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Bachelor Thesis

Extreme Value Laws for the Tent Map via Fibonacci-like Sequences

in Mathematics

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Abstract

The present paper provides an alternative proof of an Extreme Value Law for a stochastic process obtained by iterating the tent map, originally established by George Haiman (2003). The proof closely follows the methodology employed by N.B. Boer and A.E. Sterk (2021) in their work on a similar result for the Rényi map. This approach not only gives an alternative proof of the original theorem but also extends the understanding of connections between extreme value laws and recursive sequences in dynamical systems.

Contents

1 Introduction and aim of the paper

Chaotic iterative maps on an interval form a significant subclass of examples within the broader study of chaotic dynamical systems. The trajectories they produce, despite being entirely deterministic when given an initial value, exhibit behavior that closely resembles that of a classical random process. Only a few chaotic iterative maps are mathematically tractable, among which an important subclass consists of functions defined by $f_v(x) = 1 - |2x-1|^v$, where $x \in [0,1]$ and $v > 0$. These maps have been extensively studied, particularly in relation to their invariant distribution—specifically, the distribution of a random variable X_0 with values in [0, 1] such that $X_1 = f_v(X_0)$ has the same distribution [3].

In the present paper, we are concerned with the particular case $v = 1$, also called the "tent map", where

$$f(x) = 1 - |2x-1|, \quad x \in [0,1].$$

Let $X_0 \sim \mathcal{U}(0,1)$ and define the tent map process by

$$X_{m+1} = 1 - |2X_m - 1|, \quad m \geq 0. \tag{1}$$

Moreover, let

$$M_n := \max(X_0, \ldots, X_{n-1}).$$

Extreme value theory for a sequence of i.i.d. random variables $(X_i)_{i=0}^{\infty}$ examines the asymptotic behavior of the partial maximum M_n as $n \to \infty$. Since the distribution of M_n becomes degenerate in the limit, it is necessary to apply a rescaling. Under suitable conditions, there exist sequences $a_n > 0$ and $b_n \in \mathbb{R}$ such that the rescaled maximum, $a_n(M_n - b_n)$, has a non-degenerate limiting distribution [1, 4].

Consider a simple example where the random variables $X_i \sim \mathcal{U}(0,1)$ are independent. Let $a_n = n$ and $b_n = 1$. For any $\lambda \geq 0$, we have:

$$\lim_{n\to\infty} \mathbb{P}(a_n(M_n - b_n) \leq -\lambda) = \lim_{n\to\infty} \mathbb{P}\left(M_n \leq 1 - \frac{\lambda}{n}\right) = \lim_{n\to\infty}\left(1 - \frac{\lambda}{n}\right)^n = e^{-\lambda}. \tag{2}$$

For completion, we show how the uniform density on $[0,1]$ is invariant with respect to the Lebesegue measure under the transformation f:

Lemma 1.1. *If X is a random variable such that $X \sim \mathcal{U}(0,1)$, then $f(X) \sim \mathcal{U}(0,1)$.*

Proof. For $u \in [0,1]$, we have $\mathbb{P}(X \in [0,u]) = u$. The tent map function f can be rewritten as

$$f(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq \frac{1}{2}, \\ 2(1-x) & \text{if } \frac{1}{2} < x \leq 1. \end{cases}$$

We tackle both cases separately: for $0 \leq x \leq \frac{1}{2}$, $f(X) \leq u$ implies $x \leq \frac{u}{2}$. For $\frac{1}{2} < x \leq 1$, $f(X) \leq u$ implies $x \geq 1 - \frac{u}{2}$. Hence,

$$\mathbb{P}(f(X) \in [0,u]) = \mathbb{P}\left(X \in \left[0, \frac{u}{2}\right] \cup \left[1 - \frac{u}{2}, 1\right]\right) = u,$$

which implies that $f(X) \sim \mathcal{U}(0,1)$. □

Consider the stochastic process $(X_i)_{i=0}^{\infty}$ defined by (1). Lemma 1 implies that the variables X_i are identically distributed, although they are no longer independent. Let M_n be defined as above. In [3], Haiman proves the following result:

Theorem 1.2. *For any $\lambda > 0$, define $n_k := \lfloor 2^k \lambda \rfloor$. We have*

$$\lim_{k\to\infty} \mathbb{P}\left(M_{n_k} \leq 1 - 2^{-k}\right) = e^{-\lambda}.$$

See how, for $\lambda \in \mathbb{N}$, we have $\mathbb{P}(M_{n_k} \leq 1 - 2^{-k}) = \mathbb{P}(2^k\lambda(M_{2^k\lambda} - 1) \leq -\lambda)$. Thus, the result of Theorem 1.2 aligns in essence with the example in (2), although this time a subsequence of M_n is being considered.

Haiman's proof follows from studying the asymptotic behavior of a previously proven result, mainly that for any integers $n, k \geq 1$ we have

$$P\left(M_n \leq 1 - 2^{-k}\right) = 1 - \sum_{i=0}^{\lfloor n/k \rfloor} (-1)^i \binom{n - i(k-1)}{i+1} \left(2^{-k}\right)^{i+1},$$

where $\lfloor . \rfloor$ denotes the floor function [†]. For an alternative approach to this result that does not consider a subsequence of n, the reader is referred to [2].

This paper aims to provide yet another proof of Theorem 1.2, focusing on Fibonacci-like sequences arising from the tent map process. A similar approach was previously used by N. B. Boer and A. E. Sterk in [1] to establish an analogous result for the Rényi map[‡]. The present work closely follows their methodology.

2 The relation with Fibonacci-like sequences

In this section, we fix the numbers $k \in \mathbb{N}$ and $u = 2^{-k}$. For any integer $m \geq 0$ we define the set

$$E_m = \{X_m > 1 - u\}, \tag{3}$$

where $X_{m+1} = f(X_m)$ and the dependence on k is omitted in the notation for simplicity. Then

$$\begin{aligned}\mathbb{P}(M_n \leq 1 - u) &= \mathbb{P}\left(\bigcap_{m=0}^{n-1} (X_m \leq 1 - u)\right) \\ &= 1 - \mathbb{P}\left(\bigcup_{m=0}^{n-1} (X_m > 1 - u)\right) \\ &= 1 - \text{Leb}\left(\bigcup_{m=0}^{n-1} E_m\right),\end{aligned}$$

where Leb denotes the *Lebesgue measure* on $[0, 1]$. Put

$$B_n := \text{Leb}\left(\bigcup_{m=0}^{n-1} E_m\right), \quad n \geq 1,$$

[†]In [3], M_n is defined as the maximum of $X_1, \ldots, X_n$.

[‡]The Rényi map is defined as $f : [0, 1) \longrightarrow [0, 1)$, $f(x) = \beta x \mod 1$.

and thus $\mathbb{P}(M_n \leq 1-u) = 1 - B_n$ in short. Based on self-similarity arguments Haiman derives the following formulas in [3], which hold for each fixed $k \geq 1$:

$$B_n = n \cdot u \qquad \text{if } n \leq k, \tag{4}$$
$$B_{n+1} = B_n + u(1 - B_{n-k+1}) \qquad \text{if } n \geq k. \tag{5}$$

Now, analogous to [1], where the F_n's are first presented in order to tackle the recursive properties of the coverage layers arising from the Rényi map, for $n \in \mathbb{Z}$ we define the following numbers:

$$F_n = \begin{cases} 0 & \text{if } n < 1, \\ 1 & \text{if } n = 1, \\ (B_n - B_{n-1}) \cdot 2^{n+k-1} & \text{if } n > 1. \end{cases} \tag{6}$$

These numbers have the following geometric meaning. Observe that if we put $E_0 = (1-u, 1]$, the sets E_m can be written as a union of 2^{m-1} intervals:

$$E_m = \bigcup_{s=0}^{2^{m-1}-1} \left(\frac{1+2s}{2^m} - \frac{u}{2^m}, \frac{1+2s}{2^m} + \frac{u}{2^m} \right), \quad m \geq 1.$$

It is essential to note how, for $n \geq 1$, the number F_n equals the number of sub-intervals of the set E_n which need to be added to $E_0 \cup \cdots \cup E_{n-1}$ in order to obtain $E_0 \cup \cdots \cup E_n$. Figure 1 illustrates this for the case $k = 3$.

Lemma 2.1. *For any $k, n \in \mathbb{N}$ it follows that*

$$\mathbb{P}(M_n \leq 1 - 2^{-k}) = 2^{1-n-k} F_{n+k}.$$

Proof. For $n \geq k+1$ equation (5) gives

$$F_n = (B_n - B_{n-1}) \cdot 2^{n+k-1} = (1 - B_{n-k}) \cdot 2^{n-1},$$

or, equivalently,

$$B_{n-k} = 1 - \frac{F_n}{2^{n-1}}.$$

Replacing $n-k$ with n yields

$$B_n = 1 - \frac{F_{n+k}}{2^{n+k-1}}.$$

Substituting $\mathbb{P}(M_n \leq 1-2^{-k}) = 1 - B_n$ in the above equation proves the desired result. □

The following result provides the connection between the sequence (B_n) and the Fibonacci-like sequence (F_n) for each k observed in Table 1.

Lemma 2.2. *The following statements are equivalent:*

(i) Eqs. (4) and (5) hold;

(ii) For fixed $k \in \mathbb{N}$, the sequence (F_n), where $n \in \mathbb{Z}$, defined in (6) satisfies

$$F_n = \begin{cases} 0 & \text{if } n < 1, \\ 1 & \text{if } n = 1, \\ \sum_{i=1}^{k-1} F_{n-i} + 1 & \text{if } n \geq 2. \end{cases} \tag{7}$$

In particular, $F_n = 2^{n-1}$ for $1 \leq n \leq k$.

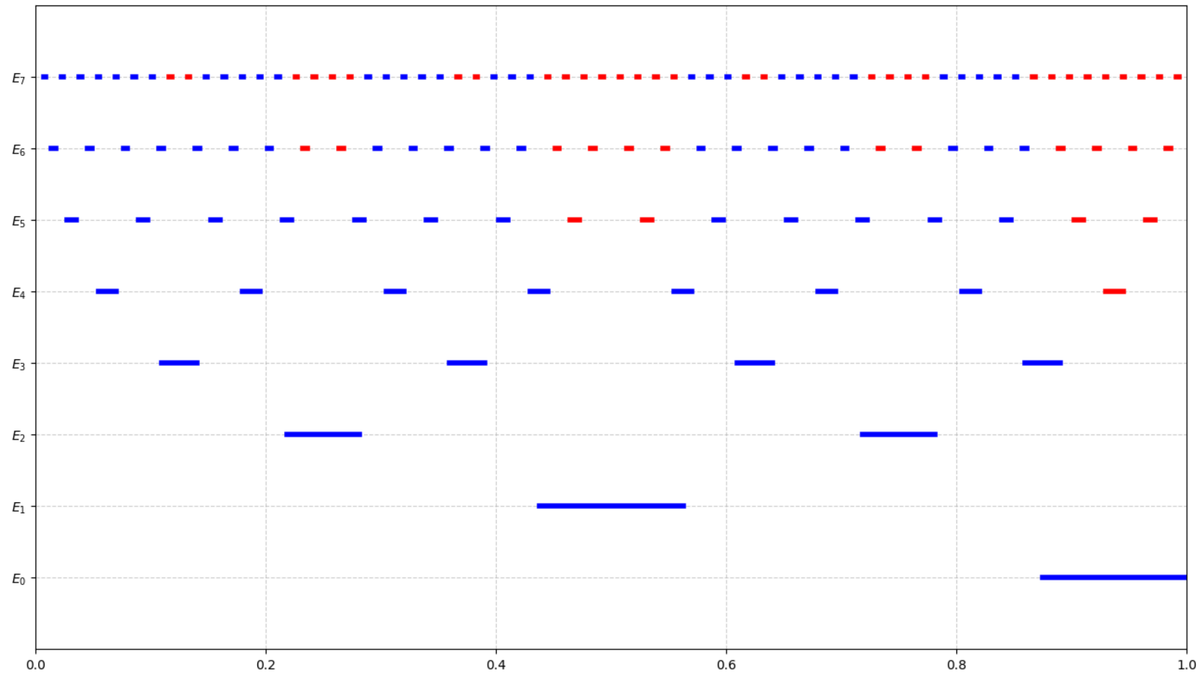

Figure 1: Illustration of the sets $E_0, \ldots, E_6$ for $k = 3$. Each set E_n is a union of 2^{n-1} intervals, $n \geq 1$. Intervals within E_n that are not fully contained within the intervals forming $E_0, \ldots, E_{n-1}$ are shown in blue, while intervals contained within the intervals forming $E_0, \ldots, E_{n-1}$ are shown in red. By definition $F_0 = 0$ and $F_1 = 1$; and the figure clearly illustrates how $F_2 = 2, F_3 = 4, F_4 = 7, F_5 = 12, F_6 = 20$ and $F_7 = 33$. It appears to be that $F_n = F_{n-1} + F_{n-2} + 1$ $(n \geq 2)$, which corresponds to the Fibonacci sequence with an additional increment of one.

Proof. (i) $\Rightarrow$ (ii): Assume that statement (i) holds. By definition $F_1 = 1$ and for $1 \leq n \leq k$ equation (4) implies that

$$F_n = (B_n - B_{n-1}) \cdot 2^{n+k-1} = (nu - (n-1)u) \cdot 2^{n+k-1} = u \cdot 2^{n+k-1} = 2^{n-1}.$$

We proceed with induction on n. For any $n \geq k$ equation (5) gives

$$F_{n+1} = (B_{n+1} - B_n) \cdot 2^{n+k} = (u(1 - B_{n-k+1})) \cdot 2^{n+k} = (1 - B_{n-k+1}) \cdot 2^n. \tag{8}$$

In particular, for $n = k$ we have

$$\begin{aligned} F_{k+1} &= (1 - B_1) \cdot 2^k \\ &= 2^k - 1 \\ &= (2-1) \sum_{i=1}^{k} 2^{k-i} \\ &= \sum_{i=1}^{k} F_{k+1-i} \\ &= \sum_{i=1}^{k-1} F_{k+1-i} + 1. \end{aligned}$$

	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}
$k=2$	1	2	3	4	5	6	7	8	9	10
$k=3$	1	2	4	7	12	20	33	54	88	143
$k=4$	1	2	4	8	15	28	52	96	177	326
$k=5$	1	2	4	8	16	31	60	116	224	432
$k=6$	1	2	4	8	16	32	63	124	244	480

Table 1: Table of values containing the first 10 nonzero numbers of the sequence F_n for $k \in \{2, \ldots, 6\}$.

Assume that for some $n \geq k$ it follows that

$$F_{n+1} = \sum_{i=1}^{k-1} F_{n+1-i} + 1.$$

First using (8), then (5), and finally (8) again yields:

$$\begin{aligned} F_{n+2} &= (1 - B_{n-k+2}) \cdot 2^{n+1} \\ &= (1 - (B_{n-k+1} + u(1 - B_{n-2k+2}))) \cdot 2^{n+1} \\ &= (1 - B_{n-k+1}) \cdot 2^{n+1} - u(1 - B_{n-2k+2}) \cdot 2^{n+1} \\ &= 2F_{n+1} - F_{n-k+2}. \end{aligned}$$

Using the induction hypothesis we get:

$$\begin{aligned} F_{n+2} &= F_{n+1} + F_{n+1} - F_{n-k+2} \\ &= F_{n+1} + \sum_{i=1}^{k-1} F_{n+1-i} + 1 - F_{n-k+2} \\ &= \sum_{i=1}^{k-1} F_{n+2-i} + 1. \end{aligned}$$

Hence, statement (ii) follows.

(ii) ⇒ (i): Assume that statement (ii) holds. In particular, $F_n = 2^{n-1}$ for $2 \leq n \leq k$ so that by (6) it follows that

$$B_n = B_{n-1} + 2^{1-n-k} \cdot F_n = B_{n-1} + u.$$

Thus, $B_n - B_{n-1} = u$ and since $B_1 = u$ then $B_n = n \cdot u$ for $2 \leq n \leq k$.

Recalling that $B_1 = u = 2^{-k}$, equation (6) implies that

$$\begin{aligned} B_{k+1} &= B_k + 2^{-2k} \cdot F_{k+1} \\ &= B_k + 2^{-2k}(2^k - 1) \\ &= B_k + 2^{-k} - 2^{-2k} \\ &= B_k + u(1 - 2^{-k}) \\ &= B_k + u(1 - B_1), \end{aligned}$$

which shows that (5) holds for $n = k$. Assume that there exists $m \in \mathbb{N}$ such that (5) holds for all $k \le n \le m$. Observe that

$$F_{m+1} = \sum_{i=1}^{k-1} F_{m+1-i} + 1 = F_m - F_{m+2-k} + \sum_{i=1}^{k-1} F_{m-i} + 1 = 2F_m - F_{m+1-k}.$$

Therefore,

$$\begin{aligned} B_{m+1} - B_m &= 2^{-m-k} \cdot F_{m+1} \\ &= 2^{-m-k}(2F_m - F_{m+1-k}) \\ &= 2^{1-m-k}F_m - 2^{-m-k}F_{m+1-k} \\ &= B_m - B_{m-1} - 2^{-m-k}(2^m(B_{m+1-k} - B_{m-k})) \\ &= B_m - B_{m-1} - 2^{-k}(B_{m+1-k} - B_{m-k}). \end{aligned}$$

The induction hypothesis gives

$$\begin{aligned} B_{m+1} - B_m &= u(1 - B_{m-k}) - 2^{-k}(u(1 - B_{m+1-2k})) \\ &= u(1 - (B_{m-k} + u(1 - B_{m-2k+1}))) \\ &= u(1 - B_{m+1-k}). \end{aligned}$$

Hence, statement (i) follows. This concludes the proof. □

3 The Binet formula: deriving a closed form

In this section we derive a closed-form expression for F_n as a function of n. The characteristic polynomial corresponding to the recursion relation is given by

$$p_{k-1}(x) = x^{k-1} - \sum_{i=0}^{k-2} x^i. \tag{9}$$

The following lemma concerns properties of the roots of this polynomial. The importance of this result will become evident later.

Lemma 3.1. *Let $k \in \mathbb{Z}_{\ge 3}$. Then,*

(i) the polynomial p_{k-1} has a real root $1 < r_{k-1,1} < 2$;

(ii) the remaining roots $r_{k-1,2} \dots r_{k-1,k-1}$ of p_{k-1} lie within the unit circle of the complex plane;

(iii) the roots of p_{k-1} are simple.

Proof. (i) *Descartes's rule of signs* implies that p_{k-1} has exactly one positive root $r_{k-1,1}$. Note how

$$p_{k-1}(1) = 1 - (k-1) < 0 \quad \text{and} \quad p_{k-1}(2) = 1$$

since $2^{k-1} - \sum_{i=0}^{k-2} 2^i = 2^{k-1} - \frac{1-2^{k+1}}{1-2} = 1$. Now, the *Intermediate Value Theorem* implies the existence of a root $1 < r_{k-1,1} < 2$.

(ii) Define the polynomial

$$q_k(x) = (x-1)p_{k-1}(x) = x^k - 2x^{k-1} + 1.$$

We establish two auxiliary claims:

Claim 1. *If $x > r_{k-1,1}$, then $p_{k-1}(x) > 0$, and if $0 < x < r_{k-1,1}$, then $p_{k-1}(x) < 0$.*

Proof Claim 1. First, assume $x > r_{k-1,1}$. Since $p_{k-1}(2) = 1$ and we showed in (i) that $r_{k-1,1} \in (1, 2)$ is the only positive root of p_{k-1}, we can conclude that $p_{k-1}(x) > 0$ for all $x > r_{k-1,1}$. Second, assume now $0 < x < r_{k-1,1}$. See how p_{k-1} cannot be positive anywhere in this open interval since otherwise we would have a second positive root. Thus, $p_{k-1}(x) < 0$ for all $0 < x < r_{k-1,1}$. ■

Claim 2. *If $x > r_{k-1,1}$, then $q_k(x) > 0$, and if $1 < x < r_{k-1,1}$, then $q_k(x) < 0$.*

Proof Claim 2. The polynomial q_k has the same roots as p_k plus the extra root $x = 1$. An analogous reasoning to the above proves our claim. ■

Note that p_{k-1} has no root r such that $|r| > r_{k-1,1}$. Indeed, if such a root exists, then $p_{k-1}(r) = 0$, or, equivalently, $r^{k-1} = \sum_{i=0}^{k-2} r^i$. The triangle inequality gives $|r|^{k-1} \leq \sum_{i=0}^{k-2} |r|^i$. Hence, $p_k(|r|) \leq 0$, which contradicts Claim 1.

In addition, p_{k-1} has no root r with $1 < |r| < r_{k-1,1}$. Indeed, if such a root exists, then $q_k(r) = (r-1)p_{k-1}(r) = 0$ so that $2r^{k-1} = r^k + 1$. The triangle inequality implies that $2|r|^{k-1} \leq |r|^k + 1$. Hence, $q_k(|r|) \geq 0$, which contradicts Claim 2.

Finally, p_{k-1} has no root r with either $|r| = 1$ or $|r| = r_{k-1,1}$ but $r \neq r_{k-1,1}$. Indeed, if such a root exists, then $q_k(r) = (r-1)p_{k-1}(r) = 0$, and just as before, the triangle inequality implies

$$2|r|^{k-1} \leq |r|^k + 1. \tag{10}$$

If the inequality in (10) is strict, then $q_k(|r|) > 0$. Since $q_k(1) = 0$ and $q_k(r_{k-1,1}) = 0$ it then follows that $|r| \neq 1$ and $|r| \neq r_{k-1,1}$. If the inequality in (10) is an equality, then $|r|^k$ must be real. Since $q_k(r) = 0$, it follows that $r^{k-1} = \frac{r^k+1}{2}$ is real as well and hence $\frac{r^k}{r^{k-1}} = r \in \mathbb{R}$ too. An application of *Descartes' rule of signs* to q_k implies that when k is odd p_{k-1} has one negative root, and when k is even p_{k-1} has no negative root. If k is odd, then $p_{k-1}(0) = -1$ and $p_{k-1}(-1) = 1$. By the *Intermediate Value Theorem* it follows that $-1 < r < 0$. We conclude that no root of p_{k-1}, except $r_{k-1,1}$ itself, has absolute value 1 or $r_{k-1,1}$.

(iii) Assume for contradiction that p_{k-1} has a multiple root. As a consequence, so has q_k. In that case, there exists r such that $q_k(r) = q_k'(r) = 0$. Note that $q_k'(r) = r^{k-2}(kr - 2k + 2)$, meaning either $r = 0$ or $r = \frac{2(k-1)}{k}$. Firstly, it is obvious that $r = 0$ cannot be a root of q_k. Secondly, $r = \frac{2(k-1)}{k} \neq \pm 1$ and by the *Rational Root Theorem* cannot be a root either. We conclude that q_k, and thus p_{k-1}, cannot have multiple roots. This finishes the proof. □

In what follows, we derive a closed form expression for the generating function of the sequence (F_n):

$$G(x) = \sum_{n=0}^{\infty} F_{n+1} x^n.$$

See how

$$\begin{aligned}
G(x) &= \sum_{n=0}^{\infty} F_{n+1} x^n \\
&= \sum_{n=0}^{k-2} F_{n+1} x^n + \sum_{n=k-1}^{\infty} \left(\sum_{i=1}^{k-1} F_{n-i+1} + 1 \right) x^n \\
&= \sum_{n=0}^{k-2} F_{n+1} x^n + \sum_{i=1}^{k-1} x^i \left(\sum_{n=k-1}^{\infty} F_{n+1-i} x^{n-i} \right) + \sum_{n=k-1}^{\infty} x^n \\
&= \sum_{n=0}^{k-2} F_{n+1} x^n + x^{k-1} \left(\sum_{n=k-1}^{\infty} F_{n+1-(k-1)} x^{n-(k-1)} \right) + \sum_{i=1}^{k-2} x^i \left(\sum_{n=k-1}^{\infty} F_{n+1-i} x^{n-i} \right) + \sum_{n=k-1}^{\infty} x^n \\
&= \sum_{n=0}^{k-2} F_{n+1} x^n + x^{k-1} G(x) + \sum_{i=1}^{k-2} x^i \left(G(x) - \sum_{m=0}^{k-i-2} F_{m+1} x^m \right) + \sum_{n=k-1}^{\infty} x^n \\
&= \sum_{n=0}^{k-2} F_{n+1} x^n + \sum_{i=1}^{k-1} x^i G(x) - \sum_{i=1}^{k-2} x^i \sum_{m=0}^{k-i-2} F_{m+1} x^m + \sum_{n=k-1}^{\infty} x^n \\
&= \sum_{n=0}^{k-2} F_{n+1} x^n + \sum_{i=1}^{k-1} x^i G(x) - \sum_{n=1}^{k-2} \sum_{i=1}^{n} F_i x^n + \sum_{n=k-1}^{\infty} x^n \\
&= 1 + \sum_{n=1}^{k-2} F_{n+1} x^n + \sum_{i=1}^{k-1} x^i G(x) - \sum_{n=1}^{k-2} \sum_{i=1}^{n} F_i x^n + \sum_{n=k-1}^{\infty} x^n \\
&= \sum_{n=1}^{k-2} \left[F_{n+1} - \left(\sum_{i=1}^{n} F_i + 1 \right) \right] x^n + \sum_{i=1}^{k-1} x^i G(x) + \sum_{n=0}^{\infty} x^n.
\end{aligned}$$

Recall that $F_n = 2^{n-1}$ for $1 \leq n \leq k$, meaning:

$$F_{n+1} - \left(\sum_{i=1}^{n} F_i + 1 \right) = 2^n - \left(\sum_{i=1}^{n} 2^{i-1} + 1 \right) = 0.$$

Putting it all together,

$$G(x) = \sum_{i=1}^{k-1} x^i G(x) + \frac{1}{1-x},$$

and finally,

$$G(x) = \frac{1}{(1-x)\left(1 - \sum_{i=1}^{k-1} x^i\right)} \tag{11}$$

where (11) is our desired closed form.

Let $d_{k-1}(x) := \left(1 - \sum_{i=1}^{k-1} x^i\right)$. The following Lemma relates the roots of $p_{k-1}(x)$ with the ones from $d_{k-1}(x)$.

Lemma 3.2. *Let $p_{k-1}(x)$ and $d_{k-1}(x)$ be defined as above. Then, $\frac{1}{r}$ is a root of $d_{k-1}(x)$ if and only if r is a root of $p_{k-1}(x)$.*

Proof. See how

$$\begin{aligned} d\left(\frac{1}{r}\right) = 0 &\iff 1 - \sum_{i=1}^{k-1} \frac{1}{r^i} = 0 \iff 1 = \frac{r^k - r}{r^k(r-1)} \\ &\iff r^{k-1} = \frac{r - r^k}{r - r^2} \iff r^{k-1} - \sum_{i=0}^{k-2} r^i = 0 \\ &\iff p_{k-1}(r) = 0 \end{aligned}$$

□

Lemma 3.3. *The sequence (F_n) as defined in (7) is given by the following Binet formula:*

$$F_n = \sum_{j=0}^{k-1} \frac{r_{k-1,j}^n}{2 + k(r_{k-1,j} - 2)},$$

where $r_{k-1,0} = 1$ and $r_{k-1,1}, \ldots, r_{k-1,k-1}$ are the roots of the polynomial p_{k-1} defined in (9).

Proof. By Lemma 3.1 part (iii) and Lemma 3.2 we can factor out the denominator of the generating function as follows:

$$\begin{aligned} G(x) &= \frac{1}{1-x} \cdot \frac{1}{\prod_{j=1}^{k-1} (x - 1/r_{k-1,j})} \\ &= -\frac{1}{x-1} \cdot \frac{1}{\prod_{j=1}^{k-1} (x - 1/r_{k-1,j})} \\ &= -\frac{1}{\prod_{j=0}^{k-1} (x - 1/r_{k-1,j})}. \end{aligned}$$

where $r_{k-1,0} := 1$. Then, using partial fraction decomposition we obtain

$$G(x) = \sum_{j=0}^{k-1} \frac{c_j}{x - 1/r_{k-1,j}}. \tag{12}$$

The coefficients c_j are given by

$$\begin{aligned}
c_j &= \lim_{x\to 1/r_{k-1,j}} \left(c_j + \sum_{\substack{t=0\\ t\neq j}}^{k-1} \frac{c_t\,(x-1/r_{k-1,j})}{x-1/r_{k-1,t}} \right) \\
&= \lim_{x\to 1/r_{k-1,j}} \left((x-1/r_{k-1,j}) \left(\sum_{j=0}^{k-1} \frac{c_j}{x-1/r_{k-1,j}} \right) \right) \\
&= \lim_{x\to 1/r_{k-1,j}} \left((x-1/r_{k-1,j}) \left(\frac{1}{(1-x)\left(1-\sum_{i=1}^{k-1} x^i\right)} \right) \right) \\
&= \lim_{x\to 1/r_{k-1,j}} \left(\frac{x-1/r_{k-1,j}}{x^k-2x+1} \right).
\end{aligned}$$

Using *L'Hôpital's rule* we get

$$c_j = \frac{1}{k\left(\frac{1}{r_{k-1,j}}\right)^{k-1} - 2},$$

where the denominator does not vanish since all roots of $x^k - 2x + 1$ are simple. Moreover, since $r_{k-1,j}^k - 2r_{k-1,j}^{k-1} + 1 = (r_{k-1,j} - 1)p_{k-1}(r_{k-1,j}) = 0$ it follows that $r_{k-1,j} - 2 = -1/r_{k-1,j}^{k-1}$ so that

$$c_j = -\frac{1}{2 + k(r_{k-1,j} - 2)}. \tag{13}$$

Finally, we have that

$$\begin{aligned}
G(x) &= \sum_{j=0}^{k-1} c_j \left(\frac{1}{x - 1/r_{k-1,j}} \right) \\
&= \sum_{j=0}^{k-1} c_j \left(-r_{k-1,j} \cdot \frac{1}{1 - r_{k-1,j}x} \right) \\
&= \sum_{j=0}^{k-1} c_j \left(-r_{k-1,j} \sum_{n=0}^{\infty} r_{k-1,j}^n x^n \right) \\
&= \sum_{n=0}^{\infty} \left(-\sum_{j=0}^{k-1} c_j r_{k-1,j}^{n+1} \right) x^n.
\end{aligned}$$

Comparing coefficients gives us

$$F_n = \sum_{j=0}^{k-1} \frac{r_{k-1,j}^n}{2 + k(r_{k-1,j} - 2)}.$$

$\square$

4 Exponentially growing sequences

The following results and their corresponding proofs are directly taken from [1] and are included here for the reader's convenience, as well as for completeness.

Lemma 4.1. *If* (a_k) *is a sequence such that* $\lim_{k\to\infty} ka_k = c$*, then*

$$\lim_{k\to\infty} (1 - a_k)^k = e^{-c}.$$

Proof. Let $\varepsilon > 0$ be arbitrary. Then there exists $N \in \mathbb{N}$ such that $|ka_k - c| \leq \varepsilon$, or, equivalently,

$$\left(1 - \frac{c+\varepsilon}{k}\right)^k \leq (1 - a_k)^k \leq \left(1 - \frac{c-\varepsilon}{k}\right)^k$$

for all $k \geq N$. Hence, we obtain

$$e^{-(c+\varepsilon)} \leq \liminf_{k\to\infty} (1 - a_k)^k \leq \limsup_{k\to\infty} (1 - a_k)^k \leq e^{-(c-\varepsilon)}.$$

Since $\varepsilon > 0$ is arbitrary, the result follows. □

Lemma 4.2. *If* $a > 1$ *and* (b_k) *is a positive sequence such that* $\lim_{k\to\infty} a^k b_k = c$*, then*

$$\lim_{k\to\infty} \frac{a^k - (a - b_k)^k}{k} = \frac{c}{a}.$$

Proof. The algebraic identity

$$x^k - y^k = (x - y) \sum_{i=0}^{k-1} x^{k-1-i} y^i$$

leads to

$$\frac{a^k - (a - b_k)^k}{k} = \frac{a^k b_k}{a} \cdot S_k \quad \text{where} \quad S_k = \frac{1}{k} \sum_{i=0}^{k-1} \left(1 - \frac{b_k}{a}\right)^i.$$

It suffices to show that $\lim_{k\to\infty} S_k = 1$. To that end, note that the assumption implies that $\lim_{k\to\infty} b_k = 0$ so that $-1 < -b_k/a < 0$ for k sufficiently large. *Bernoulli's inequality* gives

$$1 - i\frac{b_k}{a} \leq \left(1 - \frac{b_k}{a}\right)^i < 1,$$

which implies that

$$1 - \frac{k-1}{2} \cdot \frac{b_k}{a} < S_k < 1$$

for k sufficiently large. Moreover, the assumption implies that $\lim_{k\to\infty} kb_k = 0$. An application of the *Squeeze Theorem* completes the proof. □

5 Proof of the extreme value law (Theorem 1.2)

Let $\lambda > 0$ and recall $n_k := \lfloor \lambda 2^k \rfloor$. Using both Lemmas 2.1 and 3.2 gives

$$\begin{aligned}
\mathbb{P}(M_{n_k} \le 1 - 2^{-k}) &= 2^{1-n_k-k} F_{n_k+k} \\
&= 2 \sum_{j=0}^{k-1} \left[\frac{1}{2 + k(r_{k-1,j} - 2)} \left(\frac{r_{k-1,j}}{2} \right)^{n_k+k} \right] \\
&= 2 \sum_{j=0}^{k-1} a_j(k),
\end{aligned}$$

where

$$a_j(k) := \frac{1}{2 + k(r_{k-1,j} - 2)} \left(\frac{r_{k-1,j}}{2} \right)^{n_k+k}.$$

As defined previously, $r_{k-1,0} = 1$ and $r_{k-1,j}$ are the roots of p_{k-1} for $1 \le j \le k-1$. Recall that $r_{k-1,1}$ is the unique root in the interval $(1,2)$, and that $|r_{k-1,j}| \le 1$ for $i \in \{0\} \cup \{2, \ldots, k-1\}$. In the remainder of this section, Theorem 1.2 will be proved through a detailed analysis of the asymptotic behavior of the dominant root $r_{k-1,1}$.

We define the following values:

$$r_{k-1,\max} = 2 - \frac{1}{2^{k-1} - 1} \quad \text{and} \quad r_{k-1,\min} = 2 - \frac{1}{2^{k-1} - 1} \left(1 + 2^{(-k+1)/2} \right),$$

where

$$r_{k-1,\max} = 2 - \frac{p_{k-1}(2)}{p'_{k-1}(2)} = 2 - \frac{2^{k-1} - \sum_{i=0}^{k-2} 2^i}{(k-1)2^{k-2} - \sum_{i=1}^{k-2} i 2^{i-1}} = 2 - \frac{1}{2^{k-1} - 1}.$$

In words, $r_{k-1,\max}$ is obtained by applying a single iteration of Newton's method to p_{k-1} using the starting point $x_0 = 2$. The value $r_{k-1,\min}$ is a correction of $r_{k-1,\max}$ with an exponential decreasing factor. Both numbers will aid in proving partial results needed for the proof of Theorem 1.2.

Lemma 5.1. *For $k \in \mathbb{N}$ sufficiently large:*

(i) $p_{k-1}(r_{k-1,max}) > 0$;

(ii) $p_{k-1}(r_{k-1,min}) < 0$;

(iii) $r_{k-1,min} < r_{k-1,1} < r_{k-1,max}$.

Proof. (i) For $x \neq 1$ we have

$$\begin{aligned}
p_{k-1}(x) &= x^{k-1} - \sum_{i=0}^{k-2} x^i \\
&= x^{k-1} - \frac{x - x^k}{x - x^2} \\
&= \frac{1}{x(1-x)} \left(x^{k-1}(x - x^2) + x^k - x \right) \\
&= \frac{1}{1-x} \left((2-x)x^{k-1} - 1 \right).
\end{aligned}$$

In particular, for $k \geq 3$ it follows that

$$p_{k-1}(r_{k-1,\max}) = \frac{1}{2^{k-1}-2}\left[2^{k-1} - \left(2 - \frac{1}{2^{k-1}-1}\right)^{k-1} - 1\right].$$

All is left now is to show that the expression between brackets is positive for k sufficiently large. See how, for $a = 2$ and $(b_k) = \frac{1}{2^{k-1}-1}$, Lemma 4.2 gives

$$\lim_{k\to\infty} \frac{1}{k}\left(2^{k-1} - \left(2 - \frac{1}{2^{k-1}-1}\right)^{k-1}\right) = \frac{1}{2}.$$

Hence, for k sufficiently large it follows that

$$2^{k-1} - \left(2 - \frac{1}{2^{k-1}-1}\right)^{k-1} - 1 \geq \frac{1}{4}k - 1$$

and the right-hand side is positive for $k \geq 5$.

(ii) Similar to the proof of part (i), it follows that, through some algebraic manipulations,

$$p_{k-1}(r_{k-1,\min}) = \frac{1}{2 + 2^{(-k+1)/2} - 2^{k-1}}\left[\left(2 - \frac{1}{2^{k-1}-1}\left(1 + 2^{(-k+1)/2}\right)\right)^{k-1}\left(1 + 2^{(-k+1)/2}\right) - 2^{k-1} + 1\right].$$

It suffices to show that the expression between brackets is positive for k sufficiently large. Again, for $a = 2$ and $(b_k) = \frac{1}{2^{k-1}-1}\left(1 + 2^{(-k+1)/2}\right)$, Lemma 4.2 gives

$$\lim_{k\to\infty} \frac{1}{k}\left(2^{k-1} - \left(2 - \frac{1}{2^{k-1}-1}\left(1 + 2^{(-k+1)/2}\right)\right)^{k-1}\right) = \frac{1}{2}.$$

Hence, for k sufficiently large it follows that

$$2^{k-1} - \left(2 - \frac{1}{2^{k-1}-1}\left(1 + 2^{(-k+1)/2}\right)\right)^{k-1} \leq k.$$

This gives

$$\begin{aligned}
&\left(2 - \frac{1}{2^{k-1}-1}\left(1 + 2^{(-k+1)/2}\right)\right)^{k-1}\left(1 + 2^{(-k+1)/2}\right) - 2^{k-1} + 1 \\
&= 2^{(k-1)/2} + 1 - \left(1 + 2^{(-k+1)/2}\right)\left(2^{k-1} - \left(2 - \frac{1}{2^{k-1}-1}\left(1 + 2^{(-k+1)/2}\right)\right)^{k-1}\right) \\
&\geq 2^{(k-1)/2} + 1 - \left(1 + 2^{(-k+1)/2}\right)k
\end{aligned}$$

and the right-hand side is positive for $k \geq 7$.

(iii) By the *Intermediate Value Theorem* there exists a point $c \in (r_{k-1,\min}, r_{k-1,\max})$ such that $p_{k-1}(c) = 0$. Note that $c > 1$ already for $k \geq 3$. Since $r_{k-1,1}$ is the only zero of p_{k-1} that lies outside the unit circle, it follows that $c = r_{k-1,1}$. □

Lemma 5.2. *We have that*
$$\lim_{k\to\infty} a_1(k) = \frac{1}{2}e^{-\lambda}.$$

Proof. From Lemma 5.1 it follows for sufficiently large k that
$$2 - \frac{1}{2^{k-1}-1}\left(1+2^{(-k+1)/2}\right) < r_{k-1,1} < 2 - \frac{1}{2^{k-1}-1}. \tag{14}$$

The *Squeeze Theorem* implies that
$$\lim_{k\to\infty} r_{k-1,1} = 2,$$
and together with (14),
$$\lim_{k\to\infty} k(r_{k-1,1}-2) = 0.$$
Hence, we can conclude that
$$\lim_{k\to\infty} \frac{1}{2+k(r_{k-1,1}-2)} = \frac{1}{2}. \tag{15}$$

Now, the inequality $2^k\lambda - 1 \le n_k \le 2^k\lambda$ combined with (14) implies that
$$\left(1 - \frac{1}{2^k-2}\left(1+2^{(-k+1)/2}\right)\right)^{2^k\lambda-1+k} < \left(\frac{r_{k-1,1}}{2}\right)^{n_k+k} < \left(1-\frac{1}{2^k-2}\right)^{2^k\lambda+k}, \tag{16}$$
where we first divided by 2 everywhere. We tackle both bounds for $\left(\frac{r_{k-1,1}}{2}\right)^{n_k+k}$ separately.

First, see how
$$U_k := \left(1-\frac{1}{2^k-2}\right)^{2^k\lambda+k} = \left(\left(1-\frac{1}{2^k-2}\right)^{2^k}\right)^{\lambda}\cdot\left(1-\frac{1}{2^k-2}\right)^{k}$$
and thus
$$\lim_{k\to\infty} U_k = \left(\lim_{k\to\infty}\left[\left(1-\frac{1}{2^k-2}\right)^{2^k}\right]\right)^{\lambda}\cdot\lim_{k\to\infty}\left[\left(1-\frac{1}{2^k-2}\right)^{k}\right].$$

Let $f(k) := \left(1-\frac{1}{2^k-2}\right)^{2^k}$. We seek to find $\lim_{k\to\infty} f(k)$. Rewrite the expression by setting $y := 2^k$, so that as $k\to\infty$, $y\to\infty$. Then
$$f(y) = \left(1-\frac{1}{y-2}\right)^{y}$$
and we are interested in evaluating $\lim_{y\to\infty}\left(1-\frac{1}{y-2}\right)^{y}$. Taking the logarithm of $f(y)$ and expanding $\ln\left(1-\frac{1}{y-2}\right)$ using the Mclaurin series for $\ln(1-t)$:
$$\ln f(y) = y\ln\left(1-\frac{1}{y-2}\right) = -\frac{y}{y-2} - \frac{y}{2(y-2)^2} - \cdots.$$

Now consider the term $-\frac{y}{y-2}$:

$$\frac{y}{y-2} = \frac{y}{y\left(1-\frac{2}{y}\right)} = \frac{1}{1-\frac{2}{y}}.$$

As $y \to \infty$, $\frac{2}{y} \to 0$, so $\frac{y}{y-2} \to 1$, which implies

$$-\frac{y}{y-2} \to -1 \quad \text{as} \quad y \to \infty.$$

The remaining terms, such as $-\frac{y}{2(y-2)^2}$, approach zero as $y \to \infty$ because they contain higher powers of $\frac{1}{y}$. Therefore, we have

$$y \ln\left(1 - \frac{1}{y-2}\right) \to -1 \quad \text{as} \quad y \to \infty.$$

Note we can treat the Mclaurin series term-wise since it is absolutely convergent for $|t| < 1$. Exponentiating both sides, we find that $f(k) \to e^{-1}$ as $k \to \infty$.

Dealing now with $\left(1 - \frac{1}{2^k-2}\right)^k$, setting $(a_k) = \frac{1}{2^k-2}$ and using Lemma 4.1 yields

$$\lim_{k\to\infty} (1-a_k)^k = 1.$$

Combining both sub-results we get that $\lim_{k\to\infty} U_k = e^{-\lambda}$.

Second, as done for U_k, we define L_k as follows:

$$L_k := \left(\left(1 - \frac{1}{2^k-2}\left(1+2^{(-k+1)/2}\right)\right)^{2^k}\right)^{\lambda} \cdot \left(1 - \frac{1}{2^k-2}\left(1+2^{(-k+1)/2}\right)\right)^{k-1}.$$

Letting $g(k) = \left(1 - \frac{1}{2^k-2}\left(1+2^{(-k+1)/2}\right)\right)^{2^k}$ and proceeding in a similar matter as above, we get that $\lim_{k\to\infty} g(k) = e^{-1}$ too. The second limit is dealt with by letting $(a_k) = \frac{1}{2^{k+1}-2}\left(1+2^{(-k)/2}\right)$ and using Lemma 4.1.

Again, combining both sub-results we get that $\lim_{k\to\infty} L_k = e^{-\lambda}$.

Finally, (15) together with the *Squeeze Theorem* applied to (16) completes the proof. □

Lemma 5.3. *Let $\mathcal{J} := \{0\} \cup \{2, \ldots, k-1\}$. For k sufficiently large we have that*

$$|a_j(k)| < \frac{2}{|2-k|} \cdot \frac{1}{2^{n_k+k}} \quad \textit{for all } j \in \mathcal{J}$$

Proof. Using that $|r_{k-1,j}| \leq 1$ for $j \in \mathcal{J}$ gives

$$\begin{aligned} |a_j(k)| &= \frac{1}{|2+k(r_{k-1,j}-2)|}\left(\frac{|r_{k-1,j}|}{2}\right)^{n_k+k} \\ &< \frac{2}{|2+k(r_{k-1,j}-2)|} \cdot \frac{1}{2^{n_k+k}}. \end{aligned}$$

For $z \in \mathbb{C}$, let $f(z) := 2 + k(z-2)$. Writing $z = x + iy$ yields

$$\begin{aligned} |f(z)|^2 &= \left(\left((2 + k(x-2))^2 + (ky)^2 \right)^{1/2} \right)^2 \\ &= (2 + k(x-2))^2 + k^2 y^2 \\ &\geq (2 + k(x-2))^2 . \end{aligned}$$

The latter quadratic function attains its minimum value at $x_k = 2 - \frac{2}{k}$, and already for $k \geq 3$ it follows that $x_k > 1$. Using that $\mathrm{Re}(r_{k-1,j}) \in [-1, 1]$ for $j \in \mathcal{J}$ gives

$$|f(r_{k-1,j})| \geq |2 - k|.$$

This completes the proof. □

From Lemma 5.3 it follows for k sufficiently large that

$$\left| \sum_{j \in \mathcal{J}} a_j(k) \right| \leq \sum_{j \in \mathcal{J}} |a_j(k)| \leq \frac{2(k-1)}{|2-k|} \cdot \frac{1}{2^{n_k + k}},$$

so that Lemma 5.2 implies that

$$\begin{aligned} \lim_{k \to \infty} \mathbb{P}(M_{n_k} \leq 1 - 2^{-k}) &= \lim_{k \to \infty} 2 \sum_{j=0}^{k-1} a_j(k) \\ &= \lim_{k \to \infty} 2 \left(a_1(k) + \sum_{j \in \mathcal{J}} a_j(k) \right) \\ &= \lim_{k \to \infty} 2 \cdot a_1(k) \\ &= e^{-\lambda}, \end{aligned}$$

whereby Theorem 1.2 has been proven.

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