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Equitable Staking: Addressing Wealth Compounding in Proof-of-Stake Through Reward Function Optimization

Bachelor's Project Thesis

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Abstract: This thesis addresses wealth compounding in Proof-of-Stake (PoS) systems, while maintaining incentive compatibility. It examines the effect of smoothing the geometric reward function proposed by Fanti et al. (2019) on wealth compounding and incentive compatibility. Three novel reward functions based on the geometric reward function are proposed and the study quantifies and evaluates wealth compounding and incentive compatibility for each function. A PoS system is simulated in Python, and results show that smoothing the geometric reward function can improve wealth compounding and boost incentive compatibility, resulting in a more equitable and rewarding PoS system.

1 Introduction

The phenomenon of wealth compounding (Leporati, 2023) in the realm of cryptocurrencies, specifically within Proof-of-Stake (PoS) (Buterin, 2022) protocols, presents a potential challenge to the mechanism of decentralization and equitable wealth distribution (Fanti et al., 2019). Excessive wealth compounding, the process where wealth disproportionately accrues more wealth, could lead to an unfair distribution of power (Leporati, 2023). This can be explained by the mechanism that the power of participants in the base PoS protocol is directly related to the amount staked by a participant (Buterin, 2022). The probability of validating transactions and earning a reward then benefits those with significant initial resources. The consequence of considerable differences in stake is the centralization of power (Ge et al., 2022). Consequently, the core principle of the PoS protocol - decentralization - as introduced by Buterin, 2022 and King & Nadal, 2012, is directly challenged.

As a consensus mechanism in blockchain, the PoS protocol relies on the stake of participants to validate transactions and add new blocks to the chain (Buterin, 2022). Each node in the PoS network uses a hashing scheme to validate transactions and create new blocks. This resembles Bitcoin's Proof-of-Work (Nakamoto, 2008) system, although it operates in a smaller search space (King & Nadal, 2012). One important component of the PoS protocol is the reward function, which dictates how rewards are distributed among participants of the protocol, and is crucial in influencing the overall wealth distribution of the system. This will be further elaborated upon in later sections.

The research question (RQ) explored in this thesis is, "What is the effect of the geometric reward function and smoothing of future rewards on wealth compounding in PoS protocols?" This RQ is rooted in understanding and potentially mitigating the wealth-compounding effect in PoS protocols through reward function adjustments (Fanti et al., 2019). The geometric reward function (Fanti et al., 2019), as can be seen in equation (1.1), is used as a base. Here, R is the total reward, T is the number of block proposals and n is the timestep of the PoS system. The geometric reward function was developed to reduce the variations in wealth accumulation that are frequently found in blockchain networks (Nakamoto, 2008), particularly those that use a PoS consensus method (Ge et al., 2022).

$$r_g(n) := (1+R)^{\frac{n}{T}} - (1+R)^{\frac{n-1}{T}} \tag{1.1}$$

In computing the distribution of rewards to validators in a decentralized PoS network, the geometric reward function will, at each time slot, always return a constant fraction of the entire stake. This contrasts with standard linear or constant reward schemes, which reward the same amount of tokens for each block (Buterin, 2022). The geometric reward function effectively reduces the tendency of wealthier nodes to collect wealth disproportionately (Fanti et al., 2019). The essence of this function is its ability to lessen the relative advantage that larger stakes give in the growth of wealth which is critical in reducing the 'rich-get-richer' dynamic and fostering a more equitable wealth distribution across network participants (Leporati, 2023).

Fanti et al., 2019 highlighted the geometric reward function as a potential solution to the unequal wealth distribution in Proof-of-Stake (PoS) systems, ideal for the start of a PoS system with the prospect of transitioning to a smoother reward mechanism as the network matures. This thesis expands on their earlier work by proposing three new reward functions derived from the geometric model, to reduce the effect of wealth compounding and improve incentive structures. To be able to measure the performance of the reward functions, this research makes use of three metrics. The following section will discuss these metrics in detail.

1.1 Metrics

1. **Gini Coefficient:** The Gini Coefficient (Dorfman, 1979) is used for measuring the level of inequality in wealth distribution within a system, such as Proof-of-Stake (PoS) protocols. It has a scale of 0 to 1, with 0 representing perfect equality (everyone has the same wealth) and 1 representing extreme inequality (all wealth is concentrated with a single entity) (Dorfman, 1979). A lower Gini Coefficient is desired in the context of PoS systems since it suggests a more equitable distribution of wealth among participants.

$$G = \frac{\sum_{i=1}^{N} \sum_{j=1}^{N} |s_i - s_j|}{2N^2 \bar{s}} \tag{1.2}$$

This is formalized as follows: let N be the number of nodes in the system, and s_i the stake for node i. The Gini Coefficient is then given by equation (1.2). Then, $\sum_{i=1}^{N} \sum_{j=1}^{N} |s_i - s_j|$ is the sum of the absolute differences between all the pairs of the stake of every node. Additionally, $\bar{s}$ is the average of all stakes.

2. **Equitability:** Equitability, conceptualized by Fanti et al., 2019, is another metric in evaluating the effects of reward functions on wealth compounding, specifically in blockchain networks Fanti et al., 2019. It measures the normalized variance of a node's stake proportion, thus reflecting the influence of the reward function. The equitability for a node A_i at time T, denoted as ε_i, is formulated in equation (1.3).

$$\varepsilon_i = \frac{\mathrm{Var}(\gamma_{A_i,r}(T))}{\gamma_{A_i}(0)(1 - \gamma_{A_i}(0))} \tag{1.3}$$

The variance $\mathrm{Var}(\gamma_{A_i,r}(T))$, defined as the variance at time T of the fractional stake of node A_i for reward function r, is adjusted by the initial stake proportion $\gamma_{A_i}(0)$, yielding a metric that spans from 0, indicating uniform distribution, to 1, the maximum wealth compounding. Following Fanti et al., a reward function r_1 is seen as more equitable than r_2 if it results in less variance in the final stake distribution, as can be seen in equation (1.4).

$$\mathrm{Var}(\gamma_{A_i,r_1}(T)) \leq \mathrm{Var}(\gamma_{A_i,r_2}(T)) \tag{1.4}$$

Indicating that r_1 results in a more equitable distribution of wealth among the network's participants than r_2. To assess the equitability across stakeholders, and directly across reward functions, we construct the vector ε and identify the minimum variance $\tilde{\varepsilon}$ in equation (1.5).

$$\tilde{\varepsilon} = \min(\varepsilon_i), \quad i \in [m] \tag{1.5}$$

By comparing the minimum variance values $\tilde{\varepsilon}$ of two reward functions, r_1 and r_2, we can quantify and compare their equitability and determine which function better promotes a balanced distribution of wealth within the ecosystem.

3. **Reward Ratio:** To address the aspect of incentive compatibility within blockchain systems, we use the reward ratio to evaluate

the incentive to participate in the PoS system (Wang et al., 2020). This metric, denoted as rat^{inv}_{rx}, quantifies the motivation of participants to engage in the consensus mechanism. The reward ratio makes incentive tangible by comparing the initial block reward to that of the i^{th} block within a specific interval. The reward ratio is defined mathematically as:

$$rat^{inv}_{rx} = \frac{r_x(t^{inv}_1)}{r_x(t_j)} \tag{1.6}$$

Here, x is defined as the reward function, inv^i represents the i^{th} interval and t_j denotes the j_{th} block within interval i. Additionally, the initial block is defined as t^{inv}_1, which is the first in the i_{th} interval. As per (1.6), the reward ratio rat^{inv}_{rx} quantifies the degree to which a reward function maintains incentive compatibility *over time*. The incentive to participate in a PoS system and the reward ratio have a positive relationship (Wang et al., 2020). To encourage active involvement and keep the integrity of the consensus mechanism in the PoS protocol, this metric is important to optimize.

1.2 Principles

In adjusting the dynamics of wealth distribution inside Proof-of-Stake (PoS) protocols, this thesis presents a set of two principles for reward functions to adhere to. These principles seek to ensure a fair, long-term, and equal distribution of wealth within the network while maintaining an incentive to participate.

To reach this goal, reward functions must avoid extreme relative reward differences, which kick-start wealth compounding (Fanti et al., 2019). This concept is illustrated through the use of the Pareto distribution (Makoto, 2009), to represent the distribution of wealth among participants.

The probability density function of the Pareto distribution can be seen in equation (1.7), where k is a scaling constant, x is the wealth, and a the shape parameter denoting how heavy the tail of the distribution is.

$$P(x) = \frac{k}{x^{a+1}} \tag{1.7}$$

The Pareto distribution is characterized by a heavy tail, where a small number of participants can hold a large partition of the total wealth. A reduction of the shape parameter a results in greater variability and significant inequalities in wealth (Makoto, 2009). The variance can be regulated by limiting excessive growth and relative differences between rewards, thus encouraging a more equitable distribution of rewards.

Regulating the variance and reducing a in reward functions is achieved by having reward functions adhere to the following two principles:

1.1. Smooth Transitioning: The transitions in reward phases are smooth and devoid of abrupt disruptions to combat wealth compounding (Fornaro & Wolf, 2023). The principle relies on the mathematical function known as the sigmoid (1.9), which is continuously differentiable. The smooth transitioning is essential for avoiding unpredictability and instability in the system (Han & Moraga, 1995), thus reducing the shape parameter a in (1.7).

1.2. Use of Damping Functions: The natural logarithmic function can play an important role in counteracting runaway growth; the logarithmic functions ensure sub-linear growth, where the rate of increase diminishes as inputs grow (Bryant, 2014). The shape parameter a in (1.7) is reduced by preventing the disproportionate growth of rewards.

1.3 Reward Functions

The geometric reward function (1.1) was identified as a potential solution to the unequal wealth distribution in Proof-of-Stake (PoS) systems Fanti et al., 2019, ideal for the start of a PoS system with the goal of shifting to a smoother reward mechanism as the network matured (Fanti et al., 2019). This thesis builds on their previous work by proposing three new reward functions based on the geometric model and the two principles (smooth transitioning 1.1 and use of damping functions 1.2) mentioned in section 1.2, with the purpose of decreasing the influence of wealth compounding and strengthening incentive compatibility. These functions are developed with two main objectives; correcting wealth disparities and maintaining incentive compatibility. The following section will examine these functions and their construction in more detail.

1. **Logarithmically increasing geometric reward:**

$$R_{\log}(s, M) = R_{\text{geo}}(s) + 10 \cdot M \cdot \log(s + 1) \quad (1.8)$$

Where $R_{\log}$ is the logarithmically increasing geometric reward, s is the current step in the simulation, M is the multiplier to influence the magnitude of the logarithmic bonus, and Rgeo is the base geometric reward function as seen in equation (1.1).
In this function (1.8) the reward mechanism incorporates a logarithmic factor into the basic geometric reward. The logarithmic component, $10 \cdot M \cdot \log(s + 1)$, scales the reward based on the natural logarithm of the step count, incentivizing earlier participation in the blockchain system. By adding 1 to s, forming $s+1$, we prevent the natural logarithm, $\log(s+1)$, from becoming undefined. This approach aligns with scenarios where early contributions or stakes are deemed more valuable (Fanti et al., 2019), reflecting a diminishing return for later contributions.

2. **Sigmoid modulated geometric reward:**

$$R_{\text{sig}}(s, F) = R_{\text{geo}}(s) \cdot \left(1 - \frac{1}{1 + e^{-\frac{s}{F}}}\right) \quad (1.9)$$

Where R_{sig} is the sigmoid modulated geometric reward, s is the current step in the simulation, F is the scaling factor for the sigmoid function, and R_{geo} is the base geometric reward function as seen in equation (1.1).
In equation (1.9) a sigmoid function is used to modulate the reward. The sigmoid function $\frac{1}{1+e^{-\frac{s}{F}}}$, provides a smooth transition from a lower to an upper limit as the step count increases. This modulation offers a gradual increase in rewards, which then asymptotically approaches a maximum. This behavior is especially useful in preventing wealth compounding because of less abrupt changes in the reward distribution over time, explained by principle 1.1.

3. **Sinusoidal geometric reward:**

$$R_{\sin}(s, S) = R_{\text{geo}}(s) \cdot (|\sin(s)| + S) \quad (1.10)$$

Where $R_{\sin}$ is the sinusoidal modulated geometric reward, s is the current step in the simulation, S is the scaling factor for the sinusoidal part, and R_{geo} is the base geometric reward function as seen in equation (1.1).
Lastly, the sinusoidal modulation in equation (1.10), $|\sin(s)| + S$, introduces a periodic component to the reward structure. The absolute value of the sine function ensures that the reward oscillates in a positive range. The oscillation of the reward function ensures the advantage of joining a PoS early is lessened since rewards fluctuate to provide more opportunity for stakeholders throughout the system's lifespan.

This thesis aims to improve on the geometric reward function as an approach to addressing wealth compounding in traditional PoS systems, building further on the research by Fanti et al., 2019 by smoothing the geometric reward function. The subsequent section is dedicated to a methodical examination of the three previously introduced functions. This study aims to provide a thorough understanding of how these reward functions can influence and potentially reduce wealth distribution imbalances within the PoS protocol.

2 Methods

The implemented system simulates a PoS model where the probability of validating a block and receiving the associated rewards is proportional to a participant's stake in the system. For the purpose of how reward function adjustments can influence and potentially improve wealth distribution imbalances within the PoS protocol, a Python-based simulation is used to mirror this mechanism. The simulation maintains an array representing the stakes of various nodes. Each node has the potential to add a new block to the blockchain, with higher stakes increasing the likelihood of selection. This model simulates the process of choosing a validator and rewarding them for block validation and appending it to the blockchain. A pseudo-code implementation of the system can be seen in Algorithm 2.1.

2.1 Components of the System

1. **Initializer (`main.py`):** Defines the structure and initial conditions of the PoS system.
   (a) Number of nodes: Determines the total number of participants in the network.
   (b) Initial stake of each node: Sets the initial stake for each node, laying the foundation for the stake-based selection process.
   (c) Number of epochs: Represents the total number of cycles the simulation will run, each potentially affecting the state of the network.
   (d) Number of block intervals: Specifies the frequency at which blocks are validated and rewards are distributed.
   (e) Number of proposers per choice: Determines how many nodes are selected for the possibility of block validation in each round.
   (f) Number of steps: Defines the granularity of the simulation, and how many steps each block interval has.
   (g) Reward functions: Selects the type of reward function (the geometric reward function (1.1), logarithmically increasing geometric reward function (1.8), sigmoid modulated geometric reward function (1.9), or sinusoidal geometric reward function (1.10)) to be used.
   (h) Parameters for the reward functions: Specifies the parameters that fine-tune the behavior of the chosen reward function.

2. **Simulation Class (`simulation.py`):** Responsible for initializing nodes and their stakes, selecting validators, and allocating rewards. The class models three components:
   (a) Stakeholders and stakes: The simulation employs a uniform stake distribution, to simplify the model and focus on the dynamics of the reward mechanisms, where each of the 100 stakeholders possesses an equal stake of 10. This setup is designed to parallel the stake distribution in a typical PoS system. Each node in the simulation represents a stakeholder in the blockchain network. The node's stake is indicative of its wealth and potential influence within the network. Stakes are initialized at the beginning of the simulation, with each node assigned a stake that influences its likelihood of being selected as a validator.
   (b) Block validation and reward mechanism: Nodes are randomly selected based on their stake to validate blocks. Upon successful validation, a reward is assigned to the validator, thereby increasing their stake and influence in the network. The simulation offers a nuanced view of this process, allowing for different reward calculation methods.
   (c) Reward functions: Validators are rewarded for their participation in the network. The simulation incorporates the four reward mechanisms mentioned before; the geometric reward function (1.1), logarithmically increasing geometric reward function (1.8), sigmoid modulated geometric reward function (1.9), and sinusoidal geometric reward function (1.10).

3. **Stakeholders (`node.py`):** The Node class models the stakeholders and their stake update logic in the blockchain network.

4. **Metrics (`tester.py`):** The Tester class is responsible for evaluating wealth compounding and incentive compatibility of the PoS system and its reward functions by implementing the Gini coefficient (1.2), equitability (1.5), and the reward ratio (1.6).

5. **Data Writer (`experiment.py`):** The Data Writer class handles the output of simulation data. It ensures that the results of the simulation are accurately captured and stored for analysis.

6. **Plotter (`plot.py`):** The Plotter class visualizes the outcomes of the simulation.

The upcoming section covers the parameters that govern the behavior of the PoS simulation.

Table 2.1: Parameter setup for each reward function.

Reward Function	Parameter	Tested Values
Geometric	N/A	N/A
Logarithmic	M	0.1, 0.5, 1.0, 1.5, 2.0
Sigmoid	F	50, 100, 150, 200, 250
Sinusoidal	S	0.1, 0.5, 1.0, 1.5, 2.0

2.2 Parameters

The simulation model utilizes various parameters to define the behavior of the system and its reward functions. The parameters for the system influence the dynamics of the Proof of Stake (PoS) system, affecting its behavior.

1. **Node Configuration**: Defines the initial setup of nodes in the system before any staking has happened, including their number and initial stake.
   (a) Number of nodes (`nodes`) is 100.
   (b) Initial stake per node (`initial_stake`) is 10.
2. **Epochs**: Represents the number of cycles the simulation runs.
   (a) Number of epochs (`epochs`) is 10.
3. **Block Interval**: The interval between blocks, impacting reward distribution frequency and validator selection.
   (a) Block interval (`block_interval`) is 21.
4. **Number of Proposers per Choice**: The number of nodes selected as proposers in each simulation step.
   (a) Number of proposers per choice (`number_of_proposers_per_choice`) is 1.
5. **Total Simulation Steps**: The total number of steps the simulation will run.
   (a) Steps (`steps`) is set dynamically to `block_interval` · `epochs`, so here 210.

The reward functions were each evaluated with five distinct parameter values - as can be seen in Table 2.1 - to measure the impact of these variations on the system's metrics. The chosen parameters span a range that is sufficiently broad to show observable differences in system behavior, but narrow enough to facilitate meaningful comparisons between different parameter settings.

3 Results

To answer what the effect of the geometric reward function and smoothing future rewards is on wealth compounding of PoS protocols, three different reward functions (logarithmically increasing geometric reward, sigmoid modulated geometric reward, and sinusoidal geometric reward) based on the geometric reward function were evaluated on their equitability, gini coefficient, and reward ratio with varying parameter settings. The total dispensed reward per reward function can be seen in Figure A.1, showing differences in the way each function adjusted the reward distribution. To evaluate the performance of the reward functions, the median is used as a comparison metric, since the data from the simulation is not normally distributed and includes outliers. A Shapiro-Wilk test (Shapiro & Wilk, 1965) was performed to test for normality, as can be seen in Table 3.1. From the normality test results it can be observed that the data is not normally distributed, thus the results of the simulation are analyzed by utilizing the median. The following two sections will explore the results and the interpretations of these results in detail.

3.1 Logarithmically Increasing Geometric Reward

3.1.1 Equitabilities

The logarithmically modulated reward function results in a gradual increase towards higher equitability levels compared to the geometric reward, as can

Algorithm 2.1 Core of the PoS Simulation

```
function run(steps):
  set run_at_time to current time
  set initial_reward to reward_function(0)
  increment total_reward_pool by initial_reward
  set block_interval_counter to 0
  for each step from 1 to steps:
    set proposers to select_proposers()
    set reward to reward_function(step)
    increment total_reward_pool by reward
    if block_interval_counter ≥ block_interval then:
      set initial_reward to reward
      reset block_interval_counter to 0
    end if
    for each proposer in proposers:
      increment proposer.stake by reward divided by number_of_proposers_per_choice
    end for
    update total_stake
    update fractional stake for each node
    increment block_interval_counter
end function
```

Table 3.1: Normality test results for different reward functions.

Metric	Reward Function	p-value
Gini Coefficients	Geometric Reward	9.231×10^{-25}
	Logarithmically Increasing Geometric Reward	3.219×10^{-38}
	Sigmoid Modulated Geometric Reward	2.852×10^{-49}
	Sinusoidal Geometric Reward	2.356×10^{-41}
Reward Ratios	Geometric Reward	5.409×10^{-13}
	Logarithmically Increasing Geometric Reward	4.615×10^{-41}
	Sigmoid Modulated Geometric Reward	1.571×10^{-31}
	Sinusoidal Geometric Reward	5.408×10^{-40}
Equitabilities	Geometric Reward	1.308×10^{-25}
	Logarithmically Increasing Geometric Reward	2.549×10^{-35}
	Sigmoid Modulated Geometric Reward	9.871×10^{-44}
	Sinusoidal Geometric Reward	6.115×10^{-42}

be seen in Figure A.2. Higher parameters lead to higher equitability, which may be less desirable under the criteria of reducing wealth compounding. Figure A.5 shows that the distribution of the equitabilities is worse for the logarithmically increasing geometric reward function, where the median of the logarithmically increasing geometric reward function (0.079) lays above the median of the geometric reward function (0.074). The median is closer to the top of the IQR and the violin is wider at the top, indicating that the equitabilities are skewed towards the higher end of values.

3.1.2 Gini coefficient

There's a clear positive relationship between a higher parameter value and an increase in the Gini coefficient as can be seen in Figure A.3. So a stronger logarithmic modulation term results in a higher Gini coefficient, thus more inequality and higher wealth compounding. From Figure A.6 it can be seen that the median Gini coefficient over all parameters for the logarithmically increasing geometric reward (0.716) is higher than that of the other functions, indicating more wealth compounding than the geometric reward function.

3.1.3 Reward Ratio

As the time steps progress, the reward ratio balances out and approaches a constant value, due to the reduction in rewards dispensed over time, as can be seen in Figure A.1. The drop in the reward ratio per epoch is less than that of the geometric reward function. The effect of this can be seen in Figure A.7, where the logarithmically increasing geometric reward median (0.959) outperforms the geometric reward function median (0.618), thus ensuring a PoS system that's more incentive compatible. A higher parameter value leads to an increase in the reward ratio, indicating that the system becomes more incentive compatible as the logarithmic modulation increases in magnitude, as can be seen in Figure A.4.

3.2 Sigmoid Modulated Geometric Reward

3.2.1 Equitabilities

From Figure A.2 the sigmoid function's parameters appear to influence the equitabilities in a more chaotic manner, where there appears to be no clear relationship between equitabilities and the change in parameter value. Figure A.5 shows that the sigmoid modulated geometric reward has a symmetric violin shape with the median (0.062) centrally located, indicating a balanced distribution of equitabilities. The compressed interquartile range suggests similar variability compared to the geometric reward function. The equitabilities are lower than that of the geometric reward function median (0.074), indicating a more equitable system.

3.2.2 Gini coefficient

The increase of the parameter value affects the increase of the Gini coefficient minimally, which can be observed in Figure A.3. The Gini coefficient does increase incrementally when increasing the parameter value, indicating that a more prominent sigmoid modulation term increases wealth compounding. Figure A.6 shows that the interquartile range is relatively small compared to the other functions, indicating that the parameter values make less of an impact on wealth compounding. The median for the sigmoid modulated geometric reward (0.541) is lower than the other functions, indicating reduced wealth compounding and an improvement compared to the geometric reward function.

3.2.3 Reward Ratio

The distribution of the reward ratio is similar to that of the geometric reward function, though the spread is larger at peak values, as can be seen in Figure A.7, where the median of the reward ratio for the sigmoid modulated geometric is 0.298. Figure A.4 also shows that the rewards are dispensed in a similar manner, though lower parameter values lead to a higher reward ratio.

3.3 Sinusoidal Geometric Reward

3.3.1 Equitabilities

The sinusoidal geometric reward in Figure A.2 function displays a pattern of equitability increase similar to the geometric reward, though with variations across different parameters. Higher parameter values result in higher equitabilities. Figure A.5 displays a median (0.071) similar to the geometric reward, but with a wider interquartile range, implying more variability. The overall shape is more uniform than the geometric reward, suggesting a more consistent distribution of reward values. Though, a clustering of values in the high end can be observed, due to the rapid increase in equitabilities, as the parameter values grow.

3.3.2 Gini coefficient

Similar to the logarithmically increasing geometric reward function, the parameter values have a clear positive relationship with the Gini coefficient, as can be seen in Figure A.3. A higher parameter value results in a higher Gini coefficient and higher wealth compounding. From Figure A.6 it can be observed that the Gini coefficient values are similar to the geometric reward function, with a median of 0.618, except that the top quadrant is larger, indicating a higher wealth compounding compared to the geometric reward function.

3.3.3 Reward Ratio

The reward ratio fluctuates heavily over time, showing many peaks as can be seen in Figure A.4 where the performance of the reward ratio for the sinusoidal geometric reward can be seen varying wildly. Figure A.7 shows that while the distribution of the reward ratio is similar to that of the geometric reward function, with a median of 0.261, the spread is much greater, showing an outlier of above six for the reward ratio. The lower the parameter value is, the higher the fluctuations in reward.

4 Discussion

This study presents an analysis of the wealth distribution in PoS systems using three new reward functions derived from the geometric model to reduce the effect of wealth compounding and improve incentive compatibility, and answer the RQ "What is the effect of the geometric reward function and smoothing of future rewards on wealth compounding in PoS protocols?". The proposed reward functions aim to address the wealth compounding by building on the geometric reward function as introduced by (Fanti et al., 2019), while also maintaining an incentive for actors to participate in the PoS system. The results of the three metrics introduced (Gini coefficient, equitability, and the reward ratio) demonstrate that smoothing the geometric reward function can maintain a reduction in wealth compounding, while also improving incentive compatibility, ensuring a productive system with active participation.

The logarithmically increasing geometric reward function yielded the highest median reward ratio (0.959) and the highest Gini coefficient (0.716), showing that while it substantially increases the reward to highly staking participants in the system, it also leads to greater inequality and wealth compounding. The function results in a high reward ratio, suggesting that this function could promote more active participation, but the increased Gini coefficient points to a significant wealth concentration among fewer participants, indicating more wealth compounding.

In contrast, the sigmoid modulated geometric reward function resulted in the most balanced outcomes among the reward functions tested. It had the lowest median Gini coefficient (0.541) and a median equitability (0.062) lower than the geometric reward function as a baseline (0.074). These values hint towards a more equitable distribution of rewards, and less wealth compounding, making it a preferable option for PoS protocols aimed at minimizing wealth compounding, while ensuring the system maintains incentive for participation. Its reward ratio (0.298) is higher than the geometric reward function, guaranteeing that the system maintains a larger incentive to participate than that of the geometric reward function.

The sinusoidal geometric reward function, similar to the logarithmically increasing reward function, showed higher variability in the distribution of equitabilities and a Gini coefficient (0.618) identical to the geometric baseline. Its reward ratio (0.261) is slightly lower than the geometric baseline. This suggests that it does not fundamentally alter the incentive or equality structure of the reward system compared to the standard geometric function.

4.1 Future Research

While this research addresses the unequal wealth distribution issue, it is essential to acknowledge potential limitations. Specifically:

1. The total reward dispensed remains a variable unaccounted for in this study. The total reward dispensed by each reward function is not equal, while the absolute reward earned by a participant does influence wealth compounding. This might pose a problem in terms of comparison between reward functions, since the unequal total rewards dispensed could lead to misleading interpretations of which reward function is more effective in reducing wealth compounding and maintaining incentive compatibility.

2. Due to performance constraints, this PoS system is currently static and simplified; incorporating more dynamic elements to more accurately model the system could lead to a more realistic representation of the PoS network, wealth compounding, and incentive compatibility. Participation in a PoS system is not static in the real world, and variable network participation in the PoS model could lead to a more robust simulation.

Table 3.2: Medians of each reward functions for all metrics. The optimal values are highlighted.

Function	Median Equitabilities	Median Reward Ratio	Median Gini
Geometric	0.074	0.267	0.618
Log. Increasing Geom.	0.079	*0.959*	0.716
Sigmoid Mod. Geom.	*0.062*	0.298	*0.541*
Sinusoidal Geom.	0.071	0.261	0.618

3. Alternative metrics to assess wealth distribution and incentive compatibility could give rise to a new perspective on wealth compounding and incentive compatibility, since the combination of the equitabilities, Gini coefficient, and reward ratio might not be sufficient to fully quantify the economics of a PoS system. Modeling the PoS system with variable network participation adds the possibility of measuring the participation rate, providing a deeper understanding of incentive compatibility.

5 Conclusions

The findings indicate that smoothing the geometric reward function with each reward function had a measurable effect on the performance of the PoS system, in terms of wealth compounding and incentive compatibility. The logarithmically increasing geometric reward function improves on incentive compatibility, while also increasing wealth compounding. The sinusoidal geometric reward function alters the dynamics of the geometric reward function in terms of variability, while not improving the performance of the PoS system. The sigmoid modulated geometric reward function improves both wealth compounding and incentive compatibility compared to the geometric reward function. In conclusion, smoothing the geometric reward function, as suggested by (Fanti et al., 2019), could improve on both wealth compounding and incentive compatibility, if implemented in a PoS system.

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A Appendix

A.1 Explanation of Figures

A.1.1 Figure A.1

Four line graphs are illustrated representing each of the four reward functions (in top-to-bottom order of logarithmically increasing, sigmoid modulated, sinusoidal, and geometric reward), where the total reward dispensed over time per reward parameter setting is plotted in different colors. Each reward function has four different parameter settings, except for the geometric reward function. The x-axis represents the time steps of the PoS system, while the y-axis represents the amount of reward that is dispensed at the corresponding timestep.

A.1.2 Figure A.2

Four line graphs are illustrated representing each of the four reward functions (in top-to-bottom order of geometric reward, logarithmically increasing, sigmoid modulated, and sinusoidal), where the equitabilities over time per reward parameter setting is plotted in different colors. Each reward function has four different parameter settings, except for the geometric reward function. The x-axis represents the time steps of the PoS system, while the y-axis represents the equitabilities - as described in equation 1.5 - at the corresponding timestep.

A.1.3 Figure A.3

Four line graphs are illustrated representing each of the four reward functions (in top-to-bottom order of sigmoid modulated, logarithmically increasing, geometric reward, and sinusoidal), where the Gini coefficient over time per reward parameter setting is plotted in different colors. Each reward function has four different parameter settings, except for the geometric reward function. The x-axis represents the time steps of the PoS system, while the y-axis represents the Gini coefficient - as described in equation 1.2 - at the corresponding timestep.

A.1.4 Figure A.4

Four line graphs are illustrated representing each of the four reward functions (in top-to-bottom order of logarithmically increasing, sigmoid modulated, sinusoidal, and geometric reward), where the reward ratio over time per reward parameter setting is plotted in different colors. Each reward function has four different parameter settings, except for the geometric reward function. The x-axis represents the time steps of the PoS system, while the y-axis represents the reward ratio - as described in equation 1.6 - at the corresponding timestep.

A.1.5 Figure A.5

Four violin plots are illustrated representing each of the four reward functions (in left-to-right order of geometric reward, logarithmically increasing, sigmoid modulated, and sinusoidal), displaying the distribution of equitabilities over one run of the PoS simulation as described in the Methods section. The x-axis represents the different reward functions, labeled accordingly. The y-axis represents the equitability values. Each violin in the plot represents the distribution of the equitability values for a particular reward function. The width of the violin represents the number of equitability values that fall within a particular range, so a wider violin indicates a higher density of data points in that range. The dashed line with longest segments within each violin indicates the median value, while the other dashed lines indicate the interquartile range.

A.1.6 Figure A.6

Four violin plots are illustrated representing each of the four reward functions (in left-to-right order of geometric reward, logarithmically increasing, sigmoid modulated, and sinusoidal), displaying the distribution of the Gini coefficient over one run of the PoS simulation as described in the Methods section. The x-axis represents the different reward functions, labeled accordingly. The y-axis represents the Gini coefficient values. Each violin in the plot represents the distribution of the Gini coefficient values for a particular reward function. The width of the violin represents the number of Gini coefficient values that fall within a particular range, so a wider violin indicates a higher density of data points in that range. The dashed line with longest segments within each violin indicates the median value, while the other dashed lines indicate the interquartile range.

A.1.7 Figure A.7

Four violin plots are illustrated representing each of the four reward functions (in left-to-right order of geometric reward, logarithmically increasing, sigmoid modulated, and sinusoidal), displaying the distribution of the reward ratio over one run of the PoS simulation as described in the Methods section. The x-axis represents the different reward functions, labeled accordingly. The y-axis represents the reward ratio values. Each violin in the plot represents the distribution of the reward ratio values for a particular reward function. The width of the violin represents the number of reward ratio values that fall within a particular range, so a wider violin indicates a higher density of data points in that range. The dashed line with longest segments within each violin indicates the median value, while the other dashed lines indicate the interquartile range.

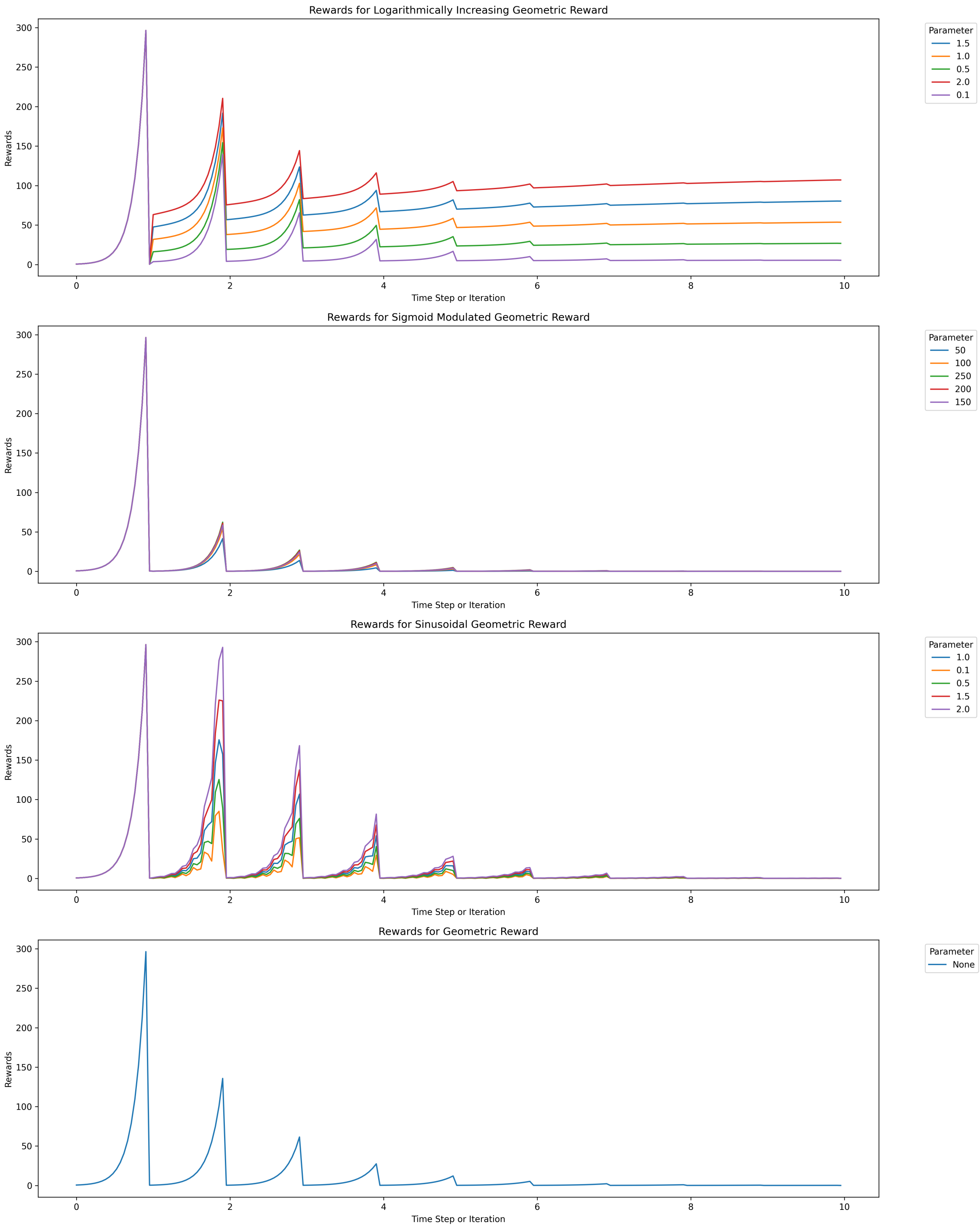

Figure A.1: Graph of the total reward dispensed for each reward function. Explained in detail in section A.1.1.

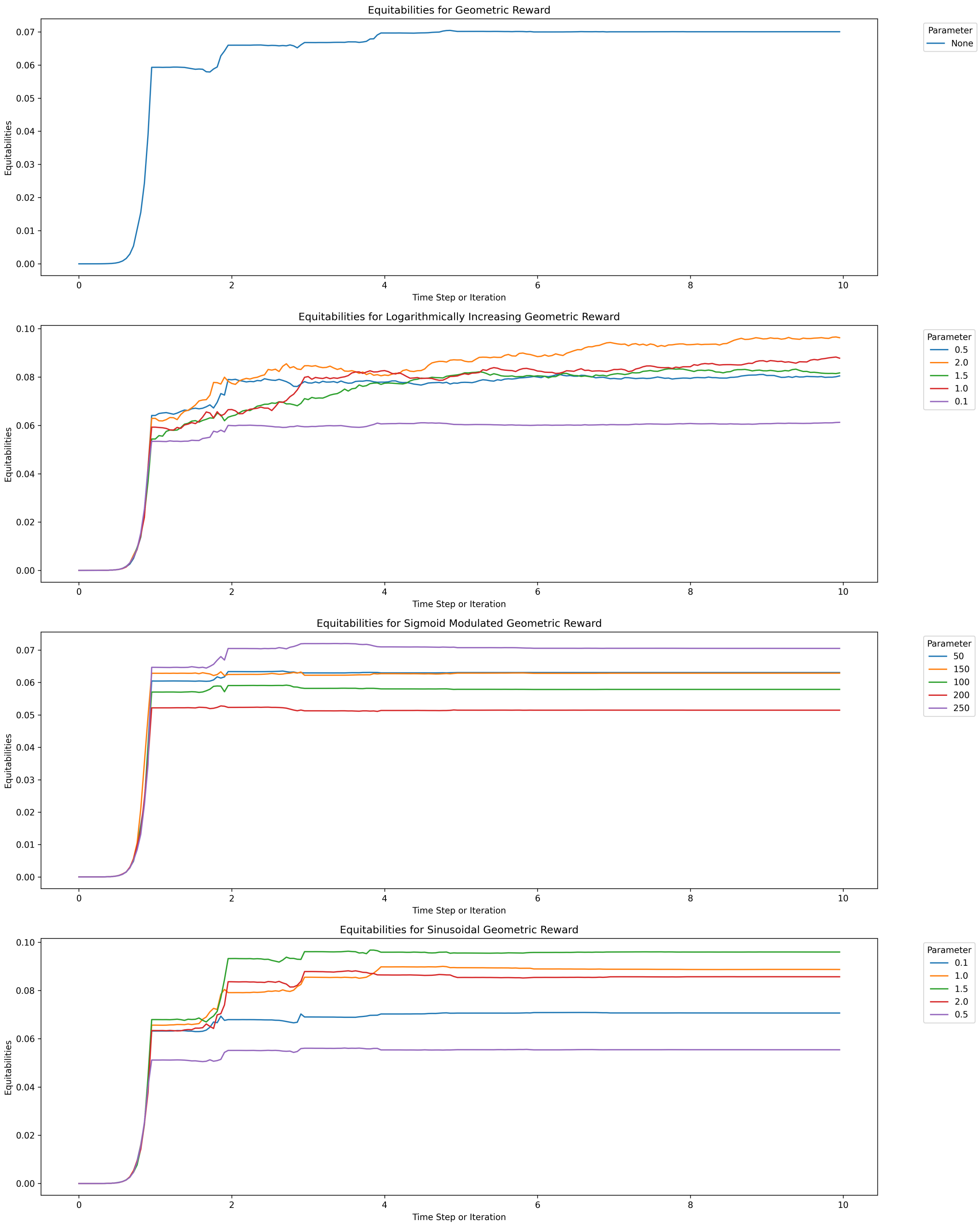

Figure A.2: Graph of the the equitabilities for each reward function. Explained in detail in section A.1.2.

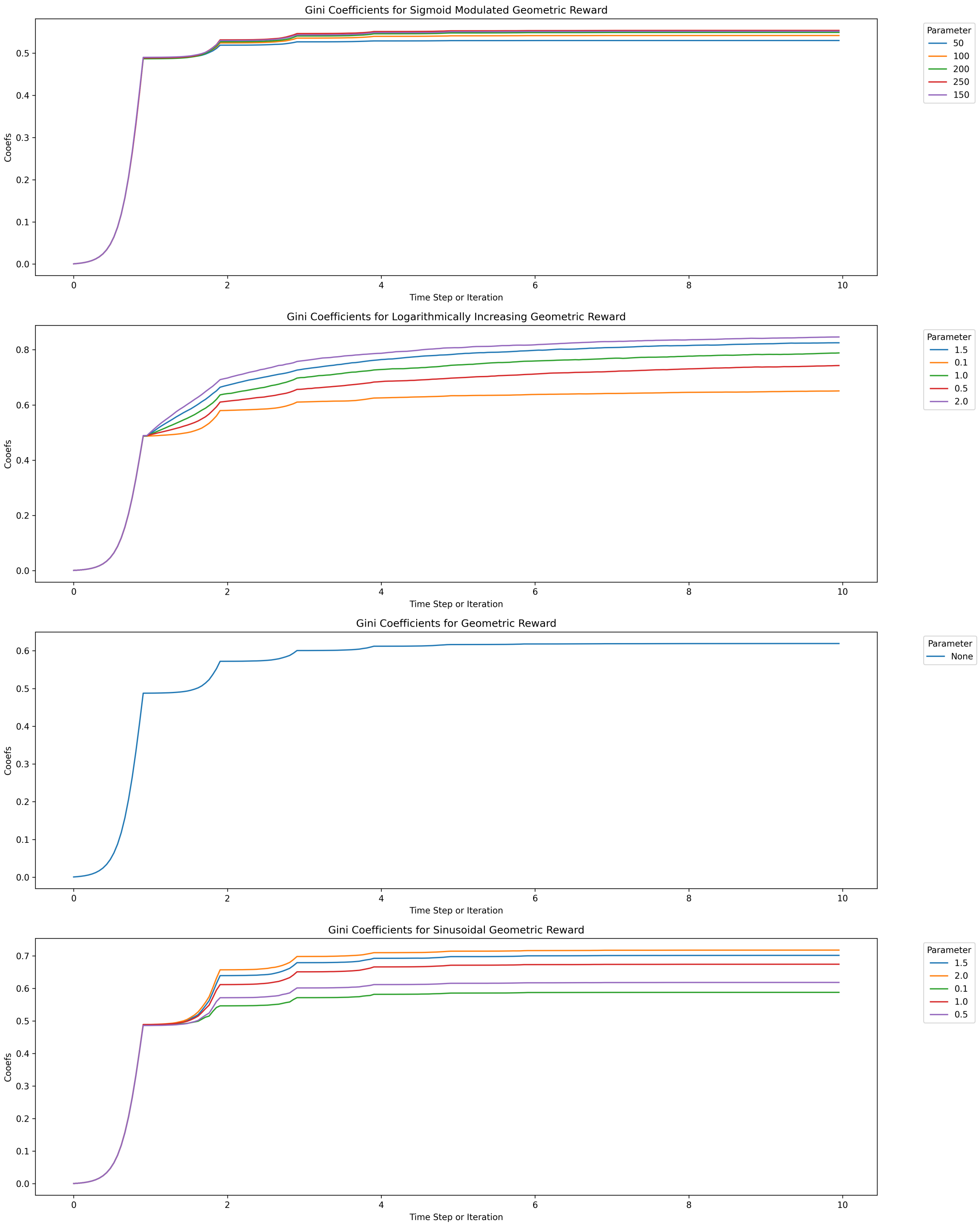

Figure A.3: Graph of the the Gini coefficient for each reward function. Explained in detail in section A.1.3.

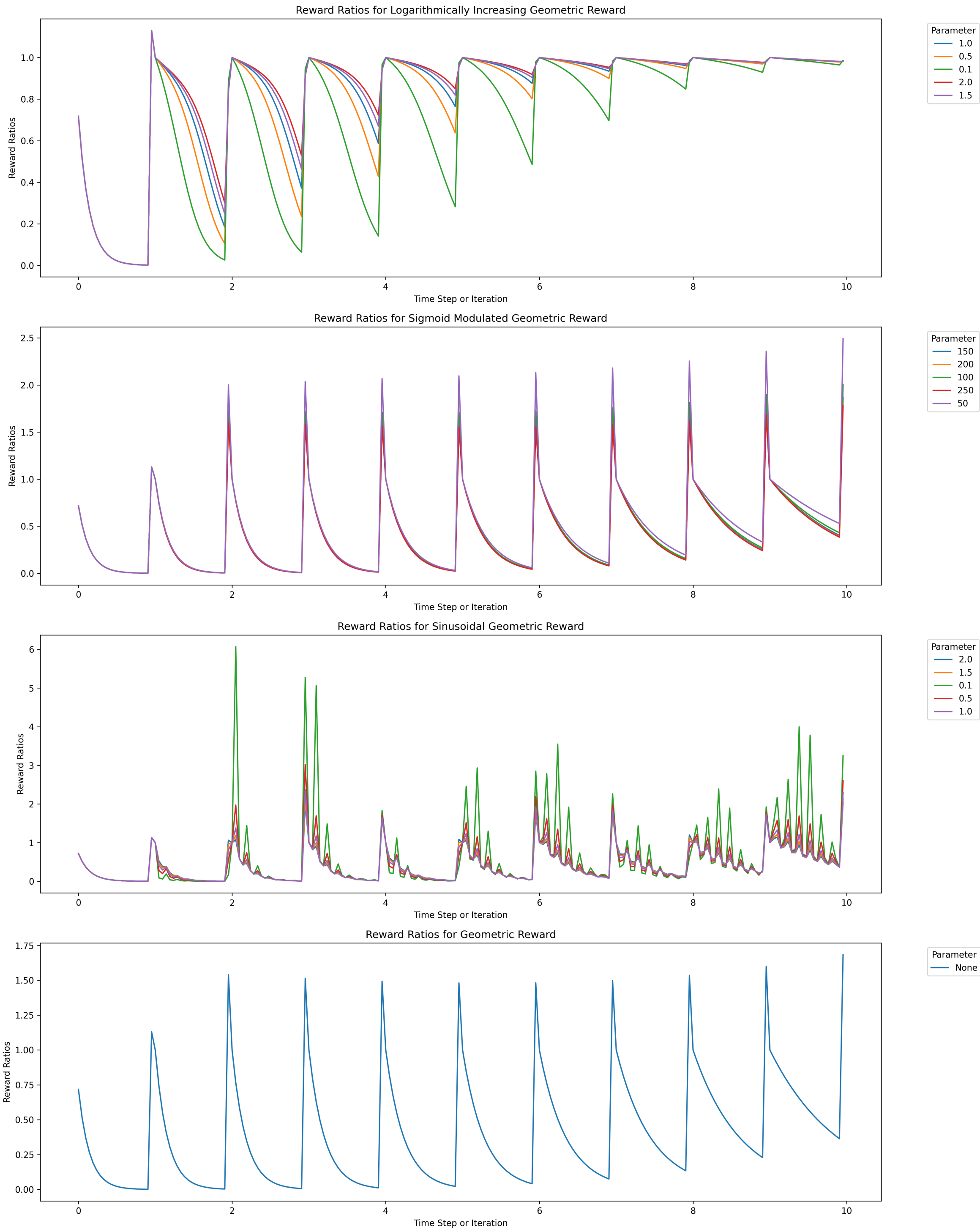

Figure A.4: Graph of the the reward ratios for each reward function. Explained in detail in section A.1.4.

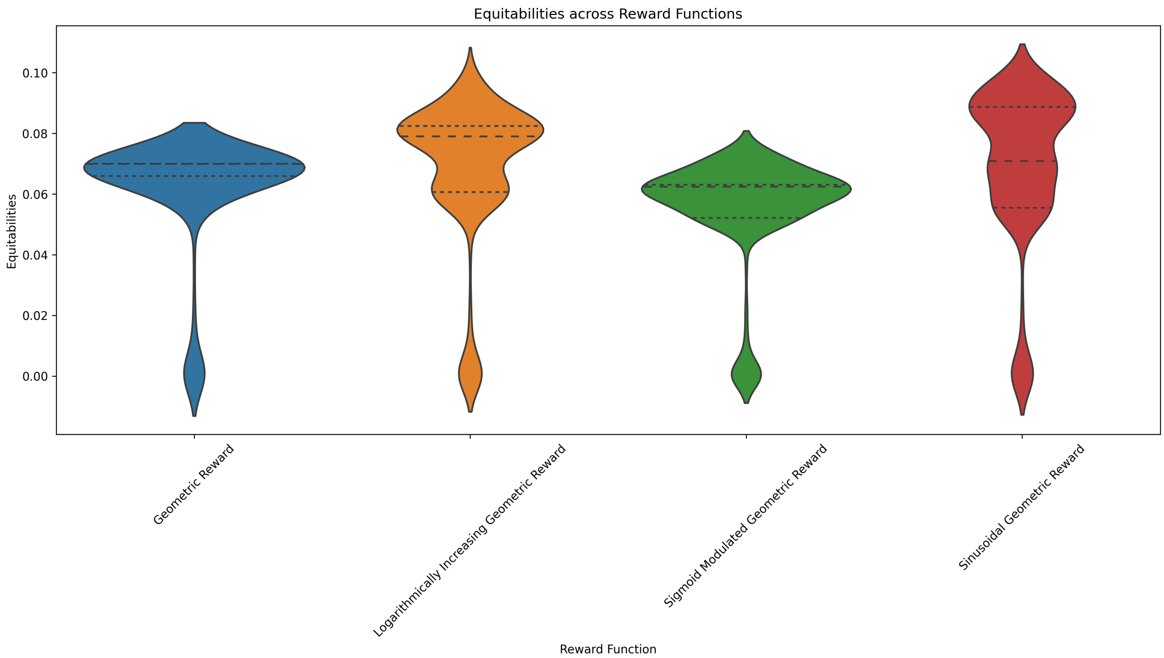

Figure A.5: Violin plot of the the aggregated equitabilities for each reward function. Explained in detail in section A.1.5.

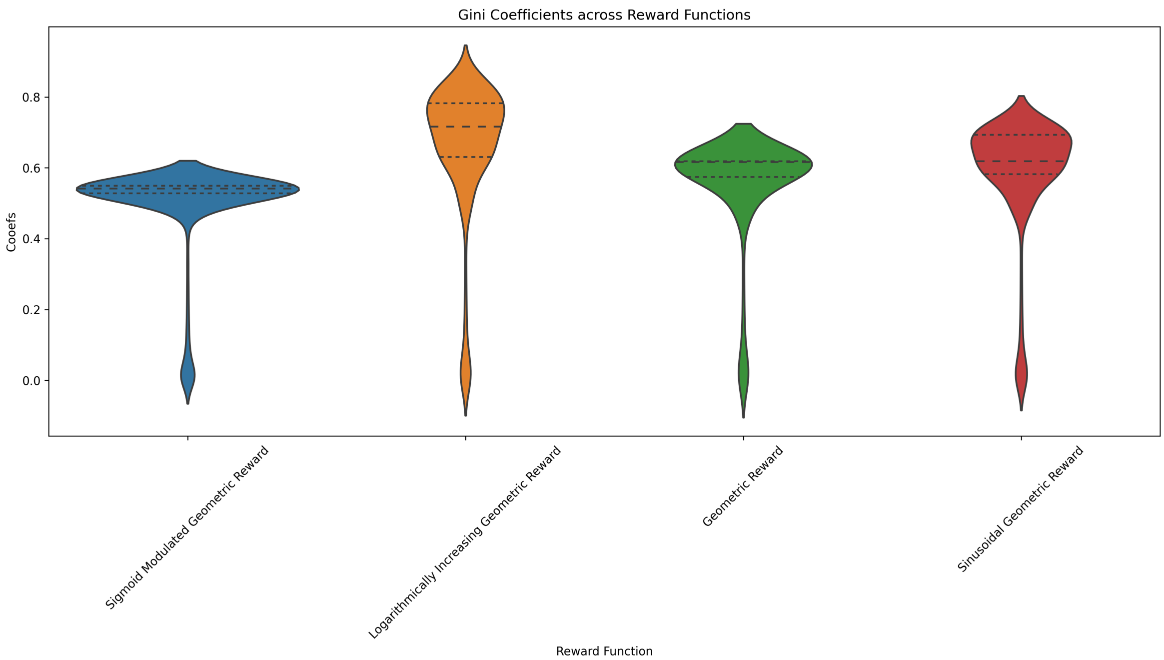

Figure A.6: Violin plot of the the aggregated Gini coefficients for each reward function. Explained in detail in section A.1.6.

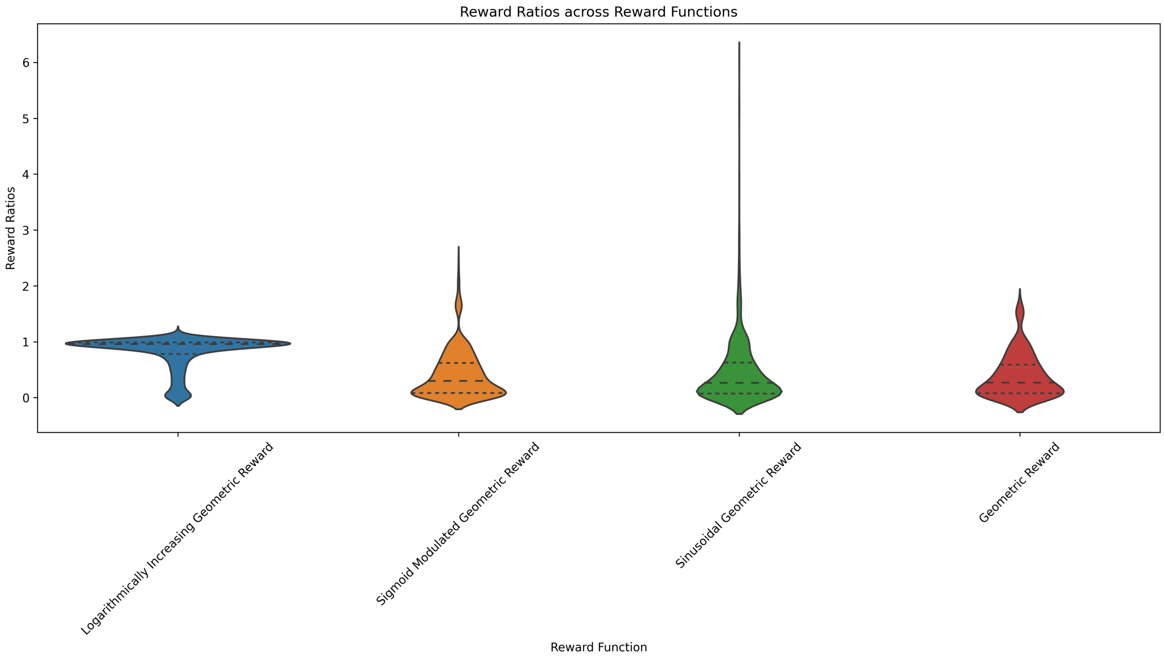

Figure A.7: Violin plot of the the aggregated reward ratios for each reward function. Explained in detail in section A.1.7.