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Flats over T-Matroids

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Abstract

In this thesis we develop a theory of flats over matroids with coefficients in a tract, and use them to give a novel cryptomorphism of T-matroids. We define T-representations for flats of the underlying matroid. We then use this to show that modular triples of hyperplane functions are linearly dependent if and only if there exists a T-matroid which is a quotient of each of the hyperplane functions. Defining T-flats as the vectors of the T-representation of the flats, we show that for a given T-matroid, the T-flats form a geometric lattice with respect to inclusion, which in fact has the same lattice structure as the lattice of flats of the underlying matroid. Using our previous results, we can give a cryptomorphic definition of a T-matroid in terms of a lattice of T-flats. We conclude the thesis by showing that over a field K, the notion of K-flats coincide with the notion of hyperplane arrangements.

Contents

1 Introduction

Matroids are a diverse and bountiful concept, lying in the intersection of different areas of mathematics. Combinatorics, linear algebra, graph theory and algebraic geometry all have direct links to matroid theory, just to name a few. Due to their many equivalent definitions, we can take varying angles when studying matroids, which makes them very interesting.

Matroids first appeared in the 1930s. Although similar objects had already been studied at the time, the first mentioning of a matroid was in Hassler Whitney's paper "On the Abstract Properties of Linear Dependence" [Whi35]. He observed that sets of linearly independent vectors had specific combinatorial properties, giving rise to the notion of independent sets. Whitney defined matroids as collections of sets which had such properties. Since then many equivalent or so-called cryptomorphic definitions of matroids have been discovered. These equivalent definitions are a part of what makes matroid theory so interesting. On the other hand, it can also make matroids difficult when coming across them for the first time. With most mathematical objects, one only needs to learn a single definition, with matroids, one must learn ten.

Over time, matroid theory became more popular and different types of matroids started to appear. This gave rise to a new unifying theory. In 2019, Matthew Baker and Nathan Bowler published their paper "Matroids over Partial Hyperstructures" [BB19]. They define the notion of matroids over tracts, which offers a generalisation of many different types of matroids. Tracts are a generalisation of fields where instead of an abelian group structure over addition, we only have a notion of an additive inverse. By construction of specific tracts, T-matroids take different forms.

In this thesis, we aim to contribute to this theory by developing flats in the domain of matroids over tracts. Flats in usual matroids can be seen as a generalisation of linear subspaces and offer a cryptomorphic definition of matroids. It turns out that the concept of flats over tracts developed in this thesis, also gives us a cryptomorphic definition of matroids over tracts.

In section 2.1 we introduce matroids and give the necessary background on matroid theory for the rest of the text. This includes independent sets, bases, rank, closure, flats, duals and minors. In section 2.2 we introduce matroids over tracts. We first look into tracts and their properties. Then we define matroids over tracts, which we call T-matroids, using hyperplane functions. This is a slightly different definition to that given by [BB19], which defines T-matroids using circuit functions, however, we see that these definitions are similar. Next, we give an equivalent definition of T-matroids as equivalence classes of Grassmann-Plücker functions. We introduce the idea of duality and minors in T-matroids. We also look into the vectors of T-matroids and how they link to the notion of quotients.

In section 3, we start on the novel concepts developed in this thesis. We use the Grassmann-Plücker functions to define the new concept of T-representations of flats. This is closely linked to the contraction. However, to define T-representations of flats in a well defined manner we first tackle a series of preliminary results, which are showcased in section 3.1. Then in section 3.2 we define T-representations of flats proper and consider their underlying matroids.

In section 4, we show one of the two novel cryptomorphism of T-matroids given in this thesis. We define T-linear representations as a set of hyperplanes such that for every modular triple there exists a rank two Grassmann-Plücker function which is quotient to the triple. This gives us a cryptomorphism of T-matroids which will be very useful in section 5.

In section 5, we introduce the main new concept of this thesis, that is T-flats. In 5.1, we define a T-flat as the collection of vectors of the T-representation of a flat. It turns out that the T-flats of a T-matroid form a geometric lattice, in fact this is exactly the lattice of flats of the underlying matroid. We give an abstract definition of lattices of T-flats, separate from the T-representations of the flats, using Anderson's results on vectors [And19]. We show that this gives another novel cryptomorphism of T-matroids.

In section 5.2, we give a study of T-flats over a field K. We show that there is a correspondence between the orthogonal complement of lattices of K-flats and hyperplane arrangements.

In Section 6, we consider an example of a lattice of T-flats over $\mathbb{R}$.

2 Background

2.1 An Introduction to Matroids

This section is mainly based on Oxley's book "Matroid Theory" [Oxl92]. In this section we discuss some of the definitions that are important to this text and how they are equivalent.

2.1.1 Independent Sets

One of the most standard definitions of a matroid is through independent sets. Indeed, this was the definition given by Whitney when he first defined matroids [Whi35]. We define a matroid as follows.

Definition 2.1. In terms of independent sets, a matroid M is a pair $(E, \mathcal{I})$, where E is a finite set and $\mathcal{I} \subseteq 2^E$ is a family of subsets of E called the independent sets, such that

(I1) the empty set is in $\mathcal{I}$,

(I2) for all $X \subseteq Y$ such that Y in $\mathcal{I}$, we have X in $\mathcal{I}$,

(I3) for any two sets X, Y in $\mathcal{I}$ such that $\#Y < \#X$, there exists an element $x \in X$ such that $Y \cup \{x\}$ in $\mathcal{I}$.

We call the subsets of E in $\mathcal{I}$ independent and the subsets of E not in $\mathcal{I}$ dependent. Here our first parallel with linear algebra arises, namely we see that sets of linear independent vectors form independent sets.

Example 2.2. Consider an $m \times n$ matrix A with entries in a field K. Let E be the multiset (that is elements can be repeated) of the m column vectors in A. Then let $\mathcal{I}$ be the collection of subsets of E that are linearly independent in the vector space K^n. Then $(E, \mathcal{I})$ forms a matroid [Oxl92, Proposition 1.1.1].

2.1.2 Circuits

In terms of independent sets, the circuits of a matroid $M = (E, \mathcal{I})$ are the minimally[1] dependent subsets of E. That is, a set S is a circuit if every proper subset of it is independent. The set of circuits of a matroid characterise the matroid completely. This is clear if we notice that a set is independent if and only if it does not contain a circuit. In this way independent sets and circuits give equivalent definitions and so we can use circuits to define matroids. We say circuits give a *cryptomorphic description* or *cryptomorphism*.

Let us first look at when sets form the circuits of a matroid:

[1]When we refer to something being maximal or minimal for some property then we mean that any proper superset or subset does not have this property respectively.

Proposition 2.3. Let M be a matroid with collection of circuits $\mathcal{C}$. Then $\mathcal{C}$ is such that

(C1) the empty set is not in $\mathcal{C}$,

(C2) for C_1, C_2 in $\mathcal{C}$, we have that if $C_1 \subseteq C_2$ then $C_1 = C_2$,

(C3) for distinct C_1, C_2 in $\mathcal{C}$ and e in $C_1 \cap C_2$, there exists C_3 in $\mathcal{C}$ such that $C_3 \subseteq (C_1 \cup C_2) - e$.

Proof. See [Oxl92, Lemma 1.1.3]. □

The following theorem tells us that circuits give an equivalent definition of matroids.

Theorem 2.4. Let E be a set and $\mathcal{C}$ a set of subsets of E satisfying (C1)–(C3). Let $\mathcal{I}$ be the subsets of E which do not contain an element of $\mathcal{C}$. Then $(E, \mathcal{I})$ forms a matroid with circuits $\mathcal{C}$.

Proof. See [Oxl92, Theorem 1.1.4]. □

Example 2.5. Let E be the edge set of a graph G, then the edge sets of the cycles in G form a set of circuits. Therefore we can construct a matroid from any graph, by taking the independent sets to be the sets not containing the edge sets of the cycles on the graph, that is exactly the edge sets of spanning forests on the graph.

2.1.3 Bases

The bases of a matroid are the maximally independent sets, that is the independent sets such that any proper superset is dependent. Bases are akin to bases of vector spaces and share many properties. We notice that the independent sets are exactly the subsets of the bases and so the bases characterise the matroid completely. Therefore bases give us another way of defining matroids.

Proposition 2.6. Let M be a matroid and $\mathcal{B}$ be a set of bases of M. Then $\mathcal{B}$ is such that

(B1) we have $\mathcal{B}$ is non-empty,

(B2) for $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 - B_2$, there exists $y \in B_2 - B_1$ such that $(B_1 - x) \cup y \in \mathcal{B}$.

Proof. See [Oxl92, Lemma 1.2.2]. □

Theorem 2.7. Let E be a set and $\mathcal{B}$ collection of subsets of E satisfying (B1)–(B2). Let $\mathcal{I}$ be the collection of subset of elements of $\mathcal{B}$. Then $(E, \mathcal{I})$ forms a matroid for which the collection of bases is $\mathcal{B}$.

Proof. See [Oxl92, Theorem 1.2.3]. □

An interesting property of bases of a matroid is that they all have the same cardinality. This is a neat similarity to the property the bases of a vector space always have the same cardinality.

Example 2.8. Consider the matroid in Example 2.2. The bases of this matroid are exactly the linearly independent sets of column vectors which span the linear subspace of K^n spanned by the m column vectors of A. Let us aim for a contradiction and assume that we have two sets of linearly independent sets of vectors V and W that span the space, but for x in $V - W$ there exists no y in $W - V$ such that $V \cup \{y\} - \{x\}$ is also linearly independent and spanning the space. This implies that there is no y in W such that y can be written as a linear combination of elements of W containing x. This implies that W is in the span of $V - \{x\}$ which contradicts V and W both being bases.

Example 2.9. Consider the matroid given in Example 2.5. The bases of this matroid are exactly the edge sets which span G but do contain a cycle, that is the spanning forests of G.

Example 2.10. An important class of matroids are the uniform matroids. A uniform matroid of rank r over E is the matroid $U_{r,E}$ with collection bases exactly every subset of E with cardinality r. The matroid $U_{0,E}$ is the matroid over E with empty bases, in other words for every e in E, we have $\{e\}$ a circuit of $U_{0,E}$.

2.1.4 Rank

The rank of a set $S \subseteq E$ is the cardinality of the maximal independent set contained in S.

If we have a matroid M then we find that the independent sets are exactly those whose ranks are equal to their cardinality. In converse, if we have a rank function on E then the collection of subsets that have rank equal to cardinality satisfy (I1)–(I3). Hence, again we have that the rank function on a matroid characterises it completely. We have yet another cryptomorphism.

Proposition 2.11. Let M be a matroid and let $r : 2^E \to \mathbb{N}_0$ be the rank function on this matroid. Then r is such that

(R1) if $X \subseteq E$, then we have $0 \leq r(X) \leq |X|$,

(R2) if $X \subseteq Y \subseteq E$, then we that $0 \leq r(X) \leq r(Y)$,

(R3) for $X, Y \subseteq E$, we have that $r(X) + r(Y) \geq r(X \cup Y) + r(X \cap Y)$.

Proof. See [Oxl92, Lemma 1.3.1]. □

Theorem 2.12. Let E be a set and let $r : 2^E \to \mathbb{N}_0$ be a function satisfying (R1)–(R3). Let $\mathcal{I}$ be the collection of $X \subseteq E$ such that $r(X) = |X|$. Then $(E, \mathcal{I})$ is a matroid with rank function r.

Proof. See [Oxl92, Theorem 1.3.2]. □

Remark 2.13. We say $M = (E, r)$ is a rank n matroid if $r(E) = n$. Sometimes this is written as $r(M) = n$.

2.1.5 Closure

The notion of the closure is similar to the notion of span in linear algebra. For a matroid M with rank function r we define the closure of $X \subseteq E$ as $\mathrm{cl}(X) = \{x \in E \mid r(x \cup X) = r(X)\}$. Let us see what properties the closure operator has.

Proposition 2.14. Let $(E, \mathcal{I})$ be a matroid and let $\mathrm{cl} : 2^E \to 2^E$ be the closure operator. Then cl has the following properties.

(Cl1) if $X \subseteq E$ then $X \subseteq \mathrm{cl}(X)$

(Cl2) if $X \subseteq Y$ then $\mathrm{cl}(X) \subseteq \mathrm{cl}(Y)$

(Cl3) if $X \subseteq E$ then $\mathrm{cl}(\mathrm{cl}(X)) = \mathrm{cl}(X)$

(Cl4) if $X \subseteq E$, $x \in E$ and $y \in \mathrm{cl}(X \cup x) - \mathrm{cl}(X)$ then $x \in \mathrm{cl}(X \cup y)$

Proof. See [Oxl92, Lemma 1.4.3]. □

Example 2.15. Let $(E, \mathcal{I})$ be the matroid defined in Example 2.2. Then the closure of a subset X of E is exactly the column vectors of A contained in the span of X .

Again we find that the closure operator allows us to define a matroid. We notice that for a matroid $(E, \mathcal{I})$ with rank function r, the independent sets are exactly the sets $X \subseteq E$ such that for all $x \in X$, $x \notin \mathrm{cl}(X - x)$. The following theorem shows us that we can define a matroid using a closure operator.

Theorem 2.16. Let E be a set and let $\mathrm{cl} : 2^E \to 2^E$ satisfying (Cl1)–(Cl4). Then let $\mathcal{I}$ be the set of sets $X \subseteq E$ such that for all x in X, $x \notin \mathrm{cl}(X - \{x\})$. Then $(E, \mathcal{I})$ forms a matroid with closure operator cl.

Proof. See [Oxl92, Theorem 1.4.5]. □

Definition 2.17. Let E be a matroid with closure operator $\mathrm{cl} : 2^E \to 2^E$. We say $X \subseteq S$ *spans* S if $\mathrm{cl}(X) = S$.

Proposition 2.18. A set S spans a set X if and only if $r(S) = r(X)$ and $\mathrm{cl}(S) = S$.

Proof. See [Oxl92, Proposition 1.4.10]. □

2.1.6 Flat and Hyperplanes

We can use the closure operator to define some more interesting sets in our matroid. For a matroid M, a flat is a set $X \subseteq E$ such that $X = \mathrm{cl}(X)$. A hyperplane is defined as a flats of rank $r(M) - 1$. Let us take a look at their properties.

Proposition 2.19. Let $\Lambda = \{X \in E \mid \text{cl}(X) = X\}$ be the collection of flats of a matroid M with closure operator $\text{cl} : 2^E \to 2^E$. Then Λ is such that

(F1) $E \in \Lambda$,

(F2) if $F_1, F_2 \in \Lambda$, then $F_1 \cap F_2 \in \Lambda$,

(F3) if $F \in \Lambda$ and $\{F_1, ..., F_n\}$ the set of minimal members of Λ properly containing F, then $F_1 - F, ..., F_n - F$ partition $E - F$.

Proof. See [Oxl92, Page 31]. □

Theorem 2.20. Let E be a finite set and Λ a set of subsets of E satisfying (F1)–(F3). Let $p : 2^E \to 2^E$ be such that $p(X) = \bigcap_{F \in \Lambda, X \subseteq F} F$. Then p satisfies (Cl1)–(Cl4), in other words Λ defines a matroid.

Proof. See [Oxl92, Page 31]. □

Theorem 2.21. A set $\mathcal{H}$ is the set of hyperplanes of a matroid if and only if

(H1) for all X, Y in $\mathcal{H}$, $X \subseteq Y$ implies $X = Y$,

(H2) for all X, Y in $\mathcal{H}$ and e not in $X \cup Y$, there is $Z \in \mathcal{H}$ such that $(X \cap Y) \cup \{e\} \subseteq Z$.

Proof. See [Oxl92, Proposition 2.1.21] □

Proposition 2.22. We say a set $X \subseteq E$ is spanning if $\text{cl}(X) = E$. We have the following properties:

- For all $X \subseteq E$, X is spanning if and only if $r(X) = r(E)$.
- For all $X \subseteq E$, X is a basis if and only if X is independent and spanning.
- For all $X \subseteq E$, X is a basis if and only if X is minimally spanning.
- For all $X \subseteq E$, X is a hyperplane if and only if X is maximally non-spanning.

Proof. See [Oxl92, Proposition 1.4.10]. □

The flats of a matroid have a nice property in that they form a special type of partially ordered set (poset) called a lattice.

2.1.7 Lattice of Flats

Let us discuss posets and lattices.

Definition 2.23. A poset is a set X along with an ordering $\leq$ satisfying the following properties:

(P1) for all $x \in X$, $x \leq x$

(P2) for all $y, x \in X$, if $x \leq y$ and $y \leq x$ then $x = y$

(P3) for $x, y, z \in X$, if $x \leq y$ and $y \leq z$ then $x \leq z$

We say that $x \geq y$ if $y \leq x$. We say $x < y$ if $x \leq y$ and $x \neq y$. We say x covers y in a poset X if $y < x$ and there does not exist a $z \in X$ such that $y < z < x$.

A lattice is a poset such that for each pair $x, y \in X$ the least upper bound and greatest lower bound exist.

Definition 2.24. A finite lattice $\mathcal{L}$ is a finite poset such that:

(L1) for all $x, y \in \mathcal{L}$ there exists $x \vee y \in \mathcal{L}$ such that for all $z \in \mathcal{L}$ such that $z \geq x$ and $z \geq y$ we must also have $z \geq x \vee y$

(L2) for all $x, y \in \mathcal{L}$ there exists $x \wedge y \in \mathcal{L}$ such that for all $z \in \mathcal{L}$ such that $z \leq x$ and $z \leq y$ we must also have $z \leq x \wedge y$

Lemma 2.25. The set of flats of a matroid M form a lattice $\mathcal{L}(E, \mathcal{I})$ with $X \leq Y$ if $X \subseteq Y$, $X \wedge Y = X \cap Y$ and $X \vee Y = \text{cl}(X \cup Y)$.

Proof. See [Oxl92, Lemma 1.7.3] □

Definition 2.26. For a poset P, a *chain* from x_1 to x_n is a collection $x_1, \cdots, x_n \in P$ such that $x_1 < \cdots < x_n$. The *length* of this chain is $n-1$ and it is maximal if for every $0 \leq m < n$, x_{m+1} covers x_m. If for all $x, y \in P$ we have that the length of all the maximal chains from x to y have the same length then P is said to satisfy the *Jordan-Dedekind chain condition.* If there exists a $x \in P$ such that $x \leq y$ for all $y \in P$ then x is said to be the *bottom* for P. If there exists a $x \in P$ such that $y \leq x$ for all $y \in P$ then x is said to be the *top* for P. An element of P that cover the bottom is said to be an *atom.* For $x \in P$ the *height* $h(x)$ is defined as the length of the chain from zero to x. A *geometric lattice* is a lattice satisfying the Jordan-Dedekind chain condition such that $h(x) + h(y) \geq h(x \wedge y) + h(x \vee y)$ and every element is a join of atoms.

Theorem 2.27. A lattice is geometric if and only if it is the lattice of flats of a matroid

Proof. See [Oxl92, Theorem 1.7.5] □

Remark 2.28. Every flat is the intersection of hyperplanes[Oxl92, Proposition 1.7.8].

Example 2.29. Consider the rank three flat over $E = 1234$ (here we use lattice notation where ijk refers to $\{i, j, k\}$) with hyperplanes $12, 13, 14$ and 234. Then the lattice of flats is as pictured below. A line here represents inclusion. We have $\emptyset$ as the bottom and E as the top. Notice that the chain between flats of different ranks is always the same. In fact the height of a flat in the lattice of flats is exactly the rank of the flat.

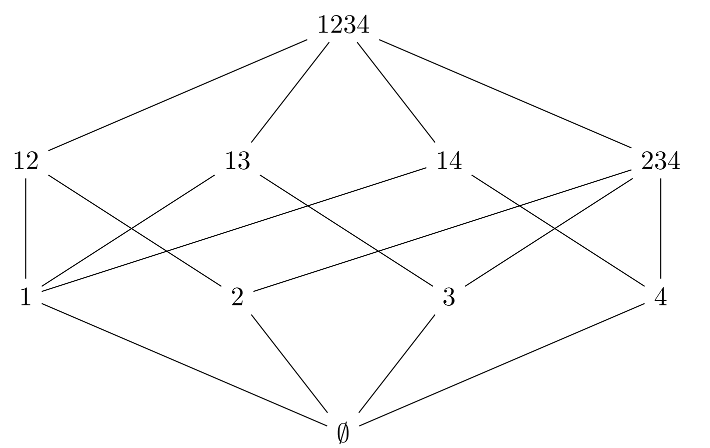

Figure 1: The rank three matroid over $E = 1234$ with hyperplanes $12, 13, 14$ and 234.

Example 2.30. The matroid over $E = 12345$ with bases all size three subset of E excluding $345, 124$ and 123.

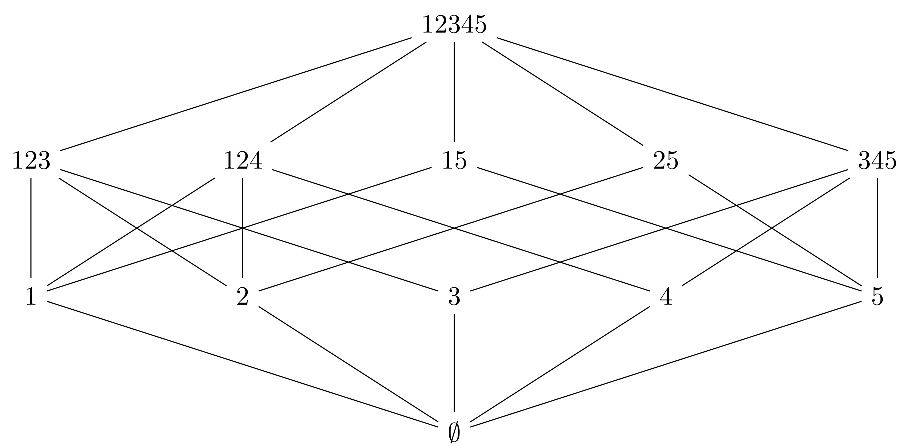

Figure 2: The lattice of flats of the matroid with ground set $E = 12345$ and bases as all size three subset of E excluding $345, 124$ and 123.

We should also touch on loops, parallel sets and simplified matroids.

Definition 2.31. A *loop* of a matroid M is an element of E which is not contained in any basis of M, that is, $\{e\}$ forms a single element circuit. A *parallel set* is a set of two elements $\{x, y\} \subseteq E$ such that any for any basis B of M such that $x \in B$, we must have $B \cup \{y\} - \{x\}$ forms a basis. We call a matroid *simple* if it contains no loops or parallel sets.

Remark 2.32. It is possible to construct a simplification of a matroid, which rids us of loops and parallel sets, however it is not necessary to discuss this here. For the interested reader, please consult [Oxl92, Page 49]. What is important is that the structure of the lattice of flats remains unchanged when the loops and parallel sets are removed. In fact, the abstract structure of the lattice determines the matroid up to simplification. That is, the atoms of the lattice correspond to the elements of E which are not loops and then the structure of the lattice gives us the flats of the matroid.

2.1.8 Duals

We define the dual of a matroid through its bases. The collection of bases of the dual of a matroid is defined as $\mathcal{B}^* = \{E - B \mid B \in \mathcal{B}\}$. Note that this means the dual of the dual of the a matroid is exactly the original matroid and that the dual of a matroid is unique. The bases of the dual are called the cobases. In fact every term we defined on a matroid also has a equivalent notion on the dual of the matroid. In this way we have cohyperplanes as the hyperplanes of the dual, cocircuits as the circuits of the duals and so forth. We refer to the dual of a matroid M as M^*.

The sets of the dual has some important properties [Oxl92, Proposition 2.16]:

- For all $X \subseteq E$, X is independent if and only if $E - X$ it is cospanning. Due to the fact that the dual of the dual is just the original matroid, this also works the other way as X is coindependent if and only if $E - X$ is spanning.
- For all $X \subseteq E$, X is a hyperplane if and only if $E - X$ is a cocircuit and equivalently X is a circuit if and only if $E - X$ is cohyperplane.

Proposition 2.33. For M a matroid with rank function $r : 2^E \to \mathbb{N}_0$, the rank function of the dual is given as $r^*(X) = r(E - X) + |X| - r(E)$.

Proof. See [Oxl92, Proposition 2.19]. □

2.1.9 Minors

Minors give us a notion of a "sub-matroid". They are obtained through sequences of two types operation. We define the deletion of $T \subseteq E$ from a matroid $M = (E, \mathcal{I})$ as $M \backslash T = (E - T, \mathcal{I} \backslash T)$ where $\mathcal{I} \backslash T = \{I \subseteq E - T \mid I \in \mathcal{I}\}$. The contraction of M onto $E - T$ is $M/T = (M^* \backslash T)^*$. Although a minor is any matroid obtained through a sequence of deletions or contractions, we will here only focus on single contractions.

Proposition 2.34. Let $M = (E, \mathcal{I})$ be a matroid. Let B_T be a basis of $M \backslash (E - T)$. Let $\mathcal{I}(M/T)$ be the collection of independent set of M/T. Then

$$\begin{aligned}\mathcal{I}(M/T) &= \{X \subseteq E - T \mid X \cup B_T \subseteq \mathcal{I}\} \\ &= \{X \subseteq E - T \mid M \backslash (E - T) \text{ has a basis } B \text{ such that } X \cup B \subseteq \mathcal{I}\}.\end{aligned}$$

Proof. See [Oxl92, Proposition 3.1.7]. □

Proposition 2.35. Let $M = (E, \mathcal{I})$ be a matroid. Let B_T be a basis of $M \backslash (E-T)$ and $\mathcal{B}(M/T)$ the collection of bases of M/T.

$$\begin{aligned}\mathcal{B}(M/T) &= \{B' \subseteq E-T \mid B' \cup B_T \in \mathcal{B}\}\\ &= \{B' \subseteq E-T \mid M\backslash(E-T) \text{ has a basis } B \text{ such that } B' \cup B \in \mathcal{B}\}\end{aligned}$$

Proof. See [Oxl92, Corollary 3.1.8]. □

Proposition 2.36. For all $X \subseteq T$ we have that $\mathrm{cl}_{M/T}(X) = \mathrm{cl}_M(X \cup T) - T$.

Proof. See [Oxl92, Proposition 3.1.13]. □

2.2 Baker and Bowler Theory

In 2019 Matthew Baker and Nathan Bowler published a paper called "Matroids over Partial Hyperstructures" [BB19]. In this paper they introduced the notion of a matroid over a tract, which generalised both linear subspaces and many types of matroids. In this thesis we build on the concepts discussed in this paper, therefore it is important to first give a overview.

2.2.1 Tracts

The central algebraic structure in this theory is a tract. Tracts are generalisations of fields. Multiplicatively they are an abelian group, hence similar to fields, however, in terms of addition they only allow for a notion of inverse. Since we only need to know when elements sum to zero when considering matroids, tracts give us the most basic possible algebraic structure we can work with.

Definition 2.37. A *tract* $T = (G, N_T)$ is an abelian group G written multiplicatively with identity element i, along with an additive relation structure on G. The additive relation structure a subset N_T of the semi-ring[2] $\bigoplus_{g \in G} \mathbb{N}_0$ such that

(T1) the zero element of $\bigoplus_{g \in G} \mathbb{N}_0$ is in N_T,

(T2) the identity element i is not in[3] N_T,

(T3) there exists a unique element $\epsilon \in G$ such that $i + \epsilon \in N_T$,

(T4) N_T is closed under the natural group action of G.

Each element in the group semi-ring $\bigoplus_{g \in G} \mathbb{N}_0$ can be viewed as an element of $\mathbb{N}_0^{|G|}$. This vector represents a sum with each integer giving the number of times the related element occurs in the

[2] A semi-ring is a set R along with two binary operations $+$ and $\cdot$ such that $(R, +)$ forms a commutative monoid and $(R, \cdot)$ forms a monoid.

[3] Here we immediately abuse notation. What is meant by i is actually the sum $\sum_{g \in G} \delta_{gi}$, where $\delta_{gi} = 1$ if $g = i$ and δ_{gi} is 0 otherwise. This will be the notation used for the rest of the text.

sum. This way $\bigoplus_{g\in G}\mathbb{N}_0$ gives all the possible sums of elements of G. Hence we can view N_T as the sums of elements of G equal to 0. Note, we often refer to G as $T^\times$. Also note that the tract also contains a zero element arising from the additive identity of $\bigoplus_{g\in G}\mathbb{N}_0$ with $g\cdot 0 = 0\cdot g \in N_T$. We sometimes say $h = 0$ instead of $h \in N_T$.

Let us state some important rules about tracts.

Proposition 2.38. Let $T = (G, N_T)$ be a tract.

1) If $x, y \in G$ are such that $x + y \in N_T$, then $y = \epsilon \cdot x$.
2) The multiplicative inverse of ϵ is itself, that is $\epsilon^2 = i$.
3) No single element of G is in the null set, that is $G \cap N_T = \emptyset$.

Proof. See [BB19, Lemma 1.1]. □

Note that this means that every element has a unique additive inverse and that no element in our group is in N_T. For ease of use, from now on we refer to i as 1, ϵ as -1 and $\epsilon \cdot x$ as $-x$.

Definition 2.39. A *tract homomorphism* $f : (G, N_T) \to (G', N_{G'})$ is a group homomorphism $f^* : G \to G'$ such that $\sum a_i g_i \in N_T$ implies $\sum a_i f^*(g_i) \in N_{T'}$ for $a_i \in \mathbb{N}_0$ and $g_i \in G$.

Example 2.40. Any field K can be viewed as a tract. We let G be the elements of $K^\times$ along with its multiplicative structure. Then N_T is the set of sums equal to zero.

Example 2.41. An important example of a tract is the *Krasner hyperfield* $\mathbb{K}$. The Krasner hyperfield is constructed using the one element group $G = \{1\}$ along with a zero element. Then $N_T = \mathbb{N}_0\backslash\{1\}$, that is, every possible sum is in N_T other than the sum containing a single 1.

2.2.2 Matroids over Tracts

Before we define matroids over tracts, we first need to define some other notions.

Let E be a finite set and T a tract. Then we refer to the set of all possible functions from E to T as T^E. The *support* of $X \in T^E$ is the set $\underline{X} = \{e \in E \mid X(e) \neq 0\}$, we define $\text{supp}(S) := \{\underline{X} \mid X \in S\}$.

We say $X_1, ..., X_k \in T^E$ are *linearly dependent* if there exists $c_1, ..., c_k \in T$ not all 0 such that

$$c_1X_1 + \cdots + c_nX_k \in N_T^E.$$

Here N_T^E is the set of maps from E to N_T, in other words, evaluating $X \in N_T^E$ at any point in E gives an element in N^T. Sets of elements in T^E which are not linearly dependent are called *linearly independent*.

We also have a notion of an inner product. For $X, Y \in T^E$, we have

$$X \cdot Y := \sum_{e \in E} X(e) \cdot Y(e).$$

We say $X, Y \in T^E$ are orthogonal if $X \cdot Y \in N_T$ and we denote this by $X \perp Y$. For a set $W \subseteq T^E$, the orthogonal complement of W is the set $W^\perp = \{X \in T^E \mid X \perp Y \text{ for all } Y \in W\}$.

For elements in T^E, we also have a notion of span. The span of a set $W = \{W_1, ..., W_n\}$ is the set $\langle W \rangle = \{X \in T^E \mid X - (a_1 W_1 + \cdots + a_n W_n) \in N_T^E, \text{for some } a_1, ...a_n \in T^\times\}$.

Definition 2.42. Let E be a finite set and $\mathcal{C}$ a set of subsets of E such that for each pair of elements C_i, C_j in $\mathcal{H}$ we have that $C_i \nsubseteq C_j$ and $C_j \nsubseteq C_i$. Then $C_1, C_2 \in \mathcal{C}$ form a *modular pair* in $\mathcal{C}$ if $C_1 \neq C_2$ and $C_1 \cup C_2$ does not properly contain a union of 2 distinct elements of $\mathcal{C}$.

Lemma 2.43. Let $\mathcal{H}$ be a collection of incomparable subset of a finite set E. The set $\mathcal{H}$ is the set of hyperplanes of a matroid if and only if for every modular pair $E - X, E - Y$ in $\mathcal{H}^c = \{E - \underline{X} \mid X \in \mathcal{H}\}$, there exists $Z \in \mathcal{H}$ such that $(X \cap Y) \cup \{e\} \subseteq Z$.

Proof. We use the fact that the hyperplanes of a matroid are exactly the complements of the cocircuits. Hence $\mathcal{H}$ is the set of hyperplanes of a matroid if and only if the complements of the elements of $\mathcal{H}$ form a set of circuits. We note that if for every modular pair X, Y in $\mathcal{H}$, there exists $Z \in \mathcal{H}$ such that $(X \cap Y) \cup \{e\} \subseteq Z$, then the set of complements of the elements $\mathcal{H}$ satisfies statement 2 of [BB19, Lemma 3.6]. Hence, the set of complements of $\mathcal{H}$ form a set of circuits and so $\mathcal{H}$ is the set of hyperplanes of a matroid. □

We have all the knowledge to define T-matroids.

Definition 2.44. Let E be a finite set and let T be a tract. A subset η of T^E is called the *set of hyperplane functions of a weak T-matroid on E* if it satisfies

(Hf1) $0 \notin \eta$,

(Hf2) if $X \in \eta$ and $\alpha \in T^\times$, then $\alpha \cdot X \in \eta$,

(Hf3) if $X, Y \in \eta$ and $\underline{X} \subseteq \underline{Y}$, then there exists $\alpha \in T^\times$ such that $X = \alpha \cdot Y$,

(Hf4) if $X, Y \in \eta$ are such that $\underline{X}, \underline{Y}$ are a modular pair in $\{E - \underline{X} \mid X \in \eta\}$ and $e \in E$ such that $X(e) = -Y(e) \neq 0$, then there exists $Z \in \eta$ such that $X + Y - Z \in N_T^E$.

Definition 2.45. A T-matroid is a set E along with a subset of T^E satisfying (H1)–(H4).

Remark 2.46. This definition is a rewording of the definition of T-circuits given in [BB19]. We use the fact that the complements of the cocircuits are hyperplanes. In fact, the set of hyperplane functions of a weak T matroid M is exactly a set of T-circuits of the dual of M (this is introduced in Section 2.2.4). However, since the dual is unique, this also characterises M completely. On anther note, although this definition uses hyperplane functions, it is different to the definition of modular systems of hyperplane functions given in [BL20]. This definition has one hyperplane function for each hyperplane, whilst the definition given above is closed under multiplication by $T^\times$.

Theorem 2.47. Let T be a tract and η a subset of T^E satisfying (Hf1)–(Hf4). Then the set $H = \{E - \underline{X} \mid X \in \eta\}$ forms a set of hyperplanes of a matroid over E.

Proof. We need H to satisfy (H1)–(H2). By (Hf1) and (Hf3), H satisfies (H1). By (Hf4) if $X, Y \in \eta$ are such that $E - \underline{X}, E - \underline{Y}$ are a complementary modular pair in H and there is $e \notin (E - \underline{X}) \cup (E - \underline{Y})$, then there is $Z \in \eta$ such that $Z(e) \notin N_T$, that is $e \in E - \underline{Z}$, and $Z(f) \in N_T$ if $X(f) \in N_T$ and $Y(f) \in N_T$, that is $(E - \underline{X}) \cap (E - \underline{Y}) \subseteq E - \underline{Z}$. Hence, H satisfies (H2). □

This implies that if we take η to be the set of hyperplane functions of a weak T-matroid M over E, then there exists an *underlying matroid* $\underline{M}$ over E with $\{E - \underline{X} \mid X \in \eta\}$ as hyperplanes. We define the *rank* of M as the rank of $\underline{M}$.

We can also define an equivalence class of hyperplane functions. That is $X_1 \sim X_2$ if $X_1 = \alpha \cdot X_2$ for some $\alpha \in T^\times$. We notice that for a T-matroid, there is a bijection between hyperplanes of the underlying matroid and the equivalence classes of the hyperplane functions.

Remark 2.48. In [BB19], strong T-matroids are also defined, however for the novel work done in this thesis, we do not need this concept and so we do not define it here.

2.2.3 Grassmann-Plücker Functions

In this section we define the so called Grassmann-Plücker functions. This allows us to give a cryptomorphic definition of T-matroids.

Definition 2.49. Let E be a finite set, let $T = (N_T, G)$ be a tract and r any positive integer. A *weak Grassmann-Plücker function of rank* r *on* E *with coefficients in* T is a function $\varphi : E^r \to T$ such that the support φ form a collection of bases of a matroid and

(GP1) the function φ does not map everything to 0,

(GP2) the map φ is alternating, in other words $\varphi(x_1, ..., x_i, ..., x_j, ..., x_r) = -\varphi(x_1, ..., x_j, ..., x_i, ..., x_r)$ and so $\varphi(x_1, ..., x_r) = 0$ if $x_i = x_j$ for some $i \neq j$,

(GP3) for any two tuples $I = \{x_1, ..., x_{r+1}\}$ and $J = \{y_1, ..., y_{r-1}\}$ of elements in E such that $|I \backslash J| = 3$ we have,

$$\sum_{k=1}^{r+1} (-1)^k \varphi(x_1, x_2, ..., \hat{x}_k, ..., x_{r+1}) \cdot \varphi(x_k, y_1, ..., y_{r-1}) \in N_T, \tag{1}$$

where $\hat{x}$ means the element x is excluded.

We say φ_1 and φ_2 equivalent if $\varphi_1 = \alpha \cdot \varphi_2$ for some $\alpha \in T^\times$.

By definition, a Grassmann-Plücker function also has an underlying matroid, just like the sets of hyperplanes of a weak T-matroid.

Theorem 2.50. Let E be a finite set, let T be a tract and let r be a positive integer. There is a natural bijection between equivalence classes of weak Grassmann-Plücker functions of rank r on E with coefficients in T and sets of hyperplane functions of a weak T-matroid.

Proof. We note that hyperplane functions are equivalent to the circuit function used to define T-matroids in [BB19]. Then the result follows from [BB19, Theorem 3.17]. □

Remark 2.51. An important note on notation. We use square brackets to refer to the equivalence class under multiplication by $T^\times$ of any function. If $f : A \to T$ for A any set and T a tract, then $[f : A \to T]$ or simply $[f]$ is the equivalence class of f. By Theorem 2.50, we can view T-matroids as equivalence classes of Grassmann-Plücker functions. This is the standard view we take in this paper.

Definition 2.52. Given a usual matroid $\underline{M}$ of rank r, a Grassmann-Plücker function $\varphi : E^r \to T$ is said to be a *T-representation of $\underline{M}$*, if the underlying matroid of φ is $\underline{M}$. That is, if the sets $\{x_1, ..., x_r\}$ such that $\varphi(x_1, ..., x_r) \neq 0$ are exactly the bases of $\underline{M}$.

Example 2.53. A weak T-matroid over the Krasner hyperfield $\mathbb{K}$ is exactly a usual matroid, that is there exists a bijection between $\mathbb{K}$-matroids and usual matroids. For a given matroid, the function from E to $\mathbb{K}$, which maps to 1 if and only if the input is a basis, gives a Grassmann-Plücker function. This function satisfies (GP1) due to the fact that every matroid has some basis. It satisfies (GP2) due to the fact that a basis must contain r distinct elements, for r the rank of the matroid and by the fact that $-1 = 1$ in $\mathbb{K}$. The function satisfies (GP3) by the fact that any sum $1 + \cdots + 1 \neq 1$ is in $N_{\mathbb{K}}$. We note that if we have two tuples I and J in E^r such that $I/J = \{x_i, x_j, x_k\}$, then if some $I - \{x_t\}$ and $J \cup \{x_t\}$ form a basis, we must have by (B2) that there is some x_s such that $I - \{x_s\}$ and $J \cup \{x_s\}$ form a basis. Hence, for any tuples I and J such that $|I/J| = 3$, we have that equation (1) is some sum containing 1 more than once, hence it must be in $N_{\mathbb{K}}$. Therefore the basis indicator function is a Grassmann-Plücker function. This function is unique for every matroid and by definition every Grassmann-Plücker function with coefficients in $\mathbb{K}$ is of this form. Hence, there exists a bijection.

2.2.4 Duality and Minors

Just like there is a notion of duality in usual matroids, there is also a dual for T-matroids.

Definition 2.54. Let E be a finite set such that $|E| = m$, let T be a tract and M a weak T-matroid of rank r represented by the weak Grassmann-Plücker function φ. Then there is a weak

T-matroid M^* of rank $m-r$ called the dual matroid represented by the Grassmann-Plücker function φ^* such that:

$$\varphi^*(x_1, ..., x_{m-r}) = \text{sign}(x_1, ..., x_{m-r}, x'_1, ..., x'_r)\varphi(x'_1, ..., x'_r),$$

where $x'_1, ..., x'_r$ is any ordering of $E - \{x_1, ..., x_{m-r}\}$. Here $\text{sign}(x_1, ..., x_{m-r}, x'_1, ..., x'_r)$ is the sign of the permuation that gives the elements of E in this order with respect to the original order of E.

The dual has some nice properties [BB19, Theorem 3.24]:

- The underlying matroid of M^* is the dual of the underlying matroid of M.
- The dual of the dual of M is M itself.

Remark 2.55. It is important to note that the hyperplane functions of a T-matroid are exactly the T-circuits (defined in [BB19]) of the dual. We can now explain the bijection in Theorem 2.50. The result [BB19, Theorem 3.17] gives a bijection between T-circuits and equivalence classes of Grassmann-Plücker functions. This bijection is given by

$$\frac{Y(x_k)}{Y(x_0)} = (-1)^k \frac{\varphi(x_0, ..., \hat{x}_k, ..., x_r)}{\varphi(x_1, ..., x_r)}$$

Where $\{x_1, ..., x_r\}$ forms a basis for the underlying matroid containing the $\underline{X}$. Since, the hyperplane functions of a T-matroid are exactly the T-circuits of the dual we have

$$\frac{Y(x_k)}{Y(x_0)} = (-1)^k \frac{\varphi^*(x_0, ..., \hat{x}_k, ..., x_{m-r})}{\varphi^*(x_1, ..., x_{m-r})} = (-1)^k \frac{\varphi(x_k, y_1, ..., y_{r-1})}{\varphi(y_1, ..., y_{r-1})},$$

where $\{y_1, ..., y_{r-1}\}$ must form a basis for the hyperplane $E - \underline{X}$ and $x_0 \notin E - \underline{X}$ fixed. Hence,

$$\frac{X(e)}{X(e')} = (-1)^k \frac{\varphi(x_1, ..., x_{r-1}, e)}{\varphi(x_1, ..., x_{r-1}, e')}$$

gives a bijection between equivalence classes of weak Grassmann-Plücker functions of rank r on E with coefficients in T and sets of hyperplane functions of a weak T-matroid.

Another aspect of usual matroids that carries over to T-matroids are minors. We formulate this in terms of Grassmann-Plücker functions. Here we only define contraction, since deletion is not of importance to this text, however a similar definition exists and is given in [BB19, Definition 4.3].

Definition 2.56. Let T be a tract $\varphi : E^r \to T$ be a Grassmann-Plücker function defining a T-matroid M, with underlying matroid $\underline{M}$. Let A be a rank l subset of $\underline{M}$ with a maximal independent set $\{a_1, ..., a_l\}$. Then the *contraction of* φ *to* $E - A$ is the function $(\varphi/A) : (E - A)^{r-l} \to T$ such that $(\varphi/A)(x_1, ..., x_{r-l}) := \varphi(x_1, ..., x_{r-l}, a_1, ..., a_l)$.

The contraction φ/A is always a Grassmann-Plücker function if φ is so and so the contraction also defines a T-matroid. The underlying matroid of this T-matroid has the desired properties. That is if $M_{\varphi/A} = [\varphi/A]$ and $M = [\varphi]$, then $\underline{M_{\varphi/A}} = \underline{M}/A$ [BB19, Lemma 4.4].

2.2.5 Vectors of T-Matroids

The vectors of a T-matroid are central to this thesis. In later sections we will use them to give a novel cryptomorphic description of T-matroids. An important reference for vectors of T-matroids is [And19] which gives a set of vector axioms. We do not discuss this here, however it is important to note that this exists.

Definition 2.57. Let M be a T-matroid with hyperplane function set η. Let M^* be the dual of M with hyperplane function set η^*. The *set of vectors of* M is $\mathcal{V} = \eta^\perp$. The *set of covectors of* M is $\mathcal{V}^* = (\eta^*)^\perp$.

Let us touch on perfect T-matroids, which are key when considering vectors.

Definition 2.58. A T-matroid is perfect if $\mathcal{V}^* \perp \mathcal{V}$.

Definition 2.59. A *perfect tract* P is a tract such that every T-matroid is perfect.

Remark 2.60. An important implication of a tract being perfect is that every weak T-matroid over the tract is also strong [BB19, Theorem 3.46]. Hence, over perfect tracts, we can use properties of weak and strong matroids interchangeably.

2.2.6 Quotients

Let us also introduce the notion of quotients in T-matroids which we take from [JL24].

Definition 2.61. Let T be a tract and E a finite set. Let M and N be two T-matroids over E. Let η_M be the set of hyperplanes of M and $\mathcal{V}_N^*$ the covectors of N. Then M is a *quotient* of N if $\eta_M \subseteq \mathcal{V}_N^*$.

Theorem 2.62. Let E be a finite set, T be a perfect tract and let M and N be T-matroids over E of respective ranks r and w. Then M is a quotient of N if and only if for some choice of Grassmannn-Plücker functions μ and υ such that $M = [\mu]$ and $N = [\upsilon]$, respectively, satisfies the Plücker flag relations

$$\sum_{k=1}^{w+1} (-1)^k \upsilon(y_1, ...\hat{y}_k, ..., y_{w+1}) \cdot \mu(y_k, x_1, ..., x_{r-1}) \in N_T,$$

for each choice of $y_1, ..., y_{w+1}, x_1, ..., x_{r-1} \in E$.

Proof. In [JL24] this is given as the definition of quotients and the definition we give is given as an equivalence. We take T to be perfect here, since this definition only holds for strong matroids. Since the cocircuits of a matroid are equivalent to the set of hyperplane functions, the equivalence follows from [JL24, Theorem 2.4]. □

Covectors give a useful equivalence to quotients.

Theorem 2.63. Let T be a perfect tract and E a finite set. Let M and N be two T-matroids over E with covectors $\mathcal{V}_M^*$ and $\mathcal{V}_N^*$ respectively. Then M is a quotient of N if and only if $\mathcal{V}_M^* \subseteq \mathcal{V}_N^*$.

Proof. See [JL24, Theorem 2.17]. □

3 T-representations of Flats

The goal of this section is to introduce the novel notion of a T-representation of a flat. However, to show that this is well defined, we need some preliminary results.

3.1 Preliminary Results

Let T be a tract and E a finite set. In this section we assume $\varphi : E^r \to T$ to be a weak Grassmann-Plücker function. We let $M = [\varphi : E^r \to T]$ be the weak T-matroid with underlying matroid $\underline{M}$. For a flat F of rank s of $\underline{M}$ and $J = \{j_1, \dots, j_s\}$ an independent set spanning F, we consider the function $\varphi_J : E^t \to T$ such that

$$\varphi_J(x_1, \dots, x_t) = \varphi(j_1, \dots, j_s, x_1, \dots, x_t)$$

where $t + s = r$.

Proposition 3.1. The function $\varphi_J(x_1, \dots, x_t) \neq 0$ if and only if $x_1, \dots, x_t \in E - F$ and $\{x_1, \dots, x_t\}$ forms a basis for $\underline{M}/F$.

Proof. Let us first show the backwards direction We have $x_1, \dots, x_t \in E - F$ and $\{x_1, \dots, x_t\}$ forms a basis for $\underline{M}/F$. By definition of $\underline{M}/F$, if $\{x_1, \dots, x_t\}$ forms a basis for $\underline{M}/F$ then for every basis B_F of F, $\{x_1, \dots, x_t\} \cup B_F$ is a basis for $\underline{M}$. By assumption, $\{j_1, \dots, j_s\}$ is independent and spans F, hence it forms a basis for F. Therefore, $\{j_1, \dots, j_s, x_1, \dots, x_t\}$ must form a basis for $\underline{M}$. The bases of $\underline{M}$ are exactly the support of φ. Hence, $\varphi_J(x_1, \dots, x_t) = \varphi(j_1, \dots, j_s, x_1, \dots, x_t) \neq 0$.

Now let us attempt the forward direction. If we have $\varphi_J(x_1, \dots, x_t) \neq 0$, then by definition $\varphi(j_1, \dots, j_s, x_1, \dots, x_t) \neq 0$ and so $\{j_1, \dots, j_s, x_1, \dots, x_t\}$ must form a basis for $\underline{M}$. Let us show that $\{x_1, \ldots, x_t\}$ forms a basis for $\underline{M}/F$. We have that $\{j_1, \ldots, j_s\}$ must form a basis for F and $\{j_1, \ldots, j_s, x_1, \ldots, x_t\}$ must form a basis for $\underline{M}$. Then by definition of $\underline{M}/F$ we have that $\{x_1, \ldots, x_t\}$ forms a basis for $\underline{M}/F$ and that $x_1, \ldots, x_t \in E - F$. □

Proposition 3.2. If φ is a weak Grassmann-Plücker function on E, then $\varphi_J : E^t \to T$ is a weak Grassmann-Plücker function on E and so $[\varphi_J]$ is a weak T-matroid.

Proof. For φ_J to be a weak Grassmann-Plücker function, we need its support to be the bases of a matroid and it must satisfy (GP1), (GP2) and (GP3) from Definition 2.49. By Proposition 3.1, we have that the supports of φ form the bases of $\underline{M}/F$. We now show that φ_J satisfies (GP1), (GP2) and (GP3).

(GP1) From claim 1 we have $\varphi_J(x_1, \ldots, x_t) \neq 0$ if and only if $x_1, \ldots, x_t \in E - F$ and $\{x_1, \ldots, x_t\}$ forms a basis for $\underline{M}/F$. Such a set must exist, hence φ_J cannot be identically zero. Hence, φ_J satisfies (GP1).

(GP2) We notice that

$$\begin{aligned}\varphi_J(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_t) &= \varphi(j_1, \ldots, j_s, x_1, \ldots, x_i, \ldots, x_j, \ldots, x_t)\\ &= -\varphi(j_1, \ldots, j_s, x_1, \ldots, x_j, \ldots, x_i, \ldots, x_t)\\ &= -\varphi_J(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_t).\end{aligned}$$

Hence, φ_J satisfies (GP2).

(GP3) We know that φ satisfies (GP3). Let $I = \{x_1, ..., x_{t+1}\}$ and $J = \{y_1, ..., y_{t-1}\}$ be two tuples of elements in E such that $|I \backslash J| = 3$. Since, φ satisfies (GP3), we have

$$\begin{aligned}&\sum_{k=1}^{s}(-1)^k \varphi(j_1, \ldots, \hat{j}_k, \ldots, j_s, x_1, \ldots, x_t) \cdot \varphi(j_k, j_1, \ldots, j_s, y_1, \ldots, y_{t-1})\\ &+\sum_{k=1}^{t+1}(-1)^{s+k} \varphi(j_1, , \ldots, j_s, x_1, \ldots, \hat{x}_k, \ldots, x_t) \cdot \varphi(x_k, j_1, \ldots, j_s, y_1, \ldots, y_{t-1}) \in N_T.\end{aligned}$$

We notice that $j_k, j_1, \ldots, j_s, x_1, \ldots, x_t$ is never independent and so cannot be a basis. Hence, $\varphi(j_k, j_1, \ldots, j_s, x_1, \ldots, x_t) = 0$ for all $1 \leq k \leq s$ and so

$$\sum_{k=1}^{t+1}(-1)^k \varphi(j_1, , \ldots, j_s, x_1, \ldots, \hat{x}_k, \ldots, x_t) \cdot \varphi(x_k, j_1, \ldots, j_s, y_1, \ldots, y_{t-1}) \in N_T,$$

implying

$$\sum_{k=1}^{t+1}(-1)^k \varphi_J(x_1, \ldots, \hat{x}_k, \ldots, x_t) \cdot \varphi_J(x_k, y_1, \ldots, y_{t-1}) \in N_T.$$

We conclude φ_J satisfies (GP3).

Hence, φ_J must be a weak Grassmann-Plücker function and $[\varphi_J]$ must be a weak T-matroid. □

Proposition 3.3. Let J be an independent set of . Let $I = \{i_1, \ldots, i_s\}$ be an independent set in E^s such that $I \neq J$. Then I spans F if and only if $\varphi_I \in [\varphi_J]$, in other words $\varphi_J = a \cdot \varphi_I$ for some $a \in T^\times$.

Proof. First the backwards direction. We have $\varphi_I \in [\varphi_J]$, in other words φ_I and φ_J have the same support. By Lemma 3.1 we have that the support of φ_J is exactly the set of bases of $\underline{M}/F$. Hence we have for $\{x_1, ..., x_t\}$ a basis of $\underline{M}/F$, that $\varphi_I(x_1, ..., x_t) \neq 0$. However, this also means $\varphi(j_1, ..., j_s, x_1, ..., x_t) \neq 0$ and so that $\{j_1, ..., j_s, x_1, ..., x_t\}$ forms a basis for E. Aiming for a contradiction, let us assume that I spans a rank s flat $F' \neq F$. Then, $\{x_1, ..., x_t\}$ form a basis for $\underline{M}/F'$. However this would imply that $\underline{M}/F = \underline{M}/F'$, which implies $F = F'$ contradicting our assumption. Hence, I spans F.

Now for the forward direction. For this direction we use the notion of a basis graph. For a matroid N with bases $\mathcal{B}$, the *basis graph* is the graph with vertex set given by the collection of bases $\mathcal{B}$ such that two bases are neighbours if they differ by one element, in other words, one can be obtained from the other through basis exchange. We denote the basis graph of N by $BG(N)$.

We use [Mau73, Theorem 2.1], which states that for any matroid, the basis graph is connected.

Consider the set of independent spanning sets of F. These are exactly the bases of $\underline{M}\backslash(E-F)$. We know that $BG(\underline{M}\backslash(E-F))$ is connected, hence, for any two bases I and J we can find a path from I to J on $BG(\underline{M}\backslash(E-F))$. If we show that for any two neighbours I and I' on $BG(\underline{M}\backslash(E-F))$ that $\varphi_I = \alpha \cdot \varphi_{I'}$ for some $\alpha \in T^\times$, then with an induction argument it must hold for any two bases in general.

Let $I = \{i_1, \ldots, i_{n-1}, i_n, i_{n+1}, ..., i_s\}$ and $I' = \{i_1, \ldots, i_{n-1}, i'_n, i_{n+1}, ..., i_s\}$, such that $i_n \neq i'_n$, be two bases of F, hence, neighbours on $BG(\underline{M}\backslash(E-F))$. Let $B = \{x_1, \ldots, x_r\}$ be a basis for $\underline{M}/F$ and let C be the circuit contained in $I \cup \{i'_n\} \cup B = I' \cup \{i_n\} \cup B$. Since C is both the fundamental circuit of i'_n with respect to $I \cup B$ (that is the unique circuit contained in $I \cup B \cup \{i'_n\}$ [Oxl92, Corollary 1.2.6]) and the fundamental circuit of i_n with respect to $I' \cup B$, we have that i_n and i'_n in C. We also have $I \cup B$ forms a basis for $\underline{M}$ containing $C - \{i'_n\}$. Therefore, by [BB19, Lemma 4.5] we have that the quantity

$$\alpha = \frac{\varphi(i'_n, i_1, \ldots, \hat{i}_n, \ldots, i_s, x_1, \ldots, x_r)}{\varphi(i_n, i_1, \ldots, \hat{i}_n, \ldots, i_s, x_1, \ldots, x_r)},$$

is independent of choice of $i_1, \ldots, \hat{i}_n, \ldots, i_s, x_1, \ldots, x_r$ such that $i_n, i_1, \ldots, \hat{i}_n, \ldots, i_s, x_1, \ldots, x_r$

forms a basis for $\underline{M}$ containing $C - \{i'_n\}$.

This implies the quantity is independent of choice of basis for $\underline{M}/F$ and so we have that

$$\alpha = \frac{\varphi(i'_n, i_1, \ldots, \hat{i}_n, \ldots, i_s, x_1, \ldots, x_t)}{\varphi(i_n, i_1, \ldots, \hat{i}_n, \ldots, i_s, x_1, \ldots, x_t)} = \frac{\varphi(i'_n, i_1, \ldots, \hat{i}_n, \ldots, i_s, y_1, \ldots, y_t)}{\varphi(i_n, i_1, \ldots, \hat{i}_n, \ldots, i_s, y_1, \ldots, y_t)},$$

for $y_1, \ldots, y_r$ any basis for $\underline{M}/F$. We note that, by definition, this means

$$\alpha = \frac{\varphi_{I'}(x_1, \ldots, x_t)}{\varphi_I(x_1, \ldots, x_t)} = \frac{\varphi_{I'}(y_1, \ldots, y_t)}{\varphi_I(y_1, \ldots, y_t)}.$$

Hence $\varphi_{I'}(y_1, \ldots, y_t) = \alpha \cdot \varphi_I(y_1, \ldots, y_t)$ for any basis for $\underline{M}/F$. By Proposition 3.1 we know that the support of φ_I and $\varphi_{I'}$ are equal and are exactly the set of bases of $\underline{M}/F$. Hence we have that for every $\{y_1, \ldots, y_t\}$ in the support $\varphi_{I'}(y_1, \ldots, y_t) = \alpha \cdot \varphi_I(y_1, \ldots, y_r)$. Therefore, we must have that $\varphi_{I'} = \alpha \cdot \varphi_I$.

Since our basis graph is connected and for every two neighbours I and I' we have exactly that $\varphi_{I'} = \alpha \cdot \varphi_I$, it follows from induction that we must have that the same holds for any two arbitrary bases of F. □

3.2 T-representations of Flats

With these results, we can define T-representations of flats.

Definition 3.4. Let T be a tract, E a finite set and $\varphi : E^r \to T$ a weak Grassmann-Plücker function. Let $M = [\varphi]$ be a T-matroid with underlying matroid $\underline{M}$. Let F be a rank s flat of $\underline{M}$. Then the *T-representation of F with respect to M* is the T-matroid $M_F = [\varphi_F]$ where $\varphi_F = \varphi_I$ for some $I \subseteq E$ independent in $\underline{M}$ and spanning F.

We note that, by Proposition 3.3, it does not matter which set spanning set I we choose. We can also consider M_F from a different perspective which allows us to easily uncover the underlying matroid of M_F. To do so we need to introduce the direct sum of matroids and of T-matroids.

Definition 3.5. Let E and A be disjoint finite sets and M and N matroids with ground sets E and A and rank r and t respectively. Then the *direct sum of M and N* is the matroid $M \oplus N$ with ground set $E \cup A$ such that a set $\{i_1, ..., i_r, j_1, ..., j_t\}$ forms a basis for $M \oplus N$ if $\{i_1, ..., i_r\}$ forms a basis for M and $\{j_1, ..., j_s\}$ forms a basis for N.

By [Oxl92, Page 124], this does actually form a matroid.

Definition 3.6. Let T be a tract and E and A be disjoint finite sets. Let $\varphi : E^r \to T$ and $\varepsilon : A^t \to T$ be two weak Grassmann-Plücker functions with $M_\varphi = [\varphi : E^r \to T]$ and

$M_\varepsilon = [\varepsilon : A^t \to T]$. Then the *direct sum of* M_φ *and* M_ε is the T-matroid $M_\varphi \oplus M_\varepsilon$ such that $M_\varphi \oplus M_\varepsilon = [\varphi \oplus \varepsilon : E^r \times A^t \to T]$ where $\varphi \oplus \varepsilon(i_1, ..., i_r, j_1, ..., j_t) = \varphi(i_1, ..., 1_r) \cdot \varepsilon(j_1, ..., j_t)$.

Proposition 3.7. Let T be a tract and E and A be finite sets. Let $\varphi : E^r \to T$ and $\varepsilon : A^t \to T$ be Grassmann-Plücker functions. Let $\underline{M}_\varphi$ be the underlying matroid of $M_\varphi = [\varphi : E^r \to T]$ and let $\underline{M}_\varepsilon$ be the underlying matroid of $M_\varepsilon = [\varepsilon : A^t \to T]$. Then the underlying matroid of $M_\varphi \oplus M_\varepsilon$ is $\underline{M}_\varphi \oplus \underline{M}_\varepsilon$.

Proof. The underlying matroid of $M_\varphi = [\varphi : E^r \to T]$ is the matroid over E with bases exactly the support of φ, the same holds for M_ε. The supports of $M_\varphi \oplus M_\varepsilon$ must therefore be the disjoint union of bases of $\underline{M}_\varphi$ and $\underline{M}_\varepsilon$. This is exactly the bases of $\underline{M}_\varphi \oplus \underline{M}_\varepsilon$. Hence $\underline{M_\varphi \oplus M_\varepsilon} = \underline{M}_\varphi \oplus \underline{M}_\varepsilon$ □

The next result shows an interesting link between T-representations of flats and contractions of T-matroids. It also lets us find the underlying matroids of T-representations easily.

Theorem 3.8. Let T be a tract, E a finite set and $\varphi : E^r \to T$ a weak Grassmann-Plücker function. Let $M = [\varphi]$ be a T-matroid with underlying matroid $\underline{M}$. Let $\varphi_{0,F} : F^0 \to T$ be the function that maps all elements of F to zero. Let F be a rank s flat of $\underline{M}$ and let $[\varphi_F]$ be the T-representation of F with respect to M. Then $[\varphi_F] = [\varphi/F] \oplus [\varphi_{0,F}]$.

Proof. If $x_1, \ldots, x_n$ are in $E - F$ then $\varphi_F(x_1, \ldots, x_t) = \varphi/F(x_1, \ldots, x_t)$ and so $(\varphi/F \oplus \varphi_{0,F})(x_1, \ldots, x_t) = \varphi/F(x_1, \ldots, x_t)$. If $x_1, \ldots, x_t \notin E - F$, then φ/F is undefined, but in this case $(\varphi/F \oplus \varphi_{0,F})(x_1, \ldots, x_t) = 0$. For $x_1, \ldots, x_t \notin E - F$, we must also have $\varphi_J(x_1, \ldots, x_t) = 0$ since $x_1, \ldots, x_t$ cannot form a basis of F. Hence, we have that $(\varphi/F \oplus \varphi_{0,F})(x_1, \ldots, x_t) = \varphi_F(x_1, \ldots, x_t)$. Hence, $[\varphi_F] = [\varphi/F] \oplus [\varphi_{0,F}]$. □

Theorem 3.9. Let T be a tract, E a finite set and $\varphi : E^r \to T$ a Grassmann-Plücker function. Let F be a rank s flat of $\underline{M}$, the underlying matroid of $M = [\varphi]$. Then the underlying matroid of $M_F = [\varphi_F]$, the T-representation of F with respect to M, is $\underline{M}/F \oplus U_{0,F}$, where $U_{0,F}$ is the uniform matroid of rank 0 over F.

Proof. This follows directly from Proposition 3.7 and Theorem 3.8. □

4 T-linear Representations

The goal of this section is to introduce a new cryptomorphism of T-matroids using T-representations of flats. Here a modular triple of hyperplanes of $\underline{M}$ is a triple of hyperplanes such that $F = H_1 \cap H_2 \cap H_3$ is a corank 2 (that is, of rank $r - 2$, for r rank of $\underline{M}$) flat. A modular triple of hyperplane functions is a triple of hyperplane functions in T^E such that the complements of the supports of the functions form a modular triple of hyperplanes. A modular system for a matroid $\underline{M}$ is a collection of functions in T^E, one for each $H \in \mathcal{H}_{\underline{M}}$ such that whenever H_1, H_2, H_3 is a modular triple of hyperplanes in $\mathcal{H}_{\underline{M}}$, the corresponding functions are linearly dependent.

Theorem 4.1. Let $\underline{M}$ be a matroid. A modular triple $f_{H_1}, f_{H_2}, f_{H_3}$ of hyperplane functions of $\underline{M}$ is linearly dependent if and only if there exists a T-matroid $[\varphi : E^2 \to T]$ such that $[f_{H_i} : E \to T]$ for $i = 1, 2, 3$ is a quotient of $[\varphi : E^2 \to T]$.

Proof. First the forward direction. We notice that since f_{H_1}, f_{H_2} and f_{H_3} are linearly dependent, $\{\alpha \cdot f_{H_1} \mid \alpha \in T^\times\} \cup \{\alpha \cdot f_{H_2} \mid \alpha \in T^\times\} \cup \{\alpha \cdot f_{H_3} \mid \alpha \in T^\times\}$ satisfy (Hf1)–(Hf4) and so form a set of hyperplane functions, defining a T-matroid N. Let $\underline{N}$ be the underlying matroid from this T-matroid, with hyperplanes exactly the supports of f_{H_1}, f_{H_2} and f_{H_3}, in other words $\underline{N}$ has as hyperplanes exactly H_1, H_2, H_3.

Since f_{H_1}, f_{H_2} and f_{H_3} form a modular triple, we have $F = H_1 \cap H_2 \cap H_3$ forms a corank 2 flat, so this implies that $\underline{N}$ is a rank 2 matroid.

Consider the rank 2 Grassmann-Plücker function $\varphi : E^2 \to T$ representing $\underline{N}$. We note f_{H_1}, f_{H_2} and f_{H_3} are also Grassmann-Plücker functions with T-matroids $N_1 = [f_{H_1}]$, $N_2 = [f_{H_2}]$ and $N_3 = [f_{H_3}]$ respectively. Then the set of hyperplane functions of N_i is going to be exactly $[f_{H_i}]$ for $i = 1, 2, 3$. We also remember that the covectors of N are orthogonal to the vectors of N. Since the vectors of N are themselves orthogonal to the hyperplanes of N, we have that the hyperplane functions of N are contained in the covectors. This implies that $\eta_{N_i} \subseteq \mathcal{V}^*(N)$. We use Definition 2.61 and conclude that $N = [\varphi]$ is a quotient of $N_i = [f_{H_i}]$ for $i = 1, 2, 3$.

For the backward direction, we want to show that f_{H_1}, f_{H_2} and f_{H_3} are part of a modular system of $[\varphi : E^2 \to T]$. If we show for each f_{H_i} that for $x \in H_i$ and $y, z \notin H_i$

$$\frac{f_{H_i}(y)}{f_{H_i}(z)} = \frac{\varphi(x, y)}{\varphi(x, z)},$$

then we have by [BL20, Theorem 2.16] that f_{H_1}, f_{H_2} and f_{H_3} are a modular triple and so are linearly dependent.

Take $x \in H_i$ and $y, z \notin H_i$. By φ being quotient to f_{H_i}, we must have by Theorem 2.62 that

$$\varphi(x, y) f_{H_i}(z) - \varphi(x, z) f_{H_i}(y) + \varphi(y, z) f_{H_i}(x) \in N_T.$$

Since $x \in H_i$, we have that $f_{H_i}(x) = 0$, hence

$$\varphi(x, y) f_{H_i}(z) - \varphi(x, z) f_{H_i}(y) \in N_T.$$

This gives exactly the result we want. Since this hold for $i = 1, 2, 3$ we must have that f_{H_1}, f_{H_2} and f_{H_3} are part of a modular system for $\varphi : E^2 \to T$ and so are linearly dependent. □

Using this result we can give a cryptomorphic description of weak T-matroids.

Definition 4.2. Let T be a tract and E a finite set. Let $\underline{M}$ be a usual matroid over E with collection hyperplanes $\mathcal{H}$. A T-linear representation of $\underline{M}$ is a family of hyperplane functions $\mathcal{R}$ with one hyperplane function for each $H \in \mathcal{H}$ and $E - \underline{f_H} = H$ for each $f_H \in \mathcal{R}$ such that for each modular triple of hyperplanes H_1, H_2 and H_3 in $\mathcal{H}$, there exists a T-matroid $[\varphi : E^2 \to T]$ such that $[f_{H_i} : E \to T]$ is a quotient of $[\varphi : E^2 \to T]$ for $i = 1, 2, 3$.

Theorem 4.3. Let T be a tract and E a finite set. Let $\underline{M}$ be a usual matroid of rank r over E with hyperplanes $\mathcal{H}$. There exists a bijection between weak T-matroids with underlying matroids $\underline{M}$ and equivalence classes by multiplication over $T^\times$ of T-linear representations of $\underline{M}$, that is the collection of equivalence classes of the functions in the T-linear representation.

Proof. By [BL20, Theorem 2.16], there exists a bijection between weak T-representations of $\underline{M}$ and modular systems of hyperplanes of $\underline{M}$. By Theorem 4.1 modular systems of hyperplanes of $\underline{M}$ are equivalent to a T-linear representation. Hence, there exists a bijection between weak T-representations of $\underline{M}$ and T-linear representations of $\underline{M}$. Since a T-matroid with underlying matroids $\underline{M}$ is just an equivalence class of T-representations of $\underline{M}$, we have that there is a bijection between weak T-matroids with underlying matroids $\underline{M}$ and equivalence classes by multiplication of $T^\times$ of T-linear representations of $\underline{M}$. □

5 T-Flats

In this section we define the notion of a T-flat and discuss its properties. We also show that we can use T-flats to give a cryptomorphism of a T-matroid. When T is a field, T-flats are strongly linked to hyperplane arrangements. In this section we assume that T is a perfect tract and so we view every T-matroid as weak.

5.1 T-flats and Their Properties

Definition 5.1. Let E be a finite set. Let $\varphi : E^r \to T$ be a T-representation of $\underline{M}$ a usual matroid over E. Let F be a flat of $\underline{M}$. The *T-flat of F with respect to* φ is the set $\mathcal{V}_F = \mathcal{V}([\varphi_F])$, with $\mathcal{V}_\emptyset = \mathcal{V}([\varphi])$ and $\mathcal{V}_E = \mathcal{V}([\varphi_E])$, that is, the vectors of the T-representation of F. The *lattice of T-flats of a T-representation* φ is the set $\{\mathcal{V}_F \mid F \in \Lambda\}$, where Λ is the lattice of flats of $\underline{M}$.

Remark 5.2. The T-matroid $[\varphi_E]$ is the T-matroid over E with underlying matroid $U_{0,E}$. It can also be viewed as having Grassmann-Plücker function $\varphi_E : E^0 \to T$.

We discuss the properties of the lattice of T-flats. To do this we prove this useful result.

Lemma 5.3. Let E be a finite set. Let $\varphi : E^r \to T$ be a T-representation of $\underline{M}$ a usual matroid over E. Let η be the set of hyperplane functions of $[\varphi : E^r \to T]$. Let F be a flat of $\underline{M}$. Then $\eta_F = \{X \in \eta \mid F \subseteq E - \underline{X}\}$ is the set of hyperplane functions of $[\varphi_F]$.

Proof. By the bijection given in Remark 2.55, we have that for all $X \in \eta$,

$$\frac{Y(e)}{Y(e')} = (-1)^k \frac{\varphi_F(x_1, ..., x_{s-1}, e)}{\varphi_F(x_1, ..., x_{s-1}, e')} = (-1)^k \frac{\varphi(i_1, .., i_{r-s}, x_1, ..., x_{s-1}, e)}{\varphi(i_1, .., i_{r-s} x_1, ..., x_{s-1}, e')}.$$

Hence, for every $X \in \eta_F$, we have $X \in \eta$. We know that $[\varphi_F]$ has underlying matroid $\underline{M}/F \oplus U_{0,F}$. The hyperplanes of this are exactly the hyperplanes of $\underline{M}$, such that F contained in H. Since η_F and $[\varphi_F]$ must have the same underlying matroid, we must have the $\eta_F = \{X \in \eta_F \mid F \subseteq E - \underline{X}\}$.

□

Proposition 5.4. Let E be a finite set and let $\varphi : E^r \to T$ be a T-representation of $\underline{M}$ a usual matroid over E. Let $\{\mathcal{V}_F \mid F \in \Lambda\}$ be the lattice of T-flats for $[\varphi]$. Then $\mathcal{V}_F = \bigcap_{S \subseteq H} \mathcal{V}_H$ for $S \subseteq E$ such that $\mathrm{cl}(S) = F$.

Proof. We have that $\mathcal{V}_F = \eta_F^\perp$ by Definition 2.57, where η_F is the set of hyperplane functions of $[\varphi_F]$. Since the only hyperplane that contains a hyperplane, is the hyperplane itself, we have by Lemma 5.3 that $\mathcal{V}_H = \{X \in \eta \mid H = E - \underline{X}\}^\perp$. Hence

$$\bigcap_{S \subseteq H} \mathcal{V}_H = \bigcap_{S \subseteq H} \{X \in \eta \mid H = E - \underline{X}\}^\perp = \left(\bigcup_{S \subseteq H} \{X \in \eta \mid H = E - \underline{X}\} \right)^\perp.$$

We note that a hyperplane is such that $S \subseteq H$ if and only $F \subseteq H$, hence

$$\bigcup_{S \subseteq H} \{X \in \eta \mid H = E - \underline{X}\} = \bigcup_{F \subseteq H} \{X \in \eta \mid H = E - \underline{X}\}$$

$$= \{X \in \eta \mid F \subseteq E - \underline{X}\} = \eta_F.$$

Hence,

$$\bigcap_{S \subseteq H} \mathcal{V}_H = \eta_F^\perp = \mathcal{V}_F.$$

□

Theorem 5.5. Let E be a finite set and T a perfect tract. The lattice of T-flats of a T-matroid over E with underlying matroid $\underline{M}$ has the same abstract lattice as the lattice of flats of $\underline{M}$.

Proof. We have $\mathcal{V}_F = \bigcap_{S\subseteq H} \mathcal{V}_H$ if and only if $F = \bigcap_{S\subseteq H} H$. Since there is also a bijection between the hyperplanes of the lattice of flats of $\underline{M}$ and the hyperplanes of the lattice of T-flats, this means that they have the same lattice structure. □

Let us give an abstract definition of a lattice of T-flats.

Definition 5.6. Let E be a finite set. Let $V = \{\mathcal{V}_k \mid k \in L\}$ be a collection of subset of T^E, partially ordered by inclusion. Then we say V is a *lattice of T-flats over E* if

(LT1) we have V forms a geometric lattice with respect to inclusion with $|E| = n$ atoms, such that the top is T^n and the bottom is $\{0\}$,

(LT2) let Λ be the lattice with the same structure as V but atoms given by the elements of E. Then the corank 1 elements of V are of the form $\mathcal{V}_H = (f_H)^\perp$ where f_H is such that $E - \underline{f_H} = H$ for $H \subseteq E$ a hyperplane of Λ with atoms E,

(LT3) the elements of the lattice with corank 2 or corank 1 form a set of vectors in the sense of [And19],

(LT4) every element of V is the intersection of corank 1 elements of V.

In [And19], necessary conditions are given for a subset of T^E to be a set of vectors of a T-matroid. Here we assume the sets of corank 1 or 2 satisfy these conditions.

Theorem 5.7. Let E be a finite set. There exists a bijection between T-matroids over E and lattices of T-flats over E.

Proof. We show that there exists a bijection between T-matroids and lattices of T-flats. This is the map that sends a T-matroid to its respective lattice of T-flats. We show injectivity and then surjectivity.

Let M be a T-matroid. Then by Theorem 5.5, the lattice of T-flats of M forms a geometric lattice and each $\mathcal{V}_F$ in the lattice are vectors for a T-matroid. Hence, lattice of T-flats of M is a lattice of T-flats in the sense of Definition 5.6. Since the lattice of T-flats of a matroid characterizes it completely (its top is the set of vectors of the matroid which characterizes the matroid completely by [And19, Theorem 2.18]), we have that every T-matroid has a unique lattice of T-flats and so the map is injective.

Let us consider the inverse map. Let V be a lattice of T-flats. We claim the functions f_H form a T-linear representation. Consider a modular triple of hyperplanes H_1, H_2, H_3 in Λ with atoms E. Then $\mathcal{V}_{H_1}, \mathcal{V}_{H_2}, \mathcal{V}_{H_3}$ and $\mathcal{V}_F = \mathcal{V}_{H_1} \cap \mathcal{V}_{H_2} \cap \mathcal{V}_{H_3}$ are of corank 1 or 2 and so forms a set of vectors for T-matroids $M_{H_1}, M_{H_2}, M_{H_3}$ and M_F respectively. Since $\mathcal{V}_F \subseteq \mathcal{V}_{H_i}$ for every $i = 1, 2, 3$, M_F is quotient to each M_{H_i}. Since the T-representation of M_{H_i} is exactly f_{H_i} for each $i = 1, 2, 3$, we have that the set of functions f_H form a T-linear representation. By Theorem 4.3, this is equivalent to a T-matroid over E. Hence, the forward map is also surjective and so a bijection. □

5.2 K-flats and Hyperplane Arrangements

We note that fields are perfect tracts. For a field K and a K-vector space $V_0 \simeq K^r$, a hyperplane arrangement is a set of hyperplanes (that is $r-1$ dimensional subspace of V_0 and not hyperplanes in the matroid sense), such that the intersection of all the hyperplanes is exactly 0. Hyperplane arrangements give rise to matroids. If we have $E = \{1, ..., n\}$ and we have $V_1, ..., V_n \subseteq V_0 \simeq K^r$ hyperplanes, then we can define a rank function $r : E \to \mathbb{N}_0$ such that $r(X) = \text{codim}(\bigcap_{i \in X} V_i)$. This satisfies the axioms of a rank function and so defines a matroid.

Lemma 5.8. Let K be a field and $M = [\varphi]$ a K-matroid with lattice of T-flats $\{\mathcal{V}_F \mid F \in \Lambda\}$, hyperplane functions η. Then

$$\mathcal{V}_F^* = \mathcal{V}_F^\perp = \langle \eta_F \rangle,$$

where η_F is the collection hyperplane functions of $[\varphi_F]$ and $\langle \eta_F \rangle$ its span. Further,

$$\mathcal{V}_F^* = \bigcap_{i \in S} \mathcal{V}_{\{i\}}^*.$$

Proof. By [And19, Proposition 2.19], we have that for any K-matroid with vectors $\mathcal{V}$, it holds that $\mathcal{V}^\perp = \mathcal{V}^*$. Hence, we also have $\mathcal{V}_F^* = \mathcal{V}_F^\perp$. Since $\mathcal{V}_F = \eta_F^\perp$, we have that $\mathcal{V}_F^* = \langle \eta_F \rangle$.

We have $\eta_F = \{X \in \eta \mid F \subseteq E - \underline{X}\}$ and so $\eta_{\{i\}} = \{X \in \eta \mid i \in E - \underline{X}\}$. We notice that

$$\bigcap_{i \in S} \mathcal{V}_{\{i\}}^* = \bigcap_{i \in S} \langle \{X \in \eta \mid i \in E - \underline{X}\} \rangle = \langle \{X \in \eta \mid i \in E - \underline{X}, \text{ for all } i \in F\} \rangle$$

$$= \langle \{X \in \eta \mid F \subseteq E - \underline{X}\} \rangle = \mathcal{V}_F^*.$$

□

Theorem 5.9. Let K be a field. The orthogonal complements of the K-flats of a K-matroid M over E form a hyperplane arrangement in the space of covectors of M. Furthermore, the matroid defined by the hyperplane arrangement is exactly $\underline{M}$, the underlying matroid of M.

Proof. It is a known fact that the covectors of a rank r K-matroids form r-dimensional linear subspaces of K^n for $\#E = n$ [And19, Proposition 2.19]. This implies that for a rank t flat of $\underline{M}$, the vectors of K-matroid given by $[\varphi_F]$ form an $r - t$ dimensional subspace of K^n. Let us define $\mathcal{V}_F^*$ as the covectors of $[\varphi_F]$. Then $\mathcal{V}_\emptyset^*$ are the covectors of M, isomorphic to K^r, and $\mathcal{V}_{\{i\}}^*$ for $i \in E$ are the covectors of $[\varphi_{\{i\}}]$, which is isomorphic to K^{r-1}. Hence, $\mathcal{V}_{\{i\}}^*$ are hyperplanes in $\mathcal{V}_\emptyset^*$. We also have that $\bigcap_{i \in E} \mathcal{V}_{\{i\}}^* = \mathcal{V}_E^* = \{0\}$. Hence, we have a hyperplane arrangement. By Lemma 5.8 we have $\mathcal{V}_F^* = \bigcap_{i \in S} \mathcal{V}_{\{i\}}^*$.

We consider the matroid formed by the hyperplane arrangement. It has ground set $\{\mathcal{V}_{\{i\}}^* | i \in E\}$ and we have that $\text{codim}(\bigcap_{i \in S} \mathcal{V}_{\{i\}}^*) = \text{codim}(\mathcal{V}_F^*) = r_{\underline{M}}(F) = r_{\underline{M}}(S)$. Hence the matroid given by the hyperplane arrangement is exactly $\underline{M}$. □

Theorem 5.10. Let K be a field and let $\{V_i \subseteq V_\emptyset | i = 1, ..., n\}$ be a set of $r-1$ dimensional hyperplanes such that $V_\emptyset$ an r-dimensional subspace of K^n and $\bigcap_{i=1,...,n} V_i = \{0\}$. Let $\tilde{M}$ be the matroid of the hyperplane arrangement of $\{V_i \subseteq V_\emptyset | i = 1, ..., n\}$. Then there exists an embedding $\phi : V_\emptyset \to K^n$ and a K-matroid M with lattice of K-flats $\{\mathcal{V}_F^* \subseteq K^n | F \in \Lambda\}$ such that $\phi\left(\bigcap_{i \in F} V_i\right) = \mathcal{V}_F^*$ and $\phi(V_\emptyset) = \mathcal{V}_\emptyset^*$.

Proof. Let $v_1, ..., v_r$ a basis of $V_\emptyset$. For each V_i let $\omega_i = \sum_{1 \le j \le r} a_{j,i} v_j$ be a vector orthogonal to V_i. Let A be an $r \times n$ dimensional matrix over K such that $A_{s,t} = a_{s,t}$. Then we claim this matrix represents $\tilde{M}$ in the sense that a set I is independent in $\tilde{M}$ if and only if the vectors $\{\omega_i | i \in I\}$ form a linear independent set of vectors in $V_\emptyset$.

Let us prove this claim. We know a set I is independent in $\tilde{M}$ if $\#I = r - \dim(\bigcap_{i \in I} V_i)$. Consider $(\bigcap_{i \in I} V_i)^\perp = \bigcup_{i \in I} V_i^\perp$. We note $V_i^\perp = \operatorname{span}(\omega_i)$. Hence $\bigcup_{i \in I} V_i^\perp = \operatorname{span}(\{\omega_i | i \in I\})$. Notice $r - \dim(\bigcap_{i \in I} V_i) = \dim(\bigcup_{i \in I} V_i^\perp)$. Since $\dim(\bigcup_{i \in I} V_i^\perp) = \#I$ if and only if $\{\omega_i | i \in I\}$ forms a set of linear independent vectors, we have shown that A represents $\tilde{M}$.

Consider the function $\varphi : E^r \to K$ for $E = \{1, ..., n\}$ such that $\varphi(i_1, ..., i_r) = \det [A_{i_1} \cdots A_{i_r}]$ where A_i is the i-th row of A. It is well known that φ is a Grassmann-Plücker function [BB19, Page 16] and so defines a K-matroid M. We note that, since $\det [A_{i_1}, ..., A_{i_r}] \neq 0$ if and only if $\omega_{i_1} \cdots \omega_{i_r}$ linearly independent, the support of φ are exactly the sets indexing sets of r linearly independent vectors. Hence $\underline{M}$, the underlying matroid of M, is exactly $\tilde{M}$. This implies $[\varphi]$ is a K-matroid with underlying matroid $\tilde{M}$.

Let $\{\mathcal{V}_F | F \in \Lambda\}$ be the K-flats of M. Let us define a map $f : V_\emptyset \to K^n$ such that

$$v = \sum c_i v_i \mapsto A^T \begin{bmatrix} c_1 \\ \vdots \\ c_r \end{bmatrix}.$$

We claim that $f(V_\emptyset) = \mathcal{V}_\emptyset^*$.

$$A^T \begin{bmatrix} c_1 \\ \vdots \\ c_r \end{bmatrix} = \begin{bmatrix} \omega_1 \cdot v \\ \vdots \\ \omega_n \cdot v \end{bmatrix}.$$

Evidently, f is a linear transformation. We claim that it is injective, in other words, the kernel is 0. For an element v to be in the kernel of f, we must have $\omega_i \cdot v$ for every $1 \le i \le n$. This would mean v is in every V_i which by assumption means that $v = 0$.

Let $e_1, ..., e_n$ be the natural basis for K^n. An important observation is that for ω_i orthogonal to $V_{\{i\}}$, we have that $\omega_i \cdot v = 0$ if and only if $v \in V_i$. We have that $\mathcal{V}_\emptyset^* = \langle f_H | H \in \Lambda_{r-1} \rangle$ that is that $\mathcal{V}_\emptyset^*$ is spanned by the hyperplane functions of M. We will show that each hyperplane function of M is contained in $f(V_0)$, implying that the span is also contained in $f(V_0)$.

Consider $X \in f(\bigcap_{i\in H} V_i - \bigcup_{i\notin H} V_i)$ for some hyperplane H. Then we must have that $X(i) = 0$ if and only if $i \in H$, hence X is a K-hyperplane of H. Since such an X must exist for every H, we have that $\mathcal{V}_\emptyset^* \subseteq f(V_0)$.

Consider $x \in V_0$. We have $x \in \bigcap_{i\in F} V_i - \bigcup_{i\notin F} V_i$, for some F in $\tilde{M}$. Then $f(x)(i) = 0$ if and only if $i \in F$. Since the hyperplanes contain F intersect exactly at F, $f(x)$ can be written as the sum of hyperplane functions of the hyperplanes containing F. Hence $f(x) \in \mathcal{V}_\emptyset^*$ and so $f(V_0) \subseteq \mathcal{V}_\emptyset^*$. Hence $f(V_0) = \mathcal{V}_\emptyset^*$.

We claim $f(V_i) = \mathcal{V}_{\{i\}}^*$. Recall that for some $x \in V_0$, $f(x)(i) = 0$ if and only if $x \in V_i$. Hence, for all $X \in f(V_i)$, $X(i) = 0$ and so $f(V_i) \subseteq \langle e_i\rangle^\perp \cap \mathcal{V}_\emptyset^*$.

Consider $X \in \langle e_i\rangle^\perp \cap \mathcal{V}_\emptyset^*$. Since $f(V_0) = \mathcal{V}_\emptyset^*$, we have $X \in f(V_0)$. Since $X \in \langle e_i\rangle^\perp$, we must have $X(i) = 0$, hence we must have $X \in f(V_{\{i\}})$.

We claim $\mathcal{V}_{\{i\}}^* = \langle e_i\rangle^\perp \cap \mathcal{V}_\emptyset^*$. First we show $\mathcal{V}_{\{i\}}^* \subseteq \langle e_i\rangle^\perp \cap \mathcal{V}_\emptyset^*$ and then $\langle e_i\rangle^\perp \subseteq \mathcal{V}_{\{i\}}^*$. Note $\mathcal{V}_{\{i\}}^* = \langle f_H | i \in H\rangle$ and $f_H(x) = 0$ if and only if $x \in H$. Hence, for all $X \in \mathcal{V}_{\{i\}}$, we have $X(i) = 0$ and so $X \perp \langle e_i\rangle$. We obtain $\mathcal{V}_{\{i\}}^* \subseteq \langle e_i\rangle^\perp \cap \mathcal{V}_\emptyset^*$. For the other direction, we show that $X \in \langle e_i\rangle^\perp \cap \mathcal{V}_\emptyset^*$ implies $X \in \mathcal{V}_{\{i\}}^*$. If $X \in \langle e_i\rangle^\perp \cap \mathcal{V}_\emptyset^*$ then X is a linear combination of hyperplanes such that $f_H(i) = 0$, in other words, such that $i \in H$. Therefore $X \in \mathcal{V}_{\{i\}}^*$. Hence, $\langle e_i\rangle^\perp \cap \mathcal{V}_\emptyset^* \subseteq \mathcal{V}_{\{i\}}^*$.

We have shown $f(V_i) = \langle e_i\rangle^\perp \cap \mathcal{V}_\emptyset^* = \mathcal{V}_{\{i\}}^*$. Let us show that it follows that $\mathcal{V}_F^* = f(\bigcap_{i\in F} V_i)$. Since f is injective, we must have that $f(\bigcap_{i\in F} V_i) = \bigcap_{i\in F} f(V_i) = \bigcap_{i\in F} \mathcal{V}_{\{i\}} = \mathcal{V}_F$. Hence, there exists embedding ϕ and K-matroid M as we claim. □

6 Example

We finish of with an example of T-flats. We shall use the rank three matroid $\underline{M}$ over four elements which was described in Example 2.29. We consider the problem over the real numbers $\mathbb{R}$. We describe an $\mathbb{R}$-matroid with underlying matroid $\underline{M}$. We then look at the $\mathbb{R}$-representations of each of the flats, the lattice of $\mathbb{R}$-flats and the hyperplane arrangement described by it.

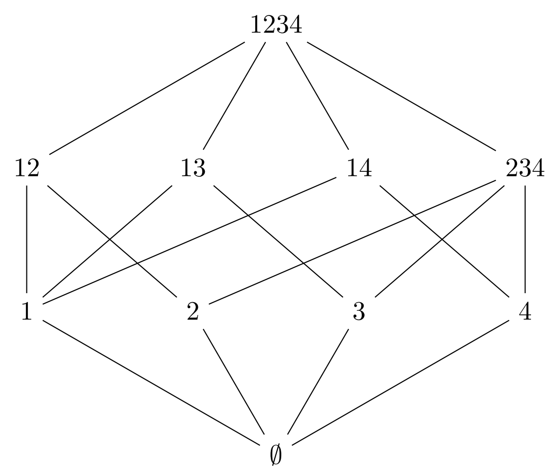

Figure 3: The rank three matroid $\underline{M}$ over $E = 1234$ with hyperplanes $12, 13, 14$ and 234.

Let us consider what such an $\mathbb{R}$-matroid would look like in terms of Grassmann-Plücker functions. The bases of $\underline{M}$ are $123, 124$ and 134. We can construct a Grassmann-Plücker function as the determinant of a minor of a matrix [BB19, Page 16]. Consider the vectors

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, v_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

then defining $\varphi : E^3 \to \mathbb{R}$ as $\varphi(i,j,k) = \det(A_{ijk})$, where A_{ijk} is the 3×3 matrix with columns v_i, v_j and v_k, will give us a Grassmann-Plücker function. We have $\det(A_{ijk}) \neq 0$ if and only if v_i, v_j and v_k are linearly independent. Hence, $\varphi(i,j,k) \neq 0$ if and only if $ijk = 123$, $ijk = 124$ or $ijk = 134$. Hence, $[\varphi]$ has underlying matroid $\underline{M}$.

Now let us look at the $\mathbb{R}$-representations of different flats. We see that

$$\varphi_1(i,j) = \det(A_{1ij}), \varphi_2(i,j) = \det(A_{2ij}), \varphi_3(i,j) = \det(A_{3ij}), \varphi_4(i,j) = \det(A_{4ij})$$

$$\varphi_{12}(i) = \det(A_{12i}), \varphi_{13}(i) = \det(A_{13i}), \varphi_{14}(i) = \det(A_{14i}),$$

$$\varphi_{234}(i) = \det(A_{23i}) = \det(A_{24i}) = -\det(A_{34i}).$$

We see that in $\underline{M}$, 234 is spanned by 23, 24 and 34. This is paralleled by $\text{span}(v_2, v_3) = \text{span}(v_2, v_4) = \text{span}(v_3, v_4)$.

Let us now describe a new $\mathbb{R}$-matroid using hyperplane functions. We consider the definition of hyperplane functions again. We need to find four vectors (note we view functions from E to $\mathbb{R}$ as vectors in $\mathbb{R}^{|E|}$), one for each of our hyperplanes, such that their zero elements correspond with the hyperplanes of $\underline{M}$. Our vectors should therefore be of the form

$$\begin{bmatrix}0\\0\\\cdot\\\cdot\end{bmatrix}, \begin{bmatrix}0\\\cdot\\0\\\cdot\end{bmatrix}, \begin{bmatrix}0\\\cdot\\\cdot\\0\end{bmatrix}, \begin{bmatrix}\cdot\\0\\0\\0\end{bmatrix}.$$

Next to this, if we have two vectors which share an element $e \in E$ in the support (this would both have a non-zero entry in the same position, in the vector sense), then we need a Z such that X, Y and Z are linearly dependent. We see that the vectors

$$\omega_1 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}, \omega_2 = \begin{bmatrix}0\\1\\0\\1\end{bmatrix}, \omega_3 = \begin{bmatrix}0\\-1\\1\\0\end{bmatrix}, \omega_4 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix},$$

satisfy this property. Hence the set $\eta = \{a \cdot \omega_i \mid \text{for } a \in \mathbb{R} \text{ and } i \in \{1,2,3,4\}\}$ forms a set of hyperplanes of an $\mathbb{R}$-matroid M with underlying matroid $\underline{M}$. We can deduce that the vector set of M is spanned by only the vector $\begin{bmatrix}0 & 1 & 1 & -1\end{bmatrix}^T$.

We can find the collection of hyperplanes of the $\mathbb{R}$-representation of the flats. By Lemma 5.3, the hyperplane functions of the $\mathbb{R}$-representation of a flat are the hyperplane functions in η for which the vanishing set contains the flats. Hence we have,

$$\eta_{12} = \{a \cdot \omega_1 \mid a \in \mathbb{R}\},\ \eta_{13} = \{a \cdot \omega_2 \mid a \in \mathbb{R}\},$$

$$\eta_{14} = \{a \cdot \omega_3 \mid a \in \mathbb{R}\},\ \eta_{234} = \{a \cdot \omega_4 \mid a \in \mathbb{R}\},$$

$$\eta_1 = \{a \cdot \omega_i \mid \text{for } a \in \mathbb{R} \text{ and } i \in \{1,2,3\}\},\ \eta_2 = \{a \cdot \omega_i \mid \text{for } a \in \mathbb{R} \text{ and } i \in \{1,4\}\},$$

$$\eta_3 = \{a \cdot \omega_i \mid \text{for } a \in \mathbb{R} \text{ and } i \in \{2,4\}\},\ \eta_4 = \{a \cdot \omega_i \mid \text{for } a \in \mathbb{R} \text{ and } i \in \{3,4\}\}.$$

Now that we have the hyperplane functions for each flat, we can find the $\mathbb{R}$-flats. Remembering the definition of an $\mathbb{R}$-flat, we have $\mathcal{V}_F = \eta_F^{\perp}$. We use $\langle v_1, .., v_n \rangle$ to be the span of the vectors $v_1, ..., v_n$. We deduce that

$$\mathcal{V}_1 = \left\langle \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\-1\\-1\\1\end{bmatrix} \right\rangle, \mathcal{V}_2 = \left\langle \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\-1\end{bmatrix} \right\rangle, \mathcal{V}_3 = \left\langle \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\-1\\0\\1\end{bmatrix} \right\rangle, \mathcal{V}_4 = \left\langle \begin{bmatrix}0\\1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix} \right\rangle,$$

$$\mathcal{V}_{12} = \left\langle \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\-1\end{bmatrix} \right\rangle, \mathcal{V}_{13} = \left\langle \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\-1\\0\\1\end{bmatrix} \right\rangle,$$

$$\mathcal{V}_{14} = \left\langle \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix} \right\rangle, \mathcal{V}_{234} = \left\langle \begin{bmatrix}0\\0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix} \right\rangle.$$

Let us put the T-flats into lattice and see its structure, see Figure 6. We notice that it has the same structure as the lattice of flats in Figure 6, and so is an example of Theorem 5.5. Since we are working over a field, we should have that the orthogonal complements of the $\mathbb{R}$-flats form a hyperplane arrangement in the space $\mathcal{V}_M^{\perp}$. We have that

$$\mathcal{V}_M^{\perp} = \left\langle \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\\1\end{bmatrix} \right\rangle$$

$$\mathcal{V}_1^{\perp} = \left\langle \begin{bmatrix}0\\1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\\1\end{bmatrix} \right\rangle, \ \mathcal{V}_2^{\perp} = \left\langle \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\1\end{bmatrix} \right\rangle, \mathcal{V}_3^{\perp} = \left\langle \begin{bmatrix}0\\1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \right\rangle, \ \mathcal{V}_4^{\perp} = \left\langle \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\-1\\0\end{bmatrix} \right\rangle$$

$$\mathcal{V}_{12}^{\perp} = \left\langle \begin{bmatrix}0\\0\\1\\1\end{bmatrix} \right\rangle, \ \mathcal{V}_{13}^{\perp} = \left\langle \begin{bmatrix}0\\1\\0\\1\end{bmatrix} \right\rangle, \mathcal{V}_{14}^{\perp} = \left\langle \begin{bmatrix}0\\-1\\1\\0\end{bmatrix} \right\rangle, \ \mathcal{V}_{234}^{\perp} = \left\langle \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \right\rangle$$

We see that the $\mathcal{V}_i^{\perp}$-s form two dimensional subspaces of the three dimensional space $\mathcal{V}_M^{\perp}$. We also notice that for every i and j we have that $\mathcal{V}_{ij}^{\perp} = \mathcal{V}_i^{\perp} \cap \mathcal{V}_j^{\perp}$ and that $\mathcal{V}_{234}^{\perp} = \mathcal{V}_2^{\perp} \cap \mathcal{V}_3^{\perp} \cap \mathcal{V}_4^{\perp}$. Further, $\mathcal{V}_1^{\perp} \cap \mathcal{V}_2^{\perp} \cap \mathcal{V}_3^{\perp} \cap \mathcal{V}_4^{\perp} = \emptyset$. Hence, we see that the $\mathcal{V}_i^{\perp}$-s form a hyperplane arrangement in $\mathcal{V}_M^{\perp}$. Also notice how the $\mathcal{V}_{12}^{\perp}, \mathcal{V}_{13}^{\perp}, \mathcal{V}_{14}^{\perp}$ and $\mathcal{V}_{234}^{\perp}$, correspond exactly with the hyperplane functions of M.

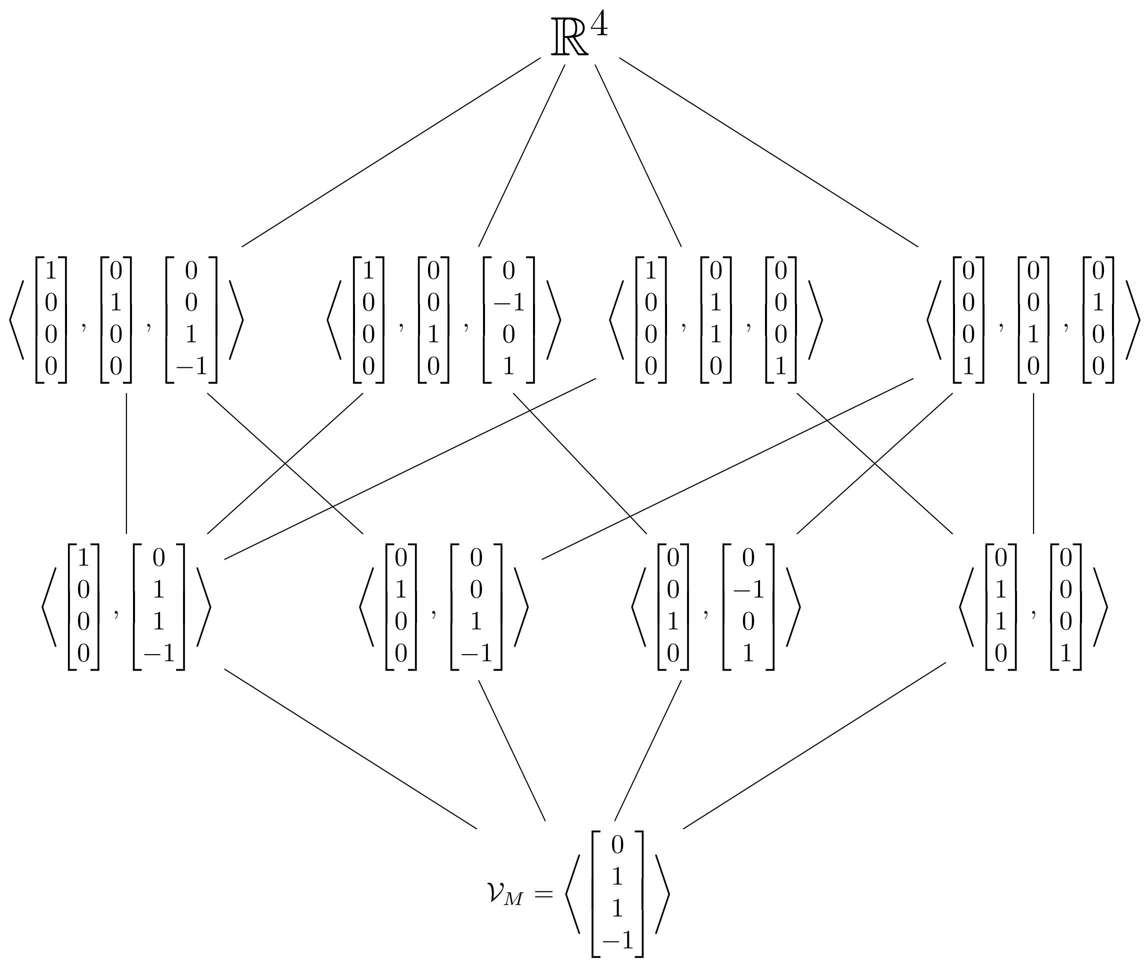

Figure 4: The lattice of T-flats of η. We can clearly see the same lattice structure here as the lattice of flats of the underlying matroid.

7 Conclusion

In conclusion, we develop a theory of flats over matroids with coefficients in a tract. We define T-representations of flats and find their underlying matroid. Then we define T-linear representations and show that these give an equivalent definition for T-matroids. We define T-flats as the vectors of the T-representations and show that they form the same lattice structure with respect to inclusion as the lattice of flats of the underlying matroid. We give an abstract definition of a lattice of T-flats and show this gives an equivalent definition of T-matroids. Finally, we show that for a field K there is a correspondence between lattices of K-flats and hyperplane arrangements. Further research suggestions would be to see when the lattice of T-flats gives a more workable definition of T-matroid. Next to this, it could also be interesting to see what types of structure a lattice of T-flats gives over different tracts.

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