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Hidden Zero behavior in the vector formulation of the Nonlinear Sigma Model

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Contents

Abstract

Pions are scalar particles that arise due to spontaneous symmetry breaking of QCD. The physics of their interactions is described by an effective field theory known as the Nonlinear Sigma Model. Alternative formulations that describe the tree-level scattering amplitudes of pions have recently been proposed. These imply using the double copy prescription to extract said amplitudes from a simpler theory, known as the biadjoint scalar theory, by observing the structural similarities between that and the chiral current formalism of the Nonlinear Sigma Model. This will be referred to as the vector formulation, due to the mathematical structures it uses. Together with the existence of the hidden zeros, which is the property that these amplitudes vanish on specific kinematic loci, this motivated the objective of this thesis to extract more detailed properties using the aforementioned formulation. The focus of the following approach is to use the vector formulation to group diagrams that contribute to a scattering amplitude into subsets that individually vanish on the kinematic locus. This will be done by considering various geometric and topological properties of similar diagrams. In practice, the lower point diagrams will be calculated first and a characterization of what constitutes a subset in those cases will be attempted. Then, a general argument that groups diagrams of arbitrarily high multiplicity for specific kinematic loci will be considered. Finally, an extension of this approach to a general splitting of the kinematic configuration will be formulated.

Acknowledgments

I would like to express my gratitude to my supervisor, Prof. Diederik Roest and his PhD student, Yang Li, for welcoming me into their research group and introducing me to the fascinating study of amplitudes. Their dedication to the research they conduct inspired me to want to learn more and contribute to this interesting topic.

I am also deeply grateful to my parents for their continuous support and encouragement while I was pursuing my studies, and to my partner for having the unending patience to always listen and talk to me about the topics that fascinate me in physics.

1 Introduction

The study of pion physics has been a topic of significant interest since the beginning of the second half of the twentieth century. Pions can be modeled as Goldstone bosons that exist due to the spontaneous symmetry breaking implied by the different symmetry groups of the Lagrangian and ground state of QCD. Since QCD is non-perturbative at low energies, there is a need to formulate the physics of these bosons in alternative ways. The effective field theory which enables us to describe pion scattering amplitudes is named the Nonlinear Sigma Model (NLSM). Two particular properties of this model that motivated the research goals of this paper are: the double-copy prescription [1, 2, 3] and the existence of hidden zeros [4, 5]. The former provides a framework in which the scattering amplitudes of the NLSM can be constructed from vectors, instead of scalars. These theoretical considerations were first illustrated in [6]. The latter refers to the observation that, under particular kinematic configurations, the pion amplitudes vanish.

In [7], the covariant formulation that describes NLSM scattering amplitudes was used to group diagrams into subsets that are null on the kinematic locus. The aim of this paper is to take the vector formalism and use it to construct a mechanism through which the hidden zeros become manifest by forming similar subgroups of diagrams. In practice, this will be done by using symmetry and geometric arguments to group contributions from different Feynman diagrams into subsets that vanish individually when the particular kinematic configuration of the hidden zeros is used.

1.1 Research Question

To summarize, this thesis focuses on a variety of approaches to provide a solution to the following problem:

How can the vector formulation of the Nonlinear Sigma Model be used to make the hidden zeros manifest on subsets of diagrams?

1.2 Thesis Outline

This paper will initially be structured around providing an extensive review of the literature that will be correlated with the theory necessary to understand the general topic. This will begin with a basic illustration of the fundamental principles of the SU(N) symmetry group and chiral theory, which is then related to the study of pions by modeling them as massless particles which appear as a consequence of spontaneous symmetry breaking. These lay a foundation for the derivation of the NLSM and the procedure of color ordering and partial amplitudes, which are paramount to further study. Then, the vector formalism and hidden zeros will be explored and an analysis of the approach of [7] will be presented.

In the second part of the paper, the 4 and 6 point amplitudes of the NLSM, which consist of 2 and 14 diagrams, will be derived using the vector formalism and then the results will be used to check whether they vanish on the kinematic configuration imposed by the hidden zeros. Further, a framework will be considered in which amplitudes can be split up into lower subsets of contributing diagrams, which have certain symmetries within the subset. The contributions of each subset will be evaluated for the kinematic configurations of the hidden zeros. A generalized approach for the splitting into subsets for amplitudes of arbitrarily high multiplicities will be explored.

2 Background Literature and Theory

2.1 Foundations of pion physics

2.1.1 SU(N) symmetry group

The Special Unitary group SU(N) is the Lie group that consists of NxN unitary matrices with determinant 1. A case of particular interest for the presented paper is SU(2), which has the following generators for its Lie algebra:

$$T_a = \frac{\sigma_a}{2} \tag{1}$$

where σ_a are the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

For SU(3) the generators are:

$$T_a = \frac{\lambda_a}{2} \tag{2}$$

where λ_a are the Gell-Mann matrices:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

The generators satisfy:

$$[T_a, T_b] = i f_{abc} T_c \tag{3}$$

Here, f_{abc} are the structure constants of the Lie algebra.

2.1.2 Chiral theory

Quantum chromodynamics is the theory that governs the behavior of the strong interaction. It is a non-Abelian gauge theory with gauge group SU(3) that couples to fermions in the fundamental representation [8]. When these fermions (quarks) are taken to be massless, the Lagrangian of a theory with N_f fermions is [9]:

$$\mathcal{L} = \frac{1}{2} \mathrm{Tr}\left[F_{\mu\nu} F^{\mu\nu}\right] + \sum_{i=1}^{N_f} \bar{\psi}_i \not{D} \psi_i \tag{4}$$

where the matrix-valued field strength is:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] \tag{5}$$

and the gauge field is:

$$A_\mu = A_\mu^a T^a \tag{6}$$

The Dirac operator is defined as:

$$\not{D} = \gamma^\mu(\partial_\mu - iA_\mu) \tag{7}$$

The sum runs over all quark flavors $1...N_f$.

The fermionic terms of this Lagrangian can be decomposed into left-handed and right-handed components, which transform in this manner under the action of the symmetry group:

$$SU(N_f)_L : \quad \psi_{-i} \mapsto L_{ij}\psi_{-j}$$

$$SU(N_f)_R : \quad \psi_{+i} \mapsto R_{ij}\psi_{+j}$$

2.1.3 Spontaneous symmetry breaking

The spontaneous breaking of a symmetry refers to the difference in symmetry groups between the Lagrangian of a theory and its ground state. For the presented chiral theory, the Lagrangian has the global symmetry:

$$U(1)_V \times SU(N_f)_L \times SU(N_f)_R$$

whereas the ground state has:

$$U(1)_V \times SU(N_f)_V$$

Goldstone's theorem [10, 11] states that for each broken generator there exists a massless scalar particle. For the SU(2) case, there are 3 of these particles, which are called pions. These will be the focus of the remaining part of this paper.

2.1.4 Color-ordering

Amplitudes of any interaction process in this theory can be expressed as:

$$\mathcal{M}^{a_1 a_2 \ldots a_n}(p_1, p_2, \ldots, p_n) = \sum_{\sigma \in S_n/\mathbb{Z}_n} Tr[t^{a_{\sigma(1)}} t^{a_{\sigma(2)}} \cdots t^{a_{\sigma(n)}}] M_\sigma(p_1, \ldots, p_n) \tag{8}$$

where the trace over the generators is taken modulo cyclic permutations and the last term is called the stripped/partial amplitude, as it only encodes information about the kinematic structure of the particles involved.

2.2 Nonlinear Sigma Model

2.2.1 Lagrangian

The chiral nonlinear sigma model [12] is the field theory used to describe pion interactions. The leading-order Lagrangian is:

$$\mathcal{L} = \frac{f_\pi^2}{4} \operatorname{Tr}\left[\partial_\mu U^\dagger \partial^\mu U\right] \tag{9}$$

where f_π is the decay constant of the Goldstone bosons and U is a unitary matrix which encodes information about the pion fields. The explicit form of U depends on the chosen parameterization. For example, in the exponential parameterization:

$$U(x) = \exp\left(\frac{i\pi^a(x)T^a}{f_\pi}\right). \tag{10}$$

where π^a are the pion fields and T^a are the broken generators.

2.2.2 Currents

Expanding the Lagrangian in terms of the pion fields in the exponential parameterization:

$$\mathcal{L} = \mathrm{tr}(\partial_\mu\pi)^2 + \frac{2}{3f^2}\,\mathrm{tr}\left[\pi^2(\partial_\mu\pi)^2 - (\pi\partial_\mu\pi)^2\right] + \dots \tag{11}$$

Then the following infinitesimal transformations can be considered:

$$\delta L = e^{i\alpha^a T^a} \approx (1 + i\alpha^a T^a)$$

$$\delta L = e^{-i\alpha^a T^a} \approx (1 - i\alpha^a T^a)$$

These correspond to $SU(N)_L$ and $SU(N)_R$, respectively.

By Noether's Theorem, the conserved currents implied by the invariance of the action under these transformations are:

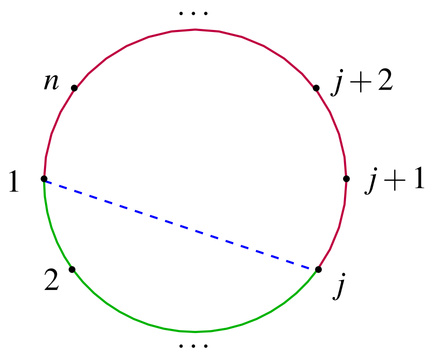
$$j^a_{\mu,L} \approx -\frac{f_\pi^2}{2}\partial_\mu\pi^a \tag{12}$$

$$j^a_{\mu,R} \approx \frac{f_\pi^2}{2}\partial_\mu\pi^a \tag{13}$$

2.2.3 Hidden zeros

One peculiar property of scattering amplitudes in the NLSM is called the hidden zero. This refers to the vanishing of the amplitude when the contributing particles have a very specific kinematic configuration, that is, their momenta are related in a particular manner.

The first zero discovered is the Adler zero [5], which indicates that under the soft limit (one pion momentum goes to 0), the amplitude vanishes. The conditions for all other zeros can be constructed by using the Mandelstam variables s_{ij}, which are Lorentz invariant and therefore suitable for generalizing the theory. Using the surfaceology formalism of [13, 14, 15], these can be represented graphically as a circle that has along its circumference all particle labels. Drawing a line between two points splits the circle into an upper and lower portion. The Mandelstam variables composed of one momentum from the lower region and another from the upper region vanish, this defines the kinematic locus for that splitting of the kinematic diagram.



These conditions can be formulated as follows: suppose that you have an n-particle interaction; pick out particle 1 and k, where $k \in \{3,\ldots,n-1\}$: this splits the set of particles into a "lower" and "upper" subset. The former refers to momenta that are between 2 and k-1, while the latter refers to momenta that are between k+1 and n-1. A kinematic locus is defined as the kinematic configuration that establishes a relation between momenta from the two subsets and fixes the following condition: $s_{ij} = 0 \quad \forall i \in \{2,\ldots,k-1\},\ j \in \{k+1,\ldots,n\}$. On the kinematic locus, the amplitude vanishes, property which is known as the hidden zero.

For the 4-point amplitude:

$$A_4 \propto s_{24} \tag{14}$$

which obviously vanishes on the $1 \to 3$ splitting of the kinematic configuration.

At 6-point:

$$A_6 \propto \left[\frac{s_{24}s_{45}}{s_{123}} + \frac{s_{25}s_{56}}{s_{234}} + \frac{s_{26}s_{61}}{s_{345}} + \frac{s_{24}s_{12}}{s_{456}} + \frac{s_{25}s_{23}}{s_{561}} + \frac{s_{26}s_{34}}{s_{612}}\right] \tag{15}$$

Written in this way, the vanishing on the $1 \to 3$ splitting is again obvious. However, this can also be rewritten by exploiting momentum conservation and masslessness such that it also vanishes immediately on the other non-trivial splitting: $1 \to 4$.

2.3 Vector formalism

All of the theoretical developments portrayed in the following section of this paper are the results of [6]. The authors show that the dual nature of certain theories enable the construction of a double copy prescription that allows the amplitudes of the NLSM to be formulated in terms of vectors, instead of scalars. This work serves as the foundation for the contributions of this thesis.

2.3.1 Biadjoint scalar

In order to develop an understanding of the vector formulation, an apparently unrelated theory has to be studied first: the biadjoint scalar theory (BAS). It serves as the simplest theory that can participate in the double copy prescription. From the analysis of its structure and how it later relates to elements in the NLSM, some properties of pion scattering amplitudes will be extracted.

The biadjoint scalar is a toy model theory and is represented by a scalar field that has two color indices. The Lagrangian of this theory is:

$$\mathcal{L}_{\text{BAS}} = \frac{1}{2}\partial_\mu \phi^{a\bar{a}} \partial^\mu \phi^{a\bar{a}} - \frac{1}{3!} f^{abc} f^{\bar{a}\bar{b}\bar{c}} \phi^{a\bar{a}} \phi^{b\bar{b}} \phi^{c\bar{c}} + \phi^{a\bar{a}} J^{a\bar{a}} \tag{16}$$

The source $J^{a\bar{a}}$ is localized at asymptotic infinity, such that it produces on shell states.

The equation of motion is:

$$\Box \phi^{a\bar{a}} + \frac{1}{2} f^{abc} f^{\bar{a}\bar{b}\bar{c}} \phi^{b\bar{b}} \phi^{c\bar{c}} = J^{a\bar{a}} \tag{17}$$

From the perturbative expansion of the E.O.M in terms of $J^{a\bar{a}}$ [16], the one-point correlation function can be written as a derivative of the partition function W[J] as:

$$\langle \phi^{a\bar{a}}(p) \rangle_J = \frac{1}{i} \frac{\delta W[J]}{\delta J^{a\bar{a}}(p)} \tag{18}$$

The n-point correlator takes the form:

$$\left\langle \phi^{a_1\bar{a}_1}(p_1) \phi^{a_2\bar{a}_2}(p_2) \cdots \phi^{a_n\bar{a}_n}(p_n) \right\rangle_{J=0} = \left(\prod_{i=1}^{n-1} \frac{1}{i} \frac{\delta}{\delta J_{a_i\bar{a}_i}(p_i)} \right) \left\langle \phi^{a_n\bar{a}_n}(p_n) \right\rangle_J \Bigg|_{J=0} \tag{19}$$

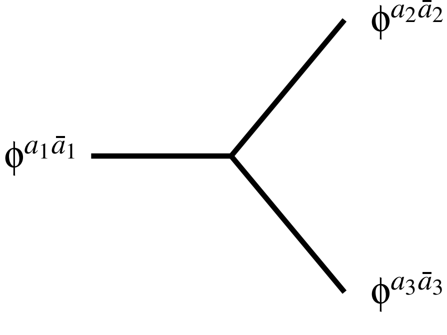
To calculate this, all diagrams where the particles connect to the one from which the one point correlator is calculated have to be summed over. This can be represented graphically as Feynman diagrams, where the first leg of a diagram is taken to be the ”root” leg and all other n-1 legs are ”leaf” legs.

This description allows for the establishment of the following Feynman rules for the amplitudes of the BAS theory:

For the BAS propagator:

$$= \frac{i\delta^{a_1a_2}\delta^{\bar{a}_1\bar{a}_2}}{p^2}$$

For the vertex:



$$= -if^{a_1a_2a_3} f^{\bar{a}_1\bar{a}_2\bar{a}_3}$$

The biadjoint scalar has a special place along the theories that can be formulated with a similar methodology. This is because it is composed only of scalar fields and the amplitude does not vary under the permutation of any legs. As will become evident later on, other theories are invariant only under the permutation of the leaf legs, while the root leg has a special place.

2.3.2 Chiral current formulation

The formula for an adjoint vector field with vanishing field strength is:

$$\partial_\mu j^a_\nu - \partial_\nu j^a_\mu + f^{abc} j^b_\mu j^c_\nu = 0 \tag{20}$$

The chiral current of the NLSM from equation 13 satisfies this equation. Since that current follows from an invariance of the action under a symmetry transformation, the following equation of motion can be constructed:

$$\partial^\mu j^a_\mu = J^a \tag{21}$$

where J^a is a source that produces NLSM scalars at infinity.

Combining equations 20 and 21:

$$\Box j^a_\mu + f^{abc} j^{b\nu} \partial_\nu j^c_\mu = \partial_\mu J^a \tag{22}$$

Inverting equation 13 such that the pion fields are written in terms of the chiral current:

$$\pi^a = -\frac{q^\mu j^a_\mu}{q\partial} \tag{23}$$

where q is an arbitrary momentum introduced in order to contract both terms in the fraction.

The one-point correlators of the scalar fields involved in the description of the NLSM can be related to those of the chiral current by:

$$\langle \pi^a(p) \rangle_J = \tilde{\varepsilon}^\mu(p) \, \langle j^a_\mu(p) \rangle_J \tag{24}$$

where $\tilde{\varepsilon}^\mu(p)$ is the polarization of the root leg and takes the form:

$$\tilde{\varepsilon}^\mu(p) = \frac{iq^\mu}{pq} \tag{25}$$

The n-point correlator is:

$$\langle \pi^{a_1}(p_1)\, \pi^{a_2}(p_2) \cdots \pi^{a_n}(p_n) \rangle_{J=0} = \left(\prod_{i=1}^{n-1} \frac{1}{i} \frac{\delta}{\delta J^{a_i}(p_i)} \right) \tilde{\varepsilon}^\mu(p_n) \left\langle j^{a_n}_\mu(p_n) \right\rangle_J \Bigg|_{J=0} \tag{26}$$

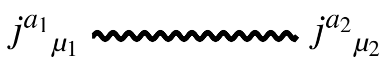
2.3.3 Double copy

There is a striking structural resemblance between the equations of motion of BAS theory and the chiral current formulation of the NLSM. This fact enables the existence of the double copy prescription, which creates a mapping between elements from the two theories. In this manner, all corresponding theoretical elements of the NLSM can be elucidated by applying these mappings to the elements of the BAS theory. The rules are as follows:

$$V^a \to V^\mu$$

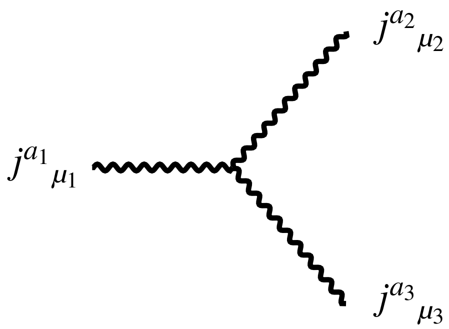
$$f^{abc}V^bW^c \to V^\nu\partial_\nu W^\mu - W^\nu\partial_\nu V^\mu$$
$$J^a \to \partial_\mu J$$

In the chiral current formalism of the NLSM, the contributions from the propagators are:



$$= \frac{i\delta^{a_1a_2}\eta^{\mu_1\mu_2}}{p^2}$$

For the vertices:



$$= -if^{a_1a_2a_3}\left(ip_2^{\mu_1}\eta^{\mu_2\mu_3} - ip_1^{\mu_2}\eta^{\mu_1\mu_3}\right)$$

3 Methods

3.1 Evaluating and checking the hidden zeros for 4 and 6-point amplitudes

In this section of the paper, the Feynman rules of [6] will be used to calculate the contributions of each Feynman diagram at 4-point and at 6-point. At 4-point, there are 2 contributing diagrams, while at 6-point, there are 14. These results will be used in an attempt to extract features of the scattering amplitudes which could point towards a general argument that shows the relation between the formalism that comes from the chiral current equation of motion and the present understanding of the hidden zeros.

These results will be used to provide a topological intuition that relates diagrams with a particular geometry. This will be used to show that their grouped contribution to the total amplitude vanishes independently once the constraints implied by the chosen kinematic locus are enforced.

3.2 Proving that the hidden zeros are manifest on subsets of n-point amplitudes

Once an intuition is built from the analysis of the lower-point amplitudes, a general argument for arbitrarily high multiplicities will be considered. The aim will be to use the patterns found to develop a systematic grouping in terms of topology that manifestly determines the hidden zero behavior of subgroups. A general and rigourous proof will be developed that shows that the structure of the Feynman rules for the scattering amplitudes in the vector formulation of the NLSM imply necessarily that such subsets exist and that they respect the hidden zero property independent of the other contributing diagrams.

Practically, the scope of this procedure will be to go gradually from the specific cases explored previously to more and more general structures. For a particular splitting, which defines the kinematic locus, the diagrammatic approach will be to draw the Feynman diagrams as a principal line that has at its ends the momenta that define the splitting. All other momenta are ramifications from that principal line. As the complexity increases, more and more ramifications will be considered until a general form is reached.

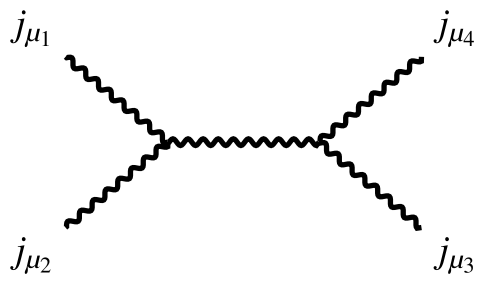
4 Results

4.1 Calculating the 4 and 6-point amplitudes

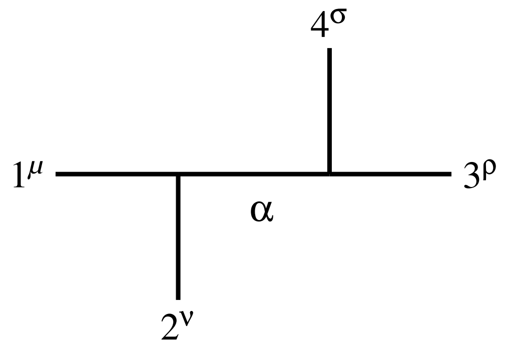
The Feynman rules for the chiral current scattering amplitudes will be used to calculate the contributions of each diagram. After summing over all contributing diagrams, the presence of the hidden zero will be checked.

4.1.1 4 point

In a similar argumentation as has been presented in [7], since the q term is non-physical and has been used only as a mathematical artifact to obtain the Feynman rules, it should drop out. Since its coefficient is p_1, the numerator should also have a $p_1 q$ term. By momentum conservation: $p_1 = -p_2 - p_3 - ... - p_n$. This means that the whole amplitude can be reconstructed by analyzing the coefficients of the $p_k q$ terms. This defines the $1 \rightarrow p_k$ "splitting" and the expectation is that the hidden zero defined by the same splitting of the kinematic configuration will become apparent. As most the later work of this thesis will be based on the kinematic locus defined by the $1 \rightarrow 3$ splitting, the chosen representation of the diagrams will be different from [6]. The color factors appearing in the Feynman rules can be added afterwards, so the focus of this paper is to calculate the kinematic numerators. Therefore, for each Feynman diagram of the form:



The following equivalent diagram with the 1-3 line on the horizontal can be drawn:



This is the first diagram that contributes to the 4-point amplitude. Applying the Feynman rules gives:

$$\frac{i[-i(ip_4^{\rho}\eta^{\sigma\alpha} - ip_3^{\sigma}\eta^{\rho\alpha})][-i(i(p_3+p_4)^{\nu}\eta^{\mu\alpha} - ip_2^{\alpha}\eta^{\mu\nu}]\eta_{\alpha\alpha}}{(p_3+p_4)^2} p_{2\nu}p_{3\rho}p_{4\sigma}\frac{iq_{\mu}}{(p_1 q)} \tag{27}$$

All constant terms such as i and -i can be factored out when summing over different diagrams, so they can be ignored. Multiplying the vertices with the polarizations gives:

$$\frac{[(p_3p_4)p_4^{\alpha}-(p_3p_4)p_3^{\alpha}][[(p_3+p_4)p_2]q_{\alpha}-(p_2q)p_{2\alpha}]}{(p_3+p_4)^2(p_1q)} \tag{28}$$

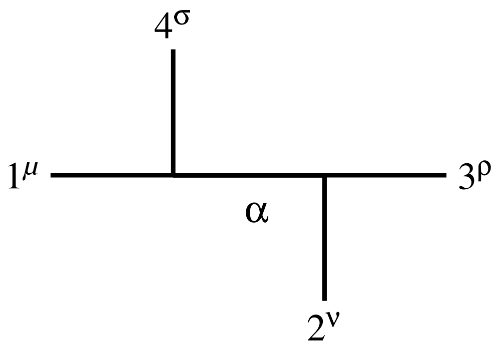
But $(p_3+p_4)^2$ can be written as p_3p_4 because $(p_{3,4})^2=0$. Therefore, the product of the first vertex cancels the propagator and the equation becomes:

$$\frac{[(p_3+p_4)p_2][(p_4-p_3)q]-(p_2q)[(p_4-p_3)p_2]}{p_1q} \tag{29}$$

In the case of the first diagram of the 4-point scattering amplitude, the coefficient of the p_3q term is:

$$-[(p_3+p_4)p_2]=p_1p_2 \tag{30}$$

The other contributing diagram is:



Calculating this gives:

$$\frac{[p_3^{\nu}\eta^{\rho\alpha}-p_2^{\rho}\eta^{\nu\alpha}][p_4^{\alpha}\eta^{\mu\sigma}-(p_2+p_3)^{\sigma}\eta^{\mu\alpha}]\eta_{\alpha\alpha}}{(p_2+p_3)^2}p_{2\nu}p_{3\rho}p_{4\sigma}\frac{q_{\mu}}{(p_1q)} \tag{31}$$

$$\frac{[(p_2p_3)p_3^{\alpha}-(p_2p_3)p_2^{\alpha}][(p_4q)p_{4\alpha}-[(p_2+p_3)p_4]q_{\alpha}]}{(p_2p_3)(p_1q)} \tag{32}$$

$$\frac{(p_4q)[(p_3-p_2)p_4]-[(p_2+p_3)p_4][(p_3-p_2)q]}{p_1q} \tag{33}$$

The p_3q coefficient is:

$$-[(p_2+p_3)p_4] \tag{34}$$

On the $1\rightarrow 3$ splitting, there are the following constraints: $p_2p_3=p_2p_4=0$, so the sum of the contributions of the two diagrams is:

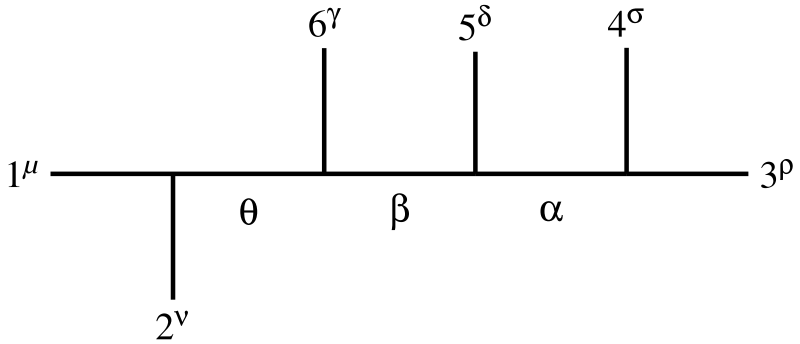
$$p_1p_2-p_3p_4=(p_1+p_2)^2-p_3p_4=(-p_3-p_4)^2-p_3p_4=p_3p_4-p_3p_4=0 \tag{35}$$

Evidently, the requirement of the hidden zero is satisfied by the 4-point scattering amplitude in the vector formulation of the NLSM.

4.1.2 6-point

There are 14 diagrams that make up the 6-point amplitude. The calculation of one of them will be shown explicitly, while the rest can be found in the appendix.

A possible diagram for a 6-point interaction is:



The terms from the vertices are:

$$[(p_4^{\rho}\eta^{\sigma\alpha} - p_3^{\sigma}\eta^{\rho\alpha})][(p_5^{\alpha}\eta^{\delta\beta} - (p_3+p_4)^{\delta}\eta^{\alpha\beta})][(p_6^{\beta}\eta^{\gamma\theta} - (p_3+p_4+p_5)^{\gamma}\eta^{\beta\theta})][(p_3+p_4+p_5+p_6)^{\nu}\eta^{\theta\mu} - p_2^{\theta}\eta^{\mu\nu}]$$

The propagators are:

$$\frac{\eta_{\theta\theta}\eta_{\beta\beta}\eta_{\alpha\alpha}}{(p_3+p_4)^2(p_3+p_4+p_5)^2(p_3+p_4+p_5+p_6)^2} \tag{36}$$

The polarizations are:

$$p_{2\nu}p_{3\rho}p_{4\sigma}p_{5\delta}p_{6\gamma}\frac{iq_{\mu}}{(p_1q)} \tag{37}$$

After performing the contraction of the indices and the multiplication, the p_3q coefficient is:

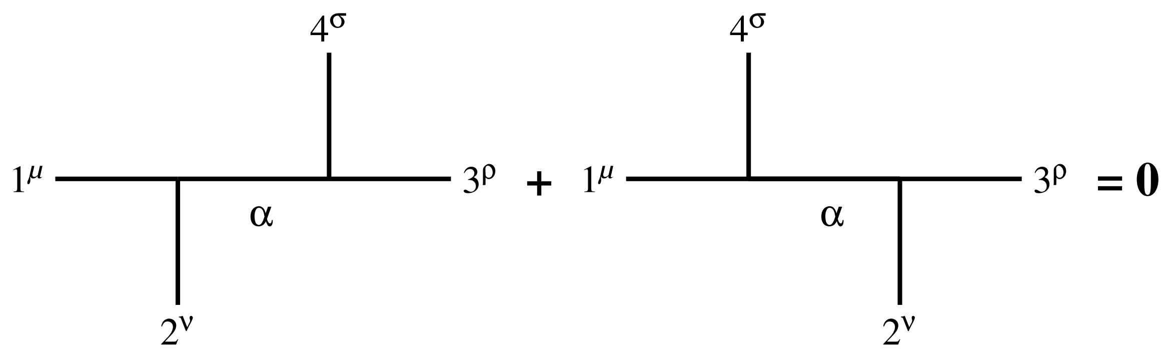
$$\frac{-[(p_3+p_4)p_5](p_3+p_4+p_5)p_6]}{(p_3+p_4+p_5)^2} \tag{38}$$

When summing over all 14 diagrams, it is evident that the Feynman rules deduced from the chiral current formulation of the NLSM predict correctly the amplitudes of the NLSM and have the property of the hidden zeros.

4.2 Subsets that individually vanish on the kinematic locus

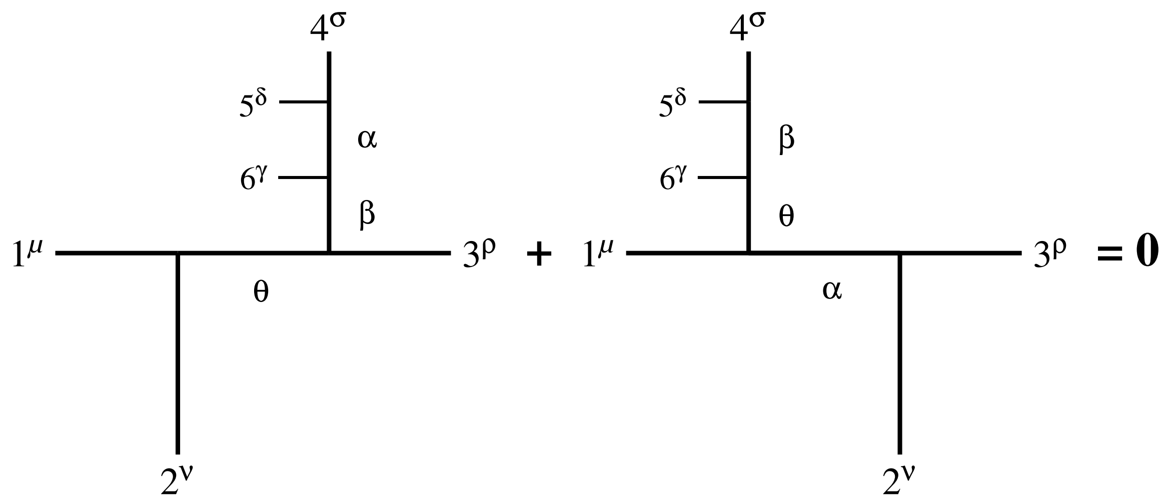
In the remaining part of this thesis an approach will be attempted to group contributions from different diagrams into subsets that are 0 on the kinematic locus. Based on the intuition built by following the work of [7], such groupings for a particular splitting may be related to the geometric structure and symmetry between diagrams that have the momenta that define the splitting as a principal horizontal line.

At 4-point:

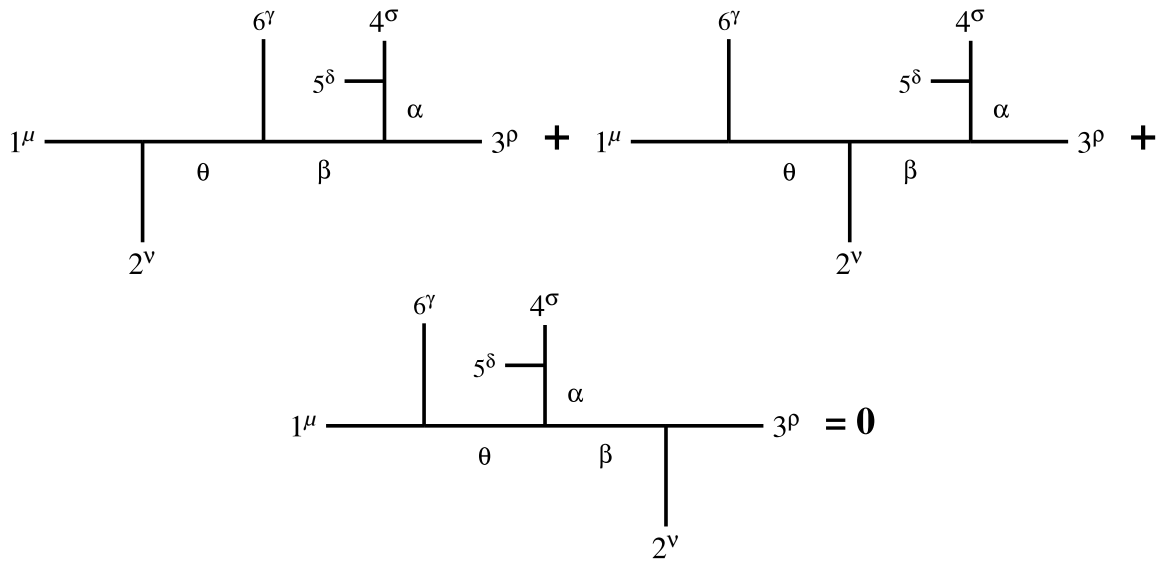


Although the result is trivial and there are no possible subgroups because only two diagrams contribute to the amplitude, there is still some insight to be gained from this. An observation that could be made is that the lower leg with momentum 2 (corresponding to the lower part of the kinematic diagram, which allows for a geometric understanding of the kinematic locus) is permuted to the left and to the right of the upper leg.

Surely enough, if this is attempted with two 6-point diagrams that have only one upper main leg (by this is understood a leg that connects directly to the principal $1 \to 3$ line), and that leg is the same between the diagrams:



If there are two main upper legs, the subgroup will have to consist of three diagrams, such that the lower leg is permuted in every position from the left to the right of the principal line:



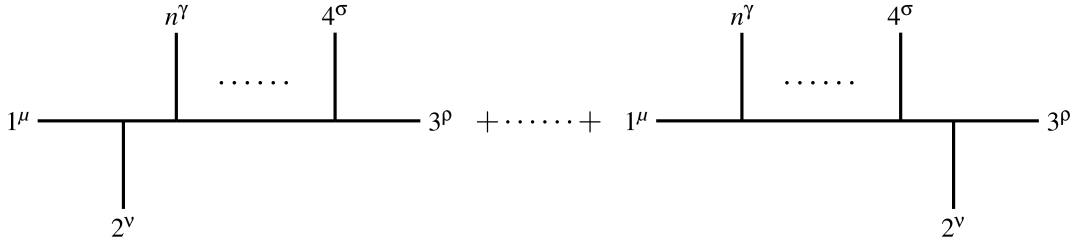
Such groupings that respect these geometric considerations have been checked for all possible combinations of 6-point diagrams and they all conform to this pattern. The rest of them (one more grouping with one main upper leg, one with two upper main legs, one with three upper main legs) can be found in the appendix.

4.3 General proof for $1 \to 3$ splitting of n particles

For a n-particle scattering amplitude, there will be contributions from diagrams that have from one to $n-3$ upper main legs, depending on how many momenta are on each of them. The aim of this section is to prove that all diagrams respecting a similar topology (same upper main legs) will manifestly vanish when summed over and applying the kinematic constraints implied by the kinematic locus that defines the $1 \to 3$ splitting.

4.3.1 Single upper legs, even n

Firstly, a proof of this will be shown for an easier case, where all upper main legs are individual (only one on-shell momentum coming from them into the principal line). Only the case where n is even will be considered, as when n is odd the amplitude is 0 trivially and the hidden zero property is not relevant. There will be $n-2$ such diagrams, where the lower leg is permuted between every position with respect to the upper legs:

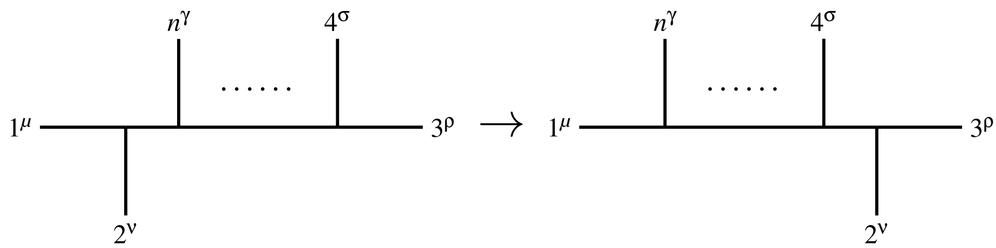


One observation that will prove useful for further calculations is the following: when computing the p_3q term, the only factor from each vertex that contributes (besides the first and last vertex), upon multiplying with the polarizations, is the one that contains an η with two propagator indices. This is because the last vertex always has a $p_{3\alpha}$ term and the first one has a q_θ term, where θ and α are the indices of the first and last propagators, respectively. So, when q_θ is multiplied by the next vertex term, only the factor with $\eta_{\theta\beta}$ (β is the index of the second propagator) matters, as the product will yield a q_β term. This goes on until the second to last vertex, where it becomes q_α and can be multiplied by the last vertex to get a p_3q term. Therefore, the contributions to the numerators of all vertices between the principal line and the upper legs, including the first and last one, are:

$$[-p_3p_4][-(p_3+p_4)p_5][-(p_3+p_4+p_5)p_6]...[-(p_3+p_4+...+p_{n-1})p_n] \tag{39}$$

This expression is independent of where the lower leg is placed on the principal line because of the fact that this calculation is done using the constraints imposed by the kinematic locus, therefore even if there was a p_2 term in the parenthesis, it would disappear because $s_{2j}=0$ for any j that is not 1 or 3. This means that the expression written above factors out between contributing diagrams of similar topology, which is incredibly helpful when calculating their sum, as it reveals that the only terms that matter is the vertex between the lower leg and the principal line and the propagators. The former is of the form $[(p_3+p_4+...p_k)p_2]$, where p_k is the momentum on the right of that vertex. Again, because of the kinematic constraints, this becomes simply p_2p_3. Therefore, only the propagators matter when attempting to check whether the sum is 0.

Another interesting observation is: for n even (odd number of upper legs with $n/2+2$ middle leg) there exists a prescription that can obtain the amplitudes of the diagrams where the lower leg is to the right of the middle upper leg from the amplitudes of the diagrams where the lower leg is to the left by establishing a 1-1 correspondence. The first pair looks like:



The prescription that takes an amplitude of a diagram and transform it into the amplitude of its mirrored equivalent is: $p_1 \to p_3, p_n \to p_4, ..., p_{n/2+3} \to p_{n/2+1}$.

The proof of this is as follows: Consider a random diagram that has middle leg "j",with the lower leg in the left half of the diagram: it has upper legs $n \to j+k$ to the left of the lower leg and $4 \to j+k-1$ upper legs to the right. It has propagator:

$$\frac{1}{(p_1+p_n)^2(p_1+p_n+p_{n-1})^2...(p_1+p_n+...+p_{j+k})^2(p_3+p_4...+p_{j+k-1})^2....(p_3+p_4)^2} \tag{40}$$

The corresponding diagram to this one is its mirrored image, with upper legs $n \to j-k+1$ to the left and $4 \to j-k$ to the right. It has propagator:

$$\frac{1}{(p_1+p_n)^2(p_1+p_n+p_{n-1})^2...(p_1+p_n+...+p_{j-k+1})^2(p_3+p_4...+p_{j-k})^2....(p_3+p_4)^2} \tag{41}$$

Therefore, the aforementioned prescription $p_{j+i} \to p_{j-i}$ manifestly relates the two amplitudes.

To prove the generality of the hidden zero on this subset, the diagrams will be numbered, such that the one with the lower leg to the left is number 1, then moving the lower leg one position to the right gives diagram number 2, etc. This way, only diagrams $1 \to n/2-1$ (with lower leg to the left of the middle upper leg) have to be summed, and the contribution from the other diagrams can be found out by applying the prescription to this sum.

Diagram 1 has propagator $\frac{1}{(p_1+p_2)^2(p_1+p_2+p_n)^2...(p_3+p_4)^2}$, and diagram 2 has $\frac{1}{(p_1+p_n)^2(p_1+p_2+p_n)^2...(p_3+p_4)^2}$. Adding them together gives:

$$\frac{(p_1+p_2)^2+(p_1+p_n)^2}{(p_1+p_2)^2(p_1+p_n)^2(p_1+p_2+p_n)^2...(p_3+p_4)^2} \tag{42}$$

$$\frac{2p_1p_2+2p_1p_n}{(p_1+p_2)^2(p_1+p_n)^2(p_1+p_2+p_n)^2...(p_3+p_4)^2} \tag{43}$$

which, due to the constraint of the kinematic locus $p_2p_n = 0$, has numerator $(p_1+p_2+p_n)^2$. This is the third term in the denominator, so it gets canceled:

$$\frac{1}{(p_1+p_2)^2(p_1+p_n)^2(p_1+p_2+p_n+p_{n-1})^2...(p_3+p_4)^2} \tag{44}$$

Diagram 3 has propagator $\frac{1}{(p_1+p_n)^2(p_1+p_n+p_{n-1})^2(p_1+p_n+p_{n-1}+p_2)^2...(p_3+p_4)^2}$. Adding this to the previous sum:

$$\frac{(p_1+p_n+p_{n-1})^2+(p_1+p_2)^2}{(p_1+p_2)^2(p_1+p_n)^2(p_1+p_n+p_{n-1})^2(p_1+p_n+p_{n-1}+p_2)^2...(p_3+p_4)^2} \tag{45}$$

Evidently, adding $(p_1+p_2)^2$ to any propagator $(p_1+p_n+p_{n-1}+...+p_l)^2$, with l more than 3, will yield $(p_1+p_n+p_{n-1}+...+p_l+p_2)^2$.. For the expression above this means that the fourth term in the denominator will be canceled, remaining with:

$$\frac{1}{(p_1+p_2)^2(p_1+p_n)^2(p_1+p_n+p_{n-1})^2(p_1+p_n+p_{n-1}+p_{n-2}+p_2)^2...(p_3+p_4)^2} \tag{46}$$

The pattern is as follows: when we add diagram "k" to the previous sum, there will be k-1 propagators after the first one that can be written as a squared sum of momenta that are only from the upper legs.

After adding all diagrams with lower leg to the left of the middle leg:

$$\frac{1}{(p_1+p_2)^2(p_1+p_n)^2...(p_1+p_n+p_{n-1}+...+p_{n/2+3})^2(p_3+p_4+...+p_{n/2+1})...(p_3+p_4)^2} \tag{47}$$

To find out the contribution from the rest of the diagrams, the prescription is applied and the result is:

$$\frac{1}{(p_3+p_2)^2(p_3+p_4)^2...(p_3+p_4+p_5+...+p_{n/2+1})^2(p_1+p_n+...+p_{n/2+3})...(p_1+p_n)^2} \tag{48}$$

All terms besides the first propagators are the same, so when adding them up:

$$\frac{(p_1+p_2)^2+(p_3+p_2)^2}{(p_1+p_2)^2(p_3+p_2)^2(p_1+p_n)^2...(p_1+p_n+p_{n-1}+...+p_{n/2+3})^2(p_3+p_4+...+p_{n/2+1})...(p_3+p_4)^2} \tag{49}$$

$$\frac{2p_2(p_1+p_3)}{(p_1+p_2)^2(p_3+p_2)^2(p_1+p_n)^2...(p_1+p_n+p_{n-1}+...+p_{n/2+3})^2(p_3+p_4+...+p_{n/2+1})...(p_3+p_4)^2} \tag{50}$$

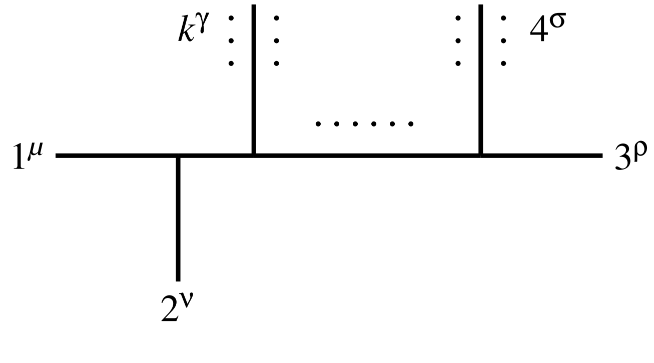
Applying momentum conservation to the sum in the numerator:

$$\frac{2p_2(-p_2-p_4-p_5-...-p_n)}{(p_1+p_2)^2(p_3+p_2)^2(p_1+p_n)^2...(p_1+p_n+p_{n-1}+...+p_{n/2+3})^2(p_3+p_4+...+p_{n/2+1})...(p_3+p_4)^2} \tag{51}$$

$(p_2)^2 = 0$ because it is on-shell and the rest of the terms in the numerator are null due to the kinematic locus constraints. Therefore, the numerator is 0 and this concludes the proof that, for even n, the contributions of the subgroups of diagrams that have individual upper main legs vanishes independently once the kinematic constraints are imposed.

4.3.2 Extending to arbitrary upper main legs

The aim of this section is to extend the argument presented previously to an arbitrary configuration of momenta on each upper main leg. The first diagram can be represented visually as (the other ones are permutations of the lower leg in every position with respect to the upper legs):



The main realization that allows for a generalization of the argument is the following: every vertex and propagators term on the upper main legs is the same between diagrams contained within a subset. This means that the vertex and propagator contributions to the amplitude factor out between diagrams, so they are irrelevant when trying to prove that the sum of the diagrams vanishes. Therefore, this situation is equivalent to the one where the upper legs are individual, with the distinction that the momenta coming into the principal line from the upper legs ($p_4 \to p_k$) are off-shell, as they represent the sum of the momenta of the particles that are on that upper leg.

The kinematic locus constraints can be applied similarly $p_2 p_j = 0$, with j being the label attributed to a upper main leg, because it is simply the sum of momenta that are on the upper section of the kinematic splitting. This means that the vertex of the principal line with the lower leg again gives a $p_2 p_3$ term. Equation 39 for the upper vertices of the principal line still holds, as its derivation did not use masslessness. Therefore, even in this formulation with arbitrary upper legs, when attempting to prove the presence of a hidden zero on a subset, only the propagator terms matter, as everything else factors out. The prescription that establishes a correspondence between the first half of diagrams within a subset to the second half still holds.

Diagram 1 has propagator $\frac{1}{(p_1+p_2)^2(p_1+p_2+p_k)^2...(p_3+p_4)^2}$, and diagram 2 has $\frac{1}{(p_1+p_k)^2(p_1+p_2+p_k)^2...(p_3+p_4)^2}$. Adding them together gives:

$$\frac{(p_1+p_2)^2+(p_1+p_k)^2}{(p_1+p_2)^2(p_1+p_k)^2(p_1+p_2+p_k)^2...(p_3+p_4)^2} \tag{52}$$

Because p_k is off-shell, this becomes:

$$\frac{2p_1p_2+2p_1p_k+p_k^2}{(p_1+p_2)^2(p_1+p_k)^2(p_1+p_2+p_k)^2...(p_3+p_4)^2} \tag{53}$$

But $p_2p_k = (p_2)^2 = (p_1)^2 = 0$, so $2p_1p_2+2p_1p_k+p_k^2 = (p_1+p_2+p_k)^2$. The third term in the denominator is canceled, remaining with:

$$\frac{1}{(p_1+p_2)^2(p_1+p_k)^2(p_1+p_k+p_{k-1}+p_2)^2...(p_3+p_4)^2} \tag{54}$$

The third diagram has propagator $\frac{1}{(p_1+p_k)^2(p_1+p_k+p_{k-1})^2(p_1+p_k+p_{k-1}+p_2)^2...(p_3+p_4)^2}$. Adding it to the previous sum gives:

$$\frac{(p_1+p_2)^2+(p_1+p_k+p_{k-1})^2}{(p_1+p_2)^2(p_1+p_k)^2(p_1+p_k+p_{k-1})^2(p_1+p_k+p_{k-1}+p_2)^2...(p_3+p_4)^2} \tag{55}$$

Every sum of the form $(p_1+p_2)^2+(p_1+p_k+p_{k-1}+...p_j)^2$, with j bigger than 3, will become, after implementing the kinematic constraints, $(p_1+p_k+p_{k-1}+...p_j+p_2)^2$. In the case of the equation above, the numerator will cancel the fourth term in the denominator, remaining with:

$$\frac{1}{(p_1+p_2)^2(p_1+p_k)^2(p_1+p_k+p_{k-1})^2(p_1+p_k+p_{k-1}+p_{k-2})^2...(p_3+p_4)^2} \tag{56}$$

This goes on for every added diagram until the last one where the lower leg is on the left of the middle upper leg, and the summed contribution is:

$$\frac{1}{(p_1+p_2)^2(p_1+p_k)^2...(p_1+p_k+p_{k-1}+...+p_{k/2+3})^2(p_3+p_4+...+p_{k/2+1})...(p_3+p_4)^2} \tag{57}$$

Applying the prescription to yield the sum of contributions of all the diagrams where the lower leg is to the right of the middle leg:

$$\frac{1}{(p_3+p_2)^2(p_3+p_4)^2...(p_3+p_4+p_5+...+p_{k/2+1})^2(p_1+p_n+...+p_{k/2+3})...(p_1+p_k)^2} \tag{58}$$

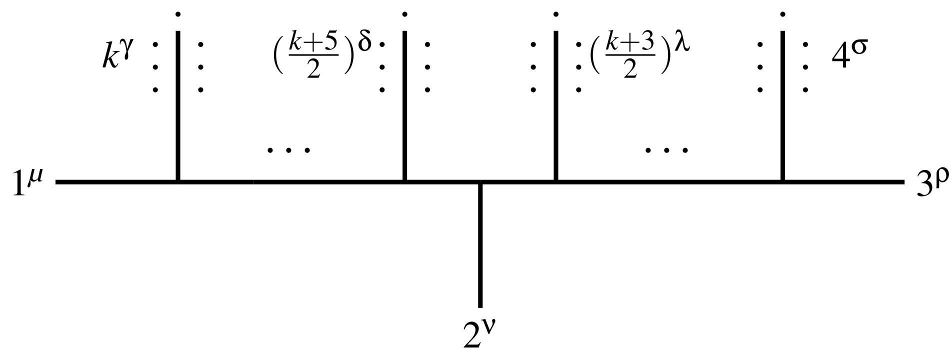
All terms in the denominator are the same between the two equations, besides the first ones. Therefore, adding them gives:

$$\frac{(p_1+p_2)^2+(p_3+p_2)^2}{(p_1+p_2)^2(p_3+p_2)^2(p_3+p_4)^2...(p_3+p_4+p_5+...+p_{n/2+1})^2(p_1+p_n+...+p_{n/2+3})...(p_1+p_n)^2} \tag{59}$$

As was the case for the individual upper legs, the numerator becomes $p_2(-p_2-p_4-p_5-...-p_k)$, which is 0 under the kinematic constraints. Therefore, in the case of an even number of arbitrary upper legs, the summed contributions of diagrams that respect the same topology is still 0.

So far, the argumentation for even individual and multiple upper legs has been very similar. However, the attention has to be turned now towards the following case: when the upper legs are not individual, k can be odd (as opposed to n) and the total number of particles can still be even, therefore not trivially 0; k being odd means that there are an even number of upper main legs, so there is no middle leg and the prescription that relates diagrams with the lower leg in the left half with the ones in the right half does not work anymore. To approach this scenario, the following situations will be considered:

Firstly, the diagram where the lower leg is in the middle, so there are an equal number of upper legs to the left and right of it:



This has propagator:

$$\frac{1}{(p_1+p_k)^2...(p_1+...+p_{\frac{k+5}{2}})^2(p_3+...+p_{\frac{k+3}{2}})^2...(p_3+p_4)^2} \tag{60}$$

Secondly, all diagrams where the lower leg is to the left of this position. The same recursive reasoning for summing them as was previously used in 57 gives:

$$\frac{1}{(p_1+p_2)^2(p_1+p_k)^2...(p_1+p_k+p_{k-1}+...+p_{\frac{k+7}{2}})^2(p_3+p_4+...+p_{\frac{k+3}{2}})^2...(p_3+p_4)^2} \tag{61}$$

For the diagrams where the lower leg is to the right:

$$\frac{1}{(p_1+p_k)^2...(p_1+p_k+p_{k-1}+...+p_{\frac{k+5}{2}})^2(p_3+p_4+...+p_{\frac{k+1}{2}})^2...(p_3+p_4)^2(p_3+p_2)^2} \tag{62}$$

Adding 61 and 62 gives:

$$\frac{(p_1+p_2)^2(p_3+p_4+...+p_{\frac{k+3}{2}})^2+(p_3+p_2)^2(p_1+p_k+p_{k-1}+...+p_{\frac{k+5}{2}})^2}{(p_1+p_2)^2(p_1+p_k)^2...(p_1+p_k+p_{k-1}+...+p_{\frac{k+5}{2}})^2(p_3+p_4+...+p_{\frac{k+3}{2}})^2...(p_3+p_4)^2(p_3+p_2)^2} \tag{63}$$

But, by momentum conservation, $(p_1+p_2)^2=2p_1p_2=-2p_2p_3=-(p_2+p_3)^2$ and $(p_3+p_4+...+p_{\frac{k+3}{2}})^2=(p_1+p_k+p_{k-1}+...+p_{\frac{k+5}{2}}+p_2)^2$:

$$\frac{(p_1+p_2)^2[(p_1+p_k+p_{k-1}+...+p_{\frac{k+5}{2}}+p_2)^2-(p_1+p_k+p_{k-1}+...+p_{\frac{k+5}{2}})^2]}{(p_1+p_2)^2(p_1+p_k)^2...(p_1+p_k+p_{k-1}+...+p_{\frac{k+5}{2}})^2(p_3+p_4+...+p_{\frac{k+3}{2}})^2...(p_3+p_4)^2(p_3+p_2)^2} \tag{64}$$

Since $(p_1+p_k+p_{k-1}+...+p_{\frac{k+5}{2}}+p_2)^2=(p_1+p_k+p_{k-1}+...+p_{\frac{k+5}{2}})^2+2p_1p_2=(p_1+p_k+p_{k-1}+...+p_{\frac{k+5}{2}})^2-(p_2+p_3)^2$:

$$\frac{-(p_1+p_2)^2(p_2+p_3)^2}{(p_1+p_2)^2(p_1+p_k)^2...(p_1+p_k+p_{k-1}+...+p_{\frac{k+5}{2}})^2(p_3+p_4+...+p_{\frac{k+3}{2}})^2...(p_3+p_4)^2(p_3+p_2)^2} \tag{65}$$

The denominator, besides the first and last term, is the same as in equation 60 for the diagram where the lower leg is in the middle. Amplifying that equation by the first and last propagators and adding them together:

$$\frac{(p_1+p_2)^2(p_2+p_3)^2-(p_1+p_2)^2(p_2+p_3)^2}{(p_1+p_2)^2(p_1+p_k)^2...(p_1+p_k+p_{k-1}+...+p_{\frac{k+5}{2}})^2(p_3+p_4+...+p_{\frac{k+3}{2}})^2...(p_3+p_4)^2(p_3+p_2)^2} \tag{66}$$

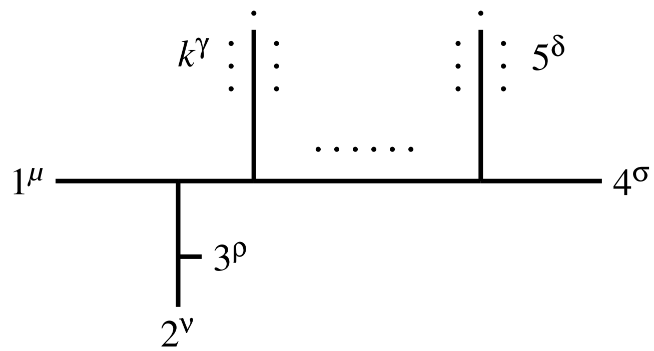
This is, evidently, 0. Therefore, the hidden zeros are manifest on subsets of diagrams of particle interactions in the NLSM to arbitrary multiplicity.

4.4 Extending the proof for an arbitrary splitting

4.4.1 $1 \to 4$ Splitting

One difficulty in establishing subsets for higher splitting is the presence of multiple lower topologies. For example, the $1 \to 4$ splitting has two.

The first one is:



The proof that the sum of diagrams with this topology vanishes on the kinematic locus is analogous to the one for the $1 \to 3$ splitting, with one distinction: consider the p_2 from the $1 \to 3$ splitting to be off shell and be equal to $p_2 + p_3$ from the $1 \to 4$ splitting (it will be denoted as p_2^*). Then move the label of each momenta $3 \to k$ by +1, such that it becomes $4 \to k+1$. For the proof to be equivalent, the following identities have to be validated:

$$(p_1 + p_2^*)^2 = -(p_4 + p_2^*)^2 \tag{67}$$

This is the equivalent to the identity $(p_1 + p_2)^2 = -(p_2 + p_3)^2$ from the $1 \to 3$ splitting, which was used to simplify equation 63. Since $(p_1 + p_2^*)^2 = 2p_1 p_2^* + (p_2^*)^2 = -2(p_2^*)^2 - 2p_2^* p_4 + (p_2^*)^2 = -(p_4 + p_2^*)^2$, the identity remains valid.

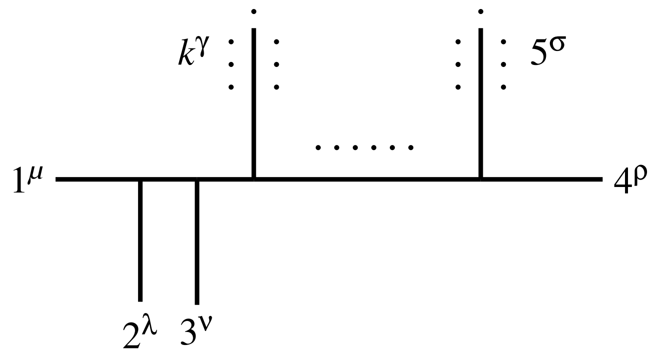
The other expression is:

$$(p_1 + p_2^*)^2 + (p_1 + p_{k+1} + ... + p_l)^2 = (p_1 + p_{k+1} + ... + p_j + p_2^*)^2 \tag{68}$$

where j is more than 4. This was used to simplify equation 55. Since $(p_1 + p_2^*)^2$ is $2p_1 p_2^* + (p_2^*)^2$ and the kinematic constraints imply $p_2^* p_{4,5...k+1} = 0$, the equation is still valid.

These were the only instances where the massless character of the p_2 momentum was used when proving the vanishing of subgroups on $1 \to 3$ splitting. Since the expression still stand for the modified off-shell momentum p_2^*, then the same proof as the one in the previous section will apply to this topology from the $1 \to 4$ splitting.

The second topology is:



In order to make manifest the vanishing of the subsets of diagrams with this lower topology, all possible permutations of the lower legs have to be considered. To see why that it, the following framework will be constructed: First, fix the position of the "2" leg and permute "3" on all positions; there will be $k-3$ contributing diagrams and the sum of them will be denoted S_1. Now, move "2" one position to the right and repeat the process to get $k-4$ diagrams that sum to S_2. Do this until the last position of "2", where there will be only one diagram, that will be denoted by S_{k-3}. To show the vanishing of the sum of diagrams on this subset means to show $S_1+S_2+...+S_{k-3}=0$.

The focus now is to find an expression for S_1. As was the case for the $1\to3$ splitting, only terms besides the propagators factor out between diagrams, so they are not relevant when trying to check whether the sum is 0. Consider the first diagram the one where the "3" leg is to the most right position, near the "4" leg. This will have propagator:

$$\frac{1}{(p_3+p_4)^2(p_3+p_4+p_5)^2...(p_3+p_4+...+p_k)^2} \tag{69}$$

The second diagram has propagator:

$$\frac{1}{(p_4+p_5)^2(p_3+p_4+p_5)^2...(p_3+p_4+...+p_k)^2} \tag{70}$$

Adding them:

$$\frac{(p_3+p_4)^2+(p_4+p_5)^2}{(p_3+p_4)^2(p_4+p_5)^2(p_3+p_4+p_5)^2...(p_3+p_4+...+p_k)^2} \tag{71}$$

The numerator becomes $2p_3p_4+2p_4p_5+(p_5)^2$ and, since $p_3p_5=0$ due to the kinematic constraints, this is equivalent to $(p_3+p_4+p_5)$. Therefore, the third term in the denominator is canceled.

Adding the third diagram will cancel the fourth term in the denominator and so on. This is due to the generality of the fact: $(p_3+p_4)^2+(p_4+p_5+...+p_j)^2=(p_3+p_4+p_5+...+p_j)^2$. So the propagator to the left of leg "3" will disappear when adding that diagram to the previous sum. This means that S_1 will be:

$$\frac{1}{(p_3+p_4)^2(p_4+p_5)^2...(p_4+p_5+...+p_k)^2} \tag{72}$$

For S_2 the reasoning works the same way, but there is one extra propagator $(p_1+p_k)^2$ which is to the left of leg "2" and therefore not canceled by summing the contributing diagrams. Since p_k is to the right of the lower legs, the terms that coincide with the ones from S_2 stop at $(p_4+p_5+...+p_{k-1})^2$. Therefore, S_2 takes the form:

$$\frac{1}{(p_3+p_4)^2(p_4+p_5)^2...(p_4+p_5+...+p_{k-1})^2(p_1+p_k)^2} \tag{73}$$

Adding S_1 to S_2:

$$\frac{(p_1+p_k)^2+(p_4+p_5+..+p_k)^2}{(p_3+p_4)^2(p_4+p_5)^2...(p_4+p_5+...+p_{k-1})^2(p_4+p_5+...+p_k)^2(p_1+p_k)^2} \tag{74}$$

By momentum conservation: $(p_4+p_5+..+p_k)^2=(p_1+p_2+p_3)^2$. The numerator becomes $p_1p_k+(p_k)^2+p_1p_2+p_1p_3+p_2p_3$. Due to the kinematic constraints: $p_2p_k=p_3p_k=0$; the numerator is: $(p_1+p_2+p_3+p_k)^2=(p_4+p_5+...+p_{k-1})^2$, which cancels the third to last term in the denominator:

$$\frac{1}{(p_3+p_4)^2(p_4+p_5)^2...(p_4+p_5+...+p_{k-2})^2(p_1+p_k)^2(p_1+p_2+p_3)^2} \tag{75}$$

S_3 is:

$$\frac{1}{(p_3+p_4)^2(p_4+p_5)^2...(p_4+p_5+...+p_{k-2})^2(p_1+p_k+p_{k-1})^2(p_1+p_k)^2} \tag{76}$$

Adding it to the previous sum will cancel the fourth to last term in the denominator $(p_4+...+p_{k-2})^2$:

$$S_1+S_2+S_3=\frac{1}{(p_3+p_4)^2(p_4+p_5)^2...(p_4+p_5+...+p_{k-3})^2(p_1+p_k+p_{k-1})^2(p_1+p_k)^2(p_1+p_2+p_3)^2} \tag{77}$$

In general terms, adding S_j to $S_1+...+S_{j-1}$ will transform the $(p_4+...+p_{k-j+1})^2$ term into $(p_1+p_k+...+p_{k-j+2})^2$.

This means that summing all S besides the last one gives:

$$S_1+...+S_{k-4}=\frac{1}{(p_3+p_4)^2(p_1+p_k)^2..(p_1+p_k+...+p_6)^2(p_1+p_2+p_3)^2} \tag{78}$$

S_{k-3} only has one contributing diagram, where both legs "2" and "3" are to the right of all upper main legs. That has propagator:

$$\frac{1}{(p_3+p_4)^2(p_1+p_k)^2..(p_1+p_k+...+p_6)^2(p_2+p_3+p_4)^2} \tag{79}$$

Adding this to the previous sum yields the total contributions of diagrams within the subset of this lower leg topology:

$$\frac{(p_1+p_2+p_3)^2+(p_2+p_3+p_4)^2}{(p_3+p_4)^2(p_1+p_k)^2..(p_1+p_k+...+p_6)^2(p_1+p_2+p_3)^2(p_2+p_3+p_4)^2} \tag{80}$$

$$\frac{p_1p_2+p_2p_3+p_1p_3+p_2p_3+p_3p_4+p_2p_4}{(p_3+p_4)^2(p_1+p_k)^2..(p_1+p_k+...+p_6)^2(p_1+p_2+p_3)^2(p_2+p_3+p_4)^2} \tag{81}$$

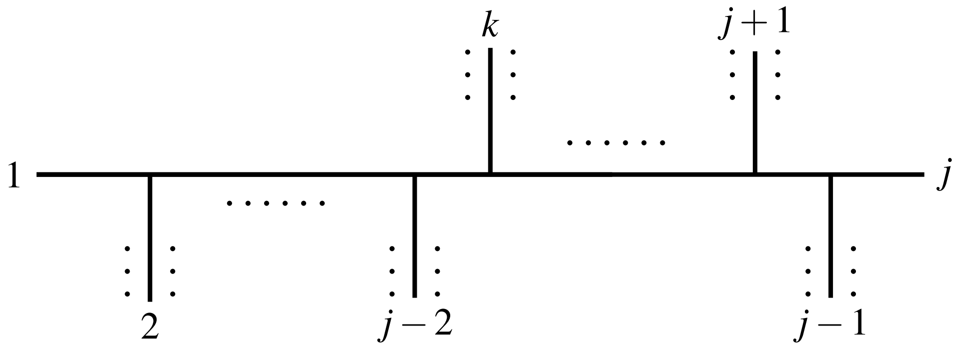
$$\frac{p_2(p_1+p_3+p_4)+p_3(p_1+p_2+p_4)}{(p_3+p_4)^2(p_1+p_k)^2..(p_1+p_k+...+p_6)^2(p_1+p_2+p_3)^2(p_2+p_3+p_4)^2} \tag{82}$$

$$\frac{-p_2(p_2+p_5+..+p_k)-p_3(p_3+p_5+...+p_k)}{(p_3+p_4)^2(p_1+p_k)^2..(p_1+p_k+...+p_6)^2(p_1+p_2+p_3)^2(p_2+p_3+p_4)^2} \tag{83}$$

Upon implementing masslessness of p_2 and p_3 and the kinematic constraints specific to the $1 \to 4$ splitting, the numerator becomes 0. Therefore, all diagrams represented by this splitting can be grouped into subsets that individually vanish on the hidden zero conditions.

4.4.2 General proof

The final step in providing an argument for the existence of hidden zeros in any subset with the same topology on the secondary legs is to consider an arbitrary splitting $1 \to j$. These will have an unknown number of lower main legs with labels that are off-shell momenta, as they constitute a sum of the momenta that branch off from that respective tree. Following the pattern observed from the two previous splittings that were analyzed, the expectation is that the hidden zero will become manifest when all lower legs are permuted on all possible positions. First, all lower legs besides the last one will be fixed on the left of the diagram and the last leg $j-1$ will be permuted. The first diagram looks like:



This has propagator:

$$\frac{1}{(p_j+p_{j-1})^2(p_j+p_{j-1}+p_{j+1})^2...(p_1+p_2)^2} \tag{84}$$

The second diagram has propagator:

$$\frac{1}{(p_j+p_{j+1})^2(p_j+p_{j-1}+p_{j+1})^2...(p_1+p_2)^2} \tag{85}$$

Adding them together gives the following term in the numerator: $(p_j + p_{j+1})^2 + (p_j + p_{j-1})^2$. Implementing masslessness of p_j and the kinematic locus constraint $p_{j-1}p_{j+1} = 0$ transforms the numerator into $(p_j + p_{j-1} + p_{j+1})^2$. This is the third term in the denominator and it gets canceled. Following the same logic, adding all diagrams where only the last leg is moved gives:

$$S_1 = \frac{1}{(p_j + p_{j-1})^2(p_j + p_{j+1})^2(p_j + p_{j+1} + p_{j+2})^2...(p_j + ... + p_k)^2(p_1 + p_2)^2...(p_1 + ... + p_{j-3})^2} \tag{86}$$

Now, moving the second lower leg $j-2$ one position to the right and permuting the last leg in all remaining positions:

$$S_2 = \frac{1}{(p_j + p_{j-1})^2(p_j + p_{j+1})^2...(p_j + ... + p_{k-1})^2(p_1 + p_2)^2...(p_1 + ... + p_{j-3})^2(p_1 + ... + p_{j-3} + p_k)^2} \tag{87}$$

Adding S_1 to S_2 will produce the following term in the numerator: $(p_j + ... + p_k)^2 + (p_1 + ... + p_{j-3} + p_k)^2$. Since $(p_j + ... + p_k)^2 = (p_1 + ... + p_{j-1})^2$, using the constraints imposed by the kinematic locus the terms can be rewritten as:$(p_1 + ... + p_{j-1} + p_k)^2 + (p_1 + ... + p_{j-3})^2$. Using momentum conservation again to rewrite the first term: $(p_j + ... + p_{k-1})^2 + (p_1 + ... + p_{j-3})^2$. Each term cancels one propagator, so the sum is split into two fractions with 1 in the numerator.

Adding S_3 to this will split the first term of the previous sum into two terms, one cancels the $(p_j + ... + p_{k-2})^2$ propagator and the other one cancels $(p_1 + ... + p_{j-3})^2$ again. Therefore, adding all S besides the last one (up to and including S_{k-j}) gives the following expansion:

$$\begin{gathered}\frac{1}{(p_{j-1}+p_j)^2(p_j+\cdots+p_k)^2(p_1+\cdots+p_{j-3}+p_k)^2\cdots(p_{j-2}\cdots p_{j+1})^2\dots(p_1+\cdots+p_{j-3})^2}+\\ \frac{1}{(p_{j-1}+p_j)^2(p_j+p_{j+1})^2(p_j+\cdots+p_k)^2(p_1+\cdots+p_{j-3}+p_k)^2\cdots(p_{j-2}\cdots p_{j+1})^2\dots(p_1+\cdots+p_{j-4})^2}+\\ \frac{1}{(p_{j-1}+p_j)^2..(p_j+p_{j+1}+p_{j+2})^2(p_j+\cdots+p_k)^2(p_1+\cdots+p_{j-3}+p_k)^2..(p_{j-2}\cdots p_{j+2})^2\dots(p_1+\cdots+p_{j-4})^2}\\ +\cdots\end{gathered} \tag{88}$$

Now, S_{k-j+1} has to be added. It consists only of one diagram where the last two lower legs are all the way to the right of the diagram. It has propagator:

$$\frac{1}{(p_{j-1} + p_j)(p_1 + \cdots + p_{j-3} + p_k + \cdots + p_{j+1})^2 \dots (p_1 + \cdots + p_{j-3})^2} \tag{89}$$

Adding this to the first term in the previous sum has the following numerator: $(p_j + \cdots + p_k)^2 + (p_1 + \cdots + p_{j-3} + p_k + \cdots + p_{j+1})^2$. By momentum conservation: $(p_1 + \cdots + p_{j-1})^2 + (p_1 + \cdots + p_{j-3} + p_k + \cdots + p_{j+1})^2$. Using the kinematic constraints, the terms can be redistributed as $(p_1 + \cdots + p_{j-1} + p_k + \cdots + p_{j+1})^2 + (p_1 + \cdots + p_{j-3})^2$. The first term is $(p_j)^2$, which is 0. Therefore, the entire sum will have $k-j$ terms that all have as the last propagator:$(p_1 + \cdots + p_{j-4})^2$.

Now, the terms in the sum can be regrouped since all terms that are next to each other vary by at most one factor in the denominator. For example, adding the first term with the second one has the numerator: $(p_{j-2}+p_{j-1}+p_j)^2+(p_j+p_{j+1})^2=(p_{j-2}+p_{j-1}+p_j+p_{j+1})^2$, which cancels one propagator and reduces the number of terms in the expansion. Adding the third term to this has numerator: $(p_{j-2}+p_{j-1}+p_j)^2+(p_j+p_{j+1}+p_{j+2})^2=(p_{j-2}+p_{j-1}+p_j+p_{j+1}+p_{j+2})^2$, which again cancels one propagator. Following this procedure, the entire sum will be:

$$\Sigma S=\frac{1}{(p_{j-2}+p_{j-1}+p_j)^2(p_{j-1}+p_j)^2(p_j+p_{j+1})^2...(p_j+\cdots+p_k)^2(p_1+p_2)^2...(p_1+\cdots+p_{j-4})^2} \tag{90}$$

This represents the sum of all diagrams where legs $j-2$ and $j-1$ are permuted in all possible positions while the others are fixed. If $j-3$ is moved one position to the right, the resulting sum will be identical besides the fact that $(p_j+\cdots+p_k)^2$ is transformed into $(p_1+\cdots+p_{j-4}+p_k)^2$, since the diagrams will have one more propagator to the left of $j-3$ and one less to the right. If $j-2$ and $j-1$ are permuted in every position possible in this configuration, the resulting contribution will be:

$$\Sigma S'=\frac{1}{(p_{j-2}+p_{j-1}+p_j)^2(p_{j-1}+p_j)^2(p_j+p_{j+1})^2...(p_j+\cdots+p_{k-1})^2(p_1+p_2)^2...(p_1+\cdots+p_{j-4}+p_k)^2} \tag{91}$$

Adding ΣS to $\Sigma S'$ will have the numerator: $(p_j+\cdots+p_k)^2+(p_1+\cdots+p_{j-4}+p_k)^2$. This is equivalent to $(p_1+\cdots+p_{j-1})^2+(p_1+\cdots+p_{j-4}+p_k)^2$. Due to the kinematic constraints, the p_k term can be transferred from the second parenthesis to the first one: $(p_1+\cdots+p_{j-1}+p_k)^2+(p_1+\cdots+p_{j-4})^2$. By momentum conservation, this equals: $(p_j+\cdots+p_{k-1})^2+(p_1+\cdots+p_{j-4})^2$. This splits the sum into two fractions with unitary numerators. This situation is highly analogous to the initial addition, which resulted in the expansion from 88. Adding the last term will again yield a zero by exploiting masslessness of p_j. Therefore, after contracting the remaining terms in the expansion, the result is:

$$\Sigma(\Sigma S)_{j-3}=\frac{1}{(p_{j-3}+\cdots+p_j)^2\cdots(p_{j-1}+p_j)^2(p_j+p_{j+1})^2...(p_j+\cdots+p_k)^2(p_1+p_2)^2...(p_1+\cdots+p_{j-5})^2} \tag{92}$$

Comparing this to S_1 (only $j-1$ is permuted) and ΣS ($j-2$ and $j-1$ permuted), the following general formula for the contribution of diagrams where only legs $j-1$ to $j-i$ are permuted:

$$\Sigma(\Sigma(...(\Sigma S)_{j-i}=\frac{1}{(p_{j-i}+\cdots+p_j)^2..(p_{j-1}+p_j)^2(p_j+p_{j+1})^2..(p_j+\cdots+p_k)^2(p_1+p_2)^2..(p_1+\cdots+p_{j-i-2})^2} \tag{93}$$

In order to assert the validity of this formula, the following inductive reasoning will be applied: assume it is true and see whether $\Sigma(\Sigma(...(\Sigma S)_{j-i-1}$ follows directly from it. Moving leg $j-i-1$ one position to the right will change the sum in 94: the term $(p_j+...+p_k)^2$ will transform into $(p_1+\cdots+p_{j-i-2}+p_k)^2$. Adding this new sum to the sum in 94 will result in: $(p_j+...+p_k)^2+(p_1+\cdots+p_{j-i-2}+p_k)^2$. Rewriting the first term using momentum conservation: $(p_1+\cdots+p_{j-1})^2+(p_1+\cdots+p_{j-i-2}+p_k)^2$. Using the kinematic constraints, the p_k term can be moved from the second parenthesis to the first one. Since each factor cancels a propagator, the addition of more contributions by moving $j-i-1$ one step further will increase the number of terms in a similar expansion as in 88.

Adding the last diagram (where legs $j-1 \to j-i-1$ are all the way to the right) to the first term in the expansion will result in: $(p_1+p_2+\cdots+p_{j-i-2}+p_k+\cdots+p_{j+1})^2+(p_j+\cdots+p_k)^2$. Using momentum conservation to rewrite the second term: $(p_1+p_2+\cdots+p_{j-i-2}+p_k+\cdots+p_{j+1})^2+(p_1+\cdots+p_{j-1})^2$. On the kinematic locus, this equals: $(p_1+p_2+\cdots+p_{j-i-2}+\cdots+p_{j-1}+p_k+\cdots+p_{j+1})^2+(p_1+\cdots+p_{j-i-2})^2$. Since the first term is $(p_j)^2=0$, all factors in the expansion will have as the last propagator $(p_1+\cdots+p_{j-i-3})^2$. Analogous to what has happened before, this means that they all differ by at most one term and the sum can be regrouped into:

$$\Sigma(\Sigma(...(\Sigma S)_{j-i-1} = \frac{1}{(p_{j-i-1}+\cdots+p_j)^2..(p_{j-1}+p_j)^2(p_j+p_{j+1})^2..(p_j+\cdots+p_k)^2..(p_1+\cdots+p_{j-i-3})^2} \tag{94}$$

This means that the formula in 94 is correct and can be used to express the contribution of diagrams where $j-i$ legs are permuted in all possible positions and the rest are kept fixed in the left most position. Permuting one more leg will eliminate one of the propagators of the form $\frac{1}{(p_1+p_2)^2...(p_1+\cdots+p_{j-i-3})^2}$. For $\Sigma(\Sigma(...(\Sigma S)_4$ only $(p_1+p_2)^2$ contributes. Permuting one more leg to get $\Sigma(\Sigma(...(\Sigma S)_3$ will cancel $(p_1+p_2)^2$ and the expression remains without any propagators of that form. That is the extent to which this general formula is valid.

Therefore, this formula can be used to calculate the contribution of diagrams where all legs besides ”2” are permuted:

$$\Sigma(\Sigma(...(\Sigma S)_3 = \frac{1}{(p_3+p_4+\cdots+p_j)^2..(p_{j-1}+p_j)^2(p_j+p_{j+1})^2..(p_j+\cdots+p_k)^2} \tag{95}$$

Moving leg ”2” one position to the right will transform the $(p_j+\cdots+p_k)^2$ term into $(p_1+p_k)^2$, resulting into:

$$\Sigma(\Sigma(...(\Sigma S)_3^* = \frac{1}{(p_3+p_4+\cdots+p_j)^2..(p_{j-1}+p_j)^2(p_j+p_{j+1})^2..(p_j+\cdots+p_{k-1})^2(p_1+p_k)^2} \tag{96}$$

Adding $\Sigma(\Sigma(...(\Sigma S)_3$ to $\Sigma(\Sigma(...(\Sigma S)_3^*$ will result in the numerator: $(p_1+p_k)^2+(p_j+\cdots+p_k)^2$. Using momentum conservation and the kinematic constraints, this becomes: $(p_j+\cdots+p_{k-1})^2$, which cancels a propagator. Summing over all positions of ”2” besides the last one results in:

$$\frac{1}{(p_3+p_4+\cdots+p_j)^2..(p_{j-1}+p_j)^2(p_j+\cdots+p_k)^2(p_1+p_k)^2...(p_1+p_k+p_{k-1}...p_{j+2})^2} \tag{97}$$

The last diagram, where all lower legs are to the right of all upper legs, has propagator:

$$\frac{1}{(p_2+p_3+\cdots+p_j)^2..(p_{j-1}+p_j)^2(p_j+\cdots+p_k)^2(p_1+p_k)^2...(p_1+p_k+p_{k-1}...p_{j+2})^2} \tag{98}$$

Adding this to the previous sum in 97 gives the entire contribution of diagrams respecting this topology on the $1 \to j$ splitting. It will have numerator: $(p_2+p_4+\cdots+p_j)^2+(p_j+\cdots+p_k)^2$. Using the kinematic constraints, these can be reunited in one term: $(p_2+p_3+\cdots+p_j+p_{j+1+}\cdots+p_k)^2$. By momentum conservation, this becomes $(p_1)^2$, which is 0. This concludes the proof for the vanishing of subsets of diagrams on the kinematic locus for an arbitrary splitting.

5 Conclusion

This thesis successfully proved the existence of subsets of diagrams that vanish individually on the kinematic locus that defines the hidden zero. In this way, the existence of hidden zeros is made manifest by observing the fact that any scattering amplitude of arbitrarily high multiplicity is composed of such subsets where the topology of the secondary legs is kept the same, but the order is switched.

This was done by using the Feynman rules implied by the vector formulation of the Nonlinear Sigma Model. Lower-point amplitudes were first analyzed and a pattern emerged that motivated the investigation and generalization of this property. Then, the proof was extended to include all possible configurations for n-point scattering amplitudes on the $1 \to 3$ splitting, starting from the more simple case of individual main upper legs and then considering arbitrary legs. Afterwards, the $1 \to 4$ splitting proof was formulated and the possibilities increased in complexity, as now there were two lower legs. In the end, the general proof for the vanishing of subsets on the $1 \to j$ splitting was constructed. In this way, all possible configurations were considered and the existence of hidden zeros on corresponding subsets was demonstrated.

The structuring of the previous argument presents a remarkable similarity to the ones in [7]. Therefore, a natural extension of the theoretical considerations of this thesis would be to attempt to link the geometry of these groupings of diagrams to a factorization channel that can express higher-point amplitudes in terms of lower-point ones in the kinematic regions near the locus.

Another interesting direction might be to consider field theories that are similar to the NLSM and also have alternative formulations, such as DBI or SG, and see whether those formulations could point to underlying symmetries or groupings that can make manifest certain important properties, such as hidden zeros.

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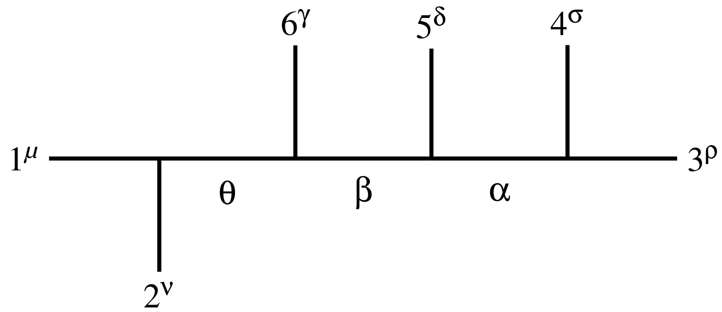
Appendices

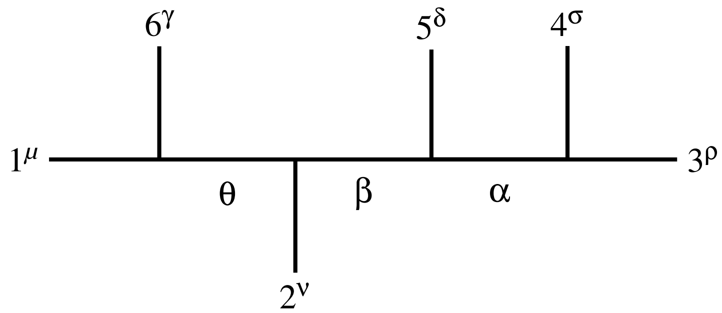
A 6-point amplitudes

In this section of the appendix, the p_3q contributions of every 6-point amplitude will be shown. Also, they will be shown in groups that form a subset that vanishes individually on the kinematic locus.

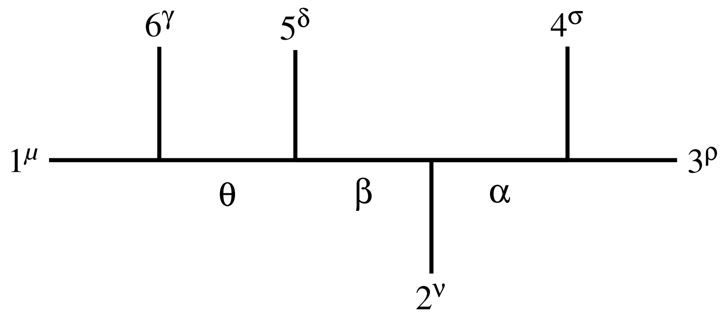
The first group is:



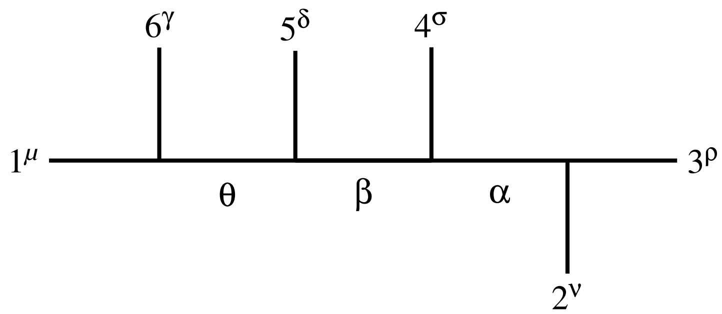
$$= -\frac{[(p_3+p_4)p_5][(p_1+p_2)p_6]}{(p_3+p_4+p_5)^2}$$



$$= - - \frac{[(p_1+p_6)p_2[(p_3+p_4)p_5]]}{(p_3+p_4+p_5)}$$

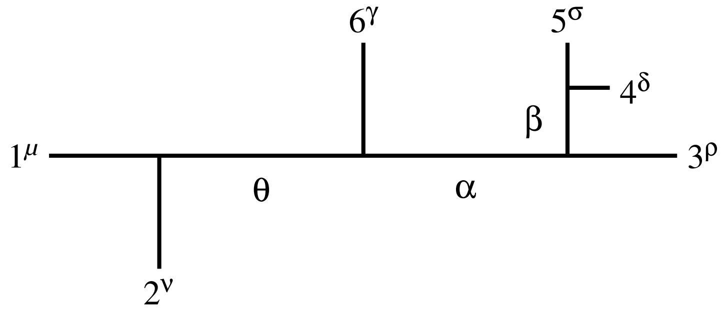


$$= -\frac{[(p_3+p_4)p_2][(p_1+p_6)p_5]}{(p_2+p_3+p_4)^2}$$

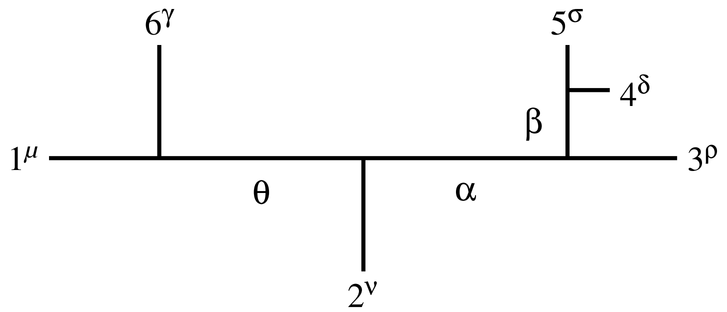


$$= -\frac{[(p_2+p_3)p_4][(p_1+p_6)p_5]}{(p_2+p_3+p_4)^2}$$

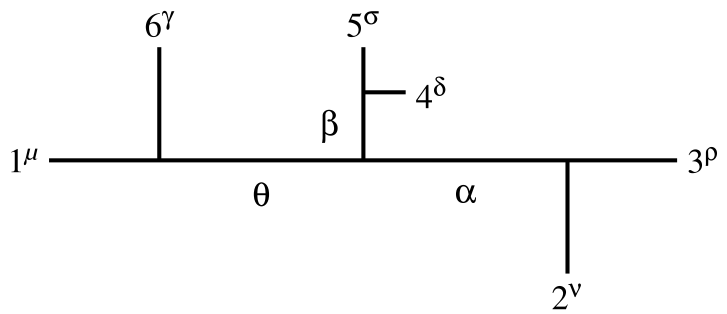
The second group is:



$$= \frac{[(p_4+p_5)p_3][(p_1+p_2)p_6]}{(p_3+p_4+p_5)^2}$$

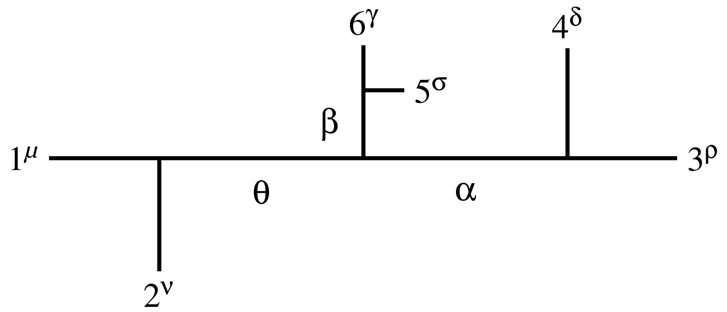


$$= \frac{[(p_1+p_6)p_2][(p_5-p_4)p_3]}{(p_3+p_4+p_5)^2}$$

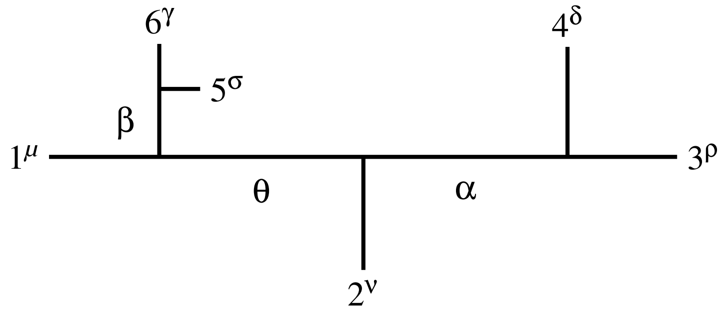


$$= -(p_2+p_3)(p_5-p_4)$$

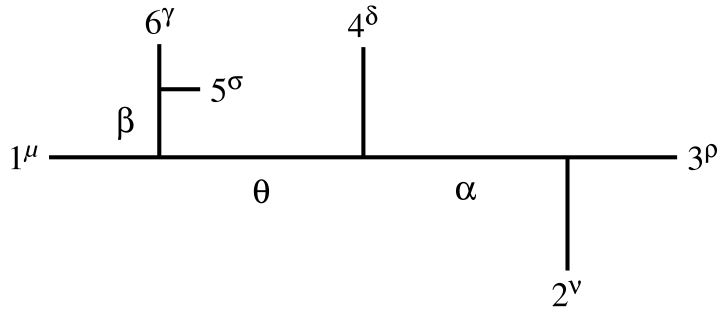
The third group is:



$$= -(p_3 + p_4)(p_6 - p_5)$$

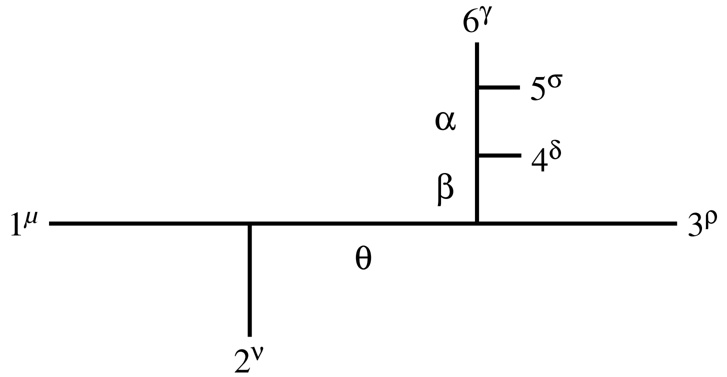


$$= -\frac{[(p_6 - p_5)p_1][(p_3 + p_4)p_2]}{(p_2 + p_3 + p_4)^2}$$

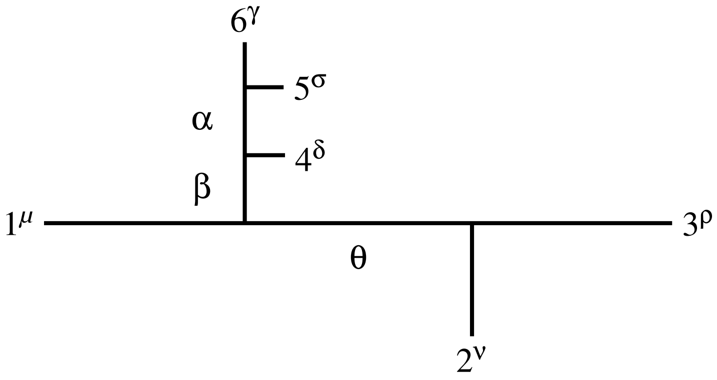


$$= -\frac{[(p_2 + p_3)p_4][(p_6 - p_5)p_1]}{(p_2 + p_3 + p_4)^2}$$

The fourth group is:

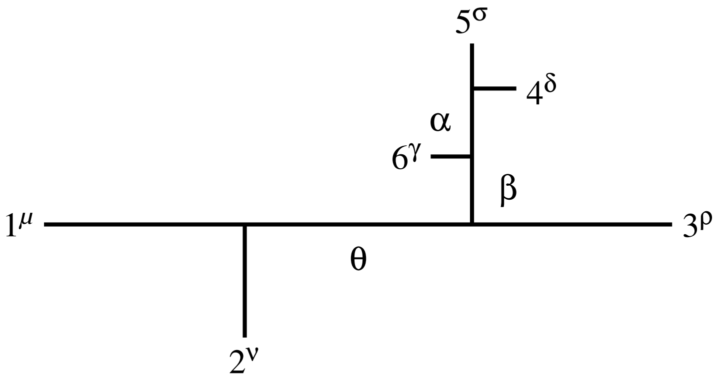


$$= \frac{[(p_5 + p_6)p_4][(p_6 - p_5)p_3] - [(p_6 - p_5)p_4](p_3 p_4)}{(p_1 + p_2 + p_3)^2}$$

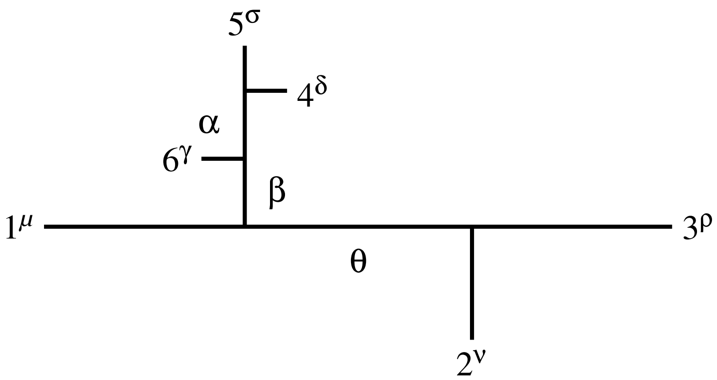


$$= \frac{-[(p_5+p_6)p_4][(p_6-p_5)(p_2+p_3)]+[(p_6-p_5)p_4][(p_2+p_3)p_4]}{(p_1+p_2+p_3)^2}$$

The fifth group is:



$$= \frac{[(p_5-p_4)p_6](p_3p_6)-[(p_4+p_5)p_6][(p_5-p_4)p_3]}{(p_1+p_2+p_3)^2}$$



$$= -\frac{[(p_2+p_3)p_6][(p_5-p_4)(p_2+p_3+p_6]+(p_1p_6)[(p_5-p_4)(p_2+p_3)]}{(p_1+p_2+p_3)^2}$$