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Random function iteration and the Cantor function

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Abstract

We examine a random function iteration that gives rise to the Cantor function. We focus on a specific case, where two linear functions $f_0(x) = 3x$ and $f_1(x) = 3x - 2$ are selected independently with equal probability to define a random sequence starting from an initial point $x_0 \in [0, 1]$. The function $\mathbf{P}(x_0)$ is defined as the probability that the resulting sequence diverges to infinity. Remarkably, plotting $\mathbf{P}(x_0)$ reveals a graph that visually resembles the classical Cantor function. The primary objective of this thesis is to explore this construction rigorously and prove that $\mathbf{P}(x)$ coincides with the standard Cantor function. To the author's knowledge, a formal proof of this equivalence has not yet appeared in the literature. Additionally, we extend the model by varying the probability distribution used to select between the two functions, leading to a family of generalized Cantor-like functions.

Contents

1 Introduction

In many mathematical frameworks, the evolution of a system over time is modeled by the repeated application of a fixed transformation or function. This forms the basis of deterministic dynamical systems, where the future state of the system is entirely determined by its current state and the governing rule. However, in many real-world scenarios, the evolution of a system is not entirely predictable and presents uncertainty at each step. Random function iteration is a mathematical framework that captures this idea by allowing the system to apply a randomly chosen function at each step, rather than the same one every time.

In this project, we focus on a particular example of random function iteration and we prove that it gives rise to the Cantor function. The Cantor function is a popular object in analysis due to its property of being continuous, but not absolutely continuous. It challenges our understanding of continuity, derivative and measure, as it is flat almost everywhere (i.e. its derivative is equal to zero almost everywhere), but also increasing. It was first introduced by Georg Cantor in 1884 (see [2]) and has been studied ever since. There are multiple constructions of this object, for example, via ternary expansion, or via function iteration using a specific recurrence relation (as done in [4]). More details about these constructions will be given in Section 2, but before that, we provide an image with the graph of the Cantor function.

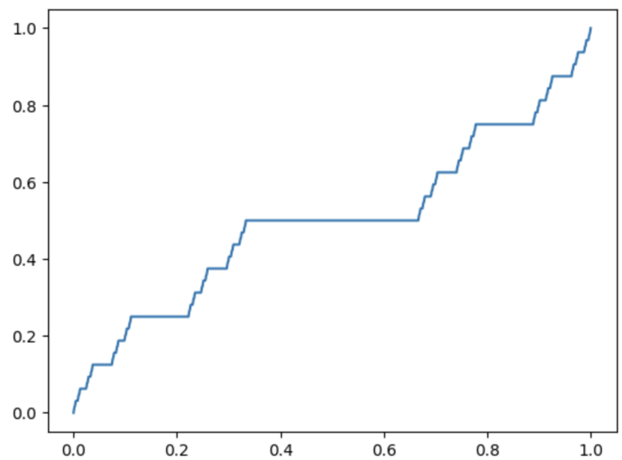

Figure 1: The Cantor function, created with the code in the Appendix.

In [1], the authors describe a way to generate a function that resembles the Cantor function through random function iteration. In this construction, two functions $f_0, f_1 : \mathbb{R} \to \mathbb{R}$ are defined: $f_0(x) = 3x$ and $f_1(x) = 3x - 2$. Starting from a real number x_0, a stochastic sequence $(x_n)_{n\geq 1}$ is built by repeatedly applying either f_0 or f_1, each chosen with equal probability at each step. The function $\mathbf{P}(x_0)$ is then defined as the probability that this sequence diverges to infinity. When this probability is plotted as a function of x_0 on the interval $[0, 1]$, the result closely resembles the Cantor function.

The main goal of this thesis is to explore this construction in detail and to prove that the resulting function $x \mapsto \mathbf{P}(x)$ is truly equivalent to the standard Cantor function. To the author's knowledge, such a proof is currently missing from the literature. Additionally, we present a generalization of this iteration, obtained by changing the probability distribution, while keeping the two functions f_0 and f_1. It appears that in this case, Cantor-like functions emerge and will be studied in Section 4.

2 The Cantor Function

In this section we give an overview of the constructions of the Cantor function, as well as the characterizations by means of functional equations. For the first two subsections, we follow the descriptions provided in [4].

2.1 Construction via ternary expansion

The first and most popular construction of the Cantor function is the one using ternary expansion. More precisely, for any $x \in [0,1]$, we expand it in base 3 as

$$x = \sum_{k=1}^{\infty} \frac{a_{kx}}{3^k}, \quad a_{kx} \in \{0,1,2\}. \tag{1}$$

Define $n(x)$ to be the smallest index k such that $a_{kx} = 1$, if such an index exists. If no such index exists, set $n(x) = \infty$. In other words, $n(x) := \inf\{k \in \mathbb{N} : a_{kx} = 1\}$ (with $\inf \varnothing = \infty$). The Cantor function $C : [0,1] \to \mathbb{R}$ is then defined by:

$$C(x) := \frac{1}{2^{n(x)}} + \frac{1}{2} \sum_{k=1}^{n(x)-1} \frac{a_{kx}}{2^k}. \tag{2}$$

Observe that C is well defined, as it does not depend on the choice of representation in base 3, if x has 2 of them (it cannot have more and it has at least one).

The Cantor set $\mathbf{C} \subset [0,1]$ consists of all points whose ternary expansion (1) uses only the digits 0 and 2. For $x \in \mathbf{C}$, i.e., when $n(x) = \infty$, the definition (2) simplifies to

$$C(x) = \frac{1}{2} \sum_{k=1}^{\infty} \frac{a_{kx}}{2^k}. \tag{3}$$

Remark 1. ([8]) It is worth mentioning the classical definition of the Cantor set as well. Start from the set $C_0 := [0,1]$ and remove the open middle-third interval, obtaining $C_1 := [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. Doing the same again for each of the 2 intervals of C_1 yields $C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$. Continuation of this process of deleting the open middle thirds of the intervals leads to a nested sequence (C_n) such that every C_n is the union of 2^n intervals, each of length $\frac{1}{3^n}$. Define the Cantor set to be $\mathbf{C} := \bigcap_{i=1}^{\infty} C_n$. Since each C_n is closed, it follows that $\mathbf{C}$ is closed. Moreover, the map $C \restriction_{\mathbf{C}}$ is surjective (any $y \in [0,1]$ can be written in base 2, say $y = \sum_{k=1}^{\infty} b_{ky} 2^{-i}$ with $b_{ky} \in \{0,1\}$; choosing x such that $a_{kx} = 2b_{ky}$ ensures $C(x) = y$), hence $\mathbf{C}$ is uncountable.

2.2 Construction via function iteration

Alternatively, one can define the Cantor function as the limit of a sequence of functions in the Banach space $\mathcal{B}([0,1]) := \{f : [0,1] \to \mathbb{R} \text{ bounded}\}$, with respect to the supremum norm, $\|f\|_\infty := \sup\{|f(x)| : x \in [0,1]\}$.

Start from any function $f_0 \in \mathcal{B}([0,1])$. Furthermore, define the iteration:

$$f_{n+1}(x) = \begin{cases} \frac{1}{2} f_n(3x) & \text{if } 0 \le x \le \frac{1}{3}, \\ \frac{1}{2} & \text{if } \frac{1}{3} < x < \frac{2}{3}, \\ \frac{1}{2} + \frac{1}{2} f_n(3x-2) & \text{if } \frac{2}{3} \le x \le 1. \end{cases}$$

Then, the following result is true:

Theorem 2.1. The sequence (f_n) converges to the Cantor function C, which is the unique element in $\mathcal{B}([0,1])$ that satisfies:

$$C(x) = \begin{cases} \frac{1}{2} C(3x) & \text{if } 0 \le x \le \frac{1}{3}, \\ \frac{1}{2} & \text{if } \frac{1}{3} < x < \frac{2}{3}, \\ \frac{1}{2} + \frac{1}{2} C(3x-2) & \text{if } \frac{2}{3} \le x \le 1. \end{cases} \tag{4}$$

Proof. We do not give the full proof of this result (see [4] for one), but we are concerned with proving that the Cantor function, as we have defined it in the previous section, truly satisfies (4).

Let $x \in [0,1]$. First of all, notice that, as multiplication by 3 shifts the digits from the base 3 representation of x to the left, we have:

$$n(3x) = n(3x-2) = n(x) - 1. \tag{5}$$

Due to the same reason, for any representation of x in base 3, we can construct one for $3x$, where the k^{th} digit of the latter is the $(k+1)^{\text{th}}$ digit of the former.

Next, we expand x as in (1) and distinguish three separate cases:

Case 1: If $x \in [0, \frac{1}{3}]$, then $a_{1x} \in \{0,1\}$.

If $a_{1x} = 1$, then $n(x) = 1$ and $a_{kx} = 0$, for any $k \geq 1$ (otherwise $x > \frac{1}{3}$). Thus $x = \frac{1}{3}$ and $C(x) = \frac{1}{2}$. Additionally, 1 can be written as $0.222\ldots$ in base 3, so

$$C(1) = \frac{1}{2}\sum_{k=1}^{\infty} \frac{2}{2^k} = \sum_{k=1}^{\infty} \frac{1}{2^k} = 1 = 2C\left(\frac{1}{3}\right).$$

If $a_{1x} = 0$, then, due to (5) and the observation stated after (5) we obtain:

$$\sum_{k=1}^{n(3x)-1} \frac{a_{k(3x)}}{2^k} = 2\sum_{k=1}^{n(x)-2} \frac{a_{(k+1)x}}{2^{k+1}} = 2\sum_{k=1}^{n(x)-1} \frac{a_{kx}}{2^k}. \tag{6}$$

Hence, combining (6) with $2^{-n(3x)} = 2 \cdot 2^{-n(x)}$ (due to (5)) yields $C(3x) = 2C(x)$, by the definition of C. In other words, C satisfies (4) in this case.

Case 2: If $x \in [\frac{1}{3}, \frac{2}{3}]$, then $a_{1x} \in \{1,2\}$.

If $a_{1x} = 1$, then $C(x) = \frac{1}{2}$, as explained before.

If $a_{1x} = 2$, then $a_{kx} = 0$, for any $k \geq 1$ (otherwise $x > \frac{2}{3}$). Thus:

$$C(x) = \frac{1}{2}\sum_{k=1}^{\infty} \frac{a_{kx}}{2^k} = \frac{1}{2}.$$

Hence (4) holds in this case as well. We are now left with one case left to check.

Case 3: If $x \in [\frac{2}{3}, 1]$, then $a_{1x} = 2$. Using (5) and the observation after (5) again yields:

$$\sum_{k=1}^{n(3x-2)-1} \frac{a_{k(3x-2)}}{2^k} = 2\sum_{k=1}^{n(x)-2} \frac{a_{(k+1)x}}{2^{k+1}} = 2\sum_{k=1}^{n(x)-1} \frac{a_{kx}}{2^k} - 2. \tag{7}$$

Therefore, combining (7) with $2^{-n(3x-2)} = 2 \cdot 2^{-n(x)}$ (due to (5)) yields:

$$C(3x-2) = \frac{1}{2^{n(x)-1}} + \frac{1}{2}\left(2\sum_{k=1}^{n(x)-1} \frac{a_{kx}}{2^k} - 2\right) = 2C(x) - 1.$$

Consequently, (4) holds true in all possible cases, which is exactly what we wanted to prove. □

2.3 Functional equations

Another way to study the Cantor function is by finding a functional equation with unique solution that has the Cantor function as a solution. Before proceeding, we introduce the following notation:

Definition 2.1. Let $f : [0,1] \to \mathbb{R}$ be arbitrary. Then define $\hat{f} : \mathbb{R} \to \mathbb{R}$ as

$$\hat{f}(x) = \begin{cases} 0 & \text{if } x < 0, \\ f(x) & \text{if } x \in [0,1], \\ 1 & \text{if } x > 1. \end{cases}$$

Definition 2.2. For any $f : \mathbb{R} \to \mathbb{R}$, define $\tilde{f}$ to be the restriction of f to the interval $[0,1]$, i.e. $\tilde{f} := f \restriction_{[0,1]}$.

In [3], Chalice gives an interesting characterization of the Cantor function, one which we will use to establish that the process described in the beginning truly gives rise to the Cantor function.

Theorem 2.2. Any nondecreasing function $f : [0,1] \to \mathbb{R}$ which satisfies $f(\frac{x}{3}) = \frac{f(x)}{2}$ and $f(1-x)+f(x) = 1$ for any $x \in [0,1]$ is the Cantor function.

However, in [11], Sumi provides another definition of the (extended) Cantor function via a functional equation. This proves to be another strategy that can be used to achieve the goal in the introduction, provided that the following definition matches our definition of the Cantor function.

Definition 2.3. A bounded function $f : \mathbb{R} \to \mathbb{R}$ is called *extended Cantor* if it satisfies $f \restriction_{(-\infty,0]} \equiv 0$, $f \restriction_{[1,\infty)} \equiv 1$ and, for all $x \in \mathbb{R}$:

$$f(x) = \frac{f(3x) + f(3x-2)}{2}. \tag{$\dagger$}$$

Before relating the two functional equations above, we observe the following symmetry of solutions.

Lemma 2.1. If f satisfies the conditions of Definition 2.3, then g also satisfies these conditions, where $g(x) := 1 - f(1-x)$.

Proof. First of all, suppose $x \leq 0$. Then $1 - x \geq 1$, so $g(x) = 1 - f(1-x) = 0$. Similarly, for $x \geq 1$ we have $1 - x \leq 0$ and thus $g(x) = 1 - f(1-x) = 1$. This means that $g \restriction_{(-\infty,0]} \equiv 0$, and $g \restriction_{[1,\infty)} \equiv 1$.

From the definition of f we deduce that for all $x \in \mathbb{R}$ we have:

$$f(1-x) = \frac{f(3-3x) + f(1-3x)}{2}.$$

Manipulating the above equality yields:

$$1 - f(1-x) = \frac{1 - f(1-(3x-2)) + 1 - f(1-3x)}{2}.$$

Hence, by definition:

$$g(x) = \frac{g(3x) + g(3x-2)}{2}.$$

Moreover, since f is bounded, suppose $|f(x)| \leq M$, $\forall x \in \mathbb{R}$, where M is a positive real number. Then using the triangle inequality yields $|g(x)| = |1 - f(1-x)| \leq 1 + |f(1-x)| \leq 1 + M$. In other words, g is bounded and this concludes the proof. □

The next result establishes the connection between the aforementioned characterizations.

Proposition 2.1. The following are true:

(i) If f satisfies the conditions of Theorem 2.2, then $\hat{f}$ satisfies the conditions of Definition 2.3.

(ii) If f satisfies the conditions of Definition 2.3, then $\tilde{f}$ satisfies the conditions of Theorem 2.2.

Proof. For part (i), suppose that f satisfies the conditions of Theorem 2.2. Then, by plugging $x = 0$ in the first equation one obtains $f(0) = \frac{f(0)}{2}$, so $f(0) = 0$. Then, since $f(0) + f(1) = 1$ (for $x = 0$ in the last equation), we have $f(1) = 1$. Hence, using the definition of the extension, we obtain $\hat{f} \restriction_{(-\infty,0]} \equiv 0$ and $\hat{f} \restriction_{[1,\infty)} \equiv 1$. Furthermore, $0 = f(0) \le f(x) \le f(1) = 1$, for all $x \in [0,1]$ due to nondecreasingness, so we can deduce $0 \le \hat{f}(x) \le 1$, for all $x \in \mathbb{R}$. In other words, $\hat{f}$ is bounded. We are now left to prove that $\hat{f}$ satisfies (†) and we distinguish the following cases:

Case 1. If $x \le 0$, then $\hat{f}(x) = 0$ and $3x - 2 < 3x \le 0$, hence $\hat{f}(3x-2) = \hat{f}(3x) = 0$. Thus (†) is true for $x \le 0$.

Case 2. If $x \in (0, \frac{1}{3})$, then $3x \in [0,1]$ and $3x - 2 \le -1 < 0$, so $\hat{f}(3x-2) = 0$. Also, $\hat{f}(x) = f(x) = f(\frac{3x}{3}) = \frac{\hat{f}(3x)}{2}$, using the properties of f. Thus $\hat{f}(x) = \frac{\hat{f}(3x)+\hat{f}(3x-2)}{2}$ in this case as well.

Case 3. If $x \in [\frac{1}{3}, \frac{2}{3}]$, then $3x \ge 1$ and $3x - 2 \le 0$. Hence $\hat{f}(3x) = 1$ and $\hat{f}(3x-2) = 0$ which implies $\frac{\hat{f}(3x)+\hat{f}(3x-2)}{2} = \frac{1}{2}$. Additionally, using the properties of f yields $f(\frac{1}{3}) = \frac{f(1)}{2} = \frac{1}{2}$ and $f(\frac{2}{3}) = f(1 - \frac{1}{3}) = 1 - f(\frac{1}{3}) = \frac{1}{2}$. Since f is nondecreasing, for any $x \in \left[\frac{1}{3}, \frac{2}{3}\right]$ we obtain:

$$\frac{1}{2} = f\left(\frac{1}{3}\right) \le f(x) \le f\left(\frac{2}{3}\right) = \frac{1}{2}.$$

Thus $\hat{f}(x) = \frac{1}{2}$ for any $x \in \left[\frac{1}{3}, \frac{2}{3}\right]$, so (†) holds in this case as well.

Case 4. If $x \in (\frac{2}{3}, 1)$, then $3x - 2 \in (0,1)$ and $3x > 1 \Rightarrow \hat{f}(3x) = 1$. Also, $1 - x \in (0, \frac{1}{3})$, so $3(1-x) \in (0,1)$. Thus, using the properties of f yields:

$$1 - f(x) = f(1-x) = \frac{f(3(1-x))}{2} = \frac{f(1-(3x-2))}{2} = \frac{1 - f(3x-2)}{2}.$$

Rewrite the above equation as:

$$f(x) = \frac{1 + f(3x-2)}{2}.$$

Thus, by the definition of $\hat{f}$:

$$\hat{f}(x) = \frac{1 + \hat{f}(3x-2)}{2} = \frac{\hat{f}(3x) + \hat{f}(3x-2)}{2},$$

so (†) is true for $x \in (\frac{2}{3}, 1)$.

Case 5. If $x \ge 1 \Rightarrow 3x > 3x - 2 \ge 1$, so $\hat{f}(x) = \hat{f}(3x) = \hat{f}(3x-2) = 1$, so (†) is true for $x \ge 1$.

Therefore (†) holds true for all $x \in \mathbb{R}$ and the first part of the result is proven.

For part (ii), suppose f satisfies the conditions of Definition 2.3.

We know that (†) is true for all $x \in \mathbb{R}$. For any $x \in [0,1]$ we know that $f(x-2) = 0$ and, substituting $x = \frac{x}{3}$ in (†) we deduce that $f(\frac{x}{3}) = \frac{f(x)+f(x-2)}{2}$, hence$\tilde{f}(\frac{x}{3}) = \frac{\tilde{f}(x)}{2}$ for any $x \in [0,1]$.

Next, we know that the (Banach) space $\mathcal{B}(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{R} \text{ bounded}\}$ is complete with respect to the metric $d(f,g) := \|f - g\|_\infty := \sup\{|f(x) - g(x)| : x \in \mathbb{R}\}$. Also, observe that the following subset is closed: $\mathcal{C} := \left\{f \in \mathcal{B}(\mathbb{R}) : f \restriction_{(-\infty,0]} \equiv 0 \text{ and } f \restriction_{[1,\infty)} \equiv 1\right\}$. Thus $\mathcal{C}$ is complete as well (with respect to d).

Let $T : \mathcal{C} \to \mathcal{C}$ be defined by:

$$Tf(x) := \frac{f(3x) + f(3x-2)}{2}.$$

We prove that T is a contraction.

We have $Tf(x) - Tg(x) = \begin{cases} \frac{1}{2}\left(f(3x) - g(3x)\right) & \text{if } x \in [0, \frac{1}{3}], \\ \frac{1}{2}\left(f(3x-2) - g(3x-2)\right) & \text{if } x \in [\frac{2}{3}, 1], \\ 0 & \text{otherwise.} \end{cases}$

Therefore, $\|Tf - Tg\|_\infty \leq \frac{1}{2}\|f - g\|_\infty$, so T is a contraction, which is what we wanted to prove.

This way, since T is a contraction and $\mathcal{C}$ is complete, we can use Banach's contraction principle to obtain that T has a unique fixed point, namely f. Moreover, this principle also provides the means to compute the fixed point, as the limit of the sequence $(f_n)_{n\geq 0} \subset \mathcal{C}$, defined by $f_{n+1} = Tf_n$, for $n \in \mathbb{N}$ and arbitrary starting point $f_0 \in \mathcal{C}$.

Put

$$f_0(x) := \begin{cases} 0 & \text{if } x < 0, \\ x & \text{if } x \in [0, 1], \\ 1 & \text{if } x > 1. \end{cases}$$

Notice that f_0 is a nondecreasing function. Moreover, for any nondecreasing function $g \in \mathcal{C}$, we have that Tg is nondecreasing (suppose $x < y$ are two real numbers $\Rightarrow g(3x) \leq g(3y)$ and $g(3x-2) \leq g(3y-2)$, since g is nondecreasing, hence $Tg(x) \leq Tg(y)$). We deduce that f_n is nondecreasing for any $n \in \mathbb{N}$, which implies that $\lim_{n\to\infty} f_n = f$ is also nondecreasing. Thus $\tilde{f}$ is also nondecreasing.

Finally, Lemma 2.1 and the uniqueness of the fixed point imply that $f(x) = 1 - f(1-x)$, for any $x \in \mathbb{R}$, hence $\tilde{f}(x) + \tilde{f}(1-x) = 1$, for any $x \in [0, 1]$ which concludes the proof.

□

Remark 2. In the previous result, we have also proven that there is a unique function which satisfies Definition 2.3. Using (4), it is clear that $\hat{C}$ satisfies Definition 2.3, so whenever we write *extended Cantor function*, we mean $\hat{C}$.

Moreover, in [10], Sterk provides the means for another characterization, one involving distribution functions. These are right-continuous, nondecreasing functions $F : \mathbb{R} \to [0, 1]$, which satisfy $\lim_{x\to-\infty} F(x) = 0$, and $\lim_{x\to\infty} F(x) = 1$. Although we do not use this result in the paper explicitly, we mention it for completion, adding to the list of functional equations for the (extended) Cantor function.

Theorem 2.3. The extended Cantor function is the unique distribution function $f : \mathbb{R} \to [0, 1]$ which satisfies $f(x) = \frac{f(3x)+f(3x-2)}{2}$ for all $x \in \mathbb{R}$.

Remark 3. The reader might observe the similarity with Definition 2.3. We do not prove this result here, but one can follow the ideas from [10] and combine them with the fact that any function satisfying Definition 2.3 is continuous (as did in Proposition 2.1), to obtain the desired equivalence of the results.

3 Standard Cantor function via Random Iteration

3.1 Construction of the iteration

In this section we present the random process described in the beginning in broad detail, following the ideas from [6]. Firstly, let $\Omega := \{0, 1\}^\infty$. In other words, our sample space is the set of infinite sequences of 0s and 1s. We will equip this space with a σ-algebra and a probability measure.

Start by defining $\Omega_n := \{0, 1\}^n$ for any $n \in \mathbb{N}$. Henceforth, for any $S \in \bigcup_{n=1}^\infty 2^{\Omega_n}$ we will identify $S \times \{0, 1\}^\infty$ as a subset of $\{0, 1\}^\infty$. With this in mind, let $\mathcal{A}_n := \{S \times \{0, 1\}^\infty : S \subseteq \Omega_n\}$. What we have just defined is the collection of events which depend only on the first n choices (it can be proven that this collection is a σ-algebra). Then, $\mathcal{A}_0 := \bigcup_{n=1}^\infty \mathcal{A}_n$ is the collection of events which only depend on finitely many choices.

We are one step away from the definition of the desired σ-algebra, but before that, we prove the following important result:

Proposition 3.1. $\mathcal{A}_0$ is an algebra.

Proof. There are 3 conditions which need to be checked. First of all, $\Omega \in \mathcal{A}_0$ since $\Omega = \{0,1\} \times \{0,1\}^\infty \in \mathcal{A}_1 \subset \mathcal{A}_0$. Second of all, if $A \in \mathcal{A}$, then there exists a positive integer n such that $A = S \times \{0,1\}^\infty$, for some $S \subseteq \Omega_n$. Then, $A^c = (\Omega_n \setminus S) \times \{0,1\}^\infty$, hence $A^c \in \mathcal{A}_n \subset \mathcal{A}_0$. Lastly, let m be a positive integer and suppose that $A_1, \ldots, A_m \in \mathcal{A}_0$. Then, $A_i = S_i \times \{0,1\}^\infty$, for all $i \in \{1, \ldots, m\}$, where $S_i \in \Omega_{n_i}$, for some positive integers $n_1, \ldots, n_m$. Let $M := \max\{n_1, \ldots, n_m\}$ and define the following sets $S_i' := S_i \times \{0,1\}^{M-n_i}$. Identify these sets as subsets of Ω_M and notice that $A_i = S_i' \times \{0,1\}^\infty$. Therefore

$$\bigcup_{i=1}^{m} A_i = \left(\bigcup_{i=1}^{m} S_i\right) \times \{0,1\}^\infty \in \mathcal{A}_M \subseteq \mathcal{A}_0$$

Hence $\mathcal{A}_0$ satisfies all of the necessary conditions for it to be an algebra and the proof is complete.

□

Remark 4. The collection $\mathcal{A}_0$ is not a σ-algebra. It can be seen that the event $\{\omega \in \Omega : \omega_n = 0, \text{ for all } n \in \mathbb{N}\} = \bigcup_{n=1}^{\infty} \{\omega \in \Omega : \omega_n = 0\}$ is not in $\mathcal{A}_0$.

Because we want to measure all events in $\mathcal{A}_n$, for any $n \in \mathbb{N}$, we need a σ-algebra that includes $\mathcal{A}_0$. Therefore, define $\mathcal{A} := \sigma(\mathcal{A}_0)$ to be the σ-algebra generated by $\mathcal{A}_0$. We have thus obtained a measurable space, namely the pair $(\Omega, \mathcal{A})$.

The next step is to equip this space with a probability measure. To this end, we first construct a pre-measure on the algebra $\mathcal{A}_0$ and then use Carathéodory's extension theorem (**Theorem 1** from Lecture 2 in [6]) to obtain the desired measure. As established earlier, any set $A \in \mathcal{A}_0$ is of the form $S \times \{0,1\}^\infty$, where $S \subseteq \Omega_n$ for some positive integer n. Since we are interested in the case where the two functions (f_0 and f_1) are chosen uniformly at random, set $\mu(A) := \frac{|S|}{2^n}$. We now prove that μ is well defined. Suppose that $A = U \times \{0,1\}^\infty = V \times \{0,1\}^\infty$, with $U \in \Omega_n, V \in \Omega_m$ and $n > m$. Thus $U = V \times \{0,1\}^{n-m}$ (identifying the right hand side as a subset of Ω_n), which yields $|U| = |V| \cdot 2^{n-m}$. Consequently, $\mu(U \times \{0,1\}^\infty) = \frac{|U|}{2^n} = \frac{|V| \cdot 2^{n-m}}{2^n} = \frac{|V|}{2^m} = \mu(V \times \{0,1\}^\infty)$, so μ is well defined.

Examples. In order to understand how to work with this construction, we compute the following:

- $\mu(\{\omega_1 = 0\}) = \mu(\{0\} \times \{0,1\}^\infty) = \dfrac{1}{2}$
- $\mu(\{\omega \in \Omega : \omega_1 + \omega_2 = 1\}) = \mu(\{(0,1),(1,0)\} \times \{0,1\}^\infty) = \dfrac{2}{2^2} = \dfrac{1}{2}$
- Let $\varepsilon \in \Omega_n$, so ε is a length n string of zeros and ones. Then:

$\mu(\omega \in \Omega : \omega_i = \varepsilon_i, \forall i \in \{1, \ldots, n\}) = \mu(\{(\varepsilon_1, \ldots, \varepsilon_n)\} \times \{0,1\}^\infty) = \dfrac{1}{2^n}$. In other words, the probability that the first n choices of functions are the ones given by ε is $\dfrac{1}{2^n}$, which is what we need for this model.

Lemma 3.1. The set function $\mu : \mathcal{A}_0 \to [0, \infty]$ is a pre-measure.

Proof. First of all, it is clear that $\mu(\varnothing) = 0$ and $\mu(\Omega) = \mu(\{0,1\} \times \{0,1\}^\infty) = \frac{2}{2} = 1$.

Second of all, μ is finitely additive. Indeed, suppose $A, B \in \mathcal{A}_0$ are disjoint. Then we can write $A = S \times \{0,1\}^\infty$ and $B = T \times \{0,1\}^\infty$, with $S \in \Omega_n$ and $T \in \Omega_m$. Without loss of generality, assume $n \geq m$. Thus, on the one hand, we have:

$$\mu(A) + \mu(B) = \frac{|S|}{2^n} + \frac{|T|}{2^m} = \frac{|S| + 2^{n-m}|T|}{2^n}.$$

On the other hand, let $T' := T \times \{0,1\}^{n-m}$ and identify it as a subset of Ω_n. Notice that $B = T' \times \{0,1\}^\infty$ and $|T'| = 2^{n-m}|T|$. Since A and B are disjoint, we deduce that S and T' are disjoint as well. Hence:

$$\mu(A \cup B) = \mu((S \cup T') \times \{0,1\}^\infty) = \frac{|S \cup T'|}{2^n} = \frac{|S| + |T'|}{2^n} = \frac{|S| + 2^{n-m}|T|}{2^n}.$$

Therefore $\mu(A \cup B) = \mu(A) + \mu(B)$, as the two expressions above match. Therefore, μ is a finitely additive measure. Repeated usage of the previous property yields:

$$\mu\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu(A_i), \text{ for all } n \in \mathbb{N} \text{ and any pairwise disjoint sets } A_1, \ldots, A_n \in \mathcal{A}_0. \tag{8}$$

Next, we prove that if $(A_i)_{i \geq 1}$ is a family of pairwise disjoint sets contained in $\mathcal{A}_0$ and $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}_0$, then $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$. Using the monotonicity under inclusion of μ and (8) gives:

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \geq \mu\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu(A_i), \text{ for any positive integer } n.$$

Sending $n \to \infty$, we obtain

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \geq \sum_{i=1}^{\infty} \mu(A_i). \tag{9}$$

Using (8) again, we obtain:

$$\sum_{i=1}^{\infty} \mu(A_i) \geq \sum_{i=1}^{n} \mu(A_i) = \mu\left(\bigcup_{i=1}^{n} A_i\right), \text{ for any positive integer } n.$$

Taking the limit as n goes to ∞ in the above inequality yields:

$$\sum_{i=1}^{\infty} \mu(A_i) \geq \lim_{n\to\infty} \mu\left(\bigcup_{i=1}^{n} A_i\right). \tag{10}$$

We now prove the following claim, which is the last ingredient we need for this proof.

Claim: $\lim_{n\to\infty} \mu\left(\bigcup_{i=1}^{n} A_i\right) = \mu\left(\bigcup_{i=1}^{\infty} A_i\right).$

Proof of the Claim: Let $B_n := \bigcup_{i=1}^{n} A_i$ and $B := \bigcup_{i=1}^{\infty} A_i$. Observe that $B_n \subseteq B_{n+1} \subseteq B$, for any positive integer n, thus $B_n \uparrow B$. Then, the claim is equivalent to $\lim_{n\to\infty} \mu(B_n) = \mu(B)$, which is equivalent to

$$\lim_{n\to\infty} \mu(B \setminus B_n) = 0.$$

Let $C_n := B \setminus B_n$. This way, we have $C_n \supseteq C_{n+1}$, for any positive integer n and $\bigcap_{i=1}^{\infty} C_n = B \setminus (\bigcup_{i=1}^{\infty} B_i) = \varnothing$, hence $C_n \downarrow \varnothing$. We are now left to prove that $\lim_{n\to\infty} \mu(C_n) = 0$ if $C_n \downarrow \varnothing$. In fact, we prove a stronger statement, namely $C_n \downarrow \varnothing$ implies that there exists an index $N \in \mathbb{N}$, such that $C_N = \varnothing$ and, since the sets are nested, $C_n = \varnothing$, for all $n \geq N$.

Using Tychonoff's theorem (see [12]), we know that $\{0,1\}^{\infty}$ is compact with respect to the product topology ($\{0,1\}$ is endowed with the discrete topology) and the sets C_n are closed with respect to this topology. For the sake of contradiction, suppose that none of the sets is empty. Thus, since the sets are nested, we deduce that the family $(C_n)_{n\geq 1}$ has the finite intersection property. Therefore $\cap_{n=1}^{\infty} C_n \neq \varnothing$ by the finite intersection theorem (**Theorem 26.9** from [5]), which is absurd. Hence there exists an index N such that $C_n = \varnothing$, for $n \geq N$, from which we can deduce that $\lim_{n\to\infty} \mu(C_n) = 0$ and this concludes the proof of the claim. ■

We can now combine (9) and (10) with the claim and obtain $\mu(\bigcup_{i=1}^{\infty} A_i) \geq \sum_{i=1}^{\infty} \mu(A_i) \geq \mu(\bigcup_{i=1}^{\infty} A_i)$, so μ is σ-additive and the lemma is proven. □

As mentioned before, we use Carathéodory's extension theorem to conclude that there exists a unique probability measure $\mathbb{P}$ on $\mathcal{A}$, that coincides with μ on $\mathcal{A}_0$. As noted in the examples before the previous lemma, this probability measure assigns probability $\frac{1}{2^n}$ to any sequence of n choices of functions. Thus, we have managed to construct a probability space suitable for our random iteration, namely $(\Omega, \mathcal{A}, \mathbb{P})$.

Remark 5. The measure we have constructed is the infinite product of Bernoulli measures. In other words, for $\mu : \{0,1\} \to \mathbb{R}$, with $\mu(\omega = 0) = \mu(\omega = 1) = \frac{1}{2}$ we have $\mathbb{P} = \bigotimes_{i=1}^{\infty} \mu$. For more details regarding infinite product measures one may consult [7].

We are ready to describe the events in which we are interested and introduce some new notations.

Definition 3.1. Let $\omega \in \Omega$, $x \in \mathbb{R}$ and n be a positive integer. Then, we use the following notation:

$$x_n = \bigcirc_{i=1}^{n} f_{\omega_i}(x) := f_{\omega_n}(f_{\omega_{n-1}}(\cdots f_{\omega_1}(x)\cdots)).$$

With this, the event that $x_n \to \infty$ as $n \to \infty$, given the first term of the sequence is x, can be described formally as:

$$\{x_n \to \infty, \text{ starting form } x\} = \left\{ \omega \in \Omega : \lim_{n\to\infty} \bigcirc_{i=1}^{n} f_{\omega_i}(x) = \infty \right\}.$$

Whenever we encounter an event concerning the created sequence, we will specify the starting point of the sequence, say x, and in this scenario, whenever we write x_n we mean $\bigcirc_{i=1}^{n} f_{\omega_i}(x)$.

Example. The event that the sequence stays bounded, given it started from x is:

$$\{(x_n)_{n\geq 1} \text{ is bounded, starting from } x\} = \left\{ \omega \in \Omega : \left(\bigcirc_{i=1}^{n} f_{\omega_i}(x) \right)_{n\geq 1} \text{ is bounded} \right\}.$$

3.2 Dynamics

Having constructed the context, we will now explore the properties that it presents. We start by examining how the sequence behaves if the starting point is outside the interval $[0,1]$. This behavior is captured by the following two results.

Proposition 3.2. $\mathbb{P}\left(x_n \to \infty, \text{ starting from } x\right) = 0$ for $x \leq 0$.

Proof. For $x \leq 0$, we have $3x - 2 \leq 3x \leq 0$, so $f_{\omega_i}(x) \leq 0$, for any $i \in \mathbb{N}$ and any $\omega_i \in \{0,1\}$. Thus we can deduce: $\bigcirc_{i=1}^{n} f_{\omega_i}(x) \leq 0$, for any $n \in \mathbb{N}$ and any $\omega_i \in \{0,1\} \Rightarrow \mathbb{P}\left(x_n \leq 0, \text{ for all } n \in \mathbb{N}\right) = 1$. Moreover, if the sequence $(x_n)_{n\geq 1}$ diverges to ∞, then there exists an index $N \in \mathbb{N}$ such that $x_N > 0$. In terms of measures, this means: (in the events below, the starting point of the sequence is x)

$$\mathbb{P}\left(x_n \to \infty\right) \leq \mathbb{P}\left(\exists N \in \mathbb{N} \text{ such that } x_N > 0\right) = 1 - \mathbb{P}\left(x_n \leq 0, \forall n \in \mathbb{N}\right) = 0 \Rightarrow \mathbb{P}\left(x_n \to \infty\right) = 0,$$

as claimed before. □

Proposition 3.3. $\mathbb{P}\left(x_n \to \infty, \text{ starting from } x\right) = 1$ for $x > 1$.

Proof. We prove that $x_n \geq 1 + 3^n(x-1)$ for all $n \in \mathbb{N}$, which is enough to deduce that $\lim_{n\to\infty} x_n = \infty$, as $\lim_{n\to\infty}\left(1 + 3^n(x-1)\right) = \infty$.

We proceed by induction. Since $3x > 3x - 2 = 1 + 3(x-1)$ and $x_1 = 3x$, or $x_1 = 3x - 2$, we observe that $x_1 \geq 1 + 3(x-1)$ and the base case is proven.

Now suppose that $x_n \geq 1 + 3^n(x-1)$ for some $n \in \mathbb{N}$. Then, since $x_{n+1} = 3x_n$, or $x_{n+1} = 3x_n - 2$ and, by the induction hypothesis, $3x_n > 3x_n - 2 \geq 3(1 + 3^n(x-1)) - 2 = 1 + 3^{n+1}(x-1)$, we conclude that $x_{n+1} \geq 1 + 3^{n+1}(x-1)$ and with this, the induction is complete.

Consequently, $\lim_{n\to\infty} x_n = \infty$ if the sequence starts from $x > 1$, so $\mathbb{P}\left(x_n \to \infty, \text{ starting from } x\right) = 1$.

□

Remark 6. The above proposition is also true for $x = 1$, but the proof is more involved and also redundant, as we will see in Section 3.4.

The next result examines what happens when we condition on the first choice of function. More precisely, we are establishing that our measure is shift invariant. To this end, we use the notation $\mathbb{P}(A \mid B) := \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$ to denote the (conditional) probability that the event A happens, given that the event B happened.

Lemma 3.2. $\mathbb{P}(\{x_n \to \infty, \text{ starting from } x\} \mid \{\omega_1 = \varepsilon\}) = \mathbb{P}(x_n \to \infty, \text{ starting from } f_\varepsilon(x))$, for any real number x and any $\varepsilon \in \{0, 1\}$.

Proof. By definition we have:

$$\mathbb{P}(\{x_n \to \infty, \text{ starting from } x\} \mid \{\omega_1 = \varepsilon\}) = \frac{\mathbb{P}(\{x_n \to \infty, \text{ starting from } x\} \cap \{\omega_1 = \varepsilon\})}{\mathbb{P}(\omega_1 = \varepsilon)}.$$

Since $\mathbb{P}(\omega = \varepsilon) = \frac{1}{2}$, the above becomes:

$$\mathbb{P}(\{x_n \to \infty, \text{ starting from } x\} \mid \{\omega_1 = \varepsilon\}) = 2\mathbb{P}\left(\left\{\omega \in \Omega : \lim_{n\to\infty} \bigcirc_{i=1}^{n} f_{\omega_i}(x) = \infty\right\} \cap \{\omega_1 = \varepsilon\}\right).$$

Observe that $\{\omega \in \Omega : \lim_{n\to\infty} \bigcirc_{i=1}^{n} f_{\omega_i}(x) = \infty\} \cap \{\omega_1 = \varepsilon\} = \{\omega \in \Omega : \lim_{n\to\infty} \bigcirc_{i=2}^{n} f_{\omega_i}(f_\varepsilon(x)) = \infty\} \cap \{\omega_1 = \varepsilon\}$, which implies that:

$$\mathbb{P}(\{x_n \to \infty, \text{ starting from } x\} \mid \{\omega_1 = \varepsilon\}) = 2\mathbb{P}\left(\left\{\omega \in \Omega : \lim_{n\to\infty} \bigcirc_{i=2}^{n} f_{\omega_i}(f_\varepsilon(x)) = \infty\right\} \cap \{\omega_1 = \varepsilon\}\right).$$

Moreover, since the events $\{\omega \in \Omega : \lim_{n\to\infty} \bigcirc_{i=2}^{n} f_{\omega_i}(x) = \infty\}$ and $\{\omega_1 = \varepsilon\}$ are independent (and $\mathbb{P}(\omega_1 = \varepsilon) = \frac{1}{2}$), we obtain:

$$\mathbb{P}(\{x_n \to \infty, \text{ starting from } x\} \mid \{\omega_1 = \varepsilon\}) = \mathbb{P}\left(\left\{\omega \in \Omega : \lim_{n\to\infty} \bigcirc_{i=2}^{n} f_{\omega_i}(f_\varepsilon(x)) = \infty\right\}\right). \tag{11}$$

Let $y := f_\varepsilon(x)$. By definition, we can rewrite the right hand side as:

$$\mathbb{P}\left(\left\{\omega \in \Omega : \lim_{n\to\infty} \bigcirc_{i=2}^{n} f_{\omega_i}(y) = \infty\right\}\right) = \mathbb{P}\left(\bigcap_{M=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{P=N}^{\infty} \bigcap_{N \le n \le P} \left\{\omega \in \Omega : \bigcirc_{i=2}^{n+1} f_{\omega_i}(y) > M\right\}\right).$$

Consequently, because the sets are nested:

$$\mathbb{P}\left(\left\{\omega \in \Omega : \lim_{n\to\infty} \bigcirc_{i=2}^{n} f_{\omega_i}(y) = \infty\right\}\right) = \lim_{M\to\infty} \lim_{N\to\infty} \lim_{P\to\infty} \mathbb{P}\left(\bigcap_{N \le n \le P} \left\{\omega \in \Omega : \bigcirc_{i=2}^{n+1} f_{\omega_i}(y) > M\right\}\right). \tag{12}$$

As any finite intersection of sets in $\mathcal{A}_0$ is still in $\mathcal{A}_0$, observe that:

$$A_{M,N,P} := \bigcap_{N \le n \le P} \left\{\omega \in \Omega : \bigcirc_{i=2}^{n+1} f_{\omega_i}(y) > M\right\} \in \mathcal{A}_0.$$

Hence $A_{M,N,P} = S \times \{0,1\}^\infty$ for some $S \subseteq \Omega_{P+1}$, as $A_{M,N,P}$ depends only on the first $P+1$ entries.

Moreover, define:

$$B_{M,N,P} := \bigcap_{N \le n \le P} \left\{\omega \in \Omega : \bigcirc_{i=1}^{n} f_{\omega_i}(y) > M\right\}.$$

Similarly, $B_{M,N,P} \in \mathcal{A}_0$, so we can write $B_{M,N,P} = T \times \{0,1\}^\infty$, for some $T \subseteq \Omega_P$, since $B_{M,N,P}$ depends only on the first P indices.

Next, let $R_0, R_1 : \Omega \to \Omega$ be defined by $R_k(\omega) = (k, \omega_1, \omega_2, \ldots)$, for $k \in \{0,1\}$. Clearly, R_0 and R_1 are injective and notice that:

$$A_{M,N,P} = R_0(B_{M,N,P}) \cup R_1(B_{M,N,P}), \text{with } R_0(B_{M,N,P}) \cap R_1(B_{M,N,P}) = \varnothing.$$

Writing $R_0(B_{M,N,P}) = U \times \{0,1\}^\infty$ and $R_1(B_{M,N,P}) = V \times \{0,1\}^\infty$, with $U, V \subseteq \Omega_{P+1}$ and using the injectivity of R_0 and R_1 yields $|U| = |V| = |T|$.

Therefore, using the above relation yields $|S| = 2|T|$. Then, we have:

$$\mathbb{P}(A_{M,N,P}) = \frac{|S|}{2^{P+1}} = \frac{|T|}{2^P} = \mathbb{P}(B_{M,N,P}).$$

The above equality implies:

$$\lim_{M\to\infty} \lim_{N\to\infty} \lim_{P\to\infty} \mathbb{P}(A_{M,N,P}) = \lim_{M\to\infty} \lim_{N\to\infty} \lim_{P\to\infty} \mathbb{P}(B_{M,N,P}). \tag{13}$$

Retracing our previous steps, but this time for $B_{M,N,P}$, we obtain:

$$\lim_{M\to\infty} \lim_{N\to\infty} \lim_{P\to\infty} \mathbb{P}(B_{M,N,P}) = \mathbb{P}\left(\bigcap_{M=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{P=N}^{\infty} \bigcap_{N\le n\le P} \left\{ \omega \in \Omega : \bigcirc_{i=1}^{n} f_{\omega_i}(y) > M \right\} \right). \tag{14}$$

However, by definition we have:

$$\mathbb{P}\left(\bigcap_{M=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{P=N}^{\infty} \bigcap_{N\le n\le P} \left\{ \omega \in \Omega : \bigcirc_{i=1}^{n} f_{\omega_i}(y) > M \right\} \right) = \mathbb{P}\left(\left\{ \omega \in \Omega : \lim_{n\to\infty} \bigcirc_{i=1}^{n} f_{\omega_i}(y) = \infty \right\} \right). \tag{15}$$

Therefore, combining (11), (12), (13), (14) and (15) finishes the proof.

□

The next result is concerning the boundedness of the sequence and will be useful in the end of the section. In addition, since it is not measure dependent, it will be useful in the general case as well.

Lemma 3.3. $\{(x_n)_{n\ge1}$ is bounded, starting from $x\} = \{(x_n)_{n\ge1} \subset [0,1]$, starting from $x\}$.

Proof. The set on the right hand side is trivially included in the set on the left hand side, so we focus on proving the other inclusion. In fact, we will prove that:

$$\{(x_n)_{n\ge1} \subset [0,1], \text{ starting from } x\}^c \subseteq \{(x_n)_{n\ge1} \text{ is bounded, starting from } x\}^c.$$

If the sequence escapes the interval $[0,1]$, we distinguish two cases:

Case 1: There exists $N \in \mathbb{N}$ such that $x_N > 1$. We prove by induction the following claim:

Claim 1: $x_{N+k} \ge 1 + 3^k(x_N - 1)$, for all $k \in \mathbb{N}$.

Proof of the Claim: Since $3x_N > 3x_N - 2 = 1 + 3(x_N - 1)$, notice that, regardless of the choice of function: $x_{N+1} \ge 1 + 3(x_N - 1)$, which proves the base case of the induction.
For the induction step, suppose that $x_{N+k} \ge 1 + 3^k(x_N - 1)$ for some $k \in \mathbb{N}$. Then, $3x_{N+k} > 3x_{N+k} - 2 \ge 3(1 + 3^k(x_N - 1)) - 2 = 1 + 3^{k+1}(x_N - 1)$. We deduce that $x_{N+k+1} \ge 1 + 3^{k+1}(x_N - 1)$, so the induction step is proven. With this, the induction is complete and so is the proof of the claim. ■

As $x_N - 1 > 0$, we know that $\lim_{k\to\infty} \left(1 + 3^k(x_N - 1)\right) = \infty$. Using *Claim 1* we obtain that $\lim_{k\to\infty} x_{N+k} = \infty$ and, by definition of the limit, that $\lim_{n\to\infty} x_n = \infty$. Hence $(x_n)_{n\geq 1}$ is not bounded, which establishes the inclusion in this case.

Case 2: There exists $N \in \mathbb{N}$ such that $x_N < 0$. Similarly, we prove the following by induction:

Claim 2: $x_{N+k} \leq 3^k x_N$, for all $k \in \mathbb{N}$.

Proof of the Claim: Since $3x_N - 2 < 3x_N$, notice that, regardless of the choice of function: $x_{N+1} \leq 3x_N$, which proves the base case of the induction.
For the induction step, suppose that $x_{N+k} \leq 3^k x_N$ for some $k \in \mathbb{N}$. Then, $3x_{N+k} - 2 < 3x_{N+k} \leq 3^{k+1}x_N$. We deduce that $x_{N+k+1} \leq 3^{k+1}x_N$, which means the induction step is proven and so is the claim. ■

As $x_N < 0$, we know that $\lim_{k\to\infty} 3^k x_N = -\infty$. Using *Claim 2* yields $\lim_{k\to\infty} x_{N+k} = -\infty$ and, by definition of the limit $\lim_{n\to\infty} x_n = -\infty$. Hence $(x_n)_{n\geq 1}$ is unbounded in this case as well, which establishes the desired result.

□

The last result of this subsection establishes that, almost surely, the sequence leaves the interval $[0, 1]$. Consequently, using Lemma 3.3 we deduce that $|x_n| \to \infty$ almost surely.

Proposition 3.4. $\mathbb{P}\left((x_n)_{n\geq 1} \subset [0, 1], \text{ starting from } x\right) = 0$, for all $x \in \mathbb{R}$.

Proof. If $x \in \mathbb{R} \setminus [0, 1]$, then clearly the sequence is not fully contained in the interval $[0, 1]$, so the above probability is indeed 0.

We now examine the case $x \in [0, 1]$. Firstly, we rewrite the event from the statement:

$$\mathbb{P}\left((x_n)_{n\geq 1} \subset [0, 1], \text{ starting from } x\right) = \mathbb{P}\left(\bigcap_{n=1}^{\infty}\left(\bigcap_{m\leq n}\left\{\omega \in \Omega : \bigcirc_{i=1}^{m} f_{\omega_i}(x) \in [0, 1]\right\}\right)\right).$$

Since $\bigcap_{m\leq n}\{\omega \in \Omega : \bigcirc_{i=1}^{m} f_{\omega_i}(x) \in [0, 1]\} \supseteq \bigcap_{m\leq n+1}\{\omega \in \Omega : \bigcirc_{i=1}^{m} f_{\omega_i}(x) \in [0, 1]\}$, the above relation can be rewritten as:

$$\mathbb{P}\left((x_n)_{n\geq 1} \subset [0, 1], \text{ starting from } x\right) = \lim_{n\to\infty} \mathbb{P}\left(\bigcap_{m\leq n}\left\{\omega \in \Omega : \bigcirc_{i=1}^{m} f_{\omega_i}(x) \in [0, 1]\right\}\right).$$

Let $p_n := \mathbb{P}\left(\bigcap_{m\leq n}\{\omega \in \Omega : \bigcirc_{i=1}^{m} f_{\omega_i}(x) \in [0, 1]\}\right)$. In order to find the desired probability, we will study the behavior of the sequence $(p_n)_{n\geq 1}$, as n approaches infinity. We have the following:

Claim. $p_{n+1} \leq \frac{1}{2} \cdot p_n$, for all $n \in \mathbb{N}$.

Proof of the Claim: The proof of this claim relies heavily on the observation that for any $x \in [0, 1]$, either $f_0(x) \notin [0, 1]$, or $f_1(x) \notin [0, 1]$. To see this, note that if $x \in [0, \frac{1}{3}]$, then $3x - 2 \leq -1 < 0$, so $f_1(x) \notin [0, 1]$. Moreover, if $x \in (\frac{1}{3}, 1]$, then $3x > 1$, hence $f_0(x) \notin [0, 1]$. Hence, for any $x \in [0, 1]$, $\mathbb{1}_{[0,\frac{1}{3}]}(x)$ is the index of one of the functions which take x out of the interval $[0, 1]$ ($\mathbb{1}_S$ denotes the indicator function of the set S). Furthermore, if $x_m \in [0, 1]$, for all $m \leq n + 1$, then x_n and x_{n+1} are both in $[0, 1]$. Hence the $(n + 1)^{\text{th}}$ choice of function is so that the next term of the sequence stays inside $[0, 1]$ (in other words, the index of the chosen function cannot be $\mathbb{1}_{[0,\frac{1}{3}]}(x_n)$). We can now deduce:

$$\bigcap_{m\leq n+1}\left\{\omega\in\Omega:\bigcirc_{i=1}^{m}f_{\omega_i}(x)\in[0,1]\right\}\subseteq\left(\bigcap_{m\leq n}\left\{\omega\in\Omega:\bigcirc_{i=1}^{m}f_{\omega_i}(x)\in[0,1]\right\}\right)\bigcap\left\{\omega_{n+1}\neq\mathbb{1}_{[0,\frac{1}{3}]}(x_n)\right\}.$$

Since the events $\bigcap_{m\leq n}\{\omega\in\Omega:\bigcirc_{i=1}^{m}f_{\omega_i}(x)\in[0,1]\}$ and $\left\{\omega_{n+1}\neq\mathbb{1}_{[0,\frac{1}{3}]}(x_n)\right\}$ are independent (the first one depends only on the first n components of ω and the second one depends only on the $(n+1)^{\text{th}}$ component), we obtain:

$$\mathbb{P}\left(\bigcap_{m\leq n+1}\left\{\omega\in\Omega:\bigcirc_{i=1}^{m}f_{\omega_i}(x)\in[0,1]\right\}\right)\leq\mathbb{P}\left(\bigcap_{m\leq n}\left\{\omega\in\Omega:\bigcirc_{i=1}^{m}f_{\omega_i}(x)\in[0,1]\right\}\right)\mathbb{P}\left(\omega_{n+1}\neq\mathbb{1}_{[0,\frac{1}{3}]}(x_n)\right).$$

However, $\mathbb{P}\left(\omega_{n+1}\neq\mathbb{1}_{[0,\frac{1}{3}]}(x_n)\right)=1-\mathbb{P}\left(\omega_{n+1}=\mathbb{1}_{[0,\frac{1}{3}]}(x_n)\right)=1-\frac{1}{2}=\frac{1}{2}$, so the above inequality becomes $p_{n+1}\leq\frac{1}{2}\cdot p_n$, hence the claim is proven. ■

Using the claim successively yields $0\leq p_n\leq\left(\frac{1}{2}\right)^{n-1}\cdot p_1$ and since $\lim_{n\to\infty}\left(\frac{1}{2}\right)^{n-1}\cdot p_1=0$, we obtain $\lim_{n\to\infty}p_n=0$, by the squeeze theorem and this concludes the proof of the proposition.

□

3.3 Symmetry

This section is dedicated to studying the symmetry exhibited by the iteration. This proves to be helpful for connecting the iteration to the Cantor function, as the Cantor function presents a similar underlying symmetry (given by $C(x)+C(1-x)=1$). We start by introducing some new terminology.

Definition 3.2. For $\omega\in\Omega$, let $\overline{\omega}\in\Omega$ be defined by $\overline{\omega}_k=1-\omega_k$, for all $k\in\mathbb{N}$.

Remark 7. A simple computation yields $\overline{\overline{\omega}}_k=1-\overline{\omega}_k=1-(1-\omega_k)=\omega_k$, so $\overline{\overline{\omega}}=\omega$.

Definition 3.3. For $A\in\mathcal{A}$, let $\operatorname{rec}(A):=\{\omega\in\Omega:\overline{\omega}\in A\}$.

The most important properties we notice about this construction are:

Proposition 3.5. The following are true:

1. $\operatorname{rec}\left(\bigcup_{i\in I}A_i\right)=\bigcup_{i\in I}\operatorname{rec}(A_i)$ for any countable index set I and any sets $A_i\in\mathcal{A}$;
2. $\operatorname{rec}(\operatorname{rec}(A))=A$ for all sets $A\in\mathcal{A}$.

Proof. 1. We prove this using double inclusion. First, let $\omega\in\operatorname{rec}\left(\bigcup_{i\in I}A_i\right)\Rightarrow\overline{\omega}\in\bigcup_{i\in I}A_i$, so there exists an index $i_\star\in I$ such that $\overline{\omega}\in A_{i_\star}\Rightarrow\omega\in\operatorname{rec}(A_{i_\star})\Rightarrow\omega\in\bigcup_{i\in I}\operatorname{rec}(A_i)$. This settles the inclusion $\operatorname{rec}\left(\bigcup_{i\in I}A_i\right)\subseteq\bigcup_{i\in I}\operatorname{rec}(A_i)$. For the other inclusion, let $\omega\in\bigcup_{i\in I}\operatorname{rec}(A_i)$, so there is an index $i^\star\in I$ such that $\omega\in\operatorname{rec}(A_{i^\star})\Rightarrow\overline{\omega}\in A_{i^\star}\Rightarrow\overline{\omega}\in\bigcup_{i\in I}A_i\Rightarrow\omega\in\operatorname{rec}\left(\bigcup_{i\in I}A_i\right)$. Thus, the second inclusion also holds and this finishes the proof of the first part.

2. Similarly, let $\omega\in\operatorname{rec}(\operatorname{rec}(A))$. Then, by definition we have $\overline{\omega}\in\operatorname{rec}(A)$, which implies $\overline{\overline{\omega}}\in A$, but since $\overline{\overline{\omega}}=\omega$ (by Remark 7), we get $\omega\in A$. Hence $\operatorname{rec}(\operatorname{rec}(A))\subseteq A$. For the other inclusion, pick an arbitrary $\omega\in A\Rightarrow\overline{\overline{\omega}}\in A\Rightarrow\overline{\omega}\in\operatorname{rec}(A)\Rightarrow\omega\in\operatorname{rec}(\operatorname{rec}(A))$, thus $A\subseteq\operatorname{rec}(\operatorname{rec}(A))$, and the proof is done. □

The objective of the next result is to prove that $\text{rec} : \mathcal{A} \to \mathcal{A}$ preserves the measure $\mathbb{P}$. We will see that this is not the case in general.

Lemma 3.4. $\mathbb{P}(\text{rec}(A)) = \mathbb{P}(A)$, for any $A \in \mathcal{A}_0$.

Proof. We will first prove the lemma for sets in $\mathcal{A}_0$ and then establish the result in general. Let $A_0 \in \mathcal{A}_0 \Rightarrow \exists n \in \mathbb{N}$ such that $A_0 = S \times \{0,1\}^\infty$, for some $S \subseteq \Omega_n$. Then, $\mathbb{P}(A_0) = \frac{|S|}{2^n}$.

Additionally, $\text{rec}(A_0) := \{\omega \in \Omega : \overline{\omega} \in A_0\} = \{\omega \in \Omega : (\overline{\omega}_1, \ldots \overline{\omega}_n) \in S\} = \{\omega \in \Omega : L(\omega) \in S\} = L^{-1}(S) \times \{0,1\}^\infty$, where $L : \Omega_n \to \Omega_n$ is defined by $L(\omega_1, \ldots, \omega_n) := (1-\omega_1, \ldots, 1-\omega_n)$.

Notice that $L(L(\omega_1, \ldots, \omega_n)) = L(1-\omega_1, \ldots, 1-\omega_n) = (\omega_1, \ldots, \omega_n)$, so L is an involution, which means that it is bijective. Hence $|L^{-1}(S)| = |S|$, which implies that:

$$\mathbb{P}(\text{rec}(A_0)) = \mathbb{P}\left(L^{-1}(S) \times \{0,1\}^\infty\right) = \frac{|L^{-1}(S)|}{2^n} = \frac{|S|}{2^n}.$$

Thus $\mathbb{P}(\text{rec}(A_0)) = \mathbb{P}(A_0)$, so the result holds for sets in $\mathcal{A}_0$.

For the general case, let $A \in \mathcal{A} \setminus \mathcal{A}_0$. Since $\mathcal{A}$ is the σ-algebra generated by $\mathcal{A}_0$, and $\mathcal{A}_0$ is an algebra (by Proposition 3.1), write:

$$A = \bigcup_{i=1}^\infty C_i, \text{ where } C_i \in \mathcal{A}_0 \text{ for all } i \in \mathbb{N}.$$

Without loss of generality, assume that the sets in the above union are pairwise disjoint (otherwise we can define the following pairwise disjoint sets: $C'_1 := C_1$ and $C'_i := C_i \setminus (C_1 \cup, \cdots \cup C_{n-1})$. We have $\bigcup_{i=1}^\infty C_i = \bigcup_{i=1}^\infty C'_i = A$). Therefore, the sets in the family $(\text{rec}(C_i))_{i \geq 1}$ are also pairwise disjoint (if there exists $\omega \in \text{rec}(C_i) \cap \text{rec}(C_j)$, for distinct indices i and j, then, by definition $\overline{\omega} \in C_i \cap C_j$, absurd) and since $\mathbb{P}(\text{rec}(C_i)) = \mathbb{P}(C_i)$, for all $i \geq 1$, we obtain:

$$\sum_{i=1}^\infty \mathbb{P}(\text{rec}(C_i)) = \sum_{i=1}^\infty \mathbb{P}(C_i).$$

By σ-additivity, we have:

$$\mathbb{P}\left(\bigcup_{i=1}^\infty \text{rec}(C_i)\right) = \mathbb{P}\left(\bigcup_{i=1}^\infty C_i\right).$$

Using Proposition 3.5 in the above equality yields:

$$\mathbb{P}\left(\text{rec}\left(\bigcup_{i=1}^\infty C_i\right)\right) = \mathbb{P}(A).$$

Therefore, $\mathbb{P}(\text{rec}(A)) = \mathbb{P}(A)$, which concludes the proof. □

The following result is the bridge between the symmetry of the iteration and the symmetry of the Cantor function. Moreover, since it does not depend on the measure, we can also use in the general case.

Proposition 3.6. Let $\omega \in \Omega$ be arbitrary. Then, for all $n \in \mathbb{N}$ and any $x \in \mathbb{R}$ we have:

$$\bigcirc_{i=1}^n f_{\omega_i}(1-x) + \bigcirc_{i=1}^n f_{\overline{\omega}_i}(x) = 1.$$

Proof. We will use mathematical induction. To prove the statement for the base case $n = 1$, we notice that there are exactly two possibilities for the value of ω_1, namely 0 and 1.

If $\omega_1 = 0$, then $f_{\omega_1}(1-x) + f_{\overline{\omega}_1}(x) = 3(1-x) + 3x - 2 = 1$.

If $\omega_1 = 1$, then $f_{\omega_1}(1-x) + f_{\overline{\omega}_1}(x) = 3(1-x) - 2 + 3x = 1$.

Hence, the base case is proven, since it is true in both scenarios.

For the induction step, suppose that: $\bigcirc_{i=1}^{n} f_{\omega_i}(1-x) + \bigcirc_{i=1}^{n} f_{\overline{\omega}_i}(x) = 1$ for an arbitrary $n \in \mathbb{N}$. Similarly, there are exactly 2 possibilities for ω_{n+1}.

If $\omega_{n+1} = 0$, then
$$\begin{aligned}\bigcirc_{i=1}^{n+1} f_{\omega_i}(1-x) + \bigcirc_{i=1}^{n+1} f_{\overline{\omega}_i}(x) &= 3 \cdot \bigcirc_{i=1}^{n} f_{\omega_i}(1-x) + 3 \cdot \left(\bigcirc_{i=1}^{n} f_{\overline{\omega}_i}(x)\right) - 2 \\ &= 3\left(1 - \bigcirc_{i=1}^{n} f_{\overline{\omega}_i}(x)\right) + 3 \cdot \left(\bigcirc_{i=1}^{n} f_{\overline{\omega}_i}(x)\right) - 2 \\ &= 1.\end{aligned}$$

If $\omega_{n+1} = 1$, then
$$\begin{aligned}\bigcirc_{i=1}^{n+1} f_{\omega_i}(1-x) + \bigcirc_{i=1}^{n+1} f_{\overline{\omega}_i}(x) &= 3 \cdot \left(\bigcirc_{i=1}^{n} f_{\omega_i}(1-x)\right) - 2 + 3 \cdot \left(\bigcirc_{i=1}^{n} f_{\overline{\omega}_i}(x)\right) \\ &= 3\left(1 - \bigcirc_{i=1}^{n} f_{\overline{\omega}_i}(x)\right) - 2 + 3 \cdot \left(\bigcirc_{i=1}^{n} f_{\overline{\omega}_i}(x)\right) \\ &= 1.\end{aligned}$$

We conclude that $\bigcirc_{i=1}^{n+1} f_{\omega_i}(1-x) + \bigcirc_{i=1}^{n+1} f_{\overline{\omega}_i}(x) = 1$, so the induction is complete.

□

From the above proposition, we can deduce that $\lim_{n\to\infty} \bigcirc_{i=1}^{n} f_{\omega_i}(1-x) = \pm\infty$, provided that $\lim_{n\to\infty} \bigcirc_{i=1}^{n} f_{\overline{\omega}_i}(x) = \mp\infty$ and vice versa. Thus, the following results hold true:

Corollary 3.1. $\operatorname{rec}(\{x_n \to \infty,\ \text{starting from } x\}) \subseteq \{x_n \to \infty,\ \text{starting from } 1-x\}^c$.

Corollary 3.2. $\operatorname{rec}(\{x_n \to -\infty, \text{starting from } x\}) \subseteq \{x_n \to \infty, \text{starting from } 1-x\}$.

3.4 Connection to Cantor's function

In this section we connect all the previous results and prove that the process indeed produces the Cantor function. To this end, we will make use of the characterizations from Theorem 2.2 and Definition 2.3.

Theorem 3.1. The function $F : [0,1] \to \mathbb{R}$, defined by $F(x) := \mathbb{P}(x_n \to \infty, \text{starting from } x)$ is the Cantor function.

Proof. We will prove that the above function satisfies all the conditions of Theorem 2.2. For the first requirement, suppose that $0 \le x \le y \le 1$. Then, since the functions f_0 and f_1 are increasing, we have that $\bigcirc_{i=1}^{n} f_{\omega_i}(x) \le \bigcirc_{i=1}^{n} f_{\omega_i}(y)$, for any $\omega \in \Omega$. Therefore, if $\bigcirc_{i=1}^{n} f_{\omega_i}(x) \to \infty$ as $n \to \infty$, then also $\bigcirc_{i=1}^{n} f_{\omega_i}(y) \to \infty$ as $n \to \infty$. We have thus proven the following inclusion of events:

$$\left\{\omega \in \Omega : \lim_{n\to\infty} \bigcirc_{i=1}^{n} f_{\omega_i}(x) = \infty\right\} \subseteq \left\{\omega \in \Omega : \lim_{n\to\infty} \bigcirc_{i=1}^{n} f_{\omega_i}(y) = \infty\right\}.$$

Hence:

$$\mathbb{P}\left(\left\{\omega\in\Omega:\lim_{n\to\infty}\bigcirc_{i=1}^{n}f_{\omega_i}(x)=\infty\right\}\right)\leq\mathbb{P}\left(\left\{\omega\in\Omega:\lim_{n\to\infty}\bigcirc_{i=1}^{n}f_{\omega_i}(y)=\infty\right\}\right).$$

The above inequality reads as $F(x)\leq F(y)$, which means that the function is nondecreasing.

For the second requirement, notice that:

$$\mathbb{P}\left(\left\{\omega\in\Omega:\lim_{n\to\infty}\bigcirc_{i=1}^{n}f_{\omega_i}\left(\frac{x}{3}\right)=\infty\right\}\right)=\mathbb{P}\left(\left\{\omega\in\Omega:\lim_{n\to\infty}\bigcirc_{i=1}^{n}f_{\omega_i}\left(\frac{x}{3}\right)=\infty\right\}\middle|\{\omega_1=0\}\right)\mathbb{P}(\omega_1=0)$$
$$+\mathbb{P}\left(\left\{\omega\in\Omega:\lim_{n\to\infty}\bigcirc_{i=1}^{n}f_{\omega_i}\left(\frac{x}{3}\right)=\infty\right\}\middle|\{\omega_1=1\}\right)\mathbb{P}(\omega_1=1).$$

By Lemma 3.2 we arrive at:

$$\mathbb{P}\left(\left\{\omega\in\Omega:\lim_{n\to\infty}\bigcirc_{i=1}^{n}f_{\omega_i}\left(\frac{x}{3}\right)=\infty\right\}\right)=\mathbb{P}\left(\left\{\omega\in\Omega:\lim_{n\to\infty}\bigcirc_{i=1}^{n}f_{\omega_i}(x)=\infty\right\}\right)\mathbb{P}(\omega_1=0)$$
$$+\mathbb{P}\left(\left\{\omega\in\Omega:\lim_{n\to\infty}\bigcirc_{i=1}^{n}f_{\omega_i}(x-2)=\infty\right\}\right)\mathbb{P}(\omega_1=1).$$

Notice that $\mathbb{P}\left(\{\omega\in\Omega:\lim_{n\to\infty}\bigcirc_{i=1}^{n}f_{\omega_i}(x-2)=\infty\}\right)=0$, due to Proposition 3.2 (as $x-2<0$) and $\mathbb{P}(\omega_1=0)=\frac{1}{2}$. We thus obtain:

$$\mathbb{P}\left(\left\{\omega\in\Omega:\lim_{n\to\infty}\bigcirc_{i=1}^{n}f_{\omega_i}\left(\frac{x}{3}\right)=\infty\right\}\right)=\frac{1}{2}\mathbb{P}\left(\left\{\omega\in\Omega:\lim_{n\to\infty}\bigcirc_{i=1}^{n}f_{\omega_i}(x)=\infty\right\}\right).$$

Rewriting the previous equality in terms of F yields $F(\frac{x}{3})=\frac{F(x)}{2}$, so the second requirement is met as well.

Using the monotonicity of measures under inclusion and Corollary 3.1 we obtain:

$$\mathbb{P}\left(\operatorname{rec}\left(\{x_n\to\infty,\ \text{starting from } x\}\right)\right)\leq\mathbb{P}\left(\{x_n\to\infty,\ \text{starting from } 1-x\}^c\right).$$

The previous inequality is equivalent to:

$$\mathbb{P}\left(\operatorname{rec}\left(\{x_n\to\infty,\ \text{starting from } x\}\right)\right)+\mathbb{P}\left(\{x_n\to\infty,\ \text{starting from } 1-x\}\right)\leq 1.$$

Using Lemma 3.4 we arrive at:

$$\mathbb{P}\left(\{x_n\to\infty,\ \text{starting from } x\}\right)+\mathbb{P}\left(\{x_n\to\infty,\ \text{starting from } 1-x\}\right)\leq 1. \qquad (\star)$$

Thus we are left to prove the reverse inequality in order to finish the proof of the theorem. Observe the following relation of events:

$$\{x_n\to\infty,\ \text{starting from } x\}^c=\{x_n\to-\infty,\ \text{starting from } x\}\bigcup\{x_n \text{ is bounded, starting from } x\}.$$

Since the events on the right hand side are disjoint, we have:

$$\mathbb{P}\left(\{x_n\to\infty,\ \text{starting from } x\}^c\right)=\mathbb{P}\left(x_n\to-\infty,\ \text{starting from } x\right)+\mathbb{P}\left(x_n \text{ is bounded, starting from } x\right).$$

Additionally, $\mathbb{P}\left(x_n \text{ is bounded, starting from } x\right)=\mathbb{P}\left((x_n)_{n\geq0}\subset[0,1],\ \text{starting from } x\right)=0$, using Lemma 3.3 in the first equality and Proposition 3.4 in the second one. Hence, using Lemma 3.4 again and Corollary 3.2 yields:

$$\mathbb{P}\left(\{x_n\to\infty,\ \text{starting from } x\}^c\right)=\mathbb{P}\left(\operatorname{rec}\left(\{x_n\to-\infty,\ \text{starting from } x\}\right)\right)\leq\mathbb{P}\left(x_n\to\infty,\ \text{starting from } 1-x\right).$$

Rewriting the above inequality gives:

$$\mathbb{P}\left(\{x_n \to \infty, \text{ starting from } x\}\right) + \mathbb{P}\left(\{x_n \to \infty, \text{ starting from } 1-x\}\right) \geq 1. \qquad (\star\star)$$

Combining ($\star$) and ($\star\star$) confirms that the last condition of Theorem 2.2 holds true and this finishes the proof of the theorem.

□

Theorem 3.2. The function $F : \mathbb{R} \to \mathbb{R}$, defined by $F(x) := \mathbb{P}\left(x_n \to \infty, \text{starting from } x\right)$ is the extended Cantor function.

Proof. We will prove that F satisfies all the conditions of Definition 2.3. By definition, $0 \leq F(x) \leq 1$, so F is bounded. We proceed by picking an arbitrary $x \in \mathbb{R}$ and conditioning on the first choice, which yields:

$$\begin{aligned}\mathbb{P}\left(\left\{\omega \in \Omega : \lim_{n\to\infty} \bigcirc_{i=1}^{n} f_{\omega_i}(x) = \infty\right\}\right) &= \mathbb{P}\left(\left\{\omega \in \Omega : \lim_{n\to\infty} \bigcirc_{i=1}^{n} f_{\omega_i}(x) = \infty\right\} \middle| \{\omega_1 = 0\}\right) \mathbb{P}\left(\omega_1 = 0\right) \\ &+ \mathbb{P}\left(\left\{\omega \in \Omega : \lim_{n\to\infty} \bigcirc_{i=1}^{n} f_{\omega_i}(x) = \infty\right\} \middle| \{\omega_1 = 1\}\right) \mathbb{P}\left(\omega_1 = 1\right).\end{aligned}$$

Using Lemma 3.2 gives:

$$\begin{aligned}\mathbb{P}\left(\left\{\omega \in \Omega : \lim_{n\to\infty} \bigcirc_{i=1}^{n} f_{\omega_i}(x) = \infty\right\}\right) &= \frac{1}{2}\mathbb{P}\left(\left\{\omega \in \Omega : \lim_{n\to\infty} \bigcirc_{i=1}^{n} f_{\omega_i}(3x) = \infty\right\}\right) \\ &+ \frac{1}{2}\mathbb{P}\left(\left\{\omega \in \Omega : \lim_{n\to\infty} \bigcirc_{i=1}^{n} f_{\omega_i}(3x-2) = \infty\right\}\right).\end{aligned}$$

Since the choice of x was arbitrary, the above equality reads as $F(x) = \frac{F(3x)+F(3x-2)}{2}$, for any $x \in \mathbb{R}$, hence F satisfies (†).

Moreover, Proposition 3.2 implies that $F \restriction_{(-\infty,0]} \equiv 0$. Similarly, Proposition 3.3 implies that $F \restriction_{(1,\infty)} \equiv 1$. Since $F(1) = \frac{F(3)+F(1)}{2}$ and $F(3) = 1$, we deduce that $F(1) = 1$. Therefore, all of the conditions are satisfied and the result is proven.

□

4 Generalized Cantor functions via Random Iteration

In this section we explore what happens if we choose the indices 0 and 1 with unequal probabilities. In the spirit of Remark 5, we can change the measure in the following way. Let $p \in (0,1)$, $q := 1-p$ and $\mu_p : \{0,1\} \to \mathbb{R}$ such that $\mu_p(\omega_1 = 0) = q$ and $\mu_p(\omega_1 = 1) = p$. Consider the measure $\mathbb{P}_p := \bigotimes_{i=1}^{\infty} \mu_p$, so our new probability space is $(\Omega, \mathcal{A}, \mathbb{P}_p)$.

As mentioned in the previous section, one may consult [7] for more details about the construction of $\mathbb{P}_p$, but we would like to mention that it is first defined on sets of the form $(\varepsilon_1, \ldots, \varepsilon_n) \times \{0,1\}^\infty$ ($\varepsilon \in \Omega_n$), called cylinder sets. More precisely

$$\mathbb{P}_p\left((\varepsilon_1, \ldots, \varepsilon_n) \times \{0,1\}^\infty\right) = p^{\sum_{i=1}^n \varepsilon_i} q^{n - \sum_{i=1}^n \varepsilon_i}.$$

It is then extended to $\mathcal{A}_0$ and afterwards extended to $\mathcal{A}$. Notice that we can define such a measure for any $p \in (0,1)$.

Example. We will work out a short example to better understand how this measure works.

$$\mathbb{P}_p\left(\{\omega \in \Omega : \omega_1 + \omega_2 = 1\}\right) = \mathbb{P}_p\left(\{(0,1)\} \times \{0,1\}^{\infty}\right) + \mathbb{P}_p\left(\{(1,0)\} \times \{0,1\}^{\infty}\right) = 2pq.$$

Having defined the new context, we can now study the behavior of the sequence x_n. Most of the results from the previous section can be generalized. We first list the ones regarding the the behavior of the sequence outside $[0,1]$. The proofs of these statements are almost identical to the ones of Proposition 3.2 and Proposition 3.3, so we omit them.

Proposition 4.1. $\mathbb{P}_p\left(x_n \to \infty, \text{ starting from } x\right) = 0$ for $x \leq 0$.

Proposition 4.2. $\mathbb{P}_p\left(x_n \to \infty, \text{ starting from } x\right) = 1$ for $x > 1$.

Another interesting property which easily generalizes is the fact that $(x_n)_{n\geq 1}$ is almost surely unbounded. Again, the proof of the following statement is the same as the proof of Proposition 3.4, except the claim is $p_{n+1} \leq \max\{p,q\}\, p_n$. As $\max\{p,q\} < 1$, the result is still true.

Proposition 4.3. $\mathbb{P}_p\left((x_n)_{n\geq 0} \subset [0,1], \text{ starting from } x\right) = 0$, for all $x \in \mathbb{R}$.

There is also a result which needs a different reasoning in some parts of its proof, namely Lemma 3.2. That is because we have chosen to define the measure $\mathbb{P}$ directly on the algebra $\mathcal{A}_0$. However, it is not too difficult to adapt the proof of Lemma 3.2 to obtain the following result, so this is left as an exercise to the reader.

Lemma 4.1. $\mathbb{P}_p\left(\{x_n \to \infty, \text{ starting from } x\} \mid \{\omega_1 = \varepsilon\}\right) = \mathbb{P}_p\left(x_n \to \infty, \text{ starting from } f_\varepsilon(x)\right)$, for any real number x and any $\varepsilon \in \{0,1\}$.

With that being said, we are ready to characterize the function $F_p : \mathbb{R} \to \mathbb{R}$, defined by $F_p(x) := \mathbb{P}_p\left(x_n \to \infty, \text{starting from } x\right)$ and we will observe that it is qualitatively similar to the Cantor function. We first introduce a new definition.

Definition 4.1. We call $C_p : \mathbb{R} \to \mathbb{R}$ a *$p-$generalized extended Cantor function* if it satisfies $C_p \restriction_{(-\infty,0]} \equiv 0$, $C_p \restriction_{[1,\infty)} \equiv 1$ and, for all $x \in \mathbb{R}$:

$$C_p(x) = qC_p(3x) + pC_p(3x-2).$$

Remark 8. The reader may observe the similarity with Definition 2.3. Also, notice that there is a unique function that satisfies Definition 4.1, due to a similar reasoning as in Proposition 2.1 (define $T_p : \mathcal{C} \to \mathcal{C}$ by $T_p f(x) := qf(3x) + pf(3x-2)$, which is a contraction as well), so we may call it *the p-generalized Cantor function.*

The following theorem is perhaps one of the most important results of this section and it is proven in a similar manner as Theorem 3.2, but instead of using the results referenced there, one can use their generalizations listed in this section. We leave this proof to the reader as well.

Theorem 4.1. The function $F_p : \mathbb{R} \to \mathbb{R}$ defined by $F_p(x) = \mathbb{P}_p\left(x_n \to \infty, \text{starting from } x\right)$ is the *p-generalized Cantor function.*

Here is a picture with (an approximation of) the graph of $F_{0.3}$ generated using the code in the Appendix.

The similarity with the Cantor function is striking. The function F_p is nondecreasing, continuous and constant almost everywhere (all of the properties follow from Definition 4.1 Theorem 4.1). What changes in this case are the intervals on which it is constant and the value it takes on those intervals.

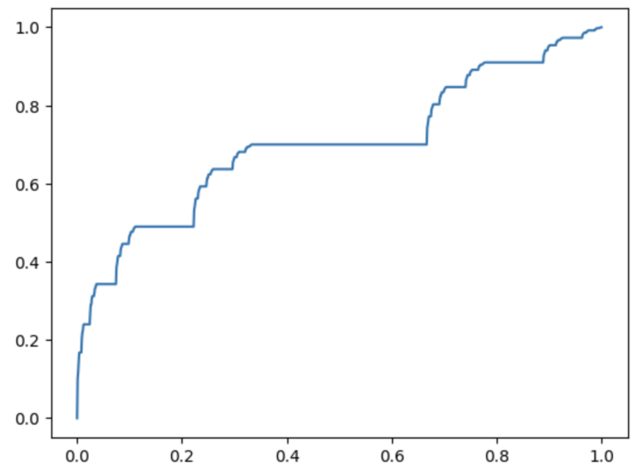

Figure 2: Generalized Cantor function; p=0.3

The last thing we want to discuss here is a generalization of Theorem 2.2. In order to do that, we first need a generalization of Lemma 3.4, as it was essential in the proof of Theorem 2.2.

Lemma 4.2. $\mathbb{P}_p\left(\operatorname{rec}(A)\right) = \mathbb{P}_q\left(A\right)$, for any $A \in \mathcal{A}$.

Proof. Notice that it is enough to prove that $\mathbb{P}_p\left(\operatorname{rec}(C)\right) = \mathbb{P}_q\left(C\right)$, for any cylinder set C. That is because any set in $\mathcal{A}_0$ is a finite union of (disjoint) cylinder sets and from this we can obtain $\mathbb{P}_p\left(\operatorname{rec}(A_0)\right) = \mathbb{P}_q\left(A_0\right)$, for any $A_0 \in \mathcal{A}_0$. From here we can proceed as in Lemma 3.4 to finish the proof of the result, as that part of the proof is not dependent on the measure, but only requires the properties of rec (contained in Proposition 3.5) and σ-additivity, properties that are valid in this case as well.

Now, without loss of generality, suppose $C = (\varepsilon_1, \ldots, \varepsilon_n) \times \{0,1\}^\infty$ for some $\varepsilon \in \Omega_n$. Then:

$$\mathbb{P}_p\left(\operatorname{rec}(C)\right) = \mathbb{P}_p\left((1-\varepsilon_1, \ldots, 1-\varepsilon_n) \times \{0,1\}^\infty\right) = p^{\sum_{i=1}^n (1-\varepsilon_i)} q^{n-\sum_{i=1}^n (1-\varepsilon_i)} = p^{n-\sum_{i=1}^n \varepsilon_i} q^{\sum_{i=1}^n \varepsilon_i} \tag{16}$$

By definition we have $\mathbb{P}_q\left(C\right) = q^{\sum_{i=1}^n \varepsilon_i} p^{n-\sum_{i=1}^n \varepsilon_i}$ and, by using (16) we obtain $\mathbb{P}_p\left(\operatorname{rec}(C)\right) = \mathbb{P}_q\left(C\right)$, which is what we wanted to show.

□

Remark 9. Although rec : $\mathcal{A} \to \mathcal{A}$ preserves the measure $\mathbb{P}_{\frac{1}{2}}$, the previous result establishes that it does not preserve any of the measures $\mathbb{P}_p$, for $p \in (0,1) \setminus \left\{\frac{1}{2}\right\}$. However, it exhibits a nice symmetry which allows us to generalize Theorem 2.2.

Theorem 4.2. Let $p \in (0,1)$. Then $G_p := \tilde{F}_p$ satisfies the following:

(i) it is nondecreasing,
(ii) $G_p(\frac{x}{3}) = qG_p(x)$,
(iii) $G_p(1-x) + G_q(x) = 1$.

Proof. We provide a sketch of the proof, as it is akin to the one of Theorem 3.1. The reader is invited to assemble all the pieces. For (i) the only thing we change is the measure. For (ii), we condition on the first choice and then we use Lemma 4.1 and Proposition 4.2. For (iii) we again follow the steps from Theorem 3.1, but we use Lemma 4.2 instead of Lemma 3.4 and Proposition 4.3 instead of Lemma 3.3.

□

5 Future research

The process studied here was restricted to the two functions $f_0(x) = 3x$ and $f_1(x) = 3x - 2$. In the following subsections we present some possible future work directions.

5.1 Different functions

The first change we can make to this model is picking other two functions. For simplicity, we can first look at other linear functions. In the spirit of [10], consider $f_0(x) = \frac{x}{\beta}$ and $f_1(x) = \frac{x}{\beta} + 1 - \frac{1}{\beta}$, where $\beta \in (0, \frac{1}{2}]$. Denote by $F_{p,\beta}(x)$ the probability that the sequence goes to ∞, where p is the probability that f_0 is chosen (at a specific step). Notice that this function is dependent on both the probability distribution and the parameter β. Clearly $F_{\frac{1}{2},\frac{1}{3}}$ is the standard Cantor function and $F_{p,\frac{1}{3}}$ is the same generalization presented above. However, in the case when $\beta \neq \frac{1}{3}$, although $F_{p,\beta}$ resembles the Cantor's function properties, such functions are yet to be explored and characterized.

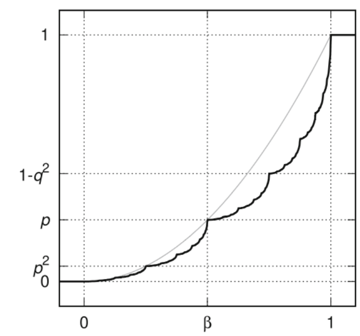

Figure 3: Graph of $F_{\frac{1}{4},\frac{1}{2}}$ from [10]

5.2 Multiple functions

Another change we can make to our setup is the number of functions, but select them with equal probability at each step. This idea is explored in [1], for linear functions with specific restrictions on the coefficients. For instance, consider $f_0(x) = 20x - 1$, $f_1(x) = 6x - 3$ and $f_2(x) = 10x - 8$. Then, let $\mathbf{P}(x_0)$ denote the probability that the sequence goes to infinity, having x_0 as its seed. Examining the graph of this function for $x \in [0, 1]$, one may observe qualitatively similar properties to the Cantor function. This is yet another generalization of this function.

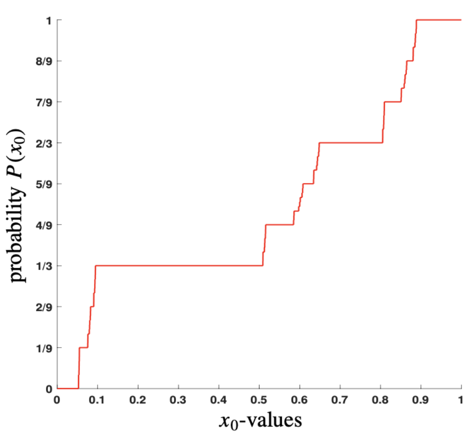

Figure 4: Graph of $\mathbf{P}(x_0)$ from [1]

5.3 Complex numbers

The last research direction we present here is working with complex numbers instead of real numbers. This changes the model drastically, but with a careful choice of functions, one can obtain similar results to the ones here. For example, in [9], the following setup is considered. Let $g_0(z) := \frac{z^2}{4}$ and $g_1(z) := z^2 - 1$ and $f_0 := g_0^2$, $f_1 := g_1^2$ be the second iterates of the previous functions. Similarly, for a seed $z_0 \in \mathbb{C}$, let $(z_n)_{n\geq 0}$ be the sequence generated by the random iteration and $\mathbf{P}(z_0)$ be the probability that $|z_n| \to \infty$. For large values of z_0 ($|z_0| > 10$ is enough), we have $\mathbf{P}(z_0) = 1$, since any sequence of function choices leads the orbit $(z_n)_{n\geq 0}$ to infinity. Near zero, $\mathbf{P}(z_0) = 0$, although the argument is more subtle. Between these regions, $\mathbf{P}(z_0)$ shows remarkable behavior. Its graph, known as the Devil's Colosseum resembles the Devil's Staircase. If one tried to climb from the bottom to the top, one's path would mirror the fractal, step-like structure of the classic Cantor function.

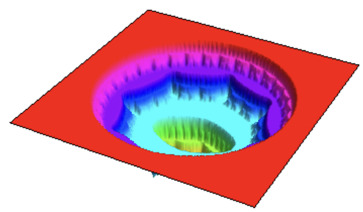

Figure 5: Devil's Colosseum from [9]

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6 Appendix

We attach below the Python code used for plotting the Cantor function and the generalized Cantor functions. It is worth mentioning that the code follows the ideas from Proposition 2.1 (the part concerning the operator T) and Remark 8.

```
import numpy as np
import matplotlib.pyplot as plt
```

```
#here we fix the number of points we use for the plotitng

pts=1000
```

```
    #Original Cantor function


#define the starting point of the iteration
def f0(x):
    return np.piecewise(x, [x<0, 0<=x and x<=1, x>1], [0, lambda x: x, 1])

#define the operator mentioned in the paper
def t(f):
    def g(x):
        return 0.5*f(3*x)+0.5*f(3*x-2)
    return g

#define the itaration recursively by repeatedly
#applying the operator, stopping it at a chosen time
def cantor(n):
    if(n==0):
        return f0
    else :
        return t(cantor(n-1))

c=cantor(10)

#plot the function
xaxis=np.linspace(0,1,pts)
yaxis=np.array(xaxis)
for i in range(len(xaxis)):
    yaxis[i]=c(xaxis[i])

plt.plot(xaxis, yaxis)
plt.show()
```

```
#we change the below values depending on the function we want to plot

p=0.9
q=1-p
```

```
    #Generalised Cantor functions

#define the starting point of the iteration,
#which is the same as in the previous case
def f0(x):
    return np.piecewise(x, [x<0, 0<=x and x<=1, x>1], [0, lambda x: x, 1])

#define the operator mentioned in the paper,
#this time depending on the parameter p
def t(f):
    def g(x):
        return q*f(3*x)+p*f(3*x-2)
    return g

#define the itaration recursively by repeatedly
#applying the operator, stopping it at a chosen time
def cantor(n):
```

```
    if(n==0):
        return f0
    else :
        return t(cantor(n-1))

c=cantor(10)

#plot the function
xaxis=np.linspace(0,1,pts)
yaxis=np.array(xaxis)
for i in range(len(xaxis)):
    yaxis[i]=c(xaxis[i])

plt.plot(xaxis, yaxis)
plt.show()
```