Rijksuniversiteit Groningen

Master Thesis

Color-Kinematics Duality of Heavy Scalars Coupled to Gluons and Higher Derivative Corrections

A Bootstrap Approach

Author:
B. D. Alferink

Supervisor:
prof. dr. D. Roest
Second examiner:
prof. dr. E. A. Bergshoeff

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Abstract

The duality between color and kinematics have shed new light on the structures of gauge theories and gravity in the recent decades. Modern amplitude methods involving the amplitude bootstrap have made a large impact on this. There are still many open questions regarding the how and why of the structures which are being studied by considering different theories. The aim of this thesis is to study the duality of massive scalars coupled to gluons, particularly for the large mass limit, utilizing the framework of modern scattering amplitude methods. The approach is to construct and constrain an ansatz based on physical principles to derive the scattering amplitudes and learn about the structures of the theories. In addition, extensions of such theories are studied in the form of higher derivative corrections. This is motivated by the infinite tower derivatives necessary for extended Yang-Mills theory to be consistent with the color kinematics duality. The thesis successfully showcases how to construct amplitudes involving massive scalars and their large mass limit up to five point using the bootstrap approach. Whether the massive scalars coupled to gluons have a structure such as the infinite tower of derivatives seen for gluons is inconclusive. In a future work, this could be further investigated by extending the methods used in this thesis to at least six point amplitudes.

Contents

1 Introduction

> "Physical science is that department of knowledge which relates to the order of nature, or, in other words, to the regular succession of events".

Is the opening sentence of the posthumous (1888) book *Matter and Motion* by James Clerk Maxwell [1]. From a modern particle physics perspective, the fundamental laws of nature are described by interactions of elementary particles ultimately describing any succession of events in spacetime.

Particularly in quantum field theory (QFT), the interactions are described by scattering amplitudes. Scattering amplitudes are the elementary building blocks which allow us to predict probabilities of particle interactions. This forms a crucial connection between theory and experiments. Predictions of the standard model and beyond can be tested in experiments such as ATLAS and CMS (e.g. the discovery of the Higgs boson [2, 3]) at the Large Hadron Collider (LHC) at CERN. As will become clear later, this is not only important for quantum field theory but also for the study of gravity and gravitational waves. Since the famous observations of gravitational waves due to merging black holes by the LIGO and Virgo collaborations [4–7], confirming more predictions from General Relativity, there has been a huge interest and need for theoretical computations and prediction to gain understanding of the physics, and to improve detections [8, 9].

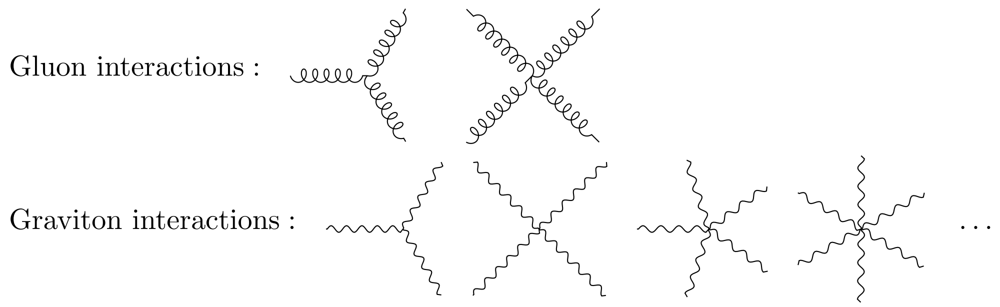
The 'modern' S-matrix, of which the scattering amplitudes are the matrix elements, relates asymptotically free states in a scattering process. The elements are usually perturbatively computed by utilizing Feynman diagrams. These diagrams are graphical representations of the perturbative expansion of the path integral formulation of quantum field theory. The contribution of each component of these graphs can be derived from the Lagrangian describing the interactions of the theory which, in case of the standard model, is a non-Abelian gauge theory.

In the 80s, the state-of-the-art computations were five-gluon scattering amplitudes at tree level (i.e. the leading order in the perturbative expansion) [10], which resulted in page long expressions. To compute scattering amplitudes with the Feynman approach, one needs to sum over all possible Feynman diagrams. At five-gluon level, there are 'only' 25 diagrams, but at higher multiplicities, the number of diagrams grows rapidly. For example, at 8 and 9 gluons in a certain spin configuration, there are 34300 and 559405 diagrams respectively [11].

A significant result which kick-started the modern amplitudes program was the discovery of the Parke-Taylor formula [12]. This reduced the page long expression to a very simple, single line result for n-gluon amplitudes. The incredible simplicity of this powerful formula was a major hint that there are more underlying structures, at least to gauge theories. The reason for this simplicity that contrasts the initial result is due to cancellations that happen between Feynman diagrams. In fact, the gauge dependent Feynman diagrams are not the only way, and perhaps often not the most efficient way to calculate scattering amplitudes.

Besides the standard model, the 'other' fundamental component of the universe responsible for the mutual attraction between masses, is gravity. It is most accurately described by Einstein's General Relativity (GR). There have been attempts to unify GR with the standard model by quantize gravity, and to find possible signatures of such theories [13], however, most approaches result in incomplete theories. At high energies, (UV) divergences will appear in the perturbative expansion rendering it non-renormalizable. At low energies, however, a quantum gravity theory will reduce to GR.

This is not stopping anyone from using a quantum field theory approach to study gravity. Quantum gravity can be studied as an effective field theory (EFT) at low energies and make consistent predictions. An important difference between the gauge theory that describes gluons, and a (quantum) gravity theory describing 'gravitons' is that gluons only have two interaction terms (three and four point), whilst gravitons have an infinite number of contact terms. The interaction diagrams are illustrated in equation (1.1).



(1.1)

The structural difference between theories governing gluons and gravitons is accentuated by these possible interaction types. For gravitons, the infinite number of contact terms would make the computation of scattering amplitudes with off-shell methods, such as Feynman diagrams, incredibly cumbersome as the number of diagrams grows even more rapidly than already the case with gauge theories. Besides this, the graviton vertices are not as simple as the gluon vertices. The cubic vertex already contains over 100 terms [14]. It should be noted, however, that this is a gauge dependent vertex, and can be made more compact by removing gauge dependence, for example, by using on-shell methods.

As part of the modern amplitudes program, a different approach of computing scattering amplitudes is usually taken. Instead of the off-shell Feynman diagrams, amplitudes are build from gauge invariant, on-shell, building blocks. Using principles such as unitarity and locality, and the analytic structures of the amplitudes, one can find recursive relations between the gauge invariant building blocks [10, 15, 16]. This allows for the computation of scattering amplitudes, both a tree and loop level, using a so called 'bootstrap' approach. The bootstrap approach is a method to construct amplitudes by first writing down the most general ansatz possible given the external states, then imposing constraints due properties such as gauge invariance, factorization, and Jacobi relations, on the amplitudes. This results in fully constrained amplitudes, which, in principle, can be done without knowledge of the Lagrangian of the theory. One certainly does not have to deal with the off-shell intricacies of a Feynman approach. Not only does this approach make the process for certain computations much simpler, it also reveals new properties that were previously hidden. This provides new insights leading to a deeper understanding of the underlying structures of the theories.

Despite the structural differences between gauge theories and gravity theories, there are some remarkable similarities. At the level of the Lagrangian of the theories, this is not immediately clear. On-shell scattering amplitudes, however, reveal a number of non-trivial connections. In particular, they both use similar kinematic building blocks [14]. Bern, Carrasco, and Johansson proved that gauge theory amplitudes can be written as a sum of cubic graphs with distinct color factors and kinematic numerators,

$$\mathcal{A}_n^{\text{tree}} = \sum_{i \in \text{cubic}} \frac{c_i n_i}{D_i}. \tag{1.2}$$

Here c_i are the color factors, n_i the kinematic numerators, and D_i the propagators of the cubic graphs. Moreover, there is enough freedom to always write the kinematic numerator in such a way that they satisfy the same algebraic relations as the color factors (e.g. symmetries and the Jacobi Identity) [17]. The duality with the color factors that the 'BCJ' numerators exhibit is referred to as the color-kinematics (CK) duality.

Even more remarkably, the CK duality can be used to replace the color factors of a gauge theory amplitude with another 'copy' of the kinematic numerator. The resulting amplitude turns out to be equivalent to the on-shell tree level graviton amplitude [18]. This process is referred to as the double copy, schematically relating Gravity theory to the square of a Yang-Mills theory,

$$\mathrm{GR} \sim \mathrm{YM}^2. \tag{1.3}$$

The relation between gauge and gravity theories is not entirely novel as a similar relation was found in string theory relating closed and open strings in the form of the KLT relations [19]. The BCJ double copy however, imply much more general relations that not only holds for Yang-Mills theory and gravity, but for a whole 'web' of theories [14, 20, 21]. It allows us to learn about the structure of theories through a new lens, paving the way for new insights and geometric interpretations such as the amplituhedron [22]. There is still much left to understand about the double copy. For example, the relations (e.g. Jacobi identities) of the color factors are imposed by the underlying Lie algebra of the gauge group. Even though the kinematic numerators can be made to satisfy the same algebraic relations, it is highly non-trivial to understand why this is the case. Is there some underlying kinematic algebra, if so, what does it look like, and why should it be there?

Another important result of the modern amplitude program and the double copy is the fact that it greatly simplifies the computation of graviton amplitudes. In essence, the double copy allows one to perform 'simple' gauge theory computations and obtain corresponding gravity amplitudes by simply isolating the kinematic numerators, removing the color structures, and 'squaring' the kinematics. It has been proven to be incredibly useful in the study of gravitational wave physics [20, 23, 24]. Theoretical modelling of gravitational wave sources is a challenging task. Typically, in this type of studies one needs to solve the Einstein field equations perturbatively. As the Einstein field equations are non-linear and multiscale, the complexity of such problems quickly spirals out of control. In recent years, there has been a new fruitful collaboration between the gravitational wave and scattering amplitudes communities [24].

One of the topics of this thesis will be the study of heavy-mass effective field theory (HEFT) amplitudes. In such a theory, a massive particle is coupled to a gauge or gravity theory, and the mass is assumed to be much larger than the exchange momenta of a process. This turns out to be a very suitable theory for the study of black hole scattering processes, as black holes can be treated as pointlike particles in the adiabatically inspiraling phase of a merger event. In certain constructions of HEFT amplitudes, the double copy is manifest, making it possible to study gravitational wave physics efficiently [25, 26]. Moreover, the loop expansion of such a construction corresponds directly with the post-Minkowskian expansion of the gravitational wave signal [20, 26, 27]. As future gravitational wave detectors such as LISA require a precise theoretical understanding of the signals to measure now phenomena [8, 9], this collaboration will continue to be of great importance.

To push the boundaries of the gauge theories and HEFT, it is possible to add higher derivative operators. This allows for the study of the high energy behavior of the theories. An interesting question is how the higher derivative corrections affect the double copy as there have been some interesting results for the pure gluon case [28].

In this thesis the aim is to answer a number of questions concerning the construction of gauge fields coupled to massive scalars, the behavior under addition of higher derivative operators, and the compatibility of the double copy. With this we attempt to answer the following research questions:

1. *Is it possible to (re)produce amplitudes of Yang-Mills coupled to massive scalars using a bootstrap approach? Both for general mass and the large mass limit?*

2. *What would a 'HEFT bootstrap' look like?*
3. *Is it possible to extend the bootstrap approach to higher derivative corrections, and what would the amplitudes look like?*
4. *Is the infinite tower of higher derivative corrections of Carrasco, Lewandowski, and Pavao preserved when the gauge theory is coupled to a massive scalar?*

To aid answering these questions, we will first discuss the textbook approach of scattering amplitudes in Section 2 by first discussing Yang-Mills theory, Massive scalar fields, and HEFT. In Section 3 we will dive into the color-kinematics duality and the double copy. We will go into detail of the BCJ relations, and at the end discus a powerful construction of the double copy specifically for HEFT amplitudes. In Section 4, we will go into the details of on shell methods and recursion relations for tree level amplitudes. In the second half of the section we will show how to construct Yang-Mills amplitudes and Yang-Mills coupled to heavy scalars using the bootstrap, and in the final subsection we discuss what a HEFT bootstrap would look like. Lastly, in Section 5, the higher derivative corrections will be discussed, including the infinite tower of higher derivative corrections necessary for the consistency of the double copy. Results of amplitudes with higher derivative theories will be given using the bootstrap and an outline is given on how to proceed, with a special focus on HEFT amplitudes.

2 Gauge Theory, Gravity, Massive Scalars and the Large Mass Limit

This section will be dedicated to explaining relevant concepts of quantum field theory in the context of scattering amplitudes. The aim is to provide a foundation for the discussion of scattering amplitudes, first from the textbook perspective of Feynman Rules and Lagrangians. Yang-Mills theory and its structures will be discussed as well as scalar QCD (Yang-Mills coupled to massive scalars). The limit of scalar QCD where the mass of the scalar is taken to be large is one of the main focuses of this thesis, hence this will also be introduced. In order to make the discussion as intuitive as possible, we will use several examples involving gluon self interactions and scalar-gluon interactions. Further details of the standard textbook approach to scattering amplitudes can be found in books such as Peskin and Schroeder 1995 [29] and Zee 2010 [30]. Some more specific details about theories involving massive scalar fields and HEFT are discussed in [25, 31, 32].

2.1 Gauge theory and Yang-Mills

Gauge theories are a class of field theories that are invariant under local transformations of a certain symmetry group. When these *gauge fields*, A_μ, are quantized, the corresponding particles are called *gauge bosons*. The most well-known example of a gauge theory is Quantum Electrodynamics (QED), which is based on the symmetry group U(1).

A quantum field theory is often formulated utilizing the action. In a gauge theory, the action (and the Lagrangian) is invariant under the gauge transformations. In this formulation, the gauge transformations represent redundancies in the description of the theory. This is necessary to restrict the unphysical degrees of freedom that would otherwise be present in the theory. For example, in QED, the photon field is invariant under the transformation,

$$A_\mu \to A_\mu + \partial_\mu \Lambda, \tag{2.1}$$

where Λ is an arbitrary function. Such a function, or *gauge*, can be chosen to make certain calculations easier, but the physics, the observables, should not depend on this choice.

Gauge theories have a number of properties, one of which is whether the symmetry group is Abelian or non-Abelian (commutative or non-commutative respectively). QED, for example, is an Abelian gauge theory. Our focus will be on a non-Abelian Yang-Mills (YM) theory based on a compact Lie group that can be used to describe gluons. Yang-Mills theories are characterized by the generators T^a of the underlying Lie algebra of the compact Lie group G. The generators are hermitian and, given a certain normalization, satisfy the following relations,

$$[T^a, T^b]_{ij} = i f^{abc} T^c_{ij}, \qquad \mathrm{Tr}(T^a T^b) = \frac{1}{2}\delta^{ab}. \tag{2.2}$$

Here $a, b, c = 1, \ldots, \dim G$ are referred to as color indices, and f^{abc} are the structure constants which are fully antisymmetric. Although the group G can be a number of different groups, we will focus on the special unitary group SU(N) which has $\dim G = N^2 - 1$. Such a theory is of great importance in Quantum Chromodynamics (QCD) as it governs the strong nuclear force described by an $SU(3)$ Yang-Mills theory.

The structure constants, f^{abc}, can be seen as the generators in the adjoint representation. It is convenient to analogously introduce the antisymmetry for the fundamental generators [33], (T^a_{ij}),

$$f^{abc} = -f^{bac}, \qquad T^a_{ij} = -T^a_{ji}. \tag{2.3}$$

The fundamental generators are a representation of the special unitary group SU(N). If we combine the commutator in equation (2.2) with the identity,

$$[T^a,[T^b,T^c]]+[T^b,[T^c,T^a]]+[T^c,[T^a,T^b]]=0, \tag{2.4}$$

the relation,

$$f^{abe}f^{ecd}=f^{ade}f^{ebc}-f^{bde}f^{eac}, \tag{2.5}$$

is obeyed. This is the Jacobi Identity, which is a defining property of a Lie algebra.

With the use of gauge fields that live in the adjoint representation, the field strength tensor can be constructed as,

$$F^a_{\mu\nu}=\partial_\mu A^a_\nu-\partial_\nu A^a_\mu+gf^{abc}A^b_\mu A^c_\nu, \tag{2.6}$$

where g is the coupling strength and A^a_μ are the gauge fields that can be contracted with generators through color indices, $A_\mu=A^a_\mu T^a$. Note that $F^a_{\mu\nu}$ is gauge invariant and the gauge fields A^a_μ represents the gluon field in QCD.

The Lagrangian of the Yang mills theory can take the form,

$$\mathcal{L}_{\text{YM}}=-\frac{1}{4}F^a_{\mu\nu}F^{a,\mu\nu}. \tag{2.7}$$

This Lagrangian is invariant under gauge transformations, which are given by,

$$A^a_\mu\mapsto A^a_\mu+\frac{1}{g}\partial_\mu\alpha^a-f^{abc}\alpha^b A^c_\mu. \tag{2.8}$$

Here α is an arbitrary function.

Before we can study the interactions in this theory we need to take care of the excess redundancies in the theory by choosing a gauge. This can be done by introducing a gauge fixing term to the Lagrangian,

$$\mathcal{L}_{\text{GF}}=-\frac{1}{2\xi}\partial_\mu A^{\mu,a}\partial_\nu A^{\nu,a}. \tag{2.9}$$

A common choice of gauge is the Feynman gauge, $\xi=1$. We can now study the possible interactions of the theory.

2.1.1 Gluon (self) interactions

Now that the Lagrangian is established, consider the kinetic term governing the self interactions of the gauge fields. Given the field strength tensor in Equation (2.6), there are two types of interactions possible. The first being a three point interaction involving two contributions of the gauge field A and one contribution from the derivative of the gauge field. The term in the Lagrangian corresponding to the three point interaction is of the form,

$$\mathcal{L}_{\text{3pt.}}\sim gf^{abc}(\partial_\mu A^a_\nu)A^{\mu,b}A^{\nu,c}. \tag{2.10}$$

The second is a four point interaction which is product of the last term which involves four copies of the gauge field and no derivatives,

$$\mathcal{L}_{\text{4pt.}}\sim g^2(f^{eab}A^a_\mu A^b_\nu)(f^{ecd}A^{\mu,c}A^{\nu,d}). \tag{2.11}$$

From this, the corresponding Feynman rules can be derived.

In contrast to gravity and higher derivative corrections to Yang-Mills, any higher point vertices do not exist in this theory. The Lagrangian in equation (2.7) produces terms containing at most 4 powers of gauge fields. Gravity is treated in Section 2.2 and the higher derivative corrections will be treated in Section 5.

Traditionally, the way to approach calculating scattering amplitudes of a given theory is by first calculating the Feynman rules of the vertices and propagators of the theory. Then, a perturbative expansion in the coupling constant (or equivalently, the number of loops) is performed. In this process we need to sum over all possible diagrams that contribute to the amplitude. Even though this is a powerful, fairly intuitive, and a very algorithmic approach, there are some issues. The immediate problems come from the fact that the "objects", the Feynman diagrams, are rather complex objects. Particularly at higher multiplicity and loop order. As one needs to sum over all possible diagrams, this turns out to be quite a tedious process. Additionally, most of the intermediate steps of the calculation are dependent on the choice of gauge. Although the physics, the final amplitude, does not depend on the gauge choice, there are many ways to get there. The biggest issue of this redundancy is that it can obscure underlying structures of the theory and make the calculations more complex than they need to be.

In Section 4, we will discuss another approach which is more in line with the modern amplitudes approach. The essence of this approach is to avoid using the action entirely. Instead, one writes down an ansatz for the on-shell amplitude based on all possible functions that could satisfy some physical principles. This is then further constrained by fundamental principles such as, symmetries, the 'color kinematics' duality (Section 3), and factorization of the amplitudes (Section 4). These concepts will be introduced in their respective sections. For now, we will stick with Feynman/Lagrangian approach and treat some examples to further build our intuition of the theories that are considered.

Unless specified otherwise, we work with all momenta ingoing. As a result, momentum conservation for some interaction with n ingoing particles is captured in the relation,

$$\sum_{i=1}^{n} p_i^{\mu} = 0. \tag{2.12}$$

With this in mind, let us consider the Feynman vertices for Yang-Mills theory.

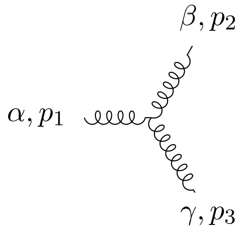
There is a neat way to derive the vertex rules corresponding to these interactions which we will do explicitly for three point. Due to the commutativity of the Fourier transform with linear operators we can do the following. Starting from the Lagrangian of the three point self interaction, we first want to go to momentum space. This results in replacing each derivative with a momentum vector p_i. Secondly, we replace the gauge fields with polarization tensors ε_i. The gauge fields can be contracted with the structure constants in 3 different ways, up to a minus sign. Effectively this means we get 3 different permutations of the first terms.

$$\sim (p_{1\mu}\varepsilon_{1\nu} - p_{1\nu}\varepsilon_{1\mu})\varepsilon_2^{\mu}\varepsilon_3^{\nu}. + \text{cycl. perm. } 1 \mapsto 2 \mapsto 3 \mapsto 1. \tag{2.13}$$

However, this relation is already contracted with the external polarizations. To recover the Feynman vertex rule, we have to take a derivative with respect to the external polarizations,

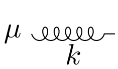
$$\begin{aligned} \text{3pt. vertex} &\sim g f^{abc} \frac{\partial\big[(p_{1\mu}\varepsilon_{1\nu} - p_{1\nu}\varepsilon_{1\mu})\varepsilon_2^{\mu}\varepsilon_3^{\nu}. + \text{cycl. perm. } 1 \mapsto 2 \mapsto 3 \mapsto 1\big]}{\partial\varepsilon_{1\alpha}\partial\varepsilon_{2\beta}\partial\varepsilon_{3\gamma}} \\ &\sim g f^{abc}\big[(p_1^{\beta}g^{\alpha\gamma} - p_1^{\gamma}g^{\alpha\beta}) + (p_2^{\gamma}g^{\alpha\beta} - p_2^{\alpha}g^{\beta\gamma}) + (p_3^{\alpha}g^{\beta\gamma} - p_3^{\beta}g^{\alpha\gamma})\big], \end{aligned} \tag{2.14}$$

where the second line is achieved by using the identity $\frac{\partial X^{\mu}}{\partial X_{\nu}} = g^{\mu\nu}$. Rearranging the terms gives us the familiar pure gauge Feynman vertex of Yang-Mills,

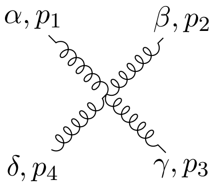


$$= g f^{abc}\left[g^{\alpha\beta}(p_1 - p_2)^\gamma + g^{\beta\gamma}(p_2 - p_3)^\alpha + g^{\gamma\alpha}(p_3 - p_1)^\beta\right]. \tag{2.15}$$

The gluon propagator and 4pt. amplitude can be derived in a similar vain and yield,



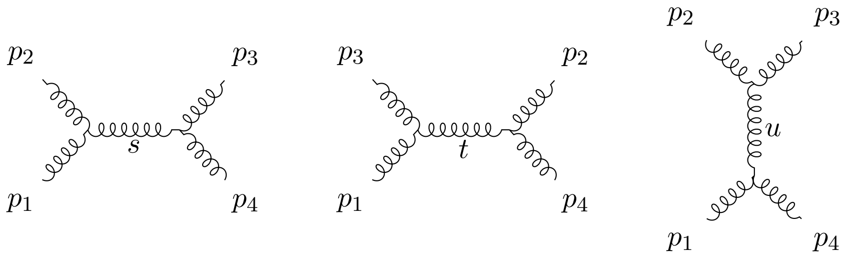
$$= i\frac{\delta^{ab}}{k^2 + i\varepsilon}\left(g_{\mu\nu} + (\xi - 1)\frac{k^\mu k^\nu}{k^2 + i\epsilon}\right) \overset{\xi=1}{=} i\frac{\delta^{ab} g_{\mu\nu}}{k^2 + i\varepsilon}(\text{Feynman Gauge}), \tag{2.16}$$



$$= g^2 \begin{bmatrix} f^{abe} f^{ecd}(g^{\alpha\gamma} g^{\beta\delta} - g^{\alpha\delta} g^{\beta\gamma}) \\ + f^{ace} f^{ebd}(g^{\alpha\beta} g^{\gamma\delta} - g^{\alpha\delta} g^{\gamma\beta}) \\ + f^{ade} f^{ebc}(g^{\alpha\beta} g^{\delta\gamma} - g^{\alpha\gamma} g^{\delta\beta}) \end{bmatrix}. \tag{2.17}$$

Note that at four point, all possible ways to contract the gauge fields with the structure constants need to be considered. Using the antisymmetry properties of the structure constants this is reduced to three different contractions which correspond to, clockwise and counterclockwise planar ordering, and non-planar ordering.

With the use of the interaction vertices and propagators it is possible to build exchange diagrams. This is done by introducing the external legs with the vertex rules and contracting the internal vertices to propagators. At four point there are four possible graphs that can be constructed. One being the contact vertex, the other three are exchange diagrams based on the ordering of the external legs,



$$. \tag{2.18}$$

The three *channels* are usually referred to as the s, t and u channel which refers to the poles. The corresponding Mandelstam invariants, which appear due to the propagators, are defined by,

$$s = (p_1 + p_2)^2, \quad t = (p_1 + p_3)^2, \quad u = (p_1 + p_4)^2. \tag{2.19}$$

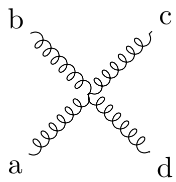
As the gluon propagator scales as $1/k^2$, the poles of the exchange diagrams are given by the Mandelstam invariants, $1/s$, $1/t$, and $1/u$. Note that the Mandelstams can be rewritten in terms of different momenta using momentum conservation (i.e. $s = (p_1 + p_2)^2 = (p_3 + p_4)^2$). In the rest of this work, sums of momenta will be denoted by $p_{i_1 i_2 \ldots i_r} = p_{i_1} + p_{i_2} + \cdots + p_{i_r}$, and Mandelstam variables as $s_{i_1 i_2 \ldots i_r} = p^2_{i_1 i_2 \ldots i_r}$. For example, at four point (Eq. (2.19)), the t-channel Mandelstam would be denoted by $s_{13} = s_{24}$. This notation generalizes to aid the need of higher point diagrams.

When three point vertices are contracted, their structure constants contract as well. For example, if we identify particles 1 to 4 with the color indices a to d, and contract the repeated indices, we get the following structure constants for each channel,

$$s\text{−channel} = f^{abe}f^{ecd}, \qquad t\text{−channel} = f^{ace}f^{ebd}, \qquad u\text{−channel} = f^{ade}f^{ebc}. \tag{2.20}$$

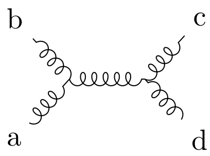
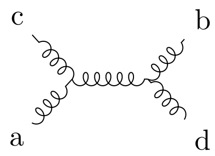
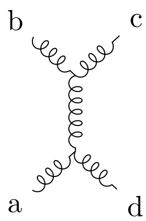
Note that these combinations of structure constants are also present in the four point vertex of Eq. (2.17).

The four point vertex, however, does not have poles as it is a contact term. The contact terms can always be written in terms of a sum of the cubic s, t and u pole structures by using the identity, $1 = \frac{s}{s} = \frac{t}{t} = \frac{u}{u}$. As a result, the kinematic contributions of the contact terms contribute to the kinematic numerators of the cubic diagrams and can be recognized as they are polynomial in the Mandelstam invariants. Explicitly, the contact term consists of three terms with the structure constants contracted in different ways multiplied by some kinematic contributions which we denote by $\tilde{n}_i$ corresponding to the channels of the structure constants. If each term is multiplied and divided by the corresponding propagator, the quartic vertex can be absorbed into the cubic graphs [34],



$$\sim f^{abe}f^{ecd}\tilde{n}_s + f^{ace}f^{ebd}\tilde{n}_t + f^{ade}f^{ebc}\tilde{n}_u$$

$$= \frac{f^{abe}f^{ecd}\tilde{n}_s s}{s} + \frac{f^{ace}f^{ebd}\tilde{n}_t t}{t} + \frac{f^{ade}f^{ebc}\tilde{n}_u u}{u} \tag{2.21}$$

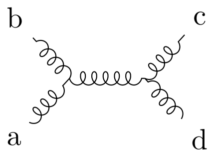
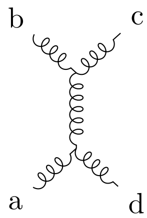
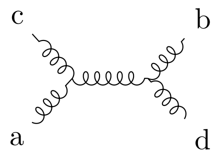
$\sim$  +  +  .

There is some freedom on how to exactly distribute the contact terms into the cubic diagrams as the structure constants are related with the Jacobi identity, Eq. 2.5.

If we now reconsider the relations between the combinations of structure constants in the form of the adjoint Jacobi identity (Eq. (2.5)), we see that they can be represented as relations between the color structures of the different channels in terms of the cubic diagrams,

$$f^{abe}f^{ecd} = f^{ade}f^{ebc} - f^{bde}f^{eac}$$

$$\Downarrow$$

$c\Big($  $\Big) = c\Big($  $\Big) - c\Big($  $\Big)$. (2.22)

Note that this is just a schematic representation of the color structures of the cubic diagrams. This relation is often written as,

$$c_i - c_j = c_k, \tag{2.23}$$

where c_i are *color factors* which represent all the structures due to the Lie Algebra, or *color*. This will prove to be a key concept in the color kinematics duality, which will be discussed in Section 3

The tree level[1] amplitude of the four point interaction can be written as just a sum of the cubic diagrams. With the use of Feynman rules this is found to be,

$$\mathcal{A}_4^{\text{YM Tree}} = \frac{c_s n_s}{s} + \frac{c_t n_t}{t} + \frac{c_u n_u}{u}, \tag{2.24}$$

where c_s, c_t, and c_u are the structure constants corresponding to the respective channels as defined before, and n_s, n_t, and n_u are the kinematic numerators of the cubic diagrams including the contact contributions due to Eq. (2.21).

From [14] we take the kinematic numerator which is,

$$\begin{aligned} n_s =& \\ &\Big((\varepsilon_1\cdot\varepsilon_2)p_1^\mu + 2(p_2\cdot\varepsilon_1)\varepsilon_2^\mu - (\varepsilon_1\cdot\varepsilon_2)p_2^\mu + 2(p_1\cdot\varepsilon_2)\varepsilon_1^\mu\Big)\Big((\varepsilon_3\cdot\varepsilon_4)p_3^\mu + 2(p_3\cdot\varepsilon_4)\varepsilon_4^\mu - (\varepsilon_3\cdot\varepsilon_4)p_4^\mu + 2(p_4\cdot\varepsilon_3)\varepsilon_3^\mu\Big) \\ &+ 2\Big((\varepsilon_2\cdot\varepsilon_3)(\varepsilon_1\cdot\varepsilon_4) - (\varepsilon_1\cdot\varepsilon_3)(\varepsilon_2\cdot\varepsilon_4)\Big) \end{aligned} \tag{2.25}$$

where the Lorentz indices can be fully contracted and the on shell conditions, $p_i^2 = 0$ and $p_i \cdot \varepsilon_i = 0$, are satisfied. The other numerators, n_t and n_u, can be found by cyclic permutation of the external labels ($1 \mapsto 2 \mapsto 3 \mapsto 1$ and $1 \mapsto 3 \mapsto 2 \mapsto 1$ for n_t and n_u respectively). The final term of the numerator is linear in the Mandelstam, hence it comes from the contact term.

Earlier we mentioned that the structures of Yang-Mills theory are very different from gravity. To illustrate this, we will now discuss the self interactions of the graviton.

2.2 Graviton (self) interactions

Although gravity can not be quantized without running into UV divergences, we can still consider the theory as an effective field theory at low energies and treat it in the context of quantum field theory. In this construction, general relativity is described by the Einstein-Hilbert action,

$$\mathcal{S}_{\text{EH}} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}R, \tag{2.26}$$

here g is the determinant of the metric tensor $g_{\mu\nu}$, R is the Ricci scalar, and G is the gravitational constant. The coupling is often written as the modified planck mass, which is given by, $M_p^2 = \frac{1}{16\pi G}$. This is the mass scale of the effective field theory.

The graviton is defined as a small perturbation of the Minkowski metric,

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{1}{M_p}h_{\mu\nu}, \tag{2.27}$$

where $\eta_{\mu\nu}$ is the Minkowski metric and $h_{\mu\nu}$ is the graviton field. We must remember that such an approach only makes sense for weak gravity due to the definition of the graviton and the fact that a 'QFT' construction of GR only makes sense for low energy as the theory is UV divergent.

To study the self interactions of the graviton, the action can be expanded in powers of $h_{\mu\nu}$ with the appropriate powers of the planck mass M_p. The Lorentz indices are suppressed for simplicity leading to the schematic expansion of the action,

$$\mathcal{S}_{\text{EH}} = \int d^4x\left((\partial h)^2 + \frac{h(\partial h)^2}{M_p} + \frac{h^2(\partial h)^2}{M_p^2} + \dots\right). \tag{2.28}$$

[1] Leading order in a loop expansion, i.e., no loops

The two derivatives of each term come from the definition of the Ricci scalar. This perturbative expansion works well for energy scales much lower than M_p.

The expanded action of Eq. (2.28) results in an infinite number of self interactions. Diagrammatically this is illustrated in Equation (1.1). In principle, one can derive Feynman rules for these interactions, but the calculations become increasingly complex. As DeWitt showed, the three point graviton vertex has “at least” 171 terms [35–37]. Higher point vertices quickly become even more complex. For more details on this topic see [30, 38–40].

This complexity can be explained by the huge amount of freedom GR has due to the diffeomorphism invariance. This is a property of the action describing gravitons and is analogous to gauge invariance in gauge theory. At linear order it is given by the transformation,

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu. \tag{2.29}$$

Where ξ_μ is an arbitrary function of spacetime.

The final type of theory we will discuss, and the main focus of this thesis, is the theory of massive scalar fields coupled to Yang-Mills theory. This can naturally be extended to massive scalars coupled to gravity as well using the double copy constructions that will be discussed in Section 3. In turn, this is important for the application to gravitational waves as discussed in the introduction. However, the details of the applications to gravitational waves and the massive scalars coupled to gravity in general is beyond the scope of this thesis.

2.3 Massive scalar fields

In this thesis, we are particularly interested in the interactions between massive fields and gauge fields. Hence, we need to understand all components to calculate amplitudes involving massive fields. The simplest model that accomplishes this is a theory with massive colorless spin 0 particles, or colorless scalars in other words. Ultimately we are interested in the limit where the mass of the scalar field is taken to be large compared to the momentum of the gauge fields. It turns out that at the leading order contribution to the amplitude in an inverse mass expansion is universal. This means the spin of the massive field does not affect the amplitude [25, 41]. A heavy quark field can simply be replaced by a heavy scalar. For this reason we focus on massive scalar fields in contrast to fermions for example. We will first discuss the massive scalar fields with a general mass. Particularly massive scalar QCD.

The Lagrangian corresponding to such a theory can be constructed as [31],

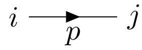
$$\mathcal{L}_{\text{sQCD}} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu,a} - \frac{1}{2}(\partial^\mu A^a_\mu)^2 + (D_\mu \varphi_i)^\dagger (D^\mu \varphi_i) - m^2 \varphi_i^\dagger \varphi_i. \tag{2.30}$$

The first term is the familiar Yang-Mills term, the second a gauge fixing term, the third term contains the interaction between scalar and gauge fields and the fourth is the mass term. In [31] there are multiple flavors for the massive scalar field which is necessary to describe the coupling of multiple different scalars with possible different masses in the theory. In our case, however, we only care for a single instance of a massive scalar and hence flavor indices can be ignored. Furthermore, a and i denote the color indices. The gauge covariant derivative is given by,

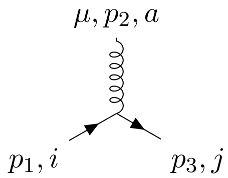
$$D_\mu \varphi_i = \partial_\mu \varphi_i - ig A^a_\mu T^a_{ij} \varphi_j, \tag{2.31}$$

where φ_i is the scalar field, g is a coupling constant, and T^a are the generators of the gauge group.

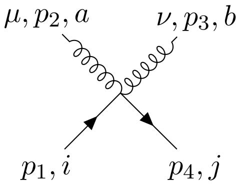
In addition to the self interactions of the gauge field generated from the first term, which were described in Subsection 2.1.1, the sQCD Lagrangian gives rise to an interaction between the gauge field and the massive scalar generated by the third term. We also have the scalar propagator and in contrast to QCD with fermions, a contact vertex through $\mathcal{L}_{\text{int}} \sim A_\mu^\dagger \phi^\dagger A^\mu \phi$.



$$= \frac{-i\delta^{ij}}{p^2 - m^2 + i\varepsilon}, \tag{2.32}$$



$$= igT^a_{ij}(p_1 + p_3)_\mu. \tag{2.33}$$



$$= ig^2(T^a_{il}T^b_{lj} + T^b_{il}T^a_{lj})g_{\mu\nu} \tag{2.34}$$

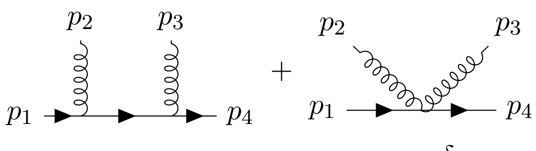
Note that the momentum p_1 of the scalar field is ingoing and p_n is outgoing. The gluon momenta are all ingoing.

Now we are ready to start building amplitudes. Note that the external particles of the amplitude have to be on-shell. Hence, for the external momenta and polarizations we can apply the on shell conditions, $p_i \cdot p_i = m_i^2$ for the momenta of massive particles and $p_i \cdot p_i = 0$ for massless momenta of massless particles. Additionally, the polarization vectors of the external gluons are transverse, $\varepsilon_i \cdot p_i = 0$. Unless mentioned otherwise we will only deal with the tree level amplitudes. In case of amplitudes with massive scalars coupled to gluons, particle '1' and 'n' are always the external massive scalars where particle '1' is ingoing and 'n' is outgoing, the other external particles (2,. . . ,n-1) are gluons.

The 3-point amplitude of two massive scalars and one gluon can be found immediately by contracting the vertex with the polarization vector of the gluon,

$$A_3(1_i, 2^a, 3_j) = igT^a_{ij}\varepsilon_2 \cdot (p_1 + p_3). \tag{2.35}$$

For the 4-point amplitude we have to do a little more work. First consider which diagrams contribute. The first channel is realized through the exchange diagram and the half of the contribution of a contact diagram. We call this the s channel,

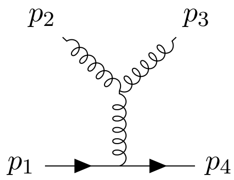


$$\begin{aligned}
&= igT^b_{kj}((p_1 + p_2) + p_4)_\mu \varepsilon_3^\mu (-i)\frac{\delta_{lk}}{s_{12} - m^2} igT^a_{il}(p_1 + (p_1 + p_2))_\nu \varepsilon_2^\nu + ig^2(T^aT^b)_{ij}g_{\mu\nu}\varepsilon_2^\mu\varepsilon_3^\nu \\
&= i2g^2(T^aT^b)_{ij}\frac{2(p_4 \cdot \varepsilon_3)(p_1 \cdot \varepsilon_2) + (\varepsilon_2 \cdot \varepsilon_3)(p_1 \cdot p_2)}{s_{12} - m^2}.
\end{aligned}$$

By using the on-shell conditions and momentum conservation, which, in the case of all particles ingoing except for '4', results in $p_1 + p_2 + p_3 - p_4 = 0$, we can simplify the expression to the second line. The contact term trivially has a cubic contribution by multiplying the term by $1 = \frac{s_{12}-m^2}{s_{12}-m^2}$. Note that $s_{12} - m^2 = (p_1 + p_2)^2 - m^2 = 2(p_1 \cdot p_2)$. This first contribution to the amplitude clearly splits up to a contribution comprised of only color factors, a numerator with only kinematic factors, and a propagator. As before, these are denoted by c_s and n_s respectively.

The second channel we will consider is the t-channel. This will yield the same kinematic factor and propagator up to the switching of the indices 2 and 3. The color factor will now be $c_t = (T^b T^a)_{ij}$, which corresponds to swapping the legs. There is another “half” of the contact term which is absorbed in the t-channel.

There is one more channel to consider, the u-channel. The contact term is already fully accounted for, hence this is just the exchange diagram with a massless propagator,



$$\begin{aligned} &= igT^d_{ij}(p_1+p_4)_\mu i\frac{\delta^{dc}g^{\mu\alpha}}{s_{23}}(-i)gf^{abc}\Big[g_{\alpha\beta}(k-p_2)_\gamma + g_{\beta\gamma}(p_2-p_3)_\alpha + g_{\gamma\alpha}(p_3-k)_\beta\Big]\varepsilon_2^\beta\varepsilon_3^\gamma \\ &= i2g^2 f^{abc}T^c_{ij}\frac{2(p_1\cdot\varepsilon_2)(p_4\cdot\varepsilon_3) - 2(p_4\cdot\varepsilon_2)(p_1\cdot\varepsilon_3) + (\varepsilon_2\cdot\varepsilon_3)[(p_1\cdot p_2)-(p_1\cdot p_3)]}{s_{23}}. \end{aligned} \tag{2.36}$$

Note that the internal momentum $k = -(p_2+p_3)$. The result in the second line can be achieved by using the on-shell conditions and momentum conservation. Similar to the other terms, this can be recognized as a color factor and a kinematic factor, c_u and n_u respectively.

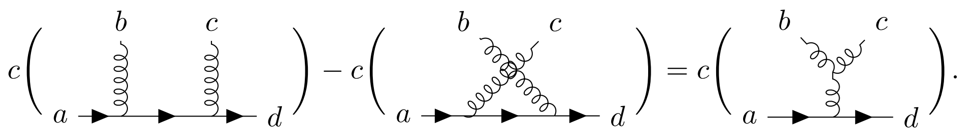
The total amplitude is then given by the sum of the contributions of the different channels,

$$\mathcal{A}_4 = \frac{c_s n_s}{s_{12}-m^2} + \frac{c_t n_t}{s_{13}-m^2} + \frac{c_u n_u}{s_{23}}. \tag{2.37}$$

Interestingly, the color factors automatically satisfy the relation of equation (2.2). This can be depicted analogously to Eq. (2.22) for diagrams that include fundamental particles as,

$$T^a_{ij}T^b_{ij} - T^b_{ij}T^a_{ij} = if^{abc}T^c_{ij}$$

$$\Downarrow$$



(2.38)

This relation is not a surprise as the color factors are a consequence of the underlying Lie algebra.

The observant reader might have noticed that the kinematic numerators written in this way satisfy a very similar relation,

$$\begin{aligned} n_s - n_t &= \big(2(p_4\cdot\varepsilon_3)(p_1\cdot\varepsilon_2) + (\varepsilon_2\cdot\varepsilon_3)(p_1\cdot p_2)\big) - \big(2\leftrightarrow 3\big) \\ &= \big(2(p_4\cdot\varepsilon_3)(p_1\cdot\varepsilon_2) + (\varepsilon_2\cdot\varepsilon_3)(p_1\cdot p_2)\big) - \big(2(p_4\cdot\varepsilon_2)(p_1\cdot\varepsilon_3) + (\varepsilon_2\cdot\varepsilon_3)(p_1\cdot p_3)\big) && (2.39) \\ &= 2(p_1\cdot\varepsilon_2)(p_4\cdot\varepsilon_3) - 2(p_4\cdot\varepsilon_2)(p_1\cdot\varepsilon_3) + (\varepsilon_2\cdot\varepsilon_3)[(p_1\cdot p_2)-(p_1\cdot p_3)] && (2.40) \\ &= n_u. && (2.41) \end{aligned}$$

In Section 3 we will see that this does not have to be a coincidence, and has some very powerful consequences. The same also applies to the Yang-Mills color factors and numerators of Eq. (2.24).

2.4 Partial amplitudes and amplitude relations

A way to reduce the complexity of calculations involving scattering amplitudes is to consider a decomposition of the color structures. One way to do this is to structure the amplitude in color factors such that the color factors are independent of each other, which can be achieved using the Jacobi identities [42]. This allows us to systematically treat the color degrees of freedom by separating them from the kinematic components, which are the *color ordered*, or *partial*, amplitudes.

By starting from equation (2.2) and taking the trace over the indices i and j. This way, we can decompose the structure constants f^{abc} in terms of traces of the generators T^a,

$$if^{abc} = \mathrm{Tr}\left(\left[T^a, T^b\right], T^c\right) = \mathrm{Tr}\left(T^a T^b T^c\right) - \mathrm{Tr}\left(T^b T^a T^c\right). \tag{2.42}$$

With the Fierz identity, $(T^a)_i^j (T^b)_k^l = \delta_i^l \delta_k^j - \frac{1}{N}\delta_i^j \delta_k^l$, we can simplify products of structure constants. For example, the product of two structure constants can be written as,

$$f^{abe} f^{ecd} = \mathrm{Tr}\left(T^a T^b T^c T^d\right) - \mathrm{Tr}\left(T^a T^b T^d T^c\right) - \mathrm{Tr}\left(T^b T^a T^c T^d\right) + \mathrm{Tr}\left(T^b T^a T^d T^c\right). \tag{2.43}$$

In principle products of any number of structure constants can be written in this form. For more details, see [14].

When this decomposition is done to the structure constants of a YM amplitude, the coefficients of the independent color factors (now traces of generators) that are left are the partial amplitudes. As described in [42, 43], an arbitrary cubic graph can be converted to a 'multi-peripheral' form or half-ladder diagram by repeatedly using the Jacobi identity. For some n-point amplitude of an arbitrary particle this would look like,

. (2.44)

The intermediate $n-2$ particles are denoted by σ which are permutations of the particles.

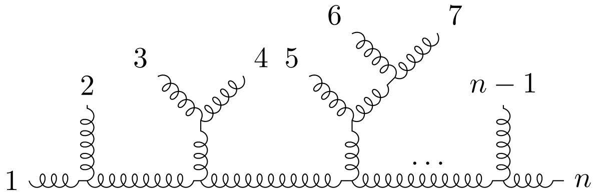
This decomposition of the color factors can be used both for the purely adjoint particles of gluon scattering and for interactions with fundamental particles as in the case of massive scalars coupled to gauge fields. For five point amplitudes of the latter, the full set of diagrams that could be written in this half-ladder form are the cubic interactions of Eq. (2.67). The diagrams for the six point amplitude are shown in equation (6.6) in appendix C. We will now give some more details for both the adjoint and fundamental case.

2.4.1 Tree level adjoint particles and amplitude relations

In a Yang-Mills theory, the particles are entirely constructed in the adjoint representation of the gauge group and hence the color factors are products of only structure constants f^{abc}. The tree level n-point amplitude can be written as a sum over all possible cubic diagrams [14]. One can show by recursively attaching new legs to every possible leg of lower multiplicity diagrams that there are $(2n-5)!!$ cubic diagrams at n-point. Note that an 'independent' cubic diagram is uniquely defined by the propagator contribution which is fixed by the order of the external legs, even when graphs have the same topology[2].

[2]We consider graphs to be of the same topology when the only difference is the order of the external legs. Hence, the s, t and u channel of YM at four point all have the same topology for example. For the theory including an adjoint particle, the s and t channel are the same topology but the u channel is different as it has a massless propagator instead of a massive one.

Another important property of the color factors is that there are only $(2n-5)!!-(n-2)!$ Jacobi relations that are independent as the color factors can be mapped to a $(n-2)!$ basis. For an n-point Yang-Mills amplitude given some diagram, the color factor can be constructed by following a path from particle '1' to 'n' and appropriately contracting color factors or commutators of color factors. For a diagram as in,



$$, \tag{2.45}$$

this results in a contraction of the form,

$$\left(f^{a_2}[f^{a_3}, f^{a_4}][f^{a_5}, [f^{a_6}, f^{a_7}] \cdots]\right)_{a_1 a_n}, \tag{2.46}$$

where the a_i are the color indices of the external particles. Note that the structure constants only have one index between the brackets as all the internal indices are contracted.

The color factors can be rewritten using the Jacobi identities This results in the multi-peripheral form where σ signifies some permutation of the external legs,

$$c_{1\sigma n} = \sum_{\sigma \in S_{n-2}} b_{i\sigma} \left(f^{a_{\sigma(1)}} f^{a_{\sigma(2)}} \cdots f^{a_{\sigma(n-2)}}\right)_{a_1 a_n}. \tag{2.47}$$

Here the parameter b_i 0 or ± 1 depending on the specific color factor. This allows us to write the gauge-theory amplitude in as a Del Duca-Dexon-Maltoni (DDM) color decomposition [43]. This is a way of writing the full amplitude in terms of partial (color ordered) amplitudes,

$$\mathcal{A}_n(1, 2, \dots, n-1, n) = \sum_{\sigma \in S_{n-2}} A_n(1\sigma_1 \dots \sigma_{n-2} n) \left(f^{a_{\sigma(1)}} f^{a_{\sigma(2)}} \cdots f^{a_{\sigma(n-2)}}\right)_{a_1 a_n}. \tag{2.48}$$

The partial amplitudes $A_n(1\sigma_1 \dots \sigma_{n-2} n)$ correspond to some fixed ordering of the color structures and only consist of kinematic objects themselves. In the trace basis the decomposition has the form,

$$\mathcal{A}_n(1, 2, \dots, n-1, n) = \sum_{\sigma \in S_{n-1}} A_n(1\sigma_1 \dots \sigma_{n-1}) \operatorname{Tr}(T^{a_{\sigma_1}} \dots T^{a_{\sigma_{n-1}}}). \tag{2.49}$$

More details on this are found in [14].

As an example, consider the four point amplitude. If we start from the YM amplitude of Eq. (2.24), we can expand the color factors in terms of traces of the generators. Using properties of the traces this can be reduced to,

$$\begin{aligned} \mathcal{A}_4(1,2,3,4) \sim & \left(\frac{n_s}{s} + \frac{n_u}{u}\right) \operatorname{Tr}(T^{a_1} T^{a_2} T^{a_3} T^{a_4}) + \left(\frac{n_t}{t} - \frac{n_u}{u}\right) \operatorname{Tr}(T^{a_1} T^{a_3} T^{a_2} T^{a_4}) \\ & + \left(\frac{n_s}{s} - \frac{n_t}{t}\right) \operatorname{Tr}(T^{a_1} T^{a_3} T^{a_4} T^{a_2}). \end{aligned} \tag{2.50}$$

Here the kinematic objects in front of the traces are the partial amplitudes $A_4(1234)$, $A_4(1324)$, and $A_4(1342)$ respectively. Note that there are only three elements whilst there should be six. This is due to the fact that half of the six could be written in terms of reversed traces with identical partial amplitudes.

Relations between (partial) amplitudes

The partial amplitudes satisfy a number of interesting relations. We will now discuss some of these, particularly in the context of tree amplitudes in YM theory. Most of the relations will also hold for theories with field in the fundamental representation, albeit in a slightly different form. We will roughly follow the enumeration of [14].

In this section we have seen the color decomposition of amplitudes. The partial amplitudes in a YM theory are functions only of the kinematics, i.e. the momenta and polarizations. The color dependence of the partial amplitudes is represented only by the ordering of the external legs, no color factors are included. Additionally, they can only have poles in planar channels. In a canonically ordered (1234) partial four point amplitude for example, there can only be poles in the s and u channels. More explicitly, it can have poles in neighboring legs, s_{12} and $s_{14} = s_{23}$ but not in s_{13}.

Due to the trace basis decomposition of the color factors we note that the partial amplitudes must also be invariant under cyclic permutations of the external legs,

$$A_n(1,2,3,\dots,n) = A_n(2,3,\dots,n,1), \tag{2.51}$$

and under order reversal (up to a sign),

$$A_n(1,2,3,\dots,n) = (-1)^n A_n(n,\dots,2,1). \tag{2.52}$$

Another relation that must be satisfied is the *photon-decoupling* identity. Holistically, the idea is that we could consider a theory where the gauge group is $U(N)$ which would be equivalent to a theory with $U(1) \times SU(N)$ where $U(1)$ is the gauge group of a photon. When considering the self interaction of the gauge fields which involves the commutator of the generators, the photon "decouples" as its generator is the identity and hence commutes with any other matrix. A consequence of this is that the amplitude should be zero when all but one leg are summed over cyclically,

$$A(123\dots n) + A(213\dots n) + \cdots + A(231\dots n) = 0. \tag{2.53}$$

Finally, the partial amplitudes must satisfy the Kleiss-Kuijf (KK) relations [43, 44],

$$A_n(1,\alpha,n,\beta) = (-1)^{|\beta|} \sum_{\sigma\in\alpha\sqcup\beta^T} A_n(1,\sigma,n), \tag{2.54}$$

where α and β are lists of external legs, the 'transpose' (i.e. β^T) represents the reverse ordering of a list, $|\beta|$ is the number of elements in β, and $\sqcup$ is the shuffle product. The shuffle product is defined as the sum of all possible ways to interleave the elements of two lists while keeping the order of the elements in each list. For example, the shuffle product of ab and xy is,

$$ab \sqcup xy = abxy + axby + axyb + xaby + xayb + xyab. \tag{2.55}$$

In other words, y is always after x and b after a, but there might be other 'stuff' in between. The KK relations originate from the fact that two decompositions (DDM and trace) utilize a different number of partial amplitudes in their basis. Therefore, there must exist a relation that maps one basis to the other which is the role of the KK relations.

We will now make some comments and give some examples for the case of fundamental particles.

2.4.2 Tree level fundamental particles

The half-ladder diagram of $n-2$ gluons and two fundamentals can also be decomposed into a basis of partial amplitudes. For this it is more convenient to utilize the fundamental 'structure constants'. In a similar approach is before, the full amplitude is then given by [43],

$$\mathcal{A}_n(1_i, 2, \ldots, n-1, n_j) = \sum_{\sigma \in S_{n-2}} (T^{a_{\sigma_1}} \ldots T^{a_{\sigma_{n-2}}})_{ij} A_n(1\sigma_1 \ldots \sigma_{n-2} n). \tag{2.56}$$

The color indices of the gluons are taken to be implicit on the right-hand side.

As an example we can take the four point amplitude. The symmetric group S_2 has only two elements. Hence, explicitly this would have the form,

$$\mathcal{A}_4(1_i, 2, 3, 4_j) = (T^a T^b)_{ij} A_4(1234) + (T^b T^a)_{ij} A_4(1324). \tag{2.57}$$

Staying with the four point example, we see that there are only two independent color factors. Therefore, the third color factor has to be written in terms of the other two. For the four point amplitude this is very straightforward by using Eq. (2.2). This allows us to decompose the u-channel as in Eq. (2.37), in terms of the s and t channel diagrams. Explicitly, the two color ordered amplitudes are,

$$A_4(1234) = \frac{n_s - n_u}{(2p_1 \cdot p_2)}, \quad A_4(1324) = \frac{n_t - n_u}{(2p_1 \cdot p_3)}, \tag{2.58}$$

At 5pt. the symmetric group S_3 has 6 elements. Explicitly this would lead to,

$$\mathcal{A}(1_i, 2, 3, 4, 5_j) = (T^a T^b T^c)_{ij} A(12345) + (T^a T^c T^b)_{ij} A(12435) + (T^b T^a T^c)_{ij} A(13245) \tag{2.59}$$

$$+ (T^b T^c T^a)_{ij} A(13425) + (T^c T^a T^b)_{ij} A(14235) + (T^c T^b T^a)_{ij} A(14325). \tag{2.60}$$

In Section 3 we will see that it is possible to check the consistency of the color-kinematics duality by comparing certain partial amplitudes using the so called (BCJ relations). For this it will be necessary to decompose certain terms of partial amplitudes in terms of other partial amplitudes. It is very useful to know what independent color factors correspond to the terms in the multi-peripheral form.

With this intermezzo on partial amplitudes, we will now return to massive scalar QCD and discuss the limit when the massive scalars have a large mass compared to the typical energy scale of the process.

2.5 Massive fields and the large mass limit

A particularly interesting limit of the massive scalar QCD amplitudes is the large mass limit. This leads to a specific class of effective field theories known as Heavy mass Effective Field Theory (HEFT). These theories are used to describe the dynamics of heavy particles in the limit where the mass of the heavy particle is much larger than the typical energy scale of the process.

The type of effective field theory relevant to our problem is familiar in the context of heavy quarks in collider/standard model physics as heavy quark effective theory (HQET) [45, 46]. We are interested in the leading order terms in an expansion in $1/m$. The focus in this thesis is on the scalar version (spin 0) of the theory.

A corresponding effective theory for gravity can be studied with the double copy. This way, tree-level amplitudes can be constructed for two heavy particles coupled to gravitons in the leading order of an inverse mass expansion [25, 47]. This is one of the main motivations in the form of an application for studying such theories as the double copy of the theory can be used to study binary black hole systems. In a binary black hole system the masses of the black holes will certainly be much larger than any momentum exchanged and in addition, the black holes can be treated as point particles. The tree-level amplitudes can be used to construct loop diagrams with unitary cuts. The HEFT gravity amplitudes can in turn be used to find classical observables in binary black hole systems [26]. This is hugely powerful for calculations necessary to study gravitational wave signals for current and future observations. The details of this are beyond the scope of this thesis.

The HEFT amplitudes have some major advantages. The leading order terms of the inverse mass expansion are the only relevant terms for classical physics [26], which is exactly of interest for black hole scattering. Additionally, it is possible to construct the numerators that can be used in a double copy such that the HEFT amplitudes are inherently gauge invariant, local w.r.t. gravitons/gluons, and satisfy Jacobi relations. This later construction is based on a 'kinematic algebra' analogous to the color algebra in gauge theories. More on this in Section 3. We first take a step back to a scalar version of Heavy Quark Effective Theory.

2.5.1 Heavy mass Effective Field Theory

The momentum of external scalar particles in this theory before an interaction (particle '1' from before) can be written as,

$$p^\mu = mv^\mu, \tag{2.61}$$

where m is the mass of the particle and $v^2 = 1$. Then, after an interaction with a 'soft' (compared to the scale of mass m) particle, the momentum of the particle (particle 'n') can be written as,

$$p^\mu = mv^\mu + k^\mu, \tag{2.62}$$

where k^μ is the momentum transfer. The momentum transfer is just the sum over the momentum of all external massless particles in the interaction. When the heavy particle is on shell, $p^2 = m^2$, we can show using the above equations that,

$$v \cdot k = \frac{-k^2}{2m}. \tag{2.63}$$

This reduces to, $v \cdot k = 0$, in the large mass limit.

Using the properties above, we can write the propagator for the heavy particle as,

$$\frac{-i\delta^{ij}}{p^2 - m^2 + i\varepsilon} = \frac{-i\delta^{ij}}{2mv \cdot k + i\varepsilon}. \tag{2.64}$$

Similarly, we could derive a HEFT Feynman vertex rule for the interaction vertices of the heavy particle with the massless particles. In principle, we can build an amplitude out of these 'new' Feynman rules and get the leading order terms in the inverse mass expansion. Alternatively, we can use the HEFT expansion to immediately simplify the sQCD amplitudes with a general mass to the HEFT version. For this, we need to first derive the sQCD amplitudes for a general mass.

As an example, let us find the HEFT amplitude for the scattering of two scalars with two gluons as was done for the general mass case in Eq. (2.37). We will use a specific color ordering for the gluons, $A_4(1234)$, hence, we will only consider the s and u channel diagrams.

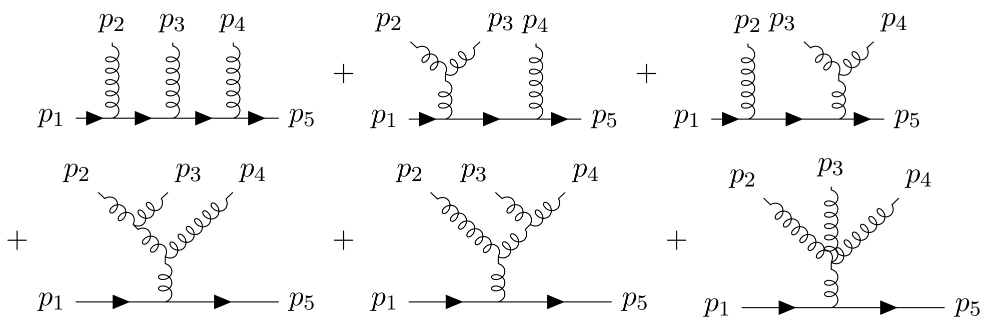
$$A_4(1234) = \frac{n_s}{s_{12} - m^2} - \frac{n_u}{s_{23}}, \tag{2.65}$$

where n_s and n_u are the kinematic numerators of the s and u channel as discussed in this section. Now substituting Eq. (2.61) and Eq. (2.62) for p_1 and p_4 respectively, and taking the leading order in the inverse mass expansions, we find that the leading order color ordered amplitude is given by,

$$A_4^{\mathrm{HEFT}}(1234) = 2m\Big(\frac{(v\cdot\varepsilon_2)(v\cdot\varepsilon_3)}{(2v\cdot p_2)} + \frac{(\varepsilon_3\cdot p_2)(v\cdot\varepsilon_2)-(\varepsilon_2\cdot p_3)(v\cdot\varepsilon_3)-(\varepsilon_2\cdot\varepsilon_3)(v\cdot p_2)}{s_{23}}\Big). \tag{2.66}$$

We used that $v\cdot p_2 = -v\cdot p_3$ due to Eq. (2.63).

In the same vain we can construct a five point amplitude consisting of two heavy scalars and three gluons. The (partial) amplitude can be constructed of 6 different topologies,



(2.67)

Sparing the details of the calculation, the HEFT amplitude is given by [25],

$$\begin{aligned}
A_5^{\mathrm{HEFT}}(12345) = 4m\Bigg(&\frac{(v\cdot e_2)(v\cdot e_3)(v\cdot e_4)}{4(v\cdot p_2)\left((v\cdot p_2)+(v\cdot p_3)\right)}\\
&+\frac{(v\cdot e_2)\left(\frac{1}{2}(e_3\cdot e_4)\left(-(v\cdot p_3)+(v\cdot p_4)\right)-(e_3\cdot p_4)(v\cdot e_4)+(e_4\cdot p_3)(v\cdot e_3)\right)}{2(v\cdot p_2)s_{34}}\\
&+\frac{(v\cdot e_4)\left(\frac{1}{2}(e_2\cdot e_3)\left(-(v\cdot p_2)+(v\cdot p_3)\right)-(e_2\cdot p_3)(v\cdot e_3)+(e_3\cdot p_2)(v\cdot e_2)\right)}{2s_{23}\left((v\cdot p_2)+(v\cdot p_3)\right)}\\
&+\frac{(e_2\cdot e_4)(v\cdot e_3)-(e_3\cdot e_4)(v\cdot e_2)}{4s_{234}}\\
&+\frac{1}{2s_{34}s_{234}}\Big(-2(e_2\cdot p_3)(e_3\cdot e_4)(v\cdot p_4)+2(e_2\cdot p_4)(e_3\cdot e_4)(v\cdot p_3)\\
&+(v\cdot e_2)\left((e_3\cdot e_4)\left(-(p_2\cdot p_3)+(p_2\cdot p_4)\right)+2(e_3\cdot p_2)(e_4\cdot p_3)-2(e_3\cdot p_4)(e_4\cdot p_2)\right)\\
&-\left((e_2\cdot p_3)+(e_2\cdot p_4)\right)\left(-(e_3\cdot p_4)(v\cdot e_4)+(e_4\cdot p_3)(v\cdot e_2)\right)\\
&+2\left((v\cdot p_3)+(v\cdot p_4)\right)\left((e_2\cdot e_3)(e_4\cdot p_3)-(e_2\cdot e_4)(e_3\cdot p_4)\right)\Big)\\
&+\frac{1}{2s_{34}s_{234}}\Big(2(e_2\cdot e_3)(e_4\cdot p_2)(v\cdot p_3)-2(e_2\cdot e_3)(e_4\cdot p_3)(v\cdot p_2)\\
&+(v\cdot e_4)\left((e_2\cdot e_3)\left((p_2\cdot p_4)-(p_3\cdot p_4)\right)+2(e_2\cdot p_3)(e_3\cdot p_4)-2(e_2\cdot p_4)(e_3\cdot p_2)\right)\\
&-\left((e_4\cdot p_2)+(e_4\cdot p_3)\right)\left((e_2\cdot p_3)(v\cdot e_3)-(e_3\cdot p_2)(v\cdot e_2)\right)\\
&+2\left((v\cdot p_2)+(v\cdot p_3)\right)\left(-(e_2\cdot e_4)(e_3\cdot p_2)+(e_2\cdot p_3)(e_3\cdot e_4)\right)\Big)\Bigg)
\end{aligned} \tag{2.68}$$

Note that one can directly read off which topology a term is coming from by looking at the propagator. To get to this amplitude, several 'trivial' relations related to the momentum conservation, on shell conditions, and the heavy mass limit were used. Although this is still manageable for a five point amplitude, it is clear that this method is not feasible for much higher point amplitudes. If we ever wish to construct higher point HEFT amplitudes, let alone for general massive fields or with higher derivative interactions, we need a more efficient method. We will get to this in Section 4.

3 Colour-Kinematics Duality and the BCJ Double Copy

Both gauge and gravity theories have an important role in the physical understanding of phenomena in the universe. At first sight, they do not have much in common. The Lagrangians look wildly different, their 'strength' (weak, strong, and the electromagnetic force vs gravity) are many orders of magnitude different, and they seem to be describing phenomena in the microscopic and macroscopic world respectively.

Despite this, there are also many features that are shared between the two theories. Some of these have been discovered before in string theory [19] which has been a large source of inspiration for the modern amplitude program. The study of 'observables' such as the on-shell scattering amplitudes instead of Lagrangians, helps to uncover deep and often non-trivial connections. The double copy and generally the color-kinematics duality offers a new perspective on gauge theory, gravity and their interpretations [14, 17, 24]. This section will be dedicated to the color-kinematics duality and its consequences.

First the general ideas of the color-kinematics duality and the double copy will be discussed. After that, we will discuss relations between (partial) amplitudes. Finally, we will discuss a tangent to this discussion in the form of a different double copy construction which proofs to be particularly useful for HEFT amplitudes.

3.1 Color kinematic duality

Very generally, the color-kinematics duality states that many theories with some Lie-algebra symmetry can be organized in such a way that the kinematic components of a set of cubic diagrams obey the same algebraic relations as the corresponding color components [14]. This is a non-trivial statement, and it is not immediately clear why this could (or even should) be the case. An important consequence of the duality is that it can not only constrain kinematics of certain theories, but also construct theories from other theories by simply replacing the color factors with kinematic factors and vice versa. This *double copy* construction alone is a large motivation to study the duality as it implies a deep connection between gauge and gravity theories. Given the lack of a consistent theory in the UV-regime for gravity, this could lead to new insights in the search for a quantum theory of gravity.

There exist proofs of the CK duality at tree level using various methods such as [48]. However, this is not the most insightful and is beyond the scope of this thesis. At loop level there are no complete proofs, but there are many examples where the duality holds at loop level as well [14]. We will focus on the construction and consequences of the duality at tree level.

At tree level, the reorganization of the amplitude such that the color contributions and kinematic components are separated is,

$$\mathcal{A}_n = g^{n-2} \sum_{i \in \Gamma} \frac{c_i n_i}{D_i}. \tag{3.1}$$

Here each c_i is a color factor that corresponds to a unique diagram, Γ is the set of all cubic diagrams, and n_i are the kinematic numerators. Note that this is a sum over only the cubic pole structures and that any higher point vertices are rewritten in terms of cubic diagrams as in Eq. (2.21) and their kinematic contributions absorbed into the respective numerators. Each cubic diagram has a corresponding denominator D_i which is the product of all internal propagators in the diagram.

It needs to be noted that this diagrammatic approach is not quite the same as Feynman diagrams due to the fact any contact terms are absorbed into the cubic diagrams. From this point onwards, diagrams will have this property unless stated otherwise. The numerators n_i are dependent on all other properties of the theory, most notably, the momenta and polarization vectors (or tensors in case of gravity) of the external particles. In general the numerators are gauge-dependent.

The color factors, which we have seen in Section 2, are not fully independent. They are related by the underlying Lie algebra of the gauge group which in the adjoint corresponds to Eq. (2.5) and for fundamental fields to Eq. (2.2) as was also noted diagrammatically in Eq. (2.22) and Eq. (2.38) respectively. In case the kinematic numerators are related by the same algebraic relations as the color factors, the amplitude (Eq. (3.1)) is said to exhibit the color-kinematics duality,

$$n_i - n_j = n_k \iff c_i - c_j = c_k. \tag{3.2}$$

Note that the signs in this equation need to be equal but depend on how the problem is defined.

At four point, the color factors simply relate to the s, t and u channel diagrams. We have seen in Section 2 that the "kinematic" Jacobi identity holds for kinematic numerators of on-shell Yang-Mills (and sQCD) amplitudes automatically. However, for higher point amplitudes, this is generally not true.

For larger multiplicity diagrams, the Jacobi identity is embedded in a larger diagram as in Figure 1. Any internal propagator is chosen to exhibit the Jacobi identity, the rest of the diagram

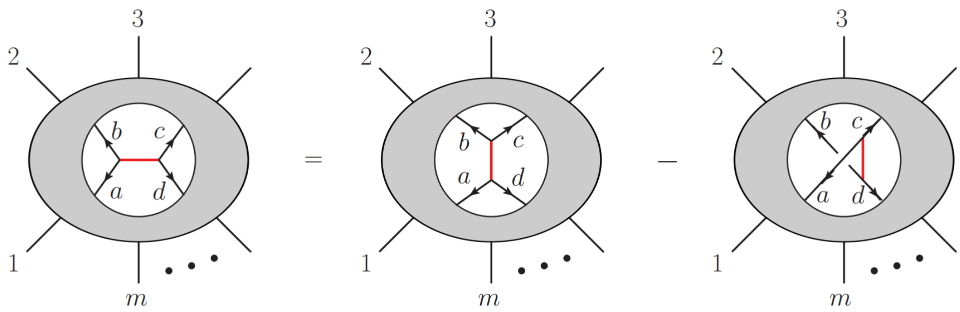


Figure 1: Diagrammatic representation of a Jacobi identity embedded in an m-point diagram. Image taken from Bern et al. [14].

is identical. Note that swapping two external legs attached to one vertex will correspond to the same color factors up to a minus sign (which is implied due to $f^{abc} = -f^{acb}$). These color factors are not independent. For the CK duality to hold, this property should also be inherited by the kinematic numerators, $c_i = -c_j \implies n_i = -n_j$.

Beyond four point amplitudes, it is a non-trivial task to actually find numerators that satisfy the CK duality of equation (3.2). When constructing amplitudes using Feynman rules for example, the numerators do not, in general[3], satisfy these relations.

One way of approaching this is to use the modern approach to on-shell scattering amplitudes where instead of using the action, one writes down all possible contributions to the amplitudes as an ansatz and then constrain this using a number of physical principles. We can require the kinematic Jacobi identity of equation (3.2) to be satisfied for all internal propagators to make sure the CK duality is manifest. This method will be introduced and thoroughly discussed using examples in Section 4.

As the algebraic properties of the color factors and the kinematic numerators are the same due to the duality, it is possible to generate new amplitudes that are consistent with properties

[3] Again, beyond 4-point.

such as gauge invariance by simply replacing the color factors with the kinematic factors,

$$c_i \to n_i. \tag{3.3}$$

This is the double copy, often dubbed BCJ duality or BCJ double copy after the pioneers, Bern, Carrasco, and Johansson. By appropriately choosing the coupling constant, the resulting amplitude gives an amplitude for graviton scattering [49],

$$\mathcal{M}_n = \left(\frac{\kappa}{2}\right)^{n-2} \sum_{i\in\Gamma} \frac{n_i \tilde{n}_i}{D_i}. \tag{3.4}$$

The tilde suggests that the numerators do not have to be identical numerators. In fact, they can have different gauge choices, they do not have to represent the same external state, and they can be from different gauge theories altogether. One implication of this is that only one of the numerators needs to exhibit the CK duality. Note that there are some more subtleties to this as discussed in [14].

3.2 BCJ relations

In addition to the relations between partial amplitudes that are discussed in Section 2.4, there are additional relations between the partial amplitudes that are a consequence of the color-kinematics duality. In this subsection we will sketch how these relations can be derived and what they mean for the double copy with the use of some examples.

We will first do this for an n-gluon amplitude (massless adjoint particles), and then show that there are similar BCJ relations for amplitudes that contain $(n-2)$ massless gluons and two massive particles in the fundamental representation. Provided the color-kinematic duality is satisfied, this should work similarly for massive fundamentals with any spin. We will be interested only in the massive scalar case.

3.2.1 BCJ relations for gluons

An n-gluon amplitude can generally be expressed as a sum over $(2n-5)!!$ terms from cubic graphs as in Eq. (3.1). If we take one connected line of propagators from external particle 1 to n, all other external lines are either connected directly to this line or connected through a side branch. Now if all external lines are connected directly to the connected line between 1 and n, the diagrams is considered a half-ladder diagram (as discussed in Section 2.4). Any generic color factor c_i corresponding to graph i can be constructed by gluing cubic (three-gluon) vertices together which all carry a structure constant f^{abc}. The color factors corresponding to the half-ladder diagrams form a subset of the full set of color factors, $c_{1\sigma n}$, which were found in equation (2.47).

Using the Jacobi identity, we can rewrite any generic color factor c_i as a sum of these half-ladder color factors. This is the DDM decomposition [43] where the half-ladder color factors form the independent Kleiss-Kuijf basis,

$$c_i = \sum_{\sigma\in S_{n-2}} M_{i,1\sigma n} c_{1\sigma n}. \tag{3.5}$$

Where $M_{i,1\sigma n}$ are certain coefficients. This can be used to write the n-gluon amplitude as,

$$\mathcal{A}_n = \sum_{\sigma\in S_{n-2}} M_{1\sigma n} A_n(1, \sigma(1), \ldots, \sigma(n-2), n), \tag{3.6}$$

$$A_n(1,\sigma(1),\ldots,\sigma(n-2),n)=\sum_i \frac{M_{i,1\sigma n}n_i}{d_i}, \tag{3.7}$$

similar to the amplitudes shown in Section 2.4.

What exactly are the coefficients $M_{i,\alpha}$? They can be read off by decomposing the color factors c_i using $f^{abc}=\mathrm{Tr}\big(T^a[T^b,T^c]\big)$ into a linear combination of traces $\mathrm{Tr}[\alpha]=\mathrm{Tr}(T^{a_{\alpha(1)}}\cdots T^{a_{\alpha(n)}})$ [50, 51]. Due to the hypothesis of the CK duality that the kinematic numerators obey the same identities as the color structures, the numerators n_i can also be written as a linear combination of half-ladder numerators,

$$n_i=\sum_{\sigma\in S_{n-2}} M_{i,1\sigma n}n_{1\sigma n}. \tag{3.8}$$

From this, it directly follows that the partial amplitude can be written as,

$$A_n(1,\sigma(1),\ldots,\sigma(n-2),n)=\sum_i \frac{M_{i,1\sigma n}M_{i,1\delta n}}{d_i}n_{1\delta n}. \tag{3.9}$$

Here the pre-factor of the kinematic numerators is often defined as the propagator matrix,

$$m(1\sigma n|1\delta n)=\sum_i \frac{M_{i,1\sigma n}M_{i,1\delta n}}{d_i}. \tag{3.10}$$

This has some interesting properties and related consequences. Firstly, as there are $(n-2)!$ independent numerators, the propagator matrix is a $(n-2)!\times(n-2)!$ matrix and as a consequence of on-shellness and momentum conservation has a rank of $(n-3)!$ [52]. This results in a set of constraints to the amplitudes in the Kleiss-Kuijf basis which also fully dictate the form of the BCJ relations independently of the kinematic numerators given that they obey the CK duality [50, 52, 53], and lead ultimately to the 'fundamental' BCJ relation,

$$\sum_{a=3}^{m}\Big(\sum_{b=a}^{m} s_{2b}\Big)A_m(1,3,\ldots,a-1,2,a,\ldots,m)=0, \tag{3.11}$$

where s_{ab} are the usual Mandelstam variables.

3.2.2 BCJ relations for amplitudes with two massive scalars and $(n-2)$ gluons

Gluon amplitudes with two massive scalars can also be expressed as a sum over cubic diagrams. We choose to fix particle 1 and n to be the massive scalars and 2 up to $n-1$ as gluons. The diagrams of the amplitudes will be identical to that of the n-gluon amplitudes except for one change. The connected line of gluon propagators from external particle 1 to n will be replaced by the massive scalar propagator. Any side branch before the diagrams are written as half-ladder diagrams will contain just external gluons and gluon propagators.

The new color factors associated with the diagrams that include massive particles can be constructed by gluing together the gluon vertices that carry a structure constant f^{abc}, and the two scalar one gluon vertices that carry T^a_{ij}. Similar to the n-gluon case, the resulting color factors can be written as a linear combination of half-ladder color factors,

$$t_{1\sigma n}=(T^{a_{\sigma_1}}\ldots T^{a_{\sigma_{n-2}}})_{ij}. \tag{3.12}$$

This is achieved by applying,

$$f^{abc}T^c_{ij}=[T^a,T^b]_{ij}, \tag{3.13}$$

to all gluon vertices emerging as side branches. The color factors can then be written as,

$$c_i = \sum_{\sigma \in S_{n-2}} M_{i,1\sigma n} t_{1\sigma n}. \tag{3.14}$$

Note that the half-ladder color factors are the same factors as previously seen in Eq. (2.56). The partial amplitudes can now be written as,

$$A_n(1, \sigma(1), \ldots, \sigma(n-2), n) = \sum_i \frac{M_{i,1\sigma n} n_i}{d_i}, \tag{3.15}$$

in terms of the propagator matrix.

It can be shown that the CK duality continues to hold, and that the propagator matrix has similar properties when there are massive particles present in the amplitude [50]. The relations for the kinematic numerators and hence the propagator matrix are the same as for the gluon case with one slight difference in the latter. The denominator, d_i, now must include the mass of the propagator of the massive scalars. If we choose to express the denominators in terms of the momenta[4], the case with gluons and massive scalars will be identical as the masses cancel. Hence, the propagator matrices have the same form and will lead to the same fundamental BCJ relation,

$$\sum_{a=3}^{m} \Big(\sum_{b=a}^{m} 2p_2 \cdot p_b \Big) A_n(1, 3, \ldots, a-1, 2, a, \ldots, m) = 0, \tag{3.16}$$

where particle 1 and n denote the massive scalar. If we want to express the fundamental BCJ relation in terms of the Mandelstam variables, we just insert the mass again,

$$\sum_{a=3}^{m} \Big(-m^2 + \sum_{b=a}^{m} s_{2b} \Big) A_n(1, 3, \ldots, a-1, 2, a, \ldots, n) = 0. \tag{3.17}$$

As an example, the BCJ relations at four point is,

$$(s_{12} - m^2) A_4(1234) = (s_{13} - m^2) A_4(1324). \tag{3.18}$$

Before we move on to the modern scattering amplitude approach, we will briefly go into a tangent to discuss the kinematic algebra of the numerators in the HEFT theory and a slightly different double copy construction using this perspective.

3.3 A kinematic algebra and HEFT

In this subsection, we would like to briefly discus a tangent in order to address some caveats and an alternative method of finding a double copy of certain theories. We will not revisit this subsection in the rest of the thesis.

Remember that we have defined the numerators of the amplitude in such a way that we are summing only over cubic vertices. This is a very specific choice where, for example, quartic vertices are also contained in the numerators with a different pole structure. Notably, it is not the only way to write the numerators. For some theories, there might be different ways of defining the numerators to make certain properties manifest such as general gauge invariance of the numerators or compactness. This can be done as long as the total amplitude is not changed. Note that not all numerators are BCJ numerators, which means that they can not be used to

[4] E.g. $s_{12} - m^2 = 2p_1 \cdot p_2$.

find the double copy in that form. In the case of the HEFT amplitudes found in Section 2.5 this is indeed a problem as sub leading terms in the heavy mass expansion are necessary for the CK duality to be manifest.

It can be approached differently, which is exactly what was done in [25]. This way of representing the BCJ double copy is based on an underlying kinematic algebra of the numerators specific to HEFT [54–56] and is shown to be massively useful to simplify loop diagrams and to calculate classical observables of binary black holes [26].

3.3.1 Gauge-invariant numerators and kinematic algebra

The previously discussed numerators are not the only way to construct amplitudes. There is a way to generate n-particle numerators from algebraic structures. This method has a few advantages over the 'traditional' double copy construction. For example, the numerators are manifestly gauge invariant and unique (given the generation mechanism), and they are sum only over a subset of the cubic diagrams. As a result, the expressions for the amplitudes will turn out to be much more compact compared to the 'traditional' construction. This particularly simplifies loop diagrams, which is one of the necessary components to study gravitational interactions of inspiraling binary systems.

This construction of the numerators is based on a fusion product between heavy mass currents [25]. The currents, inspired by tensor currents from QCD, are denoted by $J_{a_1\otimes a_2\otimes\cdots\otimes a_r}$ where a_i are either momenta or polarization vectors. For example, in HEFT, $J_a = mv\cdot a$. The tensor currents are generators of an underlying kinematic algebra and are required to satisfy a Clifford algebra and on shell conditions for external particles as in [25, 55]. The currents satisfy a fusion rule, which has the following form,

$$J_X \star J_Y = \sum_Z F^Z_{XY} J_Z, \tag{3.19}$$

where F^Z_{XY} are coefficients that depend on the kinematics. The fusion rule determines the decomposition of a tensor product of two group representations as a direct sum of irreducible representations. The coefficients can be found by building a general ansatz based on the dimensionality of the problem. Then, using the color-kinematics duality, the coefficients can be found. In this way, there is no general description and the amplitudes of the single copy need to be known to be able to find the coefficients. More recently, after realizing the uniqueness of these HEFT numerators, it was found that the coefficients of the fusion rule can be fixed completely by a quasi-shuffle Hopf algebra [56].

The fusion rule generates 'pre-numerators' for n-2 massless particles,

$$\mathcal{N}_n(23\cdots n-1, v) := J_{\varepsilon_2} \star J_{\varepsilon_3} \star \cdots \star J_{\varepsilon_{n-1}}. \tag{3.20}$$

To find the full numerators and the resulting single and double copies, it is useful to define the notion of ordered and unordered nested commutators as in [25]. These are relevant for color ordered (single copy) and graviton (double copy) amplitudes, respectively. A simple example is for n = 5, which will have the set $\{2, 3, 4\}$ as 1 and 5 are fixed. If the ordering is kept, there are two possibilities to construct nested commutators: $[[2, 3], 4]$ and $[2, [3, 4]]$. The unordered commutators have three possibilities at n=5: $[[2, 3], 4]$, $[[2, 4], 3]$, and $[[3, 4], 2]$. Any other option would only differ by a minus sign.

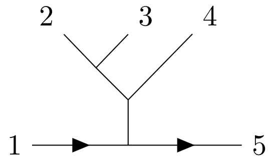
With this, we can build the amplitudes by summing over the nested commutators, denoted by Γ. This corresponds to summing over a subset of all the cubic diagrams which are connected to the heavy ingoing and outgoing particles by a single vertex. This is in contrast to the previous approach, where all diagrams are decomposed in terms of half-ladder diagrams. The amplitudes are,

$$A_n^{\text{YM HEFT}}(12\cdots n) = \sum_{\Gamma \in \text{ordered commutators}\{2,3,\ldots,n-1\}} \frac{\mathcal{N}(\Gamma, v)}{d_\Gamma}, \tag{3.21}$$

$$A_n^{\text{GR HEFT}}(12\cdots n) = \sum_{\Gamma \in \text{non-ordered commutators}\{2,3,\ldots,n-1\}} \frac{[\mathcal{N}(\Gamma, v)]^2}{d_\Gamma}. \tag{3.22}$$

The commutators that are summed over, and hence also the numerators, have the nice property that they correspond one-to-one with a cubic diagram. The propagators d_Γ can also be read of from the corresponding diagram trivially. For example, for the following BCJ numerator,

$$\mathcal{N}_5([[2,3],4], v) \quad \longleftrightarrow \tag{3.23}$$



,

the corresponding propagator would be $d_{[[2,3],4]} = s_{23}s_{234}$.

One (but not unique) way of generating the 'real' numerators from pre-numerators is using the operator $\mathbb{L}(2, 3, 4, \ldots, n-1)$ [25, 57, 58], which is defined as

$$\mathbb{L}(i_1, i_2, \ldots, i_r) := [\mathbb{I} - \mathbb{P}_{(i_2 i_1)}][\mathbb{I} - \mathbb{P}_{(i_3 i_2 i_1)}] \cdots [\mathbb{I} - \mathbb{P}_{(i_r \ldots i_2 i_1)}], \tag{3.24}$$

where $\mathbb{I}$ is the identity and $\mathbb{P}_{i_1 i_2 \ldots i_m}$ are cyclic permutations. At six point for example, the left nested commutator can be generated from the pre-numerator as follows:

$$\mathcal{N}_6([[[2,3],4],5], v) := \mathbb{L}(2,3,4,5) \circ \mathcal{N}_6(2345, v), \tag{3.25}$$

where, for example,

$$\mathbb{I} \circ \mathcal{N}_6(2345, v) = \mathcal{N}_6(2345, v), \tag{3.26}$$

$$\mathbb{P}_{(432)} \circ \mathcal{N}_6(2345, v) = \mathcal{N}_6(4235, v). \tag{3.27}$$

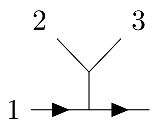
With this, the full numerators can be generated.

Finally, as an example of this construction we show the four point amplitude of HEFT using this novel double copy method as done in [25]. The four point pre-numerator is found to be,

$$\mathcal{N}_4(23, v) = 2\left(\frac{s_{23} v \cdot \epsilon_3}{4 v \cdot p_2} J_{\epsilon_2} - \frac{1}{2} J_{\epsilon_2 \otimes \epsilon_3 \otimes p_2} + \epsilon_3 \cdot p_2 J_{\epsilon_2}\right). \tag{3.28}$$

The full numerator is then found to be,

$$\mathcal{N}_4([2,3], v) =$$



$$= \mathbb{L}(2,3) \circ \mathcal{N}_4(23, v) = 2m \frac{v_\mu F_2^{\mu\nu} F_{3,\nu\rho} v^\rho}{v \cdot p_3}. \tag{3.29}$$

Here $F_i^{\mu\nu} = p_i^\mu \epsilon_i^\nu - \epsilon_i^\mu p_i^\nu$ and some rewriting has been done. Now using equation (3.21) we can straightforwardly find the four point Yang-Mills and GR amplitude of HEFT,

$$A_4^{\text{YM HEFT}} = \frac{\mathcal{N}_4([2,3], v)}{s_{23}}, \tag{3.30}$$

$$A_4^{\text{GR HEFT}} = \frac{[\mathcal{N}_4([2,3], v)]^2}{s_{23}}. \tag{3.31}$$

This process continuous analogously for higher point amplitudes. A more in depth treatment and the applications to loop amplitudes for calculations of gravitational wave observables is found in [25, 26, 59].

4 Modern Ampliutde Methods and the Amplitude Bootstrap

The methods we have considered thus far follow the "traditional" approach of first writing down some theory using the action. There are a number of implicit and explicit fundamental physical principles in this approach. Then using the action, we can ultimately derive the amplitudes using the perturbative expansion.

Inevitably these methods lead to some trouble. There are several reasons for this, but the most important issue is that the number of graphs to consider grows factorially with multiplicity. At 3 or 4 point it is easy to somewhat manageable to write down the amplitudes, at much higher multiplicities it becomes very difficult and lengthy, even for computers. Another way to derive amplitudes is through a *bootstrap*.

The basic idea of bootstrapping amplitudes is to write down the most general ansatz based on what type of terms could occur in the amplitude. Then, by using a number of physical principles such as symmetries and other constraints, we can constrain the free parameters in the ansatz. The most important physical principles that we use in this bootstrap approach are dimensional analysis, Lorentz invariance, Factorization, Gauge invariance, and the kinematics of the interaction. We will briefly introduce all of these in this section. It turns out that for many theories, the amplitudes can be fully determined this way. Additionally, by requiring that the numerators of the amplitudes satisfy the kinematic Jacobi identities, we can make the CK duality manifest.

Another important concept in the modern scattering amplitudes approach are on-shell recursion relations. This allows us to systematically build up higher point amplitudes from lower point amplitudes. Some theories can be entirely constructed recursively from three point amplitudes.

In this section we will first discuss the bootstrap approach and specifically the fundamental physical principles that underlie it. Then on-shell recursion relations are discussed holistically. The amplitude bootstrap and to a lesser extent the recursion relations will be put in practice by deriving the three and four point amplitudes for massless gluons. Ultimately we are interested in the case of gluons coupled to massive scalars. We will derive the 3, 4, and 5 point amplitude by roughly following the calculations of [34] for the lowest "mass dimensions"[5]. We will refer to the mass dimension as *powercounting* in this thesis.

As the amplitude ansatz and the algebra becomes lengthy from 4 point gluons and 5 point amplitudes with heavy scalars onwards, we utilize symbolic algebra software to perform the calculations, in this case SymPy [60]. We will discuss whether the bootstrap can be used for HEFT amplitudes specifically, and in Section 5, we will extend this to higher powercounting by including α' corrections. This will lead us towards answers to our research questions.

The issues that are posed with the methods from before are even more problematic when considering loop diagrams. The amplitude bootstrap in combination with unitarity methods can also be used to solve loop level amplitudes, however, this is beyond the scope of this thesis.

4.1 The amplitude bootstrap

The bootstrap approach is a rather flexible method. This is meant in the sense that we can use some 'known' properties of theories to constrain amplitudes, but we can also start as agnostic as possible and let the physical principles guide us.

To illustrate this, we will consider the first principle, dimensional analysis. We can consider the powercounting of an amplitude by thinking of the Lagrangian and simply count the number of derivatives per vertex and divide it by the powers of momentum due to the propagators. For

[5]Note that with mass dimension in this context we mean the power of momentum regardless whether we consider massive or massless momenta.

the four point YM amplitude, there is a single derivative for each three point vertex and a single propagator. Hence, the total powercounting of the amplitude is 0. Alternatively, we can just try a number of different options for the powercounting (i.e. different amount of momenta) and see if anything recognizable comes out. At three point amplitude, there are three polarization vectors hence we need an odd number of momenta and at least one. It would be interesting to simply try to construct amplitudes of a powercounting of 1, 3, 5, etc. and see what we get. In this section we will consider only the lowest powercounting and in Section 5 we will consider higher powercounting as they turn out to correspond to theories with extra derivatives.

This brings us to the second principle, Lorentz invariance. As we would like our amplitudes to be independent of the reference frame, the amplitudes need to be Lorentz invariant. This is easily done by using only Lorentz scalars in the ansatz for the amplitude. Hence, the amplitude will consist of three different types of building blocks,

$$\varepsilon_i \cdot \varepsilon_j, \qquad p_i \cdot \varepsilon_j, \qquad p_i \cdot p_j. \tag{4.1}$$

This explains why the three point example in the previous paragraph was required to have an odd number of momenta, as otherwise there would be no way to contract all the Lorentz indices.

A key ingredient to physical amplitudes is locality. Tree level four point amplitudes, for example, have poles that look like $\frac{1}{p^2}$ (i.e. $\frac{1}{s}$ for the s-channel), which indicates the propagation of some intermediate particle [61]. Double poles such as $\frac{1}{s^2}$ are not allowed. A fundamental property of quantum theories is that amplitudes should factorize. This means that if we tune the external momenta such that the internal momentum goes on-shell, the amplitude factorizes into two lower point amplitudes that look like two distinct processes. As an example, consider a four point amplitude in case of a massless propagator. Assume we take a specific channel, this can be chosen as the s-channel which has the pole $\frac{1}{s}$. Then, in the limit where $s \to 0$, the amplitude factorizes into two three point amplitudes,

$$\lim_{s \to 0} sA_4 = A_3 A_3. \tag{4.2}$$

If there would be a double pole, the amplitude would diverge at the factorization limit which is not allowed. This kind of factorization can be extended to higher point amplitudes and to other poles.

Lastly, we need the on-shell amplitude to be both gauge invariant, conserve momentum, and obey the on-shell conditions. With this, we can construct a *minimal kinematic basis* for the amplitude ansatz. This is a basis of Lorentz invariant objects that is reduced to an independent basis. Although this may sound obvious, it is very important to stay in this basis. If the basis is overcomplete, we “leave” the on-shell “surface” and we would no longer be able to sensibly constrain the ansatz.

Before we start applying the bootstrap approach, we will first discuss the on-shell recursion relations.

4.2 On-shell recursion relations

There are sophisticated methods that can be used to “automate” the factorizations called on-shell methods [61]. These are recursion relations that are used to build higher-point on shell amplitudes from lower point on shell amplitudes [42]. If all tree-level amplitudes can be defined recursively in terms of a finite number of lower-point ‘seed’ amplitudes, the theory is said to be ‘on-shell constructible’ [62]. This subsection contains the general formulation of the on-shell recursion relations.

In contrast to off-shell recursion, on-shell recursion relations are formulated only in terms of gauge-invariant objects. The basic idea is to apply a complex deformation to the external momenta in the on-shell amplitudes. Then, using methods from complex analysis, we can derive the relations between higher and lower point amplitudes. The most famous construction of recursion relations are the Britto, Cachazo, Feng, and Witten (BCFW) [63, 64] recursion relations. This, however, restricts to d=4 dimensions and restricts to a specific kinematic basis. We will discuss a slightly more general construction as in [42]. This will be done for the massless case, however, the massive case will be similar with appropriately modified propagators and sums over states [65].

To begin, we first consider the basic conditions for the kinematics, $p_i^2 = 0$ and momentum conservation. We will now introduce a complex vector r_i^μ with the following properties,

$$\sum_{i=1}^{n} r_i^\mu = 0, \qquad r_i \cdot r_j = 0 \quad \forall\ i, j, \qquad r_i \cdot p_i = 0. \tag{4.3}$$

With this we can define n shifted momenta,

$$\tilde{p}_i^\mu = p_i^\mu + z r_i^\mu, z \in \mathbb{C}, \tag{4.4}$$

where z parameterizes the shift of the kinematic configuration. Note that some external legs can be unshifted in case r_i^μ is vanishing for a particular leg.

The momentum shifts imply a deformation of the amplitude,

$$A \to \tilde{A}(z). \tag{4.5}$$

The unshifted amplitude is obtained by setting $z = 0$, $A = \tilde{A}(z = 0)$. Due to Eq. (4.3), momentum conservation holds for the shifted momenta. Additionally, the shifted momenta are also on shell. This implies that the deformed amplitude, $\tilde{A}(z)$, is also on-shell for all z, and hence, completely physical.

The recursion relations are derived by considering the analytic properties of the deformed amplitude. To study this, we consider a subset, I, consisting of at least two (unshifted) momenta and at most $n-2$ momenta, $\{p_i\}_{i\in I}$. If we define $P_I^\mu = \sum_{i\in I} p_i^\mu$, we can write the square of the shifted version as,

$$\tilde{P}_I^2 = P_I^2 + 2z P_I \cdot R_I, \tag{4.6}$$

where $R_I = \sum_{i\in I} r_i$. Eq. (4.6) is linear in z as $r_i^2 = 0$. We can note that $\tilde{P}_I^2$ is on shell (vanishes) when, $z = z_I = \frac{-P_I^2}{2P_I\cdot R_I}$. Hence, we can write,

$$\tilde{P}_I^2 = \frac{-P_I^2}{z_I}(z - z_I). \tag{4.7}$$

The analytic structures of $\tilde{A}(z)$ is fairly simple. Poles can only come from the poles of the shifted propagators, $\frac{1}{\tilde{P}_I^2}$, which has a simple pole at $z = z_I$. Note that generally $z_I \neq 0$ as P_I^2 is a sum of a subset of all momentum and can not vanish off-shell. Due to locality we can not have propagators with higher powers. There could be different poles, but they will not be located at the same z in the complex plane and neither at the origin.

Now let us consider $\frac{\tilde{A}(z)}{z}$. Trivially, if we integrate a contour around the pole at $z = 0$, we get the original amplitude $A(z = 0)$. Using Cauchy's residue theorem and deforming the contour such that all other poles are included gives us the amplitude in terms of residues,

$$A = \frac{1}{2\pi i}\oint_{z=0} \frac{\tilde{A}(z)}{z} dz = -\sum_I \mathrm{Res}\Big(\frac{\tilde{A}(z)}{z}, z = z_I\Big) + B_\infty. \tag{4.8}$$

Here B_∞ is a boundary contribution at infinity. Only for very special theories which have specific symmetries this term will vanish, we are interested in the theories that have this property. Therefore, we will assume it vanishes for the amplitudes we consider and not go into further detail of this term in this thesis. We know that the shifted amplitude, and particularly $\tilde{P}^2(z_I)$, is on shell. Hence, we can write the amplitude as a factorization of two lower point amplitudes,

$$\lim_{z\to z_I} \tilde{P_I}^2(z)\tilde{A}(z) = \tilde{A_L}(z_I)\tilde{A_R}(z_I). \tag{4.9}$$

Where L and R stand for the left and right side of the factorization. With the definition of the residue of a simple pole we have,

$$\text{Res}(\frac{\tilde{A}(z)}{z}, z = z_I) = \lim_{z\to z_I} (z - z_I)\frac{\tilde{A}(z)}{z}. \tag{4.10}$$

Substituting Eq. (4.7) & (4.9) and solving the limit we get,

$$\text{Res}(\frac{\tilde{A}(z)}{z}, z = z_I) = -\tilde{A}_L(z_I)\frac{1}{P_I^2}\tilde{A}_R(z_I), \tag{4.11}$$

where $\frac{1}{P_I^2}$ is the unshifted propagator.

The final recursion relation is then,

$$A = \sum_I \sum_{\text{states}} \tilde{A}_L(z_I)\frac{1}{P_I^2}\tilde{A}_R(z_I), \tag{4.12}$$

where the first sum is over all factorization channels and the second sum is over all states that can be exchanged in the propagator (e.g. polarization states of gluons). Remember that this is only valid when the boundary term vanishes.

To conclude this section, we note that recursion relations offer a sophisticated method for constructing complex amplitudes leveraging analytic properties, factorization, and symmetry. Conceptually, we explore the structures of the amplitude through momentum shifts and use the resulting insights to recursively build up the full scattering amplitude. This approach simplifies calculations in a way that aligns with modern scattering amplitude methods that emphasize physical principles and symmetries.

4.3 Color-kinematic bootstrap for massless gluons

The focus of this and the following section is to derive amplitudes for various different interactions. We choose to do this using the bootstrap approach and to a lesser extent the recursion relations of the last subsection.

The basis of a bootstrap method lies in finding a suitable ansatz for the kinematics of the amplitude. This ansatz is then constrained by symmetries, factorization, and additional relations from the CK duality if applicable. As discussed in Subsection 4.1, the ansatz must satisfy a number of properties to successfully bootstrap the amplitude. Each term in the ansatz must contain each external polarization vector exactly once. The powercounting of the numerator must be correct, e.g. for pure Yang-Mills amplitudes, the numerator has one momentum for each 3 point vertex, thus, it has a powercounting of $n - 2$ as the F^2 term in the Lagrangian has a single derivative for the self interaction. Lastly, we want to express the ansatz in terms of a minimal basis of Mandelstam variables to be able to constrain the free parameters. The minimal basis will have $\frac{1}{2}n(n - 3)$ independent Mandelstam variables for an n-point amplitude. This basis can be constructed by using on shell conditions and momentum conservation.

The bootstrap starts with the three point contribution as this is the simplest amplitude. The three point amplitudes are also required to constrain higher point amplitudes with Factorization. For gluon amplitudes, the three point amplitude is, up to the coupling constant, completely fixed by the powercounting and the antisymmetry property. Note that, unless stated otherwise, we will only consider the color ordered amplitudes in this section.

4.3.1 Three point tree level amplitudes

We start with the pure gluon amplitude. It is required to have each external polarization vector appear exactly once in each term if we want the amplitude to be gauge invariant. Additionally, we consider a powercounting of one and hence, each term contains one momentum p_i. The structure of terms in the amplitude at three point is $(\varepsilon \cdot \varepsilon)(p \cdot \varepsilon)$ where $(\varepsilon \cdot \varepsilon)$ has three different possibilities, and $(p \cdot \varepsilon)$ has only one possibility due to momentum conservation and on-shell conditions. Hence, the most general ansatz should have three terms,

$$A_{3,\mathrm{YM}}(123) = a_1(\varepsilon_1 \cdot \varepsilon_2)(p_2 \cdot \varepsilon_3) + a_2(\varepsilon_2 \cdot \varepsilon_3)(p_3 \cdot \varepsilon_1) + a_3(\varepsilon_1 \cdot \varepsilon_3)(p_3 \cdot \varepsilon_2), \tag{4.13}$$

where a_i are free parameters that we assign to each term. We can constrain the free parameters by first considering the antisymmetry of the amplitude under the exchange of the two gluons, $A(123) = -A(132)$ and $A(123) = -A(213)$. This fixes the coefficients to be $a_1 = a_2 = -a_3$.

There is one overall free parameter left, which can be taken to be the coupling constant,g, with some normalization. We can choose to absorb the coupling constant in the definition of the amplitude, which is taken to be implicit.

The final amplitude is,

$$A_{\mathrm{YM}}(123) = (\varepsilon_1 \cdot \varepsilon_2)(p_2 \cdot \varepsilon_3) + (\varepsilon_2 \cdot \varepsilon_3)(p_3 \cdot \varepsilon_1) - (\varepsilon_1 \cdot \varepsilon_3)(p_3 \cdot \varepsilon_2). \tag{4.14}$$

Note that this is indeed an antisymmetric object. In case we would use the full amplitude instead of the color ordered amplitude, we would have to multiply it with the structure constant, f^{abc}, to make it consistent. With this, it is identical to an on shell Yang-Mills amplitude.

An alternative approach to constraining the three point amplitude is to use gauge invariance. Interestingly, the property that the full amplitude needs to have an antisymmetric factor (the structure constant) then becomes a consequence. As this is the only non-zero amplitude possible using this setup, this actually proves that Yang-Mills is the only consistent theory that can be constructed for vectors with this powercounting.

4.3.2 Four point tree level amplitudes

The ansatz for a four point vector amplitude is constructed similarly as the three point amplitude. A slight difference for amplitudes beyond three point is that instead of writing down the ansatz for the amplitude, we write down the ansatz for a numerator of the amplitude. This can in turn be related to the other numerators and together with the corresponding propagators, to the color ordered amplitudes. Another consequence of this is that the powercounting of the numerators will be larger than the powercounting of the amplitude by the number of propagators times two. At four point, there are two momenta in each term for the correct powercounting. This allows for terms which include Mandelstam variables, this is to be expected as the numerator terms with Mandelstams correspond to contributions from contact terms which are absorbed into the cubic numerators.

There are two different types of terms to consider, $(\varepsilon \cdot \varepsilon)(p \cdot \varepsilon)^2$ and $(\varepsilon \cdot \varepsilon)^2(p \cdot p)$. Which contribute to $6 \times 2^2 = 24$ and $3 \times 2 = 6$ terms, respectively. Hence, the most general ansatz we can write down consists of 30 terms.

There exist a number of isomorphisms of a single topology that can be used to constrain the free parameters. One such isomorphisms, which we choose to utilize, is one where we exchange gluon 1 and 2. As a result, the ansatz for this numerator will be for the s-channel (s_{12}), the relation is,

$$n_s(1234) = -n_s(2134). \tag{4.15}$$

Solving this, results in 16 constraints to the free parameters.

By permuting the legs we can construct the two other channels. If the numerators are compatible with the double copy, they should satisfy the kinematic Jacobi relations. Hence, this constraint has to be considered. As we saw before, at four point this should be automatically satisfied by the numerators, but we can use it as a constraint nonetheless.

Next up, the color ordered amplitudes that can be constructed need to be gauge invariant. This can be used to further constrain the ansatz. The color ordered amplitudes can be written as,

$$A(1234) = \frac{n_s}{s_{12}} + \frac{n_u}{s_{23}}. \tag{4.16}$$

Where the color-kinematic duality can be used to relate the s-channel and u-channel numerators as $n_u = n_s - n_t$. Here n_t is simply n_s with the gluon 2 and 3 exchanged.

From gauge invariance we know that when we replace one of the external polarizations with itself plus the corresponding momentum, the amplitude should be invariant. In other words,

$$\varepsilon_i \to \varepsilon_i + p_i, \tag{4.17}$$

hence, applying the transformation $\varepsilon_i \to p_i$ to the amplitude, the result should vanish. Solving the free parameters for $A(1234)|_{\varepsilon_3 \to p_3} = 0$, leaves us with an ansatz that is proportional to a single free parameter.

If this is the correct amplitude, it should factorize to two 3 point amplitudes in the limit where any of the Mandelstams are taken to be on shell. We will check this using the s-channel (s_{12}) on shell. Note that when factorization of two gluon amplitudes is done, one needs to sum over the gluon polarization states. The sum over states is given by the physical state projector which in our case can be simplified to [34, 66],

$$\sum_{\text{states}} \varepsilon^\mu(-p)\varepsilon^\nu(p) = -\eta^{\mu\nu}. \tag{4.18}$$

Using this in combination with the two relevant three point amplitudes we can find the last free parameter.

The final ansatz for the s-channel numerator of the four point amplitude is,

$$\begin{aligned}
n_s(1234) = &-\frac{(e_1\cdot e_2)(e_3\cdot e_4)(p_1\cdot p_2)}{2} - (e_1\cdot e_2)(e_3\cdot e_4)(p_2\cdot p_3) + (e_1\cdot e_2)(p_1\cdot e_3)(p_2\cdot e_4) \\
&- (e_1\cdot e_2)(p_1\cdot e_4)(p_2\cdot e_3) + \frac{(e_1\cdot e_3)(e_2\cdot e_4)(p_1\cdot p_2)}{2} - (e_1\cdot e_3)(p_1\cdot e_2)(p_1\cdot e_4) \\
&- (e_1\cdot e_3)(p_1\cdot e_2)(p_2\cdot e_4) - \frac{(e_1\cdot e_4)(e_2\cdot e_3)(p_1\cdot p_2)}{2} + (e_1\cdot e_4)(p_1\cdot e_2)(p_1\cdot e_3) \\
&+ (e_1\cdot e_4)(p_1\cdot e_2)(p_2\cdot e_3) + (e_2\cdot e_3)(p_1\cdot e_4)(p_2\cdot e_1) + (e_2\cdot e_3)(p_2\cdot e_1)(p_2\cdot e_4) \\
&- (e_2\cdot e_4)(p_1\cdot e_3)(p_2\cdot e_1) - (e_2\cdot e_4)(p_2\cdot e_1)(p_2\cdot e_3) - (e_3\cdot e_4)(p_1\cdot e_2)(p_3\cdot e_1) \\
&+ (e_3\cdot e_4)(p_2\cdot e_1)(p_3\cdot e_2).
\end{aligned} \tag{4.19}$$

The amplitude factorizes correctly, and the last free parameter is fixed by the factorization limit. We now start to see the power of this method. Although the algebra and final answer is still somewhat lengthy, using these Lorentz invariant objects and physical constraints is much more straightforward than using the off-shell Feynman rules.

This process fully generalizes to higher point amplitudes, however, we stop here. Our next goal is to do the same for the theory of massive scalars coupled to gluons and ultimately the HEFT expansion of such a theory.

4.4 Color-kinematic bootstrap with a massive scalar

In this section, We consider the bootstrap method for amplitudes with massive scalars coupled to Yang-Mills gluons. Just like before, we start building up from the three point amplitude, which is also fixed by the powercounting and antisymmetry. After doing the four and five point amplitudes, we will discuss the leading order expansion of the amplitude in the heavy scalar mass in the next subsection.

4.4.1 Three point tree level amplitudes with a massive scalar

The three point amplitude that includes a pair of massive scalars, which corresponds to the vertex from Eq. (2.33), is the only amplitude we need to consider. This time, we have one polarization as there is only a single gluon. The powercounting we consider is 1, which corresponds to the single derivative in the vertex. This results in one single possible term,

$$A_3(123) = (p_1 \cdot \varepsilon_2), \tag{4.20}$$

where particle 1 and 3 are the massive scalars and particle 2 is the gluon. As before, the coupling constant is implicit. Note that this amplitude is manifestly antisymmetric due to conservation of momentum as $p_1 \cdot \varepsilon_2 = -p_3 \cdot \varepsilon_2$. This is all there is to consider at three point with this powercounting.

4.4.2 Four point tree level amplitudes with a massive scalar

There are three different amplitudes to consider at four point. One pure gluon amplitude, which was done in the previous section, one with a single pair of external massive scalars, and one with two pairs of external massive scalars. All of these can be constructed using similar methods, we are now interested in the amplitude with a single pair of massive scalars.

For this amplitude we have to consider two topologies. Previously, in Section 2.3, we discussed them as the s-channel and u-channel graphs. They differ by the fact that the former has a massive scalar propagator, which we note by m, and the latter has a massless gluon propagator, which we note by ml. We have also seen that the two topologies can be related to each other by color identities. The kinematic Jacobi identities dictate a similar relation of the numerators,

$$n^{ml}(1234) = n^m(1234) - n^m(1324), \tag{4.21}$$

where the ordering 1234 corresponds to the canonical ordering where 1 and 4 are the massive scalars. As the two topologies are related, we can focus on a single ansatz for the numerator of a massive topology. Using the relation of Eq. (4.21) to construct the amplitude makes the amplitude manifestly color dual.

As the graphs have two external gluon legs, each term in the ansatz will consist of two polarization vectors and products of the momenta. Note that again, each polarization vector must appear once in each term and the powercounting is 2. Hence, the structure of each term in the ansatz can be written either as $(\varepsilon \cdot \varepsilon)(p \cdot p)$ or $(p \cdot \varepsilon)^2$. This has $1 \times 3 = 3$ and $2^2 = 4$ terms, respectively. This results in a total of 7 terms in the ansatz,

$$\begin{aligned} n^m(1234) = &\Big(a_1(p_1 \cdot p_2) + a_2(p_2 \cdot p_3) + a_3 m^2\Big)(\varepsilon_2 \cdot \varepsilon_3) + a_4(p_1 \cdot \varepsilon_2)(p_2 \cdot \varepsilon_3) \\ &+ a_5(p_1 \cdot \varepsilon_2)(p_1 \cdot \varepsilon_3) + a_6(p_1 \cdot \varepsilon_3)(p_3 \cdot \varepsilon_2) + a_7(p_2 \cdot \varepsilon_3)(p_3 \cdot \varepsilon_2). \end{aligned} \tag{4.22}$$

We can further constrain the free parameters by considering symmetries in the form of isomorphisms of the graphs. Isomorphisms of graphs are transformations of the external legs that leave the amplitude invariant, up to a minus sign. For this particular graph, there are four isomorphisms that could be considered,

$$\begin{aligned} n^{ml}(1234) &= -n^{ml}(1324), \quad n^{ml}(1234) = -n^{ml}(4231) \\ n^{ml}(1234) &= n^{ml}(4321), \quad n^{m}(1234) = n^{m}(4321) \end{aligned} \tag{4.23}$$

As it turns out, we need only one of these for the first constraints to the free parameters. Plugging in the ansatz into the last isomorphism as it is suitable for the numerator of the massive channel and performing some straightforward algebra gives us the following relations,

$$a_4 = a_5 - a_6. \tag{4.24}$$

The other parameters are still entirely unconstrained.

Other physical constraints that can be used are gauge invariance and factorization. We consider gauge invariance first. To do this we construct the color ordered amplitude and require that the amplitude is invariant under the transformation $\varepsilon_3 \to p_3$, as before with the gluons. One of the color ordered amplitudes can be constructed as,

$$A(1234) = \frac{n^m(1234)}{s_{12} - m^2} + \frac{n^{ml}(1234)}{s_{23}}. \tag{4.25}$$

Applying gauge invariance, $A(1234)|_{\varepsilon_3 \to p_3} = 0$, to this amplitude gives us the following constraint,

$$\left\{a_1 = 0,\ a_2 = -\frac{a_5}{2},\ a_3 = -a_7,\ a_6 = 0\right\}, \tag{4.26}$$

which leaves the numerator with two undetermined parameters,

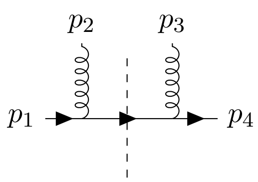
$$\begin{aligned} n^m(1234) = &-\frac{(e_2 \cdot e_3)(p_1 \cdot p_2)a_5}{2} - (e_2 \cdot e_3)(p_2 \cdot p_3)a_7 \\ &+ (p_1 \cdot e_2)(p_1 \cdot e_3)a_5 + (p_1 \cdot e_2)(p_2 \cdot e_3)a_5 + (p_2 \cdot e_3)(p_3 \cdot e_2)a_7. \end{aligned} \tag{4.27}$$

Finally, the four point amplitude should factorize properly into two three point amplitudes. This can also be used to constrain the ansatz. We will consider the factorization limit in the s_{12} channel,

$$\lim_{s_{12} \to m^2} (s_{12} - m^2) A_4 = A_3 A_3. \tag{4.28}$$

Note that there are different factorization channels that can be considered, but it turns out just one is needed to constrain this ansatz fully at this point.

The one we consider factorizes on the massive scalar propagator,



, (4.29)

where the dashed line represents the cut on which we are considering the factorization. The limit where $s_{12} \to m^2$ is equivalent to $p_1 \cdot p_2 \to 0$ as $s_{12} - m^2 = 2p_1 \cdot p_2$. This means that the first term with a_5 vanishes, luckily a_5 occurs in other terms too. As our cut is on the massive scalar, we do not have to sum over gluon states and can simply multiply the two relevant 3 point amplitudes together to find two terms which lead to $a_5 = 1$, and $a_7 = 0$.

The final numerator is given by,

$$n^m(1234) = -\frac{1}{2}(p_1 \cdot p_2)(\varepsilon_2 \cdot \varepsilon_3) + (p_1 \cdot \varepsilon_2)\big((p_2 \cdot \varepsilon_3) + (p_1 \cdot \varepsilon_3)\big) \tag{4.30}$$

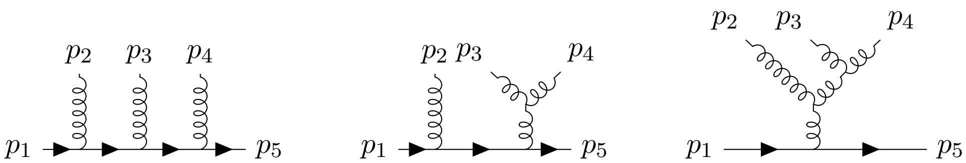
This can be written as a color ordered amplitude as,

$$\begin{aligned} A(1234) &= \frac{n^m(1234)}{s_{12} - m^2} + \frac{n^{ml}(1234)}{s_{23}} \\ &= \frac{(p_1 \cdot e_2)\left((p_1 \cdot e_3) + (p_2 \cdot e_3)\right) - \frac{1}{2}(s_{12} - m^2)(e_2 \cdot e_3)}{s_{12} - m^2} \\ &- \frac{(e_2 \cdot e_3)(p_1 \cdot p_2) - (p_1 \cdot e_2)(p_2 \cdot e_3) + (p_1 \cdot e_3)(p_3 \cdot e_2)}{s_{23}}. \end{aligned} \tag{4.31}$$

The terms proportional to the Mandelstams of the corresponding channels are the contact terms. If we rewrite this to be in terms of only the momenta p_1 and p_4, we find that it agrees with Eq. (2.58). The factorization channel on s_{23} can still be done as a check to see if the resulting amplitude correctly factorizes, which it turns out it does.

4.4.3 Five point tree level amplitudes with a massive scalar

The five point amplitude with three external gluons starts to become a bit more complicated but follows a very similar procedure as before. There are now three completely unique topologies to consider as shown in equation (4.32). They are distinguished by the types of propagators that are present in the graph. The first graph has two massive propagators denoted by (mm), the second has one massive and one massless propagator denoted by (mml) (or (mlm) if the ordering is reversed), and the third has two massless propagators denoted by $(mlml)$. The three unique topologies are,



. (4.32)

A color ordered amplitude can be build from this by considering each numerator corresponding to a topology and the corresponding propagator in the form of Mandelstams. Note that the topology with massless propagators both have two different variations which both need to be accounted for separately, hence, the amplitude has five terms.

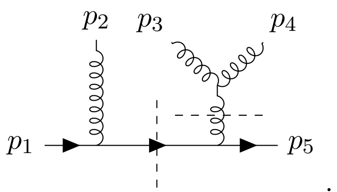
From kinematic Jacobi relations we can relate the different topologies to each other,

$$\begin{aligned} n^{mlml}(12345) &= n^{mml}(12345) - n^{mlm}(13425), \\ n^{mml}(12345) &= n^{mm}(12345) - n^{mm}(12435). \end{aligned} \tag{4.33}$$

This means that, analogously to the four point case, we only need to consider the numerator of the topology corresponding to two massive propagators, n^{mm}, as the others can be related by permuting legs.

As there will be three external gluons and a powercounting of 3, which totals to $(\varepsilon^3 p^3)$. The structure of the terms in the numerator ansatz can be written as either $(\varepsilon \cdot \varepsilon)(\varepsilon \cdot p)(p \cdot p)$ or $(p \cdot \varepsilon)^3$. The former has $3 \times 3 \times 6 = 54$ possibilities (including $p_1^2 = m^2$) and the later $3^3 = 27$ terms. Hence, the ansatz will consist of 81 terms.

As before we use an isomorphism of the graph, $n^{mm}(12345) = -n^{mm}(54321)$, and gauge invariance to constrain the ansatz. For factorization there are a number of different channels to consider, both with a single and double cut. We consider the channel corresponding to the second topology in equation (4.32), with two cuts, on s_{12} and s_{34},



. (4.34)

A double cut means that our factorization looks slightly different. It is a logical extension from a single cut. In this case,

$$\lim_{s_{12}\to m^2} \lim_{s_{34}\to 0} (s_{12} - m^2)(s_{34}) A_5 = \sum_{\text{states}} A_3(12k) A_3(k l^s 5) A_{3,YM}(l^{\bar{s}} 34), \tag{4.35}$$

where the sum is over the gluon polarization states.

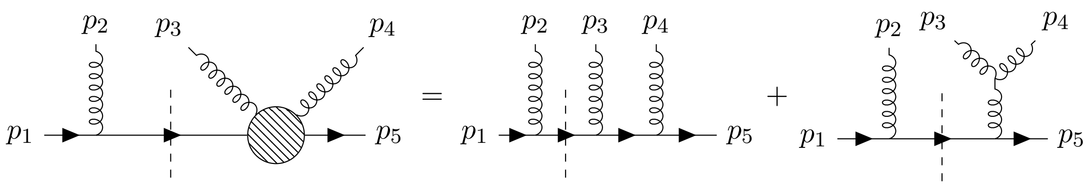
After imposing this factorization channel, there are three free parameters left in the numerator. However, if we build the color ordered amplitude, the three parameters do not appear in the amplitude. This means the remaining free parameters are gauge parameters and do not affect any physical observables.

The final numerator for the five point amplitude with one massive scalar can be found equation (6.1) of appendix A. The free gauge parameters are all set to 0 for readability. Finally, the partial amplitude can be constructed by using the kinematic Jacobi relations in Eq. (4.33),

$$\begin{aligned} A(12345) =& \frac{n^{mm}(12345)}{(s_{12} - m^2)(s_{123} - m^2)} + \frac{n^{mml}(12345)}{(s_{12} - m^2)s_{34}} \\ &+ \frac{n^{mlml}(12345)}{s_{234}s_{34}} + \frac{n^{mlm}(12345)}{s_{23}(s_{123} - m^2)} + \frac{n^{mlml}(14325)}{s_{23}s_{234}}. \end{aligned} \tag{4.36}$$

The actual expression for this is too long for this work, but can be found in the IPython notebook on the GitHub repository of the author (https://github.com/Messier104/Bootstrap_MSc_Thesis.git).

To check whether the resulting amplitude is correct, other factorization channels can be considered to see if the amplitude factorizes correctly. This is done for some other topologies, explicitly so in the notebook for the channel on the pole $s_{12} \to m^2$. The single cut factorization channel is diagrammatically a sum over two channels,



. (4.37)

The amplitude factorizes correctly.

This approach can be continued to higher point amplitudes, but in this work we stop at five point. For six point, the computational costs should still be manageable, from seven point onwards we suspect the shear number of equations to solve starts to become more troublesome. In addition, higher point computations might require some extra bookkeeping to include the right topologies and their factorization channels. The topologies necessary for the six point amplitude are shown in Eq. (6.6) of appendix C.

Now that we have all the amplitudes for a general mass we can start to consider the leading order in a HEFT expansion. It would also be interesting to consider whether we could bootstrap the HEFT amplitudes directly using the methods of this section instead of first finding the amplitudes for general mass and then expanding in the mass. The later might be not as efficient as all the sub-leading terms will not be needed in the expanded amplitude.

4.5 HEFT amplitudes and HEFT bootstrap

In Section 2.5, we introduced the HEFT expansion of the massive scalar theory. Now that we have the amplitudes for the massive scalar theory, we can start to consider the leading order in the HEFT expansion. In the minimal basis used in this section we have eliminated p_n from each amplitude, which means that only p_1 is a massive scalar. This means the mass scaling[6] can be directly read off from the numerators and propagators of the amplitudes.

We can be very brief about the three point amplitude. There is only one term which scales as m. Hence, the given amplitude is also the HEFT amplitude and this is all there is to consider.

In case of the four point amplitude of Eq. (4.31), there are two topologies. The s_{12} channel has a massive channel of which the propagator scales as $1/m$ whilst the s_{23} (u) channel has a massless channel which scales as m^0. The leading order numerator of the former scales as m^2 and the latter as m, hence, the partial amplitudes scales as m. The result is given by,

$$A_{\text{HEFT}}(1234) = \frac{(p_1 \cdot e_2)(p_1 \cdot e_3)}{-m^2 + s_{12}} + \frac{-(e_2 \cdot e_3)(p_1 \cdot p_2) + (p_1 \cdot e_2)(p_2 \cdot e_3) - (p_1 \cdot e_3)(p_3 \cdot e_2)}{s_{23}}. \tag{4.38}$$

This agrees with the literature [25]. Note that this seems to be inconsistent with the color kinematics duality, as the numerator should satisfy Eq. (4.21). If we take the numerator of the s_{12} channel and the s_{13} channel, we find that difference vanishes. This is not surprising as the mass scalings of the numerators are different. This can be resolved by including the next to leading order terms for the numerator of the massive channels. With that, the CK duality is satisfied.

The same approach can be taken for the five point amplitude. We are able to show the leading order in the inverse scalar mass expansion of the amplitude in equation (6.2) of appendix A. Checking this with the five point amplitude in [25] we find that the two results agree. This concludes the 5pt amplitude with a massive scalar.

4.6 What about a direct HEFT bootstrap?

A question that could be asked is whether the bootstrap process could be made more efficient for the calculation of HEFT amplitudes directly as many terms can be neglected when expanding for large masses. There is an issue, however, as the massive propagators affect the mass scaling, the numerators of the different topologies will scale differently. In other words, different mass scaling in numerators will be dominant for different topologies. The setup of this BCJ bootstrap

[6] With mass scaling we mean the number of massive momenta (p_1 or p_n) in the amplitude

is to start with the numerator with the most massive propagators as the other numerators can be related to this one through the BCJ relations by permuting the gluons.

The leading order of each term in the amplitude corresponding to the different topologies will have the same mass scaling. Though, as mentioned, the numerators will scale differently, and as we try to expand each numerator, we see that if we were to keep only the first term of the numerator with the most massive propagators (n^{mm} for five point), we cannot construct the other numerators (n^{mml} and beyond) any longer. In fact, the leading order of the most massive numerator will cancel entirely when constructing the next numerator. This pattern continuous for the n^{mlml} numerators if the next to leading order is included. As a result, we will have to keep all terms with in the initial numerator up to the scaling necessary to generate the leading order numerator corresponding to the topology with only massless propagators.

This means that we can very slightly simplify the bootstrap by eliminating any terms with a lower mass scaling than the leading order term of the 'least massive' numerator. In the list below we enumerate what we need to consider by looking at the mass scaling of the numerators.

- The 'strongest' potential mass scaling is equal to the powercounting (this might change when including higher derivative theories, we will come back to this in Section 5).
- The 'weakest' potential mass scaling is always equal to the actual strongest mass scaling minus the maximum number of massive propagators. For pure Yang-Mills this mass scaling is always equal to 1 (again, this can change for higher derivative theories).
- For a HEFT amplitude, any term below the 'weakest' potential mass scaling can be removed from the ansatz. At low multiplicity, this does not really simplify much, but at higher multiplicity and powercounting, this can be a significant simplification in computation time and simplicity. However, one needs to be very careful to use all the new relations that arise from HEFT amplitudes to ensure a minimal basis.

All in all, a HEFT bootstrap using this double copy construction should be possible, but the gains are minimal. Essentially, we have answered our first two research questions with varying degrees of success in this section. To move on to the other two questions, we need to consider higher derivative corrections to the theories considered so far. This will be done at the end of the next section, as we will first further introduce higher derivative corrections and its implications.

5 The UV and Higher Derivative Corrections to Yang-Mills

In Section 3, we discussed the BCJ double copy construction. The introduction very briefly mentioned there is another formulation of the KLT relations. The KLT relations were derived in string theory relating tree level amplitudes of open and closed strings [19] which are used to describe forces of gauge theory and gravity respectively. These relations from string theory explicitly depend on the only parameter of string theory [67], α', or the inverse string tension. The low energy limit of string theory is the field theory limit, where $\alpha' \to 0$. If one would like to improve the UV behavior of a quantum field theory without fully committing to string theory, one can consider a perturbative low energy expansion of string theory in this parameter α'.

A particular interest in the context of the low energy string expansion is whether this expansion also preserve the structures of the double copy. Additionally, would this hold for just for an extended Yang-Mills theory or can other particles such as the massive scalars we have considered so far also be added. Later in this section we will come back to this question, first we will briefly discuss the higher derivative additions in the context of the Lagrangian and the Feynman perspective.

5.1 Higher-derivative corrections to Yang-Mills and the Lagrangian

If we desire to add terms with more derivative to the Yang-Mills Lagrangian (F^2), we must still require that properties such as locality and gauge invariance are preserved. The lowest order higher-derivative term that can consistently be added to the Yang-Mills Lagrangian is the F^3 operator [68]. The Lagrangian that includes this term and the next order of the perterubative expansion, F^4, can be represented by,

$$\mathcal{L}_{\mathrm{YM}+\alpha' F^3+\alpha'^2 F^4} = \frac{1}{4} F^a_{\mu\nu} F^{\mu\nu,a} + \alpha' \frac{2}{3} f^{abc} F^{\nu,a}_{\mu} F^{\lambda,b}_{\nu} F^{\mu,c}_{\lambda} + \alpha'^2 \frac{1}{4} [F_{\mu\nu}, F_{\lambda\rho}][F^{\mu\nu}, F^{\lambda\rho}]. \tag{5.1}$$

Note that α' is necessarily a dimensionfull parameter as the action should be dimensionless. As there are now three field strength tensors in the Lagrangian, the three point amplitude will have three derivatives and hence a mass dimension of 3. Hence, the mass dimension of α' is -2.

To get more of a feeling for the higher-derivative terms, we will first discuss the Feynman rules for the vertices of such a theory. After that we will go back to the on shell methods we have been considering, the relation to the color kinematics duality, and some interesting properties that arise from this. Finally, we will discuss the amplitudes of the theory and how they can be derived using the bootstrap, and how this fits together with HEFT amplitudes.

5.1.1 Vertex rules from Feynman rules

In principle, the Feynman rules can be derived directly from the modified Lagrangian with the use of textbook methods. This can be a tedious process, but it is a good way to understand the theory. In the Lagrangian of Eq. (5.1), the vertices involving the adjoint particles are now modified with extra terms. Where the three point vertices for just Yang-Mills (F^2) have a single derivate and hence scale as p^1, the new vertex at order α' has 3 derivatives and hence scale as p^3. The four point vertices at order α'^2 have four derivatives and hence scale as p^4.

It is possible to construct other interactions with extra derivatives of the field strength [69, 70], these will lead to contact terms which can carry additional derivatives leading to yet higher powers of p. Note that for scalar QCD, the vertices coupling scalars to gluons are not modified by either of these types of higher-derivative terms as they are independent of the Field strength tensor.

The Feynman rule of the three point gluon vertex at α' order can be derived similarly as we previously derived the Feynman rule for the three point vertex in Yang-Mills. The terms in the Lagrangian responsible for the three point vertex at α' order is given by,

$$\mathcal{L}_{3\text{pt}} \sim g\alpha' f^{abc}(\partial^\mu A^a_\nu)(\partial^\nu A^b_\rho)(\partial^\rho A^c_\mu) \tag{5.2}$$

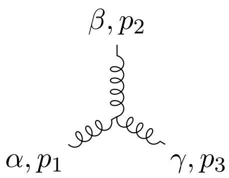
Now going to momentum space gives,

$$\sim (p_1^\mu \varepsilon_{1\nu} - p_{1\nu}\varepsilon_1^\mu)(p_2^\nu \varepsilon_{2\rho} - p_{2\rho}\varepsilon_2^\nu)(p_3^\rho \varepsilon_{3\mu} - p_{3\mu}\varepsilon_3^\rho). \tag{5.3}$$

As before, to recover the Feynman vertex rule, we have to take a derivative with respect to the external polarizations.

$$\text{3pt. vertex} \sim g\alpha' f^{abc} \frac{\partial\left[(p_1^\mu \varepsilon_{1\nu} - p_{1\nu}\varepsilon_1^\mu)(p_2^\nu \varepsilon_{2\rho} - p_{2\rho}\varepsilon_2^\nu)(p_3^\rho \varepsilon_{3\mu} - p_{3\mu}\varepsilon_3^\rho)\right]}{\partial\varepsilon_{1\alpha}\partial\varepsilon_{2\beta}\partial\varepsilon_{3\gamma}}, \tag{5.4}$$

which results in,



$$= g\alpha' f^{abc}\big[p_1^\gamma p_2^\alpha p_3^\beta - p_1^\gamma p_2 \cdot p_3 g^{\alpha\beta} - p_1 \cdot p_2 p_3^\beta g^{\alpha\gamma} + p_1^\beta p_2 \cdot p_3 g^{\alpha\gamma} \tag{5.5}$$
$$- p_1 \cdot p_3 p_2^\alpha g^{\beta\gamma} + p_1 \cdot p_3 p_2^\gamma g^{\alpha\beta} + p_1 \cdot p_2 p_3^\alpha g^{\beta\gamma} - p_1^\beta p_2^\gamma p_3^\alpha\big].$$

This vertex can be used to calculate amplitudes of higher point amplitudes.

One of the higher point vertices we could consider is that of the massive scalars coupled to gluons. The vertices coupling scalars to gluons are not modified as they are not derived from the field strength tensor[7]. Hence, there is only a single diagram contributing to the partial amplitude at $\mathcal{O}(\alpha')$. Only the u-channel of this amplitude will be present as this includes a three point gluon vertex. If the same calculation as in Eq. (2.36) is done, but with our new Feynman rule, we find that,

$$A(1_i, 2, 3, 4_j)_{\text{u-channel}} = \\ ig^2\alpha' f^{abc} T^c_{ij} \frac{(p_2 \cdot p_3 \varepsilon_2 \cdot \varepsilon_3 (p_2 - p_3)\cdot(p_1 + p_4) - p_2 \cdot \varepsilon_3 p_3 \cdot \varepsilon_2 (p_2 - p_3)\cdot(p_1 + p_4))}{s_{23}}. \tag{5.6}$$

After some algebra, the result agrees with the $\mathcal{O}(\alpha')$ contribution in equation 2.4 of Chen et al. [71].

For the α' corrections to the five point amplitude, the contributions from the four point vertex at $\mathcal{O}(\alpha')$ need to be calculated. This can be derived similarly to the three point vertex. As there are more graphs to be considered in the 5 point amplitude, we will need to take more correction terms into account. An easy way to figure out which term we need to take into account is to look at the powercounting of the amplitude. This comes down to simply counting the number of derivatives in the vertices of the graphs and subtracting p^2 for each propagator. However, at this point the number of graphs and contact terms quickly increase, and we will resort to using the bootstrap again in the following sections.

[7] In the Section 5.4 we will show that without knowing about the theory, interaction vertices with a power-counting larger than 1 can not exist for massive scalars coupled to a gluon.

5.2 Color kinematic duality and higher derivatives corrections

As discussed in [28], an open question of the double copy construction is whether it should be viewed as a technical trick or a fundamental principle of some mechanism. To study this, so-called *double copy consistency* is used to constrain an ansatz in the bootstrap approach, essentially what we have been doing in the previous sections. Consistency with the double copy can be required by demanding that the color-kinematics duality is satisfied and that the amplitudes correctly factorize.

At $\mathcal{O}(\alpha'^0)$, or pure Yang-Mills, the theory is, as we have seen in the last chapter, double copy consistent. However, once we start adding higher-derivative terms, we can not say the same for amplitudes of arbitrary multiplicity. Adding the $Tr(F^3)$ theory to the Yang-Mills theory, is consistent at three point [28, 68].

At four point, the theory is consistent at $\mathcal{O}(\alpha')$, diagrammatically speaking this corresponds to one of the vertices being a Yang-Mills vertex and the other being a $Tr(F^3)$ vertex. Inevitably, at four point specifically, we can have the situation where both vertices are $Tr(F^3)$ vertices. This term must be considered at four point and corresponds to $\mathcal{O}(\alpha'^2)$. However, this term is not compatible with the color-kinematics duality on its own. To make this order consistent, a 'contact term' must be added to the amplitude coming from $Tr(F^4)$ theory which is also at $\mathcal{O}(\alpha'^2)$ [68]. Carrasco, Lewandowski, and Pavao [28] postulated that this pattern continues and that there is no finite number of local operators that can be added to YM+$Tr(F^3)$ to make it double copy consistent.

5.2.1 Infinite tower of higher-derivative terms

To explain this, we will discuss the example of the factorization of the five point amplitude into a product of a three and four point amplitude at $\mathcal{O}(\alpha'^{n+1})$ as in [28]. The five point amplitude is factorized as,

$$\lim_{s_{45}\to 0} s_{45}A(12345) = \sum_s A_4(123l^s)A_3(-l^{\bar{s}}45) \tag{5.7}$$

The three point amplitude is fully described by the sum of the 3 point amplitudes of the Yang-Mills theory and the $Tr(F^3)$ theory, $A_3 = A_3^{\text{YM}} + \alpha' A_3^{F^3}$. This implies that, when performing the factorization with a 4 point amplitude at $\mathcal{O}(\alpha'^n)$, there is a non-vanishing five point contribution at $\mathcal{O}(\alpha'^{n+1})$. Carrasco, Lewandowski, and Pavao argued that this term can not be double copy consistent and requires the addition of an additional operator at $\mathcal{O}(\alpha'^{n+1})$. Inevitably, this term must also be combined with the TrF^3 contribution to the three point amplitude leading to a non-vanishing five point amplitude at $\mathcal{O}(\alpha'^{n+2})$. This pattern continuous with an infinite tower of higher-derivative operators that must be included to make the theory double copy consistent. It has been proven up to $\mathcal{O}(\alpha'^5)$.

The numerators compatible with the double copy construction can be found using the bootstrap approach for arbitrary order α'. At four point, Carrasco, Lewandowski, and Pavao constructed the double copy compatible (dcc) s-channel numerator up to α'^4 as follows,

$$\begin{aligned} n_s^{\text{dcc}} &= n_s^{\text{YM}} + \alpha' n_s^{YM+F^3} + \alpha'^2 n_s^{(F^3)^2+F^4} + \alpha'^3\Big[a_3\big(n_s^{D^2F^4} + \sigma_2 n_s^{\text{YM}+F^3}\big) + a_{3,\text{YM}}\sigma_3 n_s^{\text{YM}}\Big] \\ &+ \alpha'^4\Big[a_{4,1}\big(n_s^{(DF)_1^4} + \sigma_2 n_s^{(F^3)^2+F^4}\big) + a_{4,2}n_s^{(DF)_2^4} + a_{4,F^3}\sigma_3 n_s^{YM+F^3}\Big] + \mathcal{O}(\alpha'^5). \end{aligned} \tag{5.8}$$

Here a_i are unconstrained ansatz parameters. $\sigma_2 = (s^2 + t^2 + u^2)/8$ and $\sigma_3 = (stu)/8$ are kinematic building blocks that are invariant under permutations. Factorization of the four point amplitude into two three point amplitudes is required. As the only possible contribution to the three point amplitude comes from Yang-Mills and $Tr(F^3)$, all contributions beyond $\mathcal{O}(\alpha'^2)$ must be from local terms (e.g. contact terms with additional derivatives of the field strength tensor).

Imposing factorization constrains from the five point amplitude, the authors found that the parameters a_i can be fixed or non-trivially related to each other spanning different orders of α'. Additionally, when utilizing the freedom to set $a_{3,\text{YM}} = 0$, it was illustrated that the four and five point amplitudes match the expansion of 'B-Amplitudes' at $\mathcal{O}(\alpha'^4)$ [28, 72]. These amplitudes are described by a $(DF)^2$ + YM theory deformed by a massive gauge theory where the mass scale is determined by $\frac{1}{\alpha'}$ [69, 73]. This resummation into a known theory are suggestive enough to postulate that this infinite tower of higher-derivative operators can teach us something about the double copy compatibility of the UV.

In the rest of this chapter we will investigate the behavior of the higher-derivative corrections when a Yang-Mills theory is coupled to massive scalars. The bootstrap methods used so far can be used to derive the numerators and amplitudes by simply increasing the powercounting of the terms in the ansatz. We will also investigate the leading order HEFT terms and the double copy compatibility of the theory. Lastly we will address the research questions and discuss the outlook of this thesis.

5.3 Gluon amplitudes from on-shell bootstrap

Approaching the higher derivative corrections with the bootstrap method is again a two-way street. Either we look at the Lagrangian, find how many derivatives we need to add, and then use the same principles applied in Section 4, but with a larger powercounting. Alternatively, we can just start to add extra derivatives (increase powercounting) and see what kinds of theories we can construct.

We need the same properties of gauge invariance, Jacobi identities, factorization, etc. to hold. Only now we have more derivatives and hence a higher powercounting. Consequently, there are additional factorization channels to consider. We will start by considering the three point amplitude and go from there.

5.3.1 Three point tree level amplitudes with higher derivative corrections

At three point we need to contract the three gluon polarization vectors. As discussed in Section 4, we need an odd number of momenta to be able to build Lorentz invariant structures.

The next powercounting of the three gluon amplitude beyond YM is at $\mathcal{O}(\alpha')$ and has three momenta. Due to momentum conservation, any contraction of the momenta will vanish at three point, $p_i \cdot p_j = 0$. Hence, the structure must be of the form $(p \cdot \varepsilon)^3$. There can be only one term of this form in a minimal basis, and it is given by,

$$A_{3,\alpha'}(123) = \alpha'(p_1 \cdot e_2)(p_1 \cdot e_3)(p_2 \cdot e_1) \tag{5.9}$$

This is already fully constrained up to an overall factor which is set to the coupling.

Next we consider a powercounting of 5, which is at $\mathcal{O}(\alpha'^2)$. However, as we only have three polarizations, we would need to contract two momenta. As shown in the last paragraph, this will vanish. Hence, the three point amplitude at $\mathcal{O}(\alpha'^2)$ is,

$$A_{3,\alpha'^2}(123) = 0. \tag{5.10}$$

The same will be true for any powercounting beyond this. This agrees with what was expected from the Lagrangian as beyond $\mathcal{O}(\alpha')$, there are no three point vertices, only contact terms of four point and up.

5.3.2 Four point tree level amplitudes with higher derivative corrections

At $\mathcal{O}(\alpha')$, the structure of a numerator ansatz has a powercounting of 4 and hence has three different possible structures. $(\varepsilon\cdot\varepsilon)^2(p\cdot p)^2$, $(\varepsilon\cdot\varepsilon)(p\cdot p)(p\cdot\varepsilon)^2$, and $(p\cdot\varepsilon)^4$. These have $3\times 3=9$, $6\times 2\times 2^2=48$, and $2^4=16$ terms, respectively, bringing us to a total of 73 terms in the ansatz.

This can then be constrained similarly to the 4 point Yang-Mills amplitude, where we first consider isomorphisms, reversal symmetry, the kinematic Jacobi relations, and gauge invariance of the amplitude. To fully constrain this amplitude, we need to consider a factorization channel. This can be done properly by considering all the possible 'channels' within the s-channel. To illustrate this, the factorization of the s-channels looks as follows,

$$\lim_{s_{12}\to 0} s_{12}A(1234) = \sum_{\text{s}}(A_{3L,YM}(12k^s)A_{3R,\alpha'}(k^{\bar{s}}34) + A_{3L,\alpha'}(12k^s)A_{3R,YM}(k^{\bar{s}}34)). \tag{5.11}$$

This fully constrains the numerator up to two gauge parameters that are not present in any physical observables (and hence, the partial amplitude). It is still somewhat lengthy to write in full glory and can be found in equation (6.3) of Appendix B.

For the $\mathcal{O}(\alpha'^2)$, the powercounting is p^6 and has similar structures as the $\mathcal{O}(\alpha')$. Namely, $(\varepsilon\cdot\varepsilon)^2(p\cdot p)^3$, $(\varepsilon\cdot\varepsilon)(p\cdot p)^2(p\cdot\varepsilon)^2$, and $(p\cdot\varepsilon)^4(p\cdot p)$. These have $3\times 4=12$, $6\times 3\times 2^2=72$, and $2^4\times 2=32$ terms, respectively, bringing us toe a total of 116 terms in the ansatz.

The result of the bootstrap is that the $\mathcal{O}(\alpha'^2)$ amplitude is fully constrained up to 4 gauge parameters. An interesting pattern to note is that with increasing powercounting, the number of contract terms increases rapidly. This can be explained by the fact that when increasing the number of higher derivative terms, we add more possible contact interactions that contribute to these contact terms. The color ordered amplitude can be found in equation (6.4) of Appendix B.

5.4 Gluons coupled to massive scalars with higher derivate corrections

As our goal is to find HEFT amplitudes with higher-derivative corrections, their double copy compatibility, and the relation to the pure adjoint theory of Section 5.2, we need to consider the amplitudes of gluons coupled to massive scalars. Let us first consider just general massive scalars before we consider the HEFT expansion amplitudes. The process is a combination of the last subsection and Section 4.

It was noted before that vertices involving the massive scalars are not modified by the higher-derivative terms. This can be shown very straightforwardly by considering the next powercounting possibility of the three point vertex. The three point vertex would have a powercounting of 3, and hence the only possible structure is $(p\cdot\varepsilon)(p\cdot p)$. At three point we know what due to momentum conservation we have, $p_i\cdot p_j=0$. Hence, the amplitude vanishes and such a theory can not exist. Consequently, only some topologies contribute to the perturbative expansion of the amplitude in α' beyond leading order.

For $\mathcal{O}(\alpha')$, only the contributions that involve at least one massless propagator are modified. Similarly, at order $\mathcal{O}(\alpha'^2)$, only the contributions that involve at least two massless propagators are modified. For $\mathcal{O}(\alpha'^3)$ and beyond, we get additional terms coming from terms involving derivatives of the field strength tensor that need to be considered for double copy consistency. As we have already shown there is nothing left to do at three point, we will start at four point.

5.4.1 Four point tree level amplitudes with higher derivative corrections

At four point, the amplitude at $\mathcal{O}(\alpha')$ with one massive scalar corresponds to the channel with a single massless propagator, the u-channel. The powercounting of the numerator is 4. Hence, the possible structures are $(\varepsilon \cdot \varepsilon)(p \cdot p)^2$ and $(p \cdot \varepsilon)^2(p \cdot p)$. These have $1 \times 6 = 6$ and $2^2 \times 3 = 12$ terms, respectively, bringing us to a total of 18 terms in the ansatz,

$$
\begin{aligned}
n^{ml}(1234) = \alpha' \Big(&(e_2 \cdot e_3)(p_1 \cdot p_1)^2 a_1 + (e_2 \cdot e_3)(p_1 \cdot p_1)(p_1 \cdot p_2) a_2 + (e_2 \cdot e_3)(p_1 \cdot p_1)(p_2 \cdot p_3) a_3 \\
&+ (e_2 \cdot e_3)(p_1 \cdot p_2)^2 a_4 + (e_2 \cdot e_3)(p_1 \cdot p_2)(p_2 \cdot p_3) a_5 + (e_2 \cdot e_3)(p_2 \cdot p_3)^2 a_6 \\
&+ (p_1 \cdot e_2)(p_1 \cdot e_3)(p_1 \cdot p_1) a_7 + (p_1 \cdot e_2)(p_1 \cdot e_3)(p_1 \cdot p_2) a_8 + (p_1 \cdot e_2)(p_1 \cdot e_3)(p_2 \cdot p_3) a_9 \\
&+ (p_1 \cdot e_2)(p_1 \cdot p_1)(p_2 \cdot e_3) a_{10} + (p_1 \cdot e_2)(p_1 \cdot p_2)(p_2 \cdot e_3) a_{11} + (p_1 \cdot e_2)(p_2 \cdot e_3)(p_2 \cdot p_3) a_{12} \\
&+ (p_1 \cdot e_3)(p_1 \cdot p_1)(p_3 \cdot e_2) a_{13} + (p_1 \cdot e_3)(p_1 \cdot p_2)(p_3 \cdot e_2) a_{14} + (p_1 \cdot e_3)(p_2 \cdot p_3)(p_3 \cdot e_2) a_{15} \\
&+ (p_1 \cdot p_1)(p_2 \cdot e_3)(p_3 \cdot e_2) a_{16} + (p_1 \cdot p_2)(p_2 \cdot e_3)(p_3 \cdot e_2) a_{17} + (p_2 \cdot e_3)(p_2 \cdot p_3)(p_3 \cdot e_2) a_{18} \Big)
\end{aligned}
\tag{5.12}
$$

As before, symmetry, gauge invariance, and factorization constrain this ansatz fully leading to the following amplitude,

$$
A(1234) = \alpha' \Big(\frac{\frac{1}{2}(e_2 \cdot e_3)(p_1 \cdot p_2) s_{23} + \frac{1}{8}(e_2 \cdot e_3) s_{23}^2 - \frac{1}{4}(p_2 \cdot e_3)(p_3 \cdot e_2) s_{23} - (p_1 \cdot p_2)(p_2 \cdot e_3)(p_3 \cdot e_2)}{s_{23}} \Big). \tag{5.13}
$$

This amplitude only consists of the u-channel and contact contributions. Note that any products of momenta which do not have a massive particle (i.e. p_1) are written in terms of their corresponding Mandelstam variable.

If we write the color ordered amplitudes $A(1234)$ and $A(1324)$ in multi-peripheral form we find the following,

$$
A(1234) = \frac{-n^{ml}(1234)}{s_{12} - m^2}, \qquad A(1324) = \frac{+n^{ml}(1324)}{s_{13} - m^2}. \tag{5.14}
$$

Low and behold, permuting gluon 2 and 3 in $n^{ml}(1234)$ in combination with the relation $p_1 \cdot p_3 = -p_1 \cdot p_2 - \frac{1}{2} s_{23}$ tells us that $n^{ml}(1324) = -n^{ml}(1234)$. These amplitudes satisfy the BCJ relation for four point amplitudes with a massive scalars (Eq. (3.18)). Hence, the four point amplitudes at $\mathcal{O}(\alpha')$ are double copy consistent.

It might be quite unsurprising as the pure gluon three point amplitude also is double copy consistent at $\mathcal{O}(\alpha')$. At four point, the only gluon only vertices that can occur are three point. It is suspected that this pattern continuous to higher orders of α', but this is yet to be proven. Generally, to show relations that were done at n-point in [28] for the gluon amplitudes, we need to consider $n + 1$ point when including massive scalars.

The leading order in the HEFT expansion is found to be,

$$
A_{\text{HEFT}}(1234) = \alpha' \Big(\frac{\frac{1}{2}(e_2 \cdot e_3)(p_1 \cdot p_2) s_{23} - (p_1 \cdot p_2)(p_2 \cdot e_3)(p_3 \cdot e_2)}{s_{23}} \Big). \tag{5.15}
$$

After some algebra, it can be shown to agree with the leading order of the HEFT expansion in [71]. Similar to the amplitude for general mass, it can be easily shown that this amplitude satisfies the BCJ relation for four point amplitudes. Remember that for HEFT, the relation, $p_1 \cdot p_3 = -p_1 \cdot p_2$, can be used.

5.4.2 Five point tree level amplitudes with higher derivative corrections

The five point amplitudes can be derived similarly to the four point version with higher derivative corrections. However, the number of ansatz terms increases drastically. At $\mathcal{O}(\alpha')$ it can be shown there are 351 terms in the ansatz, and at $\mathcal{O}(\alpha'^2)$ 1071 terms. In this subsection we will just show the result of $\mathcal{O}(\alpha')$ case, though their approach is identical.

After constraining the ansatz, it turns out that there are 12 different gauge parameters in the numerator. The amplitude will not dependent on this as they will all cancel out. We set all of these gauge parameters to zero such that we can observe what types of terms are present. The resulting numerator for the single massless propagator case is rather manageable,

$$\begin{aligned}
n^{mml}(12345) = \alpha'\Big(&-\frac{1}{2}(e_3\cdot e_4)(p_1\cdot e_2)(p_1\cdot p_2)(p_3\cdot p_4) - (e_3\cdot e_4)(p_1\cdot e_2)(p_1\cdot p_3)(p_3\cdot p_4)\\
&- (e_3\cdot e_4)(p_1\cdot e_2)(p_2\cdot p_3)(p_3\cdot p_4) - \frac{1}{2}(e_3\cdot e_4)(p_1\cdot e_2)(p_3\cdot p_4)^2\\
&+ \frac{1}{2}(p_1\cdot e_2)(p_1\cdot p_2)(p_3\cdot e_4)(p_4\cdot e_3) + (p_1\cdot e_2)(p_1\cdot p_3)(p_3\cdot e_4)(p_4\cdot e_3)\\
&+ (p_1\cdot e_2)(p_2\cdot p_3)(p_3\cdot e_4)(p_4\cdot e_3) + \frac{1}{2}(p_1\cdot e_2)(p_3\cdot e_4)(p_3\cdot p_4)(p_4\cdot e_3)\Big).
\end{aligned} \tag{5.16}$$

From this, the partial amplitude can be build analogously to the five point amplitude of two massive scalars and three gluons. Only the term with two massive propagators does not appear as there are no three point gluon vertices present and hence no $\mathcal{O}(\alpha')$ contributions. The partial amplitude is too long to show here. The leading order HEFT amplitude is more manageable and is show in equation (6.5) of appendix B.

It is not easy to see whether the BCJ relations are satisfied for the five point amplitudes. More calculations are need to show this. We suspect that the same patterns of [28] will follow for the theory coupled to massive scalars. Or in other words, we suspect that at five point, the theory is only consistent once $\mathcal{O}(\alpha'^2)$ terms are included and from there on, an infinite tower of higher derivative corrections is build.

5.5 Discussion and future work

The results thus far show that the theory of gluons coupled to massive scalars with higher-derivative corrections is double copy consistent at four point. At five point, amplitudes can be found at $\mathcal{O}(\alpha')$ and $\mathcal{O}(\alpha'^2)$ that are consistent in the sense that the amplitudes are gauge invariant and satisfy the color kinematics duality. Although we have not managed to explicitly check the BCJ relations for the five point amplitudes, we suspect that they will be satisfied and hence, be double copy consistent.

The bootstrap approach has been a powerful tool to derive the amplitudes of the theories. Extending this to include numerators with a larger powercounting is a natural and fairly straightforward process. This partly answers our third research question. To fully answer this question, we would like to know what the resulting amplitudes would generally look like. With the limited number of amplitudes we have generated it is rather difficult to spot any patterns common through increasing multiplicity in the amplitude. Though one very clear trend is that with an increasing powercounting, the number of contact terms in the resulting numerators grows quickly. This can likely be explained by the range of possible contact terms that are added when higher derivative corrections are included.

An interesting pattern we did notice in the resulting numerators is the following. Even though the number of massive momenta in each term, p_1 in our case, is in principle only restricted by the powercounting of the numerator, the number of massive momenta in the numerators is always $n-2$ for n-point amplitudes. Hence, the maximum number of massive momenta is identical with or without higher-derivative corrections. This seems to be due to the possible ways to contract the momentum contributions from the massive vertices. If this can be shown to be always true, the property can be used to simplify the ansatz for the numerators. Which is desirable as fewer terms in the ansatz directly translates to a decrease in computation time.

Another consequence is that the 'weakest' potential mass scaling discussed in Section 4.5 is identical for any higher derivative order. Some attempts to run a HEFT bootstrap with higher derivative corrections have been made. Although this was not fully explored, it was noticed that the ansatz term could be reduced by a few hundred terms in case of the five point $\mathcal{O}(\alpha'^2)$ amplitude. As a result the computation time could be reduced by about a factor of two at this order. Though not a huge improvement, the computational cost do not scale linearly with the number of terms in the ansatz, and hence any reduction becomes more and more beneficial with increasing powercounting and multiplicity.

To answer our last question. Do we observe the infinite tower of higher-derivative corrections as discussed in [28] for the theory of gluons coupled to massive scalars? The short answer is that for our results it is inconclusive. As the properties of the n-point gluon amplitudes tend to show up at $(n+1)$-point when we consider gluons coupled to massive scalars, we suspect that the results of Carrasco et al. can be seen only when we consider the six point amplitudes. In principle, we have all the tools to do this, hence we suggest that this is a natural next step in a future study. For the six point amplitude it would also be necessary to consider many more diagrams. The set of cubic diagrams consists of 13 diagrams and is shown in equation(6.6) of appendix C. This concludes the discussion of our final research question.

Another interesting direction to consider is the relation to a kinematic algebra. It turns out that the framework discussed in Section 3.3 fits in with higher-derivative corrections [71]. The structures from the $DF^2 + \text{YM}$ theory can also be cast in this framework [74]. The α' dependencies are contained in the mapping between the generators and kinematic functions. It must be noted that the Hopf algebra construction is purely for HEFT amplitudes.

6 Conclusion

In this work, we have explored various aspects of scattering amplitudes first introducting the fundamental concepts of gauge theories and theories involving massive scalars in the Section 2. The large mass limit (HEFT) of these theories was also introduced which is a central theme in this thesis.

An extensive introduction to the color kinematics duality and the double copy followed in Section 3 explaining how gauge and gravity theories can be related. We have seen that a number of important relations between partial amplitudes can be derived from the color-kinematics duality. The double copy has many intriguing applications in physics as, for example, it allows us to study gravity from a new perspective. Though the fundamental relations and novel perspectives on scattering amplitudes themselves already make this a fascinating topic that ought to be studied and could lead to a further understanding of the most fundamental components of the universe. It was also discussed that there can be different double copy constructions such as the construction for HEFT amplitudes based on the Hopf algebra of [25].

In Section 4 we discussed on-shell approach of calculating scattering amplitudes. With this the important concept of recursion relations and Factorization. This allows to construct amplitudes recursively through the bootstrap method, which, if the kinematic Jacobi relations are imposed, can make the resulting numerators manifestly color-dual.

With this we could address the first research question:

> *Is it possible to (re)produce amplitudes of Yang-Mills coupled to massive scalars using a bootstrap approach? Both for general mass and the large mass limit?*

To clarify, this is not a novel question but more of an exercise to get familiar with the bootstrap method. We reproduced four point gluon amplitudes and five point gluon amplitudes coupled to massive scalars, including the large mass limit. The machinery of this bootstrap could then be extended to study higher derivative corrections necessary to answer the other research question.

The second research question was:

> *What would a 'HEFT bootstrap' look like?*

The BCJ bootstrap we have considered in this thesis actually benefits very little from a 'direct' HEFT bootstrap. It turns out that, at least for the double copy construction we consider, the HEFT amplitudes are already very similar to the amplitudes for a general mass. The numerators that are generated and constraint need to contain (almost) all mass orders to not miss any of the crucial terms. Only if many terms would be able to be eliminated a priori, there would be a large benefit for a HEFT bootstrap. However, the BCJ bootstrap is already very efficient in the case for a relatively low multiplicity. When going to large multiplicities, and in particular when higher derivative corrections are considered, a HEFT bootstrap is worth pursuing.

The first part of the third research question has more or less been answered in the previous paragraphs.

> *Is it possible to extend the bootstrap approach to higher derivative corrections, and what would the amplitudes look like?*

The methods we have discussed are perfectly capable of handling higher derivative corrections. A difference is that the number of terms in the ansatz will increase quite significantly, and certain graph topologies which would usually need to be considered in order to come up with constraints, can be ignored.

To answer the questions about higher derivate corrections, we have introduced the topic including a remarkable result from [28] which form another key part of this thesis. For Yang-Mills extended with higher derivative corrections, it was found that an infinite tower of higher order terms is needed to satisfy the color-kinematics duality at arbitrary multiplicity as soon as we go beyond pure YM.

The final research question wishes to address the implications of this *infinite tower* when gluons couple to massive scalars.

> *Is the infinite tower of higher derivative corrections of Carrasco, Lewandowski, and Pavao preserved when the gauge theory is coupled to a massive scalar?*

We were not able to conclusively answer this question. We suspect that due to the multiplicity of the gluon vertices in the theory where gluons couple to scalars, we need at least six point amplitudes in order to conclusively constrain the higher derivative corrections. As in [28], five point amplitudes for gluon only amplitudes were required to show this. We have discussed what steps are needed to fully answer this question at the end of Section 5.

To zoom out a bit, the main goal of this thesis was to explore the color-kinematics duality and the double copy in the context of HEFT and higher derivative corrections. Particularly the HEFT amplitudes have an important role in the calculation of graviton amplitudes through the double copy with large implications for the field of gravitational wave predictions of binary black holes.

Ultimately, studying the color-kinematics duality has shed a lot of light on the underlying structures of the theories of the standard model and beyond. As there are still large open question in the fundamental theories of physics, the novel perspective offered by the modern amplitude program provides just that to tackle these questions.

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A: F^2 numerators and amplitudes

Two massive scalar three gluon numerator

$$
\begin{aligned}
n^{mm}(12345) = & \frac{(e_2 \cdot e_3)(p_1 \cdot e_4)(p_1 \cdot p_3)}{2} + \frac{(e_2 \cdot e_3)(p_1 \cdot e_4)(p_2 \cdot p_3)}{2} - \frac{3(e_2 \cdot e_3)(p_1 \cdot p_2)(p_3 \cdot e_4)}{4} \\
& + \frac{(e_2 \cdot e_3)(p_1 \cdot p_3)(p_3 \cdot e_4)}{4} + \frac{(e_2 \cdot e_3)(p_2 \cdot p_3)(p_3 \cdot e_4)}{4} - (e_2 \cdot e_4)(p_1 \cdot e_3)(p_1 \cdot p_2) \\
& - \frac{(e_2 \cdot e_4)(p_1 \cdot e_3)(p_1 \cdot p_3)}{2} - \frac{(e_2 \cdot e_4)(p_1 \cdot e_3)(p_2 \cdot p_3)}{2} - \frac{(e_2 \cdot e_4)(p_1 \cdot p_2)(p_2 \cdot e_3)}{4} \\
& - \frac{3(e_2 \cdot e_4)(p_1 \cdot p_2)(p_4 \cdot e_3)}{4} - \frac{(e_2 \cdot e_4)(p_1 \cdot p_3)(p_4 \cdot e_3)}{4} - \frac{(e_2 \cdot e_4)(p_2 \cdot p_3)(p_4 \cdot e_3)}{4} \\
& - \frac{(e_3 \cdot e_4)(p_1 \cdot e_2)(p_1 \cdot p_3)}{2} - \frac{(e_3 \cdot e_4)(p_1 \cdot e_2)(p_2 \cdot p_3)}{2} + \frac{3(e_3 \cdot e_4)(p_1 \cdot p_2)(p_3 \cdot e_2)}{4} \\
& + \frac{(e_3 \cdot e_4)(p_1 \cdot p_3)(p_3 \cdot e_2)}{2} - \frac{(e_3 \cdot e_4)(p_1 \cdot p_3)(p_4 \cdot e_2)}{2} + \frac{(e_3 \cdot e_4)(p_2 \cdot p_3)(p_3 \cdot e_2)}{2} \\
& - \frac{(e_3 \cdot e_4)(p_2 \cdot p_3)(p_4 \cdot e_2)}{2} + (p_1 \cdot e_2)(p_1 \cdot e_3)(p_1 \cdot e_4) + (p_1 \cdot e_2)(p_1 \cdot e_3)(p_2 \cdot e_4) \\
& + (p_1 \cdot e_2)(p_1 \cdot e_3)(p_3 \cdot e_4) + (p_1 \cdot e_2)(p_1 \cdot e_4)(p_2 \cdot e_3) + (p_1 \cdot e_2)(p_2 \cdot e_3)(p_2 \cdot e_4) \\
& + (p_1 \cdot e_2)(p_2 \cdot e_3)(p_3 \cdot e_4)
\end{aligned} \tag{6.1}
$$

Two massive scalar three gluon leading order HEFT amplitude

$$
\begin{aligned}
A_{\mathrm{HEFT}}(12345) = &-\frac{(e_2\cdot e_3)(p_1\cdot e_4)(p_1\cdot p_2)}{2s_{23}\left(-m^2+s_{123}\right)}+\frac{(e_2\cdot e_3)(p_1\cdot e_4)(p_1\cdot p_3)}{2s_{23}\left(-m^2+s_{123}\right)}\\
&-\frac{(e_2\cdot e_3)(p_1\cdot e_4)s_{24}}{4s_{234}s_{34}}+\frac{(e_2\cdot e_3)(p_1\cdot e_4)s_{24}}{4s_{23}s_{234}}-\frac{(e_2\cdot e_3)(p_1\cdot e_4)s_{23}}{4s_{234}s_{34}}\\
&+\frac{(e_2\cdot e_3)(p_1\cdot e_4)}{4s_{34}}-\frac{(e_2\cdot e_3)(p_1\cdot e_4)}{2s_{234}}-\frac{(e_2\cdot e_3)(p_1\cdot e_4)s_{34}}{4s_{23}s_{234}}\\
&-\frac{(e_2\cdot e_3)(p_1\cdot p_2)(p_3\cdot e_4)}{s_{234}s_{34}}-\frac{(e_2\cdot e_3)(p_1\cdot p_2)(p_3\cdot e_4)}{s_{23}s_{234}}+\frac{(e_2\cdot e_3)(p_1\cdot p_3)(p_2\cdot e_4)}{s_{23}s_{234}}\\
&-\frac{(e_2\cdot e_4)(p_1\cdot e_3)(p_1\cdot p_2)}{2s_{23}\left(-m^2+s_{123}\right)}-\frac{(e_2\cdot e_4)(p_1\cdot e_3)(p_1\cdot p_3)}{2s_{23}\left(-m^2+s_{123}\right)}+\frac{(e_2\cdot e_4)(p_1\cdot e_3)s_{24}}{4s_{234}s_{34}}\\
&+\frac{(e_2\cdot e_4)(p_1\cdot e_3)s_{24}}{4s_{23}s_{234}}+\frac{(e_2\cdot e_4)(p_1\cdot e_3)s_{23}}{4s_{234}s_{34}}-\frac{(e_2\cdot e_4)(p_1\cdot e_3)}{4s_{34}}\\
&+\frac{(e_2\cdot e_4)(p_1\cdot e_3)}{s_{234}}+\frac{(e_2\cdot e_4)(p_1\cdot e_3)s_{34}}{4s_{23}s_{234}}-\frac{(e_2\cdot e_4)(p_1\cdot p_2)(p_2\cdot e_3)}{s_{23}s_{234}}\\
&+\frac{(e_2\cdot e_4)(p_1\cdot p_2)(p_4\cdot e_3)}{s_{234}s_{34}}-\frac{(e_2\cdot e_4)(p_1\cdot p_3)(p_2\cdot e_3)}{s_{23}s_{234}}+\frac{(e_3\cdot e_4)(p_1\cdot e_2)(p_1\cdot p_2)}{2s_{23}\left(-m^2+s_{123}\right)}\\
&-\frac{(e_3\cdot e_4)(p_1\cdot e_2)(p_1\cdot p_3)}{s_{34}\left(-m^2+s_{12}\right)}+\frac{(e_3\cdot e_4)(p_1\cdot e_2)(p_1\cdot p_3)}{2s_{23}\left(-m^2+s_{123}\right)}+\frac{(e_3\cdot e_4)(p_1\cdot e_2)s_{24}}{4s_{234}s_{34}}\\
&-\frac{(e_3\cdot e_4)(p_1\cdot e_2)s_{24}}{4s_{23}s_{234}}-\frac{(e_3\cdot e_4)(p_1\cdot e_2)s_{23}}{4s_{234}s_{34}}-\frac{(e_3\cdot e_4)(p_1\cdot e_2)}{4s_{34}}\\
&-\frac{(e_3\cdot e_4)(p_1\cdot e_2)}{2s_{234}}-\frac{(e_3\cdot e_4)(p_1\cdot e_2)s_{34}}{4s_{23}s_{234}}+\frac{(e_3\cdot e_4)(p_1\cdot p_2)(p_3\cdot e_2)}{s_{234}s_{34}}\\
&+\frac{(e_3\cdot e_4)(p_1\cdot p_2)(p_3\cdot e_2)}{s_{23}s_{234}}+\frac{(e_3\cdot e_4)(p_1\cdot p_3)(p_3\cdot e_2)}{s_{234}s_{34}}+\frac{(e_3\cdot e_4)(p_1\cdot p_3)(p_3\cdot e_2)}{s_{23}s_{234}}\\
&+\frac{(e_3\cdot e_4)(p_1\cdot p_3)(p_4\cdot e_2)}{s_{234}s_{34}}+\frac{(p_1\cdot e_2)(p_1\cdot e_3)(p_1\cdot e_4)}{\left(-m^2+s_{12}\right)\left(-m^2+s_{123}\right)}+\frac{(p_1\cdot e_2)(p_1\cdot e_3)(p_3\cdot e_4)}{s_{34}\left(-m^2+s_{12}\right)}\\
&+\frac{(p_1\cdot e_2)(p_1\cdot e_4)(p_2\cdot e_3)}{s_{23}\left(-m^2+s_{123}\right)}-\frac{(p_1\cdot e_2)(p_1\cdot e_4)(p_4\cdot e_3)}{s_{34}\left(-m^2+s_{12}\right)}+\frac{(p_1\cdot e_2)(p_2\cdot e_3)(p_2\cdot e_4)}{s_{23}s_{234}}\\
&+\frac{(p_1\cdot e_2)(p_2\cdot e_3)(p_3\cdot e_4)}{s_{234}s_{34}}+\frac{(p_1\cdot e_2)(p_2\cdot e_3)(p_3\cdot e_4)}{s_{23}s_{234}}-\frac{(p_1\cdot e_2)(p_2\cdot e_4)(p_4\cdot e_3)}{s_{234}s_{34}}\\
&-\frac{(p_1\cdot e_3)(p_1\cdot e_4)(p_3\cdot e_2)}{s_{23}\left(-m^2+s_{123}\right)}-\frac{(p_1\cdot e_3)(p_2\cdot e_4)(p_3\cdot e_2)}{s_{23}s_{234}}-\frac{(p_1\cdot e_3)(p_3\cdot e_2)(p_3\cdot e_4)}{s_{234}s_{34}}\\
&-\frac{(p_1\cdot e_3)(p_3\cdot e_2)(p_3\cdot e_4)}{s_{23}s_{234}}-\frac{(p_1\cdot e_3)(p_3\cdot e_4)(p_4\cdot e_2)}{s_{234}s_{34}}-\frac{(p_1\cdot e_4)(p_2\cdot e_3)(p_4\cdot e_2)}{s_{23}s_{234}}\\
&+\frac{(p_1\cdot e_4)(p_3\cdot e_2)(p_4\cdot e_3)}{s_{234}s_{34}}+\frac{(p_1\cdot e_4)(p_3\cdot e_2)(p_4\cdot e_3)}{s_{23}s_{234}}+\frac{(p_1\cdot e_4)(p_4\cdot e_2)(p_4\cdot e_3)}{s_{234}s_{34}}
\end{aligned}
\tag{6.2}
$$

B: Higher derivate corrections

Four point gluon α'

$$
\begin{aligned}
A^{\text{gluon}}(1234) = \alpha'\Big(& \frac{(e_1 \cdot e_2)(e_3 \cdot e_4)s_{12}}{4} + \frac{(e_1 \cdot e_2)(e_3 \cdot e_4)s_{23}}{4} - \frac{(e_1 \cdot e_2)(p_1 \cdot e_3)(p_1 \cdot e_4)}{2} \\
& - \frac{(e_1 \cdot e_2)(p_1 \cdot e_3)(p_1 \cdot e_4)s_{23}}{2s_{12}} - \frac{(e_1 \cdot e_2)(p_1 \cdot e_3)(p_2 \cdot e_4)}{2} - \frac{(e_1 \cdot e_2)(p_1 \cdot e_3)(p_2 \cdot e_4)s_{23}}{2s_{12}} \\
& - \frac{(e_1 \cdot e_2)(p_1 \cdot e_4)(p_2 \cdot e_3)}{2} - \frac{(e_1 \cdot e_2)(p_1 \cdot e_4)(p_2 \cdot e_3)s_{23}}{2s_{12}} - \frac{(e_1 \cdot e_2)(p_2 \cdot e_3)(p_2 \cdot e_4)}{2} \\
& - \frac{(e_1 \cdot e_2)(p_2 \cdot e_3)(p_2 \cdot e_4)s_{23}}{2s_{12}} + \frac{(e_1 \cdot e_3)(e_2 \cdot e_4)s_{12}}{4} + \frac{(e_1 \cdot e_3)(e_2 \cdot e_4)s_{23}}{4} \\
& - \frac{(e_1 \cdot e_3)(p_1 \cdot e_2)(p_2 \cdot e_4)}{2} - \frac{(e_1 \cdot e_3)(p_2 \cdot e_4)(p_3 \cdot e_2)}{2} + \frac{(e_1 \cdot e_4)(e_2 \cdot e_3)s_{12}}{4} \\
& + \frac{(e_1 \cdot e_4)(e_2 \cdot e_3)s_{23}}{4} - \frac{(e_1 \cdot e_4)(p_2 \cdot e_3)(p_3 \cdot e_2)s_{12}}{2s_{23}} - \frac{(e_1 \cdot e_4)(p_2 \cdot e_3)(p_3 \cdot e_2)}{2} \\
& + \frac{(e_2 \cdot e_3)(p_1 \cdot e_4)(p_2 \cdot e_1)s_{12}}{2s_{23}} + \frac{(e_2 \cdot e_3)(p_1 \cdot e_4)(p_2 \cdot e_1)}{2} + \frac{(e_2 \cdot e_3)(p_1 \cdot e_4)(p_3 \cdot e_1)s_{12}}{2s_{23}} \\
& + \frac{(e_2 \cdot e_3)(p_1 \cdot e_4)(p_3 \cdot e_1)}{2} + \frac{(e_2 \cdot e_4)(p_1 \cdot e_3)(p_3 \cdot e_1)}{2} - \frac{(e_3 \cdot e_4)(p_1 \cdot e_2)(p_2 \cdot e_1)}{2} \\
& - \frac{(e_3 \cdot e_4)(p_1 \cdot e_2)(p_2 \cdot e_1)s_{23}}{2s_{12}} - \frac{(p_1 \cdot e_2)(p_1 \cdot e_3)(p_1 \cdot e_4)(p_3 \cdot e_1)}{s_{12}} + \frac{(p_1 \cdot e_2)(p_1 \cdot e_3)(p_2 \cdot e_1)(p_2 \cdot e_4)}{s_{12}} \\
& - \frac{(p_1 \cdot e_2)(p_1 \cdot e_3)(p_2 \cdot e_4)(p_3 \cdot e_1)}{s_{12}} - \frac{(p_1 \cdot e_2)(p_1 \cdot e_4)(p_2 \cdot e_1)(p_2 \cdot e_3)}{s_{23}} - \frac{(p_1 \cdot e_2)(p_1 \cdot e_4)(p_2 \cdot e_1)(p_2 \cdot e_3)}{s_{12}} \\
& - \frac{(p_1 \cdot e_2)(p_1 \cdot e_4)(p_2 \cdot e_3)(p_3 \cdot e_1)}{s_{23}} - \frac{(p_1 \cdot e_2)(p_1 \cdot e_4)(p_2 \cdot e_3)(p_3 \cdot e_1)}{s_{12}} - \frac{(p_1 \cdot e_2)(p_2 \cdot e_3)(p_2 \cdot e_4)(p_3 \cdot e_1)}{s_{12}} \\
& + \frac{(p_1 \cdot e_3)(p_1 \cdot e_4)(p_2 \cdot e_1)(p_3 \cdot e_2)}{s_{23}} + \frac{(p_1 \cdot e_3)(p_1 \cdot e_4)(p_2 \cdot e_1)(p_3 \cdot e_2)}{s_{12}} + \frac{(p_1 \cdot e_3)(p_1 \cdot e_4)(p_3 \cdot e_1)(p_3 \cdot e_2)}{s_{23}} \\
& + \frac{(p_1 \cdot e_3)(p_2 \cdot e_1)(p_2 \cdot e_4)(p_3 \cdot e_2)}{s_{12}} + \frac{(p_1 \cdot e_4)(p_2 \cdot e_1)(p_2 \cdot e_3)(p_3 \cdot e_2)}{s_{23}} + \frac{(p_1 \cdot e_4)(p_2 \cdot e_1)(p_2 \cdot e_3)(p_3 \cdot e_2)}{s_{12}} \\
& + \frac{(p_2 \cdot e_1)(p_2 \cdot e_3)(p_2 \cdot e_4)(p_3 \cdot e_2)}{s_{23}} + \frac{(p_2 \cdot e_1)(p_2 \cdot e_3)(p_2 \cdot e_4)(p_3 \cdot e_2)}{s_{12}} + \frac{(p_2 \cdot e_3)(p_2 \cdot e_4)(p_3 \cdot e_1)(p_3 \cdot e_2)}{s_{23}}\Big)
\end{aligned}
\tag{6.3}
$$

Four point gluon $\mathcal{O}(\alpha'^2)$

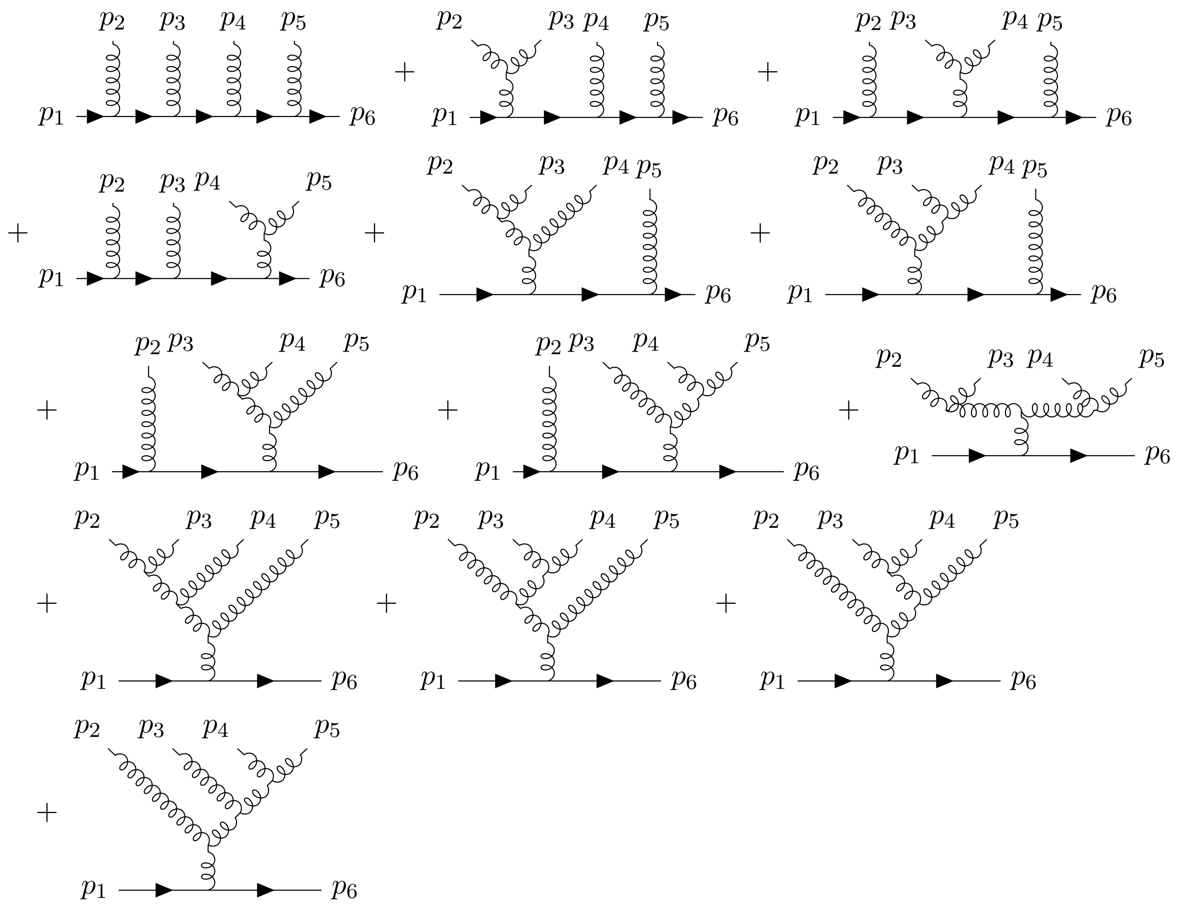
$$
\begin{aligned}
A^{\text{gluon}}(1234) = \alpha'^2\Big(&- \frac{(e_1\cdot e_2)(e_3\cdot e_4)s_{12}^2}{8} - \frac{(e_1\cdot e_2)(e_3\cdot e_4)s_{12}s_{23}}{8} + \frac{(e_1\cdot e_2)(p_1\cdot e_3)(p_1\cdot e_4)s_{12}}{4}\\
&+ \frac{(e_1\cdot e_2)(p_1\cdot e_3)(p_1\cdot e_4)s_{23}}{4} + \frac{(e_1\cdot e_2)(p_1\cdot e_3)(p_2\cdot e_4)s_{12}}{4} + \frac{(e_1\cdot e_2)(p_1\cdot e_3)(p_2\cdot e_4)s_{23}}{4}\\
&+ \frac{(e_1\cdot e_2)(p_1\cdot e_4)(p_2\cdot e_3)s_{12}}{4} + \frac{(e_1\cdot e_2)(p_1\cdot e_4)(p_2\cdot e_3)s_{23}}{4} + \frac{(e_1\cdot e_2)(p_2\cdot e_3)(p_2\cdot e_4)s_{12}}{4}\\
&+ \frac{(e_1\cdot e_3)(e_2\cdot e_4)s_{12}s_{23}}{4} + \frac{(e_1\cdot e_3)(e_2\cdot e_4)s_{23}^2}{8} - \frac{(e_1\cdot e_3)(p_1\cdot e_2)(p_2\cdot e_4)s_{12}}{4}\\
&- \frac{(e_1\cdot e_3)(p_1\cdot e_2)(p_2\cdot e_4)s_{23}}{4} - \frac{(e_1\cdot e_3)(p_2\cdot e_4)(p_3\cdot e_2)s_{12}}{4} - \frac{(e_1\cdot e_3)(p_2\cdot e_4)(p_3\cdot e_2)s_{23}}{4}\\
&- \frac{(e_1\cdot e_4)(e_2\cdot e_3)s_{12}s_{23}}{8} - \frac{(e_1\cdot e_4)(e_2\cdot e_3)s_{23}^2}{8} + \frac{(e_1\cdot e_4)(p_2\cdot e_3)(p_3\cdot e_2)s_{12}}{4}\\
&+ \frac{(e_1\cdot e_4)(p_2\cdot e_3)(p_3\cdot e_2)s_{23}}{4} - \frac{(e_2\cdot e_3)(p_1\cdot e_4)(p_2\cdot e_1)s_{12}}{4} - \frac{(e_2\cdot e_3)(p_1\cdot e_4)(p_2\cdot e_1)s_{23}}{4}\\
&- \frac{(e_2\cdot e_3)(p_1\cdot e_4)(p_3\cdot e_1)s_{12}}{4} - \frac{(e_2\cdot e_3)(p_1\cdot e_4)(p_3\cdot e_1)s_{23}}{4} + \frac{(e_2\cdot e_4)(p_1\cdot e_3)(p_3\cdot e_1)s_{12}}{4}\\
&+ \frac{(e_2\cdot e_4)(p_1\cdot e_3)(p_3\cdot e_1)s_{23}}{4} + \frac{(e_3\cdot e_4)(p_1\cdot e_2)(p_2\cdot e_1)s_{12}}{4} + \frac{(e_3\cdot e_4)(p_1\cdot e_2)(p_2\cdot e_1)s_{23}}{4}\\
&- \frac{(p_1\cdot e_2)(p_1\cdot e_3)(p_1\cdot e_4)(p_2\cdot e_1)}{2} - \frac{(p_1\cdot e_2)(p_1\cdot e_3)(p_1\cdot e_4)(p_2\cdot e_1)s_{23}}{2s_{12}} - \frac{(p_1\cdot e_2)(p_1\cdot e_3)(p_2\cdot e_1)(p_2\cdot e_4)}{2}\\
&- \frac{(p_1\cdot e_2)(p_1\cdot e_3)(p_2\cdot e_1)(p_2\cdot e_4)s_{23}}{2s_{12}} - \frac{(p_1\cdot e_2)(p_1\cdot e_3)(p_2\cdot e_4)(p_3\cdot e_1)}{2} - \frac{(p_1\cdot e_2)(p_1\cdot e_4)(p_2\cdot e_1)(p_2\cdot e_3)}{2}\\
&- \frac{(p_1\cdot e_2)(p_1\cdot e_4)(p_2\cdot e_1)(p_2\cdot e_3)s_{23}}{2s_{12}} - \frac{(p_1\cdot e_2)(p_2\cdot e_1)(p_2\cdot e_3)(p_2\cdot e_4)}{2} - \frac{(p_1\cdot e_2)(p_2\cdot e_1)(p_2\cdot e_3)(p_2\cdot e_4)s_{23}}{2s_{12}}\\
&- \frac{(p_1\cdot e_3)(p_2\cdot e_4)(p_3\cdot e_1)(p_3\cdot e_2)}{2} + \frac{(p_1\cdot e_4)(p_2\cdot e_1)(p_2\cdot e_3)(p_3\cdot e_2)s_{12}}{2s_{23}} + \frac{(p_1\cdot e_4)(p_2\cdot e_1)(p_2\cdot e_3)(p_3\cdot e_2)}{2}\\
&+ \frac{(p_1\cdot e_4)(p_2\cdot e_3)(p_3\cdot e_1)(p_3\cdot e_2)s_{12}}{2s_{23}} + \frac{(p_1\cdot e_4)(p_2\cdot e_3)(p_3\cdot e_1)(p_3\cdot e_2)}{2} + \frac{(e_1\cdot e_2)(p_2\cdot e_3)(p_2\cdot e_4)s_{23}}{4}\\
&+ \frac{(e_1\cdot e_3)(e_2\cdot e_4)s_{12}^2}{8}\Big)
\end{aligned}
\tag{6.4}
$$

Five point two scalar three gluon HEFT amplitude $\mathcal{O}(\alpha')$

$$
\begin{aligned}
A_{\mathrm{HEFT}}(12345) = \alpha' \Big(& \frac{(e_2 \cdot e_3)(p_1 \cdot e_4)(p_1 \cdot p_2)}{4\,(m^2 - s_{123})} - \frac{(e_2 \cdot e_3)(p_1 \cdot e_4)(p_1 \cdot p_3)}{4\,(m^2 - s_{123})} - \frac{(e_2 \cdot e_3)(p_1 \cdot e_4)s_{34}}{8s_{234}} \\
& + \frac{(e_2 \cdot e_3)(p_1 \cdot e_4)s_{24}}{8s_{234}} - \frac{(e_2 \cdot e_3)(p_1 \cdot p_2)(p_2 \cdot e_4)}{4s_{234}} - \frac{(e_2 \cdot e_3)(p_1 \cdot p_2)(p_3 \cdot e_4)}{4s_{234}} \\
& + \frac{(e_2 \cdot e_3)(p_1 \cdot p_3)(p_2 \cdot e_4)}{4s_{234}} + \frac{(e_2 \cdot e_3)(p_1 \cdot p_3)(p_3 \cdot e_4)}{4s_{234}} + \frac{(e_3 \cdot e_4)(p_1 \cdot e_2)(p_1 \cdot p_3)}{2\,(m^2 - s_{12})} \\
& - \frac{(e_3 \cdot e_4)(p_1 \cdot e_2)s_{23}}{8s_{234}} - \frac{(e_3 \cdot e_4)(p_1 \cdot e_2)}{8} + \frac{(e_3 \cdot e_4)(p_1 \cdot e_2)s_{24}}{8s_{234}} \\
& + \frac{(e_3 \cdot e_4)(p_1 \cdot p_2)(p_3 \cdot e_2)}{4s_{234}} + \frac{(e_3 \cdot e_4)(p_1 \cdot p_2)(p_4 \cdot e_2)}{4s_{234}} + \frac{(e_3 \cdot e_4)(p_1 \cdot p_3)(p_3 \cdot e_2)}{2s_{234}} \\
& + \frac{(e_3 \cdot e_4)(p_1 \cdot p_3)(p_4 \cdot e_2)}{2s_{234}} - \frac{(p_1 \cdot e_2)(p_1 \cdot p_3)(p_3 \cdot e_4)(p_4 \cdot e_3)}{s_{34}\,(m^2 - s_{12})} + \frac{(p_1 \cdot e_2)(p_3 \cdot e_4)(p_4 \cdot e_3)s_{23}}{4s_{234}s_{34}} \\
& + \frac{(p_1 \cdot e_2)(p_3 \cdot e_4)(p_4 \cdot e_3)}{4s_{34}} - \frac{(p_1 \cdot e_2)(p_3 \cdot e_4)(p_4 \cdot e_3)s_{24}}{4s_{234}s_{34}} - \frac{(p_1 \cdot e_4)(p_1 \cdot p_2)(p_2 \cdot e_3)(p_3 \cdot e_2)}{2s_{23}\,(m^2 - s_{123})} \\
& + \frac{(p_1 \cdot e_4)(p_1 \cdot p_3)(p_2 \cdot e_3)(p_3 \cdot e_2)}{2s_{23}\,(m^2 - s_{123})} + \frac{(p_1 \cdot e_4)(p_2 \cdot e_3)(p_3 \cdot e_2)s_{34}}{4s_{23}s_{234}} - \frac{(p_1 \cdot e_4)(p_2 \cdot e_3)(p_3 \cdot e_2)s_{24}}{4s_{23}s_{234}} \\
& + \frac{(p_1 \cdot p_2)(p_2 \cdot e_3)(p_2 \cdot e_4)(p_3 \cdot e_2)}{2s_{23}s_{234}} + \frac{(p_1 \cdot p_2)(p_2 \cdot e_3)(p_3 \cdot e_2)(p_3 \cdot e_4)}{2s_{23}s_{234}} \\
& - \frac{(p_1 \cdot p_2)(p_3 \cdot e_2)(p_3 \cdot e_4)(p_4 \cdot e_3)}{2s_{234}s_{34}} - \frac{(p_1 \cdot p_2)(p_3 \cdot e_4)(p_4 \cdot e_2)(p_4 \cdot e_3)}{2s_{234}s_{34}} \\
& - \frac{(p_1 \cdot p_3)(p_2 \cdot e_3)(p_2 \cdot e_4)(p_3 \cdot e_2)}{2s_{23}s_{234}} - \frac{(p_1 \cdot p_3)(p_2 \cdot e_3)(p_3 \cdot e_2)(p_3 \cdot e_4)}{2s_{23}s_{234}} \\
& - \frac{(p_1 \cdot p_3)(p_3 \cdot e_2)(p_3 \cdot e_4)(p_4 \cdot e_3)}{s_{234}s_{34}} - \frac{(p_1 \cdot p_3)(p_3 \cdot e_4)(p_4 \cdot e_2)(p_4 \cdot e_3)}{s_{234}s_{34}} \Big)
\end{aligned} \tag{6.5}
$$

c: Six point amplitudes

Required cubic diagrams



(6.6)