Properness of schemes and compactness of sets of rational points

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Abstract

We define the fine topology on $X(R)$ given a scheme X and a ring with topology R and outline important properties from [5] and prove new properties. We restate the theorem of equivalence of proper and compactness of schemes for local fields and prove a theorem of such equivalence for global fields in terms of adeles. It is shown that proper-to-compact results for topological fields can only hold if the field is a local field. A generalization of properness to compactness for a class of local topological rings is proven. A generalization of compactness to properness is proven for Henselian fields. Both these proofs closely follow the proof technique from [4].

Contents

1 Introduction

This thesis is based on a classical theorem for schemes over the complex numbers, that says that such a scheme is proper if and only if the set of $\mathbb{C}$-rational points is compact. Here the set of $\mathbb{C}$-rational points has a topology called the strong topology. This strong topology is defined in terms of the standard topology on $\mathbb{C}$. For general local fields there is the same statement, with the caveat that one needs to consider finite field extensions. It says that a scheme over a local field is proper if and only if for each finite extension of the base field the corresponding set of rational points is compact, again with the strong topology. We call this result the local theorem. The complex case is proven in [9]. The local theorem is proven in [4]. The general goal of the thesis is showing generalisations of these statements. Either for one direction of the implication or both. Two of the results in this thesis combine to become a proof of the local theorem itself.

Our first main result is the equivalence of properness and compactness for global fields (i.e. finite extensions of $\mathbb{Q}$ or $\mathbb{F}_p(t)$ for a prime p). This statement is in terms of the *adelic points* of a scheme, which come from the ring of adeles of a global field. This is a big ring that combines all the completions of a ring in a so-called restricted product. This remedies the fact that a global field does not have a natural topology, while the ring of adeles does have a topology defined in terms of the topologies of the completions of the global field. This topology on the ring of adeles induces a topology on the adelic points called the *adelic topology*.

The fine topology is a topology on sets of rational points, that generalises the aforementioned strong and adelic topology. We can define the fine topology on any set of rational points $X(R)$ given a scheme X and a ring with topology R. While the strong and adelic topology require algebraic and topological properties of the local fields and the ring of adeles to be defined, the fine topology requires no additional structure to define whatsoever. Henceforth we will refer to fine topologies for topologies on sets of rational points, with only incidental mentioning of the strong and adelic topology. In essence, this topology has very little structure. Permanence properties of this fine topology depend on algebraic and topological properties of the rings with topology that are considered. For example if R is a topological ring, then arbitrary closed immersions of schemes $X \to Y$ induce closed embeddings $X(R) \to Y(R)$ of topological spaces if and only if R is a Hausdorff ring. Most of this chapter is a summary of the theory from [5], with some new results.

Further generalisation splits up into two parts; proper-to-compact results, and compact-to-proper results. The former being the bulk of the remainder of the thesis. It starts with a discussion on proper-to-compact results for topological fields. We already have such a statement for local fields; the question is whether we can generalise this to, for example, higher local fields. These can be viewed as higher dimensional analogues of local fields, where regular local fields are the 1-dimensional local fields. It turns out that if a proper-to-compact result for a topological field holds, then the field is automatically a (1-dimensional) local field. This means that for more general proper-to-compact result, we need to look beyond topological fields. We start by looking at local topological rings. The advantage of working with local rings is that rational point sets of schemes are easy to describe, specifically for projective space. For projective space these point sets are described using homogeneous coordinates, just like we do for fields, whereas for more general non-local rings, such a description does not exists. Using this description we prove a set of conditions on local topological rings that give that the set of rational points of projective space are compact. This allows us to prove, for this class of local rings, a statement that generalises the proper-to-compact statement of the known local theorem. This is our second main result. It uses the proof technique for the local theorem from [4], and our result of the compactness of the rational points of projective space. The last part of the proper-to-compact part of the thesis discusses our research into an argument that can extend the previous results for local rings to classes of non-local topological rings. The core of the idea relies on embedding a ring into a restricted product of localisations at prime ideals. This would allow us to infer results for the non-local ring, by applying our theory for local rings to the restricted product of local rings. For this argument we need a theory of localisations of topological rings. We display our results for our proposed topology on these localisations. However this theory is far from complete.

This section ends with a list of questions that would need to be answered for the theory to come to full fruition.

This thesis finishes with our compact-to-proper result for Henselian fields. These are valued fields that can be defined as fields for which Hensel's lemma holds. The main components of the proof are Nagata's embedding theorem and the implicit function theorem. Nagata's theorem is a general result that says that any scheme can be embedded into a proper scheme as an open and dense subset. The implicit function theorem is a result that is equivalent to Hensel's lemma for valued fields. Hence the implicit function theorem works for Henselian fields. Using this we show the compact-to-proper result for Henselian fields by very closely following the proof technique of the same statement for local fields from [4].

2 Preliminaries

In this chapter we give preliminary results and definitions used later in the thesis. Basic knowledge about algebraic geometry (schemes), topology, category theory, rings (commutative with 1) and local fields (e.g. p-adic numbers) is assumed.

2.1 Topology and category theory

A lot of constructions in this thesis are done using filtered colimits. So here we reiterate the definition.

Definition 2.1. Let C be a category and consider a diagram X_* indexed by I. The diagram is called filtered if the following hold

1. I is non-empty,
2. for objects X_i and X_j in the diagram there exists an object X_k in the diagram, and morphisms $X_i \to X_k$, $X_j \to X_k$ in the diagram,
3. for morphisms $f, g : X_i \to X_j$, from the diagram there is a morphism in the diagram $h : X_j \to X_k$ such that $h \circ f = h \circ g$.

A colimit of a filtered diagram is called a filtered colimit

If the diagram is indexed by a partially ordered set such that a pair of elements has a common upper bound, then the diagram is filtered. In this case the diagram is also called a directed diagram. An example of this are the partially ordered set of (finite) subsets of a set S. This is usually the way filtered colimits come about.

Filtered colimits have very nice properties. In the categories we consider they can be viewed as taking the 'union' of the objects in the diagram using the suitable identifications given by the morphisms in the diagram.

The following lemma is a technical statement about when filtered colimits and finite products commute in the category of topological spaces.

Lemma 2.2. *Given finitely many filtered diagrams X_i^k of topological spaces such that all the canonical maps $f_i^k : X_i^k \to \operatorname{colim}_i(X_i^k)$ are open, then there is a natural isomorphism*

$$\operatorname*{colim}_i \left(\prod_j X_i^j \right) \cong \prod_j \operatorname*{colim}_i (X_i^j).$$

Proof. It is well known that filtered colimits and finite limits commute in the category of sets. So as sets there is a natural isomorphism. So we only need to consider the topology. We will show the statement for the case of two filtered diagrams, as the rest will follow by induction. Let X_i and Y_j be two filtered diagrams. The natural map $\operatorname{colim}(X_i \times Y_j) \to \operatorname{colim}(X_i) \times \operatorname{colim}(Y_j)$ is continuous as it is induced by the projection maps from the product. We denote $X = \operatorname{colim}(X_i)$ and $Y = \operatorname{colim}(Y_i)$

and denote $f_i : X_i \to X$ and $g_j : Y_j \to Y$ the natural maps. We identify the set of $\operatorname{colim}(X_i \times Y_j)$ with the set $X \times Y$ under the natural bijection. A subset $U \subset \operatorname{colim}(X_i \times Y_j)$ is open if and only if for each (i, j) the set $(f_i \times g_j)^{-1}U \subset X_i \times Y_i$ is open. Let $U \subset \operatorname{colim}(X_i \times Y_j)$, we show that this is open in the product topology of $X \times Y$. By the nature of colimits we can write $U = \bigcup_{(i.j)}(f_i \times g_j)(f_i \times g_j)^{-1}U$ for all (i, j). Since U is open in $\operatorname{colim}(X_i \times Y_j)$ it follows that $(f_i \times g_j)^{-1}U$ is open in $X_i \times Y_j$. So $(f_i \times g_j)^{-1}U = \bigcup_n V_n^{i,j} \times W_n^{i,j}$, for some opens $V_n^{i,j} \subset X_i$ and opens $W_n^{i,j} \subset Y_j$. Then the we have the following: $(f_i \times g_j)(f_i \times g_j)^{-1}U = \bigcup_n f_i V_n^{i,j} \times g_j W_n^{i,j}$, where the sets $f_i V_n^{i,j}$ and $g_j W_n^{i,j}$ are open because the maps f_i and g_j are open, so $(f_i \times g_j)(f_i \times g_j)^{-1}U$ is open. We conclude that U is a union of open sets in the product topology and hence open. ∎

Remark 2.3. If for a filtered diagram X_i the transition maps $t_{i,i'} : X_i \to X_{i'}$ are open and injective then the canonical maps will be open. This is as follows. Firstly if the transition maps are injective then it is an elementary property of filtered colimits in Set that the canonical maps will be injective. Furthermore given an open U in X_i, then the image f_iU can be written as $\bigcup_{i'>i} f_{i'}^{-1} f_i U$. Since $f_i = t_{i,i'} \circ f_{i'}$ we can write $f_iU = \bigcup f_j^{-1} f_{i'} t_{i,i'} U$, which is equal to $\bigcup t_{i,i'}U$ by injectivity. The $t_{i,i'}$ are open maps so f_iU is a union of open sets and hence open.

This gives us a sufficient and more easily checkable condition for when the conditions of Lemma 2.2 hold.

2.2 Algebraic geometry

We state definitions and give properties that will be of use later. We start with the definition of the functor of points, which is a central object in this thesis. We give this definition mostly to get notation and terminology down as familiarity with the concept is assumed.

Definition 2.4 (Functor of points)**.** Let Y be a scheme and let X be a scheme over Y. For a scheme T over Y we define the set of "T-rational points" $X(T)$, as the set of morphisms $T \to X$ defined over Y.

This notation leaves the base scheme Y implicit. From context is should always be clear over what base scheme the morphisms should be defined. In the case $T = \operatorname{Spec} R$ is affine, we write $X(R)$ for $X(\operatorname{Spec}(R))$.

If $R = k$ is a field then, $\operatorname{Spec} k$ has a single point. In this case we can identify a k-rational point $\operatorname{Spec} k \to X$ with the scheme theoretical point that is the image of the unique point of $\operatorname{Spec} k$ in X. Here we say a scheme theoretical point $p \in X$ is k-rational if there is a morphism $\operatorname{Spec} k \to X$ such that p is the image of the morphism.

Definition 2.5. A morphism $X \to S$ between schemes is quasi-separated if the diagonal map $X \to X \times_S X$ is quasi-compact.

Closed immersions are quasi-compact, so being quasi-separated is weaker than being quasi-compact. Quasi-separated morphisms satisfy the same cancellative property as separated morphisms; if a composition of two morphisms $f \circ g$ is quasi-separated, then g is quasi-separated. Furthermore morphisms between affine schemes are quasi-separated[1].

We call a scheme X quasi-separated if the unique morphism $X \to \operatorname{Spec} \mathbb{Z}$ is quasi separated. By the cancellative property and the fact that morphisms between affine schemes are quasi-separated it follows that a morphism $X \to \operatorname{Spec} A$ is quasi-separated if and only if X is quasi-separated. So being quasi-separated is a property of a scheme that does not depend on the base ring. So we can call as scheme X over a ring A quasi-separated if the scheme X is quasi-separated when viewed as scheme over $\mathbb{Z}$.

Definition 2.6. A scheme X over A is locally of finite presentation if there is an affine open covering $\{U_i\}$ of X such that $\mathcal{O}_{U_i}(U_i)$ is a finitely presented A-algebra. Meaning $\mathcal{O}_{U_i}$ is isomorphic to some $A[x_1, \cdots x_n]/(f_1, \cdots f_m)$ for some finite n and m.

A morphism is of finite presentation if it is locally of finite presentation, quasi-compact and quasi-separated.

Generally being (locally) of finite presentation is stronger than being (locally) of finite type, as for finite type we do not require the ideal we quotient out to be finitely generated. If the base ring A is noetherian then being of finite presentation and being of finite type is equivalent.

Being locally of finite presentation can also be characterised in a different way, which will be a useful property.

Lemma 2.7. *[2, IV, Prop. 8.14.2] Let X be a scheme over A. Then X is locally of finite presentation over A if and only if for each filtered diagram $\{R_i\}$ of A-algebras, there is a natural bijection*

$$X(\operatorname{colim} R_i) = \operatorname{colim} X(R_i).$$

If R' is a ring with a subring R we want to know when the set of rational points $X(R)$ can be viewed as a subset of $X(R')$. This works for separated schemes by the following lemma.

Lemma 2.8. *[10, Prop. 2.2.23] Let A be a ring and $R \hookrightarrow R'$ be an injective map of A algebras. Let X be a separated scheme over A, then the induced map $X(R) \to X(R')$ is injective.*

The following is a tool that can be used to take more general subsets of rational points, given a subring.

[1] This follows from the fact that they are separated.

Definition 2.9. Let $B \to A$ be a map of rings and X a scheme over A. An B-model of X is scheme Y such that the base change of Y to A is isomorphic to X.

The existence of such models will allow us to take rational points sets for subrings of the base ring. If X is a scheme over A and B is a subring of A we would like to consider the "B-rational points" of X. A priori this is not possible as $\operatorname{Spec} B$ is not a scheme over A. This can be solved using models. Assume there is a B-model Y of X, then we can regard $Y(B)$ as a subset of $X(A)$ as follows. By Lemma 2.8 there is an injection $Y(B) \to Y(A)$. There is also a natural bijection $Y(A) \cong X(A)$ so we get a natural injection $Y(B) \hookrightarrow X(A)$. Of course the existence of such models is not given. The following theorem ([2, IV, Thm. 8.10.5xii]) gives us the existence of these models for our purposes. It also gives the conditions for when these models are proper.

Theorem 2.10. *Let $\{A_i\}$ be a filtered diagram of rings and let $A = \operatorname{colim}_i A_i$. Let X be a scheme of finite presentation over A, then*

1. *There exists an i_0 with an A_{i_0}-model X_{i_0} of X,*
2. *An A_{i_0}-model X_{i_0} is proper over A_{i_0} if and only if X is proper over A.*

2.2.1 Generalisations and specialisations

Given a scheme we can partially order the points of the scheme by how 'general' or 'special' they are. This terminology is often a useful for shortcut for certain topological arguments.

Definition 2.11. Let X be a scheme and let $x \in X$ be a point. A point y is a generalisation of x if $x \in \overline{\{y\}}$. In this case x is a specialisation of y

This way we can view the points of an irreducible scheme as a pyramid, having at the top the generic point an the bottom the closed points. Each layer corresponds to the generic points of irreducible subschemes of a specific dimension.

Definition 2.12. A subset $V \subset X$ of a scheme is called closed under generalisations (resp. specialisations) if for each point $x \in V$ and each generalisation (resp. specialisation) y we have $y \in T$.

Proposition 2.13. *Let X be a scheme. The following hold.*

1. *Open sets are closed under generalisations.*
2. *Closed sets are closed under specialisations.*

Proof. Let $U \subset X$ be open and $x \in U$. If y is a generalisation of x, then $x \in \overline{\{y\}}$, so every open neighbourhood of x contains y. In particular U contains y. Let $Z \subset X$ be closed and $y \in Z$. Then $\overline{\{y\}} \subset Z$ and so every specialisation of y is in Z. ∎

Proposition 2.14. *Let $f : X \to Y$ be a morphism of schemes. Then if y a generalisation of x, then $f(y)$ is a generalisation of $f(x)$.*

Proof. If $x \in \overline{\{y\}}$, then by continuity $f(x) \in f(\overline{\{y\}}) \subset \overline{\{f(y)\}}$. So $f(y)$ is a generalisation of $f(x)$. ∎

2.3 Topological rings and modules

Recall the definition of a topological ring.

Definition 2.15. A topological ring is a ring R with a topology such that the addition map

$$\begin{aligned} + : R \times R &\longrightarrow R \\ (r, r') &\longmapsto r + r' \end{aligned}$$

and the multiplication map

$$\begin{aligned} \cdot : R \times R &\longrightarrow R \\ (r, r') &\longmapsto rr' \end{aligned}$$

are continuous, where $R \times R$ carries the product topology.

A morphism of topological rings is just a continuous homomorphism of rings. We denote $\mathrm{TopRing}$ for the corresponding category of topological rings. This category is complete and cocomplete.

Examples include $\mathbb{Z}_p$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}_p$, with their standard topologies, the inclusion of $\mathbb{R}$ into $\mathbb{C}$ is continuous and hence a morphism of topological rings. Same for the inclusion of $\mathbb{Z}_p$ into $\mathbb{Q}_p$. Maps to topological rings can be added and multiplied in a natural way by defining addition and multiplication component wise.

Proposition 2.16. *Let R be topological ring and let X be a topological space and $f : X \to R$ and $g : X \to R$ be continuous maps, then $f + g$ and fg are continuous.*

Proof. The sum $f + g$ is the composition of the natural map $(f, g) : X \to R \times R$ and the addition map $+ : R \times R \to R$. Both of which are continuous so the composition is also continuous. Similar for the product. ∎

Definition 2.17. We say a topological ring R has an open unit group if the unit group $R^\times \subset R$ is open and inversion in $R^\times$ is continuous with respect to the subspace topology in R.

Remark 2.18 (Topological R-modules). If we have a topological ring R, then we can define topological R-modules in very much the same way. Namely an R-module M with a topology is a topological R-module if the addition map $M \times M \to M$ and the scalar multiplication map $R \times M \to M$ are continuous. Of course we define a morphism of topological R-modules as being a continuous R-module homomorphism. Just like the category $\mathrm{TopRing}$, the category of topological R-modules $\mathrm{TopMod_R}$ is complete and cocomplete. We will only use this notion when discussing the definition of the topology on localisations of topological rings.

2.3.1 Limits and colimits of topological rings

Given any ring R, we can turn R into a topological ring in two natural ways. We have R with the discrete topology and R with the trivial topology. This defines two functors $D, T : \mathrm{Ring} \to \mathrm{TopRing}$ from the category of rings to the category of topological rings. Denote $F : \mathrm{TopRing} \to \mathrm{Ring}$ the forgetful functor sending a topological ring to its underlying ring. Any ring homomorphism to a trivial topological ring is continuous and and ring homomorphism from a discrete topological ring is continuous. This means that we have two adjoint pairs $D \dashv F$ and $F \dashv T$. Hence the forgetful functor is both a right and a left adjoint and hence commutes with limits and colimits. This construction is similar to how the forgetful functor $\mathrm{Top} \to \mathrm{Set}$ gives rise to two adjoint pairs, when considering the trivial and discrete topology functors. The conclusion is that computing limits and colimits of topological rings, means computing the limit or colimit of the underlying rings and subsequently computing the topology.

To describe the topology on a colimit of topological rings it is helpful to know what the underlying set it. By the discussion above we only know what the underlying *ring* is. Of course for general colimits of rings the underlying set of the colimit is not the same as the colimit of the underlying set. For example compare the coproduct of rings (tensor product) with the coproduct of sets (disjoint union). These are obviously not equal on the level of sets. In other words the forgetful functor $\mathrm{Ring} \to \mathrm{Set}$ does not commute with colimits[2]. Luckily a lot of the constructions of topological rings using colimits are done using *filtered* colimits. As it turns out the forgetful functor from Ring to Set does commute with filtered colimits. This is because the category of rings is the same as the Eilenberg-Moore category for the free ring monad on Set. Since the free ring monad commutes with filtered colimits, also the forgetful functor from the Eilenberg-Moore category commutes with filtered colimits, as such the forgetful functor from Ring to Set commutes with filtered colimits. We conclude that we can regard any filtered colimit of topological rings as a filtered colimit of sets with an induced topology and an induced ring structure. Which means that we can also apply Lemma 2.2 to collections of filtered diagrams of topological rings. The same result holds for topological R-modules.

2.4 Local and global fields

We briefly recall the definition of topological fields and local and global fields.

Definition 2.19. A topological ring k is a topological field if k is a field and if the inversion map

$$\begin{aligned} \cdot^{-1} : k^\times &\longrightarrow k^\times \\ x &\longmapsto x^{-1} \end{aligned}$$

[2]It does commute with limits as the underlying set of products of rings is just the product of the underlying set. It is also easily shown that the forgetful functor commutes with equalizers.

is continuous.

A morphism of topological fields is just a morphism of topological rings. So the category of topological fields forms a full subcategory of the category of topological rings. Often in literature being Hausdorff is added to the definition of a topological fields, which we won't do here for generality.

Remark 2.20. Any Hausdorff topological field k is with open unit group: since points are closed, the unit group $k\backslash\{0\}$ is open and inversion is continuous by definition.

Definition 2.21. A local field is a topological field K isomorphic (as a topological field) to a finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$ for some prime p.

Here $\mathbb{F}_p((t))$ denotes the field of Laurent polynomials in the variable t over $\mathbb{F}_p$. Another characterization of local fields is that they are the locally compact topological fields as shown in [12].

Definition 2.22. A global field is a field k isomorphic to a finite extension of $\mathbb{Q}$ or $\mathbb{F}_q(t)$ for some $q = p^n$.

Using this definition local fields can also be characterized by the fact that they are exactly the completions of the global fields. A global field comes with a set of places corresponding to its completions. For a global field k we will denote this set by M_k. Each place v corresponds to an equivalence class of absolute values on k and has a corresponding completion denoted by k_v. This completion is the completion of k as a metric space with the metric induces by an absolute value corresponding to v. The field operations and the absolute value on k extend naturally to k_v, see [12]. We denote $\mathcal{O}_v$ for the unit ball in k_v, which is the valuation ring of k_v in the case of a non-archimedian valuation v.

Definition 2.23. Given a global field k and a finite set of places S containing the archimedian places, we define the S-integers $\mathcal{O}_S$ as the subring $\{x \in k \mid v(x) \geq 0 \text{ for all } v \notin S\}$.

For a given place v we have that $\mathcal{O}_S \subset \mathcal{O}_v$ if and only if $v \notin S$. Furthermore we have the property for finite subsets of places S and S' that $S \subset S'$ if and only if $\mathcal{O}_S \subset \mathcal{O}_v$. This makes the set of S-integers ranging over all finite subsets of places containing the archimedian places together with the inclusion maps into a filtered diagram of rings. Since clearly k is the union of all the rings of S-integers, so we can write $k = \operatorname{colim}_S \mathcal{O}_S$, where S ranges over all the finite subsets of places. This allows us to apply Theorem 2.10 to this situation.

Let X be a scheme over a global field k. We will generally denote an $\mathcal{O}_S$-model, given a finite set of places S, by X_S. According to Theorem 2.10, a model X_S is proper if and only if X is proper.

2.5 Restricted products and the ring of adeles

The ring of adeles is a construction for global field that remedies the problem that a global field does not have a natural topology. This is fixed by considering the completions and "combining" them, using a construction called the restricted product.

2.5.1 Restricted products

Definition 2.24 (Restricted product)**.** Let X_i be a family of topological spaces with open subsets $U_i \subset X_i$ indexed by I. We define the restricted product of the X_i with respect to the U_i denoted by $\prod(X_i, U_i)$ as the set

$$\left\{(x_i) \in \prod X_i \mid x_i \in U_i \text{ for almost all } i\right\},$$

together a topology defined by a base of open sets

$$\left\{\prod V_i \,\middle|\, V_i \subset X_i \text{ is open for all } i \text{ and } V_i = U_i \text{ for almost all } i\right\}.$$

The topology of the restricted product is a finer topology than that of the normal product. In a normal product for a set to be open it has to be the whole space in almost all factors. This is not the case for the restricted product. For example if $X = \prod(X_i, U_i)$, then we have the inclusion of sets

$$\prod U_i \subset X = \prod(X_i, U_i) \subset \prod X_i.$$

Here $\prod U_i$ is open in X, but $\prod U_i$ is not open in $\prod X_i$.

Remark 2.25. The restricted product is not affected when you change finitely many U_i, meaning that $\prod(X_i, U_i) = \prod(X_i, U_i')$, whenever $U_i = U_i'$ for almost all i. So it is sufficient to define U_i for all but finitely many i.

Remark 2.26. Here we use the symbol $\prod$ for the restricted product. It is important to note that the restricted product is not a categorical product in any way. If for example one would interpret $\prod(X_i, U_i)$ as the categorical product in the category of pairs of spaces, then they would get something vastly different from the restricted product.

Given a restricted product $X := \prod(X_i, U_i)$ indexed by I, and a finite subset $S \subset I$, we define

$$X_S = \prod_{i \in S} X_i \times \prod_{i \notin S} U_i.$$

By definition this is an open subset of X. The family $\{X_S\}$, where S ranges over all finite subsets of I forms an open cover of X. Furthermore we have $X_S \subset X_{S'}$ if and only if $S \subset S'$, making the sets $\{X_S\}$ into a filtered diagram of open maps. It follows easily that we can write

$$X = \operatorname*{colim}_{S' \supset S} X_{S'},$$

for any fixed finite set S of indices.

Proposition 2.27. *There is a natural homeomorphism* $\prod(X_i^n, U_i^n) \to (\prod(X_i, U_i))^n$ *for* $n \geq 0$.

Proof. Using the description of the restricted product as a colimit and Lemma 2.2 we can write

$$\begin{aligned}\prod(X_i^n, U_i^n) &= \operatorname*{colim}_S \left(\prod_{i\in S} X_i^n \times \prod_{i\notin S} U_i^n\right) \\ &= \operatorname*{colim}_S \left(\prod_{i\in S} X_i \times \prod_{i\notin S} U_i\right)^n \\ &= \left(\operatorname*{colim}_S \prod_{i\in S} X_i \times \prod_{i\notin S} U_i\right)^n = \left(\prod(X_i, U_i)\right)^n.\end{aligned}$$

Here the third equality is the n-fold product commuting with the filtered colimit as from Lemma 2.2. ∎

2.5.2 The ring of adeles

Now we can give the definition of the ring of adeles of a global field

Definition 2.28. Let k be a global field with set of places M_k. Then we define the ring of adeles as the restricted product

$$\mathbb{A}_k := \prod(k_v, \mathcal{O}_v),$$

ranging over all places.

For a finite set $S \subset M_k$ we define the S-*adeles* as

$$\mathbb{A}_{k,S} := \prod_{v\in S} k_v \times \prod_{v\notin S} \mathcal{O}_v$$

By earlier remark we then have $\mathbb{A}_k = \operatorname{colim}_S \mathbb{A}_{k,S}$. Since the category of topological rings contains colimits and it is clear that $\mathbb{A}_{k,S}$ is a topological ring it follows that the ring of adeles $\mathbb{A}_k$ is a topological ring.

3 The fine topology

In this chapter we discuss the way to topologise the set of rational points $X(R)$ of a scheme X, in the case where R is a given ring with a topology. This topology for $X(R)$ is called the fine topology. By ring with topology we do not mean necessarily a topological ring as for the construction of the fine topology no compatibility with the addition and multiplication structure is required. This topology is functorial in both X and R. Besides this, the fine topology has, in its bare bones, very little structure; more categorical and topological properties follow from more structure on R. One should think of structure like being a topological ring, being a Hausdorff ring or being local. These properties are all lined out and discussed in detail in [5]. This chapter consists of a summary of the theory of [5] and new theory. Statements without proof come directly from [5], while statements with proof are generally new theory.

Definition of the fine topology

To define the fine topology we first define the *affine* topology for affine schemes and subsequently define the fine topology for general schemes in terms of the affine topology by universal property.

Definition 3.1 (Affine topology)**.** Let $X = \operatorname{Spec} R'$ be an affine scheme over a ring A and R a A-algebra with a topology. We define the affine topology on $X(R) = \operatorname{Hom}_A(R', R)$ as the coarsest topology such that for each $r \in R'$ the evaluation map $\operatorname{ev}_r : \operatorname{Hom}_A(R', R) \to R$ is continuous.

Observe that no compatibility of the ringstructure and the topology of R is required.

Definition 3.2 (Fine topology)**.** Let X be a scheme over a ring A and R a A-algebra with topology. We define the fine topology on $X(R)$ as the finest topology such that for each morphism $U \to R$ over A with U affine, the induced map $U(R) \to X(R)$ is continuous, where we equip $U(R)$ with the affine topology.

Remark 3.3. It is easily verified that the affine and the fine topology coincide for affine schemes.

With this definition we can describe a basis of opens for the affine topology. Namely for a ring R with topology and an affine scheme $X = \operatorname{Spec}(R')$, we have a basis of opens of $X(R) = \operatorname{Hom}(R', R)$ consisting of sets of the form

$$U_{V,a} = \{f : R' \to R | f(a) \in V\}$$

Where $V \subset R$ is open and $a \in A$.

Remark 3.4. The fine topology on $X(R)$ is functorial in both X and R. Meaning that if $X \to Y$ is a morphism of A-schemes, then $X(R) \to Y(R)$ is a continuous map of topological spaces, and if $R \to R'$ is a continuous ring homomorphism, then $X(R) \to X(R')$ is a continuous map of topological spaces.

Basic properties

Given a ring A, an A-algebra R with topology and schemes X, Y and Z over A, the following are properties of the fine topology we can desire.

- (F1) The canonical bijection $(X \times_Z Y)(R) \to X(R) \times_{Z(R)} Y(R)$ is a homeomorphism.
- (F2) The canonical bijection $\mathbb{A}^1(R) \to R$ is a homeomorphism.
- (F3) A closed immersion $Y \to X$ over A induces a closed embedding $Y(R) \to X(R)$ of topological spaces.
- (F4) A open immersion $Y \to X$ over A induces a open embedding $Y(R) \to X(R)$ of topological spaces.
- (F5) Let $\{U_i\}_i$ be an affine open cover of X. Then $X(R) = \bigcup U_i(R)$.
- (F6) Let $\{U_i\}_i$ be a finite affine open cover of X and $U := \bigsqcup U_i$. Let $\Psi : U \to X$ be the associated morphism. Then the map $\Psi_R : U(R) \to X(R)$ is surjective.

The validity of these properties depends on the algebraic and topological structure of R, characterised by the following results. Properties (F1)-(F4) follow from topological properties of R and (F5) and (F6) are properties that only rely on the algebraic properties of R.

Theorem 3.5. *Let A be a ring an R and A-algebra with topology. Then (F2) holds if and only if for each polynomial $p \in R[T]$ the corresponding evaluation map $R \xrightarrow{p} R$ is continuous.*

Proof. The natural bijection $\varphi : \mathbb{A}^1(R) = \mathrm{Hom}(R[T], R) \to R$ sends a map $f : R[T] \to R$ to $f(T)$. So the inverse image of an open $V \subset R$ under φ is just the basic open $U_{V,T}$. Hence the map is continuous. Furthermore we have that $\varphi(U_{V,g}) = \{a \in R \mid g(a) \in V\} = g^{-1}V$. This is open for all $V \subset R$ open if and only if g is continuous. Hence φ is a homeomorphism if and only only if each polynomial is continuous. ∎

Remark 3.6. If we have any ring with a topology R we can replace the topology on R by the topology on $\mathbb{A}^1(R)$, to get a new ring with topology $\hat{R}$. By the lemma above, a set in $\hat{R}$ is open if and only if it is the preimage of an open subset in R under a polynomial p. So the preimage of an open subset in $\hat{R}$ is again open in $\hat{R}$, since the composition of polynomials is again a polynomial. This means that the functor $\mathbb{A}^1$ defines a natural way of turning a ring with topology into a ring with topology where all polynomials are continuous.

The following result is one of the main results of [5]. This result fully characterizes when properties F1-F4 occur.

Theorem 3.7. *Let A be a ring and R be an A-algebra with a topology. Then the following hold:*

1. *(F1) and (F2) hold for all schemes (of finite type) over A if and only if R is a topological ring.*
2. *If R is a topological ring, then (F3) holds for all schemes (of finite type) over A if and only if R is Hausdorff.*
3. *If R is a local topological ring, then (F4) holds if and only if R has open unit group (Def. 2.17).*

Corollary 3.8. *Let R be a topological ring and X a scheme over R. If R is Hausdorff and X is separated over R and, then $X(R)$ is Hausdorff.*

Proof. Since X is separated, the diagonal morphism $X \to X \times_R X$ is closed. Since R is Hausdorff, this induces a closed immersion $X(R) \to X(R) \times X(R)$, which is equal to the diagonal map. So the diagonal is closed, hence $X(R)$ is Hausdorff ∎

Corollary 3.9. *Let A and B be rings and R a topological ring Let $A \to B \to R$ be maps of rings. Let X be a scheme over A, then the natural bijection $X(R) \to X_B(R)$ is a homeomorphism.*

Proof. Since R is a topological ring the natural map

$$X_B(R) = (X \times_{\operatorname{Spec} A} \operatorname{Spec} B)(R) \xrightarrow{\sim} X(R) \times_{\operatorname{Hom}_A(A,R)} \operatorname{Hom}_A(B, R) = X(R)$$

is a homeomorphism by property (F1).

∎

While closed immersions of affine schemes do not always induce closed embeddings on the R-rational points (this requires that R is Hausdorff), they do induce topological embeddings.

Theorem 3.10. *If R is topological ring then a closed immersion of finite type affine schemes $X \hookrightarrow Y$ induces an embedding of topological spaces $X(R) \hookrightarrow Y(R)$.*

This gives us a tool to extrapolate topological properties of affine space $\mathbb{A}^n$ to general affine schemes X by using a closed immersion $X \hookrightarrow \mathbb{A}^n$, inducing a topological embedding $X(R) \hookrightarrow \mathbb{A}^n(R) = R^n$, without requiring the ring to be Hausdorff. This will be important in the last paragraph of this chapter

Affine Patchings

We like to study the fine topology in terms of affine covers patched together. Given a scheme X and an affine open cover $\{U_i\}$, it is not generally true that $X(R) = \bigcup U_i(R)$. This can fail when $\operatorname{Spec} R$ has more than one closed point. Given a morphism $\operatorname{Spec} R \to X$, two closed points can land in separate affine opens outside of their overlap, hence this morphism is in general not in any of the $U_i(R)$. Having multiple closed points is the only situation where this can fail, as illustrated by the following lemma.

Lemma 3.11. *Let A be a ring and R be an A-algebra. Then R is a local ring if and only if for each scheme X over A and each open cover $\{U_i\}_i$ of X we have $X(R) = \bigcup U_i(R)$.*

Proof (taken from [5]). Assume R is a local ring. Any morphism $\operatorname{Spec} R \to X$ sends the closed point to a point in some $\{U_i\}$. Since generalisations are mapped to generalisations and opens are closed under generalisations (Lemma 2.13), it follows that the image of $\operatorname{Spec} R$ is fully contained in U_i. Hence $X(R) = \bigcup U_i(R)$. If R is not local, then $\operatorname{Spec} R$ has two distinct closed points. Let U_1 and U_2 be the complements of these closed points in $\operatorname{Spec} R$. Then U_1 and U_2 cover $\operatorname{Spec} R$, but the morphism $id : \operatorname{Spec} R \to \operatorname{Spec} R$ is contained in neither $U_1(R)$ nor $U_2(R)$. Hence $\operatorname{Spec}(R)(R) \neq U_1(R) \cup U_2(R)$, proving the converse. ∎

There is a more general way to describe the topology on $X(R)$ in terms of the $U_i(R)$'s. Given the affine open covering we can consider $U = \bigsqcup U_i$ and the associated map $U \to X$ and ask whether the map $U(R) \to X(R)$ is surjective. If R is local this is clearly the case. Now assume R is not local and that $\operatorname{Spec} R$ has two distinct closed points which have a common generalisation η. If we have a map $\operatorname{Spec} R \to X$ such that the closed points land in two different affine opens outside the overlap, then η must land in the intersection of these two opens. But if we want this map to factor via U then we need η to again lie inside the intersections of the two opens, but in U this is empty. This means that here the map is not surjective. We see that we need a spectrum such that each pair of distinct closed points do not have a common generalisation. We can interpret this as the spectrum being as disconnected as possible, equivalently the subspace of closed points is totally disconnected. For surjectivity of the map to hold in general we need a slightly stronger property. This called having a *maximally disconnected spectrum*.

Definition 3.12. A ring R has a maximally disconnected spectrum, if for a pair of closed points there exist respective open neighbourhoods U_1 and U_2 such that $\operatorname{Spec} R = U_1 \sqcup U_2$ as topological spaces.

Example 3.13. Examples include local rings, products of local rings and the ring of adeles of a global field k.

The following theorem formalises the above discussion.

Theorem 3.14. *Let R be a ring then the following are equivalent*

1. *R has a maximally disconnected spectrum*
2. *If X any is a scheme over R with a finite affine open covering $\{U_i\}$, denote $U = \bigsqcup U_i$ and $\psi : U \to X$ for the natural map. Then the induced map $\Psi_R : U(R) \to X(R)$ is surjective.*

Open and closed maps

Given a ring R' with a subring R, we have that for a separated scheme X we can view $X(R)$ as a subset of $X(R')$. We would like that this works well with the topology; if for example R is an open subring of R' then we would like that $X(R)$ is an open subspace of $X(R')$. Similarly for closed subrings we want that $X(R)$ is a closed subspace of $X(R')$. For affine spaces this works well without any restrictions. However in general there are some restrictions needed to make this work. This is illustrated in the following theorems.

Proposition 3.15. *Let $R \to R'$ an open (resp. closed) map of topological A-algebras. Let X be an affine scheme over A. Then the induced map $X(R) \to X(R')$ is an open (resp. closed) map.*

Proof. We show the statement for the open case. The closed case follows completely analogously. Since X is affine we can choose a closed immersion $X \hookrightarrow \mathbb{A}^n$ for some n. This induces the below commutative diagram.

$$\begin{array}{ccc} X(R) & \longrightarrow & X(R') \\ \big\downarrow & & \big\downarrow \\ \mathbb{A}^n(R) & \longrightarrow & \mathbb{A}^n(R') \end{array}$$

Since $X \to \mathbb{A}^n$ is a closed immersion the vertical maps are topological immersions. The bottom map is open since it is the same as the product map $R^n \to (R')^n$, which is open as a product of open maps. It follows that the top map $X(R) \to X(R')$ is open. ∎

Remark 3.16. Note that for the above proposition it is necessary that the map of rings is open (resp. closed). Since if $X(R) \to X(R')$ is open (resp. closed) for all schemes X. Then for $X = \mathbb{A}^1$ the map on rational points just becomes the ring homomorphism $\mathbb{A}^1(R) \cong R \to R' \cong \mathbb{A}^1(R')$.'

Proposition 3.17. *Let A be a ring and $i : R \to R'$ be an open map of topological A-algebras. Assume R has maximally disconnected spectrum and R' has open unit group. If X is a scheme over A then the induced map $X(R) \to X(R')$ is open.*

Proof. Let $\{U_i\}$ be an affine open cover, and denote $U = \bigsqcup U_i$. Then by Theorem 3.14 the induced map $\Psi_R : U(R) \to X(R)$ is a surjective continuous map. Since R' has open unit group the maps $U_i(R') \to X(R')$ are open and so $\Psi_{R'} : U(R') \to X(R')$ is open. Denote $i_U : U(R) \to U(R')$ and $i_X : X(R) \to X(R')$ the induced maps. This yields the below commutative diagram.

$$\begin{array}{ccc} U(R) & \xrightarrow{i_u} & U(R') \\ {\scriptstyle \Psi_R}\big\downarrow & & \big\downarrow{\scriptstyle \Psi_{R'}} \\ X(R) & \xrightarrow[i_X]{} & X(R') \end{array}$$

By Proposition 3.15 the map i_U is open. Let $V \subset X(R)$ be open. Since Ψ_R is surjective we can write $i_X V = i_X \Psi_R \Psi_R^{-1} U = \Psi_{R'} i_U \Psi_R^{-1} U$ by commutativity. Here $\Psi_{R'} i_U \Psi_R^{-1} U$ is open since Ψ_R is continuous and i_U and $\Psi_{R'}$ are open. Hence the induced map $X(R) \to X(R')$ is open. ∎

Proposition 3.18. *Let A be a ring and $R \to R'$ be a closed map of local topological A-algebras such that R' has an open unit group. If X is a scheme over A then $X(R) \to X(R')$ is a closed map of topological spaces.*

Proof. Let $\{U_i\}$ be some affine open cover of X. By locality of R and R' we have that $X(R) = \bigcup U_i(R)$ and $X(R') = \bigcup U_i(R')$. By the constituent maps $U_i(R) \to U_i(R')$ are closed by Proposition 3.15. It follows that $X(R) \to X(R')$ is closed. ∎

3.1 Adelic topologies

In this section we will discuss a more explicit description of the fine topology on $X(R)$ for when R is a restricted product of local A-algebras. A special case is the adele ring of a global field. The method of proof is an adaptation and generalisation of the proof that the adelic and the fine topology coincide in [5]. As a set of points there is a result that given a seperated scheme X over a global field k and the corresponding ring of adeles $\mathbb{A}_k$ that there is a bijection

$$X(\mathbb{A}_k) = \prod (X_S(k_v), X_S(\mathcal{O}_v)).$$

For an $\mathcal{O}_S$-model X_S. This bijection is also an homeomorphism, which we will show by considering the most general possible situation where such statement will hold.

We start with the following general result.

Theorem 3.19. *Let A be a ring and T_i be a family of local A algebras. Let X be a quasi-seperated, scheme of finite type over R. Then the natural map*

$$X\left(\prod T_i\right) \to \prod X(T_i)$$

induced by the projection maps a bijection. If X is affine then this is a homeomorphism.

Proof. The bijectivity assertion is written out in detail in the proof of Theorem 3.6 of [1]. If X is affine, then there exists a closed immersion $X \hookrightarrow \mathbb{A}^n$, this induces the below diagram where the vertical maps are topological embeddings and the horizontal maps are bijections.

$$\begin{array}{ccc} X\left(\prod S_i\right) & \longrightarrow & \prod X(S_i) \\ \downarrow & & \downarrow \\ \mathbb{A}^n\left(\prod S_i\right) & \longrightarrow & \prod \mathbb{A}^n(S_i) \end{array}$$

The bottom map is also a homeomorphism. This is the same as the fact that the natural map $(\prod T_i)^n \to \prod T_i^n$ is a homeomorphism. From this it follows that the top map is also a homeomorphism. ∎

For the rest of this chapter we consider a ring A and a collection of local topological A-algebras R_i, together with subsets R_i^0 and consider $R := \prod(R_i, R_i^0)$. We require R_i^0 to be an local open subring for almost all i, so that R forms a topological ring in the natural way. Furthermore we assume that for each $a \in A$, the element $a \cdot 1 \in R_i$ lies in R_i^0 for almost all i. We thus have a map $A \to R$, defined by $a \mapsto (a \cdot 1)$. This makes R into an A-algebra.

For a finite set of indices S we define

$$A_S := \{a \in A \mid a \cdot 1 \in R_i^0 \text{ for all } i \notin S\}$$

This is a subring of A.

We call the indices where R_i^0 is not a subring the archimedian indices, reminiscent of the terminology for global fields. Similarly the indices where R_i^0 is a subring we call non-archimedian. By assumption there are finitely many archimedian indices.

For a finite set of indices S containing the archimedian indices, define $R_S := \prod_{i \in S} R_i \times \prod_{i \notin S} R_i^0$. The ring R_S is naturally an A_S-algebra and we have $R \cong \operatorname{colim}_{S' \supset S} R_{S'}$ as topological rings.

Example 3.20. If $A = k$ is a global field, and the R_i's are its completions, then this satisfies all the assumptions and recovers the definition of the adele ring of k, where $R = \mathbb{A}_k$. The rings R_S are the S-adeles $\mathbb{A}_{k_S}$

Lemma 3.21. *Let X be a scheme of finite presentation over A. Then there exists a finite set S and an A_S-model X_S of X.*

Proof. Since $A = \operatorname{colim} A_S$, this is Theorem 2.10. ∎

If i is a non-archimedean index and X is a separated scheme over A we can regard $X(R_i^0)$ as an open subset of $X(R_i)$ using the open embedding induced by the open inclusion $R_i^0 \hookrightarrow R_i$ (Lemma 3.17).

Lemma 3.22. *Let X_S be a separated, scheme of finite presentation over A_S. Then the natural map*

$$h : X_S(R) = X_S\left(\prod(R_i, R_i^0)\right) \to \prod(X_S(R_i), X_S(R_i^0))$$

is a bijection. If X is affine, then this is a homeomorphism.

Proof. Since X is locally of finite presentation, we get the following identification (Proposition 2.7

$$X_S(R) = X_S\left(\operatorname*{colim}_{S' \supset S} R_{S'}\right) = \operatorname*{colim}_{S' \supset S} X_S(R_{S'}).$$

By Theorem 3.19 we have a natural bijection for $S' \supset S$

$$X_S(R'_S) \longrightarrow \prod_{i \in S} X_S(R_i) \times \prod_{i \notin S} X_S(R_i^0).$$

Applying the colimit to both sides gives us the desired restricted product.

If X is affine, then like in the proof of Theorem 3.19 we are reduced to the case of affine space itself. So we just need that $R^n = \prod(R_i^n, (R_i^0)^n)$. This is Proposition 2.27. ∎

Lemma 3.23. *Assume R_i has open unit group for each index. Let X_S be a separated scheme of finite type over A_S with finite affine open covering $\{U_{S,i}\}$. Denote $U_S = \coprod U_{S,i}$, then the induced map*

$$\Psi : \prod(U_S(R_i), U_S(R_i^0)) \longrightarrow \prod(X_S(R_i), X_S(R_i^0))$$

is an open, surjective, continuous map.

Proof. Firstly we have that for a given finite set S' of indices containing the archimedian indices, that the natural map

$$\Psi_{S'} : \prod_{i \in S'} U_S(R_i) \times \prod_{i \notin S'} U_S(R_i^0) \longrightarrow \prod_{i \in S'} X_S(R_i) \times \prod_{i \notin S'} X_S(R_i^0)$$

is open as it is a product of open maps.

Take an open $W \subset \prod(U_S(R_i), U_S(R_i^0))$. And let $Z = \Psi W$. If S' is any finite set of indices containing the archimedian indices then we have the following commutative diagram of topological spaces

$$\begin{array}{ccc} \prod_{i \in S'} U_S(R_i) \times \prod_{i \notin S'} U_S(R_i^0) & \xrightarrow{\Psi_{S'}} & \prod_{i \in S} X_S(R_i) \times \prod_{i \notin S} X_S(R_i^0) \\ {\scriptstyle i_{S'}}\downarrow & & \downarrow{\scriptstyle j_{S'}} \\ \prod(U_S(R_i), U_S(R_i^0)) & \xrightarrow{\Psi} & \prod(X_S(R_i), X_S(R_i^0)) \end{array}$$

Consider $Z_{S'} := j_{S'} \Psi_{S'} i_{S'}^{-1} W \subset \prod(X_S(R_i), X_S(R_i^0))$. This subset is open as i'_S is continuous and Φ_S and $j_{S'}$ are open. If we vary S' it is clear that $\bigcup_{S'} Z_{S'} \subset Z$. Conversely if we have any point z in Z then there exists an S' such that z is in the image of $j_{S'}$. Since Ψ_S is surjective this yields an element in $\prod(U_S(R_i), U_S(R_i^0))$ such that applying Ψ to it yields back x. This shows that $Z = \bigcup_{S'} Z_{S'}$ and hence Z is open. Thus Ψ is open. ∎

We can now prove the main result of this section.

Theorem 3.24. *Assume R_i has open unit group for each i. Then for any separated scheme X of finite type over A, and A_S-model X_S for some finite set of indices containing the archimedian indices, the natural map*

$$h : X(R) \to \prod(X_S(R_i), X_S(R_i^0))$$

is a homeomorphism.

Proof. Fix a subset S together with an A_S-model X_S of X. Let $W \subset \prod(X_S(R_i), X_S(R_i^0))$ be open. To show that $h^{-1}W$ is open we have to show for each map $f : V \to X$ from a finite type scheme V to X that $f_R^{-1}h^{-1}W$ is open. Let f be such a morphism. Let V_S be an A_S-model of V. Then we have the following commutative diagram

$$\begin{array}{ccc} V(R) & \xrightarrow{h_V} & \prod(V_S(R_i), V_S(R_i)) \\ \downarrow{\scriptstyle f_R} & & \downarrow{\scriptstyle i} \\ X(R) & \xrightarrow{h} & \prod(X_S(R_i), X_S(R_i^0)) \end{array}$$

The right vertical map i is continuous and the top map is a homeomorphism, hence pulling back W vertically then horizontally yields an open subset of $V(R)$. This is equal to $f_R^{-1}h^{-1}W$, which is hence open. This implies that $h^{-1}W$ is open. Thus h is continuous. Let $W \subset X(R)$ be open. Let $\{U_i\}$ be a finite affine open cover of X and denote $U = \bigsqcup U_i$. Consider the following commutative diagram

$$\begin{array}{ccc} U(R) & \xrightarrow{h_U} & \prod(U_S(R_i), U_S(R_i)) \\ {\scriptstyle \Phi}\downarrow & & \downarrow{\scriptstyle \Psi} \\ X(R) & \xrightarrow{h} & \prod(X_S(R_i), X_S(R_i^0)) \end{array}$$

The top map is the induced map from h and is a homeomorphism by Lemma 3.22. The map Φ is surjective and continuous and Ψ is surjective and open. It follows that $hW = \Psi h_U \Phi^{-1}W$, which is open. Hence h is an open map. We conclude that h is a homeomorphism. ■

Corollary 3.25. *Let k be a global field. Denote k_v for the completion of k at a place v and by $\mathcal{O}_v$ the corresponding ring of integers for the non-archimedian places. Let X be a finite type scheme over k. Then there is a finite set S of places containing the archimedian places and a $\mathcal{O}_S$-model X_S of S and a natural homeomorphism*

$$X(\mathbb{A}_k) \to \prod(X_S(k_v), X_S(\mathcal{O}_v)).$$

4 Properness and compactness for global fields

In this chapter we showcase two results that give an equivalence between the properness of a scheme X and the compactness of specific sets of rational points. These results are statements about local fields and global fields. The statement for local fields, we call the *local theorem*, is a known statement (see [4]). This result is a core part of the proof of the statement for global fields (the *global theorem*), which is one of the main results of this thesis.

4.1 Local theorem

We equip sets of rational points $X(K)$ with the fine topology. The local theorem reads as follows.

Theorem 4.1. *Let K be a local field and let $k \subset K$ be a subfield. Let X be a scheme of finite type over k, then X is proper if and only if for every finite field extension K'/K the set $X(K')$ is compact.*

We will not display a proof here. However the strategy for this proof was strongly followed to prove Theorem 5.11 and Theorem 6.3. Both are generalisations of the local theorem and the local theorem is a consequence of these theorems; Theorem 5.11 showing the *only if* part and Theorem 6.3 showing the *if* part.

To answer the question as to why we have to consider finite field extensions in the theorem we can consider the following examples. Firstly and trivially it is very much possible for a scheme $X(K)$ to be empty and hence compact, completely independent of properness of X. More interestingly we can consider $X = \operatorname{Spec} \mathbb{R}[X,Y]/(X^2 + Y^2 - 1)$. Since X is affine and not 0-dimensional it is not proper, but it is easily verified that $X(\mathbb{R}) \cong S^1$ (the unit circle), which is compact. However $X(\mathbb{C}) \cong \mathbb{C}^\times$, which is not compact. This highlights the importance of checking the finite extensions of the local field.

4.2 Global theorem

We will generalise the result for local fields to global fields. A priori there is the problem that global fields, like the rational numbers, do not carry a natural topology, while local fields do. To remedy this we will make the statement in terms of the adele ring $\mathbb{A}_k$ of a global field k and the corresponding set of adelic points $X(\mathbb{A}_k)$ as follows.

Theorem 4.2. *Let k be a global field and let X be a scheme of finite type over k. Then X is proper over k if and only if for each finite extension k' the set $X(\mathbb{A}_{k'})$ is compact.*

Proper-to-compact for global fields

Recall the valuative criterion for properness.

Theorem 4.3 (Valuative criterion)**.** *Let $f : X \to Y$ be quasi-separated morphism of finite type then f is proper if and only if for each solid diagram*

$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow{\scriptstyle f} \\ \operatorname{Spec} R & \longrightarrow & Y \end{array}$$

with R a valuation ring and K its field of fractions there exists a unique map $\operatorname{Spec} R \to X$, making the whole diagram commute.

This can be rephrased in terms of the functor of points. Namely if we regard X as a scheme over Y, then the valuative criterion says that X is proper over Y if and only if the natural map $\operatorname{Hom}_Y(\operatorname{Spec} R, X) = X(R) \to X(K) = \operatorname{Hom}_Y(\operatorname{Spec} K, X)$ is a bijection for each valuation ring R together with its field of fractions K.

With the valuative criterion in hand we can prove the proper-to-compact part of the global theorem.

Theorem 4.4. *Let X be a proper scheme over a global field k. Then for each finite field extension $k \subset k'$, the set of adelic points $X(\mathbb{A}_{k'})$ is compact.*

Proof. Let k' be a finite extension of k. Then the base change $X_{k'}$ is proper over k'. We show that $X_{k'}(\mathbb{A}_{k'}) \cong X(\mathbb{A}_{k'})$ is compact. Let $X_{k',S}$ be an $\mathcal{O}_{k',S}$-model of $X_{k'}$ for some finite set of places S, which is proper by Theorem 2.10. By Corollary 3.25 there is a natural homeomorphism

$$X_{k'}(\mathbb{A}_{k'}) \cong \prod (X_{k',S}(k'_v), X_{k',S}(\mathcal{O}_v)).$$

Since $X_{k',S}$ is proper, by the valuative criterion we have that the open inclusion $X_{k',S}(\mathcal{O}_p) \to X_{k',S}(k'_p)$ is a bijection and hence a homeomorphism. This means we have the following equality.

$$X_{k'}(\mathbb{A}_{k'}) \cong \prod (X_{S,k'}(k'_v), X_{S,k'}(k'_v)) = \prod X_{k'}(k'_v)$$

By the local theorem (Theorem 4.1) each of the $X_{k'}(k'_v)$ is compact and so $X_{k'}(\mathbb{A}_{k'})$ is compact as it is a product of compact sets. ∎

Compact-to-proper for global fields

Theorem 4.5. *Let X be a scheme of finite type over a global field k such that for each finite field extension $k \subset k'$ the set $X(\mathbb{A}_l)$ is compact, then X is proper over k.*

Proof. First let k' be a finite field extension of k such that $X(k')$ is non-empty. Using the map $k' \to \mathbb{A}_{k'}$, which induces a map $X(k') \to X(\mathbb{A}_{k'})$, we conclude that $X(\mathbb{A}_{k'})$ is also non-empty. Let v be a place of k'. Let K be a finite field extension of k'_v. There exists a finite field extension $k' \subset \widetilde{k'}$ such that K is the completion of $\widetilde{k'}$ at some place extending v see [12]. Let X_S be an $\mathcal{O}_{\widetilde{k'},S}$-model of $X_{\widetilde{k'}}$. Then since $X(\mathbb{A}_{\widetilde{k'}})$ is non-empty, the projection map $X(\mathbb{A}_{\widetilde{k'}}) \cong \prod(X_{\widetilde{k'}}(\widetilde{k'_v}), X_S(\mathcal{O}_{\widetilde{k'},v})) \to X_{\widetilde{k'}}(K) \cong X(K)$ is surjective. Hence $X(\widetilde{k'_v})$ is the image of a compact set under a continuous map and is itself compact. By Theorem 4.1 it follows that X is proper over k. ∎

5 Proper to compact

In this chapter we discuss properness-to-compactness results. First we consider topological fields. We show that any topological field for which a proper-to-compact result holds, it follows that the field is automatically a local field. Specifically we show that if $\mathbb{P}^1(k)$ is compact for a topological field k, then k is a local field. Afterwards we show a generalisation of the proper-to-compact result for local fields applied more generally to a class of local rings. This proof follows the same reasoning as the proof of the proper-to-compact part of the local theorem in [4]. Lastly we discuss a method meant to prove proper-to-compact results for non-local rings, by relating the non-local case to the local case, using restricted products of localisations of a non-local ring.

5.1 For topological fields

If we want a proper-to-compact result for a topological field k, we need that $\mathbb{P}^1(k)$ is compact, since the projective line is always proper. The following result shows that if this is the case, then we are reduced to the case of locally compact fields, i.e. local fields, for which the theory is already complete.

Theorem 5.1. *Let k be a Hausdorff topological field. If $\mathbb{P}^1(k)$ is compact, then k is locally compact (and hence a local field).*

We prove the theorem using the following definition and lemma.

Definition 5.2. Let X be a noncompact topological space. A map $i : X \to Y$ is called a one-point-compactification if the following hold:

- Y is compact,
- i is a topological embedding,
- the complement $Y \setminus i(X)$ is one point.

The point in the complement $Y \setminus i(X)$ is often denoted as ∞.

Lemma 5.3. *Let X be a Hausdorff topological space. If X admits a Hausdorff one-point-compactification, then X is locally compact.*

Proof. We can view X as an open subset of Y. Let $x \in X$ be a point, we have to show that x has a compact neighbourhood. Take an open U around ∞ not containing x. Then $F = Y \setminus U$ is a closed neighbourhood of x in Y, which is compact since Y is compact. Since X is open in Y, we have that F is also a compact neighbourhood of x in X, so X is locally compact. ∎

Proof of Theorem 5.1. If k is compact, then k is locally compact.

If k is noncompact, then as a set we have $\mathbb{P}^1(k) = k \sqcup \{\star\}$. Since k is Hausdorff, we have that $\mathbb{P}^1(k)$ is Hausdorff as well by Corollary 3.8. It follows that $\star$ is a closed point in $\mathbb{P}^1(k)$, hence k is open in $\mathbb{P}^1(k)$. By assumption $\mathbb{P}^1(k)$ is compact, so $\mathbb{P}^1(k)$ is a one-point-compactification of k, hence k is locally compact by Lemma 5.3. ∎

5.2 For local rings

In this section we generalise the proof of the local theorem given in [4] to a class local topological rings. The general structure of the proof of the local theorem is as follows. First it is shown that the rational points of projective space are compact. This is done using an explicit description of these sets of points. Afterwards a result called Chow's lemma is used to cover arbitrary proper schemes by projective schemes, which eventually leads to a proof of compactness of the sets of rational points of arbitrary proper schemes.

We use local rings because for a local ring R it is possible to describe the points of $\mathbb{P}^n(R)$ using homogeneous coordinates, similar to how homogeneous coordinates work for fields. We will first discuss how this description works. Then we give the statement and the proof of the proper to compactness result for local rings.

Points of projective space

The following theorem provides a categorisation of the points of projective space.

Theorem 5.4. *[11, Lemma 01NE] Let S be a scheme and $\mathbb{P}^n$ denote projective n-space. There is natural bijection between $\mathbb{P}^n(S)$ and the set of pairs $(\mathcal{L}, (s_0, s_1, \ldots, s_n))$ where*

1. *$\mathcal{L}$ is an invertable $\mathcal{O}_S$-module on S, and*
2. *$s_0, s_1, \ldots, s_n$ are generating global sections of $\mathcal{L}$.*

up to the equivalence $(\mathcal{L}, (s_0, s_1, \ldots, s_n)) \sim (\mathcal{N}, (t_0, t_1, \ldots t_n))$ if and only if there is a isomorphism $\varphi : \mathcal{L} \to \mathcal{N}$ such that $\varphi(s_i) = t_i$ for each $0 \leq i \leq n$.

In the case that $S = \operatorname{Spec} R$ is affine, then an invertable $\mathcal{O}_S$-module on S is the same as a finitely generated projective R-module of rank 1.

If R is a local ring, then Kaplansky's theorem[3][7, Thm 2.5] says that any projective module over R is free. A free module of rank 1 over R is isomorphic to R, and elements $(s_0, s_1, \ldots, s_n)$ generate R if and only if at least one of the s_0 is a unit. This is because if R is local then $R \setminus R^\times$ is an ideal. So if all the s_i's are non-units, they will all be in this ideal, and can never generate the whole ring, so we need at least one unit to generate R.

Lastly each automorphism of R corresponds to a multiplication by a unit. So pairs $(R, (s_0, s_1, \ldots, s_n)$ and $(R, (t_0, t_1, \ldots t_n))$ are equivalent if and only if there exists a unit $u \in R^\times$ such that $t_i = us_i$. This recovers the usual definition of homogeneous coordinates. Hence we will denote the point corresponding to the pair $(R, (s_0, s_1, \ldots, s_n))$ by the homogeneous coordinate $(s_0 : s_1 : \ldots : s_n)$. We summarise this in the following corollary.

[3]Kaplansky's theorem for commutative rings is a direct consequence of Nakayama's Lemma.

Corollary 5.5. *Let R be a local ring and $\mathbb{P}^n$ denote projective space. Then the points in $\mathbb{P}^n(R)$ are in natural bijection with the set of homogeneous coordinates $(s_0 : s_1 : \ldots : s_n)$, where at least one s_i is a unit.*

For points of projective space over fields we require that not all the coordinates in a homogeneous coordinate are zero. So we see that the above characterisation for local rings is the same as that what we know for fields.

The standard affine charts work here in the natural way. Given the ith standard affine chart $U_i \to \mathbb{P}^n$, we get the map on points $U_i(R) = R^n \to \mathbb{P}^n(R)$ sending a tuple $(s_1, s_2, \ldots, s_n)$ to the homogeneous coordinate $(s_1 : s_2 : \ldots : s_{i-1} : 1 : s_i : \ldots : s_n)$, where the 1 is in the ith coordinate.

Statement and proof of proper to compact for local rings

Now that we have a full grasp of the structure of the points of projective space, we can reconstruct the proof of the local theorem given in [4] for the more general case of local rings. First we display the result. The theorem will be a direct consequence of Theorem 5.7 and Theorem 5.11.

Theorem 5.6. *Let R be a local Hausdorff topological valuation ring with a compact multiplicatively closed subset R_0 such that $R \setminus R^\times \subset R_0$ and for each $r \in R^\times$, either $r \in R_0$ or $r^{-1} \in R_0$. Let X be a proper scheme with finitely many irreducible components. Then $X(R)$ is compact.*

First we remark that this result generalises the proper to compact statement in the local theorem. If K is a local field, then the unit ball is a subset satisfying the conditions above; it is compact, contains zero, and for each non-zero element r either r or r^{-1} has norm ≤ 1. Any finite extension of a local field is again a local field which again satisfies the conditions. Hence, the proper to compact result follows.

To prove the theorem, we start by showing that the set of points of projective space is compact.

Theorem 5.7. *Let R be a local topological ring with a compact multiplicatively closed subset $R_0 \subset R$ such that $R \setminus R^\times \subset R_0$ and for each $r \in R^\times$ either $r \in R_0$ or $r^{-1} \in R_0$. Then $\mathbb{P}^n(R)$ is compact for all $n \geq 0$.*

Proof. Let $\{U_i\}_i$ be the standard affine open cover of $\mathbb{P}^n$. Since R is local, we have a surjection $\bigsqcup U_i(R) \cong \bigsqcup R^n \to \mathbb{P}^n(R)$. The R-points of projective space can be described by homogeneous coordinates $(x_0 : \ldots : x_n)$, where at least one of the coordinates is a unit. The R-points of U_i correspond to the points of projective space where the ith coordinate is equal to 1.

Consider the compact subset $R_0^n \subset R^n = U_i(R)$, in $U_i(R)$ these are all points such that all coordinates are in R_0. We will argue that the restriction $R_n^0 \to \mathbb{P}^n(R)$ is still surjective. Let $x = (x_0 : \ldots : x_n) \in \mathbb{P}^n(R)$. We assume without loss of generality that it lies in $U_n(R)$, so we can rewrite it as $(x_0 : \ldots : x_{n-1} : 1)$. Assume that there are $k \geq 0$ coordinates of $(x_0, \ldots, x_{n-1})$ that are not in R_0. We assume without loss of generality that it is the first k coordinates. Then by assumption these are all units and $x_0^{-1}, \ldots, x_{k-1}^{-1}$ all lie in R_0. Now either $x_0 x_1^{-1}$ or $x_1 x_0^{-1}$ is in R_0. Assume (without loss of generality) $x_1 x_0^{-1}$ is in R_0. Then scaling $(x_0 : \ldots : x_{n-1} : 1)$ by x_0^{-1} gives us coordinates of which at most $k-1$ are not in R_0. So we can inductively reduce the number of coordinates not in R_0 to 0. So, all points lie in a $U_i(R)$ such that all coordinates are in R_0. Hence, the map is surjective as claimed. So $\mathbb{P}^n(R)$ is the image of a compact space, and hence compact. ■

This theorem gives rise to the following terminology.

Definition 5.8. Let R be a local topological ring. We call a subset $R_0 \subset R$ a *good* subset if R_0 contains $R \setminus R^\times$ and for each $r \in R^\times$ either $r \in R_0$ or $r^{-1} \in R_0$.

Example 5.9. Archimedian and non-archimedian local fields provide examples of this. In both cases we can take the unit ball around 0 to be the good subset. For non-archimedian local fields this yields the valuation ring, which is known to be compact and clearly satisfies the other properties. For archimedian local fields, for example $\mathbb{R}$ the unit ball is the closed interval $[-1, 1]$ (or the unit disc for $\mathbb{C}$), which is of course compact, and is easily verified to satisfy the other properties. These examples illustrate that a good subset can be a subring but need not be.

The following result is vital in the proof for compactness for arbitrary proper schemes.

Lemma 5.10. *[11, Lemma 0203][Chow's lemma] Let S be a quasi-compact and quasi-separated scheme. Let X be a proper scheme over S with finitely many irreducible components. Then there exists a projective scheme Y over S and a surjective morphism $\varphi : Y \to X$ over S and an open dense subscheme $U \subset X$ such that $\varphi : \varphi^{-1}(U) \to U$ is an isomorphism.*

Note that if S is affine, then it satisfies the conditions of Chow's lemma, as affine schemes are always quasi-compact and quasi-separated.

Theorem 5.11. *Let R be a topological Hausdorff valuation ring such that $\mathbb{P}^n(R)$ is compact for each $n \geq 1$. Then for any proper scheme X over R with finitely many irreducible components, the set $X(R)$ is compact.*

Proof. Consider a projective scheme Y. Since R is Hausdorff, a closed immersion $Y \hookrightarrow \mathbb{P}^n$ induces a closed embedding $Y(R) \hookrightarrow \mathbb{P}^n(R)$. Closed subsets of compact Hausdorff spaces are compact, so $Y(R)$ is compact.

The general case follows by induction on $\dim X$.

$\dim X = 0$: Since X is of finite type over R, the set $X(R)$ is finite, so it is compact

$\dim X > 0$: By Chow's lemma there exists surjective map $\varphi : Y \to X$ with Y projective and an open and dense subscheme $U \subset X$ such that $\varphi : \varphi^{-1}(U) \to U$ is an isomorphism. Define $Z = X \backslash U$. Since U is open and dense, we have $\dim Z < \dim X$ and since Z is a closed subscheme of X, it is proper and hence by induction $Z(R)$ is compact. We can view both $Z(R)$ and $\varphi_R(Y(R))$ as subsets of $X(R)$. Since φ is an isomorphism when restricted to U, it follows that $U(R) \subset \varphi_R(Y(R))$. We claim that $X(R) = Z(R) \cup \varphi_R(Y(R))$. Consider a point $f : \operatorname{Spec} R \to X$. Denote x for the closed point of $\operatorname{Spec} R$ and η for the generic point. If x lands in U then the whole of $\operatorname{Spec} R$ lands in U, so $f \in U(R) \subset \varphi_R(Y(R))$. If x lands in Z, then we consider two cases. Either the generic point η lands in Z, which means that its closure also lands in Z as Z is closed, so f is in $Z(R)$, or the generic point lies in U. In this case we can lift the generic point to a point in Y. This defines a map $\operatorname{Spec} R_\eta \to Y$. Note that the stalk R_η is the field of fractions of R. We then have the following commutative square.

$$\begin{array}{ccc} \operatorname{Spec} R_\eta & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \operatorname{Spec} R & \longrightarrow & X \end{array}$$

The map $X \to Y$ is proper as a morphism between proper schemes, so by the valuative criterion there is a lift $\operatorname{Spec} R \to Y$. This implies that $f \in \varphi_R(Y(R))$. We conclude that $X(R) = Z(R) \cup \varphi_R(Y(R))$. Both subsets being compact (by induction or being the image of a compact set, respectively) implies that $X(R)$ is compact, as desired. ■

The equality $X(R) = Z(R) \cup \varphi_R(Y(R))$, is the only place in this proof where the difficulty lies in terms of generalising the proof in [4]. If R is a field then $\operatorname{Spec} R$ is just one point. Hence the validity of the decomposition $X(R) = Z(R) \cup \varphi_R(Y(R))$ is trivial. The fact that for this equality we need that R is a valuation ring gives much less room for generalisation than expected.

Remark 5.12. If we assume that R is a noetherian valuation ring (equivalently a discrete valuation ring), then every scheme of finite type has finitely many irreducible components, allowing us to remove the condition from the statement.

5.3 For higher local fields

Some of our research went into proving a proper-to-compact result for so-called higher local fields. As we already know, the theory for proper-to-compact results for fields is complete by Chapter 5.1. Despite this we still want to share some of our insights into higher local fields, relating to this thesis. We briefly introduce the concept of higher local fields; for a more in detail introduction see [8]. Higher local fields are higher dimensional analogues of regular local fields, with their dimension defined inductively. A complete discrete valuation field is a field with a discrete valuation, such that it is complete with respect to the valuation. For a complete discrete valuation field K we write $\mathcal{O}_K$ for its valuation ring and $\overline{K}$ for the residue field of $\mathcal{O}_K$.

Definition 5.13. Let K be a field. The *complete discrete valuation dimension* $\mathrm{cdvdim} K$ is defined to be 0 if K is not a complete discrete valuation field. If K is a complete discrete valuation field, then $\mathrm{cdvdim} K$ is defined to be $\mathrm{cdvdim}\overline{K} + 1$.

This procedure gives a chain of fields

$$K^{(0)} := K \subset K^{(1)} := \overline{K} \subset K^{(2)} := \overline{\overline{K}} \subset \cdots .$$

This chain need not be finite, in that case the complete discrete valuation dimension is said to be ∞. These fields tend to be unnatural, and are generally not considered.

Example 5.14. The classical of a high dimensional complete discrete valuation field, is given an arbitrary field k. We have the field of Laurent polynomials in multiple variables

$$K = k((t_1)) \cdots ((t_n)).$$

This is a field with $\mathrm{cdvdim} K = \mathrm{cdvdim} k + n$.

Now we can define what a higher local field is.

Definition 5.15. A complete discrete valuation field K of complete discrete valuation dimension n is an n dimensional higher local field if $K^{(n)}$ (the final residue field) is a finite field.

Example 5.16. The field K as in Example 5.14 is an n-dimensional local field if the field k is finite. In particular the local field, and any finite extension of, $\mathbb{F}_p((t))$ is a 1-dimensional local field. The local fields $\mathbb{Q}_p$, are 1-dimensional local fields, as the residue field is $\mathbb{F}_\mathfrak{p}$. The same holds for any finite extension.

Of course we are interested in fields with a topology, so we'll have to define a topology on a higher local field. The obvious choice is taking the topology coming from the valuation. However as it turns out, this topology is too fine for general applications. Another topology called the higher topology is used for higher local fields, this is the topology that is generally used when studying higher local fields. The caveat is that with this topology, a higher local field is not a topological field. To apply our theory we need that our field is at least a topological ring. The higher topology does have continuous addition and multiplication by a constant is continuous. While multiplication is not continuous, it is something called *sequentially continuous*. This means that if two sequences (a_n) and (b_n) converge to some a and b respectively, then the product sequence $(a_n b_n)$ converges to ab. This property allows us to enrich the higher topology to a so called sequential topology. It is a natural way to turn a field with a topology with continuous addition and sequentially continuous multiplication into a topological field.

Let K be a higher local field, then we have multiple topologies that we can define on K. Denote K_h for K with the higher topology, denote K_s for K with the sequential topology, and denote K_v for K with the discrete valuation topology. With the discrete valuation topology being the finest topology there is a continuous map

$K_v \to K_h$. We would like to show for a proper scheme X that $X(K_h)$ is compact. We could use the continuous map $K_v \to K_h$, which induces a surjective continuous map $X(K_v) \to X(K_h)$. If $X(K_v)$ is compact, this then implies that $X(K_h)$ is compact. The obvious idea is to apply Theorem 5.6 to K, with the $\mathcal{O}_K$ as the given subset. Since $\mathcal{O}_K$ is a valuation ring of a field, it is automatically a good subset. The only question is whether it is compact. This is what poses a problem. Namely the maximal ideal $\mathfrak{m}$ is open in $\mathcal{O}_K$, so we can create an open cover of cosets $\{a\mathfrak{m}\}$ of $\mathfrak{m}$. We see that this is a cover of disjoint open sets, so this cover can only have a finite sub cover if the cover itself is finite. But this means that the residue field is finite. We conclude that $\mathcal{O}_K$ is compact if and only if K is a (1-dimensional) local field. This means we cannot apply Theorem 5.6, which is in line with the conclusion of 5.1.

The last thing of discussion is what we called the *topological field closure*, for a field with a topology, which would be constructed by naturally adjoining opens, that come from preimages of the addition map and the multiplication map. It was supposed to be the unique smallest extension of a topology on a field with topology K, that makes K into a topological field. We wanted to know if the sequential topology on a higher local field was this topological field closure of the higher topology. We had ideas on how to use this for studying proper-to-compact results for higher local fields. It turns out, however, that this 'naturally adjoining opens' to the topology to gain a unique topological field closure is not possible in general. A counterexample to the uniqueness of this topology is the following. Take the field of rational numbers $\mathbb{Q}$, along with the Zariski topology. This makes $\mathbb{Q}$ into a field with topology that is not a topological field. However, if we now consider the completions $\mathbb{Q}_p$ at every prime and the real numbers $\mathbb{R}$, then these induce topologies on $\mathbb{Q}$, that make $\mathbb{Q}$ into a topological field. Where the intersection of these topologies gives back the Zariski topology. This means that there is an infinite family of 'topological field closures'. The non-existence of this topological field closure explains why there is no mention of it in any literature.

5.4 Localisations of topological rings

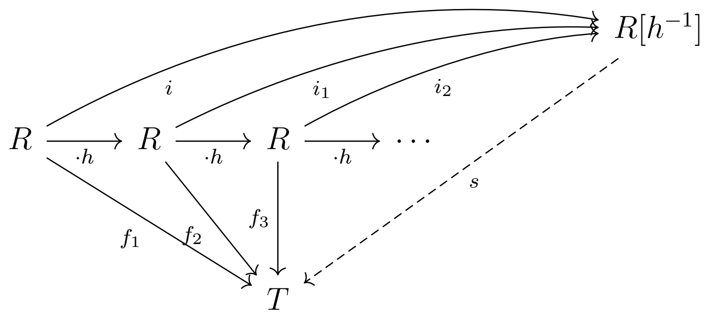
We want to use the result for local rings to prove something about more general (non-local) rings. The rest of the chapter is a discussion of a heuristic argument to show completeness to compactness for non-local rings. The idea relies on taking localisations of a topological ring R at prime ideals, and use the localisation maps $R \to R_\mathfrak{p}$ and the theory we have for the local ring $R_\mathfrak{p}$ to infer a result for the non-local ring R. To make any topological arguments we would of course need a topology on the new ring $R_\mathfrak{p}$. There does not seem to be any theory about this present in the literature. In the next section we propose a definition for a topology and prove basic properties. A main issue with the proposed definition is that there does not seem to be a clear way to show that the ring with a topology yields a topological ring in general. Studying this topology would be a good point for further research.

Given a topological ring R and a multiplicatively closed subset S, we define a topology on $S^{-1}R$. We do this in two steps. First given an $s \in S$, we define a topology on $R[s^{-1}]$ in terms of a filtered colimit. Afterwards we take the (filtered) colimit of the new $R[s^{-1}]$ to get $S^{-1}R$ with a topology. This topology turns $S^{-1}R$ into a topological R-module, but not necessarily into a topological ring. We study when this actually yields a topological ring and prove other properties of this localisation. We start with the following algebraic result.

Lemma 5.17. *Let R be a ring and let $h \in R$. Consider the diagram of R-modules and R-module homomorphisms $R \xrightarrow{\cdot h} R \xrightarrow{\cdot h} \cdots$ where each map is multiplication by h. Then $R[h^{-1}] \cong \operatorname{colim}(R \xrightarrow{\cdot h} R \xrightarrow{\cdot h} \cdots)$ as an R-module, where the canonical maps $i_n : R \to R[h^{-1}]$ are given as $i_n = h^{-n} \circ i$ and where $i : R \to R[h^{-1}]$ is the localisation map.*

Proof. Denote $i : R \to R[h^{-1}]$ for the localization map and denote $i_n = h^{-n} \circ i : R \to R[h^{-1}]$. The maps i_n define a map from the diagram $R \xrightarrow{\cdot h} R \xrightarrow{\cdot h} \cdots$ to $R[h^{-1}]$. We show that $R[h^{-1}]$ together with these maps satisfy the universal property of the colimit.

Let T be an R-module and let $f_n : R \to T$ be R-module homomorphisms such that $f_n = f_{n+1} \circ h$. Here we denote h for the multiplication by h map $R \xrightarrow{\cdot h} R$. This defines a map from the diagram to T. We have to show that there exists a unique map s such that the below diagram commutes.



Given an element $\frac{r}{h^n}$ we can define $s\left(\frac{r}{h^n}\right) = f_n(r)$. This map is easily verified to be well defined and is the unique map making the diagram commute. Hence $R[h^{-1}]$ satisfies the universal property of the colimit. ∎

If R is a topological ring, then we can view $R[h^{-1}]$ as a filtered colimit of topological R-modules. As discussed in Chapter 2.3.1 we can compute this topology as the topology obtained when viewing the diagram as a diagram of topological spaces. That means $R[h^{-1}]$ has the finest topology such that each natural map $i_n : R \to R[h^{-1}]$ is continuous. This turns $R[h^{-1}]$ into a topological R-module. In particular addition map $+ : R[h^{-1}] \times R[h^{-1}] \to R[h^{-1}]$ is continuous.

We have the following way of describing the open subsets of $R[h^{-1}]$. A subset $U \subset R[h^{-1}]$ is open if and only if $i_n^{-1}U = (h^{-n}i)^{-1}U = i^{-1}h^nU$ is open for each

$n \geq 0$. It is clear that this is the finest topology such that the maps i_n are continuous for each $n \geq 0$.

Example 5.18. If we equip R with the I-adic topology for some ideal I. And take a localisation $S^{-1}R$, with $S \subset R \setminus I$, containing no zero divisors. Then consider the subset $I^n S^{-1}$; we show that this is open for all $n \geq 1$. This subset is open precisely when $i^{-1}(sI^n S^{-1}R)$ is open in R for all s. Since I is an ideal this is equal to $i^{-1}(I^n S^{-1}R) = I^n$, which is open by definition. This implies that the topology on $S^{-1}R$ is at least finer than the $IS^{-1}R$-adic topology. The natural question to ask is if the topology is equal to the $IS^{-1}R$-adic topology. This question is a good point for further consideration.

In what comes given a ring R and a multiplicatively closed subset S and the localisation map $i : R \to S^{-1}R$, we will by abuse of notation denote s for $i(s) = \frac{s}{1} \in S^{-1}R$ for $s \in S$.

Proposition 5.19. *Let R be a topological ring and an $h \in R$. Let $f : R \to T$ be a continuous ring homomorphism to a ring with topology T with open multiplication by $f(h)$ and such that $f(h)$ is a unit in T, then the induced map $R[h^{-1}] \to T$ is continuous.*

Proof. Denote $t := f(h)$. The induced map $g : R[h^{-1}] \to T$ is defined by sending a $\frac{a}{h^n}$ to $f(a)t^{-n}$. Let $U \subset T$ be open. Then $g^{-1}U$ is open if and only if $i^{-1}h^n g^{-1}$ is open for each $n \geq 0$. It is easily verified that $i^{-1}h^n g^{-1}U = i^{-1}g^{-1}s^n U$. This is in turn equal to $f^{-1}s^n U$. Since U is open and multiplication by s is open it follows that $f^{-1}s^n U$ is open by continuity. Hence $g^{-1}U$ is open so g is continuous. ∎

To get the full localisation $S^{-1}R$ we take the colimit of the intermediate localisations $R[h^{-1}]$ by taking the diagram where the objects are the $R[s^{-1}]$ for each $s \in S$ and there is a morphism $R[s^{-1}] \to R[t^{-1}]$ precisely when $s|t$, coming from the universal property of localisations. Then we can write $S^{-1}R = \operatorname{colim}_{s \in S} R[s^{-1}]$. One can check that then the topology on $S^{-1}R$ is the finest topology such that $i_s := s^{-1} \circ i : R \to S^{-1}R$ is continuous. Thus a subset $U \subset S^{-1}R$ is open if and only if $i^{-1}sU$ is open in R. We see by the proposition above that $S^{-1}R$ is a colimit of topological R-modules and hence is itself a topological R-module and if all the $R[s^{-1}]$ are topological rings then $S^{-1}R$ will also be a topological ring.

Remark 5.20. Since $S^{-1}R$ is a topological R-module, not only is the addition map continuous, but also the scalar multiplication map $R \times S^{-1} \to R$ is continuous. Hence multiplication by a constant from R defines a continuous map $S^{-1}R \to S^{-1}$.

Theorem 5.21. *Let R be a topological ring with a multiplicatively closed subset S such that for each $s \in S$ multiplication by s is open. Then the localisation map $i : R \to S^{-1}R$ is open.*

Proof. Let $U \subset R$. We have to show that iU is open. The subset iU is open if and only if $i^{-1}siU$ is open in R for each $s \in S$. Since i is a ring homomorphism we have that $i^{-1}siU = i^{-1}isU$ (note the abuse of notation here). Furthermore by properties of ring homomorphisms, $i^{-1}isU = \ker i + sU$, which we can rewrite as $\bigcup_{x \in \ker i} x + sU$. Since multiplication by s is open, the subset sU is open and since addition is continuous it follows that $x + sU$ is open for each $x \in \ker i$. Hence $i^{-1}siU$ can be written as a union of open subsets, so it is open. It follows that the localisation map is open. ∎

Theorem 5.22. *Let R be a topological ring with a multiplicatively closed subset S, such that the localisation map $i : R \to S^{-1}R$ is open. Then $S^{-1}R$ is a topological ring.*

Proof. To show that $S^{-1}R$ is a topological ring we have to show that the multiplication map $S^{-1}R \times S^{-1}R \to S^{-1}R$ is continuous, where $S^{-1}R \times S^{-1}R$ carries the product topology. Let $U \subset S^{-1}R$ and let $s, t \in S$ be arbitrary. Then we can consider the following commutative diagram.

$$\begin{array}{ccc} R \times R & \xrightarrow{m_1} & R \\ \downarrow{\scriptstyle (i_s, i_t)} & & \downarrow{\scriptstyle i_{st}} \\ S^{-1}R \times S^{-1}R & \xrightarrow[m_2]{} & S^{-1}R \end{array}$$

Here m_1 and m_2 denote the respective multiplication maps. We have to show that $m_2^{-1}U$ is open in $S^{-1}R \times S^{-1}R$. We want to say that m_2^{-1} is open if and only if $(i_s, i_t)^{-1}m_2^{-1}U$ is open for each pair (s, t). In general it is not true that the product topology is the same as this colimit topology, however since we have that the canonical map i is open and also multiplication by $s : S^{-1}R \to S^{-1}R$ is open for each $s \in S$, it follows from Lemma 2.2 that the product and the colimit topology do coincide. Since $(i_s, i_t)^{-1}m_2^{-1}U = m_1^{-1}i_{st}^{-1}U$ by commutativity and m_1 and i_{st} are continuous we have that $(i_s, i_t)^{-1}m_2^{-1}U$ is open for any (s, t) and hence $m_2^{-1}U$ is open. So m_2 is continuous. We conclude that $S^{-1}R$ is a topological ring. ∎

Proposition 5.23. *Let R be a topological ring and let S be a multiplicatively closed subset. Assume that the localisation map $i : R \to S^{-1}R$ is injective. Then the following hold:*

- *The map i is open if and only if multiplication by s in R is an open map for each $s \in S$.*
- *The map i is closed if and only if multiplication by s in R is a closed map for each $s \in S$.*

Proof. We only show the first statement. The second statement follows completely analogously.

Let $U \subset R$ be an open subset. The image iU is open if and only if $i^{-1}isU$ is open for all $s \in S$. Since i is injective we have $i^{-1}isU = sU$. So if i is open, then iU is open for any open subset. Then this implies that sU is open for $s \in S$ and for any open U, meaning that multiplication by s is open for each $s \in S$. Conversely if multiplication by s is open then sU is open for $s \in S$, which implies that iU is open. ∎

Remark 5.24. If the localisation map $i : R \to S^{-1}R$ is injective and open/closed, then i is an embedding, meaning that we can view R as a subset of $S^{-1}R$. An embedding need not be open or closed, so we can ask ourselves if in general the map $i : R \to S^{-1}R$ is an embedding. We have no answer to this yet, so we leave it as an open question.

In the case that the localisation map is both open and injective, there is easier way to show that localisation $S^{-1}R$ is a topological ring. While this is already a consequence of Theorem 5.22, we still display it to show the technique.

Proposition 5.25. *Let R be a topological ring and S a multiplicatively closed subset, such that the localisation map $R \to S^{-1}R$ is injective and open. Then $S^{-1}R$ is a topological ring.*

Proof. We show that the multiplication map is continuous by showing it is continuous at every point. Take $x, y \in S^{-1}R$, and consider an open neighbourhood U of xy. We need to show that there exist open neighbourhoods U_a and U_b of x and y respectively such that $U_a \cdot U_b \subset U$. We can write $x = \frac{a}{s}$ and $y = \frac{b}{s'}$, with $a, b \in R$ and $s, s' \in S$. The set $ss'U$ is an open neighbourhood of ab in $S^{-1}R$, so $ss'U \cap R$ is an open neighbourhood of ab in R. Since R is a topological ring, there exist open neighbourhoods V_a and V_b in R of a and b in respectively such that $V_a \cdot V_b \subset W$. Since i is open, V_a and V_b are also open in $S^{-1}R$. Multiplying these opens by s^{-1} and $(s')^{-1}$ respectively gives open neighbourhoods the $s^{-1}V_a$ and $(s')^{-1}V_b$ of x and y, such that $s^{-1}V_a(s')^{-1}V_b \subset U$. We conclude that $S^{-1}R$ is a topological ring. ∎

We finish with two permanence properties that might be of use when studying this topology more.

It is an elementary fact that Hausdorff spaces have closed points, meaning that all the singleton sets $\{x\}$ are closed. The following proposition characterises when the property of having closed points for R is preserved when localising, in terms of the kernel of the localisation map $R \to S^{-1}R$.

Proposition 5.26. *Let R be a topological ring with a multiplicatively closed subset S and denote the localisation map $i : R \to S^{-1}R$. Then $S^{-1}R$ has closed points if and only if $\ker i$ is closed.*

In particular if i is injective and R has closed points then $S^{-1}R$ has closed points.

Proof. If $S^{-1}R$ has closed points then $\ker i = i^{-1}\{0\}$ is closed by continuity.

Assume $\ker i$ is closed. A point $x \in S^{-1}R$ is closed if and only if $i^{-1}(\{sx\})$ is closed for each $s \in S$. If sx is not in the image of i, then the preimage is empty, which is closed, otherwise the preimage of sx is of the form $r + \ker i$, where $i(r) = sx$. Since the kernel is closed also $r + \ker i$ is closed. Hence x is a closed point.

If i is injective and R has closed points, then $\ker i = \{0\}$ is closed, so $S^{-1}R$ has closed points. ∎

Proposition 5.27. *Let R be a Hausdorff topological ring and S a multiplicatively closed subset such that $R \to S^{-1}R$ is closed and surjective. Then $S^{-1}R$ is a Hausdorff space.*

Proof. We will show that the diagonal map $\Delta_{S^{-1}R} : S^{-1}R \to S^{-1}R \times S^{-1}R$ is closed. We can fit this map into the following commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{\Delta_R} & R \times R \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i \times i} \\ S^{-1}R & \xrightarrow[\Delta_{S^{-1}R}]{} & S^{-1}R \times S^{-1}R \end{array}$$

The map Δ_R is closed since R is Hausdorff and (i, i) is closed as it is a product of closed maps. Let $Z \subset S^{-1}R$. By surjectivity of i the image $\Delta_{S^{-1}R}Z$ is equal to $(i, i)\Delta_R i^{-1}Z$. This is closed by continuity of i and closedness of Δ_R and (i, i). We conclude that $S^{-1}R$ is Hausdorff. ∎

Remark 5.28. In a dual way we can show the converse in the case that $i : R \to S^{-1}R$ is injective instead of surjective. Specifically if i is injective and closed then $S^{-1}R$ is Hausdorff only if R is Hausdorff.

Proposition 5.29. *Let R be a topological ring and $S \subset R$ a multiplicatively closed subset. Assume the localisation map $i : R \to S^{-1}R$ is open. If R is locally compact, then $S^{-1}R$ is locally compact.*

Proof. Let $x \in S^{-1}R$ we can multiply x by an $s \in S$ so that $sx \in i(R)$. Finding a compact neighbourhood of x is equivalent to finding a compact neighbourhood of sx as multiplication by s is a homeomorphism. So we can assume without loss of generality that x is in the image of i Take an $r \in i^{-1}\{x\}$. Since R is locally compact there is a compact neighbourhood C of r in R. Meaning C is compact and there is an open U such that $r \in U \subset C$. The map i is open so taking images yields $x \in iU \subset iC$ with iU open and iC compact. So iC is a compact neighbourhood of x, hence $S^{-1}R$ is locally compact. ∎

Open questions

We finish this section with a list of open questions. Let R be a topological ring, and S a multiplicatively closed subset.

1. Are there more general ways to show when $S^{-1}R$ forms a topological ring, that does not necessarily rely on open multiplication by elements in S?
2. With R, I, and S as in Example 5.18. When does $S^{-1}R$ have the $IS^{-1}R$-adic topology?

3. What are general conditions under which the localisation map $R \to S^{-1}R$ is closed?
4. When is the localisation map $R \to S^{-1}R$ an embedding?
5. When does R with open unit group imply that $S^{-1}R$ has open unit group?

5.5 An argument for non-local rings

Now that we have a candidate for a topology on localisations of topological rings, we can discuss the argument for showing properness to compactness for non-local rings. We will apply this argument to the ring of adeles, which provides us with an example, albeit in some sense trivial. Besides this, we neither have other examples nor a set of practical criteria for when the argument works. The main result of this section is inconclusive. It is a result with insofar no practically verifiable conditions and no non-trivial examples. Nevertheless we outline the idea and discuss the obstacles.

Assume that R is a non-local topological ring and that X is a proper scheme over R. This last section discusses a possible method of showing that $X(R)$ is compact by relating this non-local case back to the local case. We do this by considering maps of the form $R \to \prod(R_\mathfrak{p}, R_\mathfrak{p}^0)$, which sends R into a restricted product of localisations $R_\mathfrak{p}$ at prime ideals $\mathfrak{p}$ with respect to some suitable subsets $R_\mathfrak{p}^0$.

This map of rings induces a map on rational points $X(R) \to X(\prod(R_\mathfrak{p}, R_\mathfrak{p}^0))$. The main idea is that if the map $X(R) \to X(\prod(R_\mathfrak{p}, R_\mathfrak{p}^0))$ is closed and injective and if $X(\prod(R_\mathfrak{p}, R_\mathfrak{p}^0))$ is compact, then we can conclude that $X(R)$ is a closed subset of a compact space, and is hence compact. We can use our theory for local rings to show the compactness of $X(\prod(R_\mathfrak{p}, R_\mathfrak{p}^0))$, similar to how we proved the global (theorem 4.4).

We want that the pairs $R_\mathfrak{p}^0 \subset R_\mathfrak{p}$ indexed by $\mathfrak{p} \in \mathfrak{P}$ satisfy the conditions outlined in Chapter 3.1, with archimedian and non-archimedian primes taking the place of archimedian and non-archimedian indices. This allows us to use Theorem 3.24, which says that $X(\prod(R_\mathfrak{p}, R_\mathfrak{p}^0)) = \prod(X(R_\mathfrak{p}), X_S(R_\mathfrak{p}^0))$, for some $\mathcal{O}_S$-model X_S of X, with S a finite set of primes containing the archimedian ones. Just like in the proof of the global theorem it now suffices to show that $X(R_\mathfrak{p})$ is compact for all $\mathfrak{p}$ and that $X_S(R_\mathfrak{p}^0) = X(R_\mathfrak{p})$ for almost all $\mathfrak{p}$, in order to get compactness of $X(\prod(R_\mathfrak{p}, R_\mathfrak{p}^0))$. To define the $R_\mathfrak{p}^0$ we will start with a fixed subset $R^0 \subset R$ and take $R_\mathfrak{p}^0$ to be the image of R^0 under the localisation map $R \to R_\mathfrak{p}$. To show compactness of $X(R_\mathfrak{p})$ we want that $R_\mathfrak{p}^0$ is a good compact subset. The assumption is that there are nice conditions on R^0, so that the subsets $R_\mathfrak{p}^0$ are good and compact. An easy first observation is that if R^0 is compact then so are the $R_\mathfrak{p}^0$. So we will assume the subset R^0 to be compact.

All of the arguments in the rest of of the chapter only work if each localisation $R_\mathfrak{p}$ of a topological ring R is again a topological ring. In the previous chapter we discussed some conditions for when this holds true, but this theory is far from complete. So for ease we ignore this and assume that all localisations are topological rings.

The following result gives concrete conditions for when the set $X(\prod(R_\mathfrak{p}, R^0_\mathfrak{p}))$ is compact. It uses the proper-to-compact result from this chapter (Theorem 5.6) and the same proof structure as the proof for the proper-to-compact result of the global theorem.

Theorem 5.30. *Let A be a ring and let $\{R_i\}_i$ be a family of topological A-algebras together with good compact subsets $R_i^0 \subset R_i$, such that*

1. *R_i is a valuation ring for all i,*
2. *R_i^0 is an open subring for almost all i, with the same fraction field as R_i,*
3. *for each $a \in A$, its image in R_i is contained in R_i^0 for almost all i.*

Let $R = \prod(R_i, R_i^0)$. Let X be a scheme of finite presentation over A. If X is proper over A, then $X(R)$ is compact.

Proof. We use the terminology outlined in Chapter 3.1. First take a finite set S containing the archimedian indices and an A_S-model X_S of X. Then by Theorem 3.24 there is a homeomorphism $X(R) \cong \prod(X_S(R_i), X_S(R_i^0))$. By Theorem 5.6 we have that $X_S(R_i)$ is compact for each i.

Furthermore for each non-archimedian index i we have that R_i^0 is also a valuation ring, as a subring of a valuation ring is again a valuation ring and R_i^0 has the same fraction field as R_i. The valuative criterion gives us the bijections $X_S(R_i^0) \to X_S(k_i)$ and $X_S(R_i) \to X_S(k_i)$, with k_i being the fraction field of R_i and R_i^0. Composing the first bijection with the inverse of the second bijection gives us a bijection $X_S(R_i^0) \to X_S(R_i)$. This map is naturally the same as the map induced by the inclusion of R_i^0 in R_i, which is open and continuous. Hence the bijection $X_S(R_i^0) \to X_S(R_i)$ is also open and continuous and thus a homeomorphism.

We conclude that $\prod(X_S(R_i), X_S(R_i^0)) = \prod X(R_i)$, which is a product of compact spaces and, therefore, compact. ∎

The next theorem formalises the argument outlined at the beginning of this section. Before we show it we discuss where the argument came from. The argument based on the following "trivial" case.

Example 5.31. Let k be a global field and $\mathbb{A}_k$ its ring of adeles. We consider the compact subset $R^0 = \prod \mathcal{O}_v \subset \mathbb{A}_k$. We take the set of prime ideals to be the maximal ideals of the form $\mathfrak{m}_{v_0} = \{(a_v) \in \mathbb{A}_k \mid a_{v_0} = 0\}$. It is easily verified that the localisation $(\mathbb{A}_k)_{\mathfrak{m}_{v_0}}$ gives back the completion k_{v_0} and the localisation map $\mathbb{A}_k \to k_{v_0}$ is just the projection map. The corresponding subset $R^0_{\mathfrak{m}_{v_0}}$ is just $\mathcal{O}_{v_0}$. So the map we study is the map $\mathbb{A}_k \to \prod(k_v, \mathcal{O}_v)$ induced by the projection maps. This map is of course an isomorphism of topological rings as it is just the definition of the ring of adeles. It follows that the map on rational points $X(\mathbb{A}_k) \to X(\prod(k_v, \mathcal{O}_v))$ is a homeomorphism. In particular the map is closed and injective. The right hand side $X(\prod(k_v, \mathcal{O}_v))$ is compact by Theorem 5.30. Finally we have that $X(\mathbb{A}_k)$ is a closed subset of the compact space $X(\prod(k_v, \mathcal{O}_v))$, hence $X(\mathbb{A}_k)$ is compact.

Of course in practise this argument is the same as the proof of the global theorem, in fact Theorem 5.30 is a generalisation of the argument used for the global theorem. The hope was that we could extract a more general argument, where the map is injective and closed but not a homeomorphism. While injectivity is generally not a problem (Lemma 5.35), the closedness of the map is elusive. The reason this example is so trivial is that the only reason we know that the map is closed, is because it is a homeomorphism, essentially by construction. General ways of showing such maps are closed have not been found, but is a vital step in the argument. Further research into intrinsic properties for which the map into the restricted product is closed is crucial for the following theorem to be more meaningful. This problem does not only arise for the closedness of the map but also for the other assumptions in the theorem. The other assumptions in the theorem are only found to be satisfied in this example because we know exactly what all the localisations are. In practice we would like to apply this argument to situations where we only rely on intrinsic properties of the ring R to extrapolate these properties.

Theorem 5.32. *Let R be a topological ring. Assume R has a compact subset R_0 and a set of prime ideals $\mathfrak{P}$ such that the following hold*

1. *The subset $R_{\mathfrak{p},0} \subset R_{\mathfrak{p}}$ is good for all $\mathfrak{p} \in \mathfrak{P}$,*
2. *$R_{\mathfrak{p}}$ is a valuation ring, $R_{\mathfrak{p},0}$ is an open subring, and they have the same fraction field for almost all $\mathfrak{p} \in \mathfrak{P}$,*
3. *For each $r \in R$, $\frac{r}{1} \in R_{\mathfrak{p}}$ lies in $R_{\mathfrak{p},0}$ for almost all $\mathfrak{p} \in \mathfrak{P}$,*
4. *The map $R \to \prod(R_{\mathfrak{p}}, R_{\mathfrak{p},0})$ is injective.*

Let X be a proper scheme over R. Assume that the induced map $X(R) \to X(\prod(R_{\mathfrak{p}}, R^0_{\mathfrak{p}}))$ is closed, then $X(R)$ is compact.

Proof. The closed embedding embeds $X(R)$ as a closed subset of $X(\prod(R_{\mathfrak{p}}, R^0_{\mathfrak{p}}))$. The set $X(\prod(R_{\mathfrak{p}}, R^0_{\mathfrak{p}}))$ is compact by Theorem 5.30. So $X(R)$ is closed subset of a compact space, thus is itself compact. ∎

5.6 Discussion

We finish this chapter with a discussion of the argument for non-local rings, the obstacles and points for further research.

Assume we have a topological ring R with a compact subset R^0, to apply the argument, there is the task to find a set of prime ideals for which the conditions in Theorem 5.32 hold. We want to not fully rely on knowing exactly what the localisations are, like we did in the example of the ring of adeles. Instead we want intrinsic properties. Below are questions that would need to be answered in order to obtain this. These correspond to the conditions of the theorem. We discuss these one by one. We do not have full answers to all these questions yet; it serves as a to-do list for further research.

1. When are $R_\mathfrak{p}^0 \subset R_\mathfrak{p}$ good?
2. When is a localisation $R_\mathfrak{p}$ a valuation ring?
3. When is $R_\mathfrak{p}^0$ a (sub)ring?
4. When is $R_\mathfrak{p}^0$ open in $R_\mathfrak{p}$?
5. When is the map $R \to \prod(R_\mathfrak{p}, R_\mathfrak{p}^0)$ injective?
6. When is the map $R \to \prod(R_\mathfrak{p}, R_\mathfrak{p}^0)$ closed?
7. When does a closed map of topological rings $R \to R'$ induce a closed map $X(R) \to X(R')$?

Question 1: When are $R_\mathfrak{p}^0 \subset R_\mathfrak{p}$ good?

We can give a partial answer to this question. Recall that a subset $R_\mathfrak{p}^0 \subset R_\mathfrak{p}$ is good if $R_\mathfrak{p} \setminus R_\mathfrak{p}^\times \subset R_\mathfrak{p}$, R_p^0 is a multiplicative subset, and if for each unit $r \in R_\mathfrak{p}^\times$ either $r \in R_\mathfrak{p}^0$ or $r^{-1} \in R_\mathfrak{p}^0$. Multiplicativity and the condition on the units are easy, by the following lemma.

Lemma 5.33. *Let R be a ring and let $R^0 \subset R$ be a multiplicative subset with the property that for each unit $r \in R^\times$, either r or r^{-1} is in R^0. Let $\mathfrak{p}$ be a prime ideal of R, then the subset $R_\mathfrak{p}^0 \subset R_\mathfrak{p}$ again satisfies these properties.*

Proof. let $a, a' \in R_\mathfrak{p}^0$, then $a = i(r)$ and $a' = i(r')$ for some $r, r' \in R^0$, where i denotes the localisation map $R \to R_\mathfrak{p}$. Since R^0 is multiplicative it follows that rr' is in R^0, so $aa' = i(rr')$ is in $R_\mathfrak{p}^0$.

Furthermore let $u \in R_\mathfrak{p}^\times$ be a unit. The preimage $i^{-1}u$ contains a unit v. By assumption either v or v^{-1} is in R^0. Since $v^{-1} \in i^{-1}u^{-1}$ it follows that either u or u^{-1} is in $R_\mathfrak{p}^0$. ∎

For $R_\mathfrak{p}^0$ to be a good subset we also need that the maximal ideal $\mathfrak{p}R_\mathfrak{p}$ is contained in $R_\mathfrak{p}^0$. In general this is not so easy to describe intrinsically. However if we require that the localisations $R_\mathfrak{p}$ are fields this is easy. Then we only need to require that R^0 contains 0. Intrinsically this means that we need to localise at minimal prime ideals $\mathfrak{p}$ assuming that R is reduced as illustrated by the following lemma.

Lemma 5.34. *Let R be a reduced ring and $\mathfrak{p}$ be a prime ideal. Then the localisation $R_\mathfrak{p}$ is a field if and only if $\mathfrak{p}$ is a minimal prime ideal.*

Proof. Assume that $\mathfrak{p}$ is minimal, then $\mathfrak{p}R_\mathfrak{p}$ is also minimal, so the nilradical of $R_\mathfrak{p}$ is equal to $\mathfrak{p}R_\mathfrak{p}$. The nilradical of R is 0, thus so is the nilradical of $R_\mathfrak{p}$. Hence $\mathfrak{p}R_\mathfrak{p} = 0$, so $R_\mathfrak{p}$ is a field.

Conversely if $R_\mathfrak{p}$ is a field. Let $\mathfrak{q}$ be a prime ideal such that $\mathfrak{q} \subset \mathfrak{p}$. This means that there is a localisation map $R_\mathfrak{p} \to R_\mathfrak{q}$. But since $R_\mathfrak{p}$ is a field it follows that $R_\mathfrak{p} = R_\mathfrak{q}$, hence $\mathfrak{p} = \mathfrak{q}$. So $\mathfrak{p}$ is minimal. ∎

So requiring that the localisations are all field would give a good intrinsic condition for goodness of $R^0_\mathfrak{p}$. However it is important to note that a topological fields with a good compact subset is automatically a local field. Because we know that proper to compact results for topological fields only hold for local fields, as shown in the beginning of this chapter.

Question 2: When is the localisation $R_\mathfrak{p}$ a valuation ring?

This question is either answered very easily by requiring the localisations to be fields as in the previous question, meaning we localise at minimal prime ideals. Besides this no real results have been found.

Question 3: When is $R^0_\mathfrak{p}$ a (sub)ring?

This question is of course easily solved by assuming that R^0 is a subring. However in the case of the adeles we take $R^0 = \prod \mathcal{O}_v$, which is not a subring as $\mathcal{O}_v$ is not a ring for the archimedian places. So this should not be strictly necessary.

Question 4: When is $R^0_\mathfrak{p}$ open in $R_\mathfrak{p}$?

According to Theorem 5.21, if we assume that multiplication in R is open, then for each prime ideal the localisation map $R \to R_\mathfrak{p}$ is open. If we then further require that R^0 is open in R, it follows that $R^0_\mathfrak{p}$ is open in $R_\mathfrak{p}$.

Question 5: When is the map $R \to \prod(R_\mathfrak{p}, R^0_\mathfrak{p})$ injective?

This can be easily solved using the following lemma.

Lemma 5.35. *Let R be a ring and $\mathfrak{P}$ a collection of prime ideals such that $\bigcap_{\mathfrak{p}\in\mathbb{P}} \mathfrak{p} = 0$. Then the natural map $R \longrightarrow \prod R_\mathfrak{p}$ is injective.*

Proof. Let $r \in R$, by assumption there is a $\mathfrak{p} \in \mathfrak{P}$ such that $r \notin \mathfrak{p}$. This means that $\frac{r}{1}$ is a unit in $R_\mathfrak{p}$, thus nonzero. Hence, the image of r in $\prod R_\mathfrak{p}$ is nonzero. ∎

Note that the existence of such prime ideals implies that the nilradical of R is trivial and hence R is reduced. Furthermore the condition in the lemma are not strictly necessary for the map to be injective.

Question 6: When is the map $R \to \prod(R_{\mathfrak{p}}, R_{\mathfrak{p}}^0)$ closed?

For this we basically have no result. The reason we take the map from R into the restricted product instead of just the regular product, is that topology on the restricted product is much finer than the regular product topology. This means that the map into the restricted product is much more likely to be closed. Unfortunately no such result was found.

Question 7: When does a closed map of topological rings $R \to R'$ induce a closed map $X(R) \to X(R')$?

Proposition 3.18 gives such result for closed maps between local topological rings. This is of course not useful in this case as neither R, nor $\prod(R_{\mathfrak{p}}, R_{\mathfrak{p}}^0)$ will be local. We will need a more general version of this proposition.

6 Compact to proper for Henselian fields

In this chapter we discuss a statement converse to the statements discussed in the previous chapter. We want a statement like "If the sets of rational points of a scheme X are compact, then X is proper". We have shown such a statement for global fields with regard to the sets of adelic points. In this chapter will prove such statement for so-called Henselian fields generalizing the "if"-statement of Theorem 4.1.

Henselian fields generalise the concept of local fields. Roughly speaking they are fields for which Hensel's lemma holds. More importantly is the fact that Hensel's lemma is equivalent to the generalised implicit function theorem. This theorem provides a vital step in the proof of the compact to properness theorem of this chapter.

Definition 6.1. Let K be a valued field with absolute value denoted by $|\cdot|$ and denote $\mathcal{O}_v$ the valuation ring. We call K a Henselian field if Hensel's lemma holds: Let $f_1, \ldots, f_n \in \mathcal{O}_v[X_1, \ldots, X_n]$, and $a_1, \ldots, a_n \in \mathcal{O}_v$ such that

$$|f_k(a_1, \ldots, a_n)| < \left| \det \left(\frac{\partial f_i}{\partial X_j} \right)_{i,j} (a_1, \ldots, a_n) \right|^2$$

for each $k = 1, \ldots, n$. Then there exist unique $b_1, \ldots, b_n \in O_v$ such that

$$f_k(b_1, \ldots, b_n) \quad \text{and} \quad |b_k - a_k| < \left| \det \left(\frac{\partial f_i}{\partial X_j} \right)_{i,j} (a_1, \ldots, a_n) \right|$$

for each $k = 1, \ldots, n$

Example 6.2. Examples of Henselian fields are local fields. For an arbitrary valued field there exists the *Henselisation* yielding a Henselian field. It is defined by universal property similarly to how the abelianisation is defined for groups. It can be regarded as the 'smallest' Henselian field extension of the field. Any finite extension of a Henselian field is again a Henselian field. For more information on Henselian fields see [3].

The following result is the generalization of the compact-to-proper result of the local theorem (Theorem 4.1) to Henselian fields.

Theorem 6.3. *Let K be a Henselian field. Let X be a scheme of finite type over K. Then if for each finite field extension K' the set $X(K')$ is compact, then X is proper over K*

To prove this we need the following preliminary results. First of which is the implicit function theorem. It provides one of the main large steps in the proof of the main result of this chapter. We start with a definition.

Definition 6.4. Let $f : X \to Y$ be a morphism of schemes over a base field K. Let p be a point on X. Then f is called étale at p if in an affine open around p, the map f is defined by a ring homomorphism of the form

$$K[Y_1, \ldots, Y_r] \to K[Y_1, \ldots, Y_r][X_1, \ldots, X_n]/(f_1, \ldots, f_n)$$

such that

$$\det\left(\frac{\partial f_i}{\partial X_j}\right)_{i,j}(p) \neq 0 \text{ in } \kappa(p)$$

A morphism is an étale morphism if it is locally of finite presentation and étale at every point.

Remark 6.5. It is important to note the same number of variables X_i and polynomials f_i in the definition.

The following statement is a version of the implicit function theorem for Henselian fields. Replacing the Henselian field K with an archimedean local field ($\mathbb{R}$ or $\mathbb{C}$), gives back the implicit function theorem from real analysis.

Theorem 6.6 (Implicit function theorem)**.** *Let K be a Henselian valued field. Let X be a scheme over K and let $\varphi : X \to \mathbb{A}^r_K$ be a morphism over K étale at a K-rational point $p \in X(K)$ such that $\varphi(p)$ is also K-rational. Then $\varphi_K : X(K) \to \mathbb{A}^r_K(K)$ is a homeomorphism locally around p.*

Proof. Since φ is étale at p it is defined locally around p by a ring homomorphism

$$K[Y_1, \ldots, Y_r] \to K[Y_1, \ldots, Y_r][X_1, \ldots, X_n]/(f_1, \ldots, f_n)$$

with

$$\det\left(\frac{\partial f_i}{\partial X_j}\right)_{i,j}(p) \neq 0$$

We can assume by translating that $\varphi(p)$ has coordinates $(0, \ldots, 0)$ in $\mathbb{A}^n_K(K)$ and so p has coordinates $(0, \ldots, 0, a_1, \ldots, a_n)$ in $\mathbb{A}^{n+r}_K(K) \cong \mathbb{A}^n_{K[Y_1,\ldots,Y_r]}$. We can furthermore assume that $a_i \in \mathcal{O}_v$ for each i by scaling each coordinate by a sufficiently small number. We can do the same for the f_i so that $f_i \in \mathcal{O}_v[Y_1, \ldots, Y_r][X_1, \ldots, X_n]$ for each i.

Recall that $\mathcal{O}_v = \{x \in K \mid |x| \leq 1\}$. This implies that

$$0 \neq \delta_0 := \left|\det\left(\frac{\partial f_i}{\partial X_j}\right)_{i,j}(p)\right| \leq 1,$$

as the determinant is just a sum of products of elements in $\mathcal{O}_v$ and thus lies in $\mathcal{O}_v$. Since $f_i(0, \ldots, 0, a_1, \ldots, a_n) = 0$ for each i we have that $f_i(Y_1, \ldots, Y_r, a_1, \ldots, a_n)$ has no constant terms. This means that for $y_1, \ldots, y_r \in K$ such that $|y_i| < \delta_0^2$ we have

$$|f_i(y_1, \ldots, y_r, a_1, \ldots, a_n)| < \delta_0^2$$

By Hensel's lemma there are unique $b_1, \ldots, b_n \in K$, such that $f_i(y_1, \ldots, y_r, b_1, \ldots, b_n) = 0$ and $|a_i - b_i| < \delta_0$. This defines an inverse map $\varphi_K^{-1} : B_{\delta_0^2}(0) \to B_{\delta_0}(p) \subset \mathbb{A}^{n+r}_K(K)$ sending a point $(y_1, \ldots, y_r)$ to the corresponding point $(y_1, \ldots, y_r, b_1, \ldots, b_n)$.

Now remains to show that φ_K^{-1} is continuous. Let $(y_1, \ldots, y_r) \in B_{\delta_0^2}$. Let $\varepsilon > 0$, and let $(x_1, \ldots, x_r)$ be such that

$$|(x_1, \ldots, x_r) - (y_1, \ldots, y_r)| < \frac{1}{2n+1}\varepsilon.$$

Denote $(x_1, \ldots, x_r, c_1, \ldots, c_r)$ for the image of $(x_1, \ldots, x_r)$ under φ_K^{-1}. By the triangle inequality we he have

$$\begin{aligned} |(y_1, \ldots, y_r, b_1, \ldots, b_r) - (x_1, \ldots, x_r, c_1, \ldots, c_r)| \leq{} & |(y_1, \ldots, y_r) - (x_1, \ldots, x_r)| \\ & + |(b_1, \ldots, b_r) - (c_1, \ldots, c_r)|. \end{aligned}$$

By assumption we have $|(x_1, \ldots, x_r) - (y_1, \ldots, y_r)| < \frac{1}{2n+1}\varepsilon$. And by Hensel's lemma we have $|b_i - a_i| < \frac{1}{2n+1}\varepsilon$ and $|c_i - a_i| < \frac{1}{2n+1}\varepsilon$ for each i. This means that $|b_i - c_i| < \frac{2}{2n+1}\varepsilon$ for each i. Hence $|(b_1, \ldots, b_r) - (c_1, \ldots, c_r)| < \frac{2n}{2n+1}\varepsilon$. We conclude that

$$|(y_1, \ldots, y_r, b_1, \ldots, b_r) - (x_1, \ldots, x_r, c_1, \ldots, c_r)| < \frac{1}{2n+1}\varepsilon + \frac{2n}{2n+1}\varepsilon = \varepsilon.$$

Hence φ_K^{-1} is continuous. ■

The last ingredient for the proof is the following theorem originally due to Nagata [6].

Theorem 6.7 (Nagata's embedding theorem). *Let X be a scheme over a field k. Then X can be embedded into a proper scheme over k as an open and dense subscheme.*

The following proof of Theorem 6.3 is essentially a copy of the proof in [4] for local fields as the proof only relies on Nagata's theorem and the implicit function theorem, and some general theory of curves over fields. As the implicit function theorem is essential to the proof and the implicit function theorem is equivalent to Hensel's lemma, this proof cannot be further generalised to non-Henselian topological fields.

Proof of Theorem 6.3. Let X be a scheme of finite type over K such that $X(K')$ is compact for each finite extension $K \subset K'$. By Nagata's theorem we can embed X into a proper scheme Y as an open and dense subset. If X is not proper then there is closed point in $Y \setminus X$. We will show that the existence of such point leads to a contradiction. So assume X is not proper and let p be a closed point in $Y \setminus X$.

Since $X \neq Y$ also for each finite field extension K' we have $X_{K'} \neq Y_{K'}$. This means we can enlarge the base field as needed to reach our contradiction, so we can assume that p is a K-rational point.

Consider an affine neighbourhood of p in Y and consider a curve through p that intersects X non-trivially and denote C for the closure of this curve in Y. Then $C \cap X$ is closed in X, and so $(C \cap X)(K')$ is closed in $X(K')$ for each finite field extension

K', since K' is Hausdorff. Since $X(K')$ is compact it follows that $(C \cap X)(K')$ is compact for each finite extension K'. Furthermore $C \cap X$ is open in C, since X is open in Y. The curve C is of finite type since X is, therefore $(C \setminus (C \cap X))(K)$ contains finitely many points and is an open set in $C(K)$. Since K is Hausdorff, points are closed so p is the complement of finitely many closed points in the open set $(C \setminus (C \cap X))(K)$. It follows that p is a discrete point in $C(K)$. We show that this is impossible.

Consider the normalization map $Z \to C$. The normalisation is a finite morphism, so the preimage of p in $Z(K)$ consists of finitely many discrete points of $Z(K)$. Furthermore, the normalisation of a curve over an algebraically closed field is smooth. So we can enlarge K so that Z is a smooth curve.

A smooth curve is locally étale over $\mathbb{A}^1_K$ [11, Lemma 054L]. Let $q \in Z$ be one of the discrete points in the preimage of p. Let U be an open neighbourhood of p and $\varphi : U \to \mathbb{A}^1_K$ a morphism étale at p. We can assume by extending K that q and $\varphi(q)$ are K-rational. By the implicit function theorem there exists homeomorphic neighbourhoods of q and $\varphi(q)$. So since q is discrete, $\varphi(q)$ is also discrete. However $\varphi(q)$ cannot be discrete since $\mathbb{A}^1_K(K) \cong K$ has no discrete points.

Contradiction! ⚡

■

Discussion

As stated, the proof above for fields works precisely when the field is a Henselian field. This being due to the fact that Henselian fields are exactly the fields for which the implicit function theorem holds, which is a core part of the proof. So to generalise the compact-to-proper result for fields, a new technique would be needed that does not rely on the implicit function theorem.

Further research can look into generalising this proof for Henselian rings. Similar to Henselian fields, these rings can be roughly defined as rings for which Hensel's lemma holds. It is expected that for these rings a version of the implicit function theorem holds, which can be used in this situation. Nagata's theorem for embedding a scheme into a proper scheme, which is also a vital part of the proof, generalises to arbitrary rings. Besides this, details of the proof relies on general algebraic geometry that is well known for the case of fields. Filling in these details in the case of non-fields should not pose a big problem, assuming that it works. We pose the following conjecture.

Conjecture 6.8. *Let R be a Henselian ring. Let X be a scheme of finite type over R. Assume that for each map of Henselian rings $R \to R'$, the set $X(R')$ is compact, then X is proper over R.*

For a more general result, we might want to assume that the maps $R \to R'$ are injective, or finite, or of finite type, etc. As is (automatically) the case for the statement for fields.

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