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The Bas-Serra Surface

Bachelor's Project Mathematics

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Abstract

This project analyses the Bas-Serra surface, extending the existing surface to the full projection into $\mathbb{R}^3$ when the construction is considered over the base field $\mathbb{C}$. The construction of the surface is generalised to arbitrary planar conics in the projective space and the irreducibility of these surfaces is considered. Further, the singularities and symmetries of the surface are studied.

Contents

1 Introduction

The Bas-Serra surface has its origin in an artwork by the artist Richard Serra (1938-2024), *Double Torqued Ellipse*. Seen below, this artwork consists of a monumental sculpture exhibition which sits in the Guggenheim Bilbao Museoa [1]. Serra constructed the *Double Torqued Ellipse* by placing an ellipse on either side of an axle as oblong wheels, and rotating each ellipse such that their long axes are perpendicular. Rolling this axle along a flat surface, the points of contact are exactly the points at which the tangent vectors to each ellipse are parallel. Drawing a line between each of these points, as if wrapping the plane of contact around the ellipses, the surface is ruled. Later, Bas Edixhoven (1962-2022) of Leiden further developed the surface into its current form by the addition of his so-called *Siamese Twins* [3], an extra set of lines joining the two ellipses. For each tangent vector on an ellipse, there is a point on the other side of the ellipse with a parallel tangent vector, and it is to this point that each *twin* line is drawn.

Figure 1: **Richard Serra's *Double Torqued Ellipse*** [1]

Various properties and descriptions of the surface have been worked on, and the interested reader can find these existing works in [11], [9], [10]. This thesis provides a full description of the surface over the complex numbers, following the same construction but over $\mathbb{C}$, and looking at the intersection of this extension with the real space, showing there to be additional curves in this case. Further, the set of singularities of the surface consist of three distinct sets of curves which are likewise described. Treating the surface algebraically, there exists an implicit description which, in its size and degree, becomes difficult to work with. While a goal of this work was to find an equivalent surface with a simpler description, unfortunately this has not been achieved. However, in searching for a solution we provide a detour into the symmetries of the surface which we call S. Finally, generalising the Bas-Serra construction to arbitrary conics we study under which conditions the resulting surface is irreducible.

2 Geometric Construction

The Bas-Serra surface S is defined by considering two ellipses,

$$C_1 \colon 2x^2 + y^2 = 1, \tag{2.1}$$

$$C_2 \colon x^2 + 2y^2 = 1, \tag{2.2}$$

and placing C_1 on the plane $z = -1$ and C_2 on $z = 1$ respectively. From the symmetry between x and y it is clear that these are the same ellipse, simply reflected about $y = x$. For a point $p_1 = (x_1, y_1, 1) \in C_1$, consider the tangent vector to C_1 at p_1 given by $T_1 = (-y_1, 2x_1, 0)$. There are two points on the upper ellipse C_2 which will have a tangent vector parallel to T_1. Taking one such point $p_2 = (x_2, y_2, -1)$ we have by construction the tangent vector $T_2 = (2y_2, -x_1, 0)$ such that $T_1 \times T_2 = 0$, where $\times$ is the cross product. This gives us $T_1 \times T_2 = 4x_1y_2 - x_2y_1 = 0$. We are then left with the three equations describing a curve E in, for now, $\mathbb{R}^4$, given by

$$\begin{cases} 2x_1^2 + y_1^2 - 1 = 0 & \text{(E1)} \\ x_2^2 + 2y_2^2 - 1 = 0 & \text{(E2)} \\ 4x_1y_2 - x_2y_1 = 0. & \text{(E3)} \end{cases}$$

The Bas-Serra surface is then defined by drawing straight lines between such points p_1 and p_2, as well as the twin p_1 and p_3. Thus, the surface S is given by, where $x_i, y_i \in E$,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \\ 2 \end{pmatrix}.$$

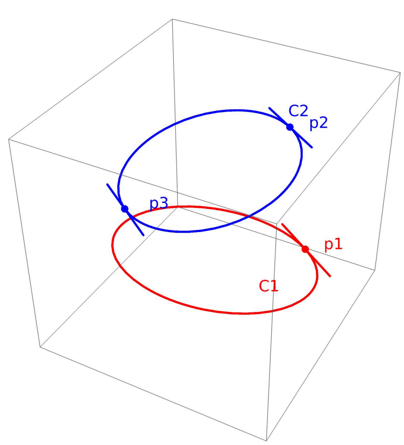


Figure 2: Points $p_1 \in C_1$ and $p_2, p_3 \in C_2$ with parallel tangent vectors.

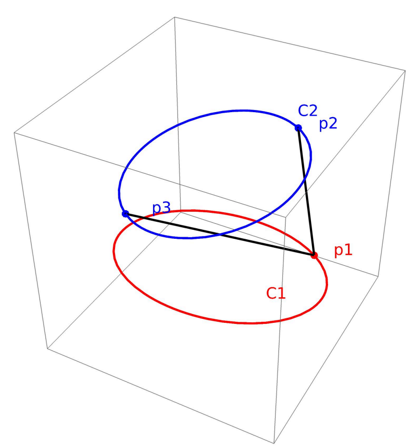


Figure 3: Drawing the lines p_1p_2, the outer surface by Serra, and p_1p_3, Bas' *twin* line.

Letting $\lambda \in [0, 1]$ develops the surface between the two ellipses, while letting $\lambda \in \mathbb{R}$ extends the surface indefinitely. While this extension is not part of the classical surface, we will study it as well. More formally, the existing surface is the projection Φ from E to $\mathbb{R}^3$,

$$\Phi : E \times [0,1] \to S \subset \mathbb{R}^3,$$
$$(x_1, y_1, x_2, y_2, \lambda) \mapsto \begin{pmatrix} x_1 \\ y_1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \\ 2 \end{pmatrix}.$$

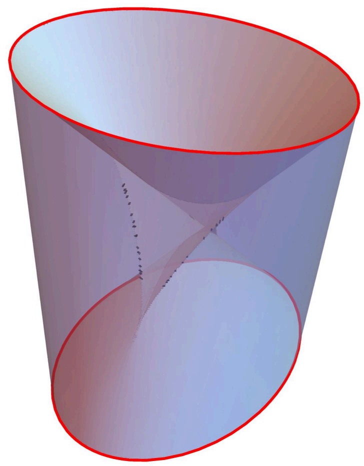


Figure 4: The Bas-Serra surface

3 Algebraic Construction

The construction given for the Bas-Serra surface has, so far, been entirely geometric. However, it proves very useful in further studying the surface to construct it through much more algebraic means. This allows us to further use computational tools such as Magma, Sagemath and Mathematica as well.

3.1 A brief introduction to Algebraic Geometry

Throughout this paper, basic knowledge regarding rings, ideals, and fields is assumed, but relevant results will be stated explicitly. For further reading and source material the lecture notes from the University of Groningen on *Group Theory* [5] by Jaap Top and Steffen Müller, *Algebraic Structures* [12] and *Advanced Algebraic Structures* [8], both by Jaap Top, are recommended. Further, for a more in depth introduction to algebraic geometry, the following texts are relevant: *Introduction to Algebraic Geometry*[7] by Justin Smith, *Undergraduate Algebraic Geometry*[6] by Miles Reid, and

Algebraic Geometry[4] by Robin Hartshorne. Here, we follow Hartshorne and Reid, and make use of Smith for sections in computational algebraic geoemtry.

Throughout this paper let k be an algebraically closed field, which we can without loss consider to be $\mathbb{C}$.

Definition 3.1 (Affine n-space)**.** *For a field k, the Affine n-space $\mathbb{A}^n := \mathbb{A}^n_k$ over k is the set of all n-tuples $p = [a_1, \dots, a_n], a_i \in k$, where p is considered a point in $\mathbb{A}^n$.*

The affine space, while similar to k^n, does not carry the structure of a vector space and is simply a point space. When convenient, we will simply write $\mathbb{R}^n$ and $\mathbb{C}^n$ rather than $\mathbb{A}^n_{\mathbb{R}}$ and $\mathbb{A}^n_{\mathbb{C}}$. A given polynomial function from $\mathbb{A}^n$ to k is then an element in the polynomial ring $R = k[x_1, \dots, x_n]$. For a subset of polynomials $T \subset R$, define the *zero set* of T in $\mathbb{A}^n$ to be all points p such that every polynomial in T vanishes at p:

$$Z(T) = \{p \in \mathbb{A}^n : f(p) = 0 \; \forall f \in T\}.$$

Definition 3.2 (Algebraic Set)**.** *A subset $X \subset \mathbb{A}^n$ is an algebraic set if there exists a subset $T \subset R$ of polynomials such that X is the zero set of T: $X = Z(T)$.*

We define similarly the *ideal* of a set $X \subset \mathbb{A}^n$ to be all polynomials that are everywhere vanishing on X:

$$\mathcal{I}(X) := \{f \in k[x_1, \dots x_n] : f(p) = 0 \quad \forall p \in X\}).$$

Definition 3.3 (Coordinate Ring)**.** *Let $X \subset \mathbb{A}^n$ be an algebraic set. The coordinate ring $k[X]$ is the quotient ring*

$$k[X] = k[x_1, \dots, x_n]/\mathcal{I}(X).$$

First, we note that the algebraic sets give rise to a (very coarse) topology on $\mathbb{A}^n$.

Definition 3.4 (Zariski Topology)**.** *A subset $X \subset \mathbb{A}^n$ is open if it is the complement of a an algebraic set; a set is closed if and only if it is an algebraic set.*

The fact that this is a well defined topology can be easily checked.

3.2 The ideal of S

Recall, we defined S as the zero set of the following polynomials:

$$E_1 = 2x_1^2 + y_1^2 - 1 \tag{E1}$$

$$E_2 = x_2^2 + 2y_2^2 - 1 \tag{E2}$$

$$E_3 = 4x_1y_2 - x_2y_1 \tag{E3}$$

$$F_1 = (1-\lambda)x_1 + \lambda x_2 - x \tag{F1}$$

$$F_2 = (1-\lambda)y_1 + \lambda y_2 - y \tag{F2}$$

$$F_3 = 2\lambda - 1 - z, \tag{F3}$$

each of which is a polynomial in the ring $R := \mathbb{R}[x_1, y_1, x_2, y_2, x, y, z, \lambda]$. Thus, we define the ideal $I_s := (E_1, E_2, E_3, F_1, F_2, F_3) \subset R$, and consider the zero set

$$Z(I_s) = \{p \in \mathbb{A}^8 : E_i(p) = F_i(p) = 0, i = 1, 2, 3\}.$$

The surface S is then precisely the intersection $Z(I_s) \cap \mathbb{A}^3$, where $\mathbb{A}^3$ corresponds to the affine space for what we will call the spatial variables x, y, z. Similarly, we can express S as the zero set of the ideal $J_s = I_s \cap k[x, y, z]$. It follows that the generators for J_s will be polynomials in $k[x, y, z]$ which define S. In order to compute J_s, we make use of Sage (or alternatively Magma, Maple, or other software). For an introduction to the mechanisms behind this, read Chapter 2.3: *Computations in Polynomials Rings: Gröbner Bases* [7] by Justin Smith. Briefly, this operates by defining an ordering on the polynomials in $k[x_1, \ldots, x_n]$ (several are possible), and using this to construct the *Gröbner Basis* for an ideal I. Then, by the Elimination Property (Proposition 2.3.23 in [7]), with $\{g_1, \ldots, g_l\}$ a Gröbner basis for I, for $1 \leq t \leq n$ the intersection $I \cap k[x_t, \ldots x_n] = \{g_1, \ldots, g_l\} \cap k[x_t, \ldots, x_n]$. This firstly shows S to indeed be an algebraic set, and further computing J_s we get the single, irreducible generator given by

$$\begin{aligned} P_8 = {} & 16384x^8 + 81920x^6y^2 + 135168x^4y^4 + 81920x^2y^6 + 16384y^8 - 6144x^6z^2 \\ & - 76800x^4y^2z^2 - 76800x^2y^4z^2 - 6144y^6z^2 - 3776x^4z^4 + 24832x^2y^2z^4 \\ & + 3776y^4z^4 - 336x^2z^6 - 336y^2z^6 + 9z^8 + 45056x^6z + 30720x^4y^2z \\ & - 30720x^2y^4z - 45056y^6z - 20736x^4z^3 + 20736y^4z^3 - 5472x^2z^5 \\ & + 5472y^2z^5 - 6144x^6 - 76800x^4y^2 - 76800x^2y^4 - 6144y^6 + 40832x^4z^2 \\ & + 77312x^2y^2z^2 + 40832y^4z^2 - 16560x^2z^4 - 16560y^2z^4 - 612z^6 \\ & - 20736x^4z + 20736y^4z + 20160x^2z^3 - 20160y^2z^3 - 3776x^4 + 24832x^2y^2 \\ & - 3776y^4 - 16560x^2z^2 - 16560y^2z^2 + 10422z^4 - 5472x^2z + 5472y^2z \\ & - 336x^2 - 336y^2 - 612z^2 + 9. \end{aligned} \tag{P_8}$$

3.3 Irreducibility of S

Firstly, this result allows us to tackle the irreducibility of S.

Definition 3.5 (Irreducible Set)**.** *In a topological space, a subset $X \neq \emptyset$ is irreducible if it cannot be written as the union $X = X_1 \cup X_2$ for two proper, closed subsets of $X_1 \cup X_2 \subset X$.*

Definition 3.6 (Algebraic Variety)**.** *An algebraic variety is an irreducible algebraic set.*

Here, the topoligcal space is $\mathbb{A}^n$ with the Zariski topology. To show that an algebraic set is irreducible, we look at the relationship between X and the ideal $\mathcal{I}(X)$. Recall that an ideal is *prime* if for any $ab \in I$, we have either $a \in I$ or $b \in I$. Corollary 1.4 in Hartshorne [4] states:

Corollary 3.7. *[4] An algebraic set X is irreducible if and only if its ideal is a prime ideal.*

Proof. Assume X is irreducible, and let $fg \in \mathcal{I}(X)$ be in its ideal. By definition, $fg(p) = f(p)g(p) = 0$ for all $p \in X$. Given k is a field, the polynomial ring $k[x_1, \ldots, x_n]$ is a domain. Thus, either $f(p) = 0$ or $g(p) = 0$ for all p, else we could write X as $X = Z(f) \cup Z(g)$ contradicting the

assumption as these subsets are proper if one is not contained in the other. So, $f \in \mathcal{I}(X)$ or $g \in \mathcal{I}(X)$. It follows that $\mathcal{I}(X)$ is prime.
Next, let I be a prime ideal and suppose that $X = Z(I) = X_1 \cup X_2$ for two proper, non-empty subsets X_i. Then $I = \mathcal{I}(X_1) \cap \mathcal{I}(X_2)$ as by definition I is precisely the polynomials which vanish on X_1 as well as X_2. However, given $\mathcal{I}(X_1) \neq \mathcal{I}(X_2)$, there exist $f \in \mathcal{I}(X_1) \setminus \mathcal{I}(X_2)$ and $g \in \mathcal{I}(X_2) \setminus \mathcal{I}(X_1)$, we have $fg \in I$ but $f, g \notin I$ which contradicts the assumption that I is prime. □

Further, by theorem V.1.4 [12], in a domain R, if an ideal $R\alpha$ is prime then α is irreducible. The following corollary is an immediate consequence of this.

Corollary 3.8. *An algebraic set $X \in \mathbb{A}^n$ is irreducible if and only if its ideal $\mathcal{I}(X)$ is generated by an irreducible polynomial.*

This leads to the result already claimed, as indeed the generator for $\mathcal{I}(S) = (P_8)$ is irreducible.

Proposition 3.9. *The Bas-Serra surface S is irreducible.*

3.4 Sanity check

In order to check that these results are consistent with our findings in section 4, as well as the geometric definition of S, we plug specific values into (P_8). First, expect $P_8(x, y, \pm 1)$ to return the ellipses C_1 and C_2 over which S is defined and indeed:

$$\begin{aligned}
P_8(x, y, 1) &= 1024(16x^4 + 16x^2y^2 + 4y^4 - 24x^2 + 12y^2 + 9)(x^2 + 2y^2 - 1)^2 \\
&= f_2(x, y, z)(C_2)^2, \\
P_8(x, y, -1) &= 1024(4x^4 + 16x^2y^2 + 16y^4 + 12x^2 - 24y^2 + 9)(2x^2 + y^2 - 1)^2 \\
&= f_1(x, y, z)(C_1)^2,
\end{aligned}$$

where f_i have no real valued solutions, thus returning the original ellipses in the real space. Note, we have C_i^2 as the ellipses are the intersection of outer and inner portion of S, and so the projection $\Phi : E \times \mathbb{C} \to S$ is $2:1$ at these points. We note that these ellipses having order two implies they are subsets of the singularities of S, and this is dealt with in section 6.

4 Complex extension

Aside from letting λ be in $\mathbb{R}$ and extending S beyond the planes $z = 1$ and $z = -1$, the most natural extension of the Bas-Serra construction is to consider the surface over the complex numbers $\mathbb{C}$ instead of the reals. That is, we let $x_i, y_i, \lambda \in \mathbb{C}$, and so with $E(\mathbb{C}) \subset \mathbb{C}^4$, the map is instead $\Phi : E(\mathbb{C}) \times \mathbb{C} \to S(\mathbb{C}) \subset \mathbb{C}^3$. However, while interesting it is fairly difficult to visualise a surface in $\mathbb{C}^3$, and so we look exclusively at the real portion $S(\mathbb{C}) \cap \mathbb{R}^3$. Effectively, we consider the subset $\overline{E} \subset E(\mathbb{C}) \times \mathbb{C}$ such that $\Phi(\overline{E}) \subset \mathbb{R}^3$. This is achieved by considering the equations (F1), (F2), (F3) defining Φ, splitting the imaginary and real parts and requiring $\text{Im}(F_i) = 0$. This defines further generators which, along with (E1), (E2), (E3) are exactly those defining $\overline{E}$.

Claim 4.1. *Considering the Bas-Serra surface over $\mathbb{C}$, the real portion $S(\mathbb{R}) = S(\mathbb{C}) \cap \mathbb{R}^3$ is given* *by*

$$S(\mathbb{R}) = \Phi(E(\mathbb{R}) \times \mathbb{R}) \cup D_x \cup D_y,$$

where D_x is a set of points contained in the plane $x = 0$, and likewise D_y is in the plane $y = 0$.

In order to prove this, we attain two small lemmas.

Lemma 4.2. *Assume $(x_1, y_1, x_2, y_2, \lambda) \in E(\mathbb{C}) \times (C)$ with $\Phi(x_1, y_1, x_2, y_2, \lambda) \in S(\mathbb{R})$. If $x_i \notin \mathbb{R}$, then $y_i \in \mathbb{R}$, and vice versa.*

Proof. First, we let

$$x_1 = a_1 + ib_1, \quad y_1 = c_1 + id_1$$
$$x_2 = a_2 + ib_2, \quad y_2 = c_2 + ib_2$$

The spatial variables x, y, z are required to be in $\mathbb{R}^3$, which immediately gives $\lambda \in \mathbb{R}$ as $z = -1+2\lambda$. Similarly, the restrictions placed on x_i, y_i are:

$$\operatorname{Im}(x_1(1-\lambda) + \lambda x_2) = 0 \tag{4.1}$$
$$\operatorname{Im}(y_1(1-\lambda) + \lambda y_2) = 0. \tag{4.2}$$

Note that $x_1 \in \mathbb{R} \iff x_2 \in \mathbb{R}$ and likewise $y_1 \in \mathbb{R} \iff y_2 \in \mathbb{R}$, and as we are looking for some extra set of complex points, we assume that at least one of the x_i, y_i is non-real. Thus, assume $x_i \notin \mathbb{R}$, that is $b_i \neq 0$. Equations (4.1) and (4.2) then imply

$$b_2 = \frac{\lambda - 1}{\lambda} b_1 \tag{4.3}$$
$$d_2 = \frac{\lambda - 1}{\lambda} d_1. \tag{4.4}$$

This division by λ can be done as $\lambda = 0$ implies the projection is $x = x_1 \in \mathbb{R}$ and similarly $\lambda = 1$ implies $x = x_2 \in \mathbb{R}$, contradicting our assumption that $x_i \notin \mathbb{R}$.

We then get by splitting the imaginary and real parts in equations (E1), (E2), and (E3):

$$2a_1^2 - 2a_1b_1 + c_1^2 - d_1^2 = 1, \tag{E_{r_1}}$$
$$2a_1b_1 + c_1d_1 = 0, \tag{E_{i_1}}$$
$$d_2^2 + 2c_2^2 - \left(\frac{\lambda - 1}{\lambda}\right)^2 (b_1^2 + 2d_1^2) = 1, \tag{E_{r_2}}$$
$$a_2b_1 + 2c_2d_1 = 0, \tag{E_{i_2}}$$
$$4a_1c_2 - a_2c_1 - 3b_1d_1\frac{\lambda - 1}{\lambda} = 0, \tag{E_{r_3}}$$
$$d_1(4a_1 - a_2) + b_1(4c_2 - c_1) = 0. \tag{E_{i_3}}$$

Next, recalling that $b_1 \neq 0$, (E_{i_1}) implies $a_1 = \frac{-c_1d_1}{2b_1}$ and (E_{i_2}) implies $a_2 = \frac{-2c_2d_1}{b_1}$. Plugging this into equation (E_{r_3}) gives us

$$4a_1c_2 - a_2c_1 - 3b_1d_1\frac{\lambda-1}{\lambda} = 4\frac{-c_1d_1}{2b_1}c_2 - \frac{-2c_2d_1}{b_1}c_1 - 3b_1d_1\frac{\lambda-1}{\lambda}$$
$$= -3b_1d_1\frac{\lambda-1}{\lambda} = 0.$$

As observed, $b_1, (\lambda - 1) \neq 0$, which implies that $d_1 = 0$. That is, $y_1 = c_1 \in \mathbb{R}$ and so $y_2 \in \mathbb{R}$, i.e. $d_2 = 0$. This shows that if the x_i are non-real, the y_i are strictly real. Similarly, if the argument that the symmetry implies the same holds for the assumption that $y_i \notin \mathbb{R}$ does not satisfy the reader, it can be explicitly shown in the same way by assuming $d_1 \neq 0$, thus equations (E_{i_1}) and (E_{i_2}) imply $c_1 = \frac{-2a_1b_1}{d_1}$, and $c_2 = \frac{-a_2b_1}{2d_1}$. Substituting into equation (E_{r_3}) as before gives us

$$4a_1c_2 - a_2c_1 - 3b_1d_1\frac{\lambda-1}{\lambda} = -3b_1d_1\frac{\lambda-1}{\lambda} = 0,$$

in turn implying that $b_1 = 0$. □

We proceed to the second lemma.

Lemma 4.3. *Assume* $(x_1, y_1, x_2, y_2, \lambda) \in E(\mathbb{C}) \times \mathbb{C}$ *with* $\Phi(x_1, y_1, x_2, y_2, \lambda) \in S(\mathbb{R})$*. If* $y_i \notin \mathbb{R}$*, then* y_i *are purely imaginary, and similarly if* $x_i \notin \mathbb{R}$ *then* x_i *are purely imaginary.*

Proof. Assume $x_i \in \mathbb{R}$ and $y_i \notin \mathbb{R}$, so $b_i = 0$ and $d_i \neq 0$. Thus,

$$2x_1^2 + y_1^2 = 2(a_1)^2 + (c_1 + id_1)^2$$
$$= (2a_1^2 + c_1^2 - d_1^2) + i(2c_1d_1) = 1$$

implies $2c_1d_1 = 0$, so $c_1 = 0$. Indeed, $y_1 = id_1$ is purely imaginary, in turn implying $y_2 = id_2$ is likewise imaginary. The same holds when switching x_i and y_i. □

We can now easily prove the Claim 4.1.

Proof of Claim 4.1. Assume x_i are non-real, then lemma 4.2 implies y_i are real in turn implying by lemma 4.3 that x_i are purely imaginary. Again, the statement holds for y_i non-real. Turning to the projection Φ into $\mathbb{R}^3$, if we let $x_i = ib_i$, we get

$$x = (1-\lambda)b_1i + \lambda b_2 i = (1-\lambda)b_1 i + \lambda\left(\frac{\lambda-1}{\lambda}b_1\right)i = 0.$$

Likewise if $y_i = \mathrm{i}d_i$ then $y = 0$. It follows that any points contributed to the real portion of S by allowing complex extensions are restricted to the planes $x = 0$ and $y = 0$. □

4.1 The additional curves

As above, we first consider the case where $x_i = ib_i$ are imaginary and $y_i = c_i$ are real. As shown this implies $x = (1-\lambda)x_1 + \lambda x_2 = 0$. Further, equations (E3) and (4.2) imply

$$4x_1y_2 - x_2y_1 = i4b_1c_2 - ib_2c_1 = ib_1(4c_2 - \frac{\lambda-1}{\lambda}c_1) = 0$$

giving $c_2 = \frac{\lambda-1}{4\lambda}c_1$, as again $b_1 \neq 0$. Thus, $y = c_1 + \lambda(c_2 - c_1) = c_1\frac{3}{4}(1-\lambda)$. From (E1) we get $2x_1^2 + y_1^2 = -2b_1^2 + c_1^2 = 1$ and from (E2)

$$\frac{1}{8}c_1^2 - b_1^2 = \left(\frac{\lambda}{\lambda-1}\right)^2,$$

giving

$$(1) - 2\left(\frac{\lambda}{\lambda-1}\right)^2 = (c_1^2 - 2b_1^2) + (-\frac{1}{4}c_1^2 + 2b_1^2) = \frac{3}{4}c_1^2$$
$$\implies c_1^2 = \frac{4}{3} - \frac{8\lambda^2}{3(\lambda-1)^2}.$$

This result, with $z = -1 + 2\lambda$ so indeed $\lambda = z/2 + 1/2$, and $y^2 = \left(c_1\frac{3}{4}(1-\lambda)\right)^2$ gives us

$$\begin{aligned} y^2 &= \frac{3}{4}(1-\lambda^2) - \frac{3}{2}\lambda^2 \\ &= -\frac{3}{4}(\frac{z}{2} + \frac{1}{2})^2 - \frac{3}{2}(\frac{z}{2} + \frac{1}{2}) + \frac{3}{4} \end{aligned}$$

so,

$$16y^2 + 3(z+3)^2 - 24 = 0. \tag{4.5}$$

The above equation (4.5) is, perhaps unexpectedly, an ellipse on the plane $x = 0$. Again by the symmetry between x and y we can see that the second case will be very similar. Indeed, proceeding as before we get

$$16x^2 + 3(z-3)^2 - 24 = 0, \tag{4.6}$$

the same ellipse but rotated by 90° and the z coordinate is flipped. Note that these ellipses meet the original real construction in some place but not in others.

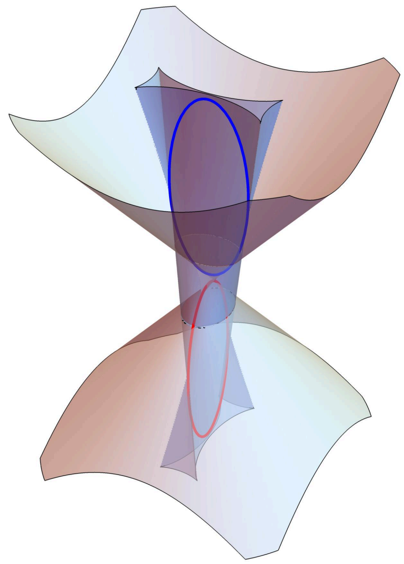


Figure 5: The Bas-Serra surface with the complex extension: $S(\mathbb{C}) \cap \mathbb{R}^3$.

5 Symmetries

A brief look at the images of the Bas-Serra surface or the defining equations suggests a great degree of symmetry is present. In this section we provide a a description of several subgroups of the group of symmetries of S, which we call G_S. First, we must define what is meant by a symmetry.

5.1 Symmetries as automorphisms of the function field

A geometric symmetry of S is a natural idea - it is some rotation, reflection or shift of S such that the surface remains unchanged following this transformation. Algebraically, this corresponds to some isomorphisms of the ideal of S, and allowing for all such isomorphisms the notion of a symmetry of S is extended past rotations and reflections to specific maps from S to S called birational maps. As will be seen shortly, these maps are not necessarily isomorphisms of S.

Definition 5.1 (Rational Map)**.** *Let $X \subset \mathbb{A}^n$ be a variety. A rational map $f : X \dashrightarrow \mathbb{A}^m$ is a map $P \mapsto [r_1(P), \ldots, r_m(P)]$ where each $r_i = \frac{f_i}{g_i}$ is in the field of fractions of $k[X_1, \ldots, X_n]$, and the g_i's are not zero divisors in the coordinate ring $k[X] = k[X_1, \ldots, X_n]/\mathcal{I}(X)$.*

Definition 5.2 (Function Field)**.** *The function field of an affine variety X is $k(X)$, the field of fractions of the coordinate ring $k[X]$.*

Definition 5.3 (Birational equivalence)**.** *A rational map $f : X \subset \mathbb{A}^n \dashrightarrow Y \subset \mathbb{A}^m$ is birational (and hence X and Y are birationally equivalent) if there exists a rational map $g : Y \dashrightarrow X$ such that $g \circ f = id_{X_0}$ and $f \circ g = id_{Y_0}$, where X_0 and Y_0 are dense open subsets of X respectively Y.*

The following theorem is then relevant.

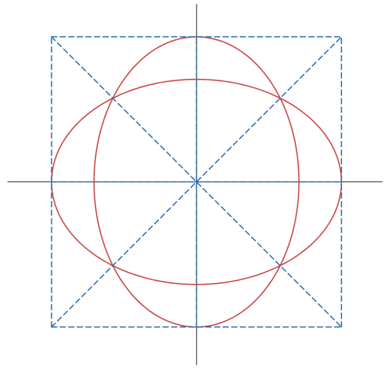


Figure 6: The symmetries $\mathbb{D}_8$ of the and C_1, C_2 projected onto $z = 0$.

Theorem 5.4 ([[6], Ch.5, pg.87)**.** *The following conditions are equivalent for a rational map* $f : X \dashrightarrow Y$*:*

1. *f is birational*
2. *$f^* : k(Y) \to k(X)$ is an isomorphism of the function fields*
3. *there exist open subsets $X_0 \subset X$ and $Y_0 \subset Y$ such that $f|_{X_0} : X_0 \to Y_0$ is an isomorphism.*

The map f^* is induced by taking $f^*(g) = g \circ f$ where $g \in k[Y]$. The consequence of theorem 5.4 is that not every automorphism of $k(S)$ corresponds to an automorphisms of S, but rather a birational map $f : S \to S$ which is not even necessarily defined everywhere on S, but rather on a dense open subset.

5.2 The Symmetries of the Bas-Serra Surface

Proposition 5.5. *The group of symmetries G_S of the Bas-Serra surface has a subgroup isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{D}_8$, as well as a subgroup isomorphic to $PGL(2, \mathbb{C}) = GL(2, \mathbb{C})/\mathbb{C}^*$.*

We begin by considering the geometric symmetries which correspond to the subgroup $\mathbb{D}_8$, the symmetries of the square. Projecting onto the plane $z = 0$ (or simply seen from above) as in Figure 6, this becomes clear. The reflections about the axes,

$$x \to -x, \quad y \to -y$$

evidently preserve S. Likewise, the reflections about $y = \pm x$,

$$\pm x \to \pm y, \quad z \to -z$$

and rotations by $\pi/2$ also preserve this projection. In terms of the spatial variables, this rotation is expressed as

$$x \to y, \quad y \to -x, \quad z \to -z$$

The reflection about the plane $z = 0$ is necessary as the C_1 and C_2 are not co-planar, and we must effectively map C_1 to C_2, and vice versa. Indeed, it can quickly be checked by plugging any of these transformations into P_8 that the polynomial is unchanged. In order to explicitly construct the group of symmetries through generators, we look at the above transformations in the coordinates x_i, y_i, λ as this allows each transformation to be written as a linear function on the vector $(x_1, y_1, x_2, y_2, \lambda)$. Treated in this way, it is the equations for the curve E: (E1), (E2), (E3) which are satisfied. The refections $x \to -x$ and $y \to -y$ correspond simply to

$$x_i \to -x_i, \quad y_i \to -y_i.$$

Further, the reflection about $y = x$ is given by

$$x_1 \leftrightarrow y_2, \quad x_2 \leftrightarrow y_1, \quad \lambda \to (1 - \lambda).$$

Similarly, the anti-clockwise rotation by $\pi/2$ is equivalent to

$$\begin{aligned} &x_1 \to y_2, \quad y_1 \to -x_2 \\ &x_2 \to y_1, \quad y_2 \to -x_1 \\ &\lambda \to 1 - \lambda. \end{aligned}$$

Note, we do not need $\lambda \to 1 - \lambda$ to satisfy E, as in fact λ does not appear, but it is rather a consequence of the projection Φ. With $\mathbb{D}_8 =< a, b | a^4 = b^2 = e, bab = a^3 >$, we let a be the rotation and b the reflection $y \to -y$.

As claimed in proposition 5.5, there is a slightly larger finite group of transformations which can be readily understood. This second set of transformations are less geometrically intuitive and rather than reflecting or rotating the surface, effectively map the outer portion of S to the inner, as seen in Figure 7. Consider the line between $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$ (we have omitted the z-coordinate for simplicity), and map this line to the line between p_1 and $p_3 = -p_2$, the twin of p_2. Certainly, we have not yet accounted for this transformation, and a simple check shows E is satisfied. Further, some inspection and composition with existing maps show there should be eight similar transformations that are yet unaccounted for. Indeed, constructing this as a matrix group we find that the group has order 16 and further that there are exactly 11 elements of order 2, implying that the group generated is in fact $\mathbb{Z}/2\mathbb{Z} \times \mathbb{D}_8$.

Further, as will be shown explicitly in section 7, S is birationally equivalent to $E \times \mathbb{A}^1$. It follows that we can look at automorphisms on the function field $k(E \times \mathbb{A}^1) = k(E)(\lambda)$. This suggests that the group $\mathrm{Aut}(\mathbb{C}(\lambda))$ is a subgroup of the symmetries. Let $\sigma \in \mathrm{Aut}(\mathbb{C}(\lambda))$, then

$$\sigma(\lambda) = \frac{p(\lambda)}{q(\lambda)} = \frac{\epsilon\lambda + \zeta}{\mu\lambda + \eta},$$

which corresponds to some invertible matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Further, we note that $p(\lambda)/q(\lambda) = \alpha/\alpha \cdot p(\lambda)/q(\lambda)$ for $\alpha \in \mathbb{C}^*$, so the group of all such transforma-

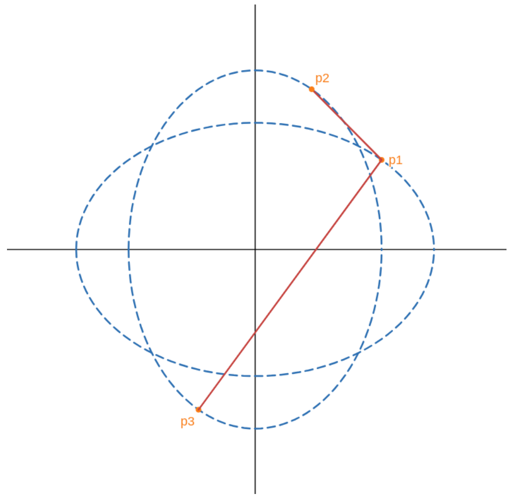


Figure 7: The "switching" of the *Siamese Twins*.

tions is indeed $PGL(2, \mathbb{C}) = GL(2, \mathbb{C})/\mathbb{C}^*$.

6 Singularities

While we have several meaningful constructions for S, in considering the singularities of S we make use of the ideal $\mathcal{I}(S) = (P_8)$ in the spatial variables. Here, we can simply consider the set of singularities $\mathfrak{s} = \mathrm{Sing}S$ as the zeros of the ideal of P_8 and its partial derivatives.

$$\mathfrak{s} = Z\left(P_8, \frac{\mathrm{d}P_8}{\mathrm{d}x}, \frac{\mathrm{d}P_8}{\mathrm{d}y}, \frac{\mathrm{d}P_8}{\mathrm{d}z}\right).$$

Naturally, this is very unwieldy, so we look at the primary decomposition of $\mathfrak{s}$.

Definition 6.1 (Primary decomposition)**.** *Let $I \subset R$ be an ideal; the primary decomposition of I is a sequence of ideals $Q_1, \ldots, Q_m$ such that $I = Q_1 \cap \cdots \cap Q_m$, and each Q_i is a primary ideal.*

We then have $Z(I) = Z(Q_1) \cup \cdots \cup Z(Q_m)$. However, computing the primary decomposition of $\mathfrak{s}$ is similarly unwieldy, so we work instead with the radical decomposition of $\mathfrak{s}$.

Definition 6.2 (Radical Decomposition)**.** *The radical decomposition of I is a sequence of prime ideals $Q_1, \ldots Q_m$ such that $\sqrt{I} = Q_1 \cap \cdots \cap Q_m$. This is equivalent to taking the primary decomposition of $\sqrt{I}$.*

By Proposition 1.2 in [4], $Z(\sqrt{I}) = \overline{Z(I)}$ which equals $Z(I)$ since $Z(I)$ is by definition closed, so the algebraic set for the radical decomposition of $\mathfrak{s}$ is simply $\mathfrak{s}$ itself.

Thus, we find $\mathfrak{s}$ has a radical decomposition into 5 ideals, $\sqrt{\mathfrak{s}} = \bigcup_{i=1}^{5} Q_i$ where the Q_i are as follows.

$$
\begin{aligned}
Q_1 &= (2x^2 + y^2 - 1, z + 1) \\
Q_2 &= (x^2 + 2y^2 - 1, z - 1) \\
Q_3 &= (16y^2 + 3z^2 + 18z + 3, x) \\
Q_4 &= (16x^2 + 3z^2 - 18z + 3, y) \\
Q_5 &= (t_1, t_2, t_3, t_4)
\end{aligned}
$$

with

$$
\begin{aligned}
t_1 &= x^4 - 35/27x^2z + 1/9y^4 + 20/27y^2z^2 + 35/27y^2z + 20/27y^2 - 85/2592z^4 \\
&\quad + 1/6z^3 + 185/1296z^2 + 1/6z - 85/2592, \\
t_2 &= x^2y^2 - 4/27x^2z + 19/18y^4 - 95/108y^2z^2 + 4/27y^2z - 95/108y^2 \\
&\quad + 343/10368z^4 - 19/96z^3 + 2341/5184z^2 - 19/96z + 343/10368, \\
t_3 &= x^2z^2 - 14/9x^2z + x^2 + 4/3y^4 - 1/9y^2z^2 + 14/9y^2z - 1/9y^2 \\
&\quad - 17/432z^4 - 1/4z^3 + 37/216z^2 - 1/4z - 17/432, \\
t_4 &= y^6 - 7/8y^4z^2 + 9/4y^4z - 7/8y^4 + 121/192y^2z^4 - 21/16y^2z^3 \\
&\quad + 139/96y^2z^2 - 21/16y^2z + 121/192y^2 - 343/13824z^6 + 49/256z^5 \\
&\quad - 2611/4608z^4 + 103/128z^3 - 2611/4608z^2 + 49/256z - 343/13824.
\end{aligned}
$$

This confirms the thought that the defining ellipses C_1 and C_2 with ideals Q_1 and Q_2 respectively are singular points as they are a crossing of lines. Further, the extra points found in section 4 have been shown to be of order two in P_8, and indeed these are singular sets corresponding to Q_3 and Q_4. The last ideal Q_5, although large, is fairly manageable.

7 Generalisations and irreducibility

The purpose and goal of this section is to investigate the construction of the Bas-Serra surface more generally and consider how changes to the defining conics, previously the ellipses C_1 and C_2, affect various properties of the resultant surface. Primarily, we look at whether or not the surface is irreducible. In order to do so, we must first outline some concepts in projective algebraic geometry.

7.1 The projective space and projective varieties

Various notions regarding affine varieties have already been introduced. This subsection will briefly introduce the necessary concepts needed for working in the projective space and a little theory regarding projective varieties following Chapter 5 in *Reid* [6].

Definition 7.1 (The Projective Space)**.** *The projective n space over a field k is*

$$
\mathbb{P}^n_k = \mathbb{P}^n = \left(\mathbb{A}^{n+1} \setminus \{0\}\right) / \sim,
$$

where the equivalence relation $\sim$ is given by

$$[X_0 : \cdots : X_n] \sim [\lambda X_0 : \cdots : \lambda X_n]$$

for $\lambda \neq 0$. That is, $\mathbb{P}^n$ is the set of equivalence classes of points in $\mathbb{A}^{n+1} \setminus \{0\}$ where two points are equivalent if and only if they lie on the same line through the origin.

In order to make use of the constructions and ideas of affine algebraic geometry, we need to look at a subset of the polynomials which are well defined on $\mathbb{P}^n$.

Definition 7.2 (Homogeneous Polynomials)**.** *A polynomial $f \in k[X_0, \ldots, X_n]$ is homogeneous of degree $d \geq 0$ if it can be expressed as*

$$f = \sum a_{i_0,\ldots,i_n} X_0^{i_0} \cdots X_n^{i_n}$$

where $i_0 + \cdots + i_n = d$. In other words, every term in the polynomial has total degree d.

Certainly, for a homogeneous polynomial of degree d we have

$$f(\lambda X_0, \ldots, \lambda X_n) = \lambda^d f(X_0, \ldots, X_n),$$

and so homogeneous polynomials are well defined in $\mathbb{P}^n$ as $f[X_0 : \cdots : X_n]$ is independent of the representative. This allows us to work with ideals similarly as in the affine case.

Definition 7.3 (Homogeneous ideals)**.** *A homogeneous ideal $I \subset k[X_0, \ldots, X_n]$ is an ideal generated by homogeneous polynomials.*

This gives us the following $Z \leftrightarrow \mathcal{I}$ correspondence

$$\left\{ \begin{matrix} \text{Homegeneous ideals} \\ J \subset k[X_0, \ldots, X_n] \end{matrix} \right\} \underset{\mathcal{I}}{\overset{Z}{\leftrightarrows}} \left\{ \begin{matrix} \text{Algebraic sets} \\ X \subset \mathbb{P}^n \end{matrix} \right\}$$

where

$$Z(J) = \{p \in \mathbb{P}^n | f(p) = 0 \quad \forall \quad \text{homog.} f \in J \in\}$$

and

$$\mathcal{I}(X) = \{f \in k[X_0, \ldots, X_n] | f(p) = 0 \quad \forall p \in X\}.$$

As in the affine case, a projective variety is then defined as an irreducible algebraic set, which are similarly in $Z \leftrightarrow \mathcal{I}$ correspondence with homogeneous prime ideals, i.e. those generated by irreducible polynomials. Further, we define rational functions and the function field as such:

Definition 7.4 (Rational Function)**.** *A rational function $f : X \to k$ is a partially defined function which can be represented as*

$$f = \frac{g}{h},$$

where $g, h \in k[X_0, \ldots, X_n]$ are homogeneous polynomials of the same degree, and h is not everywhere zero on X. Equivalently, $h \notin \mathcal{I}(X)$.

Note, a rational function need not be defined everywhere on X, but is well defined on a dense open subset of X.The rational function field then follows naturally.

Definition 7.5 (Function Field of a projective variety X)**.** *Let X be a projective variety, then the function field $k(X)$ is defined as*

$$k(X) = \{g/h | g, h \in k[X_0, \ldots, X_n], \\ g, h \quad \textit{homog. of degreed}, h \notin \mathcal{I}(X)\}$$

Definition 7.6 (Rational Map)**.** *Let $X \subset \mathbb{P}^n$ be a variety. A rational map $f : V \dashrightarrow \mathbb{P}^m$ is a map $P \mapsto [f_0(P) : \cdots : f_m(P)]$ where each $f_i \in k[X_0, \ldots, X_n]$ is a polynomial and for each $P \in X$, at least one $f_i(P) \neq 0$.*

Definition 7.7 (Birational equivalence)**.** *A rational map $f : X \subset \mathbb{P}^n \dashrightarrow Y \subset \mathbb{P}^m$ is birational (and hence X and Y are birationally equivalent) if there exists a rational map $g : Y \dashrightarrow X$ such that $g \circ f = id_X$ and $f \circ g = id_Y$.*

Further, recall Theorem 5.4 in section 5, which certainly holds over the projective space as well.

Theorem 7.8 ([[6], Ch.5, pg.87)**.** *The following conditions are equivalent for a rational map f : $X \dashrightarrow Y$ between projective varieties:*

1. *f is birational*
2. *$f^* : k(Y) \to k(X)$ is an isomorphism of the function fields*
3. *there exist open subsets $X_0 \subset X$ and $Y_0 \subset Y$ such that $f|_{X_0} : X_0 \to Y_0$ is an isomorphism.*

7.2 Irreducibility of E and S

As has been shown, the Bas-Serra surface is irreducible. However, this is a consequence of the curve E being irreducible, which in turn is determined by the choice of conics C_1 and C_2. In this section we look at the relationship between the conics, their intersections and the curve, which we continue to call E, defined by pairing their tangential points. We begin by considering two planes in $\mathbb{P}^3$, H_1 and H_2, with $H_1 \neq H_2$. Let $L = H_1 \cap H_2$ be the line of intersection, noting that we are guaranteed such a line in $\mathbb{P}^3$ as H_1 and H_2 being parallel implies they intersect at a line at infinity. Further, let C_1 and C_2 be planar conics with $C_i \subset H_i$.

Theorem 7.9 (Bezout's Theorem)**.** *Let C, D be two planar projective curves over an algebraically closed field k, with $\deg C = m$, $\deg D = n$. Then, counted with multiplicity, C and D have mn points of intersection.*

As $\mathbb{C}$ is algebraically closed, $\deg L = 1$ and $\deg C_i = 2$, Bezout's theorem guarantees that L intersects each C_i twice when counted with multiplicity. There is one point of intersection if and only if L is tangent to C_i.

Example 7.10. *Considering a standard parabola $Q : y = x^2$ in the affine space $\mathbb{A}^2$, there appear to be two "arms" which move apart as they go to infinity. However, considering the homogonised equation $YW = X^2$ with coordinates in $[X : Y : W]$, at $W = 0$, in other words in the projective* *portion at infinity, the parabola becomes $X^2 = 0$ showing there is a single point of intersection of with the line at infinity. Thus, letting this line likewise be L, L is tangent to Q.*

The curve E in the original construction of the Bas-Serra surface consists of pair of points $(P_1, P_2) \in C_1 \times C_2$ where the tangent vectors to P_1, P_2, respectively T_1, T_2 are parallel. This is equivalent to saying that they intersect line $L = H_1 \cap H_2$ at infinity at the same point: $T_1 \cap L = T_2 \cap L$. In the more general setting, we pair points in exactly this way, rather than considering their tangent vectors parallel. In order to deal with this notion more elegantly, we introduce the notion of the dual space and the dual of a curve.

Definition 7.11 (Dual Space)**.** *The dual space $\mathbb{P}^*$ of the projective plane $\mathbb{P}^2$ is the space in which "points" are the lines $Z(aX + bY + cZ)$ in $\mathbb{P}^2$. Such a line is determined by the triple $[a : b : c]$, which identifies $\mathbb{P}^*$ with $\mathbb{P}^2$.*

Definition 7.12 (Dual of a projective curve)**.** *let $C \subset \mathbb{P}^2$ be a projective curve. The dual of C, denoted by C^* is the set of all lines in $\mathbb{P}^2$ that are tangent to C. The curve C^* is likewise a projective curve in $\mathbb{P}^* = \mathbb{P}^2$.*

Example 7.13 (The dual of a conic)**.** *Let $[x : y : z]$ be coordinates in $\mathbb{P}^2$, $[X : Y : Z]$ coordinates in $\mathbb{P}^*$ and consider a general algebraic curve $C = Z(f(x : y : z))$ in $\mathbb{P}^2$. For a point $[a : b : c] \in C$, the tangent line to C is*

$$x\frac{df}{dx}(a : b : c) + y\frac{df}{dy}(a : b : c) + z\frac{df}{dz}(a : b : c) = 0$$

which is identified with $[X : Y : Z] = \left[\frac{df}{dx}(a : b : c) : \frac{df}{dx}(a : b : c) : \frac{df}{dx}(a : b : c)\right]$.
Letting $f(x : y : z) = x^2 + 2y^2 - z^2$, the conic C_2 in the Bas-Serra surface, we have all points in the dual C_2^* given by $[X : Y : Z] = [2a : 4b : -2c] \sim [a : 2b : -c]$. solving for a, b, c here and with the tangent line equation under the identification as $xX + yY + zZ$, this solves for the dual equation

$$\begin{aligned} C_2^* &= Z\left((X) \cdot X + (Y/2) \cdot Y + (-Z) \cdot Z\right) \\ &= Z\left(2X^2 + Y^2 - 2Z^2\right). \end{aligned}$$

We will use without proof the following lemma:

Lemma 7.14. *The dual of a smooth conic is again a smooth conic.*

In order to proceed we must define the map $\sigma : Q \to L$, where Q is a non-degenerate quadric and L is a line, taking a point $p \in Q$ to the intersection of the tangent T_p with L. Formally,

$$\begin{aligned} \sigma :& Q \to L \\ & p \mapsto T_p \cap L \quad \text{if} \quad p \notin L \\ & p \mapsto p \quad \text{if} \quad p \in L. \end{aligned}$$

We then have the following result. Note, by nearly everywhere we mean on a dense open susbset.

Lemma 7.15. *Let $\sigma : Q \to L$ be as above. Then σ is surjective, and is nearly everywhere a $2 : 1$*

mapping unless L is tangent. In this case σ is $1:1$.

Proof. Let L be a line and let $l \in L$ be a point on L. First, if $l \in Q \cap L$ then by definition $\sigma(l) = l$. The fact that σ is $1:1$ at this point follows as a tangent to a quadric Q intersects Q exactly once with multiplicty two, be Bezout's theorem.
Next, consider a general point $x \in \mathbb{P}^2 - Q$. A line through x is a point in the dual of x, x^*, while a tangent line to Q is likewise a point in the dual Q^*. Thus, a line through x that is tangent to Q is, in the dual, a point in the intersection $Q^* \cap x^*$. By Lemma 7.14, Q^* is a smooth conic, and as x^* is a line, Bezout's theorem guarantees that (counted with multiplicity) $Q^* \cap x^*$ has two points. Now, we are nearly done as we have shown that for a line L and $l \in L - Q$, there exist two tangents to Q through l, showing first that σ is surjective and second that it is $2:1$ almost everywhere, except on the finite set $Q \cap L$. Further, if L is tangent to Q, then clearly σ is $1:1$ as the "second" point mapping to l is always the point $p = Q \cap L$, and by definition $\sigma(p) = p$. □

Corollary 7.16. *In general, E consists of pairs of points (P_1, P_2) and its twin (P_1, P_2')*

Proof. Let $P_1 \in C_1$ be mapped by σ to $l_1 \in L$. By Lemma 7.15, there exist in general two points, $P_2, P_2^{'} \in C_2$ with tangents intersection L at l_1. Further, this is not the case if either $l_1 \in C_2$ or L is tangent to C_2. □

Corollary 7.17. *In general, the curve E is described by a polynomial of the form $s^2 = f(t)$ where s, t give rational parametrizations for C_1, C_2 respectively. Further, E is irreducible if and only if $f(t)$ is not a square in $\mathbb{C}[t]$.*

Proof. We provide only a rough sketch. The first part follows as we have $P_1(s)$ and $P_2(t)$, and as $E(P_1, P_2) \to P_1$ is $2:1$, this implies $(s, t) \mapsto s$ is likewise $2:1$. The second part follows simply by seeing that if $f = g^2$ is a square then we have $s^2 - g^2 = (s-g)(s+g)$. □

Further, we take the following lemma as a property of the construction, except for the assertion that the map is almost everywhere $1:1$, which will not be shown in this thesis.

Lemma 7.18. *The map $\Phi : E \times \mathbb{P} \to S \subset \mathbb{P}^3$ is surjective and almost everywhere $1:1$.*

Proposition 7.19. *The surface S is irreducible if and only if the curve E is irreducible.*

Proof. The map Φ is clearly a rational map, and by lemma 7.18 there are dense open subsets $U \subset E \times \mathbb{P}$ and $S_0 \subset S$ on which $\Phi|_U : U \to S_0 \subset$ is a bijection, and so Φ is birational.

Assume E is irreducible, then the coordinate ring $k[E \times \mathbb{P}]$ is a domain, and Φ induces a homomorphism showing $k[S] \subset k[E \times \mathbb{P}]$, so $k[S]$ is likewise a domain. We can then consider the field of fractions $k(S)$, i.e. the function field of S which is isomorphic to $k(E)(\lambda)$ by Lemma 7.18 and Theorem 5.4. It follows that S must be irreducible. This same argument holds in the other direction. □

7.3 The generalised Bas-Serra Construction

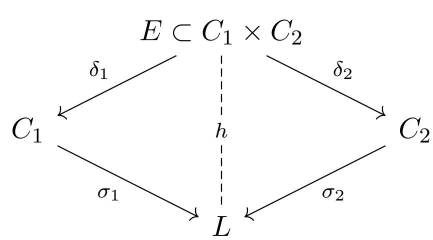
The need for the projective space becomes evident in analysing various cases, as we take the planes H_1, H_2 to be parallel, as in the Bas-Serra surface. In each case, we let the planes be $H_1 : z = -1$ and $H_2 : z = 1$, which in the projective space is $H_1 : Z + W = 0$ and $H_2 : Z - W = 0$. Clearly, $H_1 \cap H_2$ requires $W = -W$ implying the line L is indeed in the projective portion $W = 0$. This, in turn, implies the line L is the points $[X : Y : 0 : 0]$, as indeed $Z \pm 0 = 0$ implies $Z = 0$. This can be seen by considering that H_1 and H_2 are parallel to the plane $Z = 0$, and as an examples the line given by the $x-$ axis on any of these three planes will converge to the same point as infinity.

In order to work efficiently with the various curves, their partial derivatives and tangent, we work with the affine portion. This is justified as for any affine variety X and its projective closure $\overline{X}$, X is irreducible if and only if $\overline{X}$ is irreducible. To see this, assume $\overline{X}$ to be irreducible and suppose the affine portion can be written as the intersection $X = X_1 \cup X_2$ of two proper subsets. Certainly, $\overline{X_1} \cup \overline{X_2} = \overline{X}$, contradicting the assumption. Similarly, let X be irreducible and suppose $\overline{X} = Y_1 \cup Y_2$ is reducible. Then the affine portions $X_i \subset Y_i$ are proper subsets $X_i \subset X$, and $X = X_1 \cup X_2$, once again leading to a contradiction. Thus, we switch from the projective coordinates $[X : Y : Z : W]$ to the affine (x, y, z) as well as the familiar x_i, y_i coordinates for describing the curves C_1 and C_2 when necessary.

Considering the multiplicity and number of intersection points of two conics C_1 and C_2 with the line $L = H_1 \cap H_2$, we find that there are seven possible cases of intersection, each constructing a surface with varying properties.

Fact. *Consider the Bas-Serra construction over $\mathbb{C}$, with $C_1 \subset H_1$ and $C_2 \subset H_2$ two non-degenerate conics on distinct planes $H_1, H_2 \subset \mathbb{P}^3$, and let $L = H_1 \cap H_2$. The surface resulting from the Bas-Serra construction is reducible over $\mathbb{C}$ if and only if $C_1 \cap L = C_2 \cap L$ and this set consists of* 2 *points.*

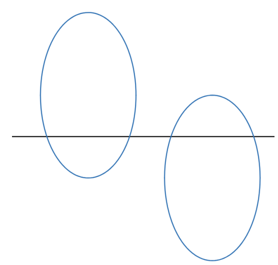
A full proof of the above fact is unfortunately beyond the scope of this paper. However, this result is analogous to the sketch in Section 7, pages 322 - 331 of the paper on *Poncelet's Closure Theorem* by H.J.M. Bos, C. Kers, F. Oort, and D. W. Raven [2]. Here, we consider all the cases of intersection of C_1 and C_2 through examples showing the above fact to at least hold true under testing, as well discussing other points of interest. In order to sketch the proof idea, consider the maps below:



where, as before, $\sigma_i : C_i \to L$ takes a point $p_i \in C_i$ to $L \cap T_{p_i}$, and $\delta_i : E \to C_i$ maps $(P_1, P_2) \to P_i$. The maps σ_i have been shown to be $2 : 1$ everywhere except at $L \cap C_i$, if $L \notin C_i^*$, and as a consequence δ_i are $2 : 1$ everywhere except where (P_1, P_2) has $P_2 \in L$. Broadly, the proof then concerns whether the points where these maps are $1 : 1$ coincide, and thus when the covering h is ramified or unramified (concepts which we do not discuss), and looks at the induced extension of the coordinate ring and function field of L to discern when E will be irreducible.

Case 1: Four distinct points of intersections
The existing Bas-Serra surface is an example of this first case. The homogenised quadrics are given by $C_1 : 2X^2+Y^2-W^2 = 0, Z+W = 0$ and $C_2 : X^2+2Y^2-W^2 = 0, Z-W = 0$. Thus, for $W = 0$, we have the two solutions to C_1 given by $2X^2+Y^2 = 0$, so $Y = \pm\sqrt{-2X^2} = \pm\mathrm{i}\sqrt{2}X$. So, $C_1 \cap L$ are the points $[1 : \pm\mathrm{i}\sqrt{2} : -1 : 0]$. Similarly, the solutions to C_2 are given by $[\pm\mathrm{i}\sqrt{2} : 1 : 1 : 0]$, showing their to be four distinct intersections. For this first case, we compute an implicit description of E by hand for illustrative purposes. For the rest we achieve this computationally (once again through ideal reduction).

First, parametrising the bottom ellipse C_1 by taking a line $x = u(y+1)$ through the point $(x,y) = (0,-1)$ and looking at the (unique) second point of intersection on C_1. This point is given by $x_1 = \frac{2u}{1+2u^2}$ and $y_1 = \frac{1-2u^2}{1+2u^2}$, providing the parametrisation. Similarly, we look at the intersection of $y = t(x+1)$ with C_2 to get $x_2 = \frac{1-2t^2}{1+2t^2}$ and $y_2 = \frac{2t}{1+2t^2}$. By construction, such x_i, y_i satisfy the first two conditions defining (**??**), and must now also satisfy the third:



Case 1 intersections

$$\begin{aligned}4x_1y_2 - x_2y_1 &= 4\frac{2u}{1+2u^2}\cdot\frac{2t}{1+2t^2} - \frac{1-2t^2}{1+2t^2}\cdot\frac{1-2u^2}{1+2u^2}\\ &= u^2(4t^2-2) - (16t)u + (1-2t^2) = 0\end{aligned}$$

Completing the square and letting $s = u(4t^2-2) - 8t$ we have

$$s^2 = 8t^4 + 56t^2 + 2 =: f(t)$$

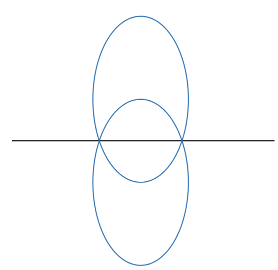
defining the curve E. As an aside, we note that this implies there are no rational points in E as for $t \in \mathbb{Q}$, f is divisible by 2 exactly once, while s^2 cannot be so for $s \in \mathbb{Q}$. Further, f is not a square and so this E is irreducible, which confirms the statement in Proposition 3.9 that S is irreducible.

Case 2: two points of multiplicity two

Considering perhaps the most obvious choice of construction, we let C_1 and C_2 be unit circle: $C_{1,2} : X^2 + Y^2 - W^2 = 0, Z \pm W = 0$. Thus, $C_{1,2} \cap L$ is computed by $X^2+Y^2 = 0$, so $X = \pm\mathrm{i}Y$. This gives

$C_1 \cap L = [\pm\mathrm{i} : 1 : 0 : 0] \sim -1 * [\pm\mathrm{i} : 1 : 0 : 0] \sim [\pm\mathrm{i} : -1 : 0 : 0]$.

Both of these points satisfy C_2 when $W = 0$, showing $C_1 \cap L = C_2 \cap L$.
Here, the parallel tangents condition is given by $x_1y_2 - x_2y_1 = 0$.



Case 2 intersections

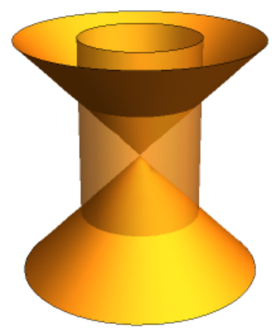
Parametrising C_1 by $s \mapsto (\frac{1-s^2}{1+s^2}, \frac{2s}{1+s^2})$ and similarly C_2 by $t \mapsto (\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2})$, we compute E in terms of s, t to be

$$s^2t^2 - s^2 - 4st - t^2 + 1 = (st - s - t - 1)(st + s + t - 1),$$

and it is clear that E is not irreducible. Computing the associated surface S, from a geometric perspective we expect S to be the union of the cone and cylinder through zero. Indeed, S is given by

$$(x^2 + y^2 - z^2)(x^2 + y^2 - 1) = 0$$

which, as predicted is both reducible and the union of the cone and cylinder:

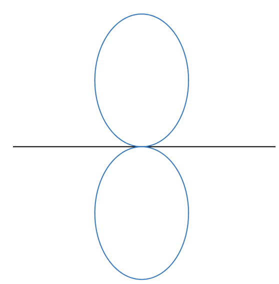


Case 2: two circles

Case 3: one point of multiplicty four

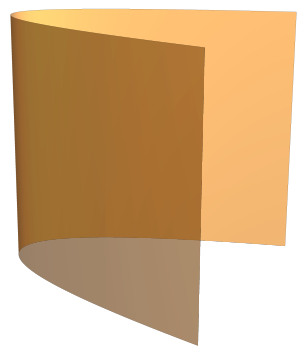
Consider $C_{1,2} : YW = X^2, Z \pm W = 0$. In this case, letting $W = 0$ gives us for both quadrics $X = 0$, and so we are left with the single point of intersection with multiplicity four:

$$(C_1 \cup C_2) \cap L = \{[0 : 1 : 0 : 0]\}.$$



Case 3 intersections

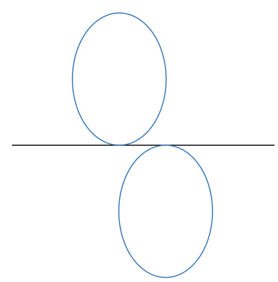
As $C_{1,2}$ are functions $y(x)$, they are trivially parametrised as $C_1 : (s, s^2), C_2 : (t, t^2)$. The parallel-tangents requirement is satisfied then by $s - t = 0$, which implies E is irreducible. Notice that this is linear in s and t, which is expected from the analysis on E and its intersection with L, as well as the mapping from the conics C_1, C_2 to L which are here one-to-one. Although somewhat boring, the surface is defined by $y = x^2$.



Case 3: identical parabolas

Case 4: two points of multiplicity two, each from a distinct conic

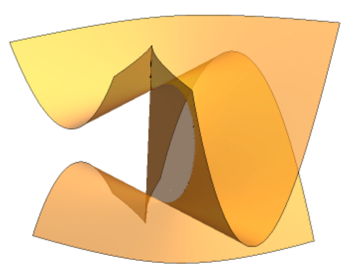
Consider $C_1 : YW = X^2$ and $C_2 : XW = Y^2$ with $C_1 \cap L = [0 : 1 : 0 : 0]$ and $C_2 \cap L = [1 : 0 : 0 : 0]$, both intersections having multiplicity two from $X^2 = 0$ respectively $Y^2 = 0$. Again, we expect E to be linear in s, t and indeed with $C_1 : (s, s^2)$ and $C_2 : (t^2, t)$, the parallel condition in coordinates is given by $4x_1y_2 - 1 = 0$ so E is given by the irreducible $4st - 1 = 0$. Further, computing the surface S leaves us with



Case 4 intersections

$$\begin{aligned} 0 =& 2048x^3z + 2048x^3 - 4096x^2y^2 + 1152xyz^2 \\ & - 1152xy - 2048y^3z + 2048y^3 + 27z^4 - 54z^2 + 27 \end{aligned}$$

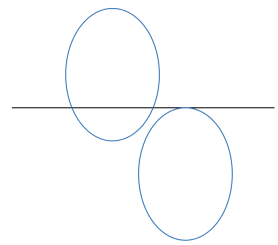
which again is irreducible.



Case 4: two parabolas with different intersections

Case 5: three points, one of multiplicity two from a single curve

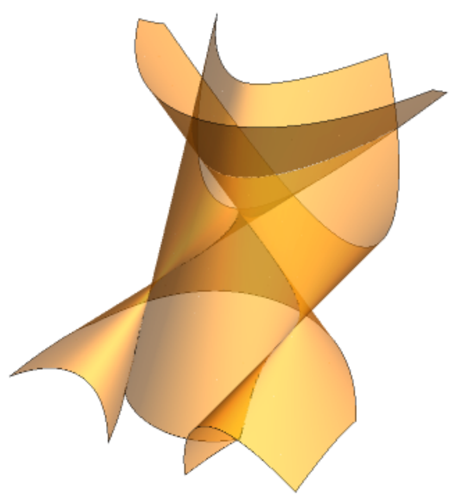
We choose the parabola such that we have three distinct points of intersection, two of multiplicity one from the circle, and one of multiplicity two from the Parabola. We choose $C_1 : X^2+Y^2-W^2=0, Z+W=0$ and $C_2 : YW = X^2, Z-W=0$. As before, $C_1 \cap L = [1 : \pm i : 0 : 0]$ and $C_2 \cap L = [0 : 1 : 0 : 0]$. With parametrisations as before, and the parallel condition by $2x_2y_1 + x_1 = 0$, we get E by $s^2 - 4ts - 1 = 0$, which is irreducible. Computing the surface S, as in the original case we find a rather larger function than we have in the other cases, with S given implicitly by



Case 5 intersections

$$\begin{aligned}0 =&1024x^6 + 1024x^4y^2 - 1280x^4yz - 1280x^4y - 752x^4z^2 + 1568x^4z - 752x^4\\&- 1024x^2y^3z - 1024x^2y^3 + 2048x^2y^2z + 48x^2yz^3 - 112x^2yz^2 - 112x^2yz\\&+ 48x^2y + 112x^2z^4 - 768x^2z^3 + 1312x^2z^2 - 768x^2z + 112x^2 + 256y^4z^2\\&+ 512y^4z + 256y^4 - 320y^3z^3 + 64y^3z^2 + 64y^3z - 320y^3 + 36y^2z^4\\&- 240y^2z^3 + 472y^2z^2 - 240y^2z + 36y^2 + 80yz^5 - 176yz^4 + 96yz^3 + 96yz^2\\&- 176yz + 80y - 25z^6 + 110z^5 - 231z^4 + 292z^3 - 231z^2 + 110z - 25.\end{aligned}$$

As expected, this polynomial, and so S, is irreducible.



Case 5: a parabola and a circle

Case 6: three points, one of multiplicity two from two different conics

We consider now the case $C_i \cap L$ has two distinct points, however one point is in both intersections: $(C_1 \cap L) \cup (C_2 \cap L) = \{l_1, l_2\} \cup \{l_1, l_3\} = \{l_1, l_2, l_3\}$ with $l_i \neq l_j$ for $i \neq j$. To this end, we consider two hyperbolas which have one shared asymptote, taking $C_1 : XY - X^2 - W^2 = 0, Z + W = 0$ and

$C_2 : Y^2 - XY - W^2 = 0, Z - W = 0$. Indeed, setting $W = 0$ gives for C_1 has $X(Y - X) = 0$, so we have

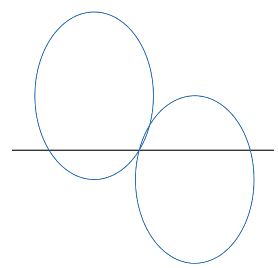
$$C_1 \cap L = \{[0:1:0:0], [1:1:0:0]\}$$

corresponding to the asymptotes $x = 0$ and $y = x$ when viewed as an affine curve. Similarly, from C_2 we have $Y(Y - X) = 0$, so

$$C_2 \cap L = \{[1:0:0:0], [1:1:0:0], \}$$

$$C_1 : \left(s, \frac{1+s^2}{s}\right) \qquad\qquad C_2 : \left(\frac{t^2-1}{t}, t\right)$$

from the asymptotes $y = 0$ and $y = x$. With C_i in coordinates x_i, y_i we get from the tangents condition the equation $y_2(2x_1 - y_1) + x_1(2y_2 - x_2) = 0$. In order to parametrise the hyperbolas, we see that for C_1 we have when $y = \frac{1+x^2}{x}$ and for C_2, $x = \frac{y^2-1}{y}$ when $x \neq 0$, $y \neq 0$ respectively. This leads to the parametrisations



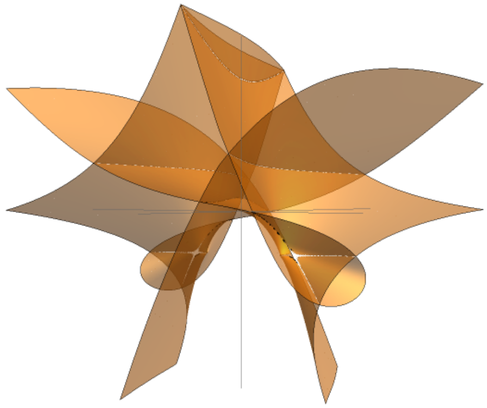
Case 6 intersections

This leads to a description of E given by $s(2st^2 + s - t^2) = 0$. However, the solution $s = 0$ corresponds to the projective portion $W = 0$ and so the affine curve E is given by the irreducible

$$2st^2 + s - t^2 = 0.$$

Computing S we have the generator

$$\begin{aligned}0 = {} & 8x^6z^2 - 16x^6z + 8x^6 - 32x^5yz^2 + 64x^5yz - 32x^5y + 40x^4y^2z^2 - 112x^4y^2z + 40x^4y^2 \\ & + 20x^4z^4 - 40x^4z^2 + 20x^4 + 128x^3y^3z - 80x^3yz^4 + 96x^3yz^2 - 80x^3y - 40x^2y^4z^2 \\ & - 112x^2y^4z - 40x^2y^4 + 120x^2y^2z^4 - 112x^2y^2z^2 + 120x^2y^2 - 2x^2z^6 + 52x^2z^5 - 30x^2z^4 \\ & - 72x^2z^3 - 30x^2z^2 + 52x^2z - 2x^2 + 32xy^5z^2 + 64xy^5z + 32xy^5 - 80xy^3z^4 + 96xy^3z^2 \\ & - 80xy^3 - 96xyz^5 + 128xyz^3 - 96xyz - 8y^6z^2 - 16y^6z - 8y^6 + 20y^4z^4 - 40y^4z^2 + 20y^4 \\ & + 2y^2z^6 + 52y^2z^5 + 30y^2z^4 - 72y^2z^3 + 30y^2z^2 + 52y^2z + 2y^2 - z^8 + 20z^6 \\ & - 102z^4 + 20z^2 - 1\end{aligned}$$



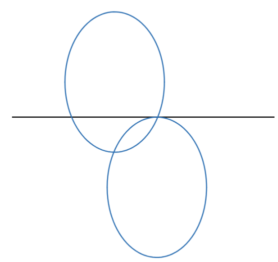
Case 6: two hyperbolas with a shared intersection

Case 7: two points, one of multiplicity three

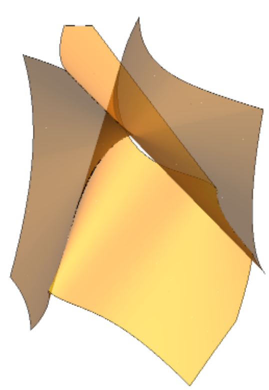
In this final case, we consider $(C_1 \cap L) \cup (C_2 \cap L) = \{l_1, l_2\})$, where l_1 has multiplicity three. We take $C_1 : XY - W^2 = 0, Z + W = 0$ and $C_2 : YW - X^2 = 0, Z - W = 0$. Thus, $W = 0$ gives $C_1 : XY = 0$, so $C_1 \cap L = \{[0 : 1 : 0 : 0], [1 : 0 : 0 : 0]\}$ and similarly for $C_2 : X^2 = 0$, we have $C_2 \cap L = \{[0 : 1 : 0 : 0]\}$.

Parametrising C_1 by $(s, 1/s)$ and C_2 by (t, t^2), together with the parallel, again in affine coordinates, $2x_2x_1 + y_1 = 0$ condition gives the curve E as $2s^2t + 1 = 0$, which is irreducible. As expected, when we compute S we get the irreducible surface given by

$$\begin{aligned} 0 =& 128x^4y - 32x^3z^2 + 64x^3z - 32x^3 - 128x^2y^2z - 128x^2y^2 \\ &+ 144xyz^3 - 144xyz^2 - 144xyz + 144xy + 32y^3z^2 + 64y^3z \\ &+ 32y^3 - 27z^5 + 81z^4 - 54z^3 - 54z^2 + 81z - 27. \end{aligned}$$



Case 7 intersections



Case 7: a parabola and hyperbola with a shared intersection

8 Conclusion

The Bas-Serra surface and its generalised construction provide an interesting example of an algebraic surface, as analysing certain properties emerge as fairly difficult problems to solve. This can be seen in the implicit definition and singularities of the surface, for neither of which a simple or neat description has been found. It was an intention of this work to simplify this implicit expression, but this proved a dead-end and has been omitted. Further, finding the real intersection of the surface $S(\mathbb{C})$ was achieved successfully, and as noted the curve that emerges intersects the purely real surface. As this was discovered fairly late into writing, further details regarding exactly where this intersection occurs have yet to be worked out. Similarly, while elegant and usable descriptions of some subsets of the singularities of S are available, a better description of all the singularities is needed. In order to gain a full understanding of the Bas-Serra construction in generality, much more theory from algebraic geometry regarding sheaves and ramification is needed and a full proof of the fact regarding irreducibility in section 7 is certainly the next step.

9 Acknowledgements

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