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Towards the Second Homology Theorem for Matroids

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Contents

Abstract

In the first part of this thesis we present the proofs of the path theorem and Tutte homotopy theorem concerning sequences of hyperplanes in a matroid. In the second part, we show the reformulations of the theorems in terms of homology groups of order complexes, following [2]. We find a replacement for the sublattice of the fourth class for the Tutte homotopy theorem and give a reformulation of the theorem. Using a computer program that we create, we find some necessary sublattices for the hypothetical second homology theorem to hold. We discuss how a second homology theorem might look like and we provide some tools that could be used in its proof.

1 Introduction

A matroid is a collection of subsets of a finite set E satisfying certain properties. It abstracts the concept of linear independence in linear algebra, the notion of cycles in graph theory and their combinatorial properties. The matroids that are obtained from a vector space over a field k are called representable over k. Over time questions arose about which matroids are representable over a particular field, or some come collection of fields. What is sought for in these kinds of representability questions is a finite list of minimal matroids which are not representable, called the excluded minors. The first question of this type was answered by Tutte who proved that a matroid is representable over $\mathbb{F}_2$ precisely when $U_{2,4}$, some matroid on the set of four elements, is not contained in it [12], so the only excluded minor is $U_{2,4}$. In the same paper he also proved under which conditions a matroid is regular, meaning it is representable over all fields. The excluded minors in this case are $U_{2,4}$, the Fano matroid and the dual of the Fano matroid.

One of the main technical tools Tutte used in his proof of the characterization of regular matroids is the homotopy theorem which is a statement about paths in a matroid [11]. The homotopy theorem is important. It is the key ingredient of the first published proof of the characterization of matroids representable over $\mathbb{F}_3$ by Bixby [5]. The homotopy theorem has been used to prove that a set of relations between generators for a certain group associated to matroids is complete [7]. Recently analogous statements have been proved for a more complicated algebraic invariant [1].

Although the proof of homotopy theorem relies on completely elementary techniques, its importance cannot be understated. A natural reformulation of the homotopy theorem and a related path theorem in terms of the 1st and 0th homology group of certain topological spaces associated to a matroid are given in [2]. This formulation offers a generalization to a hypothetical second homology theorem, concerning the 2nd homology group. For such a theorem we first need to find a finite class of matroids that build the topological space and second, we need to prove it.

The reason why the conjectural second homology theorem could be useful, is that it might provide another proof for the excluded minor characterization for the class of near-regular matroids. A matroid is called near-regular if it is representable over all fields except possibly $\mathbb{F}_2$. The list of excluded minors for the class of near-regular matroids is already known, there are ten of them [8]. Oliver Lorscheid believes that this list of excluded minors is related to the second homology theorem and formulated several concrete conjectures which were expressed in personal communication. The precise connection between the hypothetical second homology theorem and near-regular matroids is difficult to explain and is not part of our work.

The starting point of our project was understanding the conjectures by Lorscheid and developing a computer program with which we tested their validity. An example of such a conjecture is that all of the excluded minors for near-regular matroids appear as exceptional matroids for which the second homology group is non-trivial. With explicit counter-examples we disproved all variations of the conjectures. At the time of writing it is not clear how to fix or formulate a conjecture by Lorscheid that would correctly explain the relation between near-regular matroids and the second homology theorem.

The first part of the thesis is devoted to presenting the proofs of the path theorem and the homotopy theorem by Tutte in their modern formulations. In the second part, we test the ground for the second homology theorem, using the definitions developed in [2]. Our novel work is writing the computer program in SageMath with which we test for the small matroids, that need to be considered when building the topological space and we present several of such classes. In the process we discover a new formulation of the homotopy theorem by replacing one of the matroids needed for its formulation by a simpler one. We also find redundant matroids that are not needed for the homotopy theorem, but are found by the computer program. We find a counterexample to a conjecture by Lorscheid concerning the near-regular matroids and present it. Finally, we present some tools from [6] that might be useful for the proof of the second homology theorem.

Part I

Path Theorem and Homotopy Theorem

2 Background

There are many equivalent definitions of matroids, for instance, in terms of independent sets, circuits, bases, rank function or closure operator. We present the definition in terms of flats. For the definitions and basic lemmas in section 2 we follow the first four chapters of [10], except if stated otherwise. The way our exposition differs to [10] is that we take the theorems about flats as our definitions. This can be done, because it is shown in [10] how one can pass from a matroid defined in one of the above ways to another.

Definition 2.1. Let E be a finite set, a matroid is a pair $M = (E, \mathcal{F})$ where $\mathcal{F}$ is a collection of subsets of E called flats satisfying the following properties:

(F1) We have $E \in \mathcal{F}$.

(F2) For $F_1, F_2 \in \mathcal{F}$ it holds that $F_1 \cap F_2 \in \mathcal{F}$.

(F3) If $F \in \mathcal{F}$ and $F_1, \ldots, F_k$ are minimal pairwise distinct members of $\mathcal{F}$ properly containing F then $F_1 - F, \ldots, F_k - F$ partition $E - F$.

For a matroid $M = (E, \mathcal{F})$ the set E is called the ground set of M, the collection of flats of a matroid M is also denoted by $\mathcal{F}(M)$.

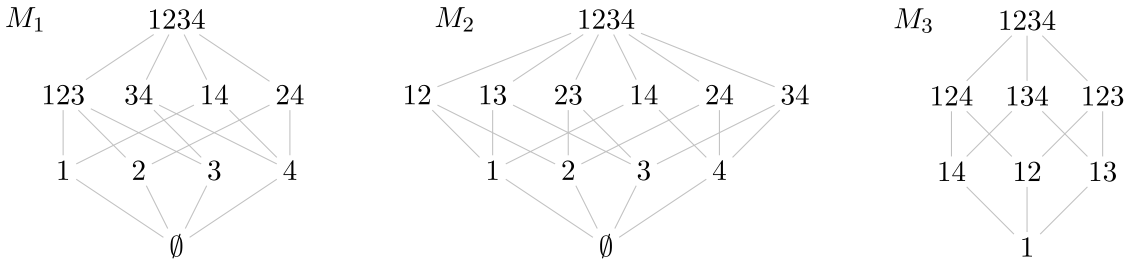


Figure 1: Matroids M_1, M_2 and M_3 on the ground set $E = \{1, 2, 3, 4\}$.

In Figure 1 there are matroids M_1, M_2 and M_3 with the corresponding flats. In the diagrams of flats we put gray lines between flats F_1 and F_2 when $F_1 \subsetneq F_2$ and there is no flat F_3 with $F_1 \subsetneq F_3 \subsetneq F_2$. Instead of the set $\{1, 2, 3\}$ etc. we write 123 for simplicity. All of the examples of matroids in the text have the ground set of cardinality at most 8, so there is no ambiguity.

Example 2.1. Let V be a vector space over a field k and $W = (v_1, \ldots, v_n)$ a finite tuple of vectors in V. Let $E = \{1, \ldots, n\}$ be the set of indices of vectors of W, in particular, the same vector might appear multiple times in W. We define a collection $\mathcal{F} \subset 2^E$ by $F \in \mathcal{F}$ if for all $i \in E - F$ we have $v_i \notin \operatorname{span}(F)$, where span of F means the vector span of all vectors with indices in F. For the collection $\mathcal{F}$ conditions (F1) and (F2) are clear.

To verify condition (F3) let $F \in \mathcal{F}$ with the minimal pairwise distinct members of $\mathcal{F}$ properly containing F denoted by $F_1, \ldots, F_k$. First, if $F_i \cap F_j \supsetneq F$ for some $1 \leq i < j \leq k$ then $F_i \cap F_j$ is an element of $\mathcal{F}$ properly containing F which goes against the definition of the set F_i.

Suppose for contradiction that there exists $j \in E - \cup_{i=1}^{k} F_i$. Let

$$\mathrm{cl}(F \cup j) = \{k \in E \ : \ v_k \in \mathrm{span}(F \cup j)\} ,$$

we have $\mathrm{cl}(F \cup j) \in \mathcal{F}$ and $F \subset \mathrm{cl}(F \cup j)$. By definition there exits i with $F_i \subset \mathrm{cl}(F \cup j)$. Let $l \in F_i - F$ then $l \in \mathrm{span}(F \cup j)$ but $l \notin \mathrm{span}(F)$. In particular, this implies that $j \in \mathrm{span}(F \cup l)$. But because $j \notin F_i$ we have $j \notin \mathrm{span}(F_i)$ which contains $\mathrm{span}(F \cup l)$, a contradiction.

Example 2.2. Let $G = (V, E)$ be a finite graph. If $I \subset E$ we define a subgraph G_I generated by I as $G_I = (V_I, I)$ where V_I is the set of endpoints of elements of I. We define a collection $\mathcal{F} \subset 2^E$ by $F \in \mathcal{F}$ if for all $f \in E - F$ the graph $G_{F \cup f}$ does not contain a cycle[1] containing f. For the collection $\mathcal{F}$ conditions (F1) and (F2) are clear.

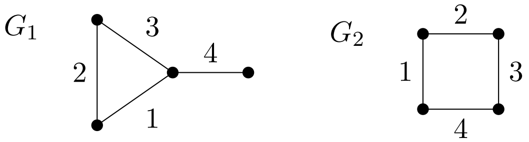
Let $I \subset E$, we claim that the set

$$\mathrm{cl}(I) = I \cup \{e \in E - I \ : \ G_{I \cup e} \text{ contains a cycle containing } e\} ,$$

is in $\mathcal{F}$. This is a consequence of the fact that if C_1 and C_2 are two cycles which share an edge e and $f \in C_1 - C_2$ then there exists a cycle C_3 with $C_3 \subset (C_1 \cup C_2) - e$, such that $f \in C_3$. Let $f \in E - \mathrm{cl}(I)$ and assume for contradiction $G_{\mathrm{cl}(I) \cup f}$ contains a cycle C_1 containing f. Because $f \notin \mathrm{cl}(I)$ we have that $G_{I \cup f}$ does not contain a cycle containing f, thus C_1 has to contain an element of $\mathrm{cl}(I) - I$. Pick among all cycles C_1 satisfying above properties the one with minimal number of members of $\mathrm{cl}(I) - I$. Let $g \in C_1 \cap (\mathrm{cl}(I) - I)$, by definition $g \in \mathrm{cl}(I)$ implying $G_{I \cup g}$ contains a cycle C_2 containing g. Notice that g is the only edge in C_2 not in I hence $f \in C_1 - C_2$. By the properties of cycles mentioned before, there exists cycle C_3 containing f such that $C_3 \subset (C_1 \cup C_2) - g$. Number of elements of C_3 which are in $\mathrm{cl}(I) - I$ is less than that of C_1 and $f \in C_3$ contradicting the choice of C_1.

To verify condition (F3) for $\mathcal{F}$ let $F \in \mathcal{F}$ and $F_1, \dots, F_k$ be the minimal members of $\mathcal{F}$ properly containing F. First, if $F_i \cap F_j \supsetneq F$ for some $1 \leq i < j \leq k$ we have that $F_i \cap F_j$ is an element of $\mathcal{F}$ properly containing F which goes against the definition of the set F_i. Suppose for contradiction that there exists $e \in E - \cup_{i=1}^{k} F_i$. We know that $\mathrm{cl}(F \cup e) \in \mathcal{F}$, by definition there exists i with $F_i \subset \mathrm{cl}(F \cup e)$. Let $f \in F_i - F$, by definition of $\mathrm{cl}(F \cup e)$ we have that $G_{F \cup e \cup f}$ contains a cycle containing f. But then $G_{F \cup f \cup e}$ contains a cycle containing e hence, by definition $e \in \mathrm{cl}(F \cup f) \subset F_i$, which is a contradiction.

We denote the matroid obtained from the graph G as described in Example 2.2 by $M(G)$.



$$A_3 = \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Figure 2: Graphs G_1 and G_2 with $M(G_i) = M_i$ and matrix A_3 whose tuple of column vectors forms the matroid M_3.

In Figure 2 there are graphs G_i with the labeling of edges such that $M(G_i) = M_i$ as described in Example 2.2 where M_i for $i = 1, 2$ are the matroids from Figure 1. The set of columns of matrix A_3 forms a matroid isomorphic to M_3 in Figure 1 as described in Example 2.1.

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[1] By a cycle C in a graph G we mean the set of edges of a non-empty closed trail in which only the first and the last vertex are the same.

Example 2.3. Let $E = \{1, \dots, n\}$ and $\mathcal{F} = \{F : \; F \subset E \text{ and } |F| \leq k-1\} \cup \{E\}$. Then $M = (E, \mathcal{F})$ is called the uniform matroid of rank r on n elements and is denoted by $U_{k,n}$.

Definition 2.2. A matroid $M = (E, \mathcal{F})$ is called simple if all subsets of E of cardinality at most 1 are flats. Matroids $(E_1, \mathcal{F}_1)$ and $(E_2, \mathcal{F}_2)$ are isomorphic if there exists a bijection $f : E_1 \to E_2$ such that for every $F \in \mathcal{F}_1$ it holds that $f(F) \in \mathcal{F}_2$ and for every $G \in \mathcal{F}_2$ we have $f^{-1}(G) \in \mathcal{F}_1$.

A matroid M is called representable over a field k if it is isomorphic to a matroid N which is defined in the same way as in Example 2.1.

In Figure 1 the matroids M_1 and M_2 are simple, while M_3 is not. The maximal flats which are not equal to the ground set are called hyperplanes.

Definition 2.3. Let $M = (M, \mathcal{F})$ be a matroid on the ground set E. A hyperplane H is a flat such that the only flat properly containing H is E. The collection of hyperplanes of a matroid M is denoted by $\mathcal{H}(M)$.

Definition 2.4. Let $M = (E, \mathcal{F})$ be a matroid, we define a function $\mathrm{cl} : 2^E \to \mathcal{F}$, called the closure operator, by

$$\mathrm{cl}(A) = \bigcap_{\substack{F \in \mathcal{F} \\ A \subset F}} F.$$

If F_1 and F_2 are flats of a matroid M with closure operator cl we write $F_1 \vee F_2 = \mathrm{cl}(F_1 \cup F_2)$. If $e \in E$ we write $F_1 \vee a = \mathrm{cl}(F_1 \cup \{a\})$.

Definition 2.5. Let $M = (E, \mathcal{F})$ be a matroid. The function $\mathrm{rk} : 2^E \to \mathbb{N}_0$ defined by

$$\mathrm{rk}(A) = \max\left\{k : \text{ there exists a chain } F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_k = \mathrm{cl}(A), \; F_i \in \mathcal{F}\right\},$$

is called the rank function of the matroid M. We define the corank function $\mathrm{crk} : 2^E \to \mathbb{N}_0$ by $\mathrm{crk}(A) = \mathrm{rk}(E) - \mathrm{rk}(A)$.

Lemma 2.1. The rank function rk of a matroid $M = (E, \mathcal{F})$ satisfies the following properties:

(R1) For $X \subset E$ we have $0 \leq \mathrm{rk}(X) \leq |X|$.

(R2) For all $X \subset Y \subset E$ it holds that $\mathrm{rk}(X) \leq \mathrm{rk}(Y)$.

(R3) For all $X, Y \subset E$ we have $\mathrm{rk}(X \cup Y) + \mathrm{rk}(X \cap Y) \leq \mathrm{rk}(X) + \mathrm{rk}(Y)$.

Property (R3) is called the submodular inequality. The set of hyperplanes of a matroid determines all of its flats.

Lemma 2.2. Let $M = (E, \mathcal{F})$ be a matroid. If $F \in \mathcal{F}$ with $\mathrm{crk}(F) = k$ where $k \geq 1$ then there exist hyperplanes $H_1, \dots, H_k$ such that $F = \cap_{i=1}^{k} H_i$.

For a matroid M, the collection of flats $\mathcal{F}$ is also called the lattice of flats due to its connection to geometric lattices which is explained in Appendix A.1. In that case, the collection of flats is usually denoted differently by Λ and not $\mathcal{F}$ to emphasize the partial order structure on the set $\mathcal{F}$ given by inclusion. For a matroid M we denote its lattice of flats by $\Lambda(M)$. In the following text we interchangeably use the language from matroid theory and terminology describing lattices. For instance, if we say a flat F is below or lies above a flat G we mean that $F \subset G$ or $G \subset F$ respectively.

Definition 2.6. ([2]) Let Λ be a geometric lattice with the top element E. We call $\Lambda' \subset \Lambda$ an upper sublattice if it is a geometric lattice with $E \in \Lambda'$ and for the bottom element $F \in \Lambda'$ we have $\operatorname{crk}_{\Lambda'}(F) = \operatorname{crk}_{\Lambda}(F)$.

Using the properties of geometric lattices, one can prove that upper sublattices of Λ are precisely the lattices of the form $\Lambda' = [F, E]_S$ (notation explained in Appendix A.1) where S is a set of elements covering F in Λ such that $\vee_{F \in S} F = E$.

Definition 2.7. Let Λ_1 and Λ_2 be geometric lattices. We call a map $f : \Lambda_1 \to \Lambda_2$ an embedding of upper sublattices if $f(\Lambda_1)$ is an upper sublattice of Λ_2 and $f : \Lambda_1 \to f(\Lambda_1)$ is a lattice isomorphism.

2.1 Minors

Deletion and contraction are ways of obtaining new matroids by restricting our attention to flats around a subset of the ground set. A matroid N that can be obtained from a given matroid M by a sequence of contractions and deletions is called a minor and is the substructure we are interested in.

Definition 2.8. Let $M = (E, \mathcal{F})$ be a matroid and $X \subset E$. We define a matroid $M \backslash X = (E - X, \mathcal{F} \backslash X)$, called the deletion of X from M, by

$$\mathcal{F} \backslash X = \{F \cap (E - X) \ : \ F \in \mathcal{F}\} .$$

An alternative notation for the matroid defined in 2.8 is $M|(E - X) = M \backslash X$. In that case the matroid $M|(E - X)$ is called the restriction of M to $E - X$.

A direct consequence of Definition 2.8 is the following lemma.

Lemma 2.3. Let $M = (E, \mathcal{F})$ be a matroid and $X \subset E$, then $\mathcal{H}(M \backslash X)$ is the set of maximal proper subsets of $E - X$ of the form $H - X$ where $H \in \mathcal{H}(M)$.

Definition 2.9. Let $M = (E, \mathcal{F})$ be a matroid and $X \subset E$. A matroid $M/X = (E - X, \mathcal{F}/X)$ is called the contraction of X from M, where

$$\mathcal{F}/X = \{F \ : F \subset E - X \text{ and } F \cup X \in \mathcal{F}\} .$$

If $e \in E$ we write M/e and $M \backslash e$ for $M/\{e\}$ and $M \backslash \{e\}$ respectively.

Definition 2.10. A matroid N is called a minor of a matroid M if it can be obtained from M by a finite sequence of deletions and contractions.

Definition 2.11. Let $M = (E, \mathcal{F})$ be a matroid and N its minor. An embedded minor is a minor of the form $N = M \backslash J / I$ with a fixed choice of the sets I and J where $I, J \subset E$ are disjoint, $\operatorname{rk}(I) = |I|$ and $\operatorname{cl}(E - J) = E$.[2]

Given a matroid $M = (E, \mathcal{F})$ we can describe any minor N as a sequence of only one contraction and one deletion.

Theorem 2.1 ([10], Lemma 3.3.2)**.** Let $M = (E, \mathcal{F})$ be a matroid, any minor N of M is isomorphic to an embedded minor.

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[2] If the reader knows the following terms, this means that I is independent and J is coindependent.

Given a matroid M, every embedded minor of M gives rise to an upper sublattice of $\Lambda(M)$ and every upper sublattice of $\Lambda(M)$ can be obtained in such a way. Let $\mathrm{USL}_{\Lambda(M)}$ be the set of upper sublattices of $\Lambda(M)$ and EMB_M the the set of embedded minors of M.

Theorem 2.2. ([3], Proposition 5.7) Let M be a matroid, the map $\Psi : \mathrm{EMB}_M \to \mathrm{USL}_{\Lambda(M)}$ sending $M \backslash J/I$ to $[\mathrm{cl}(I), E]_S$ where $S = \{\mathrm{cl}(I \cup a) : a \notin I \cup J\}$ is surjective. It holds that $\Lambda(M\backslash J/I)$ is isomorphic to $[\mathrm{cl}(I), E]_S$.

To denote the upper sublattices we use the notation $M\backslash J/I$ for the corresponding embedded minors. In examples, we use short-hand versions of this notation. For instance, we often omit the matroid M and the symbol $\backslash$ to write J/I, or if we take as the set S in Theorem 2.2 consisting of all atoms above a given flat F, we write $/F$, omitting J. As an example of the notation look at Figure 3. The upper sublattice in the matroid M_1 above 1 with solid black lines is denoted by $/1$ while the upper sublattice with dashed lines above 4 is denoted by $1/4$. If the upper sublattice contains the bottom flat $\emptyset$ we do not write $J/\emptyset$ but just J. For instance, the upper sublattice in matroid M_2 is denoted by 1246.

Therefore, any upper sublattice Λ' of a geometric Λ is obtained by first picking a flat $F \in \Lambda$ (using notation in Theorem 2.2, we have $F = \mathrm{cl}(I)$) that serves as the bottom element of Λ'. Next we pick a set J such that we we take all flats G covering F in Λ with $G \cap J = \emptyset$ as the atoms of Λ'.

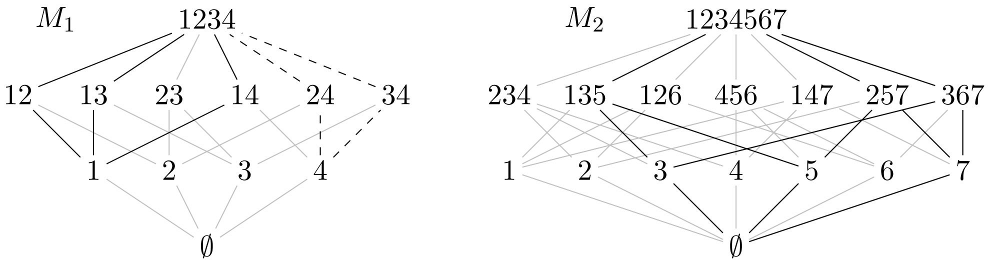


Figure 3: Illustration of the notation for upper sublattices.

2.2 Connectedness

A matroid is connected if it cannot be written as a sum of smaller matroids in a certain way.

Definition 2.12. Let $M_1 = (E_1, \mathcal{F}_1)$ and $M_2 = (E_2, \mathcal{F}_2)$ be matroids for which $E_1 \cap E_2 = \emptyset$ and let $E = E_1 \cup E_2$. Define

$$\mathcal{F} = \{F_1 \cup F_2 \ : \ F_1 \in \mathcal{F}_1 \text{ and } F_2 \in \mathcal{F}_2\} ,$$

then $M_1 \oplus M_2 = (E, \mathcal{F})$ is a matroid called the direct sum of M_1 and M_2.

An immediate consequence of Definition 2.12 is the following lemma concerning the structure of hyperplanes of the direct sum.

Lemma 2.4. Let $M_1 = (E_1, \mathcal{F}_1)$ and $M_2 = (E_2, \mathcal{F}_2)$ where $E_1 \cap E_2 = \emptyset$. We have

$$\mathcal{H}(M_1 \oplus M_2) = \{H_1 \cup E_2 \ : \ H_1 \in \mathcal{H}(M_1)\} \cup \{E_1 \cup H_2 \ : \ H_2 \in \mathcal{H}(M_2)\} .$$

Definition 2.13. A matroid M with ground set E is called disconnected if there exists a subset $T \subset E$ with $\emptyset \subsetneq T \subsetneq E$ such that

$$M = (M \backslash T) \oplus (M \backslash (E - T)).$$

A matroid is called connected if it is not disconnected. In Figure 1, the matroid M_1 is disconnected where $T = \{4\}$ with notation as in Definition 2.13, the matroid M_3 is disconnected with $T = \{1\}$ while M_2 is connected.

3 Tutte's path and homotopy theorems

We present two theorems concerning some special sequences of hyperplanes in a matroid. They are the path theorem and Tutte's homotopy theorem which were first stated and proved by Tutte [11]. The proof of the first theorem is short while the proof of the second is long and technical. We follow Tutte's original proof in content, except that our objects are dual in a certain sense. The reason why Tutte developed the path and homotopy theorem is because he wanted to provide a characterization of binary and regular matroids in terms of excluded minors [12]. A matroid is binary if it is representable over $\mathbb{F}_2$, for instance, it is not hard to see that the matroid $U_{2,4}$ is not binary. A matroid is regular if it is representable over all fields. The lemmas and the ideas of the proofs from section 3 are from [2], except if stated otherwise.

Definition 3.1. ([10, p. 266]) Let M be a matroid, a subset $\Gamma \subset \Lambda(M)$ is called a modular cut if:

1. For all $F \in \Gamma$ and $G \in \Lambda(M)$ such that $G \supset F$ we have $G \in \Gamma$.
2. For all $F_1, F_2 \in \Gamma$ such that $\mathrm{rk}(F_1)+\mathrm{rk}(F_2) = \mathrm{rk}(F_1 \cap F_2)+\mathrm{rk}(F_1 \vee F_2)$ we have $F_1 \cap F_2 \in \Gamma$.

Let M be a matroid on the ground set E. A matroid N on the ground set $E \cup e$ where $e \notin E$ is a one-element extension of a matroid M by e if $N \backslash e = M$. We are interested in modular cuts because they characterize the one-element extensions of a given matroid.

Theorem 3.1. ([10], Lemma 7.2.2 and Theorem 7.2.3) Let M be a matroid on the ground set E and $e \notin E$. For a one-element extension N of M by e we get a subset $\Gamma_N \subset \Lambda(M)$ by $\Gamma_N = \{F \in \Gamma : \mathrm{cl}_N(F) = F \cup e\}$.

The assignment $N \to \Gamma_N$ provides a bijection between the set of one-element extensions of M by e and the set of modular cuts of M.

The modular cut of a matroid is completely determined by its set of hyperplanes. We call such a subset of hyperplanes a linear subclass. The intersection of two modular cuts is a modular cut, so we can talk about the modular cut generated by a subset of the lattice of flats. This fact is important for the computer program that we implement for the second homology theorem.

Definition 3.2. ([10, p. 271]) Let M be a matroid with the set of hyperplanes $\mathcal{H}$. A subset $\mathcal{L} \subset \mathcal{H}$ is called a linear subclass if for all $H_1, H_2 \in \mathcal{L}$ such that $\mathrm{rk}(H_1 \cap H_2) = \mathrm{rk}(H_1) - 1$ we have that any hyperplane containing $H_1 \cap H_2$ is in $\mathcal{L}$.

Lemma 3.1. ([10, p. 271]) Let M be a matroid with lattice of flats Λ and collection of hyperplanes $\mathcal{H}$. A subset $\mathcal{L} \subset \mathcal{H}$ is a linear subclass if and only if the modular cut generated by $\mathcal{L}$ has precisely $\mathcal{L}$ as its set of hyperplanes.

3.1 Lemmas for the path and homotopy theorems

We need a lot of technical statements about a special type of flats in a matroid that we call indecomposable.

Definition 3.3. Let M be a matroid, a flat $F \in \Lambda(M)$ is called indecomposable if the contraction M/F is connected. If F is not indecomposable it is called decomposable.

Lemma 3.2 (Tutte's definition of connectivity)**.** Let M be a matroid on the ground set E. A flat $F \in \Lambda(M)$ is decomposable if and only if there exist subsets $X_1, X_2 \subset E$ such that $X_1 \cap X_2 = F$, $X_1 \cup X_2 = E$, $X_i \neq E$ for both $i = 1, 2$ and for all hyperplanes $H \supset F$ we have $H \supset X_1$ or $H \supset X_2$. We call $\{X_1, X_2\}$ a separation of F.

Proof. The idea of the forward direction of the proof is taken from [2]. First assume that $M/F = M_1 \oplus M_2$ is disconnected where $M_i = (E_i, \mathcal{F}_i)$, with the following properties $E_i \subset E - F$, $E_1 \cup E_2 = E - F$, $E_1 \cap E_2 = \emptyset$ and $|E_i| > 0$. We claim that $X_i = E_i \cup F$ satisfies the conditions of the sets in the statement of the lemma. Let $H \in \mathcal{H}(M)$ be a hyperplane with $H \supset F$, by Definition 2.9 we know that $H - F$ is a flat of M/F. The flat $H - F$ is also a hyperplane of M/F, because if X is a flat of M/F properly containing $H - F$ then $X \cup F$ is a flat of M properly containing H hence $X \cup F = E$, showing $X - F = E - F$. Since $H - F \in \mathcal{H}(M_1 \oplus M_2) = \mathcal{H}(M/F)$ we know by Lemma 2.4 that $H - F = H_1 \cup E_2$ or $H - F = E_1 \cup H_2$ for $H_i \in \mathcal{H}(M_i)$. If the first case we have that $H = H_1 \cup E_2 \cup F$ hence $H \supset X_2$, similarly $H \supset X_1$ in the second case and the claim follows.

For the reverse direction, assume that the sets X_i for $i = 1, 2$ exist with the properties listed in the statement of the lemma, so $X_1 \cap X_2 = F$, $X_1 \cup X_2 = E$, $X_i \neq E$ for both $i = 1, 2$ and for all hyperplanes $H \supset F$ we have $H \supset X_1$ or $H \supset X_2$. Define $E_i = X_i - F$ and write $N = M/F$. We claim that $N = (N|E_1) \oplus (N|E_2)$. We show that $\mathcal{H}(N) = \mathcal{H}((N|E_1) \oplus (N|E_2))$, which is enough by Lemma 2.2.

The reader may benefit from drawing a Venn diagram for understanding the rest of the proof. First let $H \in \mathcal{H}(N)$. This occurs precisely when $H \cup F \in \mathcal{H}(M)$. By assumption, we have either $H \cup F \supset X_1$ or $H \cup F \supset X_2$. Without loss of generality assume $H \cup F \supset X_1$, hence $(H \cup F) - F = H - F \supset X_1 - F = E_1$. Hence H is of the form $H = H_2 \cup E_1$ for a set H_2, our goal is to show that $H_2 \in \mathcal{H}(N|E_2)$. By Lemma 2.3 we know that the latter happens precisely when $H_2 - E_1$ is a maximal proper subset of $(E - F) - E_1$ of the form $H' - E_1$ where H' is a hyperplane of N. Let for contradiction H'_2 be a hyperplane of N such that

$$H_2 - E_1 \subsetneq H'_2 - E_1 \subsetneq (E - F) - E_1.$$

We then have that $H'_2 \cup F$ is a hyperplane of M hence $H'_2 \cup F \supset X_1$ or $H'_2 \cup F \supset X_2$. We can exclude the latter option because $H'_2 - E_1 \subsetneq (E - F) - E_1 = E_2$. Hence $H'_2 \cup F \supset X_1$. But then we have a strict inclusion $H_2 \cup F \subsetneq H'_2 \cup F$ of hyperplanes of M which is a contradiction.

Therefore $H_2 \in \mathcal{H}(N|E_2)$ showing that $H = H_2 \cup E_1 \in \mathcal{H}((N|E_1) \oplus (N|E_2))$ what we wanted.

Second assume $H \in \mathcal{H}((N|E_1) \oplus (N|E_2))$. Without loss of generality assume $H = H_1 \cup E_2$ for $H_1 \in \mathcal{H}(N|E_1)$. This means that $H_1 = H' - (E - F - E_1) = H' - E_2$ for $H' \in \mathcal{H}(N)$. Hence $H = H_1 \cup E_2 = (H' - E_2) \cup E_2 = H'$ where the latter set is a hyperplane of N. This is what we wanted to show. □

Lemma 3.2 shows that for a matroid on the ground set E, the top flat E and the hyperplanes are indecomposable flats.

Lemma 3.3. Let M be a matroid on ground set E with F_1 and F_2 indecomposable flats such that $F_1 \cup F_2 \neq E$. Then $F_1 \cap F_2$ is indecomposable.

Proof. If $F_1 \supset F_2$ or vice versa the statement is clear. Hence assume both $F_1 - F_2$ and $F_2 - F_1$ are non-empty. Assume for contradiction that $F_1 \cap F_2$ is a decomposable flat and let $\{X_1, X_2\}$ be the separation of $F_1 \cap F_2$. Notice that $\{X_1 \cup F_1, X_2 \cup F_1\}$ and $\{X_1 \cup F_2, X_2 \cup F_2\}$ satisfy all conditions of separations of F_1 and F_2 respectively unless one of the four sets $X_i \cup F_j$ for $i, j = 1, 2$ equals E. Assume without loss of generality that $X_1 \cup F_1 = E$. First assume for contradiction that $X_1 \cup F_2 = E$ as well. We have $X_1^c \cap F_1^c = X_1^c \cap F_2^c = \emptyset$ therefore $X_1^c \cap (F_1^c \cup F_2^c) = X_1^c \cap (F_1 \cap F_2)^c = \emptyset$. Hence $X_1 \supset (F_1 \cap F_2)^c$. But we also have $X_1 \supset F_1 \cap F_2$ hence $X_1 \supset E$ which is a contradiction, because $X_1 \subsetneq E$. Therefore $X_1 \cup F_2 \neq E$.

From $X_1 \cup F_1 = E$ we also get that $X_1 \cup (F_1 - F_2) = E$ and $X_1^c \cap (F_1 - F_2)^c = \emptyset$. Therefore $X_1^c \subset F_1 - F_2$. By definition of separation we have $X_1^c \supset (X_2 - (X_1 \cap X_2))$. Therefore we get $X_2 \cup F_2 \subset F_2 \cup (F_1 - F_2) = F_1 \neq E$.

To conclude, $\{X_1 \cup F_2, X_2 \cup F_2\}$ is thus a separation of F_2 which contradicts the assumption that F_2 is indecomposable. Thus $F_1 \cap F_2$ is indecomposable. □

A direct consequence of the above lemma is the characterization of indecomposable corank 2 flats which is used often in the text.

Lemma 3.4. Let M be a matroid. A corank 2 flat F is decomposable if and only if there are precisely two distinct hyperplanes H_1 and H_2 containing it.

Proof. If $H_1, \ldots, H_k$ are the hyperplanes containing F then $H_i - F$ partition $E - F$ by (F3). Thus, if $k > 2$ we can pick hyperplanes H_1 and H_2 which are indecomposable flats and $H_1 \cup H_2 \neq E$, implying $F = H_1 \cap H_2$ is indecomposable by Lemma 3.3. If $k = 2$, the set $\{H_1, H_2\}$ forms a separation of F. □

Lemma 3.5 (Chains of indecomposable flats)**.** Let M be a matroid with indecomposable flats F_1 and F_2 where $F_1 \supset F_2$ and $\text{rk}(F_1) > \text{rk}(F_2)$. There exists an indecomposable flat F_3 such that $F_1 \supset F_3 \supset F_2$ and $\text{rk}(F_3) = \text{rk}(F_1) - 1$.

Proof. The idea of the proof is taken from [11].

There exists a hyperplane $H \supset F_2$ such that $H \cap (F_1 - F_2)$ is nonempty. If not we have that M/F_2 is disconnected, using the notation of Lemma 3.2 there are sets $X_1 = (E - F_1) \cup F_2$ and $X_2 = F_1$, which is a contradiction.

Pick a hyperplane H such that $|H \cap F_1|$ is maximal. We claim that $F_3 = H \cap F_1$ is the desired indecomposable flat. We need to show it is indecomposable and that it has the correct rank.

If F_3 is decomposable with separation $\{X_1, X_2\}$ then the sets $X_1' = X_1 \cup (F_1 \cap X_2)$ and $X_2' = X_2 \cup (F_1 \cap X_1)$ contradict the assumption that F_1 is indecomposable.

Because F_3 is a proper subset of F_1 we know $\text{rk}(F_3) < \text{rk}(F_1)$. Assume for contradiction that $\text{rk}(F_3) < \text{rk}(F_1) - 1$. Pick $a \in F_3 - F_1$. Then

$$\text{rk}(F_3) < \text{rk}(F_3 \vee a) \leq \text{rk}(F_3) + 1 < \text{rk}(F_1).$$

If H' is any hyperplane with $H' \supset F_3 \vee a$ we have $|H' \cap F_1| > |H \cap F_1|$ hence, by maximality of $|H \cap F_1|$ we have that $H' \supset F_1$. However, this implies that the flat $F_3 \vee a$, which contains the flat F_1 as a proper subset, cannot be written as an intersection of hyperplanes, because any hyperplane containing it also contains F_1. This contradicts Lemma 2.2. □

Lemma 3.6. Let M be a matroid on ground set E with flats $F_1 \supset F_2$. There exists a flat $U \supset F_2$ such that $U \cap F_1 = F_2$, $U \vee F_1 = E$ and $\operatorname{crk}(U) = \operatorname{rk}(F_1) - \operatorname{rk}(F_2)$.

Proof. The idea of the proof is taken from [2]. We prove the statement by induction on $\operatorname{crk}(F_2)$. If $\operatorname{crk}(F_2) = 0$ then $F_1 = F_2 = U$ and the statement holds. Suppose the statement holds for some $n \geq 0$ and let $\operatorname{crk}(F_2) = n+1$. If $F_1 = E$ the take $U = E$. If not, take $a \in E - F_1$, because $F_1 \subset F_2$ we have $\operatorname{rk}(F_i \vee a) = \operatorname{rk}(F_i) + 1$, or $\operatorname{crk}(F_i \vee a) = n$.

Therefore, by the inductive hypothesis there exists a flat U such that $U \vee (F_2 \vee a) = E$, $U \supset F_1 \vee a \supset F_1$ and $\operatorname{crk}(U) = \operatorname{rk}(F_2 \vee a) - \operatorname{rk}(F_1 \vee a) = \operatorname{rk}(F_2) - \operatorname{rk}(F_1)$, thus only thing we need to check for U to be the suitable flat is whether $U \vee F_2 = E$. This holds because $a \in U$ therefore $U \vee F_2 = U \vee (F_2 \vee a)$. □

Lemma 3.7 (Indecomposable diamond)**.** Let M be a matroid and $F_1 \supset F_2$ indecomposable flats such that $\operatorname{rk}(F_1) = \operatorname{rk}(F_2) + 2$. Then there exist indecomposable flats U and V with $\operatorname{rk}(U) = \operatorname{rk}(V) = \operatorname{rk}(F_2) + 1$, such that $U \cap V = F_2$ and $U \vee V = F_1$.

Proof. We follow the proof in [11]. Let $\operatorname{rk}(F_2) = n$, we know by Lemma 3.5 that there is an indecomposable flat U with $F_1 \supset U \supset F_2$ and $\operatorname{rk}(U) = n+1$ hence we have to find one more such flat other than U. Let $a \in F_1 - U$ and $W = F_2 \vee a$. We have $\operatorname{rk}(W) = n + 1$ and $F_1 \supset W \supset F_2$.

By Lemma 3.6 there exists a flat L with $\operatorname{crk}(L) = \operatorname{rk}(F_1) - \operatorname{rk}(F_2) = 2$, $L \supset F_2$ and $L \vee F_1 = E$. We have that $L \vee U = X$ and $L \vee W = Z$ are distinct hyperplanes (their corank is at most 1 because of the submodular inequality: $\operatorname{rk}(X) \leq \operatorname{rk}(E) - 2 + n + 1 - n = \operatorname{rk}(E) - 1$, both have to be distinct hyperplanes otherwise $Z \vee X = (U \vee L) \vee (W \vee L) = (U \vee W) \vee L = S \vee L = E$ fails.) Notice that $S \cap X = U$ because $F_1 \cap X \supset U$ and F_1 is not below X otherwise $S \vee L = X$ and analogously $F_1 \cap Z = W$. We also have $U \cap Z = F_1$. If W is indecomposable we are done, so assume it is decomposable. Because $F_1 \cap Z = W$ where both F_1 and Z are indecomposable flats we know by Lemma 3.3 that $F_1 \cup Z = E$.

Assume for contradiction that $U \cup Z = E$. Because F_2 is indecomposable we know that there exists hyperplane Z' with $Z' \not\supset U$ and $Z' \not\supset Z$. This implies that there exists an element $p \in (Z' \cap U) - F_2$, otherwise $Z' \subset Z$, which implies $Z' = Z$, a contradiction. Therefore $Z' \cap U$ is a flat which is properly contained in U but it properly contains $F_2 = U \cap Z$, which is a contradiction after we calculate the ranks. Similarly, if we assume $X \cap F_1 = E$ and use assumption that U is indecomposable we get a hyperplane $Z'' \not\supset X$ and $Z'' \not\supset F_1$ and then $Z'' \cap S$ is a flat properly between F_1 and U. Hence $F_1 \cup X \neq E$.

Let us pick $b \in E - (U \cup Z)$ and $c \in E - (F_1 \cup X)$. Let $V = F_2 \vee b$, first notice that $F_1 \cup Z = E$ and $b \notin Z$, hence $b \in F_1$ hence $F_1 \supset V \supset F_2$. Notice that because $b \notin U$ and $b \notin Z \supset W$, the flat V is distinct from U and W. We claim that V is indecomposable. To prove this, first notice that $V \vee L = Y$ is a hyperplane (it has corank at least 1 because of the submodular inequality and at most 1 because if $Y \subset L$ we would get a contradiction with $F_1 \vee L = E$) distinct from X and W and $V = F_1 \cap Y$ as before. Because F_1 and Y are indecomposable we have to prove that $F_1 \cup Y \neq E$ by Lemma 3.3. But notice that $c \notin F_1 \cup X$, so assume for contradiction that $c \in Y$. Because $F_1 \cup Z = E$ we also have $c \in Z$. Hence $c \in Y \cap Z = L$. But this implies $c \in X$ which is a contradiction. Thus $c \in E - (F_1 \cap Y)$ implying V is desired second indecomposable flat. □

Lemma 3.8. Let M be a matroid with indecomposable flats F_1 and F_2 and a flat U such that $F_1 \cap U \supset F_2$ and $F_1 \vee U = E$. Then there exists an indecomposable flat V such that $F_1 \supset V \supset F_2$ with $V \vee U = E$ and $\operatorname{crk}(V) = \operatorname{rk}(U) - \operatorname{rk}(F_2)$.

Proof. Assume for contradiction that there exist flats F_1, F_2 and U as in the statement of the lemma for which the conclusion fails and let $\mathrm{crk}(U)$ be minimal among all counterexamples. We have $\mathrm{crk}(U) \neq 0$, otherwise $U = E$ and $V = F_2$ work.

Hence pick an indecomposable flat W with $F_1 \supset W \supset F_2$ such that $W \not\subset U$ and $\mathrm{rk}(W)$ is minimal. First, notice that such a flat exists because F_1 satisfies the desired properties. Second, we have $\mathrm{rk}(W) = \mathrm{rk}(F_2) + 1$. If not, we have $\mathrm{rk}(F_2) > \mathrm{rk}(F_2) + 1$ and by Lemma 3.5 there exists an indecomposable flat W' with $W \supset W' \supset F_2$ and $\mathrm{rk}(W)' = \mathrm{rk}(W) - 2$. Then by Lemma 3.7 there are distinct indecomposable flats W_1' and W_2' between W and W'. By the definition of W we have that both W_1' and W_2' are below U, but then $W = W_1' \vee W_2'$ is below U as well, which is a contradiction. Thus, $\mathrm{rk}(W) = \mathrm{rk}(F_2) + 1$.

We focus on the flat $U \vee W$. We have $U \cap W = T$, hence by the submodular inequality

$$\mathrm{rk}(U \vee W) \leq \mathrm{rk}(U) + \mathrm{rk}(W) - \mathrm{rk}(U \cap W) = \mathrm{rk}(U) + 1,$$

and U is a proper subset of $U \vee W$ because $W \not\subset U$, therefore $\mathrm{rk}(U \vee W) = \mathrm{rk}(U) + 1$. By the minimality of $\mathrm{crk}(U)$ we know that for the flats F_1, W and $U \vee W$ there exists a flat V with $F_1 \supset V \supset W$, $V \vee (U \vee W) = E$ and $\mathrm{crk}(V) = \mathrm{rk}(U \vee W) - \mathrm{rk}(W) = \mathrm{rk}(U) - \mathrm{rk}(F_2)$. But then $F_1 \supset V \supset F_2$ and $V \vee (U \vee W) = V \vee U = E$ (we have $V \supset W$) therefore V is also the desired flat for flats F_1, F_2 and U, this is a contradiction. □

When we apply Lemma 3.8 we sometimes write we apply it to flats $[F_1, U, F_2]$ using the notation as in Lemma 3.8.

The proofs of the next two theorems can be found in [11].

Lemma 3.9. ([11], (4.2)) Let M be a matroid and let L be a decomposable corank 2 flat on an indecomposable flat F with $L \supset F$. Then there exists an indecomposable corank 3 flat P with $L \supset P \supset F$.

Lemma 3.10. ([11], (4.3) and (4.4)) Let M be a matroid and P be an indecomposable corank 3 flat. If there exists a decomposable corank 2 flat $L = X \cap Y$ on P where X and Y are hyperplanes, we have that L is the unique indecomposable corank 2 flat on P and for every hyperplane Z we have that the only corank 2 flats below Z and on P are $Z \cap X$ and $Z \cap Y$.

3.2 The path theorem

We are ready to state and prove the path theorem. It is about sequences of hyperplanes such that the successive terms intersect in an indecomposable corank 2 flat.

Definition 3.4. Let Λ be a geometric lattice and Γ its modular cut. Any matroid M with lattice of flats isomorphic to Λ is called a type of τ. For an upper sublattice Λ' of Λ we denote $\Lambda' \cap \Gamma$ by $\Gamma_{\Lambda'}$.

Definition 3.5. Let Λ be a geometric lattice. A sequence $P = (H_0, \dots, H_k)$ of hyperplanes in Λ is called a Tutte path if for all $0 \leq i < k$ the intersection $H_i \cap H_{i+1}$ is an indecomposable corank 2 flat. The hyperplane X_0 is called the origin of P, the path lies on a flat F if all of its terms lie on F.

Theorem 3.2 (Path theorem, original version)**.** Let Λ be a geometric lattice with modular cut Γ. Let G_0 and G_1 be hyperplanes of Λ lying on an indecomposable flat F such that $G_1 \notin \Gamma$. Then there exists a Tutte path $G_0 = H_0, \dots, H_k = G_1$ on F such that all hyperplanes H_i except possibly H_0 are not in Γ.

Proof. The idea of the proof is taken from [11]. For contradiction let G_0, G_1 and F as in the theorem statement constitute a counterexample with $\mathrm{crk}(F)$ minimal among all counterexamples. Notice that $\mathrm{crk}(F) \geq 3$, otherwise (G_0, G_1) or (G_0) are suitable Tutte paths.

Because G_0 and F are indecomposable, by Lemma 3.5 there exists an indecomposable flat U with $F \subset U \subset G_0$ with $\mathrm{rk}(U) = \mathrm{rk}(F) + 2$. By Lemma 3.7 there exist indecomposable flats V and W with $\mathrm{rk}(V) = \mathrm{rk}(W) = \mathrm{rk}(F) + 1$ and $V \vee W = U$. The flat G_1 does not lie on V nor W, otherwise both G_0 and G_1 lie on an indecomposable flat V or W with $\mathrm{crk}(V) = \mathrm{crk}(W) = \mathrm{crk}(F) - 1$, hence by the minimality of $\mathrm{crk}(F)$ there exists a valid Tutte path from G_0 to G_1 lying on flat V or W - a contradiction.

Observe that $G_1 \cap U \supset F$ and $G_1 \vee U = E$ where G_1 and F are indecomposable. Therefore, by Lemma 3.8 there exists an indecomposable flat L such that $G_1 \supset L$, $L \vee U = E$ and $\mathrm{crk}(L) = \mathrm{crk}(U) - \mathrm{crk}(F) = 2$.

Let $L \vee V = H_V$ and $L \vee W = H_W$. We have that $L \vee V \neq L$ otherwise G_1 would lie on V, therefore $\mathrm{rk}(H_V) > \mathrm{rk}(L)$. By the submodular inequality we have

$$\mathrm{rk}(H_V) \leq -\mathrm{rk}(L \cap V) + \mathrm{rk}(L) + \mathrm{rk}(V) = \mathrm{rk}(L) + \mathrm{rk}(V) - \mathrm{rk}(F) = \mathrm{rk}(L) + 1 = \mathrm{rk}(E) - 1,$$

hence H_V and analogously H_W are hyperplanes. Hyperplanes H_V and H_W are distinct, otherwise $H_V = (L \vee V) \vee W \supset L \vee U = E$ which is a contradiction. If both H_V and H_W are in Γ we have that $H_V \cap H_W = L$ is in Γ hence $G_1 \supset L$ is in Γ, a contradiction.

Assume without loss of generality that $H_V \notin \Gamma$. It holds that $G_0 \cap H_V = V$ where V is indecomposable and $\mathrm{crk}(V) = \mathrm{crk}(F) - 1$ hence by the minimality of $\mathrm{crk}(F)$ there exists a Tutte path P from G_0 to H_V on V. Since $H_V \cap G_1 = L$ where L is indecomposable we can add G_1 to P and obtain a valid Tutte path from G_0 to G_1 on F, which is a contradiction. □

3.3 The homotopy theorem

The original version of Tutte's homotopy theorem is about decomposing an arbitrary closed Tutte path into four kinds of Tutte paths which we call elementary Tutte paths.

Definition 3.6. Let M be a matroid with $P = (H_0, \dots, H_k)$ and $Q = (H_k, H_{k+1}, \dots, H_n)$ being Tutte paths. If $H_k \cap H_{k+1}$ is an indecomposable corank 2 flat we define the product of the Tutte paths P and Q as $PQ = (H_0, \dots, H_k, H_{k+1}, \dots, H_n)$, which is a Tutte path. A Tutte path $P = (H_0, \dots, H_k)$ is called closed if $H_0 = H_k$. A Tutte path $(H_0, \dots, H_k)$ in Λ is called off a modular cut Γ if none of its terms are in Γ.

Definition 3.7. Let Λ be a geometric lattice with a modular cut Γ, by E we denote the top element of Λ. We define four kinds of closed Tutte paths off Γ as elementary paths with respect to Γ.

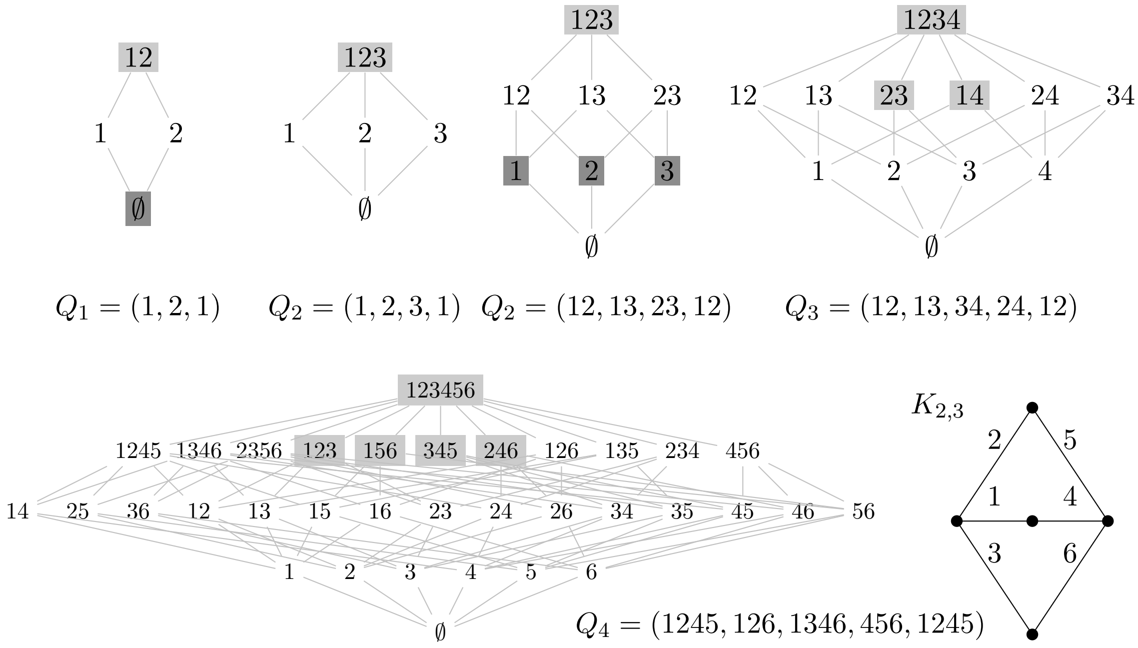
1. Let Λ' of type $U_{2,2}$ be an upper sublattice of Λ with modular cut $\Gamma_{\Lambda'} = \{E\}$. Let H_0 and H_1 denote the distinct hyperplanes of Λ'. If $H_0 \cap H_1$ is an indecomposable flat in Λ, we call $Q = (H_0, H_1, H_0)$ the elementary path of the first kind.

2.1 Let Λ' of type $U_{2,3}$ be an upper sublattice of Λ with modular cut $\Gamma_{\Lambda'} = \{E\}$. With H_0, H_1 and H_2 denoting the distinct hyperplanes of Λ', we call $Q = (H_0, H_1, H_2, H_0)$ the elementary path of the second kind.

2.2 Let Λ' of type $U_{3,3}$ be an upper sublattice of Λ with modular cut $\Gamma_{\Lambda'} = \{E\}$. Let H_0, H_1 and H_2 denote the distinct hyperplanes of Λ'. If $H_0 \cap H_1$, $H_0 \cap H_2$ and $H_0 \cap H_2$ are indecomposable corank 2 flats in Λ, we call $Q = (H_0, H_1, H_2, H_0)$ the elementary path of the second kind.

3. Let Λ' of type $U_{3,4}$ be an upper sublattice of Λ with modular cut $\Gamma_{\Lambda'} = \{E, H_4, H_5\}$, in particular $\mathrm{crk}(H_4 \cap H_5) = 3$. Let the hyperplanes not in $\Lambda' - \Gamma_{\Lambda'}$ be $\{H_0, H_1, H_2, H_3\}$ such that $\mathrm{crk}(H_0 \cap H_2) = 3$. Then $Q = (H_0, H_1, H_2, H_3, H_0)$ is an elementary Tutte path of the third kind.

4. Let Λ' of type $M(K_{2,3})$ be an upper sublattice of Λ with the modular cut $\Gamma_{\Lambda'}$ and the path Q_4 as indicated in Figure 4. Then Q_4 is the elementary path of the fourth kind.



$Q_1 = (1, 2, 1)$ $Q_2 = (1, 2, 3, 1)$ $Q_2 = (12, 13, 23, 12)$ $Q_3 = (12, 13, 34, 24, 12)$

$Q_4 = (1245, 126, 1346, 456, 1245)$

Figure 4: Elementary Tutte paths of the first kind, two of the second kind, third kind and the fourth kind with the respective upper sublattices and modular cuts.

Notice that there are two sublattices corresponding to the elementary Tutte path of the second kind. Although confusing, we pick this terminology to be consistent with [11].

The elementary path of the fourth kind is described in a different way in [11] as follows. The starting point is a corank 4 flat $E \in \Lambda$ on which there are three hyperplanes A, B and C such that $A \cap B$, $B \cap C$ and $A \cap C$ are decomposable corank 2 flats. On E there are exactly six indecomposable corank 3 flats, such that each decomposable corank 2 flats described above, lies on exactly two corank 3 flats. The flats A, B and C are in $\Lambda - \Gamma$ and there are exactly two members of Γ on each of the six indecomposable corank 3 flats. We define a path of the form $Q = (A, X, B, Y, A)$ where X and Y are on distinct indecomposable corank 3 flats below $A \cap B$ as elementary path of the fourth kind with respect to Γ.

After the definition, the explicit description of all of the flats generated by the six indecomposable corank 3 flats is given in [11]. The above description of the elementary path of the fourth kind is used in the proof of the homotopy theorem.

We see all of the elementary paths with sublattices in Figure 4, with light gray rectangles we indicate the flats which are part of the modular cut while with dark gray the flats which have to be indecomposable in Λ. They are indicated for the elementary paths of the first kind and the second kind. For the next definitions we are considering the setting with a geometric lattice Λ together with a modular cut Γ.

Definition 3.8. Let S_1, S_2 be Tutte paths off Γ. We say that S_2 can be derived from S_1 by an elementary deformation if $S_1 = PR$ and $S_2 = PQR$ where P, R are Tutte paths and Q is an elementary path with respect to Γ.

Definition 3.9. We define an equivalence relation on the set of closed Tutte paths off Γ by $P_1 \sim P_2$ if P_2 can be derived from P_1 by a finite number of elementary deformations.

Definition 3.10. A closed Tutte path P is called null-homotopic if $P \sim (H)$ where H is a hyperplane, i.e. P can be derived from a path with one term by a sequence of elementary deformations.

Definition 3.11. Let $P = (H_0, \dots H_k)$ be a Tutte path. We define the flat $F(P) = H_0 \cap \dots \cap H_k$. By corank and rank of P we mean $\mathrm{crk}(F(P))$ and $\mathrm{rk}(F(P))$ respectively.

Lemma 3.11 ($F(P)$ is indecomposable)**.** Let $P = (H_0, \dots H_k)$ be a Tutte path, then the flat $F(P)$ is indecomposable.

Proof. Because $H_i \cap H_{i+1}$ is indecomposable for all i we have that $H_i \cup H_{i+1} \neq E$ by Lemma 3.3. Define for all $0 \leq i \leq k$ the flat $F_i = \cap_{j=0}^{i} H_j$. We have that F_1 is indecomposable by assumption. Then F_2 is indecomposable by Lemma 3.3 because $F_2 = F_1 \cap H_2$ where F_1 and H_1 are indecomposable and $H_2 \cap F_1 \subset H_2 \cup H_1 \neq E$. Applying analogous reasoning k-times we get that $F_k = F(P)$ is indecomposable. ☐

Lemma 3.12. Let H_1 and H_2 be hyperplanes such that $H_1 \cap H_2$ is a decomposable flat, then $\mathrm{crk}(H_1 \cap H_2) = 2$.

Proof. Let $\{X_1, X_2\}$ be a separation for $H_1 \cap H_2$ such that $H_i \supset X_i$ (if both H_1 and H_2 contain the same set X_j of the separation then $H_1 \cap H_2 = X_j$ implying $X_i = E$ which is a contradiction.) If $H_1 - X_1$ is nonempty let $a \in H_1 - X_1$, then $a \in X_2 - X_1$ implying $a \in H_2$. Therefore $a \in H_1 \cap H_2 - X_1$ which is impossible. Therefore $X_i = H_i$, implying that the only hyperplanes above $H_1 \cap H_2$ are H_1 and H_2, thus $\mathrm{crk}(H_1 \cap H_2) = 2$. ☐

3.3.1 Statement of the homotopy theorem

We are ready to state the Tutte's homotopy theorem.

Theorem 3.3 (Tutte's Homotopy theorem)**.** Let Λ be a geometric lattice with modular cut Λ, then every closed Tutte path off Γ is null-homotopic.

The proof of the homotopy theorem is long, for the remainder of section 3 we follow the proof in [11] and translate the formulation of its proof by replacing circuits with hyperplanes of the dual matroid. When we say that a flat F is on a flat G we mean either $F \subset G$ or $G \subset F$ which is the same as the terminology used in [11].

The remainder of the part one of the thesis can be skipped if the reader wants to understand the second part.

Proof. Assume for contradiction that Theorem 3.3 is true for all closed Tutte paths P with $\text{crk}(F(P)) \leq n$, but there exists a closed Tutte path of corank $n+1$ which is not null-homotopic.

First we prove that a certain special Tutte path is null-homotopic.

3.3.2 The special lemma

Definition 3.12 (Special path)**.** Let Λ be a geometric lattice with modular cut Λ. We define a good path as $Q = (W, X, Y, Z, W)$ which is a Tutte path off Γ of corank k with $W \cap X \cap Y$ and $Y \cap Z \cap W$ indecomposable corank 3 flats and $W \cap Y$ decomposable corank 2 flat.

Lemma 3.13 (Special lemma)**.** Let Λ be a geometric lattice with modular cut Λ, and assume $n \geq 3$, then all good paths are null-homotopic.

We wish to show that good paths are null-homotopic.

The setting of all lemmas in section 3.3.1 is that we have a good path $Q = (W, X, Y, Z, W)$ of corank $n+1$ with $F_1 = W \cap X \cap Y$, $F_2 = Y \cap Z \cap W$ and we assume $n \geq 3$. We denote any null-homotopic path by 0.

Lemma 3.14. Let $Q' = (W, X', Y, Z', W)$ be a Tutte path with $X' \supset F_1$ and $Z' \supset F_2$, then $Q' \sim Q$.

Proof. Notice that

$$Q' \sim (W, X', Y)(Y, X, W)(W, X, Y)(Y, Z, W)(W, Z, Y)(Y, Z', W),$$

where the first two and the last two parts are closed Tutte paths on F_1 and F_2. Because $\text{crk}(F_1) = \text{crk}(F_2) = 3 \leq n$ the paths $(W, X', Y)(Y, X, W)$ and $(W, Z, Y)(Y, Z', W)$ are null-homotopic by assumption. ☐

We split the proof into extensive casework. The strategy is always the same; we decompose $Q \sim Q_1 \cdots Q_k$ where each Q_i lies on some indecomposable flat of corank $\leq n$. Hence each Q_i is null-homotopic by assumption which implies that Q is null-homotopic, a contradiction. We define a special type of flats above $F(Q)$ of corank n and $n-1$ which serve as flats of smaller corank on which the paths Q_i lie.

Definition 3.13 (Transversals of corank n)**.** an n-transversal is an indecomposable flat T_n of corank n on $F(Q)$ for which not both $T_n \subset W$ and $T_n \subset Y$ hold.

Lemma 3.15. If T_n is an n-transversal then $T_n \vee F_1$ and $T_n \vee F_2$ are indecomposable corank 2 flats.

Proof. Observe that $\mathrm{rk}(T_n \vee F_1) > \mathrm{rk}(F_1) = \mathrm{rk}(E) - 3$, otherwise $T_n \subset F_1$ which contradicts the definition of T_n (it would be below W and Y.) By the same reasoning we get that $T_n \cap F_1$ has to be a proper subset of T_n, hence $T_n \cap F_1 = F(Q)$. By the submodular inequality we get

$$\begin{aligned}
\mathrm{rk}(T_n \vee F_1) &\leq \mathrm{rk}(T_n) + \mathrm{rk}(F_1) - \mathrm{rk}(T_n \cap F_1) \\
&= \mathrm{rk}(E) - n + \mathrm{rk}(E) - 3 - (\mathrm{rk}(E) - (n+1)) \\
&= \mathrm{rk}(E) - 2,
\end{aligned}$$

therefore $T_n \vee F_1$ is a corank 2 flat. By Lemma 3.10 we know $T_n \vee F_1$ is indecomposable, because the unique decomposable corank 2 flat on F_1 is $W \cap Y$. ☐

Definition 3.14 (Transversals of corank $n-1$)**.** A $(n-1)$-transversal is an indecomposable flat T_{n-1} of corank $n-1$ on $F(Q)$ for which both $T_n \not\subset W$ and $T_n \not\subset Y$ hold.

Lemma 3.16. If T_{n-1} is a $(n-1)$-transversal then $T_n \vee F_1$ and $T_n \vee F_2$ are hyperplanes. We call them poles of the transversal T_{n-1}.

Proof. By the application of submodular inequality analogous to the proof of Lemma 3.15 we get that $\mathrm{rk}(T_{n-1} \vee F_1) \leq \mathrm{rk}(E) - 1$, and since $T_n \not\subset F_1$ we get $\mathrm{rk}(E) - 3 < \mathrm{rk}(T_{n-1} \vee F_1)$. Assume for contradiction that $\mathrm{rk}(T_{n-1} \vee F_1) = \mathrm{rk}(E) - 2$. We have by Lemma 3.10 that $T_{n-1} \vee F_1$ is indecomposable corank 2 flat and if H' is any hyperplane above it, then $H' \cap W$ and $H' \cap Y$ are the only corank 2 flats above F_1 and below H' which is a contradiction, because T_{n-1} is not below W or Y. Hence $\mathrm{rk}(T_{n-1} \vee F_1) = \mathrm{rk}(E) - 1$, implying that both $T_{n-1} \vee F_1$ and $T_{n-1} \vee F_2$ are hyperplanes. ☐

Lemma 3.17. For a $(n-1)$-transversal T_{n-1} at least one of its poles is in Γ.

Proof. Let the poles of T_{n-1} be X' and Z' containing F_1 and F_2 respectively, we know they exist by Lemma 3.16. Using Lemma 3.11 we know that $F(Q)$ is indecomposable. By Lemma 3.7 there exist indecomposable flats G_1, G_2 or corank n such that $F(Q) = G_1 \cap G_2$ and $T_{n-1} = G_1 \vee G_2$. If one of G_i-s is contained in both W and Y then T_{n-1} is contained in both which is a contradiction. If one of the G_i-s is contained in neither W nor Y we get

$$\mathrm{rk}(M) - (n+1) = \mathrm{rk}(G_i \cap Y \cap W) < \mathrm{rk}(G_i \cap Y) < \mathrm{rk}(G_i) = \mathrm{rk}(M) - n,$$

which is a contradiction (each inequality is obtained by the submodular inequality and using $G_i \vee Y = G_i \vee W = E(M)$.) Hence, without loss of generality, G_1 is contained in W implying G_2 is contained in Y. Thus, G_i-s are n-transversals and equal to $T_{n-1} \cap Y$ and $T_{n-1} \cap W$ respectively. Hence we know that $G_i \vee F_j$ for $i, j = 1, 2$ are $W \cap X'$, $W \cap Z'$, $Y \cap X'$, and $Y \cap Z'$ and they are indecomposable corank 2 flats, because the unique decomposable corank 2 flats on both F_1 and F_2 is $W \cap Y$ by Lemma 3.10. The path (W, X', Y, Z', W) exists because a $(n-1)$-transversal exists using Lemma 3.8 with $[S, W \cap Y, F(Q)]$, where S is any hyperplane on F_1 not equal W or Y then R with $S \supset R \supset F(Q)$ with $R \vee (W \cap Y) = E$ and $\mathrm{crk}(R) = \mathrm{rk}(W \cap Y) - \mathrm{rk}(F(Q)) = n - 1$ is the desired transversal.

Assume for contradiction that both of the poles X' and Z' are not in Γ. Because they lie on an indecomposable flat T_{n-1} there exists a path R from X' to Z' by Theorem 3.2. Hence there exists a path on $T_{n-1} \cap W$ (this is an n-transversal) given by $(W, X')R(Z', W)$ and a path

$(X', Y, Z')R^{-1}$ on $T_{n-1} \cap Y$ (this is an n-transversal). Thus because the corank of these paths is $\leq n$ we know they are null-homotopic and hence also

$$Q \sim (W, X', Y, Z', W) \sim (W, X')(X', Y, Z')(Z', W) \sim (W, X')R(Z', W) \sim 0,$$

is null-homotopic, which is a contradiction. Therefore, for any transversal of corank $n-1$ at least one of the poles has to be in Γ. $\square$

Lemma 3.18. There is an n-transversal A not on Y, it is the intersection of two $(n-1)$-transversals B and B' such that $B \vee F_1 = X' \notin \Gamma$, $B \vee F_2 = U_2 \in \Gamma$, $B' \vee F_1 = U_1 \in \Gamma$, $B' \vee F_2 = Z' \notin \Gamma$.

Proof. Using Lemma 3.8 With $[W, Y, F(Q)]$ we get an indecomposable flat A of corank n such that $W \supset A \supset F(Q)$ but $Y \not\supset A$. In particular, A is an n-transversal hence $A \vee F_i = L_i$ are indecomposable corank 2 flats by Lemma 3.15. There exists hyperplane $X' \supset L_1$ such that $X' \neq W$ and $X' \notin \Gamma$ because the assumption that L_1 is indecomposable implies there are at least three hyperplanes above it. Since Q is a Tutte path, we know that W is not in Γ, hence there can be at most one member of Γ above L_1.

Applying Lemma 3.8 with $[X', W, A]$ there is a transversal of corank $n-1$ denoted by B not on W such that $X' \supset B \supset A$. Because its pole $B \vee F_1 = X'$ is not in Γ we know that the other pole $U_2 = B \vee F_2$ has to be in Γ by Lemma 3.17.

By the analogous procedure as in the last paragraph (by applying Lemma 3.8 with $[Z', W, A]$ where Z' is above L_2 not in Γ) we get a transversal of corank $n-1$ called B' not on W on A such that its pole $Z' = B' \vee F_2$ is not in Γ and its other pole $U_1 = B' \vee F_1$ is in Γ. $\square$

Lemma 3.19. We define $T = B \vee B'$, we have $\mathrm{crk}(T) = n-2$.

Proof. By the submodular inequality we get $\mathrm{rk}(T) \leq \mathrm{rk}(B) + 1$, but because $B \vee B'$ contains both B and B' as proper subsets (we only need to check they are distinct which we can see, by observing that $B \vee F_1$ is not in Γ while $B' \vee F_1$ is in Γ) we get $B \vee B'$ is flat of corank $n-2$. $\square$

Lemma 3.20. Let the S be the set of all hyperplanes above T which are not in Γ. The set S is nonempty.

Proof. If $S = \emptyset$, we have $T \in \Gamma$ and notice that $U_1 \not\supset B$ hence $U_1 \not\supset T$ therefore (U_1, T) forms a modular pair implying $U_1 \cap T = B' \in \Gamma$. But then $Z' \supset B'$ is in Γ which is not the case. $\square$

Lemma 3.21. Let $T_i \in S$. The flat $Y \cap T_i$ is a decomposable corank 2 flat.

Proof. Assume for contradiction that $Y \cap T_i$ is indecomposable. The goal is to write $Q' = (W, X', Y, Z', W)$ as a path on A, using a sequence of deformations, which constitutes a contradiction, because $\mathrm{crk}(A) = n < n+1$. First, because $Y \cap T_i$ is indecomposable there exists a Tutte path R_0 from Y to T_i by the path theorem. By the path theorem there exist Tutte paths R_1 from X' to T_i on B (both X' and T_i are on B which is indecomposable) and R_2 from Z' to T_i on B'. Notice that the Tutte paths $(X', Y)R_0R_1^{-1}$ and $(Y, Z')R_2R_0^{-1}$ are on $B \cap Y$ and $B' \cap Y$ respectively which are transversals of rank n (a consequence of the fact that B is a transversal of corank $n-1$ as described in the paragraph on the definition of transversals.) Thus the closed

Tutte paths $(X', Y)R_0R_1^{-1}$ and $(Y, Z')R_2R_0^{-1}$ are null-homotopic because the paths have lower corank than $n+1$. Observe that

$$\begin{aligned} Q &\sim (W, X', Y, Z', W) \\ &\sim (W, X')(X', Y)(Y, Z')(Z', W) \\ &\sim (W, X')R_1R_0^{-1}R_0R_2^{-1}(Z', W) \\ &\sim (W, X')R_1R_2^{-1}(Z', W) \\ &\sim 0, \end{aligned}$$

and notice that the next to last path is on A, hence it is null-homotopic. This is a contradiction, hence $Y \cap T_i$ is decomposable and by Lemma 3.12 it is a corank 2 flat. ☐

Lemma 3.22. Let $T_i \in S$. The flat $W \cap T_i$ is a decomposable corank 2 flat.

Proof. First, following the proof of Lemma 3.17 we know $B \cap Y$ is an n-transversal because B is a $(n-1)$-transversal. We repeat the argument starting with the transversal A by replacing it with transversal $B \cap Y$. We need transversals of corank $n-1$ called B_1 and B_2 above $B \cap Y$ with the analogous properties as B and B'.

First, let $(B \cap Y) \vee F_i = L'_i$. We have $L'_1 \subset Y$ and $L'_1 \subset B \vee F_i = X'$. Notice that $X' \supset B \supset B \cap Y$ and $X' \neq Y$ (this holds because L_1 is indecomposable, so it cannot have both W and Y above it, it has W.) Hence $B_1 = B$ works.

For B_2, notice that $B_2 = B'$ does not work because $B' \not\supset B \cap Y$ otherwise $B \cap B' = A = B \cap Y$ does A is below W and below Y which is false. Therefore $(B \cap Y) \vee F_2 = L'_2 = U_2 \cap Y$ is an indecomposable corank 2 flat below $U_2 \in \Gamma$ and $Y \notin \Gamma$ hence there is $Z'' \notin \Gamma$ and B'' transversal B'' of corank $n-1$ above $B \cap Y$ with $B'' \vee F_2 = Z''$ and its $B'' \vee F_1 \in \Gamma$. As before, let $T' = B \vee B''$ and let T'_j be an arbitrary hyperplane above T'. We know by the same reasoning as for A that $T'_j \cap W$ is a decomposable corank 2 flat.

Observe that by the submodular inequality and the fact that $B'' \not\subset W$ we get $B'' \vee (T_i \cap Y)$ is a hyperplane. Since it is a hyperplane above a decomposable corank 2 flat it is either equal to T_i or W, but W is impossible because it is not above B''. Hence $B'' \subset T_i$, combining with $B \subset T_i$ which is true by definition of T_i we get $T' = B \vee B'' \subset T_i$. Therefore the flat $T_i \cap W$ is decomposable what we wanted to show. ☐

Definition 3.15. Consider an indecomposable flat G such that G is contained in a hyperplane of S, flat G is above $F(Q)$, we have $F_1 \supset G$ or $G \supset F_1$ and G has the minimal corank among flats satisfying all of the properties above. We can find such flat because $F(Q)$ satisfies all of the properties.

Lemma 3.23. We have $\operatorname{crk}(G) = 4$.

Proof. First notice that T_i is not above F_1, otherwise $T_i \cap Y$ would be a decomposable flat above F_1, but we know the unique such flat is $W \cap Y$, hence $T_i = W$ but then $T_i \cap W$ is not decomposable. Therefore we get $F_1 \supset G \supset F(Q)$ and we can bound the rank of G as follows. First we have $\operatorname{crk}(G) > \operatorname{crk}(F_1) = 3$ and

$$\operatorname{rk}(G \vee T) \leq \operatorname{rk}(G) + \operatorname{rk}(T) - \operatorname{rk}(G \cap T) \leq \operatorname{rk}(G) + 3,$$

which implies that $\operatorname{crk}(G \vee T) \geq \operatorname{crk}(G) - 3 > 0$. Therefore we can pick a hyperplane $N \supset G \vee T$, additionally, let $N \in \Gamma$ if such a hyperplane exists. Our goal is to show that $G \vee T$ is hyperplane. By Lemma 3.8 applied to $[F_1, N, G]$ we get an indecomposable flat G' not in N with $F_1 \supset G' \supset G$ and $\operatorname{rk}(G') = \operatorname{rk}(G) + 1$. First, by the submodular inequality and the fact that $N \supset G \vee T$ but $N \not\supset G' \vee T$ we get $\operatorname{rk}(G' \vee T) = \operatorname{rk}(G \vee T) + 1$. We either have $\operatorname{rk}(G' \vee T) = \operatorname{rk}(E)$ or by definition of G, because G' has smaller corank than G, that all hyperplanes above $G' \vee T$ are in Γ. In the latter case notice that this implies $G' \vee T \in \Gamma$. If we could pick $N \in \Gamma$ this leads to a contradiction because then $(N, G' \vee T)$ is a modular pair implying that $G \vee T$ and T_i are in Γ, which cannot be. If we could not pick $N \in \Gamma$ then all hyperplanes above $G \vee T$ are not in Γ which includes all hyperplanes above $G' \vee T$. This contradicts the fact that all hyperplanes above $G' \vee T$ are in Γ. Thus, we get $G' \vee T = E$ showing $\operatorname{rk}(G \vee T) = \operatorname{rk}(E) - 1$ and that $G' \vee T$ is a hyperplane, let us call it T_i.

Notice that

$$\operatorname{rk}(E) - 1 = \operatorname{rk}(G \vee T) \leq \operatorname{rk}(G) + \operatorname{rk}(T) - \operatorname{rk}(G \cap T) \leq \operatorname{rk}(G) + 3,$$

implying that $\operatorname{crk}(G) \leq 4$. Therefore combining with $\operatorname{crk}(G \vee T) \geq \operatorname{crk}(G) - 3$ we get $\operatorname{crk}(G) = 4$. □

Lemma 3.24. Let $n + 1 \geq 5$, then Q is null-homotopic.

Proof. Assume $\operatorname{crk}(F(Q)) = n + 1 \geq 5$, this implies that $F_2 \not\supset G$. If not, we have $F_1, F_2 \supset G$ hence $F(Q) = F_1 \cap F_2 \supset G$ and the coranks do not match. Applying Lemma 3.8 to $[F_2, T_i, F(Q)]$ we get an indecomposable flat G'' or rank $\operatorname{rk}(F(Q)) + 1$ such that $F_2 \supset G'' \supset F(Q)$ and it is not on T_i. Let $F_3 = G \vee G''$. By the submodular inequality and again $G \not\supset G''$ because otherwise $F_1 \supset G''$ and the coranks do not match, we get

$$\begin{aligned} \operatorname{rk}(G \vee G'') &\leq \operatorname{rk}(G) + \operatorname{rk}(G'') - \operatorname{rk}(F(Q)) \\ &= \operatorname{rk}(E) - 4 + \operatorname{rk}(E) - n - (\operatorname{rk}(E) - (n+1)) \\ &= \operatorname{rk}(E) - 3, \end{aligned}$$

therefore $\operatorname{rk}(G \vee G'') = \operatorname{rk}(E) - 3$ is a corank 3 flat. Notice that $F_3 \subset W \cap Y$ because W, $Y \supset F_1 \supset G$ and W, $Y \supset F_2 \supset G''$. Applying Lemma 3.8 with $[T_i, W \cap Y, G]$ there exists an indecomposable corank 2 flat L on G below T_i such that $L \vee (W \cap Y) = E$.

Let $i \in \{1, 3\}$ we know that $L \cap F_i \supset G$ hence

$$\operatorname{rk}(L \vee F_i) \leq -\operatorname{rk}(L \cap F_i) + \operatorname{rk}(L) + \operatorname{rk}(F_i) \leq 2\operatorname{rk}(E) - 5 - \operatorname{rk}(E) + 4 = \operatorname{rk}(E) - 1,$$

and if $L \vee F_i = L$ we see that L is either below W or Y which is a contradiction. Hence let $L \vee F_i = X_i$ be the hyperplanes for $i \in \{1, 3\}$. Notice that neither X_1 nor X_3 are equal to T_i, if not, we get for $i = 1$ contradiction with the definition of G because $F_1 \subset T_i$ with lower corank. With $i = 3$ we get a contradiction because $T_i \supset F_3 = G \vee G''$ but G'' is not contained in T_i. Because X_3 is above L and L is not in W nor Y we also get $F_3 = W \cap Y \cap X_3$. Because X_1 is on F_1 which is indecomposable we know by Lemma 3.10 that $X_1 \cap W$ and $X_1 \cap Y$ are indecomposable corank 2 flats. Therefore $X_1 \cup W \neq E$ and $X_1 \cup Y \neq E$. Because $W \cap T_i$ and $Y \cap T_i$ are decomposable we get $W \cup T_i = E$ and $Y \cup T_i = E$. Finally, notice that $L = X_3 \cap T_i = X_1 \cap T_i \neq \emptyset$

Suppose for contradiction that F_3 is decomposable with separation $\{P_1, P_2\}$ such that $W \supset P_1$ and $Y \supset P_2$. We then have either $X_3 \supset P_1$ or $X_3 \supset P_2$, hence either $X_3 \cup W = E$ or $X_3 \cup Y = E$. We prove that both options are impossible.

Let $a \notin X_1 \cup W$, from $W \cup T_i = E$ we see that $a \in T_i$. From $L = X_1 \cap T_i = T_i \cap X_3$ we get $a \notin L$ from the first equality and thus $a \notin X_3$ from the second. Finally $a \notin W \cup X_3$ or in other words $W \cup X_3 \neq E$. Similarly for the set $X_3 \cap Y$.

To finish off, notice that W and Y are on an indecomposable flat F_3 hence there exists a Tutte path R from Y to W off Γ by Theorem 3.2. Notice that $G \supset F_1$ and $G'' \supset F_2$ where both G and G'' are indecomposable and have corank $\leq n$. Therefore we can decompose Q as follows

$$Q \sim (W, X, Y)RR^{-1}(Y, Z, W) \sim 0,$$

Where the first path $(W, X, Y)R$ is null-homotopic because it is on G and the second path $R^{-1}(Y, Z, W)$ because it is on G''. Hence Q is null-homotopic which is a contradiction. $\square$

From now on we assume $n + 1 = \text{crk}(F(Q)) = 4$

Lemma 3.25. Assume that $n + 1 = 4$, then Q is null-homotopic.

Our goal is determining the structure of the lattice above $F(Q)$, we prove that it is either the same as the lattice in the definition of elementary path of the fourth kind or that Q is null-homotopic.

First we have $\text{crk}(T) = 4 - 3 = 1$, hence T is a hyperplane and $T = T_i \notin \Gamma$ is the unique hyperplane above T. Remember that the corank 2 flats $W \cap Y$, $W \cap T$ and $Y \cap T$ are decomposable and by Lemma 3.10 we get $W \cap Y \cap T$ is not indecomposable corank 3 flat. If P is any corank 3 flat and $L \in \{W \cap Y, W \cap T, Y \cap T\}$, we see by the submodular inequality that $\text{rk}(P \vee L) \leq \text{rk}(E) - 1$ and $P \vee L$ is contained in two of the hyperplanes $\{W, Y, T\}$ so in one whole flat L. In particular, because any corank 2 flat G' is on a corank 3 flat O we see that each corank 2 flat is in one of the hyperplanes W, Y or T (because if O is contained in $W \cap Y$, for instance, we get that the only flats above O and below G' are $G' \cap W$ and $G' \cap Y$, hence G' is equal to one of them.)

Lemma 3.26. Any 3-transversal is below two hyperplanes of Γ.

Proof. Consider an arbitrary n-transversal F, which is in this case of corank 3. Let $L_i = F \vee F_i$ be the indecomposable corank 2 flats and let hyperplanes above L_1 other than W or Y be $X_1, \ldots, X_k$. By Lemma 3.7 applied to X_i and F there exist indecomposable flats C_i, D_i and one of them has to be a $(n-1)$-transversal otherwise both are contained in the same hyperplane from $\{W, Y\}$ as F which would imply $C_i \vee D_i = W$ or Y which is not X_i. Pick the one that is not contained in such a hyperplane and call it B_i. It is a corank 2 flat that is not contained in W nor Y hence, because we know each corank 2 flat is contained in W or Y or T, it has to be $B_i \subset T$ and therefore $B_i = X_i \cap T$ (observe that none of X_i-s can be equal to T. This is because L_1 is either contained in W or Y by the proof of Lemma 3.17 and if it would also be contained in T it would be equal to $W \cap T$ or $Y \cap T$ hence decomposable.) We define $X'_i = B_i \vee L_2$ and remember that one of the poles X_i, X'_i has to be in Γ. If $k \geq 3$ we get that above one L_1 or L_2 there are at least two hyperplanes of Γ which leads to contradiction because it would follow that $T \in \Gamma$. Notice that $k \leq 1$ is impossible because L_i-s are indecomposable and they have only one of $\{W, Y\}$ above other than the X_i-s. Therefore $k = 2$ and without loss of generality let X_1, $X'_2 \in \Gamma$ and X_2, $X'_1 \notin \Gamma$. Next, notice that any two indecomposable corank 2 flats between T and F intersect L_1 in distinct hyperplanes. This is because $L_1 \vee L' = L_1 \vee L''$ then $L' = (L_1 \vee L') \cap T = ((L_1 \vee L'') \cap T) = L''$. Therefore, because all hyperplanes above L_1 are X_1, X_2 and one of the Y, W we see that there the only indecomposable corank 2 flats between T and F are $B_1 = X_1 \cap T$ and $B_2 = X_2 \cap T$. Because F is indecomposable and it is contained in decomposable corank 2 flat $W \cap T$ or $Y \cap T$ we know that each hyperplane P on F is on two indecomposable corank 2 flats $P \cap T$ and ($P \cap W$ or $P \cap Y$). Therefore $P \cap T = B_1$ or B_2 implying that if P is a third point of Γ above F we get that P and X_i are above B_i – an indecomposable flat and hence $T \in \Gamma$ which is a contradiction. Thus each transversal of corank 3 is below two hyperplanes of Γ. □

Lemma 3.27. For $i = 1, 2$ we have F_i is below two hyperplanes of Γ.

Proof. Let L be an indecomposable corank 2 flat on F_1. Since F_1 and L are indecomposable, by Lemma 3.7 we know that there exist indecomposable flats K_1 and K_2 in between of them. One of them is an n-transversal because L is not on both W and Y. Therefore any indecomposable corank 2 flat L on F_1 can be written as $L = F_1 \vee K_1$ where K_1 is an n-transversal. By the proof of Lemma 3.26, this means that L is contained in exactly three hyperplanes, and one of them is in Γ. Also, if L is a fixed indecomposable corank 2 flat on W and F_1 and have two distinct indecomposable corank 2 flats L', L'' on Y and F_1 we know that $L' \vee L$ and $L'' \vee L$ both of them are distinct hyperplanes because $L' = (L' \vee L) \cap W$. Therefore there are at most two indecomposable corank 2 flats on F_1 and W (in fact exactly two by Lemma 3.7 applied to W and F_1) and at most two indecomposable corank 2 flats between F_1 and Y (again exactly 2) because there are at most 2 points on L other than Y. Suppose for contradiction there are at least three hyperplanes H_1, H_2 and H_3 of Γ above F_1. We then have that at least two of them intersect with W in the same indecomposable corank 2 flat below W. Hence $W \in \Gamma$ which is a contradiction. Therefore there are exactly two members of Γ above F_1. □

In particular, looking at the proof of Lemma 3.27 we notice that there are precisely two indecomposable corank 2 flats between F_i and Y and between F_i and W, and that each of those indecomposable corank 2 flats lies on an n-transversal. Hence we have in total at least four n-transversals (distinct transversals intersect F_i in distinct indecomposable corank 2 flats because $(T_n \vee F_1) \cap T = T_n$.)

If these are all of the indecomposable corank 3 flats on $F(Q)$ then we are done, namely all of the conditions of elementary path of the fourth kind are satisfied: we have $E = F(Q)$ is a corank 4 flat, hyperplanes $A = W$, $B = Y$ and $C = T$ such that pairwise intersections are decomposable corank 2 flats, there are six indecomposable corank 3 flats on $F(Q)$, namely F_1, F_2 and four n-transversals W, Y, $T \notin \Gamma$, but on each indecomposable corank 3 flat there are exactly two members of Γ by Lemmas 3.27 and 3.26. Therefore Q is an elementary path of the fourth kind, implying it is null-homotopic and we have a contradiction.

Hence assume for contradiction there are more than 6 indecomposable corank 3 flats on $F(Q)$.

Because any transversal of corank 3 is on T we know that there are at most two transversals of corank 3 on W and at most two transversals of corank 3 on Y. This is because $T_n = (T_n \vee F_1) \cap T$ and there are at most four indecomposable corank 2 flats $T_n \vee F_1$ between F_1 and W or Y.

Therefore the 'seventh' indecomposable corank 3 flat F_3 is not a transversal of corank 3, hence it has to be both on W and Y. In particular F_3 is on $W \cap Y$. Remember that B_i is a transversal of corank 2 on X_i and F from the beginning of the proof of $n = 4$. By the submodular inequality we get $B_i \vee F_3 = X_i''$ are hyperplanes and none of them is in Γ. The latter statement holds because if $B_1 \vee F_3 = X_1$ (the only hyperplane above B_1 out of $\{X_1, X_2, T\}$ in Γ) we get that $B_1 \vee F_1 = B_1 \vee F_3 \supset F_1 \vee F_3 = Y \cap W$, hence because $Y \cap W$ is decomposable we see $X_1 \in \{W, Y\}$ - a contradiction, similarly for X_2''.

For the final contradiction, notice that $(W, X, Y, X_1'', W) = (W, X, Y)(Y, X_1'', W)$ is null-homotopic because we can repeat the whole proof of special lemma with F_3 replacing F_2 and notice that we have all of the conditions met (F_1, F_3 are indecomposable corank 3 flats and $W \cap Y$ is a decomposable corank 2), but the transversal of corank $n-1$ called B_2 has the property that $B_2 \vee F_1 = X_1'$ and $B_2 \vee F_3 = X_2''$ neither of its poles are in Γ, hence the path is null-homotopic.

By the same logic we get $(Y, Z, W, X_1'', Y) = (Y, Z, W)(W, X_1'', Y)$ is null-homotopic. Therefore

$$Q = (W, X, Y)(Y, Z, W) = (W, X_1'', Y)(Y, X_1'', W) \sim 0,$$

is null-homotopic which is the final contradiction.

3.3.3 The final proof

We still follow the proof in [11]. We assume Theorem 3.3 is false and the closed Tutte path P with $\text{crk}(F(P)) = n+1$ and the X_0 in a geometric lattice Λ with modular cut Γ is not null-homotopic. By Lemma 3.11 we know $F(P)$ is indecomposable and by Lemma 3.5 there is an indecomposable flat G with $X_0 \supset G \supset F(P)$ with $\text{crk}(G) = n$. For any closed Tutte path $P'' = (X_0, \ldots, X_m, X_0)$ on $F(P)$ with origin X_0 we define $u(P'')$ as the number of indices j such that $X_j \not\supset G$. If $u(P'') > 0$ with i the smallest number such that $X_i \notin G$ we define $v(P'') = \text{crk}(X_{i-1} \cap X_i \cap X_{i+1})$ where $X_{m+1} = X_0$. Let us pick a closed Tutte path R on $F(P)$ with origin X_0 such that:

(a) We have $R \sim P$.

(b) For all paths satisfying (a) we have that $u(R)$ is minimal.

(c) For all paths satisfying (b) we have $v(R)$ that is minimal.

We split the proceeding proof into cases and in each case derive a contradiction.

1. Assume $u(R) = 0$. In that case R lies on G which is an indecomposable flat of corank n hence R is null-homotopic by assumption, implying P is null-homotopic which is a contradiction.
2. Assume $u(R) > 0$ which implies $v(R) > 0$.
    2.1. Assume $v(R) = 2$.
        i. If $X_{i-1} = X_{i+1}$ we have that (X_{i-1}, X_i, X_{i+1}) is the elementary Tutte path of the first kind, implying that $Q = R_1(X_{i-1}, X_i, X_{i+1})R_2 \sim R_1R_2$. But R_1R_2 satisfies condition (a) with $u(R_1R_2) \leq u(R) - 1 < u(R)$ which is a contradiction.
        ii. If $X_{i-1} \neq X_{i+1}$ then $Q = (X_{i-1}, X_i, X_{i+1}, X_{i-1})$ is an elementary Tutte path of the second kind and

$$\begin{aligned} R &= R_1(X_{i-1}, X_i, X_{i+1})R_2 \\ &\sim R_1(X_{i-1}, X_i, X_{i+1}, X_{i-1})(X_{i-1}, X_{i+1})R_2 \\ &\sim R_1(X_{i-1}, X_{i+1})R_2. \end{aligned}$$

But $R_1(X_{i-1}, X_{i+1})R_2$ satisfies (a) and $u(R_1(X_{i-1}, X_{i+1})R_2) = u(R) - 1 < u(R)$ which is a contradiction.

2.2. Assume $v(R) = 3$. We have that F is an indecomposable corank 3 flat by Lemma 3.11 and $G \vee F = L$ is a corank 2 flat because F does not contain G and

$$\begin{aligned}\operatorname{rk}(G \vee F) &\leq \operatorname{rk}(G) + \operatorname{rk}(F) - \operatorname{rk}(G \cap F)\\ &= \operatorname{rk}(E) - n + \operatorname{rk}(E) - 3 - (\operatorname{rk}(E) - (n+1))\\ &= \operatorname{rk}(E) - 2.\end{aligned}$$

Let $Z = L \vee (X_i \cap X_{i+1})$ which is a hyperplane because $X_i \cap X_{i+1}$ is not on G.

i. Assume $Z \notin \Gamma$. If $Z = X_{i+1}$ we define $Q = (Z)$, if not, let $Q = (Z, X_{i+1})$, either way Q is a Tutte path. Notice that if L is indecomposable, the path $(X_{i-1}, X_i, Z, X_{i-1})$ is elementary of the second kind (X_{i-1} is on L because $X_{i-1} \supset F$ and $X_{i-1} \supset G$.) Thus we get

$$\begin{aligned}R &= R_1(X_{i-1}, X_i, X_{i+1})R_2\\ &\sim R_1(X_{i-1}, X_i, Z, X_{i-1})(X_{i-1}, Z)QR_2\\ &\sim R_1(X_{i-1}, Z)QR_2.\end{aligned}$$

Notice that R_1QR_2 satisfies (a) and $u(R_3QR_2) \leq u(R) - 1$, which is a contradiction.

If $L \supset G$ is not indecomposable we have by Lemma 3.9 that there is indecomposable corank 3 flat F' such that $L \supset F' \supset G$ We know by Lemma 3.10 that there is an indecomposable corank 2 flat L' on F' and X_{i-1} (X_{i-1} at least on two corank 2 flats and L unique decomposable corank 2 flat on F by Lemma 3.10) there is a hyperplane $T \notin \Gamma$ above L' which is not equal to X_{i-1} because $X_{i-1} \notin \Gamma$. We know $X_{i-1} \cap T$ and $Z \cap T$ are indecomposable corank 2 flats by Lemma 3.10. But then $(X_{i-1}, X_i, Z, T, X_{i-1})$ is a good path, and $X_{i-1} \cap X_i \cap Z = F$ which is distinct from $X_{i-1} \cap Z \cap T = F'$ thus $\operatorname{crk}(F(P)) = n + 1 \geq 4$ (because it is below F and F') therefore by Lemma 3.13 $(X_{i-1}, X_i, Z, T, X_{i-1})$ is null-homotopic. We get

$$\begin{aligned}R &= R_1(X_{i-1}, X_i, X_{i+1})R_2\\ &\sim R_1(X_{i-1}, X_i, Z, T, X_{i-1})(X_{i-1}, T, Z)QR_2\\ &\sim R_1(X_{i-1}, T, Z)QR_2\\ &\sim R_3QR_2,\end{aligned}$$

where R_3 is a closed Tutte path on G, because $T, Z \supset G$. Therefore we have R_3QR_2 satisfies (a) and $u(R_3QR_2) \leq u(R) - 1$, which is a contradiction.

ii. Assume $Z \in \Gamma$. By Lemma 3.7 there there exists an indecomposable corank 2 flat L' between X_{i+1} and F other than $X_i \cap X_{i+1}$ (we know that there exist two such flats and at most one is equal to $X_i \cap X_{i+1}$.) If L' is below X_{i-1}, then $(X_{i-1}, X_i, X_{i+1}, X_{i-1})$ is an elementary path of the second kind and

$$\begin{aligned}R &= R_1(X_{i-1}, X_i, X_{i+1})R_2\\ &\sim R_1(X_{i-1}, X_i, X_{i+1}, X_{i-1})(X_{i-1}, X_{i+1})R_2\\ &\sim R_1(X_{i-1}, X_{i+1})R_2,\end{aligned}$$

meaning $R_1(X_{i-1}, X_{i+1})R_2$ satisfies condition (a) but $u(R_1(X_{i-1}, X_{i+1})R_2) = u(P) - 1 < u(P)$ which is a contradiction. Therefore $(X_i \cap X_{i-1}) \vee L' = U$ and $L \vee L' = V$ are distinct hyperplanes not equal to any of X_{i-1}, X_i or X_{i+1}. Notice that $V \notin \Gamma$ because $Z \supset L$ is in Γ but $X_{i-1} \supset L$ is not in Γ. First assume $U \notin \Gamma$. Because $(X_{i-1}, U, X_{i-1}), (X_{i-1}, V, X_{i-1})$ and $(X_{i+1}, X_i, U, X_{i+1})$ are elementary Tutte paths we have

$$\begin{aligned}
R &= R_1(X_{i-1}, X_i, X_{i+1})R_2 \\
&\sim R_1(X_{i-1}, U, X_{i-1})(X_{i-1}, U, X_i, X_{i+1})R_2 \\
&\sim R_1(X_{i-1}, U, X_i, X_{i+1})R_2 \\
&\sim R_1(X_{i-1}, V, X_{i-1})(X_{i-1}, U, X_i, X_{i+1})R_2 \\
&\sim R_1(X_{i-1}, V, U, X_i, X_{i+1})R_2 \\
&\sim R_1(X_{i-1}, V, U, X_i, X_{i+1})(X_{i+1}, X_i, U, X_{i+1})R_2 \\
&\sim R_1(X_{i-1}, V, X_{i+1})R_2.
\end{aligned}$$

but $V \supset G$ hence $R_1(X_{i-1}, V, X_{i+1})R_2$ satisfies (a) and

$$u(R_1(X_{i-1}, V, X_{i+1})R_2) = u(R) - 1 < u(R),$$

which is a contradiction.

Therefore assume $U \in \Gamma$. Notice that if all indecomposable corank 2 flats on F are either on U or Z we have that $(X_{i-1}, X_i, X_{i+1}, V, X_{i-1})$ is an elementary path of the third kind because we have two hyperplanes $Z, U \in \Gamma$ on F a corank 3 flat, such that all indecomposable corank 2 flats are on either Z or U, $X_{i-1} \cap X_i, X_{i+1} \cap V \subset U$ and $X_i \cap X_{i+1}, V \cap X_{i-1} \subset Z$. Therefore

$$\begin{aligned}
P &= R_1(X_{i-1}, X_i, X_{i+1})R_2 \\
&\sim R_1(X_{i-1}, X_i, X_{i+1}, V, X_{i-1})(X_{i-1}, V, X_{i+1})R_2 \\
&\sim R_1(X_{i-1}, V, X_{i+1})R_2,
\end{aligned}$$

which again means $R_1(X_{i-1}, V, X_{i+1})R_2$ satisfies with

$$u(R_1(X_{i-1}, V, X_{i+1})R_2) < u(P)$$

a contradiction.

Therefore assume there exists another indecomposable corank 2 flat L'' on F which is neither on U nor Z. We would like to derive contradictions after assuming various relations of L'' with respect to flats X_i, X_{i-1} and X_{i+1}.

First assume that $X_{i+1} \supset L''$. We can then repeat the argument after we defined L' for the second time with L'' replacing L' (because it is an indecomposable corank 2 flat between X_{i+1} and F not equal to $X_i \cap X_{i+1}$ because the latter lies on Z) and we get that $L'' \vee (X_{i-1} \cap X_i) \neq U$ hence it cannot be in Γ because U above $X_i \cap X_{i-1}$ is in Γ. Therefore the same argument as for $U \notin \Gamma$ leads us to contradiction.

Therefore $L'' \not\subset X_{i+1}$.

Assume $L'' \subset X_i$, then $L'' \vee L = W_1$ is a hyperplane above G (by the submodular inequality and sets not being equal) not equal to X_{i-1} and Z because $X_{i-1} \cap X_i, Z \cap X_i \neq L''$. Notice that $W_1 \notin \Gamma$ because it is above L the same as Z but $L \neq Z$. We then have, because (X_{i-1}, W_1, X_{i-1}) is an elementary Tutte path that

$$\begin{aligned} R &= R_1(X_{i-1}, X_i, X_{i+1})R_2 \\ &\sim R_1(X_{i-1}, W_1, X_{i-1})(X_{i-1,W_1,X_i,X_{i+1}})R_2 \\ &\sim R_1(X_{i-1}, W_1, X_i, X_{i+1})R_2 \\ &= R'. \end{aligned}$$

Now observe that $R_1(X_{i-1}, W_1, X_i, X_{i+1})R_2$ satisfies (a), (b) because $u(R) = u(R')$ and $v(R_1(X_{i-1}, W_1, X_i, X_{i+1})R_2) = \mathrm{crk}(W_1, X_i, X_{i+1}) = \mathrm{crk}(L'') = 2$. Therefore we can replace R with R', and get $U' = L \vee \mathrm{rk}(W_1 \cap X_i) = L \vee L'' = W_1 \notin \Gamma$ which we know leads to contradiction, after applying the argument for $U \in \Gamma$ to R'. Hence $L'' \not\subset X_i$.

We then get $L'' \vee (X_i \cap X_{i+1}) = W_2$ is a hyperplane (by the submodular inequality and the sets not being equal) where $W_2 \neq X_i, X_{i+1}, Z$ (because L'' not below X_i, X_{i+1} and $Z \cap X_i \neq X_{i+1} \cap X_i$.) Assume that $L'' \supset X_{i-1}$ we then get, because (X_{i-1}, W_2, X_{i-1}) is an elementary Tutte path (both X_{i-1} and W_2 on indecomposable Tutte path L'') that

$$\begin{aligned} R &= R_1(X_{i-1}, X_i, X_{i+1})R_2 \\ &\sim R_1(X_{i-1}, W_2, X_{i-1})(X_{i-1}, W_2, X_i, X_{i+1})R_2 \\ &\sim R_1(X_{i-1}, W_2, X_i, X_{i+1})R_2 \\ &= R'. \end{aligned}$$

As before, notice that R' satisfies (a), (b) and (c) with $U'' = L \vee (W_2 \cap X_i) = L'' \cap L = W_2 \notin \Gamma$ hence we can replace the the argument with R' which we know leads to contradiction.

Hence $L'' \not\subset X_{i-1}$. In this case, $L'' \vee L = W_1, L'' \vee (X_i \cap X_{i+1}) = W_2$ and $L'' \vee \mathrm{rk}(X_{i-1} \cap X_i) = W_3$ are hyperplanes, because they are not subsets and submodular inequality. They are pairwise distinct, for instance if $W_1 = W_2$ then W_1 is above $X_{i+1} \cap X_i$ and L which is Z, but L'' is not below Z, or if $W_1 = W_3$ then W_1 is above L and $(X_{i-1} \cap X_i)$ which is X_{i-1} but L'' is not below X_{i-1}.

Also notice that W_j-s are not in Γ because each of them lies on corank 2 flats which have both elements of Γ and not in Γ above it (i.e. $U, Z \in \Gamma$.) Notice that $(W_1, W_3, W_1), (W_2, X_i, W_2), (X_{i-1}, W_1, X_{i-1}), (X_{i-1}, W_3, X_{i-1})$ and (W_3, W_2, W_3) are elementary Tutte paths of the first kind. Therefore define path R' and deform it as follows

$$\begin{aligned}R' &= R_1(X_{i-1}, W_1, W_2, X_{i+1})R_2\\ &\sim R_1(X_{i-1}, W_1)(W_1, W_3, W_1)(W_1, W_2)(W_2, X_i, W_2)(W_2, X_{i+1})R_2\\ &= R_1(X_{i-1}, W_1, W_3, W_2, X_i, X_{i+1})R_2\\ &\sim R_1(X_{i-1}, W_1, X_{i-1})(X_{i-1}, W_3)(W_3, W_2, W_3)(W_3, X_i, X_{i+1})R_2\\ &= R_1(X_{i-1}, W_3, X_i, X_{i+1})R_2\\ &= R_1(X_{i-1}, W_3, X_{i-1})(X_{i-1}, X_i, X_{i+1})R_2\\ &\sim R.\end{aligned}$$

Therefore notice that R' satisfies (a), (b) (W_3 is on G) and (c) because $v(R') = 2$. Hence we can replace R with R' in the argument starting at 2.2 (ii) and get that $U''' = L \vee (W_3 \cap X_i) = L \vee L'' = W_1 \notin \Gamma$, which we know leads to contradiction.

2.3. Assume $v(R) > 3$. Because $X_{i-1} \cap X_i$ and $F = X_{i-1} \cap X_i \cap X_{i+1}$ are indecomposable, there exists an indecomposable corank 3 flat K with $X_{i-1} \cap X_i \supset K \supset F$ by Lemma 3.10. We have that $K \vee G = L$ is an indecomposable corank 2 flat because $K \not\supset G$ (because $X_i \not\supset G$) and the submodular inequality because $\mathrm{rk}(K \cap G) = \mathrm{rk}(F(P)) = \mathrm{rk}(G) - 1$. Notice that $X_{i-1} \supset L$ since $X_{i-1} \supset G$ and K. Pick a hyperplane T above L that is not equal to X_{i-1} and, if possible, is in Γ.

By Lemma 3.8 applied to $[X_i \cap X_{i+1}, T, F]$ we get an indecomposable corank $v(P) - 1$ flat F' such that $X_i \cap X_{i+1} \supset F' \supset F$ and $F' \vee T = E$. Observe that $X_{i-1} \not\supset F'$ because otherwise $F = X_{i-1} \cap X_i \cap X_{i+1} \supset F'$ which is a contradiction. Therefore $F' \vee L = T'$ is a hyperplane (because of submodular inequality and proper subsets, because $X_{i-1} \supset L$) not equal to T nor X_{i-1}. Additionally, L is indecomposable and $T' \notin \Gamma$ holds, because if we could pick $T \in \Gamma$ then it is the only hyperplane above L in Γ because $X_{i-1} \notin \Gamma$, and if we could not, then there are no members of Γ above L. By the submodular inequality and not proper subsets we get that $K \vee F' = L'$ is a corank 2 flat. Notice that $T' \supset L'$ because $T' \supset F'$ and $T' \supset L \supset K$ as well as $X_i \supset F'$ because $X_i \cap X_{i+1} \supset F'$ and $X_i \cap X_{i-1} \supset K$.

First assume L' is indecomposable. We get (X_{i-1}, T', X_{i-1}) is an elementary path of the first kind and

$$\begin{aligned}R &\sim R_1(X_{i-1}, T', X_{i-1})(X_{i-1}, X_i, X_{i+1})R_2\\ &= R_1(X_{i-1}, T', X_i, X_{i+1})R_2\\ &= R'.\end{aligned}$$

We have $T' \supset G$ hence R' satisfies (a), (b) and $v(R') = \mathrm{crk}(T' \cap X_i \cap X_{i+1}) = \mathrm{crk}(F') < \mathrm{crk}(F)$ which is a contradiction.

Thus L' is decomposable. Using Lemma 3.9 we get an indecomposable corank 3 flat K' with $L' \supset K' \supset F'$ because F' is indecomposable. By the submodular inequality and because $K' \subset X_i$ we get $K' \vee G = L''$ is a corank 2 flat which is indecomposable by Lemma 3.10 because $X_i \cap T'$ is the unique decomposable corank 2 flat on K'. Therefore we can pick a hyperplane U above L'' below T' (T' is on two indecomposable corank 2 flats on K') not equal to T' and $U \notin \Gamma$ because $T' \notin \Gamma$.

Observe that $(T', U, X_i, X_{i-1}, T')$ is a good path because $T' \cap U = L'$ is decomposable, $T' \cap U \cap X_i = K'$ and $X_i \cap X_{i-1} \cap T' = K$ are indecomposable corank 3 flats.

Hence it is null-homotopic by Lemma 3.13 (again, $n+1 \geq 4$ in this case because $n+1 = \mathrm{crk}(F(P)) > v(R) > 3$.) Observe

$$\begin{aligned} R &\sim R_1(X_{i-1}, T', X_{i-1})(X_{i-1}, X_i, X_{i+1})R_2 \\ &= R_1(X_{i-1}, T', X_{i-1}, X_i, X_{i+1})R_2 \\ &\sim R_1(X_{i-1}, T')(T', X_{i-1}, U, X_i, X_{i-1}, T')(T', X_i, X_{i+1})R_2 \\ &= R_1(X_{i-1}, T', U, X_i, X_{i-1}, T', X_{i-1}, X_i, X_{i+1})R_2 \\ &= R_1(X_{i-1}, T', U, X_i)(X_i, X_{i-1}, T', X_{i-1}, X_i)(X_i, X_{i+1})R_2 \\ &= R_1(X_{i-1}, T', U, X_i, X_{i+1})R_2 \\ &= R_1(X_{i-1}, T')(T', U, T')(T', U, X_i, X_{i+1})R_2 \\ &\sim R_1(X_{i-1}, T', X_i, X_{i+1})R_2 \\ &= R'. \end{aligned}$$

But then R' satisfies (a) and (b) with $v(R') = \mathrm{crk}(T' \cap X_i \cap X_{i+1}) = \mathrm{crk}(F') = v(R) - 1 < v(R)$, which is a contradiction. The homotopy theorem is proved. $\square$

Part II

Towards the Second Homology Theorem

4 Order complex

Theorems 3.2 and 3.3 are topological in nature. Namely, the path theorem is about the existence of paths between hyperplanes. Hence, we expect that it can be rephrased as a statement about path-connectedness of some topological space built from hyperplanes. In the homotopy theorem, we talk about closed paths between hyperplanes being null-homotopic, so we might reformulate this result as a statement that some topological space built from hyperplanes is simply connected. In section 4, this is done with the help of the order complex, leading to new versions of path theorem and homotopy theorem. We follow [2] for the definitions and proofs in section 4. It is assumed that the reader knows basic algebraic topology, including the definition of abstract simplicial complex, simplicial homology groups and barycentric subdivision. What is needed for our purposes can be found in [9, Chapter 2] and [13].

Definition 4.1 (Order complex)**.** Let $(P, \leq)$ be a poset, a collection of subsets of P defined by

$$\Sigma(P) = \{\{p_0, p_1, \ldots, p_k\} \ : \ p_0 < p_1 < \ldots < p_k\},$$

is called the order complex of P.

The order complex is an abstract simplicial complex, so we can talk about its topological properties. All homology groups $H_i(\Sigma, \mathbb{Z})$ appearing in the following text are simplicial homology groups with coefficients in $\mathbb{Z}$ associated to a simplicial complex Σ. Therefore we omit $\mathbb{Z}$ from the notation and write $H_i(\Sigma)$ for $H_i(\Sigma, \mathbb{Z})$.

The term constellation as we define it differs from [2] where such an object is called a marked constellation.

Definition 4.2. A constellation is a triple $\tau = (\Lambda, \Gamma, \Theta)$, where Λ is a geometric lattice with a modular cut Γ and Θ is a collection of decomposable corank 2 flats in $\Lambda - \Gamma$. Constellations $\tau_1 = (\Lambda_1, \Gamma_1, \Theta_1)$ and $\tau_2 = (\Lambda_2, \Gamma_2, \Theta_2)$ are isomorphic if there exists a lattice isomorphism $f : \Lambda_1 \to \Lambda_2$ such that $f(\Gamma_1) = \Gamma_2$ and $f(\Theta_1) = \Theta_2$.

Definition 4.3. We define a poset on the set of isomorphism classes of constellations by $(\Lambda_1, \Gamma_1, \Theta_1) \leq (\Lambda_2, \Gamma_2, \Theta_2)$ if there exists an embedding of upper sublattices $f : \Lambda_1 \to \Lambda_2$ such that:

1. We have $f(\Gamma_1) = f(\Lambda_1) \cap \Gamma_2$.
2. The set Θ_1 consists precisely of those decomposable corank 2 flats in $\Lambda_1 - \Gamma_1$ for which the elements of $f(\Theta_1)$ are either in Θ_2 or indecomposable in Λ_2.

In Definition 4.3 it does not matter which representative from the isomorphism class of constellations we take. The second condition implies that all of the decomposable corank 2 flats in $\Lambda_1 - \Gamma_1$ that are not in Θ_1, stay decomposable when embedded in Λ_2.

The idea of Definition 4.3 is that Θ includes precisely the decomposable corank 2 flats in the constellation σ that need to be indecomposable when σ is viewed as a subconstellation of a constellation τ. For instance, we would like to say that the upper sublattice of type $U_{2,2}$ connecting two hyperplanes H_1 and H_2 is a subconstellation of a constellation τ. But this requires some extra information that cannot be obtained from the poset structure of the set $\{E, H_1, H_2, H_1 \cap H_2\}$. Namely the fact that the bottom of $U_{2,2}$, when viewed as an upper sublattice of τ, has to be indecomposable flat in the larger subconstellation, meaning that it has to lie below a third hyperplane as shown in Figure 5. This happens despite the fact that the bottom flat is a decomposable flat in the matroid $U_{2,2}$.

Similarly, we would like that the upper sublattice $U_{3,3}$ corresponding to the elementary path of type 2 has all of its corank 2 flats indecomposable, so Θ includes all of them, this is shown in Figure 5. As before, flats which are part of the modular cut are indicated with light gray, while the flats in Θ are indicated by dark gray.

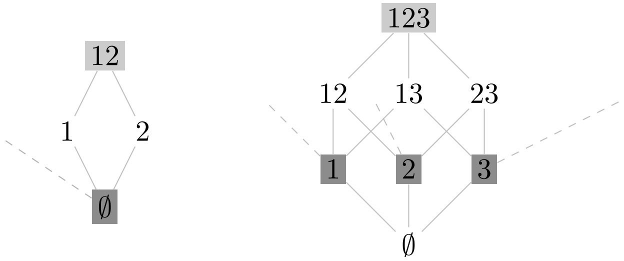


Figure 5: Subconstellations corresponding to $U_{2,2}$ and $U_{3,3}$ with the third flats above decomposable flats indicated.

Definition 4.4. A constellation σ is called a good constellation if there exists a constellation τ with empty Θ such that σ is a subconstellation of τ.

We are mostly interested in good constellations because the path and homotopy theorems are about constellation with empty Θ, i.e. all decomposable corank 2 flats are assumed to be decomposable with no embedding into another constellation. An example of a constellation which is not good is given in section 8.

The goal is to construct a topological space Σ^τ associated to a constellation τ such that $H_0(\Sigma^\tau) \simeq \mathbb{Z}$ if the geometric lattice of τ is connected. We construct Σ^τ as an order complex associated to a certain poset.

Definition 4.5 (Class 0 and class 1 constellations)**.** A constellation $\sigma_0 = (\Lambda(U_{1,1}), \{1\}, \{\})$ is called the constellation of class 0. A constellation of class 1 is $\sigma_1 = (\Lambda(U_{2,2}), \{12\}, \{\emptyset\})$.

Figure 6: Constellations of class 0 and 1.

We see the constellations of class 0 and class 1 on Figure 6. In particular a constellation of class 1 contains precisely two constellations of class 0 as subconstellations, hence its role in the order complex is to connected them.

Definition 4.6 (0th poset)**.** Let τ be a constellation. We define a poset $\mathcal{X}^\tau$ of all subconstellations of τ. We define a subposet $\mathcal{X}_0^\tau \subset \mathcal{X}^\tau$ which includes all subconstellations of τ that are of class 0.

Given a constellation τ, the order complex $\Sigma(\mathcal{X}_0^\tau)$ is a discrete set of points, one for each hyperplane of τ which is not in Γ. Second, we have to add 1-simplices to the order complex.

Definition 4.7 (1st poset)**.** Let τ be a constellation. We define a poset $\mathcal{X}_1^\tau = \mathcal{X}_0^\tau \cup \mathcal{X}_{\sigma_1}^\tau$ where $\mathcal{X}_{\sigma_1}^\tau$ includes all subconstellations of τ of class 1.

The order complex $\Sigma(\mathcal{X}_1^\tau)$ can be 0-dimensional or 1-dimensional. The 0-simplices (points) correspond to all subconstellations of class 0 and 1, while 1-simplices (edges) correspond to chains $\sigma_0 < \sigma_1$ where σ_0 is of class 0 and σ_1 of class 1. An example for the matroid $M = U_{1,1} \oplus U_{2,3}$ can be seen in Figure 7, in particular notice that the order complex $\Sigma(\mathcal{X}_1^\tau)$ is not connected as a topological space.

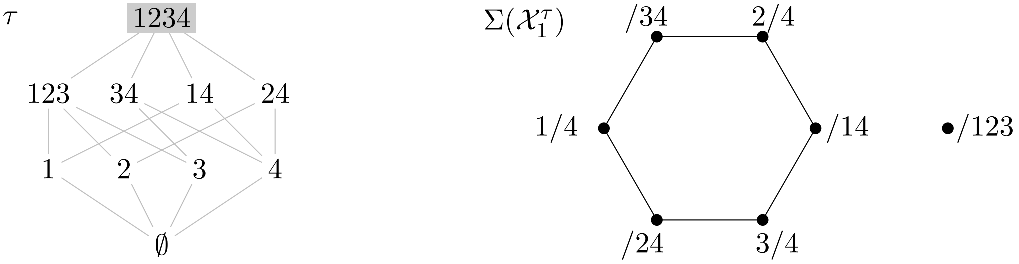


Figure 7: The constellation $\tau = (\Lambda(U_{1,1} \oplus U_{2,3}), \{1234\}, \{\})$ and its order complex $\Sigma(\mathcal{X}_1^\tau)$.

Theorem 3.2 is equivalent to saying that the order complex $\Sigma(\mathcal{X}_1^\tau)$ is path-connected as a topological space. For the homotopy and path theorem we are interested in constellations with $\Theta = \{\}$, so viewing them as constellations in which every decomposable flat is considered to be decomposable, without any embedding in larger constellation.

Theorem 4.1 (Path theorem, order complex version)**.** Let $\tau = (\Lambda, \Gamma, \{\})$ be a constellation where the bottom flat of Λ is indecomposable, then $H_0(\Sigma(\mathcal{X}_1^\tau)) \simeq \mathbb{Z}$.

Proof. For a finite simplicial complex Σ, we have $H_0(\Sigma) \simeq \mathbb{Z}^n$ where n is the number of path-connected components of Σ [13]. Thus, it is enough to show that $\Sigma(\mathcal{X}_1^\tau)$ is path-connected. Let σ_0 and σ_1 be 0-simplices in $\Sigma(\mathcal{X}_1^\tau)$. Let H_0 and H_1 be the hyperplanes contained in the upper sublattices of constellations σ_0 and σ_1 respectively. If $H_0 = H_1$ we have either that $\sigma_0 = \sigma_1$ in which case we are done, or that one of σ_0 and σ_1 is contained in the other, in which case we are also done, because they are connected by a 1-simplex.

Therefore assume that $H_0 \neq H_1$. Because the bottom flat of Λ is indecomposable and contained in both H_0 and H_1 we know by Theorem 3.2 that there exists a Tutte path $H_0 = G_0, \dots, G_k = H_1$ off Γ such that $G_i \cap G_{i+1}$ is an indecomposable corank 2 flat for all $0 \leq i < k$. In particular, for all $0 \leq i < k$, let Λ'_i be the upper sublattice above $G_i \cap G_{i+1}$ with atoms G_i and

G_{i+1}. Then $\rho_i = (\Lambda_i', \{E\}, \{G_i \cap G_{i+1}\})$ is a constellation of class 1, where E is the top element of Λ. Similarly, for all $0 \leq i \leq k$, there is a constellation of class 0 given by $\phi_i = (\Lambda_i'', \{E\}, \{\})$ where Λ_i'' is the lattice above G_i of type $U_{1,1}$.

Therefore we have a path from ϕ_0 to ϕ_k given by $\phi_0\rho_0\phi_1\rho_1 \ldots \rho_{k-1}\phi_k$. If either $\phi_0 \neq \sigma_0$ or $\phi_k \neq \sigma_1$ we add either another term σ_0 at the beginning or σ_1 at the end. This is the desired path from σ_0 to σ_1. □

To show that Theorem 4 is actually a reformulation of Theorem 3.2 we also need to prove the latter by assuming validity of the former. For the path theorem this is relatively simple, we only need to use the definition of the order complex.

Theorem 4.2 (Path theorem, deduced from order complex version)**.** Let Λ be a geometric lattice with modular cut Γ and let G_0 and G_1 be hyperplanes of Λ lying on an indecomposable flat F such that $G_1 \notin \Gamma$. Then there exists a Tutte path $G_0 = H_0, \ldots, H_k = G_1$ on F such that all hyperplanes H_i except possibly H_0 are not in Γ.

Proof. Let Λ_F denote the upper sublattice of Λ of all flats above F, and let $\Gamma_F = \Gamma \cap \Lambda_F$. Consider the constellation $(\Lambda_F, \Gamma_F, \{\})$. By Lemma 3.2 we see that the fact that F is indecomposable in Λ implies that F is indecomposable as the bottom flat of Λ_F. Therefore the constellation $\tau = (\Lambda_F, \Gamma_F, \{\})$ satisfies the assumptions of Theorem 4 implying $\Sigma(\mathcal{X}_1^\tau)$ is path connected. We split the proof into two cases.

First, assume that $G_0 \notin \Gamma$, this implies $G_0 \notin \Gamma_F$. If we denote by σ_0 and σ_1 the constellations corresponding to upper sublattices above G_0 and above G_1 of Λ_F respectively, we know that σ_0, $\sigma_1 \in \Sigma(\mathcal{X}_1^\tau)$. Therefore, because $\Sigma(\mathcal{X}_1^\tau)$ is path connected, we know there is a path $\sigma_0 = \rho_0, \rho_1, \ldots, \rho_k = \sigma_1$. By definition of the order complex $\Sigma(\mathcal{X}_1^\tau)$ we know that the even indices correspond to class 0 constellations and odd indices to class 1 constellations. Thus, the even indices give hyperplanes of the desired Tutte path from G_0 to G_1. This is because the class 1 constellation between two successive class 0 constellations in the path in $\Sigma(\mathcal{X}_1^\tau)$ imply that the intersection of successive hyperplanes is an indecomposable corank 2 flat.

Second, assume that $G_0 \in \Gamma$. We have to show G_0 can be connected via an indecomposable corank 2 flat to an element of $\Lambda_F - \Gamma_F$. First, because $G_1 \notin \Gamma$ we have that $F \notin \Gamma$. Therefore we can pick an indecomposable flat I between $G_0 \supset I \supset F$ such that $I \notin \Gamma_F$ and $\mathrm{crk}(I)$ is minimal. We claim that $\mathrm{crk}(I) = 2$. If not, by Lemma 3.5 and 3.7 there exist two distinct indecomposable flats I_1, I_2 such that $G_1 \supset I_1, I_2 \supset I$ and $\mathrm{crk}(I_1) = \mathrm{crk}(I_2) = \mathrm{crk}(I) - 1$. By the minimality of $\mathrm{crk}(I)$ we have that I_1, $I_2 \in \Gamma$. But then I_1, I_2 has the property that $\mathrm{rk}(I_1 \vee I_2) + \mathrm{rk}(I_1 \cap I_2) = \mathrm{rk}(I_1) + \mathrm{rk}(I_2)$ implying that $I = I_1 \cap I_2 \in \Gamma$ which is a contradiction. Therefore $I \notin \Gamma$ and $\mathrm{crk}(I) = 2$ implying that G_1 is the unique member of Γ above I. Therefore there is a Tutte path from I to J where $J \supset I$ and $J \notin \Gamma$. By the first case, there is a Tutte path from J to G_1, hence concatenating it with the Tutte path from I to J we get the desired result. □

4.1 Rephrased homotopy theorem

We wish to obtain a similar statement as Theorem 4 but for the homotopy theorem. In particular, for a constellation τ with empty Θ we need to construct an order complex from a certain poset $\mathcal{X}_2^\tau$ such that for $\Sigma(\mathcal{X}_2^\tau)$ we have $H_1(\Sigma(\mathcal{X}_2^\tau)) = 0$. Intuitively, the poset $\Sigma(\mathcal{X}_2^\tau)$ should contain the smallest constellations in which the elementary Tutte paths of of type 2, 3 and 4 lie. This is because the elementary Tutte paths would then lie in a ‘small’ contractible subcomplex. We can decompose closed paths, which are the generators for H_1, as the sum of elementary closed paths. If every elementary closed path lies in a contractible subcomplex, we are done as the path represents a trivial element in H_1.

Definition 4.8. We define four classes of constellations that build the order complex for the homotopy theorem:

1. Class $2a$ is $\sigma_{2a} = (\Lambda(U_{2,3}), \{123\}, \{\})$.
2. Class $2b$ is $\sigma_{2b} = (\Lambda(U_{3,3}), \{123\}, \{1, 2, 3\})$.
3. Class $2c$ is $\sigma_{2c} = (\Lambda(U_{3,4}), \{1234, 23, 14\}, \{\})$.
4. Class $2d$ is $\sigma_{2d} = (\Lambda(M(K_{2,3})), \{123456, 123, 156, 246, 345\}, \{\})$ with edges of $K_{2,3}$ labeled as in Figure 4.

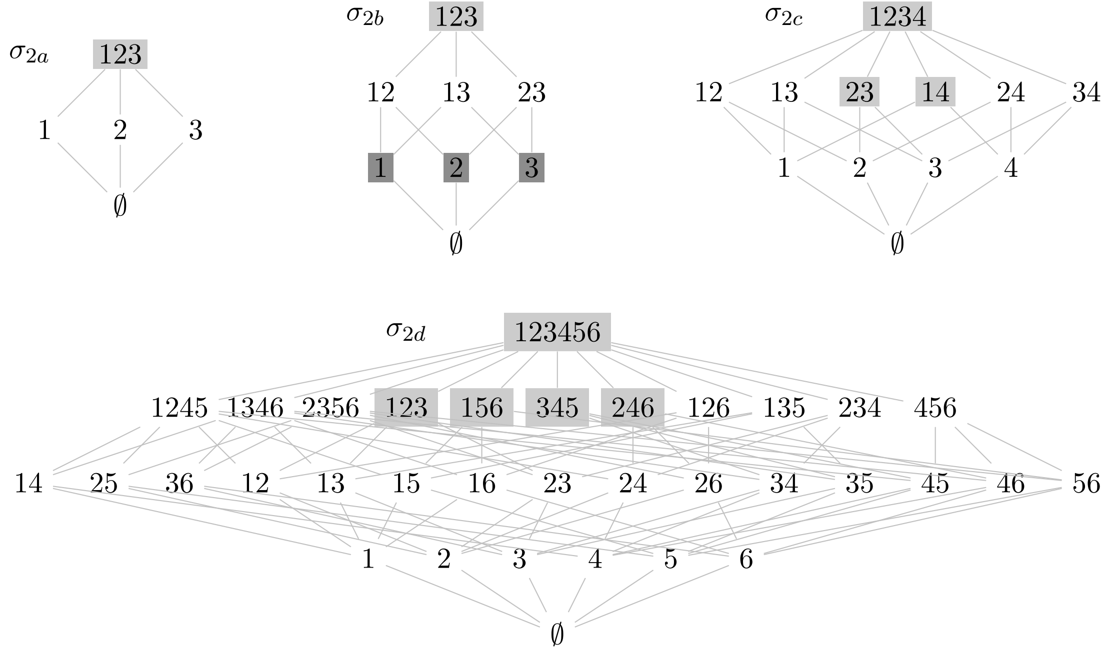


Figure 8: Constellations of classes $2a$, $2b$, $2c$, and $2d$ respectively.

Definition 4.9. Let τ be a constellation. We define a subposet $\mathcal{X}_2^\tau = \mathcal{X}_1^\tau \cup \mathcal{X}_{2ad}^\tau$ where $\mathcal{X}_{2ad}^\tau$ includes all subconstellations of τ of class $2a$–$2d$.

Given a constellation τ consider the order complex $\Sigma(\mathcal{X}_2^\tau)$. Notice that the new constellations build 2-simplices $\sigma_0 < \sigma_1 < \sigma_2$ where σ_0, σ_1 are constellations of classes 0 and 1 and σ_2 is a constellation of class $2a$–$2d$. The definition of the order complex is set up in such a way that all of the elementary Tutte paths are homotopic to a constant path, which is shown in the proof of the following theorem.

Theorem 4.3 (Homotopy theorem, order complex version)**.** Let $\tau = (\Lambda, \Gamma, \{\})$ be a constellation, then $H_1(\Sigma(\mathcal{X}_2^{\tau})) \simeq 0$.

Before the proof we verify that it is necessary to add the constellations of classes $2a$–$2d$ for the Theorem 4.3 to hold. This is done by calculating the first homology group of $\Sigma(\mathcal{X}_{2,r}^{\sigma_i})$ where $\mathcal{X}_{2,r}^{\sigma_i}$ is the poset of all subconstellations of σ_i of classes 0, 1, and $2a$–$2d$ except that we exclude the unique subconstellation of σ_i of class i. For classes $2a$, $2b$ and $2c$ the order complex $\Sigma(\mathcal{X}_{2,r}^{\sigma_i})$ is homeomorphic to $\mathbb{S}^1$ as shown in Figure 9 implying that $H_1(\Sigma(\mathcal{X}_{2,r}^{\sigma_i})) \simeq \mathbb{Z}$. For the class $2d$, the constellation σ_{2d} has six subconstellations of class $2c$, each corresponds to an embedded minor i/j where ij is a decomposable corank 2 flat in $M(K_{2,3})$ (on the lattice in Figure 8 they are 14, 25 and 36.) By considering how the six discs corresponding to subconstellations of class $2c$ glue together, we determine that the order complex $\Sigma(\mathcal{X}_{2,r}^{\sigma_{2d}})$ is homeomorphic to the projective plane. One can see this considering the identification of the boundary as shown in Figure 9. Thus $H_1(\Sigma(\mathcal{X}_{2,r}^{\sigma_{2d}})) \simeq \mathbb{Z}/2\mathbb{Z}$.

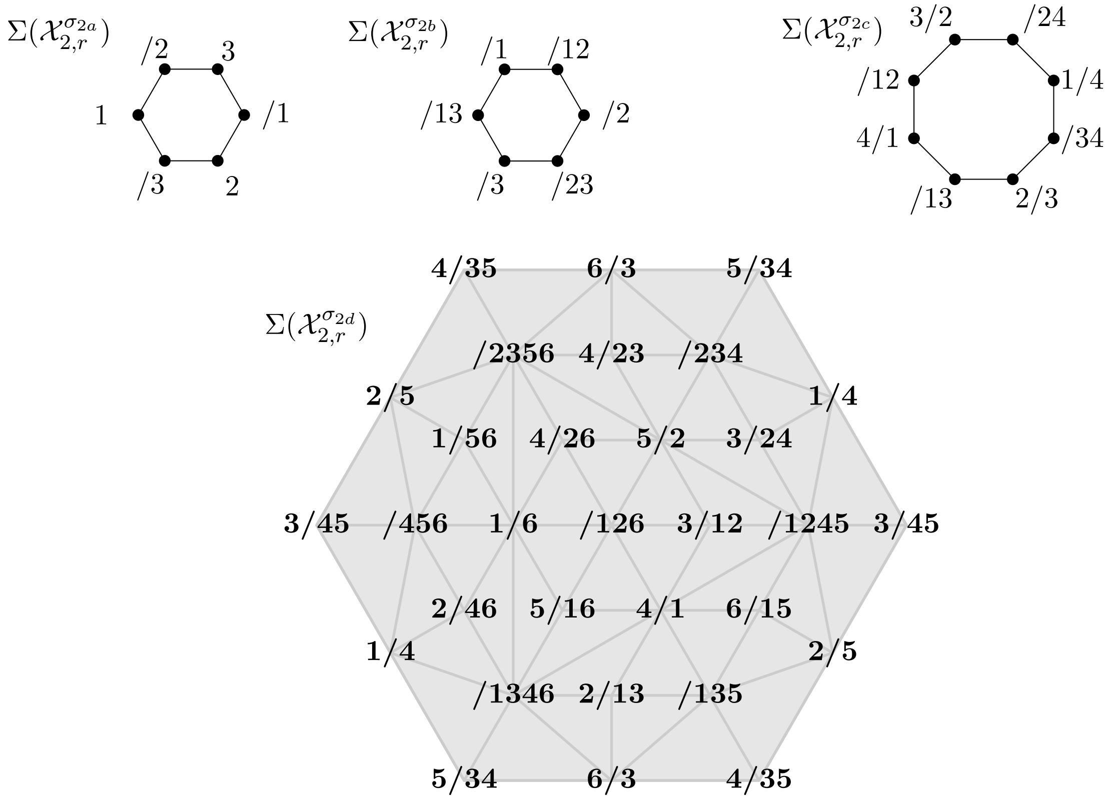


Figure 9: Order complexes $\Sigma(\mathcal{X}_{2,r}^{\sigma_i})$ for class $2a$, $2b$, $2c$, and $2d$ respectively, the figure for the class $2d$ is taken from [2].

Proof. We follow the proof in [2]. The group $H_1(\Sigma(\mathcal{X}_2^{\tau}))$ is generated by closed 1-chains. Every closed 1-chain can be represented by a sequence $S = (\sigma_1, \ldots, \sigma_n, \sigma_1)$ where σ_i and σ_{i+1} for $i = 1, \ldots, n$ are connected by a 1-simplex, where $\sigma_{n+1} = \sigma_1$. The sequence S represents an element $C \in H_1(\Sigma(\mathcal{X}_2^{\tau}))$ by $C = \sum_{i=1}^{n} [\sigma_i, \sigma_{i+1}]$. Our goal is to show that in $H_1(\Sigma(\mathcal{X}_2^{\tau}))$ we have

$C = 0$. Two sequences S_1 and S_2 representing an element of $H_1(\Sigma(\mathcal{X}_2^\tau))$ in the above way called equivalent if they represent the same element of $H_1(\Sigma(\mathcal{X}_2^\tau))$.

First we simplify the sequence S by replacing it with by an equivalent sequence S' that has constellations of class 0 at every second position. If for some $i = 1, \ldots, n$ we have that σ_i and σ_{i+1} are both not of class 0, it holds that either $\sigma_i \subset \sigma_{i+1}$ or $\sigma_{i+1} \subset \sigma_i$. Assume without loss of generality that $\sigma_i \subset \sigma_{i+1}$. There exists a subconstellation $\sigma_{i+\epsilon}$ of class 0 such that $\sigma_{i+\epsilon} \subset \sigma_i \cap \sigma_{i+1}$ (this is an abuse of notation, we mean that the inclusion holds true for the respective lattices.) The constellation $\sigma_{i+\epsilon}$ corresponds to a hyperplane which is inside the constellation σ_i. By the definition of the order complex, $[\sigma_{i+\epsilon}, \sigma_i, \sigma_{i+1}]$ is a 2-simplex hence in $H_1(\Sigma(\mathcal{X}_2^\tau))$ we have

$$[\sigma_{i+\epsilon}, \sigma_i] - [\sigma_{i+\epsilon}, \sigma_{i+1}] + [\sigma_i, \sigma_{i+1}] = 0,$$

implying $[\sigma_i, \sigma_{i+1}] = [\sigma_i, \sigma_{i+\epsilon}] + [\sigma_{i+\epsilon}, \sigma_{i+1}]$, the same holds if $\sigma_{i+1} \subset \sigma_i$. Therefore we can replace $(\sigma_1, \ldots, \sigma_n, \sigma_1)$ with $(\sigma_1, \ldots, \sigma_i, \sigma_{i+\epsilon}, \sigma_{i+1}, \ldots \sigma_n, \sigma_1)$. We do this for every other $i = 1, \ldots, n$ such that not both σ_i and σ_{i+1} are of class 0. Notice that it is impossible that both σ_i and σ_{i+1} are of class 0 because there has to be a 1-simplex between σ_i and σ_{i+1}, which means that one of them has to be properly contained in the other. At the end of this procedure (that has to be done at most n times because the number of successive members of the sequence with both not of class 0 decreases by one in each step) we get a sequence $S' = (\rho_1, \ldots, \rho_1, \rho_m)$ equivalent to S which has either a constellation of class 0 at every even index or constellation of class 0 at every odd index. Without loss of generality, assume it has class 0 constellation at every odd index, because we can shift the sequence by one place to the left and get an equivalent sequence.

Second, we replace S' by equivalent sequence S'' which has only constellations of class 0 and 1. Let ρ_i, ρ_{i+1} and ρ_{i+2} be consecutive terms such that ρ_i and ρ_{i+1} are of class 0 and ρ_{i+1} is not of class 1. Thus ρ_{i+1} has to be of class $2a$, $2b$, $2c$, or $2d$. Since ρ_i and ρ_{i+2} and contained in ρ_{i+1} they correspond to hyperplanes H_1 and H_k respectively. By inspection of the lattices of constellations of type $2a$, $2b$, $2c$ and $2d$ we see that they are connected as topological spaces, i.e. there exists a Tutte path $H_1, \ldots, H_k$ such that all hyperplanes lie in ρ_{i+1}. Let ϕ_j for $1 \leq j \leq k$ be a constellation of class 0 corresponding to H_i and τ_j for $1 \leq j \leq k$ a class 1 constellation corresponding to $H_j \cap H_{j+1}$. In particular, we have $\rho_i = \phi_1$ and $\rho_{i+2} = \phi_k$. It holds that $[\phi_j, \tau_j, \rho_{i+1}]$ and $[\phi_{j+1}, \tau_j, \rho_{i+1}]$ are 2-simplices for every $1 \leq j \leq k$ implying that

$$\begin{aligned}
[\rho_i, \rho_{i+1}] + [\rho_{i+1}, \rho_{i+2}] &= [\phi_1, \rho_{i+1}] - [\phi_k, \rho_{i+1}] \\
&= \sum_{j=1}^{k-1} \left([\phi_j, \rho_{i+1}] - [\phi_{j+1}, \rho_{i+1}]\right) \\
&= \sum_{j=1}^{k-1} ([\phi_j, \tau_j] + [\tau_j, \rho_{i+1}]) - ([\phi_{j+1}, \tau_j] + [\tau_j, \rho_{i+1}]) \\
&= \sum_{j=1}^{k-1} [\phi_j, \tau_j] + [\tau_j, \phi_{j+1}].
\end{aligned}$$

This means that S' is equivalent to $(\rho_1, \ldots, \rho_i, \tau_1, \ldots, \tau_k, \rho_{i+1}, \ldots \rho_1, \rho_m)$ where all the terms between ρ_i and ρ_{i+1} are of class 0 and 1. We repeat the procedure for every term which is not of class 0 or 1. Therefore S' is equivalent to $S'' = (\upsilon_1, \ldots, \upsilon_{2l}, \upsilon_1)$ with odd indices corresponding to class 0 constellations and even indices to class 1 constellations.

To conclude, let $P = P_1P_2 = (H_1, \dots, H_p)$ be any closed Tutte path and $R = P_1EP_2$ where E is an elementary path. We claim that the sequence $(H_1, H_1 \cap H_2, \dots, H_p \cap H_1, H_1)$ (by abuse of notation H_i corresponds to a class 0 constellation and $H_i \cap H_{i+1}$ to a class 1 constellation) is equivalent to the analogous sequence for R. This is because the closed Tutte path $E = (H'_1, \dots H'_e, H'_1)$ lies in a constellation of class $2a$, $2b$, $2c$ or $2d$, we call it σ. Let the constellations of class 0 and 1 in σ corresponding to H'_i and $H'_i \cap H'_i$ be ϕ_i and τ_i respectively. Let $\phi_{e+1} = \phi_1$, we have

$$\begin{aligned}\sum_{j=1}^{e} [\phi_j, \tau_j] + [\tau_j, \phi_{j+1}] &= \sum_{j=1}^{e} ([\phi_j, \tau_j] + [\tau_j, \sigma]) - ([\phi_{j+1}, \tau_j] + [\tau_j, \sigma]) \\ &= \sum_{j=1}^{e} [\phi_j, \sigma] - [\phi_{j+1}, \sigma] \\ &= [\phi_1, \sigma] - [\phi_{e+1}, \sigma] \\ &= 0,\end{aligned}$$

which is what we wanted to show.

Let the hyperplanes corresponding to υ_{2i+1} for $1 \leq i \leq l$ where $\upsilon_{2l+1} = \upsilon_1$ be $H_1, \dots H_l, H_1$. Notice that $Q = (H_1, \dots, H_l, H_1)$ is a closed Tutte path, hence by Theorem 3.3 it is null-homotopic to a constant path. By what we described above, it is equivalent to a constant path by a finite sequence of elementary deformations, hence the same is true for the sequences generating the corresponding closed paths. Hence S'' represents the element 0 in $H_1(\Sigma(\mathcal{X}_2^\tau))$ and the same is true for C. ☐

When we add constellations of classes $2a$–$2c$ we have to add the whole order complex to make the homology vanish since their order complexes homeomorphic to $\mathbb{S}^1$. For $2d$ the situation is more tricky because the simplicial complex is 2-dimensional, so making a cone makes it 3-dimensional. In section 5 we will show that there is, in fact, a proper subconstellation of constellation $2d$ that contains the generator for the homology.

We have to show that the order complex version of the homotopy theorem implies the original version.

Theorem 4.4 (Homotopy theorem, deduced from order complex version)**.** Let Λ be a geometric lattice with modular cut Γ, then every closed Tutte path off Γ is null-homotopic.

Proof. Let $\tau = (\Lambda, \Gamma, \{\})$. First, we notice that if $[\sigma_1, \sigma_2, \sigma_3]$ is any 2-simplex in $\Sigma(\mathcal{X}_2^\tau)$, such that $\sigma_1 \subset \sigma_2 \subset \sigma_3$ we have that the cycle $C = \partial_2([\sigma_1, \sigma_2, \sigma_3]) = [\sigma_1, \sigma_2] - [\sigma_1, \sigma_3] + [\sigma_2, \sigma_3]$ is in H_1 equal to a cycle $E = \sum_{i=0}^{k} [\rho_i, \rho_{i+1}]$ corresponding to an elementary Tutte path (which means that it contains only subconstellations of class 0 and 1.) This can best be seen by first noting that σ_3 is of class $2a$–$2d$ and both σ_1 and σ_2 are its subconstellations. Looking at the order complexes in Figure 9 we use the fact that in $\Sigma(\mathcal{X}_2^\tau)$ they become contractible because we add another point corresponding to constellation of class $2a$–$2d$. Therefore any cycle can be made equivalent to a cycle corresponding to elementary Tutte path by adding some boundaries.

Let $P = (H_0, \dots, H_k, H_0)$ be an arbitrary closed Tutte path of Γ. It corresponds to a cycle P'' in $H_1(\Sigma(\mathcal{X}_2^\tau))$ in a natural way with only subconstellations of class 0 and 1 by inserting subconstellation of class 1 corresponding to the sublattice $\{E, H_i, H_{i+1}, H_i \cap H_{i+1}\}$ between two hyperplanes. By Theorem 4.3 $H_1(\Sigma(\mathcal{X}_2^\tau)) \simeq 0$, or in other words, $P'' \in \operatorname{im} \partial_2$ where ∂_2 is the boundary operator. Thus $P'' = \sum_{i=0}^{l} \partial_2(a_i[\sigma_{1,i}, \sigma_{2,i}, \sigma_{3,i}])$. By what was explained in the first paragraph we have for all i that $\partial_2([\sigma_{1,i}, \sigma_{2,i}, \sigma_{3,i}]) = E_i$ where E_i is a cycle corresponding to an elementary Tutte path. Therefore we can write the closed Tutte path P as the sum of elementary Tutte paths which is analogous to the original statement of the homotopy theorem. □

5 Computational search for the first homology

Our goal is to find a theorem of similar flavor as Theorems 4 and 4.3 but for the group $H_2(\Sigma^\tau)$ of a certain simplicial complex Σ^τ associated to a constellation τ. Before we present necessary constellations for this extension in section 6 we explain how we might arrive at the constellations of classes $2a$–$2d$ on our own, without prior knowledge of Theorem 3.3, with a computational search. This leads us to a new formulation of the homotopy theorem. The inductive procedure here is inspired by inductive procedure in [2].

For a given constellation τ, our starting poset is $\mathcal{X}_1^\tau$ consisting of subconstellations of class 0 and class 1 which provide the starting 1-skeleton of the order complex. We want to find the isomorphism classes of subconstellations that we need to add to $\mathcal{X}_1^\tau$ to get a hypothetical poset $\mathcal{X}_{2,n}^\tau$ such that $H_1(\Sigma(\mathcal{X}_{2,n}^\tau)) \simeq 0$. We already know the answer – subconstellations of class $2a$–$2d$ suffice because of the Theorem 4.3. They are also the minimal family, by calculating the H_1 of classes $2a$–$2d$ without including a fixed class in the poset leading to a nontrivial H_1, as explained in section 4.1.

However, if the answer is not known, we can inductively search over the poset of isomorphism classes of constellations. First we need to order the set of isomorphism classes of constellations.

Lemma 5.1. *We can order the set of all isomorphism classes of constellations by $\tau_{-1}, \tau_0, \tau_1, \ldots$ such that if $\tau_i \leq \tau_j$ then $i \leq j$.*

Proof. The set of isomorphism classes of all constellations with fixed rank r and n atoms is finite. Given a constellation τ with rank r and n atoms, any proper subconstellation τ' with rank r' and n' atoms has rank $r' \leq r$, number of atoms $n' \leq n$ and at least one of the inequalities is strict. First define $\tau_{-1} = (\{\emptyset\}, \{\emptyset\}, \{\})$. Second, list all isomorphism classes of constellations with $(n, r) = (0, 0)$, continuing with $(1, 1)$, $(2, 2)$, $(3, 2)$, $(3, 3) \ldots$ Precisely, suppose for some $N \geq 1$ all constellations with rank $r \leq N$ and number of atoms $n \leq N$ are already listed. We then list all constellations with $(n, r) = (N+1, 2), (N+1, 3), \ldots, (N+1, N+1)$ in this order. Therefore we have listed all constellations with rank $r \leq N+1$ and number of atoms $n \leq N+1$. We continue like that indefinitely, the ordering has the desired property. □

With the ordering of isomorphism classes of constellations as in the proof of Lemma 5.1 we do the following inductive procedure. We define a sequence $\{\mathcal{L}_{2,i}\}_{i \geq 0}$ by $\mathcal{L}_{2,0} = \{\sigma_0, \sigma_1\}$ where σ_0 and σ_1 are the constellations of class 0 and 1 respectively. Next, if $\mathcal{L}_{2,i}$ is known for some $i \geq 0$ we define a poset $\mathcal{X}^{\tau_{i+1}}$ which includes all subconstellations of τ_{i+1} isomorphic to a member of $\mathcal{L}_{2,i}$. If $H_1(\Sigma(\mathcal{X}^{\tau_{i+1}})) \simeq 0$ we define $\mathcal{L}_{2,i+1} = \mathcal{L}_{2,i}$, if not $\mathcal{L}_{2,i+1} = \mathcal{L}_i \cup \{\tau_{i+1}\}$. Let $\mathcal{L}_2 = \cup_{i \geq 0} \mathcal{L}_{2,i}$ and define for any constellation τ a poset $\mathcal{X}_{2,i}^\tau$ which includes all of the subconstellations of τ of types in $\mathcal{L}_2$. For order complex $\Sigma(\mathcal{X}_{2,i}^\tau)$ we by construction have $H_1(\Sigma(\mathcal{X}_{2,i}^\tau)) \simeq 0$.

The constellations in $\mathcal{L}_2$ are candidates for the classes of constellations that need to be added to the order complex for the H_1 to vanish.

5.1 Commentary on constellations in $\mathcal{L}_2$

5.1.1 Redundant constellations

We expect that using the inductive procedure described in section 5 we find constellations in $\mathcal{L}_2$ which are not of class $2a$–$2d$. Indeed, the first such example with respect to ordering described in the proof of the Lemma 5.1 is the constellation $\tau_{4,4} = (\Lambda(U_{4,4}), \{1234\}, \{13, 23, 24, 14\})$ shown in Figure 10 with the corresponding order complex $\Sigma(\mathcal{X}^{\tau_{4,4}})$ which is homeomorphic to $\mathbb{S}^1$.

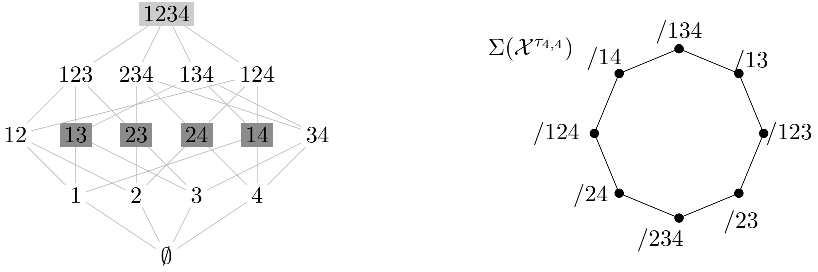


Figure 10: Constellation $\tau_{4,4} = (\Lambda(U_{4,4}), \{1234\}, \{13, 23, 24, 14\})$
and the associated order complex $\Sigma(\mathcal{X}^{\tau_{4,4}})$.

By the inductive search only, we cannot determine whether $\tau_{4,4}$ is a subconstellation of a constellation with empty Θ, which is what we are ultimately interested in. This is indeed the case as the extension τ_e of $\tau_{4,4}$ on 8 atoms with empty Θ shows in Figure 11. In particular, this means that the flats 12 and 34 stay decomposable when viewed as flats of τ_e while the flats 13, 23, 24 and 14 become indecomposable.

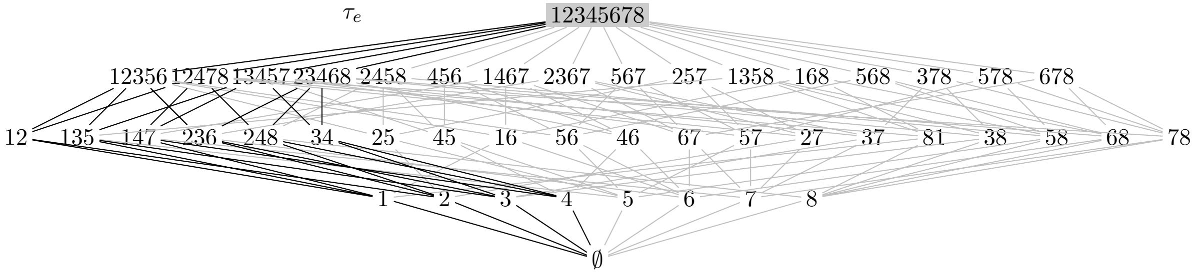


Figure 11: Constellation τ_e which is a minimal extension of τ with empty Θ.

As before, let $\mathcal{X}_2^{\tau_e}$ the poset comprised of constellations of classes 0, 1 and $2a$–$2d$ in τ_e. We know by Theorem 4.3 that $H_1(\Sigma(\mathcal{X}_2^{\tau_e}))) \simeq 0$.

This implies that although $\tau_{4,4}$ does have a nontrivial H_1, it does not generate a nontrivial cycle when viewed as a subconstellation of τ_e. By Theorem 4.3 we, in fact, know that it does not cause a non-trivial cycle in homology in any constellation τ with empty Θ.

The example of $\tau_{4,4}$ indicates that the constellations in the list $\mathcal{L}_2$ are merely good candidates for constellations that we need to include to the poset $\mathcal{X}_{2,n}^{\tau}$ to get trivial H_1 and not necessary to add. Namely, there appear exceptional constellations in $\mathcal{L}_2$ with non-empty Θ that are redundant in the sense that if we view them as as subconstellations in a constellation with empty Θ the non-trivial cycle in the first homology group gets trivial because of the way the exceptional constellation is embedded in the larger one (meaning that the other simplicies that are in the larger constellation make the cycle trivial.)

One possible way to test which of the constellations in $\mathcal{L}_3$ are redundant is as follows. For each constellation $\sigma \in \mathcal{L}_3$ with nonempty Θ we determine all of the minimal extension τ with empty Θ and calculate the H_1 of the order complex associated to the poset of all subconstellations isomorphic to a member in $\mathcal{L}_3 - \sigma$. If it has trivial H_1 for every such minimal extension, we do not add σ to the list of constellations that need to be added to the 1-dimensional order complex to make H_1 trivial.

By the above procedure we determine that the constellation of class $2b$, which has nonempty Θ, is necessary to add. A minimal extension of σ_{2b} with empty Θ is $U_{3,4}$ with trivial modular cut. Then the only proper subconstellations of $(\Lambda(U_{3,4}), 1234, \{\})$ are of classes 0, 1, $2a$ and $2b$. If we do not add the class $2b$, the first homology group is isomorphic to $\mathbb{Z}^4$, hence σ_{2b} does generate a non-trivial cycle in H_1.

5.1.2 Different classes for the homotopy theorem

Another insight that the inductive search in section 5 provides is that the list of classes needed to state the Theorem 4.3 can be modified. In particular, we can replace the constellation of class $2d$ with a certain constellation with 5 atoms and rank 4.

First, the family of classes 0, 1, and $2a$–$2d$ is not unique with the property that $H_1(\Sigma^\tau) \simeq 0$, when we construct an order complex Σ^τ associated to the family. One way to see this is to replace the constellation of class $2b$ with non-empty Θ with all of the minimal extensions of σ_{2b} with empty Θ as it is done in [2], there are six of them. We see that the proof of Theorem 4.3 still goes through because we are interested that the elementary Tutte paths of the second kind lie in a contractible subcomplex. One way that this can occur is that we make a cone over a minimal subcomplex of Σ_1^τ in which the elementary paths occur, for instance upper sublattices of types $U_{2,3}, U_{3,3}, \ldots$ But for the elementary type of the second kind lying in the upper sublattice of type $U_{3,3}$ to exist, we need all of the corank 2 flats to be indecomposable. Hence the elementary path corresponding to $U_{3,3}$ has to lie in a minimal extension of $U_{3,3}$ with all of the corank 2 flats indecomposable.

Thus, if we add all minimal extensions to the order complex (and not add constellation of class $2b$ with empty Θ), the proof of Theorem 4.3 is still valid because elementary Tutte paths of the second kind lie in a contractible sub-complex.

Therefore we are led to consider different families of classes of constellations that can be used to prove a theorem of similar type as 4.3. Classes $2a$–$2c$ shall not be modified as the order complexes corresponding to them are homeomorphic to $\mathbb{S}^1$ and if we consider any proper subcomplex, they have trivial H_1.

However, inspecting the order complex $\Sigma_{2,r}^{\sigma_{2d}}$ as in Figure 9 we see that $\Sigma_{2,r}^{\sigma_{2d}}$ is a relatively 'large' simplicial complex with respect to a cycle that generates H_1. Therefore there might be a subcomplex corresponding to a subconstellation which contains the non-trivial cycle.

Considering the subconstellations of σ_{2d} that appear in $\mathcal{L}_2$, (other than of class 0 and 1), the latter observation proves correct. Namely there are two isomorphism classes. The first is the constellation of class $2c$ which is already known. The second is the new constellation $\sigma_{2e} = (\Lambda(U_{3,4} \oplus U_{1,1}), \{12345, 234, 135\}, \{12, 24, 45, 15\})$, where $U_{1,1}$ is the matroid on the ground set $\{3\}$, for which $H_1(\Sigma(\mathcal{X}^{\sigma_{2e}})) \simeq \mathbb{Z}$.

The constellation is presented in Figure 12 together with the order complex. We check that there are 6 subconstellations of σ_{2d} isomorphic to σ_{2e}. For every i such that $1 \leq i \leq 6$ we get one such constellation by taking the lattice of flats Λ_i of $M(K_{2,3})\backslash i$ with the modular cut given by $\Gamma \cap \Lambda_i$ when we look at the natural embedding of upper sublattices $\Lambda_i \to \Lambda(M(K_{2,3}))$. The Θ is given by all of the decomposable corank 2 flats in Λ_i that are indecomposable in $\Lambda(M(K_{2,3}))$ under this inclusion.

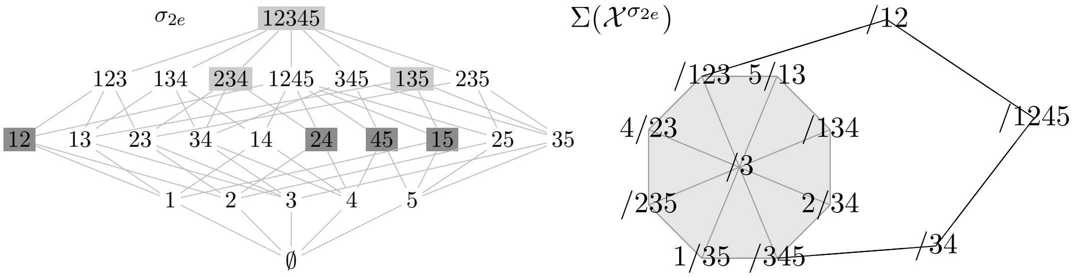


Figure 12: Constellation $\sigma_{2e} = (\Lambda(U_{3,4} \oplus U_{1,1}), \{12345, 234, 135\}, \{12, 24, 45, 15\})$ and the associated order complex $\Sigma(\mathcal{X}^{\sigma_{2e}})$.

We construct a poset $\mathcal{X}_{2,e}^{\sigma_{2d}}$ consisting of all subconstellations of classes 0, 1, $2a$, $2b$, $2c$ and $2e$. We calculate that $H_1(\Sigma(\mathcal{X}_{2,m}^{\sigma_{2d}})) \simeq 0$. This is significant because we get another version of Tutte's homotopy theorem where the constellation of class $2d$ is replaced by the constellation of class $2e$ which has less atoms.

Theorem 5.1. (Homotopy theorem, modified version) Let $\tau = (\Lambda, \Gamma, \{\})$ be a constellation and $\mathcal{X}_{2,m}^{\tau} = \mathcal{X}_1^{\tau} \cup \mathcal{X}_{2,e}^{\tau}$ where $\mathcal{X}_{2,e}^{\tau}$ consists of all subconstellations of τ of classes $2a$, $2b$, $2c$ and $2e$. We have that $H_1(\Sigma(\mathcal{X}_{2,m}^{\tau})) \simeq 0$.

Proof. We follow the proof of Theorem 4.3, the only part that we need to check is whether the elementary Tutte paths of the fourth kind still represent trivial cycles in H_1. Before, this was the case because they lied in contractible subcomplex corresponding to constellation of class $2d$. With the addition of class $2e$, the subcomplex corresponding to a subconstellation of class $2d$ has trivial $H_1 \simeq 0$ because of the constellations of classes $2e$, which is verified computationally. In particular any elementary Tutte path of the fourth kind represents a trivial cycle. □

6 Second homology theorem

The path and homotopy theorems are rephrased as statements about H_0 and H_1 of certain simplicial complexes. We wonder how could the hypothetical second homology theorem about H_2 look like. Ideally we would construct a poset $\mathcal{X}_3^{\tau} = \mathcal{X}_2^{\tau} \cup \mathcal{X}_{3,h}^{\tau}$ where $\mathcal{X}_{3,h}^{\tau}$ consists of all subconstellations of τ isomorphic to a some constellation from a finite family, such as class 1 for $\mathcal{X}_{\sigma_1}^{\tau}$ and classes $2a$–$2d$ for $\mathcal{X}_{\sigma_{2ad}}^{\tau}$. For the order complex $\Sigma(\mathcal{X}_3^{\tau})$ we would like that the second homology group is trivial, i.e. $H_2(\Sigma(\mathcal{X}_3^{\tau})) \simeq 0$. The inductive procedure and the classes $3a$–$3d$ are defined as in [2].

Our starting point is almost the same inductive procedure as the one described in section 5 except that we test for H_2 instead of H_1. As in section 5 we order the set of isomorphism classes of constellations by $\tau_0, \tau_1, \ldots$ such that if τ_i is a subconstellation of τ_j we have $i \leq j$.

We define a sequence $\{\mathcal{L}_{3,i}\}_{i\geq 0}$ by fixing the initial family $\mathcal{L}_{3,0}$. We have some options on what to pick for the initial family and discuss it later.

Next, if $\mathcal{L}_{3,i}$ is known for some $i \geq 0$ we define a poset $\mathcal{X}^{\tau_{i+1}}$ which includes all subconstellations of τ_{i+1} isomorphic to a member or $\mathcal{L}_{3,i}$. If $H_2(\Sigma(\mathcal{X}^{\tau_{i+1}})) \simeq 0$ we define $\mathcal{L}_{3,i+1} = \mathcal{L}_{3,i}$, if not we set $\mathcal{L}_{3,i+1} = \mathcal{L}_{3,i} \cup \{\tau_{i+1}\}$. Let $\mathcal{L}_3 = \cup_{i\geq 0}\mathcal{L}_{3,i}$ and define for any constellation τ a poset $\mathcal{X}_3^\tau$ which includes all of the subconstellations of τ of types in $\mathcal{L}_3$. As before, we by construction have $H_2(\Sigma(\mathcal{X}_{3,i}^\tau)) \simeq 0$.

The family $\mathcal{L}_3$ is the source of candidate constellations that we would need to insert in an order complex for the hypothetical second homology theorem. The second homology theorem would be useful if we could pick a finite number of constellations from $\mathcal{L}_3$ to build the simplicial complex, such as the family of classes 0 and 1 for the path theorem and of classes 0, 1, $2a-2d$ for the homotopy theorem.

6.1 Computational results

Our first initial family is $\mathcal{L}_{3,0} = \{\sigma_0, \sigma_1, \sigma_{2a}, \sigma_{2b}\}$ where σ_i is a constellation of class i. The reason why we do not include constellations of class $2c$, $2d$ or $2e$ is that different topological spaces might lead the validity of conjecture by Lorscheid described in section 6.2.

The first four constellations (with respect to the ordering of all constellations as in the proof of Lemma 5.1) that appear in $\mathcal{L}_3$ have four atoms, they are as follows.

($3a$) The constellation of type $3a$ is $\sigma_{3a} = (\Lambda(U_{2,4}), \{1234\}, \{\})$.

($3b$) The constellation of type $3b$ is $\sigma_{3b} = (\Lambda(U_{2,3} \oplus U_{1,1}), \{1234\}, \{1,2,3\})$, where $U_{1,1}$ is a matroid on ground set $\{4\}$.

($3c$) The constellation of type $3c$ is $\sigma_{3c} = (\Lambda(U_{3,4}), \{1234\}, \{\})$.

($3d$) The constellation of type $3d$ is $\sigma_{3d} = (\Lambda(U_{4,4}), \{1234\}, \{12, 13, 14, 23, 24, 34\})$.

For each σ_i the order complex $\Sigma(\mathcal{X}^{\sigma_i})$, corresponding to the poset of all subconstellations of classes 0, 1, $2a$, and $2b$ is homeomorphic to a sphere $\mathbb{S}^2$, thus the second homology of the order complex is $H_2(\Sigma(\mathcal{X}^{\sigma_i})) \simeq \mathbb{Z}$. The illustration of the constellations is given in Figure 13, while the order complexes $\Sigma(\mathcal{X}^{\sigma_i})$ are given in Figure 14. In Figure 14 we do not show all of the subconstellations for the constellation of class $3c$ but only the hyperplanes. One can see that every constellation of class 0, 1, $2a$–$2b$ in $\Lambda(U_{3,4})$ is completely determined by the hyperplanes lying inside of it. Therefore the classes 1 and $2a$–$2b$ correspond to the 1-simplices and 2-simplices of the octahedron respectively.

First, for the rank 2 constellations, adding the class $3a$ to $\mathcal{L}_{3,0}$ makes the second homology group trivial.

Lemma 6.1. Let τ be a constellation of rank 2 and let $\mathcal{X}_{3,1}^\tau$ the poset of all subconstellations of τ of classes in $\mathcal{L}_{3,0}$ and classes $3a$. Then $H_2(\Sigma(\mathcal{X}_{3,1}^\tau)) \simeq 0$.

Proof. As explained in section 7, in this case $\Sigma(\mathcal{X}_{3,1}^\tau)$ is, up to relabeling the vertices, the same as the first barycentric subdivision of $\Sigma = \{S \ : \ S \subset \mathcal{H} - \Gamma \text{ and } |S| = 4\}$. Thus Σ_3^τ and Σ have isomorphic homology groups. For the latter we have that Σ is the 3-skeleton of the ball $\mathbb{B}^N$ on the set $\mathcal{H} - \Gamma$ where $N = |\mathcal{H} - \Gamma|$. Because the group H_2 only depends on the 3-skeleton of a given simplicial complex we have that $H_2(\Sigma) = H_2(\mathbb{B}^N) \simeq 0$. □

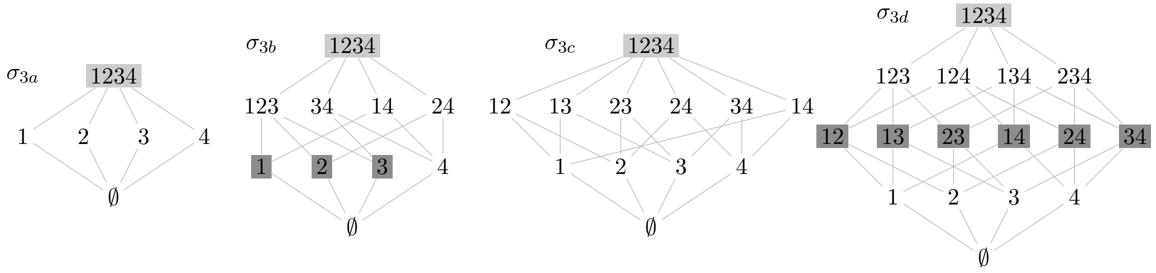


Figure 13: Constellations $\sigma_{3a}, \sigma_{3b}, \sigma_{3c}$ and σ_{3d} on 4 atoms.

We verified computationally that the order complex corresponding to the poset of constellations of classes $0, 1, 2a, 2b$ and $3a - 3d$ has trivial H_2 for all constellations with $\Gamma = \{E\}$, empty Θ, rank 3 and up to 8 atoms. With non-trivial modular cut, the first new constellation in $\mathcal{L}_3$ with empty Θ is $\tau_{6,1}$ as shown in Figure 16.

The analogous search for rank 4 constellations with trivial modular cut and classes 0, 1, $2a$, $2b$, $3a - 3d$ gives constellations with empty Θ that have non-trivial H_1. The subconstellation which causes the homology is τ_5 which is shown in Figure 15. We have $\tau_5 = \Lambda(U_{3,4} \oplus U_{1,1}), \{12345\}, \{13, 23, 24, 14\})$. The corresponding order complex $\Sigma(\mathcal{X}_3^{\tau_5})$ is homeomorphic to a ball given by the constellation of class $3c$ (in Figure 15 it is /5) together with a $\mathbb{S}^2$ glued to a part of its boundary. One also gets $\tau_{5,1}$ which has non-trivial modular cut.

Some more constellations in $\mathcal{L}_3$ are presented, the three rank 3 constellations on 6 atoms with non-empty modular cut show that with non-trivial modular cut we need to add more constellations to the list $3a$–$3d$ to make H_2 trivial for rank 3 constellations.

6.1.1 Different starting list

Our second initial family is $\mathcal{L}_{3,1} = \{\sigma_0, \sigma_1, \sigma_{2a}, \sigma_{2b}, \sigma_{2c}, \sigma_{2d}\}$ where σ_i is a constellation of class i. The order complex corresponding to this family has more 2-simplices as a result of the constellation of class $2c$, therefore there is the possibility for more closed 2-chains to be formed. This is indeed what happens, in Figure 17 there are two such constellations τ_1 and τ_2 with are not part of the $\mathcal{L}_3$ if the starting family is $\mathcal{L}_{3,0}$, for both of them $\Sigma(\mathcal{X}_3^{\tau_i})$ is homeomorphic to $\mathbb{S}^2$.

The initial family $\mathcal{L}_{3,1}$ is also the one suitable for the argument in section 9. In particular, we need that the first homology group of the order complex is trivial, so we know by Theorem 4.3 that adding classes $2a$–$2c$ is sufficient.

6.2 Relation to near-regular matroids

A matroid M is called near-regular if it is representable over all fields, except possibly $\mathbb{F}_2$. The excluded minors for the class of near-regular minors are known, there are ten of them [8]. In personal communication with Oliver Lorscheid he expressed a potential link between these excluded minors and a conjectural second homology theorem.

We start with the set $\mathcal{L}_3$. For every constellation $\tau \in \mathcal{L}_3$ we can consider a minimal extension of τ with empty Θ. Let the set of all such minimal extensions be $\mathcal{L}_3^{\text{ext}}$. Every modular cut determines a one-element extension by Theorem 3.1. Let $\widehat{\mathcal{L}_3}$ be the set of all such one-element extensions of elements of $\mathcal{L}_3^{\text{ext}}$. The conjecture by Lorscheid is that every excluded minor for near-

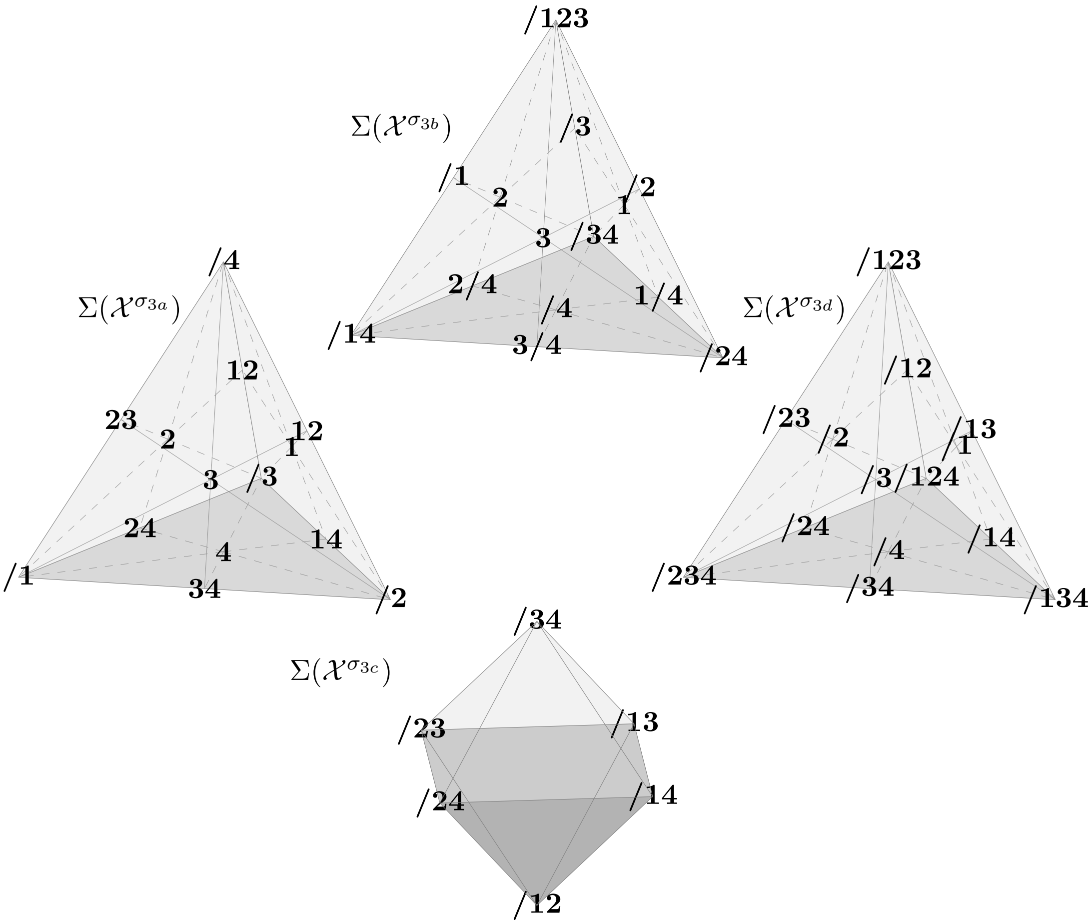


Figure 14: Order complexes corresponding to constellations $\sigma_{3a}, \sigma_{3b}, \sigma_{3c}$ and σ_{3d} on 4 atoms.

regular matroids appears in $\widehat{\mathcal{L}_3}$ and is motivated by the calculation of certain algebraic invariant called the foundation for a certain non-Fano matroid [3]. Unfortunately, the conjecture is false in this formulation. A certain matroid M on 8 elements shown in Figure 18 would have to be the one-element extension of a matroid on 7 elements with empty Θ which is minimal extension of constellation of class $3b$. We have shown this is impossible by computational search.

7 Different topological spaces

In our discussion of building topological spaces from marked constellations we deal with order complexes, which are simple to define and are suitable for computations with software. A drawback of this approach is that the vertex set of the simplicial complex corresponding to a constellation τ is given by all of the subconstellations that are isomorphic to a constellation from a certain family. Thus, we are usually led to large number of vertices.

A different approach that is valid and closer in spirit to [11] is to consider a simplicial complex built on the vertex set of hyperplanes. There are two ways this can be done, either with simplicial complexes or with CW-complexes. First we explain the approach with simplicial

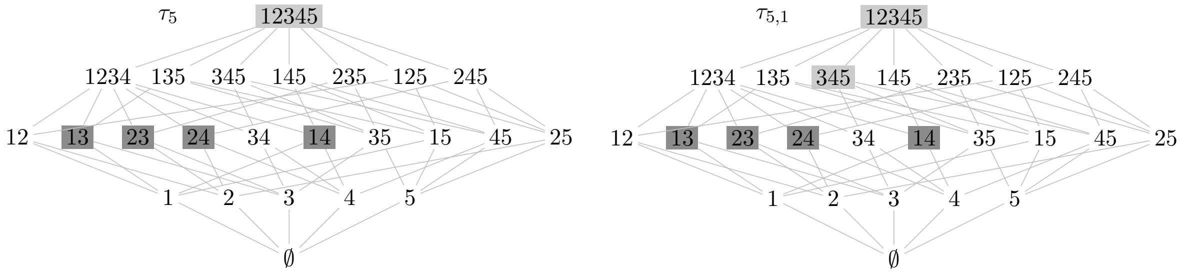


Figure 15: Two constellations on 5 atoms.

complexes.

Definition 7.1. Let τ be a constellation and $\mathcal{K} \subset \mathcal{H}$ some collection of hyperplanes. If the collection $\Lambda(\mathcal{K}) = \{\cap_{H \in S} H \ : S \subset \mathcal{K}\} \cup \{E\}$ is an upper sublattice of Λ, we call it the upper sublattice generated by $\mathcal{K}$. By the constellation generated by $\mathcal{K}$ we mean $(\Lambda(\mathcal{K}), \Lambda(\mathcal{K}) \cap \Gamma, \Theta')$, where $\Theta' = (\Lambda(\mathcal{K}) \cap \Theta) \cup \{\text{decomposable corank 2 flats in } \Lambda(\mathcal{H}) \text{ that are indecomposable in } \Lambda\}$. It is a subconstellation of τ.

Consider the initial family $\mathcal{L}_{3,0} = \{\sigma_0, \sigma_1, \sigma_{2a}, \sigma_{2b}\}$ where σ_i is a constellation of class i. If σ_j is a constellation of such class and one considers any subset S of hyperplanes in σ_j, we have the constellation that S generates is isomorphic to constellation in $\mathcal{L}_{3,0}$.

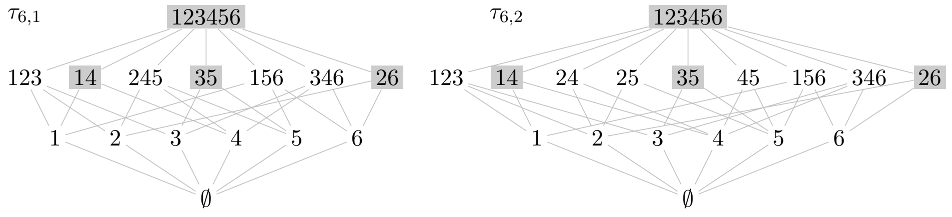
Therefore, given a constellation $\tau = (\Lambda, \Gamma, \Theta)$ and its set of hyperplanes $\mathcal{H}$ one can construct a simplicial complex Φ^τ on $\mathcal{H} - \Gamma$ by declaring that $S \subset \mathcal{H} - \Gamma$ is in Φ^τ if S generates a constellation of type σ_0–σ_{2b}.

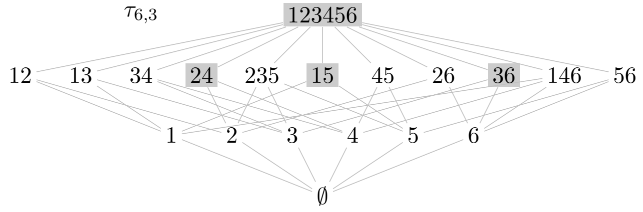
By definition of the order complex Σ^τ with respect to the family $\mathcal{L}_{3,0}$ we see that Σ^τ is in fact the first barycentric subdivision of Φ^τ up to relabeling of vertices. By [13] the simplicial complexes Σ_τ and Φ^τ are homotopy equivalent therefore share isomorphic homology groups. This is good because the simplicial complex Φ^τ has only hyperplanes in $\mathcal{H} - \Gamma$ as its vertex set and not upper sublattices hence much less simplices in general.

The Theorem 3.3 would in this context be about $H_1(\Phi^\tau)$ being generated by elementary paths of type 3 lying in the constellations of class $2c$ and type 4 lying in the constellations of class $2d$.

For the second homology theorem we could add the classes $3a$, $3b$ and $3d$ to $\mathcal{L}_{3,0}$ because the property that any subset of hyperplanes generates a constellation in the list is preserved. The hypothetical second homology theorem would be a statement about the H_2 generated by closed 2-chains which are part of subconstellations from a finite family.

Another topological space Σ^τ which can be built on the set of hyperplanes in $\mathcal{H} - \Gamma$ and has the property that $H_1(\Sigma^\tau) \simeq 0$ is a certain CW-complex. The 0-cells of the Σ^τ are given by elements of $\mathcal{H} - \Gamma$, while the 1-cells are between hyperplanes H_1 and H_2 such that $H_1 \cap H_2$ is an indecomposable corank 2 flat. The 2-cells are inserted for each elementary Tutte path. The details of this approach can be found in [6]. The difficulty with using CW-complexes is that one has no clear canonical way of gluing the 3-cells, such as for the class $3c$ into the 2-skeleton.





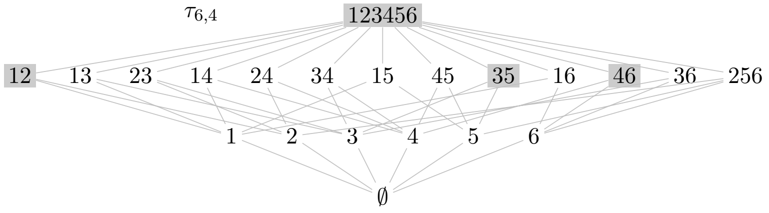


Figure 16: Constellations $\tau_{6,1}, \tau_{6,2}, \tau_{6,3}$ and $\tau_{6,4}$ on 6 atoms.

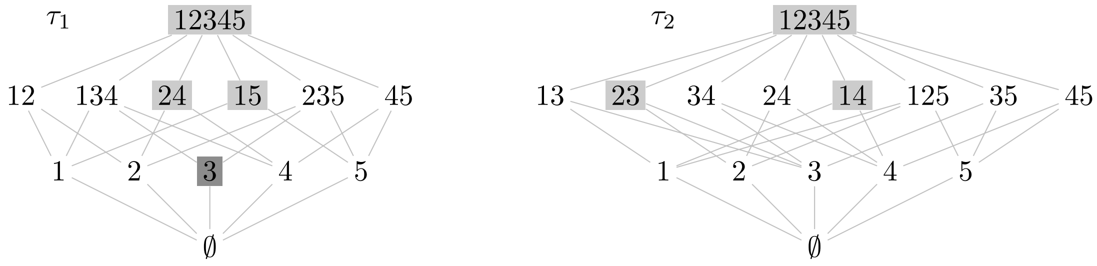


Figure 17: Two constellations on 5 atoms with the starting family $\mathcal{L}_{3,1}$.

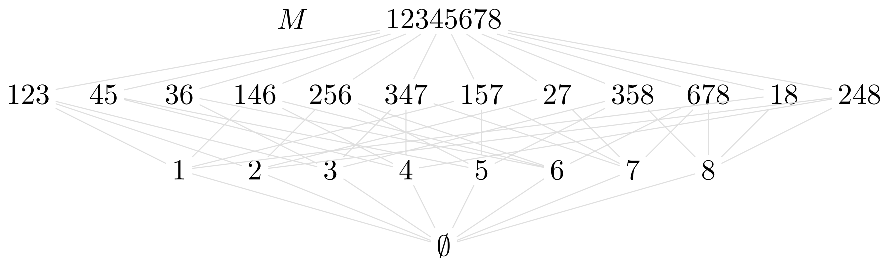


Figure 18: Excluded matroid M for the near-regular representability.

8 Computational search

In practice, we do not implement the inductive search exactly as described in sections 5 and 6. This is because not every constellation σ in $\mathcal{L}_3$ is a good constellation.

An example of a constellation σ which is not good and a member of $\mathcal{L}_2$ is given by $\sigma = (\Lambda(U_{2,3} \oplus U_{1,1}), \{1234\}, \{2\})$, where $U_{1,1}$ is a matroid on the ground set $\{4\}$. Suppose there is a suitable constellation τ with empty Θ. There is an atom 5 such that $5 \vee 2 \notin \sigma$ and $5 \vee 1$, $5 \vee 3 \in \sigma$. If either $5 \vee 1$ or $5 \vee 3$ is the image of 123 then $2 \vee 5$ is the image of 123 as well which is a contradiction. Hence $5 \vee 1$ is the image of 14 and $5 \vee 3$ is the image of 34. But then 5 is below 34 and 14 implying it is below $34 \cap 14$, i.e. the image of 4 in τ, which is a contradiction.

Instead we can find all good constellations in $\mathcal{L}_3$ that are subconstellations of a fixed constellation τ with the following algorithm.

- Fix a constellation τ with empty Θ, rank r and fix the initial class $\mathcal{L}_{3,0}$. Let $P_\tau = \{\}$ be a poset and $\mathcal{L}_{3,\tau} = \mathcal{L}_{3,0}$.
  - For coranks $r' = 1, \dots, r$ in increasing order go over all flats F of corank r'.
    - For each flat F of coank r' consider the set C_F of flats covering it. Go over all of the subsets $S \subset C_F$, by listing the subsets in such a way that their cardinality is non-decreasing.
      - If $\Lambda' = [F, E]_S$ is an upper sublattice consider $\tau' = (\Lambda', \Gamma', \Theta')$ which is a subconstellation generated by Λ'. That means that under the inclusion $i : \Lambda' \to \Lambda$ we have $\Gamma' = \Gamma \cap \Lambda'$ and $\Theta' = \{$decomposable corank 2 flats in Λ' that get indecomposable in $\Lambda\}$.
        - If τ' is isomorphic to a subconstellation in $\mathcal{L}_{3,0}$ we add τ to P.
        - Else, we consider the poset P'' of all constellations σ of P_τ which are the subconstellation of τ' under the natural embedding of $\sigma \to \tau$.
          - If $H_2(\Sigma(P'')) \simeq 0$ we continue.
          - Else, we add τ to $\mathcal{L}_{3,0}$ and to P_τ and we continue.

The set $\mathcal{L}_{3,\tau}$ once the program terminates consists precisely of the isomorphism classes of subconstellations of τ in $\mathcal{L}_3$ (with the same initial set $\mathcal{L}_{3,0}$). To see this, notice that in both cases, the poset P consists precisely of those subconstellations σ of τ for which the order complex Σ_σ has non-trivial H_1 if we exclude σ from the poset.

Using SageMath and its Matroid Theory library we implement the above algorithm. The implementation is given in Appendix A.2.

9 Strategy for proving the second homology theorem in a special case

This is the final section of the thesis and we lustrate a potential proof of a second homology theorem in the special case.

In [6] an outline of an alternative proof of the homotopy theorem in a special case is presented. Using the analogous reasoning we show how the second homology theorem might be proved in a special case by reducing the situation of general matroids to matroids of rank $r \leq 4$. The terms we use in this section are defined in [6].

Let $\tau = (\Lambda, \Gamma, \{\})$ be a constellation of rank r such that $\Gamma \neq \Lambda$. Then the poset $P'' = \Lambda - \Gamma$ is shellable and, in particular, pure by [6, 11.10 (iv)]. By definition the dimension of $\Sigma(P'')$ is $r-1$ (this holds true because the dimension of the order complex is one less than the length of the maximal chain in the poset, and maximal chain in $\Lambda - \Gamma$ has length r becasue $\Gamma \neq \Lambda$ and the top element of Λ is not in P''.) Because P'' is a pure poset, the order complex $\Sigma(P'')$ is a $(r-1)-$dimensional shellable completely balanced complex [6, p. 1858] and $(r-2)-$connected [6, p. 1854].

Pure shellable complexes are constructible [6, p. 1854] and constructible complexes are homotopy-CM [6, p. 1855]. Thus P'' is a pure $(r-1)-$dimensional completely balanced complex which is homotopy-CM, therefore by [6, Theorem 11.14] we have that $\Sigma(P'')_{(J)}$ is $(|J|-2)-$connected where $J \subset \{1, 2, \ldots, r-2\}$ (the definition of induced subcomplex is given in [6, p. 1858]). What is important for our purposes is taking $J = \{1, 2, 3, 4\}$, that is the poset of the top 4 levels of P''. Let us denote this poset by P', we have that $\Sigma(P')$ is $2-$connected. This means that $\Sigma(P')$ is connected and $\pi_1(\Sigma(P'))$ and $\pi_2(\Sigma(P'))$ are trivial, therefore by Hurewicz theorem, $\widetilde{H}_i(\Sigma(P')) \simeq 0$, for $i \in \{0, 1, 2\}$ where $\widetilde{H}_i$ are the reduced homology groups.

The most important tool for the next step is relating the topology of the poset $\mathcal{X}_3^\tau$ to the poset P', we will discuss which classes of subconstellations are in $\mathcal{X}_3^\tau$ later. Let P denote the poset on the same set as poset P' but the relations are reversed. The simplicial complexes $\Sigma(P)$ and $\Sigma(P')$ are the same.

Define a map $f : \mathcal{X}_3^\tau \to P$ by

$$\{\sigma_1 < \cdots < \sigma_i\} \xrightarrow{f} \text{the bottom flat of } \sigma_i.$$

The map f is an order preserving map. By Theorem 6.4 in [4] we know that if for every $F \in P$ and every $0 \leq i \leq 2$ the reduced homology groups of the fiber $\widetilde{H}_i(\Sigma(f^{-1}(P_{\geq F})))$ is trivial, then the groups $\widetilde{H}_i(\Sigma(\mathcal{X}_3^\tau))$ and $\widetilde{H}_i(\Sigma(P))$ are isomorphic for $0 \leq i \leq 2$. Because $\widetilde{H}_i(\Sigma(P))$ is trivial for $0 \leq i \leq 2$ this would imply that $\widetilde{H}_2(\Sigma(\mathcal{X}_3^\tau)) \simeq H_2(\Sigma(\mathcal{X}_3^\tau))$ is trivial as well, i.e. we have the second homology theorem.

We list assumptions for the above reasoning to work.

1. The classes of constellations that we insert in $\mathcal{X}_3^\tau$ have rank at most 4, otherwise the map f is not well-defined.
2. The fibers have trivial $\widetilde{H}_i(\Sigma(f^{-1}(P_{\geq F})))$ for all $F \in P$, and $0 \leq i \leq 2$.

The second condition implies that we have already proved the second homology theorem for matroids of rank 2, 3 and 4 because the fiber $f^{-1}(P_{\geq F}))$ consists of all subconstellations above a given flat in Λ of corank ≤ 4.

Another restriction that the second condition imposes is that there are no decomposable flats in $\Lambda - \Gamma$. If not, and there is such a flat F, the fiber of F would consist of a constellation which is disconnected by extended version of path theorem from [2], hence $\widetilde{H}_0(f^{-1}(P_{\geq F})) \not\simeq 0$. We can show that there are no decomposable flats in $\Lambda - \Gamma$ if an only if there are no decomposable corank 2 flats in $\Lambda - \Gamma$. For the non-trivial direction, assume that F is a decomposable flat in Λ, so $M/F = M_1 \oplus M_2$ is a disconnected matroid. By definition of the direct sum of matroids notice that corank 2 flats of M/F of the form $H_1 \cup H_2$ where H_i is a hyperplane of M_i have precisely two hyperplanes of M/F above it, so they are disconnected corank 2 flats.

Therefore, it is enough that there are no decomposable corank 2 flats in $\Lambda - \Gamma$ for the argument to work.

Lastly, the second condition also implies that we would like classes $2a$–$2c$ ($2d$ is excluded because it has disconnected corank 2 flats) in $\mathcal{X}_3^\tau$, to guarantee $\widetilde{H}_1(f^{-1}(P_{\geq F}))$ is trivial for all F, by Theorem 4.3.

To conclude, for any constellation with empty Θ, no decomposable corank 2 flats and only rank 4 constellations in $\mathcal{X}_3^\tau$ we can reduce the proof of the second homology theorem for general rank r to the rank ≤ 4.

A Appendices

A.1 Geometric lattices

For the appendix A.1 we follow the definitions and theorems of Chapter 1.7 of [10].

Definition A.1. A partially ordered set or a poset is a pair $(P, \leq)$ where P is a set and $\leq$ is a relation on P (meaning a subset of $P \times P$) such that:

1. We have $x \leq x$.
2. If $x \leq y$ and $y \leq z$ then $x \leq z$.
3. If $x \leq y$ and $y \leq x$ then $x = y$.

Posets P_1 and P_2 are isomorphic if there exists a bijection $f : P_1 \to P_2$ with the property that if $x, y \in P_1$ and $x \leq y$ then $f(x) \leq f(y)$.

If $x \leq y$ and $x \neq y$ we write $x < y$. If $x < y$ and there exists no z with $x < z$ and $z < y$ we say that y covers x. If there exists an element $0 \in P$ with the property that $0 \leq x$ for all $x \in P$ we call it a zero of P. An atom is an element of P covering 0. A chain in a finite poset is a subset $\{x_1, \ldots, x_k\}$ of P if $x_1 < \ldots < x_k$ and it is maximal if x_i covers x_{i-1} for all i. If for all a, b in a finite poset all of the maximal chains from a to b have the same length, then P satisfies the Jordan-Dedekind chain condition. If P is a finite poset having a 0 then for any x we define the height of x or $h(x)$ as the maximum length of a chain from 0 to x.

Given a poset P and $x, y \in P$ an interval $[x, y]$ is the set

$$[x, y] = \{z \ : \ x \leq z \leq y\} .$$

If $x, y \in P$ and S is a set of elements covering x such that $y = \vee_{s \in S} s$ then $[x, y]_S$ is the set

$$[x, y]_S = x \cup \left\{\vee_{s \in S'} s \ : \ S' \subset S\right\} .$$

Definition A.2. Let $(P, \leq)$ be a poset and $x, y \in P$. If they exist, we define the greatest lower bound or join of x and y denoted by $x \wedge y$ and least upper bound or meet of x and y denoted by $x \vee y$ as:

1. We have $x \wedge y \leq x$ and $x \wedge y \leq y$. If $z \leq x$ and $z \leq y$ then $z \leq x \wedge y$.
2. We have $x \leq x \vee y$ and $y \leq x \vee y$. If $x \leq z$ and $y \leq z$ then $x \vee y \leq z$.

We see that if meet or a join exists for a certain pair, then it is unique. A lattice is a poset for which join and meet exist for every pair of elements.

Definition A.3. A finite lattice is called semimodular if it satisfies the Jordan-Dedekind chain condition and for all x, y we have $h(x \wedge y) + h(x \vee y) \leq h(x) + h(y)$.

Definition A.4. A finite lattice is called atomistic if every element is a join of atoms.

Definition A.5. A finite lattice is called geometric if it is semimodular and atomistic.

Theorem A.1. A lattice is geometric if and only if it is isomorphic to a lattice of flats of a matroid.

Proof. The details of the proof can be found in [10]. Given a matroid $M = (M, \mathcal{F})$ we can prove that the poset $(\mathcal{F}, \subseteq)$ satisfies the conditions of a geometric lattice. In the reverse direction, given a geometric lattice Λ we define E to be the set of atoms. For $x \in \Lambda$ we define $F_x = \{a \in E \ : \ a \leq x\}$. We prove that the collection $\mathcal{F} = \{F_x \ : \ x \in \Lambda\}$ satisfies the conditions (F1), (F2) and (F3). □

If the matroids M_1 and M_2 are simple, then they are isomorphic if and only if they have isomorphic lattices of flats. Otherwise, a matroid M has isomorphic lattice of flats to the lattice of flats of its simplification.

Let P and Q be posets. Then $P \times Q$ can be made into a poset with $(x_1, y_1) \leq (x_2, y_2)$ if $x_1 \leq x_2$ and $y_1 \leq y_2$.

A simple matroid M is disconnected if and only if $\Lambda(M) \simeq \Lambda_1 \times \Lambda_2$ as poset where $|\Lambda_i| > 1$ for both i.

A.2 Computer program

```python
def getting_labels(geometric_lattice):
    """
    input: geometric lattice
    output: a list containing flats of the geometric lattice but the bottom flat is empty and atoms are singletons
    """
    new_lattice = []

    atoms =  geometric_lattice.atoms()

    for flats in geometric_lattice:
        new_thing = ''
        for atom in atoms:
            if geometric_lattice.is_lequal(atom, flats):
                new_thing += str(atoms.index(atom)+1)

        new_lattice.append(new_thing)

    return new_lattice


def simplification(geometric_lattice, linear_class, old_theta):
    """
    input: geometric lattice, linear class (list), theta (list)
```

```
    output: simplification of all three things
    """
    new_lattice = [frozenset()]
    new_linear_class = ([]).copy()
    new_theta = ([]).copy()

    atoms =  geometric_lattice.atoms()

    for flats in geometric_lattice:
        new_thing = []
        for atom in atoms:
            if geometric_lattice.is_lequal(atom, flats):
                new_thing.append(atoms.index(atom))

        new_thing = frozenset(new_thing)
        new_lattice.append(new_thing)
    for hyperplanes in linear_class:
        new_thing = []
        for atom in atoms:
            if geometric_lattice.is_lequal(atom, hyperplanes):
                new_thing.append(atoms.index(atom))
        new_thing = frozenset(new_thing)
        new_linear_class.append(new_thing)

    for thetass in old_theta:
        new_thing = []
        for atom in atoms:
            if geometric_lattice.is_lequal(atom, thetass):
                new_thing.append(atoms.index(atom))
        new_thing = frozenset(new_thing)
        new_theta.append(new_thing)


    new_lattice_poset = LatticePoset((new_lattice, lambda x, y: x < y))

    return new_lattice_poset, new_linear_class, new_theta


def tikz_magic(marked):
    """
    input: a tuple of a marked constellations
    output: tikz code for the lattice
    """
    M, gamma, theta = simplification(marked[0].lattice_of_flats(), marked[1], marked[2])

    n = M.rank()

    flats_stratified = [[] for k in range(n+1)]

    for x in M:
        flats_stratified[M.rank(x)].append(x)


    my_labels_list = getting_labels(M)

    element_labelsf = {}
```

```python

for x in M:
    element_labelsf[x] = my_labels_list[[y for y in M].index(x)]

print('\\begin{{tikzpicture}}[N/.style = {{inner sep = 2pt}}]'.format())

for k in range(n+1):
    g =  flats_stratified[k]
    m = len(g)
    s = -(m-1)/2
    for flat in g:
        label = element_labelsf[flat]
        if k == 0:
            if flat not in theta:
                 print(f'\\draw ({s}, {k}) node (n) [N]'.format() +'{{'.
                 format()+f'$\emptyset$'+'}};'.format())
            else:
                print(f'\\draw ({s}, {k}) node (n) [N, fill=gray!90]'.format
                () +'{{'.format()+f'$\emptyset$'+'}};'.format())

        elif k == n:
            print(f'\\draw ({s}, {k}) node ({label}) [N, fill=gray!40]'.
            format() +'{{'.format()+f'{label}'+'}};'.format())

        elif flat in gamma:

            print(f'\\draw ({s}, {k}) node ({label}) [N,fill=gray!40]'.
            format() +'{{'.format()+f'{label}'+'}};'.format())

        elif flat in theta:
             print(f'\\draw ({s}, {k}) node ({label}) [N,fill=gray!90]'.
             format() +'{{'.format()+f'{label}'+'}};'.format())

        else:
            print(f'\\draw ({s}, {k}) node ({label}) [N,]'.format() +'{{'.format()+f'{label}'+'}};'.format())

        s +=1
    big_string = ''
    for flat in M:
        for s in M.upper_covers(flat):
            if flat == frozenset({}):
                label1 = 'n'
            else:
                label1 = element_labelsf[flat]

            label2 = element_labelsf[s]
            big_string+=f'({label1})--({label2})'

print('\draw[gray!25]' + big_string + ';')

print('\\end{{tikzpicture}}'.format())



def computation_3(M, gamma, theta, dimension):
"""
```

```
    implementation of the computer program described in section 8
    ----
    input: a tuple of a marked constellations, with dimension of the homology group
    output: calculation of the homology group of the fixed dimension and tikz code with any exceptional classes of constellations found along the way
    """

    n = M.rank()

    E = M.flats(n)[0]
    L = M.lattice_of_flats()

    poset = ([]).copy()

    counter_1 = 0

    for increasing_corank in IntegerRange(n,-1,-1):
        for F in M.flats(increasing_corank):

            rank_F = L.rank(F)

            covers_of_F = L.upper_covers(F)

            for subset in Subsets(covers_of_F):

                if subset.cardinality() < (n - rank_F):
                    continue

                subset = frozenset(subset).union({frozenset(F)})

                sublattice = L.subjoinsemilattice(subset)

                if not (E in sublattice):
                    continue

                small_gamma = set(gamma).intersection(sublattice)

                for x in vanishing_homology_2:
                    counter_1 = 1

                    if not(x[0].rank() == n - rank_F):
                        continue


                    value = sublattice.is_isomorphic(x[0].lattice_of_flats(),
                    certificate = True)

                    if value[0]:

                        G = x[0].automorphism_group()


                        for elements in G:
                            the_mapping = elements.dict()

                            empty_list = set()
```

```
                            for y in small_gamma:

                                new_list = set()
                                AB = (value[1])[y]

                                for humus in AB:

                                    new_list.add(the_mapping[humus])

                                new_set = frozenset(new_list)
                                empty_list.add(new_set)

                            if empty_list == x[1]:

                                inv_map = {v: k for k, v in value[1].items()}
                                inv_map_mapping = {v: k for k, v in the_mapping.
                                items()}

                                empty_list_theta_1 = set()
                                for humus in x[2]:
                                    new_list = set()
                                    for h in humus:
                                        new_list.add(inv_map_mapping[h])

                                    new_set = frozenset(new_list)
                                    empty_list_theta_1.add(new_set)

                                empty_list_theta = set()
                                for corank_2_haha in empty_list_theta_1:
                                    empty_list_theta.add(inv_map[corank_2_haha])

                                value_for_thetas = 1

                                for z in empty_list_theta:

                                    if (not (z in theta or len(L.upper_covers(z)
                                    ) >2.5 )) or (set(L.upper_covers(z)).
                                    issubset(gamma)):
                                        value_for_thetas = 0
                                        continue

                                if value_for_thetas == 1:
                                    poset.append(sublattice)
                                    counter_1 = 0
                                    continue

                if counter_1 == 1:
                    flats = [new_flats for new_flats in sublattice]
                    L_M = LatticePoset((flats, lambda x, y: x < y))

                    corank2_disconnected2 = ([]).copy()

                    for disconected2 in flats:
                        if M.rank(disconected2) == n-2:
                            if (len(L.upper_covers(disconected2))>2.5) and len(
                            L_M.upper_covers(disconected2)) <2.5 and (not set(L.
                            upper_covers(disconected2)).issubset(gamma)):
```

```
                                corank2_disconnected2.append(disconected2)

                    modified_lattice, modified_linear_class, modified_theta =
                    simplification(L_M, small_gamma, corank2_disconnected2)

                    new_matroid = Matroid(modified_lattice)

                    small_poset = [].copy()

                    for thing_in_poset in poset:
                        my_third_counter = 1
                        for things_in_things in thing_in_poset:
                            if not (things_in_things in flats):
                                my_third_counter = 0
                                continue

                        if my_third_counter == 1:
                            small_poset.append(thing_in_poset)

                    P2 = Poset((small_poset, lambda x, y: x.is_induced_subposet(
                    y)))

                    cplx = P2.order_complex()

                    homologosss = cplx.homology(dim = dimension)

                    if (homologosss).order() >1.5 and not (sublattice.
                    is_isomorphic(L)):
                        print(homologosss)


                        poset.append(sublattice)
                        counter_ = 1
                        vanishing_homology_2.append((new_matroid, set(
                        modified_linear_class), set(modified_theta)))
                        print()
                        tikz_magic((new_matroid, set(modified_linear_class), set
                        (modified_theta)))
                        print()


                        continue

    actual_poset = Poset((poset, lambda x, y: x.is_induced_subposet(y)))

    cplx = actual_poset.order_complex()

    homology = cplx.homology(dim = dimension)


    print('Final matroid for this case')
    print()
    tikz_magic((M, gamma, theta))
    print()

    return homology

```

```

vanishing_homology_uniform = [
                    (matroids.Uniform(1,1), set(), set()),
                    (matroids.Uniform(2,2), set(), set([frozenset({})])),
                    (matroids.Uniform(2,3), set(), set()),
                    (matroids.Uniform(3,3), set(), set([frozenset({2}),
                    frozenset({1}), frozenset({0})])),
 ]

vanishing_homology_mk23 = [
                    (matroids.Uniform(1,1), set(), set()),
                    (matroids.Uniform(2,2), set(), set([frozenset({})])),
                    (matroids.Uniform(2,3), set(), set()),
                    (matroids.Uniform(3,3), set(), set([frozenset({2}),
                    frozenset({1}), frozenset({0})])),
                    (matroids.Uniform(3,4), {frozenset({1, 2}), frozenset({0,
                    3})}, set()),
                    (Matroid(graphs.CompleteBipartiteGraph(2, 3)), {frozenset
                    ({(0, 2), (0, 3), (1, 4)}), frozenset({(1, 3), (0, 2), (0,
                    4)}), frozenset({(1, 2), (0,
                    4), (0, 3)}), frozenset({(1, 2), (1, 3), (1, 4)})}, set())
]

vanishing_homology_2 = vanishing_homology_uniform.copy()

print(computation_3(matroids.Uniform(3,4), {frozenset({1, 2}), frozenset({0, 3})
}, set(), 1))
```

Listing 1: All of the programming involved

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