frac{n_s}{s}, \tag{1.33}$$

From these relations we can confirm that the partial amplitudes indeed satisfy the fundamental BCJ relations of (1.26)

$$stA_4^{\text{tree}}(1,2,3,4) = utA_4^{\text{tree}}(1,3,2,4) = suA_4^{\text{tree}}(1,2,4,3). \tag{1.34}$$

The trace-decomposition of 4-point can be generalised to m-point. The decomposition of any colour factor permutation can be written as

$$\left(\tilde{f}^{a_{\sigma(2)}}\tilde{f}^{a_{\sigma(3)}}\cdots\tilde{f}^{a_{\sigma(m-1)}}\right)_{a_1 a_m} = \mathrm{Tr}\left(T^{a_1}T^{a_2}\cdots T^{a_m}\right) + (-1)^m\left(T^m\cdots T^{a_2}T^{a_1}\right) + \ldots, \tag{1.35}$$

and the *trace basis decomposition* of the total amplitude can be denoted as

$$\mathcal{A}_m^{\text{tree}} = g^{m-2}\sum_{\sigma\in S_{m-1}} A_m^{\text{tree}}(1,\sigma(2),\sigma(3),\ldots,\sigma(m-1),m)\,\mathrm{Tr}\left(T^{a_1}T^{a_{\sigma(2)}}T^{a_{\sigma(3)}}\cdots T^{a_{\sigma(m)}}\right). \tag{1.36}$$

This is a sum over $(m-1)!$ terms, which is a larger basis than the $(m-2)!$ basis of (1.22). The two are mapped into one another using the KK relations

$$A_m^{\text{tree}}(1,\{\alpha\},m,\{\beta\}) = (-1)^{|\beta|}\sum_{\sigma\in\{\alpha\}\sqcup\!\sqcup\{\beta\}^T} A_m^{\text{tree}}(1,\sigma,m). \tag{1.37}$$

The notation appears cumbersome, but an example at five-point should clarify what is happening. For starters, $\{\alpha\}$ and $\{\beta\}$ are lists of the external legs. If we wish to denote the partial amplitudes related to for example $A_5(1,2,5,4,3)$, we construct the sets $\{\alpha\}=2$ and $\{\beta\}=3,4$. The object $\{\beta\}^T$ represents reverse ordering of the list $\{\beta\}$, which is therefore $\{4,3\}$. Next, $\{\alpha\}\sqcup\!\sqcup\{\beta\}^T$ denotes the shuffle product, or ordered permutations, of these lists which in our case will be $\{\alpha\}\sqcup\!\sqcup\{\beta\}^T=\{2\}\sqcup\!\sqcup\{4,3\}$ and gives rise to the permutations $\sigma=\{243\},\{423\},\{432\}$. Finally, the exponent $|\beta|$ denotes the number of elements in the list $\{\beta\}$. The KK relations for this ordering are then

$$A_5(1,2,5,4,3) = A_5(1,2,4,3,5) + A_5(1,4,2,3,4) + A_5(1,4,3,2,5). \tag{1.38}$$

These linear relations reduce the basis at five-point from 24 partial amplitudes down to six independent partial amplitudes, as desired.

1.2 Pions - Nonlinear Sigma Model

Now that we have discussed the detail of gauge theories, we will turn our attention to a different particle sector: scalars. Such theories typically offer a great introduction to the principles of QFT and scattering amplitudes due to their simple properties such as the absence of colour structure and being spin-0.

In this section we will investigate how we can extend and refine scalar theories to derive a more comprehensive and intriguing theory of pions. Pions we originally thought to be the mediator of the strong force, before the discovery of the gluons of section 1.1 [27]. However, it turned out that pions were only an *effective* description of these interactions at short distances. Their field theory description is therefore formulated in the form of an *effective field theory* (EFT).

Effective field theories describe the dynamics of a field theory in a certain energy regime such as the infrared (IR) or the ultraviolet (UV) regime. Typically, the 'effective' behaviour comes from additional derivative terms in the Lagrangian, which at the level of the amplitude induce higher-order momentum dependence, and therefore energy dependence.

A familiar example of an EFT are the Nambu-Goldstone bosons (NGSB's) which arrises due to spontaneous symmetry breaking, e.g. the Higgs boson [24]. The effective field theory of pions is often referred to as the *nonlinear sigma model* (NLSM) [28]. In this model, pions arise due to the spontaneous breaking of the chiral-flavour symmetry [29] in QCD. In reality they are "pseudo" NGSb's because they obtain small masses from the underlying quarks, but for our purposes we will treat them as massless scalar fields.

1.2.1 Lagrangian

Qualitatively, the NGSb arise from the symmetry breaking of a global group G into a subgroup H. Every broken generator of G gives rise to a NGSb. These NGSb's are then said to live in the coset space[3] G/H. In the context of the NLSM we will be using the symmetry breaking pattern $U(N) \times U(N) \mapsto U(N)$ or more specifically we can discuss $SU(N_f) \times SU(N_f)$, where N_f denotes the number of quark flavours of the theories in the context of low-energy QCD [30]. The discussion is the same, keeping in mind that $SU(N)$ has one fewer generator than $U(N)$ which has N^2 generators.

The unitary matrix $U(\phi)$ can be used to represent the Goldstone field, which contains the scalar field ϕ that transforms according to the adjoint representation of G as $\phi = \phi^a T^a$. One of such parametrisations, along with the Lagrangian that describe it's dynamics is

[3]'If you want to show off your mastery of mathematical jargon you can say that the Nambu-Goldstone bosons live in the coset space G/H' - A. Zee [24]

given by

$$\mathcal{L} = \frac{F^2}{4}\,\mathrm{Tr}\left(\partial^\mu U \partial_\mu U^\dagger\right), \qquad U = \exp\left(\frac{i\phi}{F}\right), \tag{1.39}$$

where F is the NLSM coupling strength, which can be seen as an expansion parameter. Expanding

The symmetry of the group is spontaneously broken by a a non-linear shift of the field, $\phi \to \phi + a$ at first order in the field (not considering terms of $\mathcal{O}(\phi^2)$). It can be shown that the Lagrangian in (1.39) is invariant under this non-linear shift, which is where the name 'non-linear sigma model' comes from.

Being an effective field theory, the full Lagrangian in (1.39) can be written as a progressive expansion in F that contains only an even number of fields.

$$\mathcal{L} = \sum_{n=1}^{\infty} \mathcal{L}_{2n}. \tag{1.40}$$

Each derivative of ϕ will contribute a factor of p at the level of the amplitude, as can be seen from a fourier transformation of the derivative in momentum-space. Every $\mathcal{L}_{2n}$ contains $2n$ derivatives and is therefore of $\mathcal{O}(p^{2n})$. As an effective field theory, we can therefore consider only terms up to a certain order in p, or equivalently, a certain order in n.

Typically, we consider the $\mathcal{O}(p^2)$ Lagrangian $\mathcal{L}_2$ and it's expansion in terms of the scalar fields ϕ:

$$\mathcal{L}_2^{\mathrm{NLSM}} = -\frac{1}{2}\partial_\mu \phi_a \partial^\mu \phi^a + \frac{1}{6F^2} f_{abe} f^{cde} \phi^a \partial_\mu \phi^b \phi^c \partial^\mu \phi^d + \dots. \tag{1.41}$$

From this Lagrangian, we can construct the amplitudes of the NLSM, which has interesting properties in soft limit due to it's momentum scaling.

1.2.2 Amplitudes

Similarly to the colour ordering in YM theory, due to their flavour structure, the NLSM amplitudes can be decomposed using the familiar trace structure of Equation 1.36 as

$$\mathcal{A}_n^{\mathrm{NLSM}} = \sum_{\sigma \in S_n/\mathbb{Z}_n} Tr\left(T^{a_1} T^{a_{\tau(1)}} \cdots T^{a_{\sigma(n)}}\right) A_n(\sigma_{a_1}, \sigma_{a_2}, \dots, \sigma_{a_n}). \tag{1.42}$$

The colour stripped amplitudes of the NLSM are cyclically symmetric and contain poles only in adjacent factorisation channels. This is a property we will explore in more detail for a different scalar theory, namely the bi-adjoint scalar in subsection 1.5.2.

The first non-vanishing amplitude of the NLSM can be found at $n = 4$. The 3-point amplitude vanishes, due to the fact that momentum conservation all Mandelstam invariants

s_{ij} must vanish at 3-point. The four-point partial amplitude is generated by a single type of diagram, namely the 4-point contact diagram.

The 4-point partial amplitude with the trace ordering [1234] can be denoted as

$$A_4(1,2,3,4) = \frac{i}{2F^2}(s+t), \tag{1.43}$$

where in most literature only the kinematic part of this amplitude is of interest, as the constant factors can be conveniently redefined.

It can be seen that even though the NLSM has two derivatives per vertex, the amplitudes of the NLSM are linear in all momenta and scale as p^1, which would typically be observed if all fields in the Lagrangian contain a derivative. This is what is called an 'enhanced soft limit', which is defined by the *soft degree* that is higher than what we would expect from the number of derivatives per field [31].

The p^1-scaling of the NLSM amplitudes was first found by Stephen Adler and it implies the vanishing of the amplitudes of the NLSM in the soft limit [32]. Specifically, the amplitude vanishes in the IR regime with a *soft degree* of $\sigma = 1$ which we will further investigate in section 1.4. This is widely known as the *Adler zero* condition, or *Adler's zero*:

$$\lim_{p\to 0} A_{\text{NLSM}} \propto p^1. \tag{1.44}$$

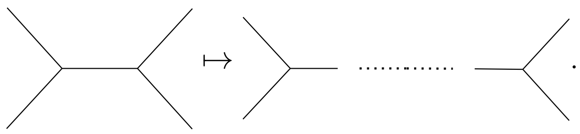
The Adler zero turns out to be an incredibly strong constraint on the amplitudes, and is sometimes regarded as a 'gauge symmetry' for such scalar EFTs [33]. One of the useful properties that can be derived from the Adler zero is the fact that the NLSM amplitudes can be unique formulated by imposing Adler's zero [34], similarly to how YM and gravity amplitudes can be uniquely formulated by imposing gauge invariance.

Moreover, higher-point NLSM amplitudes can be constructed from lower-point amplitudes through the process of *soft recursion* [35], which will be discussed in subsection 1.4.4.

1.3 Factorisation - Recursion relations

In a typical amplitude Feynman-diagram calculation of amplitudes, we construct the diagrams out of the basic vertices of the theory. At the level of diagrams, it is clear that higher-point diagrams can be built from lower-point diagrams. However, this is not a statement about amplitudes. In this section we will see that lower-point amplitudes *recursively* appear in higher-point amplitudes. There are different approaches to deriving this recursive behaviour of amplitudes. In this section we will describe the method of *on-shell recursion* [36].

On-shell recursion relies on the fact that tree-level amplitudes exhibit specific pole structures that are limited to forms such as $\frac{1}{s}$. Physically, this singularity can be interpreted as an intermediate state propagating over a physical distance in spacetime, hence the term 'propagator'. Imagine a scenario where the initial and final states are separated by a significant distance in spacetime, as illustrated for a four-particle diagram



(1.45)

In this limit, the intermediate state must be on-shell, and the initial and final vertices can be interpreted as separate, lower-point diagrams. This decomposition of higher-point amplitudes into lower-point amplitudes is known as 'factorisation'. We will see that the four point amplitude factorizes into a sum over products of 3-point amplitudes for the s, t and u factorisation channels.

1.3.1 On-Shell Recursion

A concise formulation of on-shell recursion was presented by Britto, Chachazo and Feng and Witten (BCFW)[37, 38]. The proposed BCFW recursion relations were derived by shifting two external legs by some complex factor z:

$$p_i \to p_i + zq, \qquad\qquad p_j \to p_j - zq. \tag{1.46}$$

This shift preserves the total momentum conservation as both momenta are oppositely shifted. By construction the vector q is defined such that $q^2 = p_i \cdot q = p_j \cdot q = 0$. Now that the momenta have been shifted onto the complex plane, the amplitude becomes dependent on the complex factor, implying $A \to A(z)$. We can then apply Cauchy's formula

$$f(a) = \frac{1}{2\pi i} \oint \frac{f(z)}{z-a} dz, \tag{1.47}$$

to find an expression for the original unshifted amplitude by integrating around the pole at $z = 0$. We then apply Cauchy's residue theorem to find

$$A(z=0) = \frac{1}{2\pi i} \oint \frac{A(z)}{z} dz = -\sum_I \underset{z=z_I}{Res}\left[\frac{A(z)}{z}\right] + B_\infty. \tag{1.48}$$

Here I denotes a subset of external particles. Their sum of momenta is given by

$$P_I(z) = \sum_{i \in I} p_i(z). \tag{1.49}$$

The sum over z_I denotes a summation over the residues at all other poles. These poles are the kinematic singularities (propagators) of the amplitude and hence indicated by the external legs I that contribute to the propagator. At such a singularity ($z \to z_I$), the propagator and the sub-amplitudes that connect to it are on-shell. The amplitude should therefore factorize into a product of lower-point amplitudes

$$\lim_{z \to z_I} P_I^2(z) A(z) = A_L(z_I) A_R(z_I). \tag{1.50}$$

The sum over all possible *factorisation channels* I then gives rise to the general formula for on-shell recursion, which denotes that the amplitude decomposes into the summation over all possible decompositions of lower-point amplitudes that have shifted momenta on both sides of the factorisation channel

$$A(z=0) = \sum_I A_L(z_I) \frac{1}{P_I^2} A_R(z_i) + B_\infty. \tag{1.51}$$

The boundary term that denotes the residue at $\hat{\infty}$ can cause problems for the on-shell constructibility of a theory. A theory is considered on-shell constructible if a momentum shift can be found such that $B_\infty = 0$ for every tree-level scattering amplitude. In [38] it was proven that $A(z)$ vanishes at infinity for Yang-Mills, but the story is a bit more complicated for the NLSM [39]. For pions, it is then more convenient to use a different method, namely *soft recursion*. To explore this, we will have to investigate the IR behaviour of amplitudes, which will be done in the next section.

Now that we are equipped with shell-on-shell recursion relations, it is possible to start with a *seed* amplitude such as a 3-particle amplitude and build higher multiplicity amplitudes by imposing factorisation. For example, a 4-point amplitude can be seen to split up into the product of two 3-point amplitudes.

1.4 Soft Theorems and Soft Factors

In the discussion of recursion we have seen that shifting the external legs of the amplitude gives rise to the factorisation of the amplitude in the limit of the internal propagators becoming on-shell. It would also be interesting to probe what would happen if we tune one, or more, of the external legs to vanish. Such a limit is called the *soft limit*, and is explored by assigning a dimensionless parameter λ to the momentum as λp_μ, and sending $\lambda \to 0$. When probing this limit, the amplitude will behave according to a specific *soft degree* σ as

$$\lim_{p \to 0} A \propto p^\sigma, \tag{1.52}$$

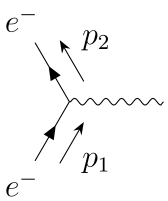
where $\sigma = -1, 0, 1, \ldots$.

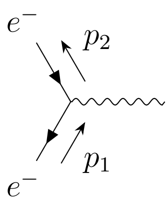
An amplitude is said to satisfy a *soft theorem* when the right-hand side of the equation features a universal structure, such as vanishing of the amplitude or as some factor multiplied by a lower-point amplitude [40]. It is possible to find an amplitude to satisfy a soft theorem as a byproduct of the action, but the inverse is also possible: impose the soft theorem in order to derive the S-matrix of the theory that satisfies this constraint.

We will start our discussion by exploring amplitudes that exhibit the soft degree $\sigma = -1$. It turns out that gauge theory and graviton amplitudes have this scaling. The soft behaviour of these theories has been famously studied by Steven Weinberg in the 1960s [41, 42], resulting in the *soft factors* of the amplitudes being named in his honour. We will examine both of these and see how, together with the other symmetries of the theories, we can find interesting soft theorems.

1.4.1 Soft factor of gauge theory

We will analyse the behaviour of gauge theory amplitudes by looking at scalar QED interactions, following the arguments of [3] to demonstrate the soft factors conceptually. The scalar QED derivation can be straightforwardly extended to QED interactions by denoting the electrons as spinors with the correct contractions of γ^μ. The (relevant) Feynman rules of scalar QED are



$= -iQe(p_1 + p_2)^\mu$ and


$= +iQe(p_1 + p_2)^\mu$, (1.53)

where $Q_i e$ denotes the electric charge of the scalar particle ($= \pm 1$ for e^+/ e^-).

There are two scenarios to consider. The first describes an outgoing photon attached to one of the incoming external legs (e^- or e^+), while the second will have the photon attached to one of the outgoing legs. This is depicted in figure Figure 1.3.

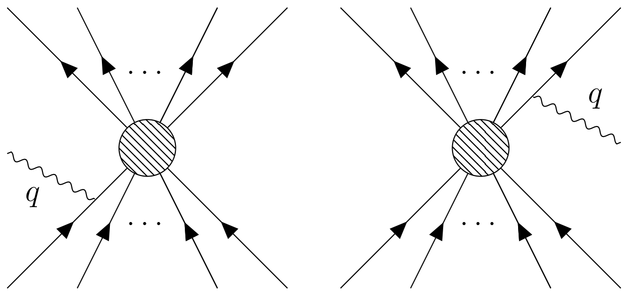
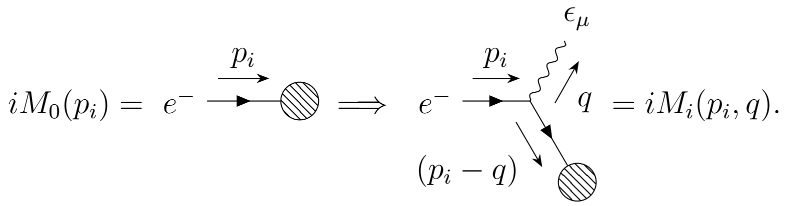


Figure 1.3: Attaching the soft photon to incoming or outgoing e^+/e^-.

In the first scenario, taking the external leg to be an e^- for now, the electron leg is attached to some scalar QED process, possibly involving many external legs and even loops. The total amplitude for the entire process is then denoted by $iM_0(p_i)$. Next, we attach an outgoing photon of momentum q and polarisation ϵ_μ to this e^- leg and describe the amplitude as $iM_i(p_i, q)$. Diagrammatically this is shown as



(1.54)

Explicitly the amplitude now receives an additional contribution from the 3 point vertex and an additional propagator contribution

$$M_i(p_i, q) = (-iQ)\frac{i\left(p_i^\mu + (p_i^\mu - q^\mu)\right)}{(p_i - q)^2 - m^2}\epsilon_\mu M_0(p_i - q). \tag{1.55}$$

After applying the on-shell conditions for the external legs $p_i^2 = m^2$ and $q^2 = q \cdot e = 0$, equation (1.55) simplifies to

$$M_i(p_i, q) \approx -Q\frac{p_i \cdot \epsilon}{p_i \cdot q}M_0(p_i), \tag{1.56}$$

at leading order in the soft limit where the photon momentum is taken to be much smaller than the external electron/position momenta, $|q \cdot p_i| \ll |p_j \cdot p_k|$.

A similar derivation can be performed for incoming e^+ and outgoing e^-/e^+. For the total amplitude we have to sum over all possible incoming/outgoing external electron/positron legs that the external photon can be attached to, giving

$$M \approx eM_0\left[\sum_{i \in \text{incoming}} Q_i \frac{p_i \cdot \epsilon}{p_i \cdot q} - \sum_{i \in \text{outgoing}} Q_i \frac{p_i \cdot \epsilon}{p_i \cdot q}\right]. \tag{1.57}$$

By the Ward identity (1.6) this amplitude should be invariant under the transformation $\epsilon_\mu \to \epsilon_\mu + \alpha q_\mu$, which implies that total charge should be conserved

$$\sum_{i\in\text{incoming}} Q_i = \sum_{i\in\text{outgoing}} Q_i. \tag{1.58}$$

Gauge invariance therefore requires the conservation of whatever coupling constant, like electric charge, governs the interaction of these particles at low energies for all amplitudes involving a massles spin-1 particle.

The pre-factor that is multiplied by the lower-point amplitudes of (1.57) is called the Weinberg soft factor [42]

$$S^\mu_{\text{QED}} = \sum_i^n Q_i \frac{p_i^\mu}{p_i \cdot q}. \tag{1.59}$$

This soft factor is the leading order contribution to the amplitude in the soft limit and allows us to relate higher point amplitudes at higher points to lower point amplitudes.

For Yang-Mills theories the soft factor is given by

$$S^{(ijk)}_{\text{YM}} = \frac{p_i e_j}{p_i p_j} - \frac{p_k e_j}{p_k p_j}, \tag{1.60}$$

where we take leg i to be the soft leg. Note that the soft factor explicitly depends on the momenta of the adjacent particles i and k.

Subleading gluon soft factor

It is interesting that at leading order, the soft factor does not depend on the spin of the soft particle. This is not the case for the subleading soft factor, where explicit dependence on angular momentum appears. For gluons in Yang-Mills theory this subleading factor is given by [43, 44]

$$S^{(ijk)}_{\text{YM},sl} = \frac{p_j J_k e_j}{p_k p_j} - \frac{p_j J_i e_j}{p_i p_j} = -\frac{p_j J_i e_j}{p_i p_j} + (i \leftrightarrow k), \tag{1.61}$$

where particle j is taken to be the soft particle. Here J_i denotes the total angular momentum of the particle i and the notation $p_j J_i e_j = p_j^\mu (J_i)_{\mu\nu}\, e_j^\nu$ denotes the summation over the Lorentz indices. It can be seen that the factor depends on the angular momentum of the legs adjacent to the soft leg.

1.4.2 Soft gravitons

Although graviton amplitudes are not the main focus of this thesis, we would like to emphasise the interesting soft theorems that can be derived from the soft behaviour of gravity. Instead of describing the interaction of spin-1 particles, we can examine a soft spin-2 particle by substituting the polarisation ϵ^μ for $\epsilon_i^{\mu\nu}$. The deduction relies on the same arguments, however the coupling of the interaction is different than scalar QED, it is governed by the gravitational coupling κ_i. The soft factor is then given by [45]

$$S^{\mu\nu} = \sum_i \kappa_i \frac{p_i^\mu p_i^\nu}{p_i \cdot q}. \tag{1.62}$$

The argument based on gauge invariance has an analogue in gravity amplitudes, which are invariant under diffeomorphisms of the form $\epsilon_{\mu\nu} \to \epsilon_{\mu\nu} + \Lambda_\mu q_\nu + \Lambda_\nu q_\mu$. Just as gauge invariance ensures that electromagnetic or Yang-Mills fields remain consistent under local transformations, diffeomorphism invariance ensures that the equations governing gravity are consistent under arbitrary smooth changes of the spacetime coordinates. Applying this argument we can derive the following soft theorem from soft gravitons

$$\sum_{\text{in}} \kappa_i p_i^\mu = \sum_{\text{out}} \kappa_i p_i^\mu. \tag{1.63}$$

This equation is severely constrained by the conservation of momentum, $\sum_{\text{in}} p_i^\mu = \sum_{\text{out}} p_i^\mu$. There can be no more constraints on the momenta p_i, as this would give only a trivial solution $p_i = 0$. Therefore, the only possible amplitudes all have $\kappa_i = \kappa$. Hence, the soft theorem of massless spin-2 particles implies that gravity is universal, which is the foundation of the theory of general relativity.

Subleading graviton soft factor

As discussed, the soft factor is a pertubative expansion relating higher multiplicity amplitudes to lower-point amplitudes. For graviton amplitudes this expansion can be written as [45]

$$M_{n+1}(k_1, k_2, \ldots, k_n, q) = \left(S^{(0)} + S^{(1)} + S^{(2)}\right) M_n(k_1, k_2, \ldots, k_n) + \mathcal{O}\left(q^2\right). \tag{1.64}$$

With $S^{(0)}$ being the leading soft factor, $S^{(1)}$ the subleading factor and so forth. We have already seen the leading order $S^{(0)}$ in our analysis. The subleading soft factors require a more complicated analysis, as shown in [45] and are given by

$$S^{(1)} \equiv -i \sum_{i=1}^{n} \frac{e_{\mu\nu} k_i^\mu q_\rho J_i^{\nu\rho}}{k_i \cdot q}, \tag{1.65}$$

$$S^{(2)} \equiv -\frac{1}{2} \sum_{i=1}^{n} \frac{e_{\mu\nu} \left(q_\rho J_i^{\mu\rho}\right) \left(q_\sigma J_i^{\nu\sigma}\right)}{k_i \cdot q}, \tag{1.66}$$

With $e_{\mu\nu} = e_\mu e_\nu$ being the soft graviton polarisation tensor satisfying $e_{\mu\nu} q^\nu = 0$ and J_i being the total angular momentum of the i-th particle.

1.4.3 Higher-spin soft theorem

Can we go one step higher and examine the soft factor of some massless spin-3 particle? Certainly! The equation in (1.63) will transform into

$$\sum_{\text{in}} \beta_i p_i^\nu p_i^\mu = \sum_{\text{out}} \beta_i p_i^\nu p_i^\mu, \tag{1.67}$$

where β_i corresponds to the coupling of a generic spin-3 form vertex.

Consider the $\mu = \nu = 0$ index of the momentum vectors, which are equal to the energy of the particles $p_i^0 = E_i$. The soft theorem on conservation changes to

$$\sum_{\text{in}} \beta_i E_i^2 = \sum_{\text{out}} \beta_i E_i^2, \tag{1.68}$$

which tells us that the sum of 'charge'$\times E^2$ is conserved. However, this is too many constraints! The only possible solution would be if all $\beta_i = 0$, resulting in a boring non-interacting theory.

The soft theorem then tells us that there are no interacting theories of *massless* particles with $s > 2$ [42]. Theories of *massive* spin-3 particles, e.g. composite particles, are allowed and have been observed at collider experiments [46].

1.4.4 Soft recursion

The above exceptional scalar EFT are constructible via on-shell recursion. This can be shown through a *soft shift* of one of the external legs, by making use of the soft degree σ of the theory. The shift that we consider,

$$p_i \to p_i(1 - z a_i), \tag{1.69}$$

conserves the on-shell kinematics only if the number of particles $n > d + 1$, where d is the dimension in which we are scattering. The dimensionality gives us a constraint on the number of particles in order to make a_i distinct for each leg. This constraint arises from the fact that if $n = d$, then the momenta p_i are linearly independent and total momentum conservation is not preserved. The case where $n = d + 1$ is not interesting, as the only solution would be to have all legs equally rescaled with $a_i = 1$ which does not probe any interesting features.

To probe the soft limit where the momentum p_i goes to zero we send $z \to \frac{1}{a_i}$. Given the soft degree σ, the amplitude then scales as

$$\lim_{z \to 1/a_i} A(z) \propto \left(z - \frac{1}{a_i} \right)^\sigma. \tag{1.70}$$

By construction, $A(z)$ now contains multiple zeros of degree σ in the complex plane and we can apply Cauchy's residue theorem, as before in the discussion of on-shell recursion;

$$A(z=0) = \frac{1}{2\pi i}\oint \frac{A(z)}{z}dz = -\sum_I \underset{z=z_I}{Res}\left[\frac{A(z)}{z}\right] + B_\infty. \tag{1.71}$$

Next we construct a function $F(z)$ which vanishes in the soft limit, but whose unshifted value is equal to 1, so that we can relate it to the unshifted amplitude $A(0)$.

$$F(z) = \prod_i^n (1 - a_i z)^\sigma, \qquad F(0) = 1. \tag{1.72}$$

We will insert this function into the denominator of (1.71) which would in general introduce new poles into the expression, which are not the factorisation channels responsible for the original poles. However, by clever construction of (1.72) these poles are cancelled by the zeroes that arise from the soft degree of (1.70).

$$A(z=0) = \frac{A(z=0)}{F(0)} = \frac{1}{2\pi i}\oint \frac{A(z)}{zF(z)}dz = -\sum_I \underset{z=z_I}{Res}\left[\frac{A(z)}{zF(z)}\right] + B_\infty. \tag{1.73}$$

The recursion into a summation over factorisation channels of lower-point amplitude then occurs, in a similar manner as the BCFW recursion relations. There is still the crucial detail of the boundary term B_∞ which occurs for the limit $z \to \infty$, which sends our momenta to the high-energy spectrum. In general, the UV behaviour of EFTs is poor, as they are nonrenormalisable theories. The only way for B_∞ to vanish is for $F(z)$ to grow faster

$$\lim_{z\to\infty} \frac{A(z)}{F(z)} = 0. \tag{1.74}$$

This will vanish precicely for the theories with an enhanced soft limit. First, as $F(z)$ is a polynomial of at most degree $n\sigma$, it will scale as $z^{n\sigma}$ for large z. The amplitude $A(z)$ scales at most as z^m, where m is the number of derivatives for the n particle vertex. Therefore B_∞ vanishes for $\sigma > m/n$, proving that the exceptional EFTs are on-shell constructible.

As an example of soft recursion for EFTs, we will calculate the 6-point NLSM amplitude from the 4-point amplitudes [47]. First we note that odd-multiplicity pion amplitudes always vanish, therefore there is no 5-point amplitude to build from. We saw that the 4-point flavour-ordered amplitude $A_4^{\text{NLSM}}(1234)$ is given by

$$A_4(1,2,3,4) = s + t = s_{12} + s_{23}, \tag{1.75}$$

up to a normalisation. We can identify three factorisation channels, namely s_{123}, s_{234} and s_{345}, which are depicted in Figure 1.4. The three factorisation channels are cyclic permutations of eachtother. The total 6-point, flavour-ordered amplitude is the sum over these three channels. We can calculate this amplitude to be

$$A_6^{\text{NLSM}}(123456) = \left[\frac{(s_{12}+s_{23})(s_{45}+s_{56})}{s_{123}} + \text{cyclic}\right] - (s_{12} + \text{cyclic}), \tag{1.76}$$

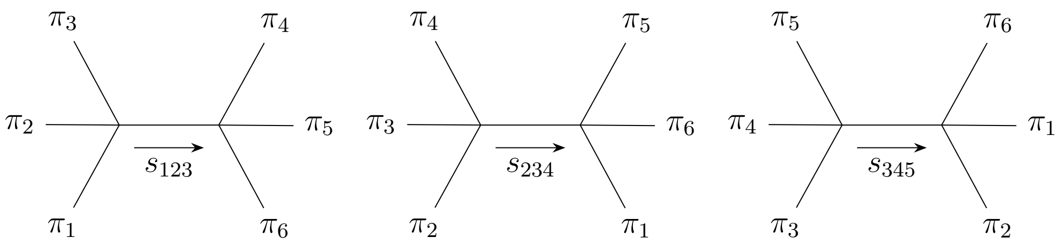


Figure 1.4: The three possible factorisation channels of the 6-point NLSM amplitude into 4-point amplitudes.

where the second term comes from the residue

$$\sum_{i=1}^{6} \underset{z=z_I}{Res} \frac{s_{12} + \text{cyclic}}{zF(z)} = -(s_{12} + \text{cyclic}). \tag{1.77}$$

It can be clearly seen that the amplitude splits up into the left-side amplitude $A_L(123x)$ and the right-side amplitude $A_R(x456)$ over the exchange channel s_{123}, in agreement with the calculation from Feynman diagrams [47].

1.5 The Double Copy

Now that we have familiarised ourselves with the properties of colour factors and the decompositions of amplitudes, we are prepared to dive deeper into the concept of colour-kinematics duality.

In this chapter, we will investigate the BCJ double copy method, which is one of the methods used to construct graviton amplitudes from gluon amplitudes by utilising the duality between the colour and kinematic numerators. We will elaborate on the 4-point gluon amplitude, as an example that can be extended to higher-multiplicity gluon amplitudes.

During our investigation of the double copy, we will also encounter the 'single copy' and the scalar amplitudes it creates. We will also briefly touch upon a different approach to the BCJ double copy, namely the KLT double copy.

Finally, we will investigate how adding higher-derivative correction terms to the Lagrangians of the theories under consideration can break the duality between colour and kinematics of the amplitudes, and what needs to be done to fix this.

1.5.1 BCJ Duality between Colour and Kinematics

The Four-Point Gluon Amplitude

One example that beautifully demonstrates the BCJ duality is the 4-gluon amplitude described in Yang-Mills theory. The formulation of this amplitude is compact enough to manipulate by hand such that we are able to understand what is going on before we dive into m-point amplitudes. Moreover it is a familiar amplitude for those of us who have studied QFT before.

As we have seen in the previous chapter, the full tree-level amplitude at 4-point can be written as a sum over the three exchange channels s, t and u

$$i\mathcal{A}_4^{tree} = g^2 \left(\frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u} \right). \tag{1.78}$$

These channels are shown diagrammatically in **figure**. The Mandelstam variables are defined as $s = (p_1 + p_2)^2$, $t = (p_2 + p_3)^2$ and $u = (p_1 + p_3)^2$. We have seen that the colour factor for the s-channel is constructed from the product of structure constants that correspond to the vertices of the diagram:

$$c_s = -2 f^{a_1 a_2 b} f^{b a_3 a_4}, \tag{1.79}$$

where the structure constants are normalised as $\left[t^a, t^b\right] = i f^{abc} t^c$.

Through Feynman rule computations, we can also find the expression for the kinematic numerator for the s-channel, n_s,

$$n_s = -\frac{1}{2}\{[(e_1\cdot e_2)p_1^\mu + 2(e_1\cdot p_2)e_2^\mu - (1\leftrightarrow 2)]\,[(e_3\cdot e_4)p_{3\mu} + 2(e_3\cdot p_4)e_{4\mu} - (3\leftrightarrow 4)] + s[(e_1\cdot e_3)(e_2\cdot e_4) - (e_1\cdot e_4)(e_2\cdot e_3)]\}, \quad (1.80)$$

where e_i denotes the polarisation of the external gluon legs.

The other combinations of colour and kinematic factors are obtained through cyclic permutations:

$$c_t n_t = c_s n_s|_{1\to 2\to 3\to 1} \qquad c_u n_u = c_s n_s|_{1\to 3\to 2\to 1}. \quad (1.81)$$

The contribution of the contact vertex is absorbed into the kinematic numerators as discussed in subsection 1.1.2.

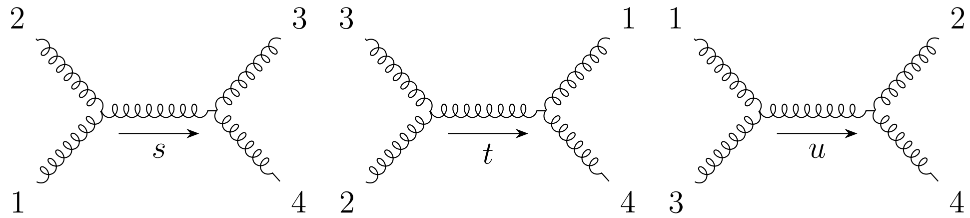


Figure 1.5: Feynman diagrams that depict the three possible 4-point cubic diagrams. The momenta of the propagators are indicated by the Mandelstam invariants s, t and u.

The colour factors of the 4-gluon amplitude

The colour factors are fully encoded by the Lie-algebra associated to the gauge group under which the field transform. In the case of gluons, the Lie-algebra we are considering is $su(3)$, of which the f_{abc}'s are the structure constants. These structure constant satisfy the Jacobi identity:

$$c_s + c_t + c_u = -2\left(f^{a_1a_2b}f^{ba_3a_4} + f^{a_2a_3b}f^{ba_1a_4} + f^{a_3a_1b}f^{ba_2a_4}\right) = 0. \quad (1.82)$$

This identity also appears as a requirement for the amplitude to be invariant under linearised gauge transformations. By utilising the Ward identity of (1.6), upon replacing for example $e_4 \to p_4$ we find

$$n_s|_{e_4\to p_4} = -\frac{s}{2}\left[(e_1\cdot e_2)((e_3\cdot p_2) - (e_3\cdot p_1)) + \mathrm{cyclic}(1,2,3)\right] \equiv s\alpha(e,p). \quad (1.83)$$

Applying this transformation to the t and u channel, we find that the total amplitude transforms as

$$\left.\frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u}\right|_{e_4\to p_4} = (c_s + c_t + c_u)\alpha(e,p). \quad (1.84)$$

Therefore, in order for the amplitude to be invariant under gauge transformation, the Jacobi identity must be satisfied.

A kinematic Jacobi identity

If, we would decide to calculate the sum of the kinematic numerators n_i, we would see that under the on-shell conditions this sum gives rise to an identity analogous to the Jacobi identity for colour factors. This is also known as the '*kinematic Jacobi identity*':

$$n_s + n_t + n_u = 0. \tag{1.85}$$

This analogy between the colour factors and kinematic numerators is referred to as the 'duality between colour and kinematics' and the n_i that together satisfy this duality are referred to as BCJ numerators. In literature, the shorthand 'CK duality' or similar terms are often used to refer to this statement.

Gravitons from Gluons

Knowing that these factors satisfy the same algebraic relations, a natural question to ask would be "what would happen if we swap out one for the other?". It turns out that by swapping colour factors for kinematic factors in the Yang-Mills four-point amplitude, we obtain a new gauge-invariant object:

$$i\mathcal{A}_m^{\text{tree}} = g^2\left(\frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u}\right) \xrightarrow[g\to\kappa/2]{c_i\to\tilde{n}_i} \left(\frac{\kappa}{2}\right)^2\left(\frac{n_s^2}{s} + \frac{n_t^2}{t} + \frac{n_u^2}{u}\right) \equiv i\mathcal{M}_4^{\text{tree}}. \tag{1.86}$$

This new amplitude, $\mathcal{M}_4^{\text{tree}}$, effectively 'doubles' the kinematic numerators, which is where the term "double copy" originates. This expression has several key properties that indicate that this amplitude is in fact a valid graviton amplitude. First of all, the external states are described by symmetric polarisation tensors $e_{\mu\nu} = e_\mu e_\nu$. Secondly, the interactions involve two derivatives. Furthermore, the amplitude remains invariant under linearised diffeomorphism transformations

$$\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu u, \tag{1.87}$$

which is the analog of gauge theory amplitudes being invariant under linearised gauge transformations. In momentum space, this implies that the amplitude should vanish upon the transformation $e^{\mu\nu} \to p^\mu e^\nu + p^\nu e^\mu$. Calculating this explicitally gives:

$$\left.\frac{n_s^2}{s} + \frac{n_t^2}{t} + \frac{n_u^2}{u}\right|_{e_4^{\mu\nu}\to p_4^\mu e_4^\nu + p_4^\nu e_4^\mu} = 2(n_s + n_t + n_u)\alpha(e,p) = 0. \tag{1.88}$$

Finally, when we choose the inital gluon polarisation vectors to be circularly polarised, implying $e^2 = 0$, the resulting polarisations $e_{\mu\nu}$ are traceless. All of this brings us to the conclusion that this amplitude describes the scattering of four gravitons in Einsteins general relativity, up to an overall normalization factor.

The example at 4-point can be generalised to higher-multiplicity amplitudes. Here the crucial argument is that it is always possible to find and relate a triplet of diagrams (i, j, k) that satisfy Jacobi relations of the form

$$c_i + c_j + c_k = 0, \tag{1.89}$$

and similarly for the kinematic numerators n_i [15]. The identification of these triplets follow from the investigation of diagrams discussed in section 1.1.

1.5.2 Zeroth Copy - Biadjoint Scalar (BAS) Theory

So far, we have seen that the duality between colour and kinematics allows us to replace the colour factors c_i of tree-level YM amplitudes by another kinematic numerator $\tilde{n}_i$ to produce tree level graviton amplitudes. This double copy procedure is denoted as

$$i\mathcal{A}_m^{\text{tree}} \sim \sum_{\text{trivalent}} \frac{n_i c_i}{D_i} \xrightarrow{c_i \to \tilde{n}_i} \sum_{\text{trivalent}} \frac{n_i \tilde{n}_i}{D_i} \sim i\mathcal{M}_4^{\text{tree}}. \tag{1.90}$$

By the same logic, we should be able to do the inverse; replace the kinematic numerator by a colour factor that obeys the same algebraic identities. This is known as the *zeroth copy* and the amplitudes that we land on are that of the exotic *biadjoint scalar* (BAS) theory

$$i\mathcal{A}_m^{\text{tree}} \sim \sum_{\text{trivalent}} \frac{n_i c_i}{D_i} \xrightarrow{n_i \to \tilde{c}_i} \sum_{\text{trivalent}} \frac{c_i \tilde{c}_i}{D_i} \sim i\mathcal{A}_m^{\text{BAS,tree}}. \tag{1.91}$$

This theory consists of a scalar field with two charges, $\phi = \phi^{a\bar{a}} T^a \tilde{T}^{\bar{a}}$, that transforms according to adjoint representation of 2, possibly different gauge groups G and $\tilde{G}$ whose generators are T^a and $\tilde{T}^{\bar{a}}$ [48]. The Lie algebras of these groups define the structure constants:

$$if^{abc}T^c = \left[T^a, T^b\right], \qquad\qquad if^{\bar{a}\bar{b}\bar{c}}\tilde{T}^{\bar{c}} = \text{Tr}\left[\tilde{T}^{\bar{a}}, \tilde{T}^{\bar{b}}\right]. \tag{1.92}$$

The Lagrangian of this theory is denoted by

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi^{a\bar{a}} \partial^\mu \phi^{a\bar{a}} + \frac{\lambda}{3!} f^{abc} \tilde{f}^{\bar{a}\bar{b}\bar{c}} \phi^{a\bar{a}} \phi^{b\bar{b}} \phi^{c\bar{c}}, \tag{1.93}$$

and results in ϕ^3 interactions. The principle of a bi-adjoint scalar that transforms according to two gauge group can be generalised beyond double colour ordering to a higher number of colour orderings. An extensive discussion of this topic can be found in [49].

At tree level, the amplitudes contain only (massless) propagators, as there are no kinematic factors in the decomposition of equation (1.91), nor can they arrise from the Lagrangian. The amplitudes of BAS theory can also be decomposed in a similar manner to (1.36). Due to the dual colour charge, the amplitudes decompose into partial amplitudes which obey a dual trace structure, denoted as

$$\mathcal{A}_m^{\text{BAS}} = \sum_{\sigma}\sum_{\sigma'} \text{Tr}\left[T^{a_{\sigma(1)}}\cdots T^{a_{\sigma(n)}}\right] \text{Tr}\left[T^{\overline{a}_{\sigma'(1)}}\cdots T^{\overline{a}_{\sigma'(n)}}\right] A_{BAS}(\sigma_1,\cdots,\sigma_n|\sigma'_1,\cdots,\sigma'_n). \tag{1.94}$$

The two orderings σ and σ' are seperated in the double colour-ordered partial amplitudes A_{BAS} by a vertical | and refer to the specific ordering of the trace structures that accompany this specific ordering.

A convenient method of calculating these partial amplitudes can be found in [50] and is briefly summarised in Appendix A.

1.5.3 Higher Derivative Corrections

We have extensively discussed the BCJ double copy, but there is another formulation of the double copy, namely the Kawai-Lewellen-Tye (KLT) formulae [21]. Similar to the BCJ formulation, it allows us to construct tree-level gravity amplitudes $\mathcal{M}^{tree}$ by taking the (KLT) product of purely-adjoint gauge theory amplitudes. For 3- and 4-point they explicitly give the graviton amplitudes in the form of

$$\mathcal{M}_3^{tree}(1,2,3) = iA_3^{tree}(1,2,3)\tilde{A}_3^{tree}(1,2,3), \tag{1.95}$$

$$\mathcal{M}_4^{tree}(1,2,3,4) = -is_{12}A_4^{tree}(1,2,3,4)\tilde{A}_4^{tree}(1,2,3,4), \tag{1.96}$$

where the amplitudes A^{tree} and $\tilde{A}^{tree}$ are the colour-ordered partial amplitudes discussed in subsection 1.1.4. As the KLT double copy is not the main focus of this work, we leave a brief discussion of this formalism to Appendix B.

Originally the KLT relations were derived in the field of string theory as relations between colour-stripped disk amplitudes of open-strings. In these relations, there is an explicit dependence on the string tension α'. In the low energy limit, where α' goes to zero, the string-theory KLT kernel reduces to the field theory KLT kernel described. Interestingly the string tension α' can be used to formulate a pertubative expansion of higher-order corrections to the field theories. In the following sections we will explore constructing such higher-order order corrections by starting from field theory amplitudes instead of string theory amplitudes.

We now return to the discussion of QFTs, where we can improve the UV behaviour of a theory by adding higher-derivative corrections, in order to create an effective action, in the spirit of section 1.2. The ultimate goal is to remove UV divergences that occur in loop amplitudes. We can do a pertubative expansion in some parameter α' and investigate the properties of the amplitudes. Specifically we are interested whether or not the colour-kinematics duality is still present after adding such terms. We will see that in order to satisfy CK-duality for some order in α' at a specific multiplicity it will be necessary to include an operator at one order higher in α' [51].

CK-Duality of Higher Order Yang-Mills

For a pure-gluon Yang-Mills theory we need to consider that the higher order corrections that we include are gauge-invariant and conserve locality. The simplest operator, i.e. lowest order in derivatives, that can then be constructed for YM is a correction to the familiar F^2 operator of subsection 1.1.2 and is denoted by the F^3 operator

$$F^3 \equiv \mathrm{Tr}\left(F_\mu^\nu F_\nu^\rho F_\rho^\mu\right) = \frac{1}{2}\,\mathrm{Tr}\Big([T^a, T^b]T^c\Big) F_\mu^{a\nu} F_\nu^{b\rho} F_\rho^{c\mu} = \frac{1}{2}\tilde{f}^{abc} F_\mu^{a\nu} F_\nu^{b\rho} F_\rho^{c\mu}. \tag{1.97}$$

The effective Lagrangian produced by this deformation then contains the first order in α' correction

$$\mathcal{L}_{\mathrm{YM}+F^3} = \frac{1}{4}\,\mathrm{Tr}\left(F^2\right) + \frac{\alpha'}{3}\,\mathrm{Tr}\left(F^3\right), \tag{1.98}$$

From which gluon amplitudes can be constructed. Double copy consistent means that the amplitudes obey the CK duality *and* consistently factorize into the correct lower-point amplitudes.

The Lagrangian in (1.98) has been shown to produce amplitudes that satisfy the colour-kinematics (CK) duality for specific multiplicities, but not universally. Specifically, at $n = 3$, the amplitudes satisfy CK duality at $\mathcal{O}\left(\alpha'^0\right)$ and $\mathcal{O}\left(\alpha'^1\right)$. Additionally, at $n = 4$, the operator satisfies CK duality at $\mathcal{O}\left(\alpha'^1\right)$. However, at the four-point level, there is also a contribution from an amplitude that arises from a combination of two cubic $\alpha' F^3$ vertices. To ensure that the amplitude maintains colour-kinematics duality at this order in α', an additional term, $\mathrm{Tr}\left(F^4\right)$, is required [52].

It was shown by Carrasco and collaborators that this pattern continues [51]. Double copy consistency at any given multiplicity and order in α' requires the addition of an operator that is one order higher in α'. This results in an infinite tower of higher-order corrections. This is different from a scalar theory, such as the NLSM, which only requires a finite number of higher-derivative operators in order for a specific multiplicity to satisfy CK duality.

An Infinite Tower of Higher Derivative Corrections

Consider the five-point amplitude. Similar to the example at $n = 4$ we note that this amplitude can be constructed out of combination of 3- and 4-point amplitude factorisation.

$$\mathcal{A}_5(12345)|_{(k_4+k_5)^2-cut} = \sum_{states} \mathcal{A}_4\left(123l^s\right) \mathcal{A}_3\left(-l^{\bar{s}}45\right). \tag{1.99}$$

This is schematically depicted in Figure 1.6. As explained before, the amplitude $\mathcal{A}_3$ comes from both F^2 and F^3 contributions through $A_3 \equiv A_3^{YM} + \alpha' A_3^{F^3}$. The five-point will then consist of a combination a four-point at $O\left(\alpha'^n\right)$ combined with the three point at $O\left(\alpha'^1\right)$ resulting in $O\left(\alpha'^{n+1}\right)$ for the five-point. This contribution can only be colour dual by addition of specific 'four-field operator' of order $O\left(\alpha'^{n+1}\right)$ that combines with the $Tr\left(F^2\right)$ term. This new $O\left(\alpha'^{n+1}\right)$ term then also has to be contracted with the $\alpha'\,\mathrm{Tr}\left(F^3\right)$ term,

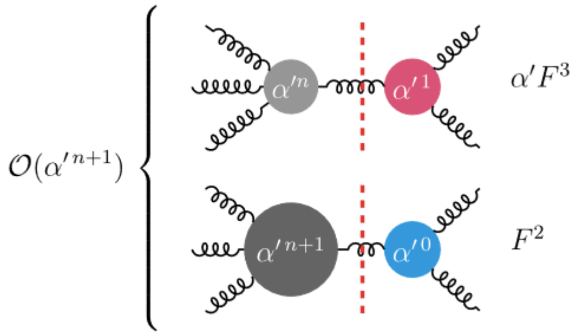


Figure 1.6: Contributions to the factorisation of the five-point tree-level amplitude at $\mathcal{O}(\alpha'^n)$. [51].

inducing a contribution at $O\left(\alpha'^{n+2}\right)$. We can continue this pattern to construct a ladder of higher-order terms, and it has been proven up to order $O\left(\alpha'^5\right)$ by [51].

Given that the CK-duality holds for these parameters, we can apply the double copy procedure by replacing the colour factors with a set of kinematic numerators. As before, this result in gravity amplitudes, which in this case also arise from an effective action. Specifically, the amplitudes of bosonic closed strings at order $\mathcal{O}(\alpha')$ and $\mathcal{O}\left(\alpha'^2\right)$. At the level of the action these higher order contributions come partly from a R^2 and R^3 operator respectively. Through KLT relations the double copy procedure can be denoted as

$$\mathcal{A}_n^{F^3} \overset{\text{KLT}}{\otimes} \mathcal{A}_n^{\text{YM}} \mapsto \mathcal{M}_n^{R^2} \text{ at } \mathcal{O}(\alpha'), \tag{1.100}$$

$$\mathcal{A}_n^{F^3} \overset{\text{KLT}}{\otimes} \mathcal{A}_n^{F^3} \mapsto \mathcal{M}_n^{R^2} \text{ at } \mathcal{O}\left(\alpha'^2\right), \tag{1.101}$$

where the details are left to [53].

Resummation to $(DF)^2$+YM

Carrasco and collaborators went back to the four-point amplitude and constructed a double-copy compatible (dcc) numerator of the s channel up to order $\mathcal{O}(\alpha'^5)$ in terms of kinematic building blocks σ that are permutation invariant such that

$$\begin{aligned} n_s^{\text{dcc}} = n_s^{\text{YM}} &+ \alpha' n_s^{\text{YM}+F^3} + \alpha'^2 n_s^{(F^3)^2+F^4} \\ &+ \alpha'^3 \left[a_3 \left(n_s^{D^2F^4} + \sigma_2 n_s^{\text{YM}+F^3}\right) + a_{3,\text{YM}} \sigma_3 n_s^{\text{YM}}\right] \\ &+ \alpha'^4 \left[a_{4,1} \left(n_s^{(DF)^4} + \sigma_2 n_s^{(F^3)^2+F^4}\right) + a_{4,2} n_s^{(DF)^4} + a_{4,F^3} \sigma_3 n_s^{\text{YM}+F^3}\right] + \mathcal{O}(\alpha'^5). \end{aligned} \tag{1.102}$$

Here σ_2 and σ_3 denote quadratic and cubic kinematic building blocks consisting of Mandelstam invariants and the a_i's are unfixed coefficients.

The question then arises whether the infinite tower resums into a known theory. It was shown that by setting $a_{3,\text{YM}} = 0$ that A_4 and A_5 precisely match the $O\left(\alpha'^4\right)$ expansion of the *'B-amplitudes'* in [54]. These amplitude correspond to a YM theory with an additional $(DF)^2$ operator deformed by a massive gauge theory whose mass is set by $1/\alpha'$ [55, 56] and which contains propagators that are constructed from the higher-order operators. This indicates that, by imposing double-copy consistency, the infinite tower of higher-order terms resums into $(DF)^2 + \text{YM}$.

Higher Derivative NLSM

In section 1.2, the NLSM Lagrangian we consider is the $\mathcal{O}(p^2)$ Lagrangian of (1.41). It is also possible to consider the terms with a greater amount of derivatives, such as the $\mathcal{L}_4$ or $\mathcal{L}_6$ which are of $\mathcal{O}(p^4)$ and $\mathcal{O}(p^6)$ respectively.

It was calculated that for 6-point amplitudes, up to $\mathcal{O}(p^{10})$, additional terms are also required in order to satisfy CK-duality due to the contact term not allowing for duality [57]. It is argued here that requiring DC consistency to arbitrary multiplicity involves adding an infinite chain of operators with fixed coefficients.

Another candidate for calculating the higher-derivative corrections for the NLSM amplitudes can be found in string theory [58]. Specifically, in a double-copy-like approach, it was found that it was possible to replace the Yang-Mills factors of Abelian open-string amplitudes by gauge theory colour factors. This replacement gives rise to Abelian Z-amplitudes [59]. A bief overview of these Z-theory amplitudes can be found in Appendix C.

In the field theory limit, the leading order α' contributions then give rise to NLSM amplitudes of the $\mathcal{O}(p^2)$ Lagrangian:

$$A_{\text{NLSM}}(1,2,\ldots,n) = \lim_{\alpha'\to 0} (\alpha')^{2-n} \sum_{\sigma\in S_{n-1}} Z_{1\sigma(2,3,\ldots,n)}(1,2,\ldots,n). \tag{1.103}$$

These amplitudes then denote the flavour-ordered NLSM amplitudes. Explicitely at $n = 4$ and $n = 6$ for example they give rise to the correct NLSM amplitudes:

$$A_{\text{NLSM}}(1,2,3,4) = \pi^2(s_{12} + s_{23}), \tag{1.104}$$

$$A_{\text{NLSM}}(1,2,\ldots,6) = \pi^2 \left[s_{12} - \frac{1}{2}\frac{(s_{12}+s_{23})(s_{45}+s_{56})}{s_{123}} + \textit{cyclic}(1,2,3,4,5,6) \right], \tag{1.105}$$

up to a normalisation factor. It is also possible to calculate the subleading terms. These correspond to the higher-derivative corrections of the NLSM.

Chapter 2

BCJ Bootstrap - Hybrid Soft Behaviour

In the previous chapters, we explored the BCJ double copy and observed that the color factors exhibit interesting identities and properties derived from the Lie algebra of the theory under consideration. However, this raises the question: What about the kinematic numerators n_i? In the 4-point example, we noted that there is no unique formulation of the numerators due to gauge invariance and the possibility of redistributing the contribution of the 4-point contact term into the numerators of the cubic exchange diagrams.

The aim of this chapter is to investigate whether there is a general formulation for the kinematic numerators, specifically for scalar theories. To achieve this, we will delve into the novel approach developed by Yang Li, Diederik Roest, and Tonnis ter Veldhuis, which classifies kinematic numerators that satisfy the BCJ colour-kinematics duality [19]. In this approach, they classify the irreducible representations (irreps) of the symmetric group S_n that meet the conditions of CK duality. With these irreps established, one can match them to the irreps of (products of) Mandelstam invariants out of which kinematic numerators are built. The constructed kinematic numerators will be used to construct amplitudes of the Gauged Nonlinear Sigma Model (GNLSM). The amplitudes of the full theory derived in this context will exhibit a hybrid soft behaviour, producing amplitudes that are characterised by a soft degree of $\sigma = 0$ and $\sigma = 1$.

2.1 A group Theory Approach to Numerators

As stated in subsection 1.5.1, we can formulate amplitudes as a sum over distinct cubic diagrams as

$$A_n = \sum_{\text{trivalent}} \frac{N\tilde{N}}{D}, \tag{2.1}$$

where the numerator factors N and $\tilde{N}$ can be referred to as BCJ numerators. These numerators satisfy the CK duality. In this notation, the numerator factors can be choosen to be either a color or kinematic numerator. As an example we have extensively considered the colour factors consisting of a product of structure constants:

$$N_{abc\dots} = f_{ab}{}^{x_1} f_{x_1 c}{}^{x_2} f_{x_2\dots} \cdots . \tag{2.2}$$

These colour numerators N are subject to the following constraints if they are to satisfy the CK duality [60]:

1. Anti-symmetric under the exchange of the first two indices $N_{abcd\dots} = -N_{bacd\dots}$.
2. Even and odd under reflections $N_{abcd\dots} = (-1)^n N_{\dots dcba}$.
3. The numerators satisfy Jacobi-like nested commutator identities

$$-N_{abcd\dots} = N_{bacd\dots} = N_{c[ab]d\dots} = N_{d[[ab]c]\dots}. \tag{2.3}$$

By satisfying the above properties, the colour numerators can then be associated with some of the representations of the symmetric group S_n. In fact, the factors then correspond to several irreducible representations (irreps) of S_n. In order to classify these specific irreps, Young diagrams can be used, which schematically depict the irrep as a collection of boxes with a specific number of rows and columns. Let us go over a brief overview of what these diagrams are and how they are used to characterise the symmetric group.

2.1.1 Young Diagrams and Representations of S_n

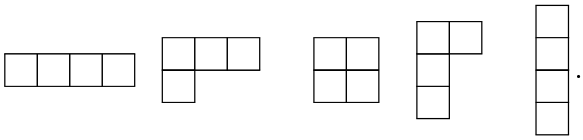
The symmetric group S_n is the group of all permutations of a set. The group can tell us how objects such as tensors transform under permutations of the tensor indices. Because the BCJ conditions on numerators discuss properties of numerators, that can be seen as tensors with indices $a, b, \dots$, we can view them from a group theory perspective. Specifically, we are interested in the group's irreducible representations.

To classify the irreducible representation of S_n, we first define young diagrams by specifying a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ of the natural number n to be a sequence of positive integers with

$$\sum_{i=1}^{r} \lambda_i = n \qquad \text{and} \qquad \lambda_i \geq \lambda_{i+1}. \tag{2.4}$$

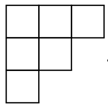
Young diagrams can then be used to depict partitions graphically, as a collection of n boxes arranged in r rows that are left-aligned. The ith row then consists of λ_i boxes. Left-aligned means that diagrams are structured such that the boxes within each row are non-increasing as you move downward, and similarly, non-increasing within each column as you move to the right. These diagrams are also known as "legitimate" diagrams.

Each legitimate diagram corresponds to a unique irreducible representation. Distinct diagrams represent inequivalent irreps, regardless of whether the dimensions of the irreps are identical. For $n = 4$, there are 5 distinct partitions:



(2.5)

Each partition for n is equal to the number of conjugacy classes[1] of S_n and is therefore equal to the number of distinct irreducible representations. We will use the shorthand notation $[3, 2, 1]$ to describe a diagram with 3 boxes in the first row, 2 boxes in the second row, and 1 box in the third row:

$[3, 2, 1] =$  . (2.6)

The dimension of an irrep S^λ can be calculated using the hook length formula. The hook length h_i of a box i in a Young tableau is the number of boxes either below or to the right of that box in the Young diagram, including the box itself. The dimensionality is then given by

$$\dim S^\lambda = \frac{n!}{\prod_i h_i}. \tag{2.7}$$

If the diagram in (2.6) were to correspond to an irrep, its dimensionality would be $\dim[3, 2, 1] = \frac{6!}{5\cdot3\cdot1\cdot3\cdot1\cdot1}$.

There are two noteworthy 1-dimensional representations, namely the *singlet* and the *sign* representation. The singlet representation, whose Young-diagram $[n]$ is displayed in Figure 2.1a, is commonly referred to as the 'trivial' representation of S_n due to the fact that it is a one-dimensional representation that sends every element of the group to the identity matrix of order 1.

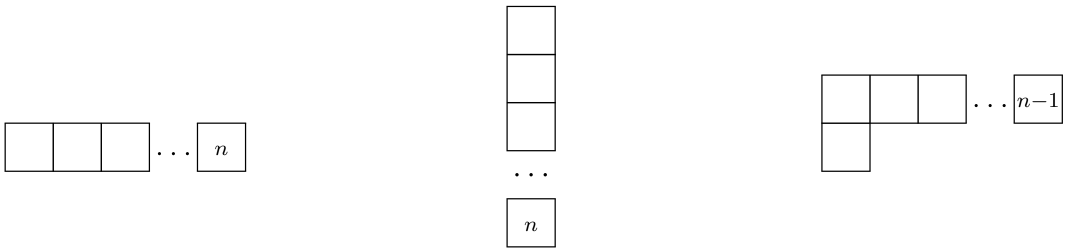
The sign representation, sometimes referred to as the 'alternating' representation, is denoted by the Young diagram $[1, 1, \ldots, 1]$ in Figure 2.1b. It sends all even number of permutations of two elements to 1 and all odd number of permutations to -1.

Returning to the discussion of the BCJ conditions on numerators, the irreps of S_n which satisfy these constraints have been classified by [19] and are denoted in Table 2.1. For convenience we display the irreps of S_4, S_5 and S_6 that satisfy the constraints explicitely:

(2.8)

$n = 4$ $\qquad$ $n = 5$ $\qquad$ $n = 6$

[1]A conjugacy class characterises a cycle structure for permutations. For example, cyclic permutations in S_3 are denoted by the conjugacy class $(abc \to bca, abc \to cab)$.



(a) (b) (c)

Figure 2.1: Young diagrams of the trivial irrep $[n]$, the sign irrep $[1, 1, \ldots, 1]$ and the standard representation $[n-1, 1]$.

n	Young diagram
4	$[2, 2]$
5	$[3, 1, 1]$
6	$[4, 2]$, $[3, 1, 1, 1]$, $[2, 2, 2]$
7	[5, 1, 1], [4, 2, 1], [3, 3, 1], [3, 2, 1, 1], [2, 2, 1, 1, 1]
8	[6, 2], [5, 2, 1], [5, 1, 1, 1], [4, 4], [4, 3, 1], 2 × [4, 2, 2], [4, 2, 1, 1], [4, 1, 1, 1, 1], 2×[3, 3, 2, 2], [3, 2, 2, 1], [3, 2, 1, 1, 1], [2, 2, 2, 2], [2, 2, 1, 1, 1, 1]

Table 2.1: Irreps of S_n that obey the BCJ constraints. [61]

Now we know from a group theory perspective what the irreps are that correspond to objects that satisfy the BCJ constraints. From here, we would like to see how these relate to numerators that are made out of kinematic objects such as Mandelstam invariants.

2.1.2 Mandelstam Representations

We have seen that the kinematic numerators n_i of YM theory are made up of products of momenta and polarisations. For this discussion, we instead consider the numerators for (coloured) scalars. The reason is that for scalar field theories the only information of the external legs are the momentum vectors. There is no polarisation to consider and therefore the kinematic numerator can only consist out of Mandelstam invariants. These are then purely constructed through Lorentz invariant products of momenta $p_i p_j$. We define these Mandelstam invariants as

$$s_{ij\ldots k} = (p_i + p_j + \cdots + p_k)^2 = 2p_i p_j + 2p_i p_k + 2p_j p_k + \ldots, \tag{2.9}$$

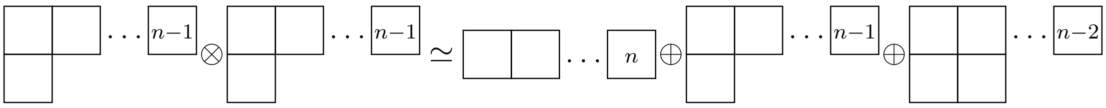
where contraction of the Lorentz indices of the momenta is implied $p_i p_j = (p_i)_\mu (p_j)^\mu$.

For the theories we consider in this section, the colour factors are all contained in either N or $\tilde{N}$, leaving the other numerator to solely depend on momentum Mandelstams. Similarly to the colour numerators, the kinematic numerators can be constructed to obey symmetries of permutations of the external legs. The same language of the symmetric group can then be applied to the kinematic numerators.

For starters, due to momentum conservation, the sum of all momenta vanishes. This summation is invariant under permutation of the momenta and can be said to live in the singlet representation $[n]$, indicating that anything we build with this irrep gives rise to vanishing contributions.

The set of all n external momenta, $\{p_1, \dots p_n\}$, lives in the *standard* representation of S_n denoted by $[n-1, 1]$ and shown in Figure 2.1c. There are $(n-1)$ independent momentum vectors due to conservation of momentum, which is consistent with the dimension of the representation $\dim = \frac{n!}{n\cdot(n-2)\cdot(n-3)\cdots 1} = n-1$.

Mandelstam invariants are products of momenta $p_i p_j$. This product can be translated to representation language to the tensor product of standard $[n-1, 1]$ representations



, (2.10)

which produces the singlet irrep $[n]$ in addition to another $[n-1, 1]$ irrep and the irrep $[n-2, 2]$. The contribution of the singlet $[n]$ vanishes due to momentum conservation, while the second leads to vanishing contributions due to the on-shell conditions $p_\mu p^\mu = 0$. We can therefore associate the Mandelstam invariants consisting of $p_i p_j$ to the irrep $[n-2, 2]$.

Now that we have identified the irrep corresponding to Mandelstam invariants, we note that the kinematic numerators N can be made out of any combinations of (different) powers p of these Mandelstam invariants. We therefore also need to find how the tensor product of $[n-2, 2]$ irreps decomposes.

Before we construct the tensor product of Mandelstam irreps, we note that there are certain numerators that satisfy the BCJ conditions, but which result in vanishing amplitudes. These were classified as 'gauge solutions' and have in common the property that they can be written as the product of Mandelstam variables $s_{i\dots j}$ and a specific irrep of S_n. The shape of this gauge irrep depends on the multiplicity of the amplitude. The example is given that at $n = 4$ the gauge numerator is denoted as

$$N_{abcd} = s_{ab} G_{abcd}, \tag{2.11}$$

where G_{abcd} is a fully anti-symmetric tensor which transforms according to the adjoint $[1, 1, 1, 1]$ irrep of S_n. For $n = 4$ through $n = 7$ the irreps that give rise to a vanishing amplitude are denoted in Table 2.2.

2.1.3 Powers of Mandelstam Invariants

The construction of kinematic BCJ numerators has been translated into a representation theory problem by investigating the irreps that arise from tensor products of the $[n-2, 2]$

n	Gauge irreps
4	$[1,1,1,1]$
5	$[2,2,1]$
6	$[3,2,1]$, $[3,1,1,1]$
7	$[4,3]$, $[4,2,1]$, $[4,1,1,1]$, $[3,2,2]$, $[3,2,1,1]$, $[3,1,1,1,1]$, $[2,2,2,1]$

Table 2.2: Irreps of S_n that give rise to a vanishing (gauge) contribution to the amplitude [62].

irreps. This product can be taken p times, corresponding to taking a Mandelstam variable to the power p:

$$s^p \cong \left(\begin{ytableau} \; & \; & \none[\cdots] & n-2 \\ \; & \; \end{ytableau} \right)^p = \begin{ytableau} \; & \; & \none[\cdots] & n-2 \\ \; & \; \end{ytableau} \otimes \cdots \otimes \begin{ytableau} \; & \; & \none[\cdots] & n-2 \\ \; & \; \end{ytableau} . \tag{2.12}$$

The specific calculation of these irrep products starts at $n = 4$ due to the fact that there exist no Mandelstams for $n = 3$ scalars. For $n = 4$ and $n = 5$ the decomposition of the product has been shown to obey a systematic classification, as we will see shortly. However, this classification does not prove useful for $n \geq 6$ due to intricate relations between invariants referred to as syzygys[2] [63, 64]. Nevertheless, by imposing that higher multiplicity amplitudes are bootstrappable from 4-point interactions it remains possible to characterize these amplitudes through powers of Mandelstam invariants.

2.2 Numerators for 4-point Interactions

2.2.1 Linear Order s^1 BCJ parameter

Out of the four external momenta p_i there are only 3 independent momenta. These momenta live in the $[3,1]$ representation of S_4. The Mandelstam invariants then live in the $[2,2]$ irrep, which is the so-called "window" irrep. Upon comparison of this irrep to the irreps of S_4 that are compatible with the BCJ constraints in Table 2.1 it becomes clear that the $[2,2]$ irrep allows for BCJ compatible expression of Mandelstam invariants linear order s^1. The combination of Mandelstams given by

$$N_4^{(1)} = N_{abcd}^{(1)} = s_{bc} - s_{ac} = t - u, \tag{2.13}$$

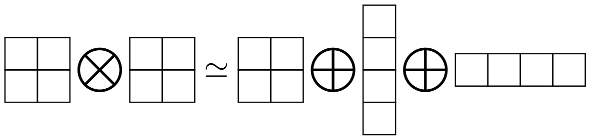
is linear in Mandelstam variables and satisfies the same BCJ constraints as the colour factors. At the level of the amplitude, this kinematic numerator will give rise to exchange

[2]A mathematical description of syzygys goes beyond the scope of this work, but a heuristic argument of such relations would be that, as you increase the number of momentum vectors at some point not all of the vectors are linearly independent. The point at which this happens would depend on the number spacetime dimensions D that these momenta live in. When the momenta are no longer linearly independent there are relations between momenta that, in the context of Mandelstams s_{ij} could result in complicated relations between invariants.

diagrams, it is therefore called the 'exchange factor'. This can be seen by the the fact that the amplitude $A \sim N/D$ will have $D \sim s^1$ for any trivalent 4 point amplitude. Explicitely, the s-channel exchange diagram will give rise to an amplitude proportional to $\frac{t-u}{s}$. We will explore the construction of amplitudes from the numerators we find in this section in more detail in section 2.5.

2.2.2 Quadratic Order s^2 BCJ Parameter

At quadratic order in Mandelstams, s^2, we need to take the symmetric product of 2 $[2,2]$ irreps. This product decomposes as



(2.14)

Here the [1,1,1,1] irrep is again the gauge factor which does not contribute to the four-point amplitude. The [4] irrep is the the trivial representation which does not satisfy the BCJ conditions. This leaves us with another contribution by the $[2,2]$ window again. We already saw that this is compatible with the BCJ conditions, therefore we can denote

$$N_4^{(2)} = s_{ab}(s_{bc} - s_{ac}) = s(t-u) \tag{2.15}$$

as the quadratic order factor. Note that it is the same as the exchange factor of (2.13) multiplied by a propagor of the half ladder diagram. It therefore gives rise to contact interactions at the level of the amplitude as we will see in section 2.5.

2.2.3 Higher-Order Mandelstam Invariants: Series Expansions

At higher orders in p the $[2,2]$ window irrep will continuously appear as the only contributing irrep in the decomposition of the product of irreps. For this reason, a pattern occurs; the BCJ factor N can be written as the product of an invariant Mandelstam expression and one of the two building blocks $N_4^{(1)}$ and $N_4^{(2)}$.

It is not immediately clear how often the BCJ irrep will appear in the irrep decomposition. We can use the Molien series given by

$$M(x) = \frac{1}{n!} \sum_{g \in S_n} \det\left(1 - xg|V^*\right)^{-1}, \tag{2.16}$$

to count the number of invariants for a specific order [65]. Here the term $1/n!$ comes from the cardinality of S_n, which is $n!$. We therefore sum over the determinants of the $(1-xg)$ for some complex representation V of S_n.

The expression will result in a product of factors $\left(1-x^{b_i}\right)$ in the denominator due to the inverse determinant,

$$M(x)=\frac{1}{\left(1-x^{b_1}\right)\cdots\left(1-x^{b_r}\right)}, \tag{2.17}$$

which can be expanded using the properties of the geometric series to obtain a polynomial whose coefficients a_i correspond to the number of independent symmetric polynomials in n variables.

$$M(x)=\sum_{n=0}^{\infty} a_{i,n}x^n = a_0 + a_1 x + a_2 x^2 + \dots . \tag{2.18}$$

This then corresponds to the number of invariants of S_n at a specific order in irrep products.

Returning to our 4-point interactions, for S_4 the Molien series is denoted as

$$M_4(x)=\frac{1}{\left(1-x^2\right)\left(1-x^3\right)}. \tag{2.19}$$

From (2.19) we can already heuristically infer that there are two 'primary' invariants of order x^2 and x^3 out of which higher order invariants can be made. These are primary, because when $M(x)$ is expanded through use of a geometric series (2.18), we will find new invariants at higher orders in primary invariants

$$M_4(x)=\frac{1}{\left(1-x^2\right)\left(1-x^3\right)}=\left(1+a_2x^2+a_4x^4+a_6x^6+\dots\right)\left(1+b_3x^3+b_6x^6+\dots\right) \tag{2.20}$$

$$= 1+a_2x^2+b_3x^3+a_4x^4+a_2b_3x^5+(a_6+b_6)x^6+\dots . \tag{2.21}$$

The two primary invariants are referred to as the 'quadratic' and 'cubic' primary invariants, $I_4^{(2)}$ and $I_4^{(3)}$ respectively. In terms of Mandelstams, these are given by

$$I_4^{(2)} = s_{ab}s_{bc}+s_{ac}s_{bc}+s_{ab}s_{ac} = st+ut+su, \tag{2.22}$$

$$I_4^{(3)} = s_{ab}s_{ad}s_{ac} = stu. \tag{2.23}$$

Specifically, the number of times the window irrep $[2,2]$ occurs at any order in powers of Mandelstams is given by [64]

$$H_4^{BCJ}(x)=\left(x+x^2\right)M_4(x)=\frac{\left(x+x^2\right)}{\left(1-x^2\right)\left(1-x^3\right)}, \tag{2.24}$$

which in turn implies that all BCJ invariants are written as $N_4^{(1)}$ or $N_4^{(2)}$ multiplied by any combination of primary invariants $I_4^{(2)}$ and $I_4^{(3)}$.

2.2.4 Gauge Parameters at 4-point

Similar to the BCJ window irrep factor, we can calculate the number of times the gauge parameter irrep $[1,1,1,1]$ occurs in the expansion. This is given by the Hilbert series

$$H_4^{\text{Gauge}}(x) = x^3 M_4(x) = \frac{x^3}{(1-x^2)(1-x^3)}. \tag{2.25}$$

The coefficients of the expansion then count the number of gauge parameters at one order higher. In this Hilbert series we recognise the two primary invariants and a cubic secondary invariant. Explicitely, the higher order combination

$$2N_4^{(2)} I_4^{(2)} - 3N_4^{(1)} I_4^{(3)}, \tag{2.26}$$

is of the order of s^4. In terms of Mandelstams this term can be written as

$$\begin{aligned} 2N_4^{(2)} I_4^{(2)} - 3N_4^{(1)} I_4^{(3)} &= 2s(t-u)(st+ut+su) + 3(t-u)(stu) \\ &= s(t-u)(5tu + 2s(t+u)). \end{aligned} \tag{2.27}$$

This term satisfies the BCJ conditions, but when we construct amplitudes from this numerator we will find that it vanishes.

2.3 Numerators for 5-point Interactions

The Mandelstam invariants live in the $[3,2]$ irrep. A quick glance at Table 2.1 reveals that there are therefore no BCJ compatible irreps at the linear order in mandelstam for interactions of 5 scalar legs. At quadratic order we find the decomposition of the product of Mandelstams [66]

$$[3,2] \otimes [3,2] \cong [5] + [4,1] + 2[3,2] + [3,1,1] + 2[2,2,1] + [2,1,1,1]. \tag{2.28}$$

Therefore the matching BCJ irrep is 'hook' tableau$[3,1,1]$, which is present in the decomposition. It turns out that the gauge parameter irrep $[2,2,1]$ occurs twice in this decomposition. For higher order invariants the Hilbert series are given by

$$H_5^{\text{Inv}}(x) = \left(1 + x^6 + x^7 + x^8 + x^9 + x^{15}\right)/D_5(x) \tag{2.29}$$

$$H_5^{\text{BCJ}}(x) = \left(x^3 + 2x^4 + 4x^5 + 5x^6 + 6x^7 + 6x^8 + 5x^9 + 4x^{10} + 2x^{11} + x^{12}\right)/D_5(x) \tag{2.30}$$

$$H_5^{\text{Gauge}}(x) = (x^2 + x^3 + 3x^4 + 3x^5 + 3x^6 + 4x^7 + 4x^8 + 3x^9 + 3x^{10} + 3x^{11} + x^{12} + x^{13})/D_5(x) \tag{2.31}$$

$$D_5(x) = \left(1-x^2\right)\left(1-x^3\right)\left(1-x^4\right)\left(1-x^5\right)\left(1-x^6\right). \tag{2.32}$$

As before, the denominator terms $\left(1-x^{b_i}\right)$ indicate the presence of primary invariants and the numerator terms point to the presence of secondary invariants. Therefore at 5-point there are primary invariant from quadratic order to quintic order for H_5^{Inv}. The sextic order invariant appears in both the denumerator and numerator, pointing to the syzygys at this order that were mentioned before.

Such relations also occur for H_5^{BCJ}, but it is stated that there are 29 independent $[3,1,1]$ hooks which generate all BCJ compatible solutions. Other hooks that appear in the Molien series can be written as products of these independent hooks with primary and secondary inariants. For H_5^{Gauge} it is argued that due to the fact that the $[2,2,1]$ irrep shows up multiple times in the decomposition of the product of BCJ irreps, the number of distinct irreps at one order higher is given by:

$$H_5^{\text{Gauge, } dist} = \left(x^3 + x^4 + 2x^5 + 3x^6 + 2x^7 + 3x^8 + 4x^9 + 3x^{10} + 2x^{11} + 2x^{12} + x^{13}\right)/D_5(x). \tag{2.33}$$

The resulting series for the number of physical BCJ parameters at every order is then given by

$$H_5^{\text{Phys}}(x) = \frac{(x^4 + 2x^5 + 2x^6 + 4x^7 + 3x^8)}{D_5(x)} + \frac{(x^9 + x^{10})}{(1-x^2)(1-x^4)(1-x^5)(1-x^6)}, \tag{2.34}$$

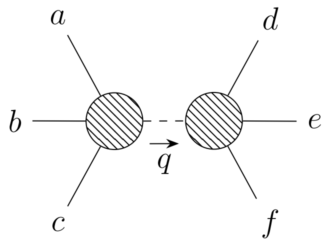
which presents 14 different hook structures, out of which 13 are independent as one of them can be written as the product of a secondary invariant. These 13 hook structures can be used as seed interactions for 5-point amplitudes.

2.4 Numerators for 6-point and Higher: BCJ Bootstrap

The Molien series methodology of 4-point and 5-point interactions can in theory be used to construct the independent BCJ structures for 6-point interactions, but in practice this is a cumbersome approach. For starters, there are 3 possible BCJ structures are 6-point and their Molien series contain terms up to $\mathcal{O}\left(x^{24}\right)$ in the numerator. This would require explicit irrep product decompositions of these secondary invariants with products of 24 BCJ irreps. While the calculation of the irreps is cumbersome, the counting of the parameters was done by [19] and is shown in Table 2.3.

Alternatively, it is much more convenient to consider the 6-point amplitudes which can be constructed out of 4-point seed interactions similar to the approach of [67]. This utilises the factorization property of the amplitude discussed in section 1.3, which can be schematically depicted as

$$A_6 = \sum_{s_{abc}}$$



$$\Longrightarrow A = \sum_{s_{abc}} A_L \frac{1}{s_{abc}} A_R. \tag{2.35}$$

p	BCJ	Gauge	Phys	Inv	Bootstrap
1	1	0	1	0	0
2	3	0	3	1	1
3	9	1	8	2	2
4	23	5	18	4	3
5	54	16	38	6	8
6	121	42	79	13	24
7	246	95	151	19	53

Table 2.3: Overview of the classification of 6-point irrep counting for the BCJ, gauge, physical and bootstrap-compatible parameters up to $\mathcal{O}(s^7)$ [62].

Such a factorisation condition can then be imposed on the 6-point numerator, giving rise to the following condition for the BCJ numerator at order s^p

$$\lim_{s_{abc}\to 0} N_6^{(p)} = \sum_q c_q N_4^{(p-q)}(abcx) N_4^{(p+q)}(xdef), \tag{2.36}$$

where the summation runs over all possible factorization combinations as s^p can be written as a product of s^{p-q} with s^q, c_q is some coefficient specific to the channel and x denotes the internal propagator.

2.4.1 Quadratic Order Numerator

The first possible 6-point BCJ factor will consist of a product of two linear $N_4^{(1)}$ and is therefore of $\mathcal{O}\left(s^2\right)$ denoted by:

$$\lim_{s_{abc}\to 0} N_6^{(2)} = N_4^{(1)} N_4^{(1)} = (s_{ac} - s_{bc})(s_{de} - s_{df}). \tag{2.37}$$

The general numerator that satisfies this factorization is the above expression plus an expression of the form s_{abc} multiplied by a factor L_{abcdef} that is linear in all 15 possible Mandelstams. Later when we calculate our amplitudes we can express these into the basis of 9 independent Mandelstam invariants. We can make a generic ansatz for such a linear factor as

$$L_{abcdef} = c_1 s_{ab} + c_2 s_{ac} + \cdots + c_{14} s_{df} + c_{15} s_{ef}. \tag{2.38}$$

We then require that L_{abcdef} satisfies the generalised Jacobi identity conditions of (2.3) which impose

$$-L_{abcdef} = L_{eabcdf} - L_{ebacdf} - L_{ecabdf} + L_{ecbadf} \tag{2.39}$$

$$- L_{edabcf} + L_{edbacf} + L_{edcabf} - L_{edcbaf}, \tag{2.40}$$

$$-L_{abcdef} = L_{fedcba}. \tag{2.41}$$

This system of equations can be solved the coefficients c_i, which reduces the system to one overall coefficient, e.g. c_5, for four of the 15 Mandelstams while the rest of the coefficients vanish. This leads to the reduced expression for the linear term

$$L_{abcdef} = -c_5(s_{ae} - s_{af} - s_{be} + s_{bf}). \tag{2.42}$$

Through this term, the quadratic 6-point amplitude is thereby uniquely fixed with the numerator

$$N_6^{(2)} = (s_{ac} - s_{bc})(s_{de} - s_{df}) + \frac{1}{2}(s_{ae} - s_{af} - s_{be} + s_{bf}). \tag{2.43}$$

2.4.2 Cubic Order Numerator

Next, we turn our attention to the numerator at $\mathcal{O}\left(s^3\right)$. While $N_6^{(2)}$ is unique, this is not the case for $N_6^{(3)}$ and $N_6^{(4)}$ as is shown in Table 2.3. In this case there are two possible combinations that contribute through factorization, explicitly

$$\lim_{s_{abc}\to 0} N_6^{(3)} = c_1 N_4^{(2)} N_4^{(1)} + c_2 N_4^{(1)} N_4^{(2)} = c_1(s_{ac} - s_{bc})s_{ab}(s_{de} - s_{df}) + c_2(s_{ac} - s_{bc})(s_{de} - s_{df})s_{ef}. \tag{2.44}$$

The solution for $N_6^{(3)}$ will be the above factor plus s_{abc} multiplied by some factor that is a linear combination of quadratic Mandelstams denoted by Q_{abcdef}. We can construct a generic ansatz for Q_{abcdef} as a sum over all possible products of s_{ij}

$$Q_{abcdef} = c_{1,1}s_{ab}s_{ab} + c_{1,2}s_{ab}s_{ac} + \cdots + c_{15,15}s_{ef}s_{ef}, \tag{2.45}$$

where the coefficients are related to elements of our basis of Mandelstams $\{s_{ab}, s_{ac}, \ldots, s_{ef}\}$ are multiplied together.

After constraining the ansatz with the generalised Jacobi identities of (2.3) we find that there are a total of 3 unfixed parameters, $c_{5,9}$, $c_{6,6}$ and $c_{6,8}$. We can further constrain this equation by imposing a soft limit of $\sigma = 0$ or $\sigma = 1$ on the amplitude.

The final generalised numerator $N_{abcdef}^{(3)}$ with $\sigma = 0$ imposed is then given by the expression

$$N_{abcdef}^{(3)} = (s_{ac} - s_{bc})s_{ab}(s_{de} - s_{df}) + (s_{ac} - s_{bc})(s_{de} - s_{df})s_{ef} + s_{abc}Q_{abcdef}. \tag{2.46}$$

These numerators can then be used to construct amplitudes, as we will see shortly in section 2.5.

2.4.3 Quartic and Higher Order Numerators

Finally at $\mathcal{O}\left(s^4\right)$ the factorization can be denoted as

$$\lim_{s_{abc}\to 0} N_6^{(4)} = c_1 N_4^{(3)} N_4^{(1)} + c_2 N_4^{(2)} N_4^{(2)} + c_3 N_4^{(1)} N_4^{(3)}. \tag{2.47}$$

We recall that the BCJ numerators can be constructed out of linear and quadratic invariant building blocks, therefore higher order numerators such as $N_4^{(3)}$ can be constructed from the building blocks $I_4^{(2)}$ and $I_4^{(3)}$. However it is stated that higher order combinations only factorize in the combinations $N_4^{(1)}(I_4^{(3)})^n$ and $N_4^{(2)}I$, where I is an invariant of arbitrary power. It has not been verified that this behaviour continues at higher order in Mandelstam. For this reason the terms with $N_4^{(3)} = N_4^{(1)}I_4^{(2)}$ do not contribute to the numerator of the quartic order. The resulting expression is therefore

$$\lim_{s_{abc}\to 0} N_6^{(4)} = N_4^{(2)}N_4^{(2)} = c_2(s_{ac} - s_{bc})s_{ab}(s_{de} - s_{df})s_{ef}. \tag{2.48}$$

We can use a similar method to finding the full expression of the numerator as for the cubic numerators; starting with an ansatz that is cubic in Mandelstam invariants and then constraining the coefficients of this ansatz using the BCJ conditions and imposing the soft degree of $\sigma = 1$ to fully determine the theory.

We have constructed the numerators $N_6^{(2)}$, $N_6^{(3)}$ and $N_6^{(4)}$, which contain the three possible combinations of $N_4^{(1)}$ and $N_4^{(2)}$. Due to the property that the higher-order numerators at 4-point are built up out of these building blocks, we can construct higher-order $N_6^{(p)}$ recursively.

2.5 Colour x Kinematics: The Gauged NLSM

Now that we have found expressions for kinematic numerators at 4- and 6-point that satisfy the BCJ constraints, we return to the starting point of the discussion. The aim is to use the constructed numerators to build amplitudes. Using the amplitude decomposition

$$A_n = \sum_{\text{trivalent}} \frac{N\tilde{N}}{D}, \tag{2.49}$$

amplitudes for different theories can be considered by filling in N and $\tilde{N}$ with colour and (or) kinematic numerators.

We will consider the single copy amplitude for a gauged scalar field, where one numerator describes a colour factor C and the other a kinematic numerator N. The scalar Goldstone field ϕ^a transforms according the adjoint representation a of gauge group G, whose generators give rise to the colour factor C.

We will see that by imposing the soft limit $\sigma_{min} = 0$ and $\sigma_{max} = 1$ at 6-point, the resulting amplitudes describe amplitudes of the gauged NLSM pions interacting with gluons.

2.5.1 4 Point Interactions

Starting with the 4-point kinematic numerator in terms of (2.13) and (2.15) we see that the total amplitude of the theory separates into two terms of different powers in Mandelstam invariants

$$A_4^{\text{GNLSM}} = \sum_{\text{trivalent}} \frac{C_4 N_4}{D} = \sum_{\text{trivalent}} \frac{C_4 N_4^{(1)}}{D} + \sum_{\text{trivalent}} \frac{C_4 N_4^{(2)}}{D}. \tag{2.50}$$

We will treat these terms seperately, beginning with the term 'linear term' generated by $N_4^{(1)}$. This term generates the three possible 4-point gluon exchange channels:

$$\sum_{trivalent} \frac{C_4 N_4^{(1)}}{D} = C_4^s \frac{(t-u)}{s} + C_4^t \frac{(u-s)}{t} + C_4^u \frac{(s-t)}{u}, \tag{2.51}$$

with $C_4^s = \tilde{f}^{abx}\tilde{f}^{xcd}$, $C_4^t = \tilde{f}^{bcx}\tilde{f}^{xad}$ and $C_4^u = \tilde{f}^{cax}\tilde{f}^{xbd}$ satisfying the usual Jacobi identity for colour factors. The Feynman diagrams depicting this exchange are shown in Figure 2.2.

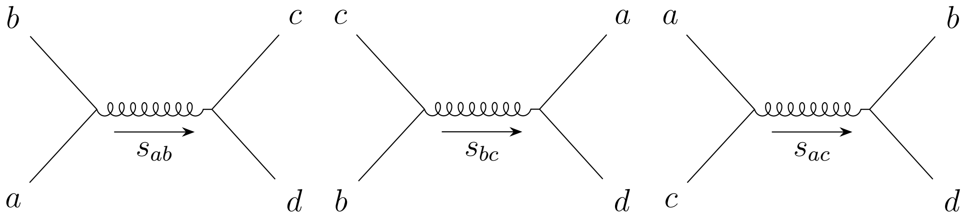


Figure 2.2: The three distinct gluon exchange channels produced by $N_4^{(1)}$.

The derivative dependence of the amplitudes can be used to determine a Lagrangian that is able to produce these exchange diagrams. Firstly, the numerator scales as $N_4^{(1)} \sim s^1 \sim p^2$, which indicates the contribution of a $(\partial\phi)^2$ term. The interaction Lagrangian that is able to produce the diagrams of Figure 2.2 can then be denoted as

$$\mathcal{L}_{int}^{(1)} \sim f^2 \phi(\partial\phi) A_\mu. \tag{2.52}$$

Since we are considering a theory of gauged scalars, we will later use the covariant derivative $(D_\mu\phi)_a = \partial_\mu\phi_a - igf^{abc}A_\mu^b\phi^c$ in order for the total Lagrangian to be gauge invariant.

Equation (2.51) can be rewritten by using $t = -s - u$. After this, we can set $p_a \to \lambda p_a$, recalling that $s = 2p_a p_b$ and $u = 2p_a p_c$ to find

$$-\frac{C_4^s \lambda(s+2u)}{\lambda s} + \frac{C_4^t \lambda(s-u)}{\lambda(s+u)} + \frac{C_4^u \lambda(2s+u)}{\lambda u}, \tag{2.53}$$

where it is clear that the dependence on λ cancels. This indicates a soft degree of $\sigma = 0$. Identical analysis can be done for the other external legs.

Next, we turn our attention the quadratic numerator $N_4^{(2)}$. The contribution to the total amplitude in (2.50) generated by this numerator is given by

$$\sum_i \frac{C_4^i N_4^{(2)}}{D_i} = C_4^s \frac{s(t-u)}{s} + C_4^t \frac{t(u-s)}{t} + C_4^u \frac{u(s-t)}{u} \tag{2.54}$$

$$= C_4^s(t-u) + C_4^t(u-s) + C_4^u(s-t). \tag{2.55}$$

Remarkably, the exchange channels have been removed through the introduction of the addition order in Mandelstams, leaving us with only contact interactions. The Feynman diagram for such a contact term is denoted in Figure 2.3.

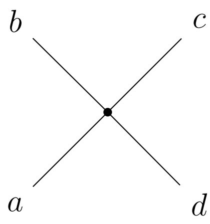


Figure 2.3: The contact term produced by $N_4^{(2)}$. The other two contact terms can be found through cyclic permutations of a, b and c.

The introduction of the additional factor of s_{ab} that cancels the gluon exchange in the amplitude indicates an extra derivative coupling at the Lagrangian level. From this, we can deduce that the contact interaction term can be associated with the interaction term

$$\mathcal{L}_{int}^{(2)} \sim f^2\phi^2(\partial\phi)^2, \tag{2.56}$$

which corresponds to the a familiar 4 point interaction of the NLSM, with the addition of structure constants that indicate the colour structure similar to (1.41). Further confirmation of NLSM-like behaviour can be seen from from the fact that the amplitude contains the Adler zero property, as the amplitude generated by $N_4^{(2)}$ is linear in all four momenta p_i, indicating a soft degree of $\sigma = 1$.

Both scalar interactions can be combined into the total Lagrangian

$$\mathcal{L}_4^{\text{GNLSM}} = -\frac{1}{2}(D\phi)^2 - \frac{1}{4}F^2 + \frac{1}{6}f^2\phi^2(D\phi)^2, \tag{2.57}$$

which includes the free Lagrangians of both the adjoint scalars and the gauge fields. This theory then satisfies gauge invariance, the nonlinear shift symmetry of the NLSM and produces CK-duality satisfying amplitudes at 4-point. The specification *at four point* will become apparent when we analyse the the 6-point amplitudes, where we will need to add additional terms in order to produce a theory that satisfies these same properties, which is familiar from Equation 1.5.3. The difference here is that only a finite number of correction terms are needed.

2.5.2 6-Point Interactions

We can go one step further and describe the amplitudes generated by the six-point numerators of section 2.4. The amplitude can then be constructed as

$$A_6^{\text{GNLSM}} = \sum_{\text{trivalent}} \frac{C_6 N_6}{D} = \sum_{\text{trivalent}} \frac{C_6\left(N_6^{(2)} + N_6^{(3)} + N_6^{(4)}\right)}{D}. \tag{2.58}$$

When we are considering cubic interactions, there are two types of diagram structures that can occur at 6-point. The first is the half-ladder diagrams we encountered in our discussion of gluon amplitudes and the second are 'snowflake' diagrams, both types of diagramsare displayed in Figure 2.4. There are a total of $(2 \cdot 6 - 5)!! = 105$ possible cubic diagrams at 6-point. Out of these, there are 90 half-ladder diagrams and 15 cubic diagrams [60]. The snowflake diagrams have distinctly different amplitude contributions. For starters, both

Figure 2.4: Half-ladder and snowflake diagram for a 6-point amplitude.

types of diagrams have denominators $D \sim s^3$, but they are made up of different interal propogators. The half-ladder diagrams will have propagators of the form

$$D_{\text{HL}} = \frac{1}{s_{ab}s_{abc}s_{abcd}}, \tag{2.59}$$

where it is possible to use momentum conservation to relate $s_{abcd} = s_{ef}$. The snowflake diagrams have denominators of the form

$$D_{\text{snowflake}} = \frac{1}{s_{ab}s_{cd}s_{ef}}. \tag{2.60}$$

Secondly, the colour factors for both diagram types are product of 4 structure constants, e.g. $C_6^{HL} = \tilde{f}^{abx}\tilde{f}^{xcy}\tilde{f}^{ydz}\tilde{f}^{zef}$ and $C_6^{snowflake} = \tilde{f}^{abx}\tilde{f}^{xyz}\tilde{f}^{ycd}\tilde{f}^{zef}$ for the diagrams in Figure 2.4.

Finally, we turn our attention to the numerators of the two diagrams. Initially, one might think that our assumption of the factorization of the numerator N_6 into a product of four-point numerators N_4N_4 does not allow for snow-flake diagrams, as these cannot be factorised in this way. However, the numerators of the snowflake diagrams are related to the numerators of the half-ladder diagrams through Jacobi identities by

$$N^{\text{snowflake}}_{abcdef} = N^{\text{half-ladder}}_{abcdef} - N^{\text{half-ladder}}_{abdcef}, \tag{2.61}$$

The total amplitude can then be said to be a sum over half-ladder and snowflake diagrams

$$A_6 = \sum_{\text{half-ladder}} \frac{N_{abcdef}C^{\text{hl}}_{abcdef}}{s_{ab}s_{abc}s_{ef}} + \sum_{\text{snow-flake}} \frac{(N_{abcdef} - N_{abdcef})C^{\text{snowflake}}}{s_{ab}s_{cd}s_{ef}}. \tag{2.62}$$

Now that we have determined how to construct amplitudes from the numerators, we will investigate the properties of these amplitudes. The quadratic numerator gives rise to amplitudes of the form

$$\sum_{\text{trivalent}} \frac{C_6 N_6^{(2)}}{D}, \tag{2.63}$$

which are completely fixed by the BCJ conditions on the numerators. In terms of degree of Mandelstams these amplitudes scale as s^{-1} due to the cubic propagators of (2.59) and (2.60). Due to the pole structure of these amplitudes, we can identify them with gluon exchange diagrams similar to Figure 2.2.

The cubic numerators had unfixed constraints after requiring factorization, but these are fixed by imposing the soft degree $\sigma = 0$ on the amplitudes.

Finally, the quartic numerators $N_4^{(3)}$ are completely fixed by imposing the soft degree $\sigma = 1$, leading to the 6-point NLSM pion contact amplitude, whose diagram is shown in Figure 2.5.

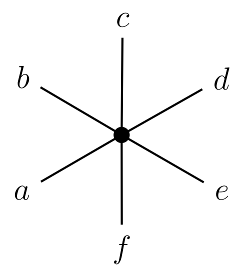


Figure 2.5: The 6-piont GNLSM contact diagram produced by $N_6^{(4)}$.

As indicated before, we need to add addition terms to the Lagrangian in order to generate the above amplitudes. Requiring the conditions of gauge inariance, nonlinear shift symmetry and BCJ compatibility, the Lagrangian can be denoted as

$$\mathcal{L}_6 = \mathcal{L}_4 + \frac{1}{45} f^4 \phi^4 (D\phi)^2 - 2f^2 F^3 + \frac{1}{6} f^2 \phi^2 F^2. \tag{2.64}$$

The coefficients of these additional terms are tuned to satisfy the symmetry conditions.

The new terms include the higher derivative F^3 operator discussed in Equation 1.5.3, the six-pion contact term $\phi^4(\partial\phi)^2$ and a new type of interaction between gluons and pions through $f^2\phi^2F^2$. This term generate a 5-point interaction of the form $\phi^2(\partial A)\phi$ that does not contribute to the 6-point amplitude, a 6-point amplitude of the form $\phi^2 A^4$ that can not contribute to a 6-point amplitude with only scalar external legs at tree level, and a 4-point interaction of the form $\phi^2(\partial A)^2$. This last interaction can contribute to the amplitudes through diagrams of the form shown in Figure 2.6. However, these amplitudes do not conform to the requirement of factorising in 4-point amplitudes on $s_{abc} \to 0$.

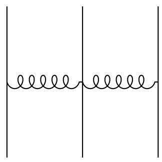


Figure 2.6: The additional 6-point interaction type generated by the interaction $f^2\phi^2F^2$ that contributes to A_6.

2.5.3 Higher Multiplicities

Moving on to higher-point amplitudes, it is expected that the above pattern continues; the Lagrangian will consist of the lower point Lagrangians plus additional terms. This is still subject to current research by Yang Li and colleagues and could potentially be an interesting thesis topic for future students reading this work.

As we have seen at 4- and 6-point, the numerator with lowest order in Mandelstam generates the gluon exchange channel and the highest order in Mandelstam generates the pion contact term (after imposing the soft limit). For general (even) multiplicity n, it is expected that the numerator of $\mathcal{O}(s^{(n-2)/2})$ will generate the gluon exchange channels, while the term with the highest order in derivatives, which is $\mathcal{O}(s^{n-2})$ generates the pion contact term. The expected pattern of degrees of s^p is depicted in Table 2.4.

			n		
p	4	6	8	10	12
1	GE				
2	PC	GE			
3		x	GE		
4		PC	x	GE	
5			x	x	GE
6			PC	x	x
7				x	x
8				PC	x
9					x
10					PC

Table 2.4: Schematic depiction of the amplitudes generated at every order in Mandelstam s^p. The gluon exchange (GE) and pion contact (PC) terms follow the pattern of the soft degree of the amplitude. The gluon exchange amplitudes all obey $\sigma = 0$ while the pion contact terms obey the Adler zero of $\sigma = 1$. The amplitudes with ambiguous soft degree are in between these lines, and are denoted by x's.

We would like to emphasise the fact that the above Lagrangian describes a theory of gauged NLSM pions amplitudes with a *hybrid soft degree*. This is quite unusual compared to ‘regular’ theories such as YM and the NLSM, where the entire theory has a singular fixed soft degree. Instead, the theory described by (2.64) has a minimum soft degree of $sigma_{min} = 0$, while a section of the theory has $\sigma = \sigma_{min} + 1$.

2.6 Key Findings and Implications

The initial goal of this chapter was to find a general expression for colour-kinematics (CK) duality satisfying numerators in scalar theories. We explored the existence of such numerators from a group theory perspective and successfully constructed the 4-point numerators, which served as the building blocks for higher-multiplicity numerators. In the final sections, we investigated the amplitudes arising from the single copy of these numerators.

These amplitudes can be interpreted as those of the Gauged Nonlinear Sigma Model (GNLSM) with lower derivative corrections, generated by a theory exhibiting a hybrid soft degree. The theory and the resulting amplitudes are highly non-trivial, and the correction terms required in the Lagrangian are also uncommon.

This raises an interesting question about the uniqueness of these amplitudes and whether alternative Lagrangians could yield similar results. Addressing this question is key to better understanding the underlying structures and seeing if there are other theories that can produce the amplitudes we discussed in this chapter.

To further our investigation, we will now turn our attention to a different method for generating CK-duality satisfying amplitudes: Unifying Relations. This approach enables the generation of amplitudes for various interacting fields, including biadjoint scalars, pions, and gluons. In the following chapter, we will thoroughly explore this subject to determine if and how we can construct amplitudes akin to those produced by the BCJ bootstrap method discussed here. This exploration will provide further insights into the versatility and applicability of the CK-duality framework in scalar theories.

Chapter 3

Unifying Relations - Transmutation Operators

The approach of this chapter is based on the concept of 'Transmutation Operators', which was initially proposed by Clifford Cheung, Chia-Hsien Shen and Congkao Wen in [22]. In this work, the authors proposed a set of differential operators which 'transmute' tree-level scattering amplitudes of one type of theory, (a), into amplitudes of a different theory, (b). In an abstract schematic sense we can depict this as follows:

$$\mathcal{T} \cdot A_a \to A_b. \tag{3.1}$$

With these operators, a web of theories related through transmutation of amplitudes can be created. At the heart of this unified web are the scattering amplitudes of *extended gravity*, which describes gravitons coupled to a dilaton and two-form[1]. By applying different types of transmutation on these amplitudes, we can derive the amplitudes of several theories, including:

- Yang-Mills (YM): Gluon self interaction as describe in section 1.1.
- Maxwell: Photons described by Maxwell's equations.
- Born-Infeld (BI): An effective field theory describing a nonlinear generalisation of electromagnetism.
- Special Galileons (SG): Derivatively coupled scalar effective field theory characterised by an enhanced soft limit of $\sigma = 3$.
- Dirac-Born-Infeld (DBI): Derivatively coupled scalar effective field theory characterised by an enhanced soft limit of $\sigma = 2$.
- Nonlinear sigma-model (NLSM): Pion self interactions as described in section 1.2.

[1]The tensor product of two polarisation $e^i_\mu e^j_\nu$ can be decomposed into three combinations representing a graviton polarisation $e^h_{\mu\nu}$, a B-field polarisation $e^B_{\mu\nu}$ and the dilaton polarisation $e^\phi_{\mu\nu}$. Respectively, these are the symmetric-traceless, the antisymmetric, and the scalar part of the polarisation tensor. The theory describing these field coupled together is sometimes referred to as $\mathcal{N} = 0$ supergravity.

- Biadjoint scalar (BAS) theory: Scalars with two colour indices that arises in the zeroth copy of YM and GR in subsection 1.5.2.

During the process of transmuting amplitude we can find 'hybrid' or 'mixed' amplitudes of all of the above theories, for example

- Einstein-Yang-Mills (EYM); describing gluons coupled to (extended) gravitons,
- Einstein-Maxwell (EM); photons coupled to (extended) gravitons
- Yang-Mills-Scalar (YMS): gluons coupled to scalars

We will extensively discuss the construction of transmutation operators, starting with a discussion on the necessary conditions and the basis of operators that arise from this. Using this basis, we create specific combinations of operators that take us to the amplitudes we desire, e.g. pion amplitudes from gluon amplitudes and amplitudes of scalars interacting with gluons. We will use the 3- and 4-point gluon amplitudes as an example of this process. Finally, we discuss the transmutation of important properties of the amplitudes, such as the soft factors, factorisation, and UV completion.

While the Unifying Relations are incredibly useful relations for building the partial amplitudes of a wide web of theories, the physical meaning of transmutation is still unclear. Two of the operators we will construct, namely the 'trace' and the 'longitudinal' operator, can also be seen as a *dimensional reduction*, where the kinematics of the external legs of e.g. gluons can be constructed as higher-dimensional objects, which reduce to D-dimensional pions by construction. A discussion on this dimensional reduction can be found in section D.3.

3.1 Conditions and Basis for Transmutation Operators

The main reasoning in the construction of transmutation operators is that amplitudes are functions of Lorentz-invariant products of the external, on-shell, kinematic data; e_ie_j, p_ie_j and p_ip_j [40]. In this formulation we use a shorthand notation for summations over Lorentz-indices $e_ie_j = e_i \cdot e_j$ where $i \neq j$. With the idea that amplitudes are functions of these kinematic products, we can construct operators that take derivatives with respect to these objects. The next step is to consider that it is not very useful to build operators that trasnsmute amplitudes into nonphysical amplitudes which violate momentum conservation and gauge invariance. Therefore there are two conditions these operators need to satisfy:

1. Transmutation operators need to preserve the on-shell kinematics.
2. Transmutation operators need to preserve the gauge invariance.

These conditions can also be translated into differential operators. It can be shown that a vanishing commutator with respect to these differential operators then guarantees conditions 1 and 2. Here we will simply define these operators, and the basis of transmutation operators that follow from them. An in-depth discussion of how these operators satisfy the claims of condition 1 and 2 and how to form the basis can be found in Appendix D.

The momentum conservation can be enforced by defining a *total momentum operator* $\mathcal{P}_v$:

$$P_v \equiv \sum_i p_i v = (p_1 + p_2 + \ldots + p_n)v, \tag{3.2}$$

where the summation runs over all external legs i and v denotes any momentum or polarisation vector. The requirement for momentum conservation can then be cast into the following requirement:

$$[P_v, \mathcal{T}] = 0. \tag{3.3}$$

The condition of gauge invariance is translated into a differential operator through the Ward identity of (1.6). The differential Ward operator can therefore be defined as:

$$W_i \equiv \sum_v p_i v \partial_{v e_i}. \tag{3.4}$$

The summation over v runs over all external momentum and polarisation vectors in the amplitude. The polarisation of leg i appears in the amplitude in the Lorentz invariants $e_i p_j$ and $e_i e_j$. What this operator essentially does is replace every $e_i \to p_i$ wherever it appears in the amplitude, turning $e_i p_j \to p_i p_j$ and $e_i e_j \to p_i e_j$.

Any gauge invariant amplitude satisfies the Ward identity and is therefore annihilated by the Ward operator:

$$W_i \cdot A = 0. \tag{3.5}$$

Requiring that gauge invariance be preserved for a transmuted amplitude then implies a vanishing result when the commutator $[\mathcal{T}, W_i]$ acts on an amplitude:

$$[W_i, \mathcal{T}] = 0. \tag{3.6}$$

Starting from the most general form that a transmutation can take, we can apply our condition for momentum conservation and gauge invariance to constrain the general form to form basic elements for the transmutation operators. The operators take the form of first-order differential equations which act on the objects $e_i e_j$, $e_i p_j$ and $p_i p_j$. The general form can be written as:

$$\mathcal{T} \equiv \sum_{i,j} A_{ij} \partial_{p_i p_j} + B_{ij} \partial_{p_i e_j} + C_{ij} \partial_{e_i e_j}. \tag{3.7}$$

Here, A_{ij}, B_{ij} and C_{ij} are general functions of external kinematic data.

We then impose the constraint of momentum conservation upon this ansatz. In terms of transmutation operators this leaves a set of commuting operators which will form the basic building blocks for more complicated operators. The operators are denoted as

$$T_{ij} \equiv \partial_{e_i e_j}, \tag{3.8}$$
$$I_{ijk} \equiv \partial_{p_i e_j} - \partial_{p_k e_j}, \tag{3.9}$$
$$T_{ijkl} \equiv \partial_{p_i p_j} - \partial_{p_k p_j} + \partial_{p_k p_l} - \partial_{p_i p_l}. \tag{3.10}$$

These operators have the following symmetry properties:

$$T_{ij} = T_{ji}, \tag{3.11}$$
$$I_{ijk} = -I_{kji}, \tag{3.12}$$
$$T_{ijkl} = -T_{kjil} = T_{klij} = -T_{ilkj}. \tag{3.13}$$

Next, we turn our attention to the constraint of gauge invariance. We can see that T_{ij} is intrinsically gauge invariant, as $[T_{ij}, W_k] = \sum_v [\partial_{e_i e_j}, p_i v \partial_{v e_i}] = 0$.

Furthermore, we can see that I_{ijk} is not intrinsically gauge invariant. However, this does not pose a problem for as it can be shown to be *effectively* gauge invariant. The commutator is $[I_{ijk}, W_l] = \delta_{il} T_{ij} - \delta_{kl} T_{jk}$, which can be combined with supplementary operators that allow the vanishing of the amplitude.

The third operator strips of pairs of polarisation vectors and will therefore give rise to an object with double poles in momenta. The object generated after transmutation is therefore not a physical scattering amplitude. Regardless, this operator does appear in the calculation of the subleading soft factor of Born-Infeld theory.

In the next sections, we will see how these operators are the building blocks to connecting the amplitudes of various theories in a unified web.

3.2 Trace and Insertion Operators

3.2.1 Trace Operator $T_{ij} = \partial_{e_i e_j}$

First we will turn our attention to the effect of one T_{ij} operator on the amplitude: $T_{ij} \cdot A$. What this effectively does is strip the amplitude of the polarisations e_i and e_j except for the combination $e_i e_j$ which transforms as $e_i e_j \to 1$. Therefore, it reduces the spin of particles i and j by one. Moreover, these particles are placed within a new 'dual colour' trace structure, hence the name 'trace operator'. This trace structure will become more apparent with the next operator I_{ijk}.

The reduction of spin by one unit turns spin-2 particles into spin-1 particles, which in turn can be turned into spin-0 particles by applying another trace operator. For example, the trace operator transmutes gluons into biadjoint scalars:

$$T_{ij} \cdot A(\cdots, g_i, g_j, \cdots) = A(\cdots, \phi_i \phi_j, \cdots). \tag{3.14}$$

We should take a moment to unpack the notation used in this equation. The scalars ϕ_i and ϕ_j that have been created carry the original colour index from their origin as gluons, plus an additional 'dual colour index' obtained from the transmutation. This is denoted through the use of commas to indicate sets of particles which are ordered according to this dual colour. The gluons carry only the single colour index and are therefore all separated by commas.

Finally, the ellipses $\cdots$ denote any 'spectator' particles present in the scattering process which we leave untouched. Using this operator, at the level of the amplitude, we can transmute gravitons into photons, gluons into (biadjoint) scalars and BI photons into DBI scalars.

3.2.2 Insertion Operator: $I_{ijk} = \partial_{p_i e_j} - \partial_{p_k e_j}$

As mentioned before, the insertion operator I_{ijk} is not intrinsically gauge invariant. Luckily we can combine it with the trace operator to create a combination which is effectively gauge invariant. We calculate

$$[T_{ik} \cdot I_{ijk}, W_l] = \delta_{il} T_{ik} \cdot T_{ij} - \delta_{kl} T_{ik} \cdot T_{jk}, \tag{3.15}$$

which vanishes upon hitting the amplitude because of the multi-linearity of the amplitude in e_i and e_k. By multi-linearity we mean that the polarisation e_i only exists in the combination of one pair $e_i e_l$ for example.

This transmutation reduces the spin of the particle j by one unit and *inserts* this particle between the particle i and k in the dual-colour trace structure. At the level of colour-ordered amplitudes. This is denoted as

$$I_{ijk} \cdot A(\cdots \phi_i \phi_k \cdots, g_j, \cdots) = A(\cdots \phi_i \phi_j \phi_k \cdots, \cdots). \tag{3.16}$$

This operator then extends the dual colour ordering of the resulting amplitude. Repeated application of the insertion operator allows us to transmute all the remaining gluons into scalars and place them in the dual colour trace.

3.2.3 Combination of Trace and Insertion Operators $\mathcal{T}[\alpha]$

The repeated use of the insertion operator I_{ijk} after initial application of a trace operator T_{ik} can be denoted by use of the following *single-trace* operator. The name is earned by creating a new trace-ordering $[\alpha]$ in the resulting amplitude, this operator is defined as

$$T[\alpha] \equiv T_{\alpha_1 \alpha_n} \cdot \prod_{i=2}^{n-1} I_{\alpha_{i-1} \alpha_i \alpha_n}. \tag{3.17}$$

Here it is important to note when dealing with a product of transmutation operators such as $T[\alpha]$ we should read this as the operators acting from left to right: first we apply the trace operator to particles α_1 and α_n, then we insert particle α_2 between α_1 and α_n followed by particle α_3 between α_2 and α_n and so forth. Applied to a Yang-Mills amplitude we will find the following resulting amplitude

$$\mathcal{T}[i_1 \cdots i_n] \cdot A(g_{i_1}, \cdots, g_{i_n}, \cdots) = A(\phi_{i_1} \cdots \phi_{i_n}, \cdots). \tag{3.18}$$

Because this is an amplitude of biadjoint scalars interacting with gluons (denoted by the $\cdots$) through gauge interactions, the theory this amplitude corresponds to is called 'gauged biadjoint scalar theory'.

On the other hand, if α is the set of all gluons in the initial amplitude then all gluons are transmuted and the amplitude is purely a biadjoint scalar amplitude.

The operator can also be applied to extended gravity amplitudes to form amplitudes of Einstein Yang-Mills theory which describes gravitons coupled to gluons [68]:

$$\mathcal{T}[i_1 \cdots i_n] \cdot A(h_{i_1}, \cdots, h_{i_n}, \cdots) = A(g_{i_1} \cdots g_{i_n}, \cdots). \tag{3.19}$$

It is essential to note that the amplitudes that are created are the partial amplitudes of subsection 1.1.4, which do not contain the associated trace factor. As an example, the four-particle graviton amplitude does not have any colour structure and, therefore, does not have a decomposition into colour-stripped partial amplitudes. When we apply the single trace operators for a to this amplitude we create the colour-ordered amplitude $A[1234]$

$$T[1234] \cdot A_{\mathrm{G}}(h_{i_1}, h_{i_2}, h_{i_3}, h_{i_4}) = A_{\mathrm{YM}}[1234], \tag{3.20}$$

which in the trace decomposition of the full amplitude $\mathcal{A}_4^{\mathrm{YM}}$ would be accompanied by the factor $\mathrm{Tr}[T^{a_1}T^{a_2}T^{a_3}T^{a_4}]$. During the process of transmutation we do not add any colour factors, but we only act on and construct the (kinematic) partial amplitudes.

3.2.4 Multi-Trace Amplitudes

Naturally, the single-trace operator can be extended to multi-trace transmutation operators which generate multiple colour-trace structures in the resulting amplitudes. These operators and amplitudes are formed through the products of the operators $T[\alpha_i]$.

We denote an amplitude which is colour ordered with ordering β as $A(\beta)$. Such an amplitude can be constructed through the single trace operator of (3.17):

$$T[\beta] \cdot A = A(\beta). \tag{3.21}$$

Next we can introduce a dual colour-ordering by acting with $T[\alpha]$ on the amplitude such that

$$T[\alpha] \cdot A(\beta) = A(\alpha|\beta). \tag{3.22}$$

Furthermore we can act with multiple different trace operators to find

$$T[\alpha_1] \ \cdots T[\alpha_m] \ \cdot A(\beta) = A(\alpha_1, \ldots, \alpha_m|\beta). \tag{3.23}$$

A particularly interesting combination of trace operators consist of ordering all particles into pairs by acting with $\mathcal{T}[i_1 j_1] \cdots \mathcal{T}[i_m j_m] = \prod_{i,j \in pairs} T_{ij}$ on an amplitude such as a gluon amplitude:

$$\mathcal{T}[i_1 j_1] \cdots \mathcal{T}[i_m j_m] \cdot A(g_{i_1}, g_{j_1}, \cdots, g_{i_m}, g_{j_m}, \cdots) = A(\phi_{i_1}\phi_{j_1}, \cdots, \phi_{i_m}\phi_{j_m}, \cdots). \tag{3.24}$$

Due to the absence of insertion operators in this case, the dual colour ordering trace reduces to products of δ_{ij}'s corresponding to the pairs $[ij]$ through the structure constant identity $\mathrm{Tr}\left(T^a T^b\right) = \delta^{ab}$. This Yang-Mills amplitude is transmuted into a Yang-Mills-Scalar amplitude, describing gluons coupled to scalars.

3.2.5 Examples: Transmutation of YM into BAS and YMS aAplitudes.

Three Point Gluon Amplitude

As an example of the above transmutations, let us consider the three point colour-ordered partial Yang-Mills amplitude:

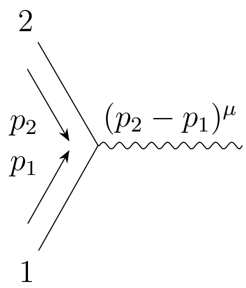
$$A(g_1, g_2, g_3) = \tfrac{1}{2} e_1 e_2 (p_2 e_3 - p_1 e_3) + \text{cyclic}(123), \tag{3.25}$$

where the colour ordering $\{123\}$ is implied for the gluon amplitude.

We can apply T_{ij} to the partial amplitude in order to turn a pair of gluons into scalars:

$$\begin{aligned} T[12] \cdot A(g_1, g_2, g_3) = T_{12} \cdot A(g_1, g_2, g_3) &= \frac{1}{2} \partial_{e_1 e_2} (e_1 e_2 (p_2 e_3 - p_1 e_3) + \text{cyclic}(123)), \\ &= A(\phi_1 \phi_2, g_3) = \tfrac{1}{2}(p_2 e_3 - p_1 e_3), \qquad (3.26) \\ T[23] \cdot A(g_1, g_2, g_3) = A(\phi_2 \phi_3, g_1) &= \tfrac{1}{2}(p_3 e_1 - p_2 e_1), \qquad (3.27) \\ T[31] \cdot A(g_1, g_2, g_3) = A(\phi_3 \phi_1, g_2) &= \tfrac{1}{2}(p_1 e_2 - p_3 e_2). \qquad (3.28) \end{aligned}$$

The transmuted amplitudes $A(\phi_i \phi_j, g_k)$ correctly correspond to the three-point amplitude of YMS theory, as can be seen by the vertex rule of YMS theory:



$$\propto g_{\mathrm{YM}} (p_2 - p_1)^\mu. \tag{3.29}$$

We can also transmute all three gluons and place them in a specific dual colour ordering of our choice to obtain a BAS amplitude:

$$T[123] \cdot A(g_1, g_2, g_3) = A(\phi_1\phi_2\phi_3) = 1, \tag{3.30}$$
$$T[321] \cdot A(g_1, g_2, g_3) = A(\phi_3\phi_2\phi_1) = -1. \tag{3.31}$$

Here, the anti-symmetry factor for the reverse ordering of the biadjoint scalar colour factors is correctly represented in the anti-symmetry of the BAS amplitude.

Four Point Gluon Amplitude

The four gluon amplitude with implied colour ordering $\{1234\}$ can be denoted as

$$A(g_1, g_2, g_3, g_4) = \frac{n_{[12][34]}}{p_1p_2} + \frac{n_{[23][41]}}{p_2p_3} - \frac{1}{2}[(e_1e_2)(e_3e_4) + (e_2e_3)(e_4e_1)] + (e_1e_3)(e_2e_4), \tag{3.32}$$

where $n_{[12][34]} = n_{1234} - n_{2134} - n_{1243} + n_{2143}$ and

$$n_{1234} = \frac{1}{4}(e_1e_2)(e_3e_4)(p_1p_3) - (e_2e_3)(p_3e_4)(p_2e_1) - \frac{1}{2}(e_1e_2)(p_2e_3)(p_1e_4) - \frac{1}{2}(e_3e_4)(p_4e_1)(p_3e_2). \tag{3.33}$$

Upon usage of the multi-trace operator which orders all gluons into pairs and transforms them into scalars we find the following three combinations of orderings into pairs for 4 gluons:

$$A(\phi_1\phi_2, \phi_3\phi_4) = T_{12} \cdot T_{34} \cdot A(g_1, g_2, g_3, g_4) = \frac{p_1p_3}{p_1p_2} = \frac{u}{s}, \tag{3.34}$$
$$A(\phi_1\phi_3, \phi_2\phi_4) = T_{13} \cdot T_{24} \cdot A(g_1, g_2, g_3, g_4) = 1, \tag{3.35}$$
$$A(\phi_1\phi_4, \phi_2\phi_3) = T_{14} \cdot T_{23} \cdot A(g_1, g_2, g_3, g_4) = \frac{p_1p_3}{p_2p_3} = \frac{u}{t}. \tag{3.36}$$

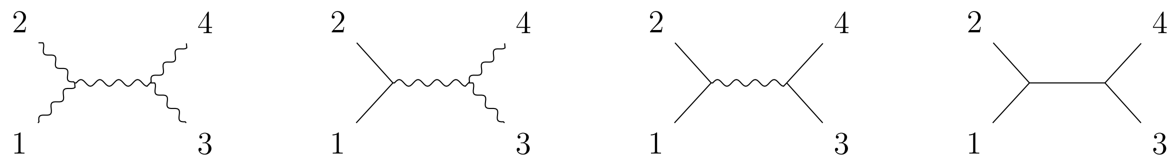
Which correspond to the three scattering diagrams that contribute to the 4-point colour-stripped YMS amplitude. This would be accompanied by the trace structure of the initial YM partial amplitude we started with, which is $\mathrm{Tr}[T^{a_1}T^{a_2}T^{a_3}T^{a_4}]$

$$A(\phi_1\phi_2, \phi_3\phi_4) + A(\phi_1\phi_3, \phi_2\phi_4) + A(\phi_1\phi_4, \phi_2\phi_3) = -\left(\frac{t}{s} + \frac{s}{t} + 1\right) = \frac{u}{s} - \frac{s}{t}. \tag{3.37}$$

When using the insertion operator we can transmute 3 and 4 gluons into scalars that are adjacent in both single and dual colour trace. These are the partial amplitudes of the BAS. for example:

$$T[123] \cdot A(g_1, g_2, g_3, g_4) = A(\phi_1\phi_2\phi_3, g_4) = \frac{p_3e_4}{p_1p_2} - \frac{p_1e_4}{p_1p_4}, \tag{3.38}$$
$$T[1234] \cdot A(g_1, g_2, g_3, g_4) = A(\phi_1\phi_2\phi_3\phi_4) = \frac{1}{p_1p_2} + \frac{1}{p_2p_3} = \frac{1}{s} + \frac{1}{t}. \tag{3.39}$$

The latter will have the dual-colour trace structure of $\mathrm{Tr}[T^{a_1}T^{a_2}T^{a_3}T^{a_4}]\,\mathrm{Tr}[T^{\bar{a}_1}T^{\bar{a}_2}T^{\bar{a}_3}T^{\bar{a}_4}]$.



(a) $A(g_1, g_2, g_3, g_4)$ **(b)** $A(\phi_1\phi_2, g_3, g_4)$ **(c)** $A(\phi_1\phi_2, \phi_3\phi_4)$ **(d)** $A(\phi_1\phi_2\phi_3\phi_4)$

Figure 3.1: Feynman diagram depiction of the transmutation of the partial gluon amplitude **(a)** into the YMS amplitude **(b)** by use of the operator T_{12}, the YMS amplitude **c** which is created by use of the combination $T_{12}T_{34}$ and a pure BAS amplitude **(d)** created by $T[1234]$. The particles ϕ_i and g_i that are not seperated by a ',' are adjacent in both the original and the dual-colour ordering.

3.3 Longitudinal Operators

3.3.1 Definition and gauge invariance of L_i and L_{ij}

Now that we have a firm grasp on the trace and insertion operator we can turn our attention to more exotic transmutation operators. We consider the *longitudinal operators* L_i and L_{ij} which Cheung et. al. define as:

$$L_i \equiv \sum_j p_i p_j \partial_{p_j e_i} \qquad \text{and} \qquad L_{ij} \equiv -p_i p_j \partial_{e_i e_j}. \tag{3.40}$$

These operators replace polarisations by polarisations by momenta by acting on the object $p_j e_i$ and $e_i e_j$. Effective this number of derivatives per transmuted particle, converting the states to a longitudinal mode. The resulting particles are, for example, pions and (special) Galileons.

We will not be combining these operators with the insertion operator, and hence not placing particles in a specific dual-colour trace. We do, however, transmute pairs of polarisations $e_i e_j$ which will introduce the same δ^{ij} that were discussed for the YMS amplitudes.

The longitudinal operators (3.40) are not guaranteed to conserve the on-shell conditions and gauge invariance of the amplitude due to the new momentum pair $p_i p_j$ introduced. We will explain how to deal with the gauge invariance of these operators in section D.4.

After this discussion, we can extend the longitudinal operator that acts on pairs to a longitudinal operator $\mathcal{L}$ that conserves the on-shell condition and the gauge-invariance of the amplitudes on which it acts. This operator acts on all the particles present in a partial amplitude and is defined as

$$\mathcal{L} \equiv \prod_i L_i = \sum_\rho \prod_{i,j \in pairs} L_{ij} + \cdots . \tag{3.41}$$

Here the summation i runs over all gluons in the amplitude and ρ denotes the partitions into pairs. The ellipses denote terms that vanish upon acting on any amplitude as discussed in section D.4.

3.3.2 Remedying Vanishing Amplitudes

There is one caveat to the procedure of transmuting all the gluons in the initial amplitude: it gives rise to a vanishing amplitude. We will show this explicitly for 3- and 4- point gluon amplitudes in the example section of this chapter.

Luckily all our work is not for nothing and we can still connect the derivatively coupled theories to our web of transmutation amplitudes. The way around this problem is found in recursion. The recursion relations of the 'extended NLSM' [35] which describes mixed amplitudes of biadjoint scalars ϕ interacting with NLSM pions π give rise to the property that the amplitude described by 2 biadjoint scalars interacting with $(n-2)$ pions are exactly equal to a pure n-point pion amplitude:

$$A_n(\phi_i, \phi_j, \pi_1, \ldots, \pi_{n-2}) = A_n(\pi_i, \pi_j, \pi_1, \ldots \pi_{n-2}). \tag{3.42}$$

This property of the mixed amplitudes was also separately shown in [69], where it was deduced that this property had to be satisfied for any mixed theory of Nambu-Goldstone bosons and biadjoint scalars. Furthermore they deduce that no vertices containing only one biadjoint scalar $\phi^{a\bar{a}}$ can exist.

While we have used the example of transmuting gluons into pions for the discussion of longitudinal operators, the same operators can be applied to an amplitude of Born-Infeld photons. When transmuted, this amplitude becomes an amplitude describing derivatively coupled Special-Galileons. The equality of a pure SG amplitude and an amplitude of SG coupled to two NLSM pions is also proven in section 3.2 of [35].

More recently, several other relations for mixed amplitudes of coloured scalars with and NLSM amplitudes were found by Arkani-Hamed and collaborators in [70, 71, 72]. For starters, in their 2024 paper entitled 'NLSM $\subset$ Tr(ϕ^3)' the authors employ kinematic shifts of planar propagators to show that mixed amplitudes with an odd number of pions vanishes. Moreover, mixed amplitudes with a single coloured scalar leg ϕ also vanish due to the Adler zero property of taking any of the pion momentum to zero. They also derive the property of amplitudes with two ϕ's combined with an even number $(n = 2m)$ of π's gives rise to a $(2m+2)$-point pure NLSM amplitude.

The identities of mixed amplitudes can be used to construct pure pion amplitudes from gluons amplitudes in the following way. First, we transmute two gluons into biadjoint scalars using the trace operator T_{ij}. Next, we apply $\mathcal{L}$ on the obtained YMS amplitude to transmute the remaining gluons into NLSM pions. The final object will be a mixed amplitude with two scalars and $(n-2)$ gluons. This procedure can be denoted as

$$\mathcal{L} \cdot T_{ij} \cdot A(g_1, \ldots, g_n) = A(\pi_1, \ldots, \pi_n). \tag{3.43}$$

As the specific pair of gluons i, j does not influence the outcome, we will use the shorthand notation $T\mathcal{L}$ for this operation. The resulting amplitude is a permutation invariant pion amplitude, therefore it is not important which pair of gluon we choose to transmute into biadjoint scalars. We can also choose to create more than two biadjoint scalar to more general mixed amplitudes of the extended NLSM. The same discussion can be applied to Born-Infeld amplitudes transmuted into (mixed) Special-Galileon amplitudes.

3.3.3 Examples: Transmuting Gluons into Pions

3-Point amplitudes

For starters, we recall that any odd-point NLSM amplitude vanishes. Especially at 3-point , where there are no non-zero Mandelstam variables due to the on-shell conditions combined with momentum conservation. For these reasons, we expect the transmuted gluon amplitude to also vanish.

Starting again from the 3 point, colour-ordered gluon partial amplitude

$$A(g_1, g_2, g_3) = \frac{1}{2} e_1 e_2 (p_2 e_3 - p_1 e_3) + (\text{cyclic}). \tag{3.44}$$

First we check that transmutation of all gluons into pions directly gives rise to a vanishing amplitude. This can be shown explicitly through summation of the different orderings of the transmutation:

$$\begin{aligned} L_3 \cdot L_{12} \cdot A(g_1, g_2, g_3) &= -\frac{1}{2} (p_3 p_1 \partial_{p_1 e_3} + p_3 p_2 \partial_{p_2 e_3}) \left(p_1 p_2 (p_2 e_3 - p_1 e_3) \right) \\ &= \frac{1}{2} (p_1 p_2)(p_3 p_1) - \frac{1}{2} (p_1 p_2)(p_3 p_2), \qquad (3.45) \\ (1 \to 2 \to 3 \to 1) &= \frac{1}{2} (p_2 p_3)(p_1 p_2) - \frac{1}{2} (p_2 p_3)(p_1 p_3), \qquad (3.46) \\ (1 \to 3 \to 2 \to 1) &= \frac{1}{2} (p_3 p_1)(p_2 p_3) - \frac{1}{2} (p_3 p_1)(p_2 p_1), \qquad (3.47) \\ (L_3 \cdot L_{12} + L_2 \cdot L_{13} + L_1 \cdot L_{23}) \cdot A(g_1, g_2, g_3) &= 0. \qquad (3.48) \end{aligned}$$

We note that this summation over the different pair arrangements of transmutation directly into pions is corresponds to a gauge transformation where all the polarisation vectors are set to their corresponding momentum vector, i.e. $e_i \to p_i$:

$$\begin{aligned} A(g_1, g_2, g_3) \xrightarrow{e_i \to p_i} &\frac{1}{2} p_1 p_2 (p_2 p_3 - p_1 e_3) + \text{cyclic} \\ &= \frac{1}{2} [(p_1 p_2)(p_2 p_3) - (p_1 p_2)(p_1 e_3) + (p_2 p_3)(p_3 p_1) \\ &- (p_2 p_3)(p_2 e_1) + (p_3 p_1)(p_1 p_2) - (p_3 p_1)(p_3 e_2)] \\ &= 0. \end{aligned} \tag{3.49}$$

As stated before in section subsection 3.3.2, we get around this gauge transformation by first transmuting a pair of gluons into biadjoint scalars before applying the longitudinal operator L_i to transmute the third gluon into a pion

$$\begin{aligned}\mathcal{L}\cdot T[12]\cdot A(g_1,g_2,g_3) &= \frac{1}{2}L_3\cdot(p_2e_3-p_1e_3)\\ &= \frac{1}{2}\sum_j p_3p_j\partial_{p_je_3}(p_2e_3-p_1e_3)\\ &= \frac{1}{2}(p_3p_2-p_3p_1)=p_3p_2,\end{aligned}\tag{3.50}$$

where we have used that $p_1=-p_2-p_3$, $p_i^2=0$ and $p_3p_2=\frac{1}{2}(p_2+p_3)^2$. Depending on which initial pair of gluons is chosen to be transmuted into gluons we will find the identical result for the three-point pion amplitude:

$$\left.\begin{aligned}\mathcal{L}\cdot T[12]\cdot A(g_1,g_2,g_3) &= A(\phi_1\phi_2,\pi_3)\\ \mathcal{L}\cdot T[23]\cdot A(g_1,g_2,g_3) &= A(\phi_2\phi_3,\pi_1)\\ \mathcal{L}\cdot T[31]\cdot A(g_1,g_2,g_3) &= A(\phi_3\phi_1,\pi_2)\end{aligned}\right\}=A(\pi_1,\pi_2,\pi_3)=0.\tag{3.51}$$

This is in agreement with fact that odd-point pion amplitudes should vanish and in agreement with the findings of [72] that mixed amplitudes with an odd number of π's should vanish.

4-Point amplitudes

We start by naively applying a transmutation of all four gluon legs into pions by using using the product of pairs of L_{ij}. For convenience, we can utilise our earlier result of the multi-trace amplitude, such as

$$T[12]\cdot T[34]\cdot A(g_1,g_2,g_3,g_4)=\frac{p_1p_3}{p_1p_2},\tag{3.52}$$

seeing as the longitudinal operator $L_{ij}=-p_ip_jT_{ij}$ is a linear combination of these trace operators.

In this naive calculation for the complete pion amplitude, we sum over all distinct orderings of 4 particles into pairs: $\mathcal{L}=L_{12}L_{34}+L_{13}L_{24}+L_{14}L_{23}$. Explicitely, these different arrangements are given by

$$\begin{aligned}L_{12}L_{34}\cdot A(g_1,g_2,g_3,g_4) &= (-p_1p_2)(-p_3p_4)T[12]\cdot T[34]\cdot A(g_1,g_2,g_3,g_4)\\ &= (-p_1p_2)(-p_3p_4)\frac{p_1p_3}{p_1p_2}=(p_3p_4)(p_1p_3)\\ &= -\frac{1}{4}u(u+t)=\frac{1}{4}us,\end{aligned}\tag{3.53}$$

$$L_{13}L_{24}\cdot A(g_1,g_2,g_3,g_4)=\frac{1}{4}uu,\tag{3.54}$$

$$L_{14}L_{23}\cdot A(g_1,g_2,g_3,g_4)=\frac{1}{4}ut,\tag{3.55}$$

where we have used the notation $s = s_{12} = (p_1 + p_2)^2 = 2p_1p_2$, $t = s_{23} = (p_2 + p_3)^2 = 2p_2p_3$ $u = s_{13} = (p_1 + p_3)^2 = 2p_1p_3$ for the Mandelstam invariants.

The summation over these three arrangements into pairs then results in

$$\mathcal{L} \cdot A(g_1, g_2, g_3, g_4) = \frac{1}{4}u(s + t + u) = 0, \tag{3.56}$$

where we have used momentum conversation to show that $p_3p_4 = p_3(-p_1 - p_2 - p_3) = -p_1p_3 - p_2p_3$. In the last equality we use the fact that the Mandelstams sum to zero: $s + t + u = 0$. Therefore, this naive application of the transmutation operator on all gluon legs results in a vanishing amplitude, which we already conjectured due to the gauge invariance of the Yang-Mills amplitude.

To obtain the correct 4-point gluon amplitude we thus resort to utilising the property of the mixed BAS and NLSM amplitudes. We apply the operator $T\mathcal{L}$ to first transmute a pair of gluons into biadjoint scalars before transmuting the remaining gluons to pions. The final result will be a permutation invariant pure pion amplitude:

$$\begin{aligned} T_{12}L_{34} \cdot A_{YM} &= (-p_3p_4)T[12] \cdot T[34] \cdot A(g_1, g_2, g_3, g_4) = p_1p_3 = \frac{u}{2} \\ &= \frac{(-p_3p_4)u}{s} = \frac{(p_3p_1 + p_3p_2)u}{s} = \frac{(u+t)u}{2s} = \frac{-su}{2s} = -\frac{u}{2}. \end{aligned} \tag{3.57}$$

For completeness we also explicitely calculate the other permutations[2]

$$T[13] \cdot \mathcal{L} \cdot \cdot A(g_1, g_2, g_3, g_4) = (-p_2p_4) \cdot 1 = p_2p_1 + p_2p_3 = \frac{s+t}{2} = -\frac{u}{2}, \tag{3.58}$$

$$T[14] \cdot \mathcal{L} \cdot \cdot A(g_1, g_2, g_3, g_4) = (-p_2p_3)\frac{p_1p_3}{p_2p_3} = -\frac{u}{2}. \tag{3.59}$$

We can therefore conclude that this procedure generates the correct pion amplitude

$$\left.\begin{aligned} \mathcal{L}\cdot T[12] \cdot A(g_1, g_2, g_3, g_4) &= A(\phi_1\phi_2, \pi_3, \pi_4) = p_1p_3 \\ \mathcal{L}\cdot T[13] \cdot A(g_1, g_2, g_3, g_4) &= A(\phi_1\phi_3, \pi_2, \pi_4) = p_1p_3 \\ \mathcal{L}\cdot T[14] \cdot A(g_1, g_2, g_3, g_4) &= A(\phi_1\phi_3, \pi_2, \pi_4) = p_1p_3 \end{aligned}\right\} = A(\pi_1, \pi_2, \pi_3, \pi_4). \tag{3.60}$$

The result is a permutation invariant pion amplitude, which is identical regardless of the choice of initial pair of biadjoint scalars that is created.

3.4 Web of theories connected through transmutation

In the above chapter, we have identified several building blocks with which we can transmute amplitudes:

[2] Recall the ordering of the transmutation operators, we should read this as the operators acting from left to right. First the trace operator acts on the amplitude followed by the longitudinal operator

- The single trace transmutation $T[i_1 \cdots i_n]$ which introduces a new colour ordering to the resulting partial amplitude.
- The multi trace transmutation $T[\alpha] \cdots T[\beta]$ which results in multiple colour traces in the obtained amplitude. Specifically the transmutation of all particles ordered into pairs with the operator combination $T[i_1 j_1] \cdots T[i_n j_n]$ is interesting as it results in a special colour structure of products of δ_{ij}'s.
- The combination of a trace and longitudinal operators $T\mathcal{L}$, which adds derivatively coupled interactions to the amplitude.

There are therefore three *pure* objects (of interactions of only 1 type of particle) we can construct out of given a starting spin-1 or spin-2 amplitude. If we choose to not transmute all of the external legs, we land on a mixed theory of transmuted legs interacting with untransmuted legs. Starting from a extended gravity amplitudes we can find the following pure amplitudes:

$$A_{\mathrm{YM}} = T[\alpha] \cdot A_{\mathrm{EG}}, \tag{3.61}$$
$$A_{\mathrm{BI}} = T\mathcal{L} \cdot A_{\mathrm{EG}}, \tag{3.62}$$
$$A_{\mathrm{EM}} = T[i_1 j_1] \cdots T[i_n j_n] \cdot A_{\mathrm{EG}} \tag{3.63}$$

The hybrid theory amplitudes of external gravitons coupled to external gluons/photons/BI photons can be computed by leaving some external graviton legs untouched by transmutation. Applying transmutation to Yang-Mills amplitudes gives

$$A_{\mathrm{BAS}} = T[\alpha] \cdot A_{\mathrm{YM}}, \tag{3.64}$$
$$A_{\mathrm{NLSM}} = T\mathcal{L} \cdot A_{\mathrm{YM}}, \tag{3.65}$$
$$A_{\mathrm{YMS}} = T[i_1 j_1] \cdots T[i_n j_n] \cdot A_{\mathrm{YM}}. \tag{3.66}$$

Finnaly we can find the following amplitudes from Born-Infeld amplitudes

$$A_{\mathrm{NLSM}} = T[\alpha] \cdot A_{\mathrm{BI}}, \tag{3.67}$$
$$A_{\mathrm{SG}} = T\mathcal{L} \cdot A_{\mathrm{BI}}, \tag{3.68}$$
$$A_{\mathrm{DBI}} = T[i_1 j_1] \cdots T[i_n j_n] \cdot A_{\mathrm{BI}}. \tag{3.69}$$

The theories that are connected through transmutation are depicted in the tetrahedron of Figure 3.2. In this visualisation it is clear that transmutation is a type of 'top-down' process which creates lower spin partial amplitudes from the spin-2 extended gravity amplitudes.

The tetrahedron also reinstates that the order of applying transmutation operators does not matter, as these lead to the partial amplitudes of the same theory. This was already apparent from the fact that the operators commute. As an example, we can calculate the NLSM amplitudes from the extended gravity amplitudes through the combination $T[\alpha] \cdot T\mathcal{L}$ and $T\mathcal{L} \cdot T[\alpha]$ as indicated by the red and orange arrows in Figure 3.2.

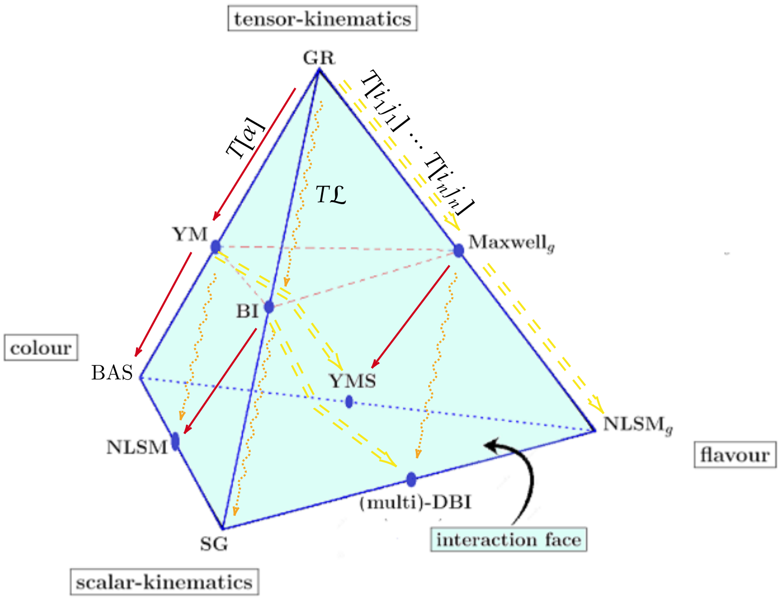


Figure 3.2: Visualisation of Unifying Relations between amplitudes of different theories, based on the tetrahedron of [60] that displays the theories that are connected through the BCJ double copy of different combination of BCJ numerators.

3.5 Infrared properties of Transmutation

As discussed in subsection 3.5.1, the amplitudes of gluon and graviton scattering take special form when taking one of the external particles to be soft; $p_i \to 0$. The amplitude factorises as a soft factor multiplied by a lower point amplitude

$$A^{(n)} \xrightarrow{p_i \to 0} S^{(i)} \cdot A'^{(n-1)}. \tag{3.70}$$

The soft factor is a pertubative expansion and can be computed to different orders. At leading order in this expansion the gluon and graviton soft factors are simple factors [42]. However, at higher orders these factors will become more elaborate; depending on momenta and polarisations of adjacent legs and containing higher order differential operators with respect to these variables [45]. We are therefore interested whether transmutation preserves the soft factorisation of the amplitude at all, or perhaps only at leading order.

To investigate whether this behaviour persists, we apply transmutation on the soft factor relation by first factorising a general product of transmutation operators as $\mathcal{T}^{(n)} = \mathcal{T}^{(i)} \cdot \mathcal{T}^{(n-1)}$, where the operator $\mathcal{T}^{(i)}$ is the operator responsible for transmuting leg i, which will be taken to be soft and $\mathcal{T}^{(n-1)}$ transmutes the remaining $(n-1)$ external legs. We can show that then show that upon transmutation we can find an expression for the transmuted soft factor $\tilde{S}^{(i)}$ which acts on the transmuted lower point amplitude denoted by $\tilde{A}^{(n-1)}$ by calculating

$$\begin{aligned}\mathcal{T}^{(n)} \cdot A^{(n)} \xrightarrow{p_i \to 0} \; & \mathcal{T}^{(i)} \mathcal{T}^{(n-1)} S^{(i)} A^{(n-1)} \\ &= \left[\mathcal{T}^{(i)} \mathcal{T}^{(n-1)}, S^{(i)}\right] A^{(n-1)} \\ &= \mathcal{T}^{(i)} \left[\mathcal{T}^{(n-1)}, S^{(i)}\right] A^{(n-1)} + \left[\mathcal{T}^{(i)}, S^{(i)}\right] \mathcal{T}^{(n-1)} A^{(n-1)} \\ &= \left[\mathcal{T}^{(i)}, S^{(i)}\right] \left(\mathcal{T}^{(n-1)} A^{(n-1)}\right) \\ &= \left[\mathcal{T}^{(i)}, S^{(i)}\right] \tilde{A}^{(n-1)}.\end{aligned} \tag{3.71}$$

This indicates that the transmuted soft factor can be denoted as

$$\tilde{S}^{(i)} \equiv \left[\mathcal{T}^{(i)}, S^{(i)}\right]. \tag{3.72}$$

The necessary arguments in this derivation are as follows. In the second line we can denote the commutator identity due to the fact that the commutator is defined as

$$\left[\mathcal{T}^{(i)} \mathcal{T}^{(n-1)}, S^{(i)}\right] A^{(n-1)} = \mathcal{T}^{(i)} \mathcal{T}^{(n-1)} S^{(i)} A^{(n-1)} - S^{(i)} \mathcal{T}^{(i)} \left(\mathcal{T}^{(n-1)} A^{(n-1)}\right), \tag{3.73}$$

where the second term vanishes because $\mathcal{T}^{(i)}$ depends on derivatives with respect to e_i and p_i, whereas $\mathcal{T}^{(n-1)} A^{(n-1)}$ by construction is independent of the variables of leg i.

Next we argue that the commutator $\left[\mathcal{T}^{(n-1)}, S^{(i)}\right]$ vanishes for the trace operator T_{jk} and the insertion operator I_{jkl} due to the dependence of $\mathcal{T}^{(n-1)}$ on combination $e_j e_k$, $e_j p_k$ or $p_j p_k$ where $j, k \neq i$ while $S^{(i)}$ only depends on combinations $e_i e_j$, $e_i p_j$ or $p_i p_j$ at leading and subleading order as we will see shortly.

This condition also holds at leading order for the Longitudinal operator L_{jk}, however we will need to revise this statement for the subleading order soft factor due to the dependence of the soft factor on adjacent legs.

We will calculate explicitly the transmutation of the leading order soft factor for the theories that originate from gravity and Yang-Mills amplitudes, as these lead to the leading order soft factors YM, YMS and the NLSM. The transmutation of the subleading soft factors is left to section D.5 of the appendix.

These soft factors are interesting to us because we aim to create amplitudes with interactions of scalars, pions and gluons similar that may align with the amplitudes that have $\sigma = 0$ created by the cubic numerator $N_6^{(3)}$ of section 2.4.

3.5.1 Transmutation of the leading order soft factors

Extended Gravity

Recall that the leading-order soft graviton factor given by Weinberg is given by [45]

$$S_G^{(i)} = \sum_{j \neq i} \frac{p_i e_j p_i \tilde{e}_j}{p_i p_j}. \tag{3.74}$$

As stated in the definition of the web of related theories, we are able to transmute the extended gravity amplitudes into EM, YM and BI amplitudes through our three different transmutation operators T_{ij}, I_{ijk} and L_{ij}. By the fact that the graviton soft factor is independent of products of polarisation e_ie_j it is straightforward that the soft factor of Einstein-Maxwell vanishes:

$$S_{\text{EM}}^{(ij)} = \left[T_{ij}, S_G^{(i)}\right] = 0. \tag{3.75}$$

The same argument holds for the computation of the leading order soft factor of BI amplitudes

$$S_{\text{BI}}^{(i)} = \left[L_i, S_G^{(i)}\right] = 0. \tag{3.76}$$

This vanishing leading order soft factor is in agreement with the fact that Born-Infeld photons are derivatively coupled which should lead to vanishing amplitudes when this leg is taken to be soft. Finally, the soft factor of Yang-Mills amplitudes can be found through the commutator with the insertion operator,

$$S_{\text{YM}}^{(ijk)} = \left[I_{ijk}, S_G^{(j)}\right] = \left[\partial_{p_ie_j} - \partial_{p_ke_j}, \sum_{j\neq i} \frac{p_ie_jp_i\tilde{e}_j}{p_ip_j}\right], \tag{3.77}$$

where in this case the leg j is taken to be soft with legs i and k adjacent to leg j. This then gives rise to the familiar leading order soft factor

$$S_{\text{YM}}^{(ijk)} = \frac{p_ie_j}{p_ip_j} - \frac{p_ke_j}{p_kp_j}, \tag{3.78}$$

which is consistent with the original soft factor by Weinberg [41].

Yang-Mills

Continuing with the leading order soft factor of Yang-Mills we can calculate commutators of (3.78) with the transmutation operators to find the soft factors of Yang-Mills-Scalar, biadjoint scalar and NLSM amplitudes.

When denoting leg j as the soft leg, the commutator with the trace operator results in the soft factor of YMS theory:

$$S_{\text{YMS}}^{(ijk|jl)} = \left[T_{jl}, S_{\text{YM}}^{(ijk)}\right] = 0. \tag{3.79}$$

the vertical bar indicates that particle j is in the same dual colour trace as particle l. Next the commutation with the insertion operator gives the soft factor of BAS theory

$$S_{BAS}^{(ijk|ljm)} = \left[T_{ljm}, S_{\text{YM}}^{(ijk)}\right] = \frac{\delta_{li} - \delta_{im}}{p_ip_j} - \frac{\delta_{lk} - \delta_{km}}{p_kp_j}. \tag{3.80}$$

Finally, at leading order the commutation with the longitudinal operator results in the vanishing of the NLSM soft factor

$$S_{\text{NLSM}}^{(ijk)} = 0, \tag{3.81}$$

as is to be expected from the Adler zero property exhibited by pion amplitudes.

3.6 On-shell recursion, factorisation and the Double Copy

It is further investigated by Cheung and collaborators whether the objects obtained after transmutation are in fact always scattering amplitudes which obey the appropriate properties of physical scattering amplitudes: on-shell construction, proper factorisation the Double Copy prescription. This section briefly touches upon the argumentation of these properties.

3.6.1 On-shell Constructibility

For starters, the discussion of the on-shell constructibility of the obtained amplitudes investigates whether higher-point amplitudes can be constructed recursively from lower-point amplitudes, as discussed in section 1.3.

By use of the BCFW recursion relation [38], it was shown through an -all-line shift that for amplitudes with more than four particles the deformed amplitude goes to zero in the case of large shifts z as a consequence of dimensional analysis.

It is further argued that for the insertion operator I_{ijk}, the resulting amplitude will have fewer derivatives as momenta are stripped off, and by using a power counting method this transmuted amplitude will be on-shell constructable.

This argumentation does not apply to the longitudinal operator L_{ij}, as it increases the number of derivatives of the amplitude. The on-shell constructibility of these theories is then saved if the resulting amplitude has suitable IR properties such as the Adler zero, which was shown in subsection 3.5.1.

3.6.2 Factorisation

In general, on the residue where the factorisation channel is propagator $1/p_I$ is tuned to 0, the the original amplitude will factorise as

$$A(\cdots) \sim \sum_I A_L(\cdots I_L)A(I_R\cdots). \tag{3.82}$$

We can investigate what happens to the factorisation of gluon amplitudes after transmutation. The summation over I runs over all possible internal gluon and scalar states. The discussion of proper factorisation is different for each transmutation operator and is discussed in [22]. Here we will simply state in what manner the amplitudes factorise depending on the initial amplitude.

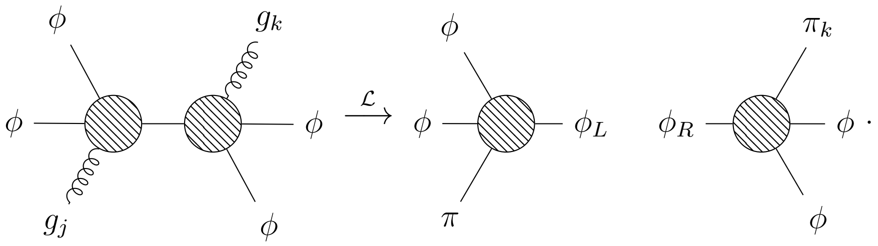
The factorisation that will later be relevant to our discussion is that of the longitudinal operator and trace operators in pairs. The latter of these transmutes gluon amplitudes into YMS amplitudes, leaving the intermediate state unaltered (gluon exchange) and therefore needs no seperate discussion.

The factorisation of the longitudinal operator $\mathcal{L} = \mathcal{L}_L \cdot \mathcal{L}_R$ knows two scenario's. In the first, the amplitude that is transmuted has scalars on both sides of the factorisation channel, requiring the internal particle to also be a scalar. The amplitude then factorises as

$$\mathcal{L} \cdot A(\cdots \phi_i \phi_l \cdots, g_j, g_k, \cdots) \sim \mathcal{L}_L \cdot A_L(\cdots \phi_i \phi_L, g_j, \cdots) \mathcal{L}_R \cdot A_R(\phi_R \phi_l \cdots, g_k, \cdots) \tag{3.83}$$

$$\sim A_L(\cdots \phi_i \phi_L, \pi_j, \cdots) A_R(\phi_R \phi_l \cdots, \pi_k, \cdots). \tag{3.84}$$

This is consistent with the factorisation of how the amplitude $A(\cdots \phi_i \phi_l \cdots, \pi_j, \pi_k, \cdots)$ would factorise. The residue of the factorisation can be schematically depicted at 6-point as



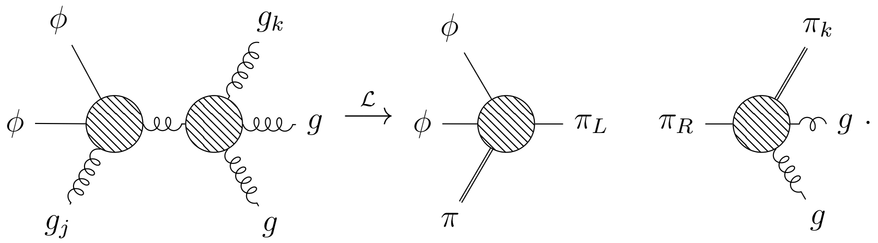
(3.85)

If the gluons are on the opposite side of the factorization channel from the all scalars, then the only possible internal state of the original amplitude is a gluon, and the transmuted amplitude factorises with an internal pion

$$\mathcal{L}_L \cdot A(\cdots \phi_i \phi_l \cdots, g_j, g_k, \cdots) \sim \mathcal{L}_L \cdot \mathcal{L}_R \cdot A_L(\cdots \phi_i \phi_l \cdots, g_j, g_L, \cdots) A_R(g_R, g_k, \cdots) \tag{3.86}$$

$$\sim A_L(\cdots \phi_i \phi_l \cdots, \pi_j, \pi_L, \cdots) A_R(\pi_R, \pi_k \cdots). \tag{3.87}$$

We can display this schematically as



(3.88)

We will later use these factorisation properties to analyse the amplitudes that we construct from *generalised transmutation*, where we define a new type of transmutation operator with the aim of landing on the amplitudes of BCJ bootstrap of section 2.4.

3.6.3 Transmutation and The Double Copy

As we have seen in subsection 1.5.1, we can use the double copy to relate the amplitudes of different theories. Specifically, the BCJ double copy relates all the amplitudes in the tetrahedron of Figure 3.2, which therefore solidifies the unifying relations of between different these theories. The transmutation of amplitudes can be shown to preserve the duality of the amplitudes through the use of the KLT relations.

The application of transmutation operators on the double copy amplitudes can be done in the KLT representation (Appendix B), due to the fact that the transmutation operators commute with the KLT relations.

In general, the KLT product allows for the crossing of amplitudes of different duality-satisfying theories. In general we can then find the following amplitudes from the KLT product of YM amplitudes with other theories:

$$\text{GR} = \text{YM} \overset{\text{KLT}}{\otimes} \text{YM}, \tag{3.89}$$

$$\text{EM} = \text{YMS} \overset{\text{KLT}}{\otimes} \text{YM}, \tag{3.90}$$

$$\text{EYM} = \text{gBAS} \overset{\text{KLT}}{\otimes} \text{YM}, \tag{3.91}$$

$$\text{BI} = \text{NLSM} \overset{\text{KLT}}{\otimes} \text{YM}. \tag{3.92}$$

Furthermore, the double copy of NLSM amplitudes results in the following amplitudes:

$$\text{BI} = \text{YM} \overset{\text{KLT}}{\otimes} \text{NLSM}, \tag{3.93}$$

$$\text{DBI} = \text{YMS} \overset{\text{KLT}}{\otimes} \text{NLSM}, \tag{3.94}$$

$$\text{BI} + \text{NLSM} = \text{gBAS} \overset{\text{KLT}}{\otimes} \text{NLSM}, \tag{3.95}$$

$$\text{SG} = \text{NLSM} \overset{\text{KLT}}{\otimes} \text{NLSM}. \tag{3.96}$$

Due to the fact that transmutation operators commute with the KLT relations, transmutation can be applied to either of the amplitudes in the product. For example, the amplitudes of BI theory can be constructed through by applying the longitudinal operator to extended graviton amplitudes. Applying this operator to either of the amplitudes in the double copy KLT product we find

$$A_{\text{BI}} = T\mathcal{L} \cdot A_{\text{EG}} = A_{\text{YM}} \overset{\text{KLT}}{\otimes} T\mathcal{L} \cdot A_{\text{YM}} \tag{3.97}$$

$$= A_{\text{YM}} \overset{\text{KLT}}{\otimes} \tilde{A}_{\text{NLSM}}, \tag{3.98}$$

which agrees with (3.93). The same argument can be applied for the different types of transmutation operators and different combinations of amplitudes in the KLT product.

3.7 UV completion

The process of transmutation extends to the UV completion of theories within the interconnected web of theories related by transmutation. However, it's important to note that the resulting amplitudes might not always be consistent since the consistency check of section 3.6 assumed a massless spectrum for the intermediate particles.

As discussed in Equation 1.5.3, the UV completion of Yang-Mills is obtained through dimensionful α' corrections in the Lagrangian. Because the YM $+\alpha'$ amplitude is gauge invariant, after transmutation, the object will remain gauge invariant. The single trace operator produces BAS amplitudes from YM, therefore we can speculate that the amplitudes created by

$$T[i_1 \cdots i_n] \cdot A_{\mathrm{YM}+\alpha'}(g_{i_1}, \cdots, g_{i_n}) = A_{\mathrm{BAS}+\alpha'}(\phi_{i_1} \cdots \phi_{i_n}), \tag{3.99}$$

where $A_{BS+\alpha'}(\phi_{i_1} \cdots \phi_{i_n})$ are amplitude obtained through some theoretical UV completion of BS theory with α' corrections.

It would be interesting if we could relate these amplitudes to some known theory or description. An example candidate for the UV complete YM theory is said to be open superstring theory [73]. The details of this theory go beyond the scope of this work, but we can nevertheless relate the amplitudes as

$$A_{\mathrm{YM}+\alpha'} \sim A_{\mathrm{open}}. \tag{3.100}$$

Applying transmutation on this open superstring amplitude is then argued to result in

$$T[i_1 \cdots i_n] \cdot A_{\mathrm{open}}(g_{i_1}, \cdots, g_{i_n}) = A_Z(\phi_{i_1} \cdots \phi_{i_n}), \tag{3.101}$$

where A_Z denotes an amplitude of biadjoint scalars described by the framework of Z-theory. A short description of these amplitudes is found in Appendix C, which is based on [58, 59, 74].

A more explicit application of transmutation of amplitudes of UV-completed theories can be done in the KLT representation. In this framework, the amplitudes of open superstrings can be described as the product of amplitudes in Z-theory and YM theory through the KLT relations:

$$A_{\mathrm{open}} = A_Z \overset{\mathrm{KLT}}{\otimes} A_{\mathrm{YM}}. \tag{3.102}$$

Here $\otimes$ denotes the KLT product of the amplitudes. The single trace operator can then be applied to the product (with which it commutes), giving

$$T[i_1 \cdots i_n] \cdot A_{\mathrm{open}} = T[i_1 \cdots i_n] \cdot (A_Z \otimes A_{\mathrm{YM}}) \tag{3.103}$$

$$= A_Z \otimes T[i_1 \cdots i_n] \cdot A_{\mathrm{YM}} \tag{3.104}$$

$$= A_Z \otimes A_{BAS}. \tag{3.105}$$

Since BAS amplitudes function as an ‘identity’ element in the KLT framework, the single trace operator therefore produces

$$T[i_1 \cdots i_n] \cdot A_{\mathrm{open}} = A_Z. \tag{3.106}$$

Similarly, applying the longitudinal transmutation procedure result in the KLT product of Z-theory amplitudes and NLSM amplitudes:

$$L \cdot T[i_1 i_n] \cdot A_{\text{open}} = L \cdot T[i_1 i_n] \cdot (A_Z \otimes A_{\text{YM}}) \tag{3.107}$$
$$= A_Z \otimes (L \cdot T[i_1 i_n] \cdot A_{\text{YM}}) \tag{3.108}$$
$$= A_Z \otimes A_{\text{NLSM}}. \tag{3.109}$$

The amplitudes $A_Z \otimes A_{\text{NLSM}}$ describe a scalar amplitude which corresponds to the amplitudes of the NLSM at low energies. It does not correspond to Abelian Z-theory [59].

Finally we note that the original paper on transmutation relations [22] does not comment on the UV completion of YMS amplitudes obtained through the pair-wise application of trace operators such as $T[i_1 i_2] \cdots T[i_{n-1} i_n]$. Presumably the reason for this is that the UV completion of YM+ϕ amplitudes is not as well-defined as the above theories. More information on higher derivative corrections to these theories can be found in e.g. [75] and [76]. The transmutation of open-string amplitudes could therefore be an interesting approach to investigating the UV completion of such theories.

Chapter 4

Generalised Transmutation of Yang-Mills

In chapter 2, we demonstrated the generation of hybrid amplitudes through the use of BCJ numerators, which specifically led to interactions involving adjoint scalars mediated by gauge interactions. On the other hand, chapter 3 introduced hybrid scalar amplitudes derived from the application of transmutation operators. This chapter aims to explore the intersection of these methodologies to address several key questions: Do these approaches yield the same amplitudes? Are the resulting amplitudes governed by interactions originating from analogous Lagrangians? Regardless of whether the methods converge or diverge, understanding the underlying reasons for their (dis)connection is of significant interest.

4.1 Defining the Generalised Transmutation Operator

The proposed amplitudes of the NLSM gauged described in section 2.5 are found by defining the kinematic numerator N through the powers of Mandelstam invariants s^p. The numerators at different orders give rise to scalars interacting with gluons and derivatively coupled scalars, therefore giving rise to YMS and NLSM amplitudes. In 3.2.4 and 3.2.5 it was discussed that the multi-trace amplitudes that transmute gluons into pairs of scalars through the differential operator T_{ij} also give rise to YMS amplitudes. Furthermore, the longitudinal operator combined with a trace operator $T \cdot \mathcal{L}$ was shown to transmute the colour-ordered gluon amplitudes into NLSM amplitudes in 3.3.

Therefore, a combination of these two transmutations would be expected to land on mixed theories describing scalars coupled to gluons and possibly gluons depending on whether or not all the external legs are transmuted and whether gluons will show up as intermediate particles. More importantly, we will construct the new operator to keep track of the order of Mandelstam invariants, allowing us to construct an amplitude with $\sigma = 0$. We would like to know whether the amplitude constructed through this method coincides with the amplitude generated by $N_6^{(3)}$.

To achieve this, the polarisations of the gluons need to be stripped off through the trace operator in pairs. When this is applied to all legs, we get a pure YMS amplitude. In order to transmute create derivatively coupled scalar modes, some pairs of polarisations are replaced by pairs of momenta $p_i p_j$. Each pair that is transformed contributes an additional factor of Mandelstams s^1 to the numerator of the amplitude.

The proposed *generalised transmutation* (GT) operator, designed to lead us to the aforementioned hybrid amplitudes, is as follows. Consider the operator $\mathbb{T}^{(1\ldots n)}$, which transmutes the external leg 1 through n. When this operator acts on an n-point Yang-Mills amplitude A_n^{YM}, it transforms all external legs and outputs a non-trivial scalar amplitude:

$$\mathbb{T}^{(1\ldots n)} \cdot A_n^{\text{YM}} = A_n^{\text{scalar}}. \tag{4.1}$$

This operator will be constructed out of a new type of longitudinal operator that strips off pairs of polarisations $(e_i e_j)$ and replaces them by $(1 + s_{ij})$ is denoted by

$$\mathbb{L}_{ij} \equiv (1 + \tau p_i p_j)\partial_{e_i e_j} = T_{ij} + \tau L_{ij}. \tag{4.2}$$

The dimensionless parameter τ will keep track of number of pairs of gluons that have been transformed into pions. Where we use the original definitions $T_{ij} = \partial_{e_i e_j}$ for the trace operator and $L_{ij} = p_i p_j \partial_{e_i e_j}$ for the longitudinal operator modulo a sign difference to split the operator into its component parts.

Similarly to the process of the original longitudinal operator, $\mathbb{L}_{ij}$ is applied for all arrangements ρ of the particle legs into distinct pairs. The full transmutation is therefore defined as

$$\mathbb{T}^{(1\ldots n)} \equiv \sum_{\rho} \prod_{i,j \in pairs} \mathbb{L}_{ij} = \sum_{\rho} \mathbb{L}_{\rho(1)\rho(2)} \cdots {}_{\rho(n-1)\rho(n)} \tag{4.3}$$

When applied to a gluon amplitude, this transmutation spits out two new amplitudes

$$\mathbb{L}_{ij} \cdot A_n^{\text{YM}}(g_1, \ldots, g_i, g_j, \ldots g_n) = A(\phi_i \phi_j, g_1, \ldots, g_n) + \tau A(g_1, \ldots, \pi_i, \ldots, \pi_j, \ldots g_n). \tag{4.4}$$

Because it contains the product of $\frac{n}{2}$ $\mathbb{L}_{ij}$ operators, the total transmutation $\mathbb{T}^{(1\ldots n)}$ which transmutes all external legs contains contributions starting at τ^0 which will be the pure YMS amplitudes of subsection 3.2.4 up to order $\tau^{n/2}$. At this order the longitudinal operator L_{ij} has been applied to all legs. However, in subsection 3.3.2 we saw that this $\mathcal{O}\left(\tau^{n/2}\right)$ transmutation will result in a vanishing amplitude and that the actual NLSM pion amplitude is constructed by first setting one pair $(e_i e_j)$ to 1. Fortunately, this is exactly what happens at $\mathcal{O}\left(\tau^{n/2-1}\right)$.

The amplitudes in between these orders will therefore be non-trivial scalar amplitudes of pions interacting with scalars. Due to the vanishing amplitudes, the first encounter with these mixed amplitudes will be at 6-point.

The spectrum of contributions can then be depicted as

$$\mathbb{L}_{12}\mathbb{L}_{34}\cdots\mathbb{L}_{(n-1)(n)}\cdot A_n^{\text{YM}} = \tau^0 A\left(\phi^n\right) \tag{4.5}$$
$$+\,\tau^1 A\left(\phi^{n-2},\pi^2\right) \tag{4.6}$$
$$+\,\tau^2 A\left(\phi^{n-4},\pi^4\right) \tag{4.7}$$
$$+\,\dots \tag{4.8}$$
$$+\,\tau^{n/2-1} A\left(\pi^n\right) \tag{4.9}$$
$$+\,\tau^{n/2}\cdot 0, \tag{4.10}$$

where we have used superscript notation to indicate the number of scalars and pions. In the following sections we will perform explicit calculations of the transmutation applied to different multiplicities.

Pair ordering and computational analysis

Although at four points there exist only three distinct arrangements of pairs of polarisation, at n-point the number of distinct combinations of divisions of n numbers into pairs is $\frac{n!}{(\frac{n}{2})!2^{(n/2)}}$. The order in τ ranges from τ^0 to $\tau^{n/2}$, where the latter again results in a vanishing amplitude. This results in 15 arrangements at 6-point, 105 arrangements at 8-point and 945 at 10-point. The pattern is that if we go from n point to $n+2$ point, the number of arrangements increases by a factor of $n+1$. For example, the number of arrangements at 12-point will be $945 * 11 = 10395$ distinct arrangements.

In the next section, we will explicitly apply the GT operator analytically for only one specific arrangement of the 4-point amplitude. The other arrangements can be calculated analogously and will be performed using a Mathematica algorithm. First of all Mathematica will be used to calculate the distinct arrangements into pairs. After this, an algorithm can be written that applies the GT-operator. Computationally this can be done either through the application of derivatives, but it was quickly seen that taking these nested derivatives with Mathematica was computationally heavier than simply setting pairs of polarisation of $e_i e_j \to 1$ using a simple replacement rule and thereafter multiplying by the pairs of momenta $p_i p_j$ in the needed arrangement. Due to the linearity of the amplitudes in polarisation pairs, there is no ambiguity as to whether this replacement is identical to (4.1).

As input we will use colour-stripped partial amplitudes of pure Yang-Mills provided by [77]. Along with this publication, the authors Song He, Linghui Hou, Jintian Tian, and Yong Zhang provided a Mathematica package that contains an algorithm that makes use of the calculation of the so-called Cayley functions [78] in the CHY formalism of amplitude calculations. This algorithm is able to compute all tree-level amplitudes of YM, GR and BI. Through use of the double copy and dimensional reduction (in the CHY formalism [79]) the algorithm can also calculate all tree-level amplitudes of the NLSM, SG, YMS, EYM and DBI theories. This package also allows for the generation of kinematic numerators that satisfy Jacobi identities by default.

4.2 Generalised transmutation at 4-point

We begin our investigation with the tree-level colour-ordered Yang-Mills amplitude $A_4^{\text{YM}}[1234]$ given by (3.32). There are 3 distinct arrangements of 4 particles into pairs, namely (12)(34), (13)(24) and (14)(23). The GT-operator therefore leads to the summation of three ordered contributions.

$$\mathbb{T}^{(1234)} \cdot A(g_1, g_2, g_3, g_4) = \sum_{\rho} \prod_{i,j \in pairs} L_{ij} \cdot A(g_1, g_2, g_3, g_4) \tag{4.11}$$

$$= (\mathbb{L}_{12}\mathbb{L}_{34} + \mathbb{L}_{13}\mathbb{L}_{24} + \mathbb{L}_{14}\mathbb{L}_{23}) \cdot A(g_1, g_2, g_3, g_4). \tag{4.12}$$

Let us first consider only one of the arrangements, namely the first ordering into pairs $\mathbb{L}_{12}\mathbb{L}_{34}$. This produces three amplitudes, two of which are non-vanishing:

$$\begin{aligned}\mathbb{L}_{12}\mathbb{L}_{34} \cdot A(g_1, g_2, g_3, g_4) &= (T_{12} - \tau L_{12})(T_{34} - \tau L_{34}) A(g_1, g_2, g_3, g_4) \\ &= \big(T_{12} \cdot T_{34} + \tau T_{12} \cdot L_{34} + \tau L_{12} \cdot T_{34} + \tau^2 L_{12} \cdot L_{34}\big) \cdot A(g_1, g_2, g_3, g_4) \\ &= A(\phi_1\phi_2, \phi_3\phi_4) + \tau A(\phi_1\phi_2, \pi_3, \pi_4) + \tau A(\pi_1, \pi_2, \phi_3\phi_4) \\ &\quad + \tau^2 L_{12} \cdot L_{34} \cdot A(g_1, g_2, g_3, g_4).\end{aligned} \tag{4.13}$$

The calculations of these 4 point transmutations were already determined in (3.52), (3.53) and (3.57):

$$\tau^0 T_{12} T_{34} \cdot A_{\text{YM}} = A(\phi_1\phi_2, \phi_3\phi_4) = \frac{p_1 p_3}{p_1 p_2} = \frac{u}{s}, \tag{4.14}$$

$$\tau T_{12} L_{34} \cdot A_{\text{YM}} = A(\phi_1\phi_2, \pi_3, \pi_4) = A_{\text{NLSM}}(\pi_1, \pi_2, \pi_3, \pi_4) = p_1 p_3 = u/2, \tag{4.15}$$

$$\tau^2 L_{12} L_{34} \cdot A_{\text{YM}} = us/4. \tag{4.16}$$

It was explicitly shown in (3.53) that the sum of all τ^2 contributions transforms all polarisations into momenta, which is a gauge transformation and therefore this contribution vanishes. In the same section we also calculated the other arrangements of T_{ij}. The final transmutation at the different orders in τ are therefore

$$\mathcal{O}\left(\tau^0\right): \quad \tau^0[A(\phi_1\phi_2, \phi_3\phi_4) + A(\phi_1\phi_3, \phi_2\phi_4) + A(\phi_1\phi_4, \phi_2\phi_3)] = \tau^0 \left[\frac{u}{s} - \frac{s}{t}\right], \tag{4.17}$$

$$\mathcal{O}\left(\tau^1\right): \quad 3\tau^1 \cdot A_{NLSM}(\pi_1, \pi_2, \pi_3, \pi_4) = \tau^1[3u], \tag{4.18}$$

$$\mathcal{O}\left(\tau^2\right): \quad 3\tau^2 \cdot 0 = \tau^2[0]. \tag{4.19}$$

These results were verified computationally using Mathematica, up to an overal minus sign, due to the difference in definition between the original longitudinal operator $L_{ij} = -p_i p_j T_{ij}$ and the GT-operator which has $\mathbb{L}_{ij}$. The YMS partial amplitude at $\mathcal{O}\left(\tau^0\right)$ is also the correct YMS amplitude, as it agrees with the YMS amplitude from the provided package up to a total sign difference.

Soft limit of generalised transmutation at 4-point

We are interested in whether the amplitudes adhere to familiar behaviour in the limit where one of the external legs is taken soft. Based on the results of (4.17), the soft behaviour of the amplitude at different orders in τ is denoted as follows:

$\mathcal{O}(\tau^0)$: The contributions at this order for 4-point are purely Yang-Mills scalar amplitudes. All polarisations pair have been set to 1, resulting in the soft $\sigma = 0$. Similar calculations can be applied to the other arrangements in (3.52) and have been confirmed to adhere to the same soft limit using Mathematica. $\mathcal{O}(\tau^1)$: The amplitudes at this order are the pure 4-point pion amplitudes. Explicitely the soft limit at this order is $\lim_{p_i \to 0} A_{\text{NLSM}}(\pi_1, \pi_2, \pi_3, \pi_4) \sim p^1$, which corresponds to the Adler zero as expected for such amplitudes. Finally, the $\mathcal{O}(\tau^2)$ amplitudes will individually have a soft limit of one order higher, $\sim p^2$. However, when all terms are collected the total amplitude at this order vanishes.

As expected, the GT-operator at four point holds relatively few suprises. The non-trivial hybrid amplitudes we are interested in will first be encountered in the six-point calculation.

4.3 Generalised Transmutation at 6-point

Transmutation of a 6-point Yang-Mills partial amplitude will lead to more interesting results as this is where the mixed theories with ambiguous soft behaviour start to appear. This is due to the fact that there are three pairs of polarisations in each arrangements to consider when splitting into pairs. The distinct arrangements are calculated to be

$$\begin{aligned}
&(12)(34)(56),\ (12)(35)(46),\ (12)(36)(45),\\
&(13)(24)(56),\ (13)(25)(46),\ (13)(26)(45),\\
&(14)(23)(56),\ (14)(25)(36),\ (14)(26)(35),\\
&(15)(23)(46),\ (15)(24)(36),\ (15)(26)(34),\\
&(16)(23)(45),\ (16)(24)(35),\ (16)(25)(34).
\end{aligned} \tag{4.20}$$

We start by consider one of such arrangements into pairs, the ordering (12)(34)(56). The contribution to the transmutation of this ordering is $\mathbb{L}_{12}\mathbb{L}_{34}\mathbb{L}_{56}$. When applied to the 6-point gluon amplitude we find contributions of $\mathcal{O}(\tau^0)$ to $\mathcal{O}(\tau^3)$, where the latter gives rise to a vanishing amplitude per the gauge transformation argument when all contributions of this order in τ are summed. We have explicitely verified that this vanishes for $n = 6$ by computational analysis.

$$\begin{aligned}
\mathbb{L}_{12}\mathbb{L}_{34}\mathbb{L}_{56} \cdot A_6^{\text{YM}} = (&T_{12}T_{34}T_{56} - \tau T_{12}T_{34}L_{56} - \tau T_{12}L_{34}T_{56} - \tau L_{12}T_{34}T_{56}\\
&+ \tau^2 T_{12}L_{34}L_{56} + \tau^2 L_{12}T_{34}L_{56} + \tau^2 L_{12}L_{34}T_{56} - \tau^3 L_{12}L_{34}L_{56}) \cdot A_6^{\text{YM}}
\end{aligned} \tag{4.21}$$

At the amplitude level, the GT-operator then creates the following amplitude contributions.

$$T_{12}T_{34}T_{56}\cdot A_6^{\mathrm{YM}} = A_{\mathrm{YMS}}(\phi_1\phi_2,\phi_3\phi_4,\phi_5\phi_6) \tag{4.22}$$

$$\tau T_{ij}T_{kl}L_{mn}\cdot A_6^{\mathrm{YM}} = \tau(A(\pi_1\pi_2,\phi_3\phi_4,\phi_5,\phi_6)+A(\phi_1\phi_2,\pi_3\pi_4,\phi_5,\phi_6)+A(\phi_1\phi_2,\phi_3\phi_4,\pi_5,\pi_6)) \tag{4.23}$$

$$\tau^2 T_{12}L_{34}L_{56}\cdot A_6^{\mathrm{YM}} = \tau^2(A(\phi_1\phi_2,\pi_3,\pi_4,\pi_5,\pi_6) = A_{NLSM}(\pi_1,\pi_2,\pi_3,\pi_4,\pi_5,\pi_6)) \tag{4.24}$$

$$\tau^3 L_{12}L_{34}L_{56}\cdot A_6^{\mathrm{YM}} + (\text{ permutations}) \implies 0 \tag{4.25}$$

For the complete amplitude we must take into account that the complete transmutation sums over different pair arrangements of these transmutations.

4.3.1 τ^0: Yang-Mills-Scalar

Starting from $\mathcal{O}(\tau^0)$, we find the 6 point YMS amplitude. An example calculation of one such an arrangement is given by the arrangement (12)(34)(56) which results in the contribution

$$\begin{aligned}4A_{\mathrm{YMS}}(\phi_1\phi_2,\phi_3\phi_4,\phi_5\phi_6) &= \frac{s_{2,3}}{s_{1,2}s_{1,2,3}}+\frac{s_{4,5}s_{2,3}}{s_{1,2}s_{1,2,3}s_{1,2,3,4}}+\frac{1}{s_{1,2,3}}+\frac{-s_{2,4}s_{3,5}+s_{2,3}s_{4,5}+s_{2,5}s_{4,5}}{s_{1,2}s_{3,4}s_{3,4,5}}\\
&+\frac{s_{2,3}s_{4,5}+s_{2,5}s_{4,5}}{s_{1,2}s_{4,5}s_{3,4,5}}+\frac{-s_{1,4}s_{2,5}-s_{2,4}s_{2,5}-s_{2,4}s_{3,5}+s_{2,3}s_{4,5}}{s_{1,2}s_{3,4}s_{1,2,3,4}}+\frac{s_{4,5}}{s_{1,2,3}s_{1,2,3,4}}+\frac{s_{4,5}}{s_{2,3,4}s_{1,2,3,4}}\\
&+\frac{-s_{2,4}s_{2,5}-s_{2,4}s_{3,5}+s_{2,3}s_{4,5}}{s_{3,4}s_{2,3,4}s_{1,2,3,4}}+\frac{1}{s_{2,3,4,5}}+\frac{s_{4,5}}{s_{2,3,4}s_{2,3,4,5}}+\frac{-s_{2,4}s_{2,5}-s_{2,4}s_{3,5}+s_{2,3}s_{4,5}}{s_{3,4}s_{2,3,4}s_{2,3,4,5}}\\
&+\frac{-s_{2,4}s_{3,5}+s_{2,3}s_{4,5}+s_{2,5}s_{4,5}}{s_{3,4}s_{3,4,5}s_{2,3,4,5}}+\frac{s_{2,3}s_{4,5}+s_{2,5}s_{4,5}}{s_{4,5}s_{3,4,5}s_{2,3,4,5}}\end{aligned} \tag{4.26}$$

up to a factor of 4 due to the replacement of the momentum factor $p_i p_j$ that was present in numerator of the gluon amplitude to $p_i p_j \to s_{ij}/2$. The complete amplitude is the summation over all 15 arrangements and is given in Appendix E.

4.3.2 Mixed Amplitudes

We then turn our attention to the $\mathcal{O}(\tau^1)$ contribution, supposedly describing pions coupled to scalars. The arrangement (12)(34)(56) contributes the following types of amplitudes to the full amplitude

$$\tau[A(\phi_1\phi_2,\phi_3\phi_4,\pi_5,\pi_6)+A(\phi_1\phi_2,\pi_3,\pi_4,\phi_5\phi_6)+A(\pi_1,\pi_2,\phi_2\phi_3,\phi_5\phi_6)], \tag{4.27}$$

and similarly for the other 14 arrangements of the 6 legs. These are calculated starting from the YMS amplitude of every arrangement. For every arrangement of legs, there are three possible mandelstams s_{ij} with which the YMS can be multiplied. This results in a total of 45 contributions at this order in τ. This calculation requires an algorithm in order to correctly keep track of the combinatorics. The full expression is too large to display here. We can however analyse the contributions that appear in the amplitude, e.g. by collecting terms that share a propagator.

The propagators for 6-point diagrams are products of the Mandelstams that enter the cubic vertices. As an example we consider the half-ladder and snowflake diagrams of Figure 4.1. The half-ladder diagram will have a propagator of the form $D_{\text{hl}} = s_{ab}s_{abcc}s_{abcd}$ and the

Figure 4.1: Half-ladder and snowflake diagram for a 6-point amplitude.

snowflake diagram will have a propagator contribution of the form $D_{\text{snowflake}} = s_{ab}s_{cd}s_{ef}$. The terms that share the propagator D_{hl} can be collected, note that we have to be mindful that we also have to account for terms that contain $D = s_{1234}s_{123}$, as these are the result of a cancellation of the term $s_{12}/(s_{12}s_{1234}s_{123})$. Collected these give

$$\frac{s_{45}\left(s_{12}^2 + s_{23}(s_{34} + s_{56}) + s_{12}(s_{23} - 2s_{13} - s_{24} + s_{34})\right)}{s_{12}s_{123}s_{1234}}. \tag{4.28}$$

For the snowflake diagrams we have to consider that by construction of the initial YM amplitudes there are Mandelstams present of the form s_{i6}. However, the package that provides the YM amplitudes will have replaced the momentum p_6 using momentum conservation $p_6 = -(p_1 + p_2 + p_3 + p_4 + p_5)$. We have to use a similar replacement fo the propagators of the snowflake diagrams. These will be altered to $D_{\text{snowflake}} = s_{ab}s_{cd}s_{ef} = s_{ab}s_{cd}s_{abcd}$. As an example, the terms that share the propagator $D = s_{12}s_{34}s_{1234}$ are

$$-\frac{(s_{14}s_{25} + s_{24}(s_{25} + s_{35}) - s_{23}s_{45})s_{56}}{s_{12}s_{34}s_{1234}} = -\frac{s_{14}s_{25} + s_{24}(s_{25} + s_{35}) - s_{23}s_{45}}{s_{12}s_{34}}. \tag{4.29}$$

In order to compare all the terms of the the 6 point partial amplitude to the amplitude obtained from the BCJ bootstrap, we will compare the factorization channels of the partial amplitude. This comparison will be done in section 5.1.

4.3.3 Mandelstam Order of 6-point Amplitude

The counting paramater τ allows us to keep track of how many orders of Mandelstam invariants we introduce in the generalised transmutation process. Here we summarise the order of Mandelstams at different orders in τ:

$\mathcal{O}(\tau^0)$: The contributions at this order for 6-point are purely Yang-Mills scalar amplitudes. All polarisation pairs have been set to 1 according to the arrangements of (4.3). The resulting soft limit $\lim_{p_i \to 0} A_{\text{YMS}}(\phi_1\phi_2, \phi_3\phi_4, \phi_5\phi_6) \sim s^{-1}$. This is in agreement with theory, where every vertex contributes $\propto p$ and every propagator contributes $\propto s^{-1}$. The half-ladder diagrams contain 4 vertices and 3 propagators, together this makes $1/s$ scaling of the amplitude.

$\mathcal{O}(\tau^1)$: The amplitudes of this order are created from the above YMS amplitude by multiplying with one Mandelstam s_{ij}. By this reasoning and by inspection of the amplitude the resulting scaling in Mandelstams can be seen to be combination of s^3/s^3, leading to a scaling of s^0.

$\mathcal{O}(\tau^2)$: These amplitudes gain an additional mandelstam s_{ij} to the above, therefore the scaling is of one order higher, s^1. we have explicitly checked that the amplitude generated by $\mathbb{T}$ at this order give rise to the correct NLSM amplitude, up to a rescaling factor of $-\frac{16}{5}$ that arises from the fact that the same permutation invariant amplitude arises multiple times in the amplitude and some factors of 2 from the definitions of $s_{ij} = 2p_ip_j$.

$\mathcal{O}(\tau^3)$: The composite amplitudes of each arrangement will have a scaling of s^2. We have verified explicitly that the summation over the GT of all arrangements results in a vanishing amplitude.

Chapter 5

Discussion

5.1 Partial 6-point Amplitude of BCJ Bootstrap

We wish to compare the amplitude obtained from the GT-operator with the amplitude that is obtained from the BCJ bootstrap of chapter 2. The latter however, contains colour factors in the amplitude, while the former does not. In order to make our comparison, we recall that we applied the GT-operator on a colour-ordered partial Yang-Mills amplitude $A_6^{\mathrm{YM}}[123456]$. Each partial amplitude is the kinematic weight of a specific trace colour factor according to the trace decomposition

$$\mathcal{A}_m^{\mathrm{tree}} = g^{m-2} \sum_{\sigma \in S_{m-1}} A_m^{\mathrm{tree}}(1, \sigma(2), \sigma(3), \ldots, \sigma(m-1), m) \operatorname{Tr}\left(T^{a_1} T^{a_{\sigma(2)}} T^{a_{\sigma(3)}} \cdots T^{a_{\sigma(m)}}\right). \tag{5.1}$$

Due to this decomposition we therefore know that in the full amplitude, this partial amplitude $A_6^{\mathrm{YM}}[123456]$ was initially accompanied by the colour trace $\operatorname{Tr}\left[T^{a_1}T^{a_2}T^{a_3}T^{a_4}T^{a_5}T^{a_6}\right]$. We can therefore compare the resulting partial amplitude $\mathbb{T}^{(123456)} \cdot A_6^{\mathrm{YM}}[123456]$ to the partial amplitude from the BCJ bootstrap that has the same colour trace $A_6^{\mathrm{BCJ}}[123456]$.

In order to to calculate this partial amplitude, we first need to calculate an expression for the BCJ numerators in the latter method. To do this, we only need to find one numerator, N_{abcdef}, and the other numerators will be permutations of the indices. The numerator N_{abcdef} can be found by decomposing the total amplitude into partial amplitudes in the, by now familiar, way

$$\mathcal{A}_6 = \frac{c_{123456} N_{123456}}{D_{123456}} + \frac{c_{123465} N_{123465}}{D_{123465}} + \ldots. \tag{5.2}$$

However we know that c_{123456} is not the only colour factor that will contain trace ordering [123456] in it's decomposition. Therefore we first decompose the total amplitude in terms of the partial amplitude in the trace decomposition of the colour factors. :

$$\mathcal{A}_6 = \mathrm{Tr}\left[T^{a_1}T^{a_2}T^{a_3}T^{a_4}T^{a_5}T^{a_6}\right]\left[\frac{N_{123456}}{D_{123456}} + \ldots + \frac{N_{123465}}{D_{123465}} + \ldots\right] + \ldots. \tag{5.3}$$

To calculate which diagrams contribute to the partial amplitude, we start from the possible 6-point colour factor for a half ladder diagram.

$$c_1 = \tilde{f}^{a_1 a_2 b}\,\tilde{f}^{b a_3 c}\,\tilde{f}^{c a_4 a_d}\,\tilde{f}^{d a_5 a_6}. \tag{5.4}$$

We have written a Mathematica algorithm that performs the process of (1.29) at n point. This decomposes the colour factor c_1 into following 16 trace contributions:

$$\begin{aligned} c_1 = {} & \mathrm{Tr}\left[T^{a_1}T^{a_2}T^{a_3}T^{a_4}T^{a_5}T^{a_6}\right] - \mathrm{Tr}\left[T^{a_1}T^{a_2}T^{a_3}T^{a_4}T^{a_6}T^{a_5}\right] - \mathrm{Tr}\left[T^{a_4}T^{a_1}T^{a_2}T^{a_3}T^{a_5}T^{a_6}\right] \\ & + \mathrm{Tr}\left[T^{a_4}T^{a_1}T^{a_2}T^{a_3}T^{a_6}T^{a_5}\right] + \mathrm{Tr}\left[T^{a_3}T^{a_1}T^{a_2}T^{a_4}T^{a_5}T^{a_6}\right] - \mathrm{Tr}\left[T^{a_3}T^{a_1}T^{a_2}T^{a_4}T^{a_6}T^{a_5}\right] \\ & - \mathrm{Tr}\left[T^{a_4}T^{a_3}T^{a_1}T^{a_2}T^{a_5}T^{a_6}\right] + \mathrm{Tr}\left[T^{a_4}T^{a_3}T^{a_1}T^{a_2}T^{a_6}T^{a_5}\right] - \mathrm{Tr}\left[T^{a_2}T^{a_1}T^{a_3}T^{a_4}T^{a_5}T^{a_6}\right] \\ & + \mathrm{Tr}\left[T^{a_2}T^{a_1}T^{a_3}T^{a_4}T^{a_6}T^{a_5}\right] + \mathrm{Tr}\left[T^{a_4}T^{a_2}T^{a_1}T^{a_3}T^{a_5}T^{a_6}\right] - \mathrm{Tr}\left[T^{a_4}T^{a_2}T^{a_1}T^{a_3}T^{a_6}T^{a_5}\right] \\ & - \mathrm{Tr}\left[T^{a_3}T^{a_2}T^{a_1}T^{a_4}T^{a_5}T^{a_6}\right] + \mathrm{Tr}\left[T^{a_3}T^{a_2}T^{a_1}T^{a_4}T^{a_6}T^{a_5}\right] - \mathrm{Tr}\left[T^{a_4}T^{a_3}T^{a_2}T^{a_1}T^{a_5}T^{a_6}\right] \\ & + \mathrm{Tr}\left[T^{a_4}T^{a_3}T^{a_2}T^{a_1}T^{a_6}T^{a_5}\right]. \end{aligned} \tag{5.5}$$

We can perform permutations of the indices of the structure constants of c_1 to get the trace decomposition of all other possible colour factor. We then construct the total amplitude in terms of the unspecified numerators and denominators and extract the coefficient of the terms that contain $Tr\left[T^{a_1}T^{a_2}T^{a_3}T^{a_4}T^{a_5}T^{a_6}\right]$ and it's cyclic permutations and reflections in order to obtain all the combination of N/D that contribute to the partial amplitude. A similar analysis needs to be done for the colour factors of the snowflake diagrams. Finally, we would need to consider that these diagrams are not all distinct, but linked by Jacobi relations. Therefore, there is a redundancy in the above calculation.

An alternative method is to use FeynCalc to calculate the full amplitude, which already takes care of the redundancy and output the amplitude in terms of the master colour factors. After this, we need to calculate the trace decomposition of these colour factors to calculate the partial amplitude [1].

[1]Kind regards to Yang Li for providing us with this partial amplitude.

The partial amplitude we are interested in is then denoted as

$$\begin{aligned}A_6[123456] =& \frac{N_{abcdef}}{s_{ab}s_{ef}\left(s_{ab}+s_{ac}+s_{bc}\right)} + \frac{N_{abcdef}-N_{abdcef}}{s_{ab}s_{cd}s_{ef}} + \frac{N_{afbecd}}{s_{af}s_{cd}\left(s_{cd}+s_{ce}+s_{de}\right)} \\ &+ \frac{N_{afebcd}}{s_{af}s_{cd}\left(s_{bc}+s_{bd}+s_{cd}\right)} + \frac{N_{cdefab}}{s_{ab}s_{cd}\left(s_{cd}+s_{ce}+s_{de}\right)} + \frac{N_{defabc}}{s_{bc}s_{de}\left(s_{ab}+s_{ac}+s_{bc}\right)} \\ &+ \frac{N_{efabcd}}{s_{cd}s_{ef}\left(s_{bc}+s_{bd}+s_{cd}\right)} - \frac{N_{bcdeaf}}{s_{af}s_{bc}\left(s_{bc}+s_{bd}+s_{cd}\right)} - \frac{N_{afbcde}-N_{afcbde}}{s_{af}s_{bc}s_{de}} \\ &- \frac{N_{abcfde}}{s_{ab}s_{de}\left(s_{ab}+s_{ac}+s_{bc}\right)} - \frac{N_{abfcde}}{s_{ab}s_{de}\left(s_{cd}+s_{ce}+s_{de}\right)} - \frac{N_{afbcde}}{s_{af}s_{de}\left(s_{cd}+s_{ce}+s_{de}\right)} \\ &- \frac{N_{efdabc}}{s_{bc}s_{ef}\left(s_{ab}+s_{ac}+s_{bc}\right)} - \frac{N_{efadbc}}{s_{bc}s_{ef}\left(s_{bc}+s_{bd}+s_{cd}\right)},\end{aligned} \tag{5.6}$$

where the numerators of the snowflake diagram are clearly taken into account.

The cubic 6-point numerator $N_6^{(3)}$ is obtained through the methodology of subsection 2.4.2, where the general expression for $N_6^{(3)}$ is given by the expression

$$N^{(3)}_{abcdef} = (s_{ac}-s_{bc})s_{ab}(s_{de}-s_{df}) + (s_{ac}-s_{bc})(s_{de}-s_{df})s_{ef} + s_{abc}Q_{abcdef}, \tag{5.7}$$

where Q_{abcdef} is the ansatz of a term that is quadratic in Mandelstams. As discussed, after imposing the BCJ conditions there are 3 unfixed parameters. The resulting Q_{abcdef} is used to construct the numerator $N^{(3)}_{abcdef}$ with which te partial amplitude is calculated. We scale the momentum of leg a through scaling of the Mandelstams $s_{ai} \mapsto xs_{ai}$. The resulting amplitude will have contain terms that are of $\mathcal{O}(x^{-1})$. We impose the soft limit of $\sigma = 0$ by requiring that the coefficients of these terms are tuned such that the $\mathcal{O}(x^{-1})$ contribution vanishes. This leaves an amplitude with leading soft behaviour of $\mathcal{O}(x^0$. This imposing of the soft limit fixes all of the remaining coefficients.

Using these numerators we can construct the partial amplitude of (5.6) by summing over all the permutations of the indices $abcdef$. We should check that this amplitude correctly factorises into the desired 4-point partial amplitudes on the imposed factorisation channel $s_{abc} \to 0$. On this factorisation channel, the amplitude factorises as

$$A_6[123456] \to A_4^{(1)}[123x]\frac{1}{s_{abc}}A_4^{(2)}[x456] + A_4^{(2)}[123x]\frac{1}{s_{abc}}A_4^{(1)}[x456], \tag{5.8}$$

where $A^{(p)}[ijkl]$ denotes the 4-point partial amplitude constructed from the numerator $N_4^{(p)}$. These partial amplitudes are given by Equation 1.31 as

$$A_4^{(p)}[1234] = \frac{N_s^{(p)}}{s} - \frac{N_t^{(p)}}{t}. \tag{5.9}$$

We have checked that the residue of the partial amplitude on the factorisation channel reduces to (5.8) up to a minus-sign. Computationally it is beneficial to calculate the residue by setting $s_{abc} \to \lambda$ and doing a series expansion around $\lambda = 0$. Once this is done, the coefficient of the $\mathcal{O}(\lambda^{-1})$ term is equal to the residue.

5.2 Factorization Comparison

The amplitude obtained from the generalised transmutation has been denoted in terms of the 9 independent Mandelstam basis $\{s_{a,b}, s_{a,c}, s_{a,d}, s_{a,e}, s_{b,c}, s_{b,d}, s_{b,e}, s_{c,d}, s_{c,e}\}$. The residue of this amplitude on the factorisation channel $s_{abc} = s_{ab} + s_{ac} + s_{bc}$ is given by

$$\begin{aligned}\lim_{s_{abc}\to 0} s_{abc} A_6^{\mathrm{GT}} = &-\frac{3\left(s_{ab}+s_{bc}\right)\left(s_{ae}+s_{be}+s_{ce}\right)}{s_{ad}+s_{bd}+s_{cd}} - \frac{s_{cd}\left(s_{ab}+s_{bc}\right)\left(s_{ae}+s_{be}+s_{ce}\right)}{s_{ab}\left(s_{ad}+s_{bd}+s_{cd}\right)} \\ &- \frac{s_{ad}\left(s_{ab}+s_{bc}\right)\left(s_{ae}+s_{be}+s_{ce}\right)}{s_{bc}\left(s_{ad}+s_{bd}+s_{cd}\right)} + \frac{3\left(s_{ab}+s_{bc}\right)\left(s_{ae}+s_{be}+s_{ce}\right)}{s_{ad}+s_{ae}+s_{bd}+s_{be}+s_{cd}+s_{ce}} \\ &+ \frac{\left(s_{ab}+s_{bc}\right)\left(s_{ae}+s_{be}+s_{ce}\right)\left(s_{ab}-s_{cd}-s_{ce}\right)}{s_{ab}\left(s_{ad}+s_{ae}+s_{bd}+s_{be}+s_{cd}+s_{ce}\right)} - \frac{\left(s_{ab}+s_{bc}\right)\left(s_{ae}+s_{be}+s_{ce}\right)\left(s_{ad}+s_{ae}-s_{bc}\right)}{s_{bc}\left(s_{ad}+s_{ae}+s_{bd}+s_{be}+s_{cd}+s_{ce}\right)} \\ &+ \frac{s_{bd}\left(s_{ae}+s_{be}+s_{ce}\right)}{s_{ad}+s_{bd}+s_{cd}} + \frac{\left(s_{ae}+s_{be}+s_{ce}\right)\left(s_{ab}+s_{bc}+s_{bd}+s_{be}\right)}{s_{ad}+s_{ae}+s_{bd}+s_{be}+s_{cd}+s_{ce}} - \frac{s_{ce}\left(s_{ab}+s_{bc}\right)}{s_{ab}} - \\ &\frac{3\left(s_{ab}+s_{bc}\right)\left(s_{ae}+s_{be}+s_{ce}\right)}{s_{ab}} - \frac{s_{ae}\left(s_{ab}+s_{bc}\right)}{s_{bc}} - \frac{3\left(s_{ab}+s_{bc}\right)\left(s_{ae}+s_{be}+s_{ce}\right)}{s_{bc}} \\ &+ 3\left(s_{ae}+s_{be}+s_{ce}\right) - 3\left(s_{ab}+s_{bc}\right) + s_{be}. \end{aligned} \tag{5.10}$$

The amplitude from the BCJ bootstrap A_6^{BC} was confirmed to factorise as (5.8). Explicitly the residue is denoted as

$$\begin{aligned}\lim_{s_{abc}\to 0} s_{abc} A_6^{\mathrm{BCJ}} = &-\frac{\left(s_{ab}+2s_{bc}\right)\left(s_{ad}-s_{ae}+s_{bd}-s_{be}+s_{cd}-s_{ce}\right)}{s_{ab}} \\ &- \frac{\left(2s_{ab}+s_{bc}\right)\left(s_{ad}-s_{ae}+s_{bd}-s_{be}+s_{cd}-s_{ce}\right)}{s_{bc}} + \frac{\left(2s_{ab}+s_{bc}\right)\left(s_{ad}-s_{ae}+s_{bd}-s_{be}+s_{cd}-s_{ce}\right)}{s_{ad}+s_{ae}+s_{bd}+s_{be}+s_{cd}+s_{ce}} \\ &+ \frac{\left(s_{ab}+2s_{bc}\right)\left(s_{ad}-s_{ae}+s_{bd}-s_{be}+s_{cd}-s_{ce}\right)}{s_{ad}+s_{ae}+s_{bd}+s_{be}+s_{cd}+s_{ce}} + \frac{\left(2s_{ab}+s_{bc}\right)\left(s_{ad}+2s_{ae}+s_{bd}+2s_{be}+s_{cd}+2s_{ce}\right)}{s_{bc}} \\ &+ \frac{\left(s_{ab}+2s_{bc}\right)\left(s_{ad}+2s_{ae}+s_{bd}+2s_{be}+s_{cd}+2s_{ce}\right)}{s_{ab}} + \frac{\left(2s_{ab}+s_{bc}\right)\left(s_{ad}+2s_{ae}+s_{bd}+2s_{be}+s_{cd}+2s_{ce}\right)}{s_{ad}+s_{bd}+s_{cd}} \\ &+ \frac{\left(s_{ab}+2s_{bc}\right)\left(s_{ad}+2s_{ae}+s_{bd}+2s_{be}+s_{cd}+2s_{ce}\right)}{s_{ad}+s_{bd}+s_{cd}}, \end{aligned} \tag{5.11}$$

where it can be seen that there are multiple contributions with the same denominator, but which cannot be expressed together as this would not fit on the page. In total there are 4 terms, which corresponds to the four terms obtained from the product of (5.9). We can distinctly identify the four-point

On the factorsiation pole s_{abc} these residues differ. There are several key aspects that can be identified. The first being that there are more exchange channels present in A_6^{GT} that are not present in A_6^{BCJ}. check factorisation into YMS partial amplitude times pion amplitude

We also note that there are some similarities, which become clearer once we take further take the residue on the channel $s_{ab} \to 0$,

$$\lim_{s_{ab}\to 0}\lim_{s_{abc}\to 0} A_6^{\mathrm{GT}} = -\frac{s_{bc}s_{cd}\left(s_{ae}+s_{be}+s_{ce}\right)}{4\left(s_{ad}+s_{bd}+s_{cd}\right)} - \frac{\left(s_{cd}+s_{ce}\right)s_{bc}\left(s_{ae}+s_{be}+s_{ce}\right)}{4\left(s_{ad}+s_{ae}+s_{bd}+s_{be}+s_{cd}+s_{ce}\right)} - \frac{3}{4}s_{bc}\left(s_{ae}+s_{be}+s_{ce}\right) - \frac{1}{4}s_{bc}s_{ce}, \tag{5.12}$$

$$\lim_{s_{ab}\to 0}\lim_{s_{abc}\to 0} A_6^{\mathrm{BCJ}} = 6s_{bc}\left(s_{ae}+s_{be}+s_{ce}\right). \tag{5.13}$$

In this limit, it becomes apparent that both partial amplitudes have a common contribution $s_{bc}\left(s_{ae}+s_{be}+s_{ce}\right)$ up to a factor. The BCJ-bootstrap amlitude has no remaining exchange channels, but the GT amplitude clearly does.

It would be interesting to investigate what kind of diagrams contribute to these factorisation limits. We first note that we can recognise two partial amplitudes in the factorisation procedure. The first is the 4-point Yang-Mills-Scalar gluon exchange channel

$$A_4^{\mathrm{YMS}}(\phi_a\phi_b,\phi_c\phi_x) = \frac{\left(s_{ab}+s_{bc}\right)}{s_{ab}}, \tag{5.14}$$

which is can be found in both (5.10) and (5.11). The diagram corresponding to this amplitude is shown in Figure **??** Specifying leg x is not necessary due to momentum conservation. We will therefore identify leg x as an internal leg.

The second type of contribution we encounter in both of the residues is the 4-point NLSM amplitude consisting of the legs d, e and f and an 'internal' leg x

$$A_4^{\mathrm{NLSM}}(\pi_d,\pi_e,\pi_f,\pi_x) = (s_{de}+s_{ef}) = (s_{ae}+s_{be}+s_{ce}). \tag{5.15}$$

Note that this type of amplitude is equal to an amplitude of 2 scalars interacting with 2 pions of the form $A(\phi,\phi,\pi,\pi)$ that is shown in Figure 5.1c.

The only factor that remains to be identified is the factors of s_{bc} in (5.13). We find that this factor is precisely the residue of the YMS amplitude in (5.14) on the channel $s_{ab} \to 0$

$$\lim_{s_{ab}\to 0} s_{ab}A_4^{\mathrm{YMS}}(\phi_a\phi_b,\phi_c\phi_x) = s_{bc}. \tag{5.16}$$

We can therefore conclude that a diagram such as the diagram displayed in Figure 5.1a is responsible for the residue in (5.13) and for a section of the residue of (5.12). It is not suprising that these are the factorisation structures that occur. After all, one of the key assumptions that we imposed on the numerator $N_6^{(3)}$ is that it factorises into the product of two four-point amplitudes. The 4-point numerators that are combined to make the cubic order numerator are $N_4^{(1)}$ and $N_4^{(2)}$, which individually created the 4-point gluon exchange channel and the 4-point NLSM contact term respectively. This exactly matches the factorisation we have found.

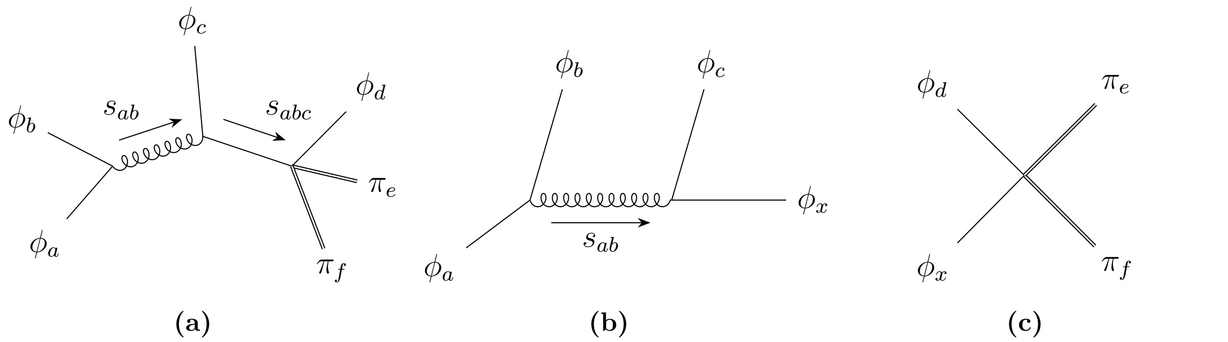


Figure 5.1: Figure **(a)** shows the type of diagram that can contribute to the residue of both A_6^{GT} and A_6^{BCJ}. On the factorisation limit where $s_{abc} \to 0$ this diagram splits into the product of the 4-point YMS amplitude shown in **(b)** and the 4-point NLSM amplitude shown in **(c)**.

The remaining contributions to the residue of A_6^{GT} are interesting to study aswell, as these are apparently not produced through the BCJ bootstrap method. We can see that after imposing the factorisation of s_{abc} there are a few terms that do not have any remaining propagation channels. These contributions can also be seen to of the form of the 4-point NLSM amplitudes in (5.15) multiplied by a constant, which could indicate a ϕ^4 scalar contact term. Such contributions could then potentially arise from the factorisation of the diagram in Figure 5.2.

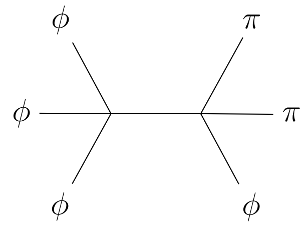


Figure 5.2: Diagram that factorises into a quartic ϕ^4 contact term amplitude and a 4-point NLSM contact term.

It is interesting that such contact terms would arise through transmutation for multiple reasons. The first being that such contact terms do not arise during the transmutation at 4-point partial amplitudes that we discussed in subsection 3.2.5.

The second reason is that such ϕ^4 contact terms are not discussed in the original publication by Cheung and collaborators [22], presumably because they do not occur in typical YMS formulations [18, 80]. However, these are explicitly mentioned to occur when discussing transmutation operators in the CHY representation [81]. It should be noted that these contact terms also deviate from the 'extended NLSM' described in [35] and [69], where mixed amplitudes of pions and scalar are obtained by extending the $U(N)$ NLSM to include a cubic biadjoint scalar self-interaction. It should be noted that these theories do not include gauge interactions.

It is not a suprise that these quartic contact terms do not arise from the BCJ bootstrap method. The numerator in this framework that gives rise to a contact diagram at four point is $N_4^{(2)}$, but this gives rise to a NLSM $\phi^2(\partial\phi)^2$ contact term, not a ϕ^4 term. The only other 4-point amplitudes that we can construct in this framework are the gluon exchange amplitudes.

There is another combination of factorisation channels that is interesting to check. We will again investigate the limit where two internal propagators go on-shell, but this time we will set $s_{ab} \to 0$ and $s_{ef} \to 0$. The residue of both approaches on these limits is

$$\begin{aligned}\lim_{s_{ab}\to 0}\lim_{s_{ef}\to 0} A_6^{\text{GT}} &= \frac{s_{b,c}s_{c,d}\left(s_{a,e}+s_{b,e}+s_{c,e}\right)}{s_{a,d}+s_{b,d}+s_{c,d}} - s_{b,c}\left(s_{a,e}+s_{b,e}+s_{c,e}\right), \\ &\quad - s_{a,d}s_{b,e} - s_{b,d}s_{c,e} - s_{b,d}s_{b,e}, \end{aligned} \tag{5.17}$$

$$\lim_{s_{ab}\to 0}\lim_{s_{ef}\to 0} A_6^{\text{BCJ}} = \frac{3\left(s_{a,d}+s_{b,d}\right)\left(s_{a,e}\left(s_{b,c}+s_{b,d}\right)+s_{b,e}\left(s_{b,c}+s_{b,d}+s_{c,d}\right)\right)}{s_{c,d}}. \tag{5.18}$$

We find that there on this limit, both amplitudes still allow for an exchange channel. As the only exchange channel left in the BCJ amplitude is of the form s_{ij}, we conclude that this residue is derived from a snowflake diagram that has $D \sim \frac{1}{s_{ab}s_{cd}s_{ef}}$.

For the amplitude obtained through generalised transmutation, there is also a contribution to the residue with no exchange channels left, identical to the contributions to the residue of (5.11) and a new type of residue that could arise from a diagram factorised on two 4-point YMS exchange channels in a snowflake diagram. The remaining amplitude should contribute no additional momenta. This is possible if the contributions from the pions cancel the remaining exchange pole, similar to how the longitudinal operator turns an amplitude with an exchange channel of the form $A(\phi,\phi,g,g)$ into a contact diagram of the form $A(\pi,\pi,\pi,\pi)$. A proposal for such a diagram is shown in Figure 5.3. This interaction would require a Lagrangian that includes the interaction $(D\phi)^2F^2$, hinting at a higher-derivative correction to the gNLSM theory that differs from the correction considered in section 2.4. It would be interesting to study this in more detail, but we leave this for future work.

Figure 5.3: Proposed diagram before (left) and after (right) longitudinal transformation, that gives rise to the $s_{ij}s_{kl}$ residues on the factorisation limits $s_{ab} \to 0$ and $s_{ef} \to 0$

Chapter 6

Conclusion and Outlook

In conclusion, this thesis provides an in-depth investigation into the Unifying Relations for scattering amplitudes and the recent BCJ bootstrap that can be used to construct kinematic numerators that conform to the duality between colour and kinematics. Prior to this, we performed an extensive analysis of gluon (Yang-Mills) amplitudes and pion (NLSM) amplitudes, exploring the duality between colour and kinematics, as well as examining the behaviour of these amplitudes in special kinematic limits.

We have extensively discussed the construction of amplitudes of the gauged nonlinear sigma-model using the BCJ bootstrap method. This theory appeared to have a hybrid soft degree, where a subsector of the amplitudes generated from the theory obey a soft degree of $\sigma = 0$, and another subsector generates pion contact terms that obey the Adler zero of $\sigma = 1$.

Using the BCJ bootstrap, we have constructed the $\sigma = 0$ amplitude that occurs at 6-point. To perform this derivation, we have imposed the BCJ conditions on an ansatz for numerators composed of Mandelstam variables in a specific order. The coefficients of the ansatz were fixed by imposing the soft degree of $\sigma = 0$. The amplitudes that are constructed describe adjoint Goldstone scalars ϕ^a interacting with gluons.

We aimed to answer the question

> *How can we construct a transmutation operator that generates amplitudes for comparison with those derived from the BCJ bootstrap?*

This was achieved by generalising the transmutation relations proposed by Cheung et al. such that we were able to generate mixed amplitudes of scalars and pions that exchange gluons through transmutation of Yang-Mills amplitudes. This generalised transmutation (GT) operator was constructed such that we were able to keep track of the order of Mandelstam variables that were introduced to the transmuted amplitudes, allowing us to determine whether we had achieved the $\sigma = 0$ soft degree that we aimed to construct.

Using this GT-operator, explicit calculations of the transmutation of 6-point Yang-Mills partial amplitude were performed using computational tools. At different orders in Mandelstam invariants, tracked by the parameter τ, these gave rise to the correct expressions for Yang-Mills-Scalar (YMS) at $\mathcal{O}(\tau^0)$ and Nonlinear Sigma Model (NLSM) amplitudes at $\mathcal{O}(\tau^2)$. The vanishing of the amplitude at $\mathcal{O}(\tau^3)$,when the longitudinal operator $\mathcal{L}$ is applied to all external gluons, i.e., when all pairs of polarisations are set to pairs of momenta, was also verified computationally at 6-point.

This leaves the $\mathcal{O}(\tau^1)$ amplitude that contains the $\sigma = 0$ soft degree to be analysed, which brings us to our second research question

> *What types of amplitudes can be derived from a generalised transmutation?* operator?

The answer to this question can be found by studying the factorisation of the resulting amplitude. In the limit where the internal exchange particles are taken to be on-shell we found the GT-amplitudes to mostly adhere to familiar lower point amplitudes. Examples of these lower-point amplitudes are the 4-point YMS amplitude, where two pairs of scalars exchange a gluon and the 4-point NLSM contact term. Beyond this, we unexpectedly encountered the 4-point ϕ^4, which does not typically occur in YMS theories but which can occur in the CHY formalism. We also interpreted that a $(\partial\phi)^2A^2$ contact term is necessary for the factorisation of 6-point snowflake-diagrams.

> *How can generalized transmutation operators inform us about the uniqueness of the theory developed via the BCJ bootstrap?* operator?

In generality the generalised transmutation operator fails to land on the same partial amplitude as the amplitude obtained from the BCJ bootstrap method. However, there are common structures in the amplitudes of the two different methods. This indicates that generalised transmutation is able to reconstruct some of the interactions that occur in the BCJ bootstrap in the $\sigma = 0$ sector, namely the set of amplitudes that factorise into the 4-point YMS exchange channel and the 4-point pion contact term. The snowflake diagrams of the 6-point BCJ bootstrap-amplitude were also identified, but cannot be matched to amplitudes obtained from GT due to the cancellation of one of the gluon exchange channels when two external gluons are transmuted into pions.

This leads us to conclude that the amplitudes of the BCJ bootstrap are not uniquely fixed by their soft degree of $\sigma = 0$, as opposed to amplitudes such as the NLSM and SG that are fully determined through its soft behaviour.

It would be interesting to study how the amplitudes obtained from GT compare to the higher-derivative corrections to Yang-Mills discussed in Equation 1.5.3. Investigating the behaviour of generalised transmutation on these higher-derivative corrections could also provide valuable insights into CK-duality compatibility between numerators of different theories. It would also be intriguing to explore the transmutation of these amplitudes and see if this contributes additional terms to the $\sigma = 0$ GT amplitude. Finally, comparing the obtained amplitudes to the field theory limit of the amplitudes of Z-theory [59] could strengthen the approach.

On a broader scale, it would be valuable to extend the investigation of transmutation operators. First, extending the transmutation calculations to include higher-order Mandelstam variables, i.e. a transmutation of $(e_i e_j) \mapsto (1 + s + s^2 + \dots)$ instead of purely $(1 + s)$ would be an interesting avenue to explore. Additionally, applying generalised transmutation techniques to the amplitudes of the other theories in Figure 3.2 presents an intriguing possibility for further study. Such transmutations could lead to mixed amplitudes of Born-Infeld photons interacting with Maxwell photons that interact through graviton exchange through GT of graviton amplitudes.

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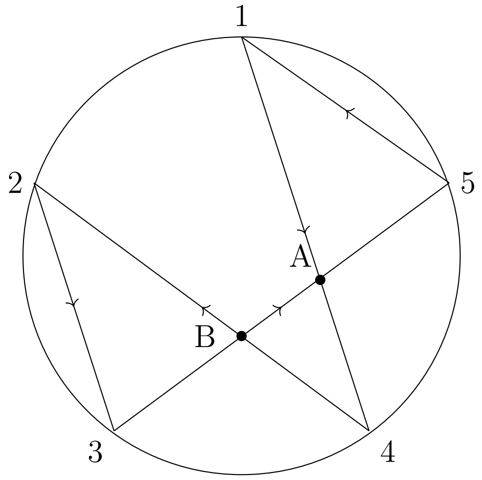
Appendix A

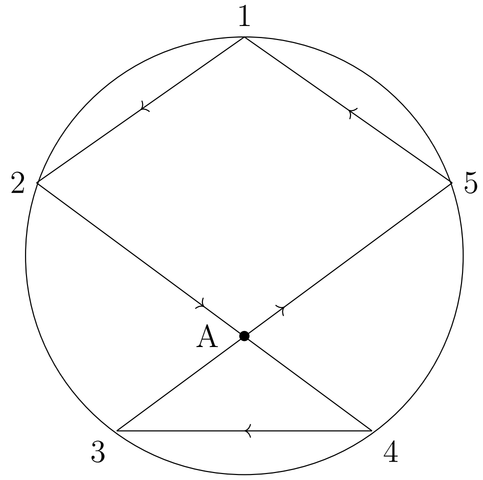
Diagrammatic calculation of double color ordered BAS amplitudes

A convenient method proposed by Cachazo, He and Yuan [82] to calculate the double color use diagrams with the partial orderings from which we can read off the propagators s_{ij} that make up the partial amplitudes. The 5-point partial amplitude $A_{BAS}(12345|14235)$ serves as a good example to demonstrate this method [83].

We construct a disk with the first ordering (12345) depicted counter-clockwise on it's boundary. Next we draw aloop of lines from point to point following the second ordering (14235), as one would with a 'connect the dots' drawing. The result is depicted in figure Figure A.1.



(a)



(b)

Figure A.1: Diagrams for $A_5(1,2,3,4,5|1,4,2,3,5)$ and $A_5(1,2,3,4,5|1,2,4,3,5)$

This diagram is the only such diagram that can contribute to this partial amplitudes.

Different diagrams, such as figure Figure A.1 correspond to different partial amplitudes. To calculate the amplitude, we consider the borders that are shared by both the outer disk and the internal lines. For figure Figure A.1b the borders $\{1,5\}$ and $\{2,3\}$ indicate the exchange channels s_{15} and s_{23}. These shared borders are also present in the ordering of A_{BAS}. The partial amplitude is then the product of these two channels

$$A_{BAS}(12345|14235) = \frac{1}{s_{23}} \frac{1}{s_{51}}, \tag{A.1}$$

up to an overall sign. Similarly, for the diagram in Figure A.1b, the amplitude $A_{BAS}(12345|12435)$ shares the boundaries {3,4} and {5,1,2}. These indicate the channel $1/s_{34}$ and $1/s_{512} = 1/(s_{12} + s_{51})$, which gives us (up to an overall sign)

$$A_{BAS}(12345|12435) = \frac{1}{s_{34}} \left(\frac{1}{s_{12}} + \frac{1}{s_{51}} \right). \tag{A.2}$$

In the diagrammatic approach, the sign is determined through the following rules

- For polygons with an odd number of vertices, a positive sign is assigned when their orientation aligns with that of the disk, and a negative sign when it does not.
- Polygons with an even number of vertices consistently receive a negative sign.
- Every intersection point results in the addition of a negative sign.

It should be noted that different assignment of signs can be used, as long as we stay consistent in applying the same rules to all amplitudes. This way, the relative signs are conserved. The resulting double-ordered amplitudes are then given by

$$A_{BAS}(12345|12435) = -\frac{1}{s_{23}} \frac{1}{s_{51}}, \tag{A.3}$$

$$A_{BAS}(12345|14235) = -\frac{1}{s_{34}} \left(\frac{1}{s_{12}} + \frac{1}{s_{51}} \right). \tag{A.4}$$

Appendix B

KLT double copy

We have extensively discussed the BCJ double copy, but there is another formulation of the double copy, namely the Kawai-Lewellen-Tye (KLT) formulae [21]. Similar to the BCJ formulation, it allows us to construct tree-level gravity amplitudes $\mathcal{M}^{tree}$ by taking the (KLT) product of purely-adjoint gauge theory amplitudes. For 3- and 4-point they explicitly give the graviton amplitudes in the form of

$$\mathcal{M}_3^{tree}(1,2,3) = iA_3^{tree}(1,2,3)\tilde{A}_3^{tree}(1,2,3), \tag{B.1}$$

$$\mathcal{M}_4^{tree}(1,2,3,4) = -is_{12}A_4^{tree}(1,2,3,4)\tilde{A}_4^{tree}(1,2,3,4), \tag{B.2}$$

where the amplitudes A^{tree} and $\tilde{A}^{tree}$ are the colour-ordered partial amplitudes discussed in subsection 1.1.4. Through the double copy, there exists a mapping between the on-shell gluon polarisation vectors e^i_μ and the polarisations of the fields corresponding to the resuting amplitude $e^i_\mu e^j_\nu$. Namely, these can be decomposed into three combinations representing a graviton polarisation $e^h_{\mu\nu}$, a B-field polarisation $e^B_{\mu\nu}$ and the dilaton polarisation $e^\phi_{\mu\nu}$. Respectively these are the symmetric-traceless, the antisymmetric and the scalar mode. An action that describes these three fields is

$$S = \int d^Dx\sqrt{-g}\left[-\frac{1}{2}R + \frac{1}{2(D-2)}\partial^\mu\phi\partial_\mu\phi + \frac{1}{6}e^{-4\phi/(D-2)}H^{\lambda\mu\nu}H_{\lambda\mu\nu}\right], \tag{B.3}$$

where $H_{\lambda\mu\nu}$ is field strength of the two-index antisymmetri tensor $B_{\mu\nu}$ that describes the B-field. This action can be truncated to Einstein gravity by requiring $\mathbb{Z}_2$ symmetries. If we wish to describe graviton amplitudes as a product of gluon amplitudes through the KLT product instead of from the action, we will have to pick the gluon polarisations such that they are in the symmetric-traceless combination.

At m-point, the KLT product between gauge theory amplitudes is given by

$$\mathcal{M}_m^{tree} = -i\sum_{\sigma,\rho\in S_{m-3}} A_m^{tree}(1,\sigma,m-1,m)S[\sigma|\rho]\tilde{A}_m^{tree}(1,\rho,m,m-1). \tag{B.4}$$

The constituent tree-level amplitudes are colour-ordered with leg 1, leg m and leg $(m-1)$ fixed. This fixing then leaves $(m-3)!$ permutations of external legs (specified by σ and ρ) that are summed over. The permutation also specify the elements of the 'KLT kernel' $S[\sigma|\rho]$ that are the amplitudes are multiplied with. These elements are polynomials of products of p_i's, which produce the factors of s_{ij} seen in (B.1).

The KLT kernel solves several problems that would occur in a naive product of gauge theory amplitudes. Firstly it cancels the double poles that occur, which would make for an unphysical amplitude. Secondly it provides the 'missing' poles that have to be included due to the gravity amplitude not being colour-ordered. Consider the 4 point example. The colour-stripped gluon tree amplitude $A_4[1234]$ has simple poles in s_{12} and s_{14}, but none in s_{13} as these legs are not adjecent in the colour-ordering. Naively squaring results in $A_4[1234]^2$ which has double poles in $s_{12}=0$ and $s_{14}=0$. However, the 4-graviton tree amplitude $M_4(1234)$ has a pole in all three channels s_{12}, s_{13} and s_{14} (the s, t, and u channel). The double copy can be written as the KLT copy in the following way:

$$M_4(1234) = A_4[1234]S_4[1234|1234]A_4[1234]. \tag{B.5}$$

The KLT kernel $S_4[1234|1234]$ can be calculated to be

$$S_4[1234|1234] = -\frac{s_{12}s_{14}}{s_{13}}, \tag{B.6}$$

which provides exactly the cancellation of the double poles in s_{12} and s_{14} and introduces the new pole in s_{13}.

Originally the KLT relations were derived in the field of string theory as relations between colour-stripped disk amplitudes of open-strings. In these relations, there is an explicit dependence on the string tension α'. In the low energy limit, where α' goes to zero, the string-theory KLT kernel reduces to the field theory KLT kernel described. Interestingly the string tension α' can be used to formulate a pertubative expansion of higher-order corrections to the field theories. In the following sections we will explore constructing such higher-order order corrections by starting from field theory amplitudes instead of string theory amplitudes.

Appendix C

Z-Theory

It was found that the amplitudes of the open-superstring can be written as the double copy of color-ordered YM amplitudes together with certain functions F^σ [58, 84]

$$A(1,\ldots,N){=}\sum_{\sigma\in S_{N-3}} A_{YM}(1,2_\sigma,\ldots,(N-2)_\sigma,N-1,N)\ F^\sigma_{(1,\ldots,N)}(\alpha') \tag{C.1}$$

The F functions are later re-named as Z functions Z_P that give rise to the name Z-theory. The function is an integral over the boundary of a disk worldsheet[1] with two orderings Q and P. The first denotes the ordering in the integrand and is responsible for dictating which color-ordering of subamplitude is being computed. The latter the defines the ordering of the integration domain. Explicitely the functions are denoted as

$$Z_P(q_1,\ldots,q_n)\equiv(\alpha')^{n-3}\int_{D(P)}\frac{dz_1dz_2\cdots dz_n}{\mathrm{vol}(SL(2,\mathbb{R}))}\frac{\prod_{i<j}^n|z_{ij}|^{\alpha' p_i p_j}}{z_{i_1i_2}z_{i_2i_3}\cdots z_{i_ni_2}}, \tag{C.2}$$

where $z_{ij}=z_i-z_j$ give rise to the propagators as seen in the denominator. The domain $D=\{-\infty<z_1<z_2<\cdots<z_n<+\infty\}$. The functions behave like scalar amplitudes, and exhibit a double ordering. The Z-function obey KK-relations similar to (1.21) and BCJ relation similar to (1.26) along the ordering Q. Additionally they obey the string-theory-monodromy relations along integration domain P

$$0=\sum_{j=2}^{n-1}\exp\left[i\pi\alpha'(k_{p_1}\cdot k_{p_2p_3\ldots p_j})\right]Z_{p_2p_3\ldots p_jp_1p_{j+1}\ldots p_n}(Q), \tag{C.3}$$

which leaves a basis of $(n-3)!$ integration domains.

[1] In string theory, a worldsheet is the two-dimensional surface traced out by a string as it propagates through spacetime. This concept generalizes the idea of a particle's worldline to higher dimensions, providing a framework for analyzing the dynamics and interactions of strings.

After applying all these relations we are left with the Q-ordered functions that simply obey field-theory relations. [59] conjectures that the resulting functions are the color-ordered scattering amplitude for *some* EFT.

The fact that Z-functions obey the Kleiss-Kuijf (KK) and Bern-Carrasco-Johansson (BCJ) relations suggests they correspond to scattering amplitudes for a conjectural, yet unidentified, doubly-colored theory, hereby referred to as Z-theory. Consequently, the Z-functions will be referred to as Z-amplitudes.

It is also shown [85] that the $\alpha' \to 0$ limit of Z-functions aligns with the inverse of the KLT matrix

$$\lim_{\alpha \to 0} Z_P(Q) = m[P|Q] \tag{C.4}$$

This is later demonstrated to correspond to the (doubly-partial) tree-level amplitudes of Born-Infeld-Skyrme (BAS) theory.

$$M_{\phi^3} = \sum_{\sigma,\rho \in S_{n-1}} \mathrm{Tr}\big(T_1 T_{\sigma(2)} \ldots T_{\sigma(n)}\big)\, \mathrm{Tr}\Big(\tilde{T}_1 \tilde{T}_{\rho(2)} \ldots \tilde{T}_{\rho(n)}\Big)\, m[1,\sigma(2,\ldots,n)|1,\rho(2,\ldots,n)] \tag{C.5}$$

It is intriguing that even though string theory does not include a BAS, the tree-level amplitudes of this theory are nevertheless contained within string tree-level amplitudes.

Z-amplitudes serve as double-copy factors for open-superstring scattering amplitudes, containing all orders in α' and obeying field-theory scattering amplitude relations. The low-energy α' expansion should be identifiable as the scattering amplitudes of known theories.

NLSM from Abelian Z-theory

It was then shown that it is possible to replace Yang-Mills factors of Abelian open-string amplitudes by gauge-theory colour factors. This replacement gives rise to Abelian Z-amplitudes. In the field theory limit the leading order α' contributions then give rise to NLSM amplitudes

$$A_{NLSM}(1,2,\ldots,n) = \lim_{\alpha' \to 0} (\alpha')^{2-n} \sum_{\sigma \in S_{n-1}} Z_{1\sigma(2,3,\ldots,n)}(1,2,\ldots,n). \tag{C.6}$$

Unlike the amplitudes in BAS ar theories, NLSM amplitudes come from specific nonzero orders $(\alpha')^{n-2}$. These amplitudes then denote the color-ordered NLSM amplitudes. Explicitely at $n=4$ and $n=6$ for example they give rise to the correct NLSM amplitudes.

$$A_{NLSM}(1,2,3,4) = \pi^2(s_{12}+s_{23}) \tag{C.7}$$

$$A_{NLSM}(1,2,\ldots,6) = \pi^2\left[s_{12} - \frac{1}{2}\frac{(s_{12}+s_{23})(s_{45}+s_{56})}{s_{123}} + cyclic(1,2,3,4,5,6)\right] \tag{C.8}$$

Besides the leading order surviving contributions in α' there are also the subleading terms. These correspond to the 'stringy' higher-derivative corrections to NLSM amplitudes.

Appendix D

Transmutation operators

D.1 Kinematics

The initial amplitude A which we will be transforming is defined with the on-shell conditions $p_i p_i = p_i e_i = e_i e_i 0$ and conserves momentum $\sum_i p_i = 0$. The on-shell conditions are already preserved by defining the amplitude to strictly be a function of the kinematic data $e_i e_j$, $p_i e_j$ and $p_i p_j$. The momentum conservation can be enforced by defining a *Total momentum operator* $\mathcal{P}_v$:

$$P_v \equiv \sum_i p_i v = (p_1 + p_2 + \ldots + p_n)v,, \tag{D.1}$$

where the sum runs over all external legs i and v denotes any momentum or polarisation vector. The inclusion of the somewhat arbitrary vector v is necessary for the operator to be Lorentz invariant. This total momentum operator annihilates the amplitude due to the presence of a momentum-conserving delta-function $\delta^{(D)}(p_1 + \ldots + p_n)$. Momentum conservation of the transmuted amplitude then has to satisfy:

$$P_v \cdot (\mathcal{T} \cdot A) = 0 = \mathcal{T} \cdot (P_v \cdot A) \tag{D.2}$$

The requirement for momentum conservation can then be cast into the following requirement:

$$[P_v, \mathcal{T}] = 0. \tag{D.3}$$

This is not satisfied for every possible differential operator which contains derivatives with respect to $p_i p_j$ and $p_i e_j$, therefore the condition constraints the spectrum of possible operators. The total momentum operator satisfies the condition $[P_v, P_w] = 0$.

D.2 Gauge invariance

The condition of gauge invariance is translated into a differential operator through the Ward identity. Recall that for a given scattering amplitude with an external gauge boson of momentum k, the amplitude can be written as $A(k) = e_\mu(k)A^\mu$. The ward identity then gives rise to a vanishing amplitude when replacing the polarisation vector e_μ by the corresponding momentum vector: $k_\mu A^\mu(k) = 0$. The differential Ward operator can therefore be defined as:

$$W_i \equiv \sum_v p_i v \partial_{ve_i}. \tag{D.4}$$

The summation over v runs over all external momentum and polarisation vectors in the amplitude. The polarisation of leg i appears in the amplitude in the Lorentz invariants e_ip_j and e_ie_j. What this operator essentially does is replace every $e_i \to p_i$ wherever it shows up in the amplitude, turning $e_ip_j \to p_ip_j$ and $e_ie_j \to p_ie_j$. To demonstrate this operator we explicitely denote the Ward operator for a three point amplitude:

$$W_1 = p_1(p_1\partial_{p_1e_1} + p_2\partial_{p_2e_1} + p_3\partial_{p_3e_1} + e_1\partial_{e_1e_1} + e_2\partial_{e_2e_1} + e_3\partial_{e_3e_1}) \tag{D.5}$$

Any gauge invariant amplitude then satisfies the Ward identity and is therefore annihilated when the Ward operator acts on it:

$$W_i \cdot A = 0. \tag{D.6}$$

Requiring that gauge invariance be preserved for a transmuted amplitude then implies a vanishing result when the commutator $[\mathcal{T}, W_i]$ acts on an amplitude:

$$W_i(\mathcal{T} \cdot A) = \mathcal{T}(W_i \cdot A) = 0 \tag{D.7}$$

In the case of W_i, we can check that the operator itself preserves the gauge invariance of the amplitude as seen by calculating

$$[W_i, W_j] = \sum_{v,w} [p_i v \partial_{we_i}, p_j w \partial_{we_j}] = 0. \tag{D.8}$$

Moreover, it can be shown to satisfy our previous requirement of momentum conservation:

$$[W_i, P_v] \cdot A = \sum_{j,w} [p_i w \partial_{we_i}, p_j v] \cdot A = \delta_{ve_i} P_{p_i} \cdot A = 0. \tag{D.9}$$

D.2.1 Basis for Transmutation Operators

Starting from the most general form that a transmutation can take we can apply our condition for momentum conservation and gauge invariance in order to constrain the general form to form basic elements for the transmutation operators. The operators take the form of first order differential equations which act on the objects e_ie_j, e_ip_j and p_ip_j. The general form can be written as:

$$\mathcal{T} \equiv \sum_{i,j} A_{ij}\partial_{p_ip_j} + B_{ij}\partial_{p_ie_j} + C_{ij}\partial_{e_ie_j} \tag{D.10}$$

Here, A_{ij}, B_{ij} and C_{ij} are general functions of external kinematic data and for later notation it is useful to choose $A_{ii} = B_{ii} = C_{ii} = 0$. We then impose the constraints of momentum conservation and gauge invariants upon this ansatz. First, momentum conservation implies

$$[T, P_v]= \sum_{i,j,k}[A_{ij}\partial_{p_ip_j} + B_{ij}\partial_{p_ie_j}, p_kv] = 0, \tag{D.11}$$

where we can choose either $v = p$ or $v = e$ respectively to optain:

$$\sum_i A_{ij} + A_{ji}= \sum_i B_{ij} = 0 \tag{D.12}$$

This in turn implies that the rows and colums of A_{ij} sum to zero, the columns of B_{ij} sum to zero and C_{ij} remains unconstrained. In terms of transmutation operators this leaves a set of commuting operators which will form the basic building blocks for more complicated operators. The operators are denoted as

$$T_{ij} \equiv \partial_{e_ie_j} \tag{D.13}$$
$$I_{ijk} \equiv \partial_{p_ie_j} - \partial_{p_ke_j} \tag{D.14}$$
$$T_{ijkl} \equiv \partial_{p_ip_j} - \partial_{p_kp_j} + \partial_{p_kp_l} - \partial_{p_ip_l} \tag{D.15}$$

These operators have the following symmetry properties:

$$T_{ij} = T_{ji} \tag{D.16}$$
$$I_{ijk} = -I_{kji} \tag{D.17}$$
$$T_{ijkl} = -T_{kjil} = T_{klij} = -T_{ilkj} \tag{D.18}$$

Next we turn our attention to the constraint of gauge invariance. We can see that T_{ij} is intrinsically gauge inviariant, as $[T_{ij}, W_k] = \sum_v[\partial_{e_ie_j}, p_iv\partial_{ve_i}] = 0$.

Furthermore, we can see that I_{ijk} is not instrinsically gauge invariant. However, this does not pose a problem for as it can be shown to be *effectively* gauge invariant. The commutator is $[I_{ijk}, W_l] = \delta_{il}T_{ij} - \delta_{kl}T_{jk}$, which can be combined with supplemental operators allowing the vanishing of the amplitude.

The third operator strips of pairs of polarisation vectors and will therefore give rise to an object with double poles in momenta. The object generated after transmutation is therefore not a physical scattering amplitude. Regardless, this operator does appear in the calculation of the subleading soft factor of Born-Infeld theory.

In the next sections, we will see how these operators are the buiding blocks to connecting the amplitudes of various theories in a unified web.

D.3 Dimensional Reduction: Pions from higher dimensional gluons

The processes of the trace operator $T[\alpha]$ and the Longitudinal operator $T\mathcal{L}$ also have a more physical interpretation, namely a dimensional reduction [86]. Here we will discuss the variation of a dimensional reduction that is identical to the longitudinal operator.

In this mechanism, massless gluons are constructed to live in a $(2d+1)$-dimensional space which will be reduced to a d-dimensional subspace. This construction implies that the momentum vectors $\mathcal{P}_i^M$ are $(2d+1)$-dimensional and are denoted with capital letters as Lorentz indices:

$$\mathcal{P}_i^M = (p_i^\mu, 0, 0) \tag{D.19}$$

The first and third entry of this vector are both d-dimensional and have Greek indices, while the middle entry is 1 dimensional. The external polarizations can also be seen to live in a $(2d+1)$-dimensional space. We will pick the external polarizations to be

$$\mathcal{E}_1^M = \mathcal{E}_n^M = (0, 1, 0) \qquad \mathcal{E}_i^M = (p_i^\mu, 0, ip_i^\mu) \qquad i \neq 1, n. \tag{D.20}$$

Here we have set leg 1 and n to be polarized in their own 1-dimensional space while the other legs live in the d-dimensional subspaces. The reasoning for this specific choice of polarizations apparant as we compute the Lorentz invariant kinematic products of these vectors. The products of higher dimensional momenta gives rise to the product of d-dimensional momenta

$$\mathcal{P}_i\mathcal{P}_j = p_ip_j \quad . \tag{D.21}$$

Furthermore, the product of $\mathcal{E}_1^M$ and $\mathcal{E}_n^M$ will return 1 (in the 1-dimensional subspace).

$$\mathcal{E}_1\mathcal{E}_n = 1. \tag{D.22}$$

Moreover, by construction these two polarizations are orthogonal to the polarizations of the other external legs as seen in (D.3). The other internal polarization products vanish:

$$\mathcal{E}_i\mathcal{E}_j = 0. \tag{D.23}$$

Finally the product of momenta and polarizations in $(2d+1)$-dimensions returns the product of d-dimensional momenta for $j \neq 1, n$

$$\mathcal{P}_i\mathcal{E}_j = p_ip_j \quad , \tag{D.24}$$

while the products $\mathcal{P}_\rangle\mathcal{E}_1$ and $\mathcal{P}_i\mathcal{E}_n$ vanish. This choice of external kinematics works identically to a differential operator which, at the level of the amplitude, acts as a longitudinal operator that replaces products of e_ip_j by p_ip_j and sets the pair of polarizations $e_1e_n \to 1$:

$$\overline{T}_{1n}\overline{\mathcal{L}} = \frac{\partial}{\partial(e_1e_n)} \prod_{i=2}^{n-1} \left(\sum_{j\neq i} p_ip \frac{\partial}{\partial(e_ip_j)} \right) \tag{D.25}$$

This version of the longitudinal operator is formulated slightly different from the $T\mathcal{L}$ operator in (3.43). However, as shown in [87] the two dimensional reductions

$$\text{Dim Red 1} \qquad e_a \cdot p_b \to 0, \qquad e_a \cdot e_b \to -p_a \cdot p_b, \qquad e_a, e_b \notin \{e_1, e_n\} \tag{D.26}$$

$$\text{Dim. Red. 2} \qquad e_a \cdot e_b \to 0, \qquad e_a \cdot p_b \to p_a \cdot p_b \tag{D.27}$$

yield the same results at the level of the amplitude.

This higher dimensional interpretation has also been shown at the level of the action. The Lagrangian for YM in $2d+1$ dimension with Feynman gauge is simplified to be

$$\mathcal{L}_{\text{YM}} = Tr\left(-\frac{1}{2}\partial_M \mathcal{A}_N \partial^M \mathcal{A}^N + i\sqrt{2}\partial_M \mathcal{A}_N \left[\mathcal{A}^M, \mathcal{A}^N\right] + \frac{1}{2}[\mathcal{A}_M, \mathcal{A}_N]\left[\mathcal{A}^M, \mathcal{A}^N\right]\right) \tag{D.28}$$

The gluon fields transform under the adjoint representation $\mathcal{A}_M = \mathcal{A}^a_M T^a$. The field can be split into three component field

$$\mathcal{A}_M = \mathcal{X}_M + \mathcal{Y}_M + \mathcal{Z}_M, \tag{D.29}$$

where these $(2d+1)$-dimensional component fields are split into a $(d,1,d)$-dimensional component fields through the parametrisation

$$\begin{aligned} \mathcal{X}_M &= \frac{1}{\sqrt{2}}(X_\mu, 0, -iX_\mu) \\ \mathcal{Y}_M &= (0, Y, 0) \\ \mathcal{Z}_M &= \frac{1}{\sqrt{2}}(X_\mu, 0, +iX_\mu) \end{aligned} \tag{D.30}$$

Here X_μ and Z_μ are d-dimensional vector fields and Y is a scalar field. In this parametrization the following inner product relations are established

$$\begin{aligned} \mathcal{A}_M \mathcal{A}^M &= X_\mu Z^\mu + Z_\mu X^\mu + Y^2 \\ \mathcal{X}_M \mathcal{X}^M &= \mathcal{Z}_M \mathcal{Z}^M = 0 \\ \mathcal{X}_M \mathcal{Y}^M &= \mathcal{Z}_M \mathcal{Y}^M = 0 \end{aligned} \tag{D.31}$$

It can then be shown through weight counting of the interactions in the Lagrangian and the tranverse properties of the fields that the Lagrangian in (D.28) truncates to a Lagrangian that describes the NGSb of the NLSM through as the fields X_μ and Y_μ

$$\mathcal{L}_{\text{NLSM}} = Tr\left(X_\mu \Box Z^\mu + \frac{1}{2}Y\Box Y + i\left(X_{\mu\nu}\left[Z^\mu, Z^\nu\right] + Z^\mu[Y, \partial_\mu Y]\right)\right) \tag{D.32}$$

Where the field strength tensor for X_μ is defined as $X_{\mu\nu} = \partial_\mu X_\nu - \partial_\nu X_\mu$. This gives rise to pion amplitudes

$$A(\pi_1, \pi_2, \ldots, \pi_{n-1}, \pi_n) = A(Y_1, Z_2, \ldots, Z_{n-1}, Y_n), \tag{D.33}$$

in agreement with the findings of [88]. [86] Further shows that a similar dimensional reduction can be applied to gravitons to derive the action and amplitudes of Born-Infeld and Special Galileon theory. Last but not least the higher-dimensional perspective the double copy is manifest at the level of the action. This can be done by dropping flavor indices of the fields and doubling the kinematic structures, e.g. $X_{\mu\bar{\mu}}$.

The connection between higher-dimensional gluons and scalars was also demonstrated in [89], where it was shown that the scattering amplitudes of gluons are intricately embedded within 'stringy' deformations of pure scalar amplitudes. This paper introduces a new approach to formulating Yang-Mills scattering amplitudes in any number of dimensions and loop orders, using combinatorial and geometric ideas in kinematic space. The authors show that by using a "scalar scaffolding" method (Figure D.1), where every pair of coloured scalars produces a gluon, the shifted amplitudes of $\mathrm{Tr}[\phi^3]$ theory match those of gluons. This scaffolded method preserves essential properties of gluon amplitudes such as multilinearity, gauge invariance, and factorization, effectively connecting scalar and gluon amplitudes through a shift in kinematic variables. It was shown in [72] that deformation of the "stringy" $\mathrm{Tr}[\phi^3]$ amplitudes secretly contain pion amplitudes, extending the connection between scalars, gluons and pions.

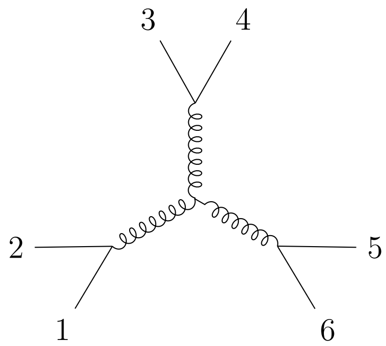


Figure D.1: A diagram depicting the 3-point gluon amplitude that is scaffolded from a 6-point scalar amplitude.

D.4 Gauge invariance of longitudinal operator

Let us consider the conditions of conservation of the on-shell kinematics and the gauge invariance of the longitudinal operators. For starters, these operators can be written as linear combinations of these operators:

$$L_i = \textstyle\sum_{j\neq k} p_i p_j I_{jik} \qquad \text{and} \qquad L_{ij} = -p_i p_j T_{ij}. \tag{D.34}$$

The (effective) vanishing of the commutation with the total momentum operator P then guarantee the conservation of the on-shell kinematics. Next, while L_{ij} is intrinsically gauge invariant due to it being a linear combination of the gauge invariant trace operator T_{ij}, this is not the case for L_i. For this operator we can argue again that it can be shown to be effectively gauge invariant if the commutator $[L_i,\ W_j] = -L_{ij}$ annihilates the amplitude. This can be shown by rewriting the Ward identity operator as

$$W_i = L_i - \Delta_i, \qquad \text{where } \Delta_i = \textstyle\sum_j p_i e_j \partial_{e_j e_i}. \tag{D.35}$$

With this we can rewrite a product $L_i \cdot L_j$ by inserting $L_i = W_i + \Delta_i$ and commute this operator to the right:

$$\begin{aligned} L_i \cdot L_j &= (W_i + \Delta_i) L_j & \text{(D.36)} \\ &= [W_i, L_j] + L_j \cdot W_i + \Delta_i L_j & \text{(D.37)} \\ &= L_{ij} + \cdots & \text{(D.38)} \end{aligned}$$

To finalise the argument we note that all the terms which are contained in $\cdots$ will annihilate the amplitude. The term with W_i will annihilate the amplitude by definition of the Ward identity operator. The term with Δ_i will annihilate the amplitude due to the multi-linearity of the amplitude in polarisations. This argument can be extended to any product of L_i operators; by keeping track of all the commutation relations it can be shown that the four-particle longitudinal operator can be written as

$$L_i \cdot L_j \cdot L_k \cdot L_l = L_{ij} \cdot L_{kl} + L_{ik} \cdot L_{jl} + L_{il} \cdot L_{jk} + \cdots , \tag{D.39}$$

where it becomes visible that the product of L_i will result in a sum of products of L_{ij} where i, j are grouped into distinct pairs.

D.5 Transmutation of the subleading order soft factors

YMS and BAS subleading soft factor from Yang-Mills

Since the soft factor is a pertubative expansion, we can apply transmutation to the subleading factor aswell. The leadling+subleading factor is given by:

$$S^{(ijk)}_{textYM} = \frac{p_i e_j - p_j J_i e_j}{p_i p_j} - \frac{p_k e_j - p_j J_k e_j}{p_k p_j} \tag{D.40}$$

where we consider the soft particle j adjacent to i and k. For this discussion we only consider the subleading factor denoted by

$$S^{(ijk)}_{\text{YM},sl} = \frac{p_j J_k e_j}{p_k p_j} - \frac{p_j J_i e_j}{p_i p_j} = -\frac{p_j J_i e_j}{p_i p_j} + (i \leftrightarrow k), \tag{D.41}$$

where the notation $p_j J_i e_j = p_j^\mu (J_i)_{\mu\nu}\ e_j^\nu$ denotes the summation over the Lorentz indices. In fact, J_i denotes the total angular momentum of the particle i, which can be split into the orbital and spin contribution:

$$(J_i)_{\mu\nu} = p_{i[\mu} \frac{\partial}{\partial p_i^{\nu]}} + e_{i[\mu} \frac{\partial}{\partial e_i^{\nu]}} = \underbrace{\left(p_{i\mu} \frac{\partial}{\partial p_i^\nu} - p_{i\nu} \frac{\partial}{\partial p_i^\mu} \right)}_{L_{\mu\nu}} + \underbrace{\left(e_{i\mu} \frac{\partial}{\partial e_i^\nu} - e_{i\nu} \frac{\partial}{\partial e_i^\mu} \right)}_{\Sigma_{\mu\nu}} \tag{D.42}$$

Here it becomes clear that the angular momentum operator is in fact a differential operator with respect to p_i and e_i, which allows for nontrivial commutation with the transmutation operators.

$$(J_i)_{\mu\nu} = [p_{i\mu}(\partial_{p_i})_\nu - p_{i\nu}(\partial_{p_i})_\mu] + [e_{i\mu}(\partial_{e_i})_\nu - e_{i\nu}(\partial_{e_i})_\mu] \tag{D.43}$$

The soft factor for YMS theory can again be deduced from the commutation with the trace operator

$$S^{(ijk|jl)}_{\text{YMS,sl}} = \left[T_{jl}, S^{(ijk)}_{\text{YM,sl}}\right] = -\sum_{m\neq i,k} \delta_{lm} I_{ilk} + \left\{\frac{N_{ijkl}}{p_i p_j} - (i \leftrightarrow k)\right\}, \tag{D.44}$$

with $N_{ijkl} = \sum_{m\neq i}(\delta_{il} p_j e_m - \delta_{lm} p_j e_i) T_{im} + \sum_{m\neq i,k} \delta_{il} p_j p_m I_{mik}$. The commutator with the insertion operator results in the subleading soft factor for BAS theory

$$S^{(ijk|ljm)}_{\text{BAS,sl}} = \left[T_{ljm}, S^{(ijk)}_{\text{YM,sl}}\right] = \left\{\frac{(\delta_{li} - \delta_{im})(1 + p_j \partial_{p_i})}{p_i p_j} - \frac{p_j e_i}{p_i p_j} I_{lim} - (i \leftrightarrow k)\right\} - T_{ilkm}. \tag{D.45}$$

NLSM subleading soft factor

In the argumentation of (3.72) we discussed how we could use that effectively $\left[T^{(n-1)}, S^{(j)}\right] = 0$, as the leading order only involved derivatives with respect to pairs which do not contain soft leg j. For the subleading order, the soft factor does contain terms proportional to adjacent legs. Explicitely the factor is given by

$$S^{(ijk)}_{YM,sl} = \frac{(p_j p_i)(\partial_{p_i} e_j) - (p_j p_i)(\partial_{p_i} e_j) + (p_j e_i)(\partial_{e_i} e_j) - (p_j e_i)(\partial_{e_i} e_j)}{p_i p_j} + (i \leftrightarrow k), \tag{D.46}$$

which contains derivatives with respect to the soft leg. When considering the commutator with the longitudinal factor $L_{ij} = -p_i p_j \partial_{e_i e_j}$, the argumentation therefore has to be altered to incorporate this. First, consider the transmutation of the non-soft legs $T^{(n-1)} \propto (-p_l p_m \partial_{e_l e_m})$ with $l, m \neq j$. If $T^{(n-1)}$ acts on the Lorentz invariants (pp), (pe) or $(e_m e_j)$ it results in a vanishing contribution

$$T^{(n-1)} \cdot S^{(ijk)}_{\text{YM,sl}} = 0 \tag{D.47}$$

Part of the commutator therefore already vanishes, the other part $S^{(ijk)}_{YM,sl} \cdot T^{(n-1)}$ contains several derivative terms. Derivatives of ∂_{e_i} will vanish, derivatives in the term $\partial_{p_i}(p_l p_m \partial_{e_l e_m}) e_j$ will only be nonzero if $l, m = i$ so only $p_m e_j \partial_{e_i e_m}$ and $p_l e_j \partial_{e_l e_i}$ survive. The only surviving terms from $\partial_{p_k}(p_l p_m \partial_{e_l e_m})$ are those with $l, m = k$, which are $p_m e_j \partial_{e_k e_m}$ and $p_l e_j \partial_{e_l e_k}$. What remains of this contribution to the commutator is

$$S^{(ijk)}_{YM,sl} \cdot T^{(n-1)} = \frac{(p_j p_i)(p_m e_j)\partial_{e_i e_m} - (p_j p_i)(p_m e_j)\partial_{e_i e_m}}{p_i p_j}, \tag{D.48}$$

which will be hit by another longitudinal operator $L \sim \partial_{(ee)}$, resulting in an effectively vanishing contribution to the commutator as it does not contain any pairs of (ee). Therefore it is shown that $\left[T^{(n-1)}, S^{(j)}\right]$ effectively vanishes for products of L_{ij} operators, which is neccesary to obtain amplitudes for the NLSM. Therefore, to obtain the subleading soft factor of the NLSM, the commutator with the Trace operator can be considered instead, due to the fact that $L_{ij} = -p_i p_j T_{ij}$:

$$S^{(ijk)}_{\text{NLSM,sl}} = \left[-p_j p_l T_{jl}, S^{(ijk)}_{YM}\right] \tag{D.49}$$

$$= \sum_{m \neq i,k} p_j p_l \delta_{lm} I_{ilk} - p_j p_l \left\{ \frac{N_{ijkl}}{p_i p_j} - (i \leftrightarrow k) \right\} \tag{D.50}$$

$$S^{(ijk)}_{\text{NLSM,sl}} = \sum_{l \neq i,k} p_j p_l I_{ilk}. \tag{D.51}$$

It is noted that this subleading soft factor contains an insertion operator, indicating that soft behaviour is dependent on a lower point BAS amplitude, which is in agreement with the proposal of Cachazo and collaborators in the same publication that argued on the mixed amplitudes of pions and biadjoint scalars [35].

Appendix E

Mathematica computations of amplitudes

E.1 6-point YMS amplitude

At $\mathcal{O}(\tau^0)$ we encounter the YMS amplitude that is generated by summing up the contributions of the GT-operator for different arrangements of 6 particles. An example of one such an ordering is given by the ordering (12)(34)(56) which results in the contribution :

$$\begin{aligned}4A_{\mathrm{YMS}}(\phi_1\phi_2,\phi_3\phi_4,\phi_5\phi_6) &= \frac{s_{2,3}}{s_{1,2}s_{1,2,3}}+\frac{s_{4,5}s_{2,3}}{s_{1,2}s_{1,2,3}s_{1,2,3,4}}+\frac{1}{s_{1,2,3}}+\frac{-s_{2,4}s_{3,5}+s_{2,3}s_{4,5}+s_{2,5}s_{4,5}}{s_{1,2}s_{3,4}s_{3,4,5}}\\
&+\frac{s_{2,3}s_{4,5}+s_{2,5}s_{4,5}}{s_{1,2}s_{4,5}s_{3,4,5}}+\frac{-s_{1,4}s_{2,5}-s_{2,4}s_{2,5}-s_{2,4}s_{3,5}+s_{2,3}s_{4,5}}{s_{1,2}s_{3,4}s_{1,2,3,4}}+\frac{s_{4,5}}{s_{1,2,3}s_{1,2,3,4}}+\frac{s_{4,5}}{s_{2,3,4}s_{1,2,3,4}}\\
&+\frac{-s_{2,4}s_{2,5}-s_{2,4}s_{3,5}+s_{2,3}s_{4,5}}{s_{3,4}s_{2,3,4}s_{1,2,3,4}}+\frac{1}{s_{2,3,4,5}}+\frac{s_{4,5}}{s_{2,3,4}s_{2,3,4,5}}+\frac{-s_{2,4}s_{2,5}-s_{2,4}s_{3,5}+s_{2,3}s_{4,5}}{s_{3,4}s_{2,3,4}s_{2,3,4,5}}\\
&+\frac{-s_{2,4}s_{3,5}+s_{2,3}s_{4,5}+s_{2,5}s_{4,5}}{s_{3,4}s_{3,4,5}s_{2,3,4,5}}+\frac{s_{2,3}s_{4,5}+s_{2,5}s_{4,5}}{s_{4,5}s_{3,4,5}s_{2,3,4,5}}\end{aligned} \tag{E.1}$$

with a factor 1/4 due to the replacement of the momentum factor p_ip_j that was present in numerator of the gluon amplitude to $p_ip_j \to s_{ij}/2$.

The complete amplitude is the summation over all 15 arrangements and is given by

$$A_{\text{YMS}}\frac{s_{1,5}s_{2,3}+s_{2,5}s_{2,3}+s_{3,5}s_{2,3}}{s_{2,3}s_{4,5}s_{1,2,3}}+\frac{-s_{1,5}s_{2,3}-s_{2,5}s_{2,3}-s_{3,5}s_{2,3}}{s_{1,2}s_{4,5}s_{1,2,3}}+\frac{-s_{1,5}s_{2,3}-s_{2,5}s_{2,3}-s_{3,5}s_{2,3}}{s_{2,3}s_{4,5}s_{1,2,3}}+$$
$$\frac{s_{1,3}s_{1,5}+s_{1,3}s_{2,5}+s_{1,3}s_{3,5}}{s_{2,3}s_{4,5}s_{1,2,3}}+\frac{-s_{2,4}s_{3,5}+s_{2,3}s_{4,5}+s_{2,5}s_{4,5}}{s_{1,2}s_{3,4}s_{3,4,5}}+\frac{s_{2,5}s_{3,4}-s_{2,3}s_{3,5}-s_{2,4}s_{3,5}}{s_{1,2}s_{4,5}s_{3,4,5}}+$$
$$\frac{s_{2,3}s_{4,5}+s_{2,5}s_{4,5}}{s_{1,2}s_{4,5}s_{3,4,5}}+\frac{-s_{2,3}s_{4,5}-s_{2,4}s_{4,5}}{s_{1,2}s_{4,5}s_{3,4,5}}+\frac{-s_{1,4}s_{2,5}-s_{2,4}s_{2,5}-s_{2,4}s_{3,5}+s_{2,3}s_{4,5}}{s_{1,2}s_{3,4}s_{1,2,3,4}}+$$
$$\frac{s_{2,3}s_{4,5}}{s_{1,2}s_{1,2,3}s_{1,2,3,4}}-\frac{s_{1,3}s_{4,5}}{s_{2,3}s_{1,2,3}s_{1,2,3,4}}+\frac{s_{2,5}s_{3,4}+s_{4,5}s_{3,4}-s_{2,4}s_{3,5}}{s_{2,3}s_{2,3,4}s_{1,2,3,4}}+\frac{s_{2,5}s_{3,4}+s_{4,5}s_{3,4}}{s_{3,4}s_{2,3,4}s_{1,2,3,4}}$$
$$+\frac{-s_{2,5}s_{3,4}-s_{3,5}s_{3,4}}{s_{3,4}s_{2,3,4}s_{1,2,3,4}}+\frac{-s_{2,4}s_{2,5}-s_{2,4}s_{3,5}+s_{2,3}s_{4,5}}{s_{3,4}s_{2,3,4}s_{1,2,3,4}}+\frac{s_{1,3}s_{2,5}-s_{1,5}s_{3,4}+s_{1,3}s_{3,5}+s_{1,4}s_{3,5}}{s_{2,3}s_{4,5}s_{2,3,4,5}}$$
$$+\frac{s_{2,3}s_{2,5}+s_{2,3}s_{3,5}}{s_{2,3}s_{4,5}s_{2,3,4,5}}+\frac{-s_{2,3}s_{2,5}-s_{2,3}s_{3,5}}{s_{2,3}s_{4,5}s_{2,3,4,5}}+\frac{s_{3,4}s_{4,5}+s_{3,5}s_{4,5}}{s_{2,3}s_{4,5}s_{2,3,4,5}}$$
$$+\frac{-s_{3,4}s_{4,5}-s_{3,5}s_{4,5}}{s_{2,3}s_{4,5}s_{2,3,4,5}}+\frac{-s_{2,5}s_{3,4}-s_{4,5}s_{3,4}+s_{2,4}s_{3,5}}{s_{2,3}s_{2,3,4}s_{2,3,4,5}}$$
$$+\frac{s_{2,5}s_{3,4}+s_{4,5}s_{3,4}-s_{2,4}s_{3,5}}{s_{2,3}s_{2,3,4}s_{2,3,4,5}}-\frac{s_{1,5}s_{3,4}}{s_{2,3}s_{2,3,4}s_{2,3,4,5}}+\frac{s_{2,5}s_{3,4}+s_{3,5}s_{3,4}}{s_{3,4}s_{2,3,4}s_{2,3,4,5}}$$
$$+\frac{s_{2,5}s_{3,4}+s_{4,5}s_{3,4}}{s_{3,4}s_{2,3,4}s_{2,3,4,5}}+\frac{-s_{2,5}s_{3,4}-s_{3,5}s_{3,4}}{s_{3,4}s_{2,3,4}s_{2,3,4,5}}+\frac{-s_{2,5}s_{3,4}-s_{4,5}s_{3,4}}{s_{3,4}s_{2,3,4}s_{2,3,4,5}}$$
$$+\frac{-s_{2,4}s_{2,5}-s_{2,4}s_{3,5}+s_{2,3}s_{4,5}}{s_{3,4}s_{2,3,4}s_{2,3,4,5}}+\frac{s_{2,4}s_{2,5}+s_{2,4}s_{3,5}-s_{2,3}s_{4,5}}{s_{3,4}s_{2,3,4}s_{2,3,4,5}}+\frac{s_{1,5}s_{2,4}}{s_{3,4}s_{2,3,4}s_{2,3,4,5}}$$
$$+\frac{-s_{2,4}s_{3,5}+s_{2,3}s_{4,5}+s_{2,5}s_{4,5}}{s_{3,4}s_{3,4,5}s_{2,3,4,5}}+\frac{s_{3,4}s_{3,5}+s_{3,4}s_{4,5}}{s_{3,4}s_{3,4,5}s_{2,3,4,5}}+\frac{s_{1,4}s_{3,5}-s_{1,3}s_{4,5}-s_{1,5}s_{4,5}}{s_{3,4}s_{3,4,5}s_{2,3,4,5}}$$
$$+\frac{s_{2,4}s_{3,5}-s_{2,3}s_{4,5}-s_{2,5}s_{4,5}}{s_{3,4}s_{3,4,5}s_{2,3,4,5}}+\frac{-s_{3,4}s_{3,5}-s_{3,4}s_{4,5}}{s_{3,4}s_{3,4,5}s_{2,3,4,5}}+\frac{-s_{1,5}s_{3,4}+s_{1,3}s_{3,5}+s_{1,4}s_{3,5}}{s_{4,5}s_{3,4,5}s_{2,3,4,5}}$$
$$+\frac{-s_{2,5}s_{3,4}+s_{2,3}s_{3,5}+s_{2,4}s_{3,5}}{s_{4,5}s_{3,4,5}s_{2,3,4,5}}+\frac{s_{2,5}s_{3,4}-s_{2,3}s_{3,5}-s_{2,4}s_{3,5}}{s_{4,5}s_{3,4,5}s_{2,3,4,5}}+\frac{s_{1,3}s_{4,5}+s_{1,4}s_{4,5}}{s_{4,5}s_{3,4,5}s_{2,3,4,5}}$$
$$+\frac{s_{2,3}s_{4,5}+s_{2,4}s_{4,5}}{s_{4,5}s_{3,4,5}s_{2,3,4,5}}+\frac{s_{2,3}s_{4,5}+s_{2,5}s_{4,5}}{s_{4,5}s_{3,4,5}s_{2,3,4,5}}+\frac{s_{3,4}s_{4,5}+s_{3,5}s_{4,5}}{s_{4,5}s_{3,4,5}s_{2,3,4,5}}$$
$$+\frac{-s_{1,3}s_{4,5}-s_{1,5}s_{4,5}}{s_{4,5}s_{3,4,5}s_{2,3,4,5}}+\frac{-s_{2,3}s_{4,5}-s_{2,4}s_{4,5}}{s_{4,5}s_{3,4,5}s_{2,3,4,5}}+\frac{-s_{2,3}s_{4,5}-s_{2,5}s_{4,5}}{s_{4,5}s_{3,4,5}s_{2,3,4,5}}+\frac{-s_{3,4}s_{4,5}-s_{3,5}s_{4,5}}{s_{4,5}s_{3,4,5}s_{2,3,4,5}}. \tag{E.2}$$

From a quick inspection it is clear that there are quite a few terms that share the same propagator which could be collected. It is chosen to not simplify this calculation at this stage for two reasons. The first being that the *Simplify* function of mathematica will collect all terms and simplify them to 1 term with a new denominator that is common to all elements of the sum. This does not give us any meaningful insight. The second reason is that we will expand on this result in later computations, therefore it is not yet necessary to simplify at this stage.