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Analysis of sequences on the Kirby fractal

Bachelor's Project Mathematics

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Abstract

In this paper we analyse a sequence recurrence that is based on the Fibonacci recurrence. We classify sequence behaviours with only convergence on the odd or even index subsequences. These classifications give rise to the 'Kirby fractal' when viewed in the complex plane. We then show that this Kirby fractal is bounded. We conclude by analysing the number of periodic sequences on the boundary fractal.

Contents

1 Introduction

In the field of dynamical systems an often studied subject is the iteration of maps[6]. One way of analysing the iteration of maps is by colouring the space of initial values depending on the iteration behaviour it generates[4]. To get a better grasp of the behaviour distinctions we will make for our main recurrence, we will first have a look at an example: the iteration of the square map on the complex plane[3]. Let $S(z) := z^2$ be the square map and (a_n) be a sequence such that $a_{n+1} = S(a_n)$. Then for any $a_0 \in \mathbb{C}$ such that $|a_0| < 1$, we have

$$\lim_{k\to\infty} |a_k| = \lim_{k\to\infty} \left|S^k(a_0)\right| = \lim_{k\to\infty} |a_0|^{2^k} = 0.$$

For $|a_0| > 1$, we have

$$\lim_{k\to\infty} |a_k| = \lim_{k\to\infty} \left|S^k(a_0)\right| = \lim_{k\to\infty} |a_0|^{2^k} = \infty.$$

If we plot this in the complex plane with initial values that converge to 0 in pink and initial values that diverge to ∞ in light blue, we get Figure 1.

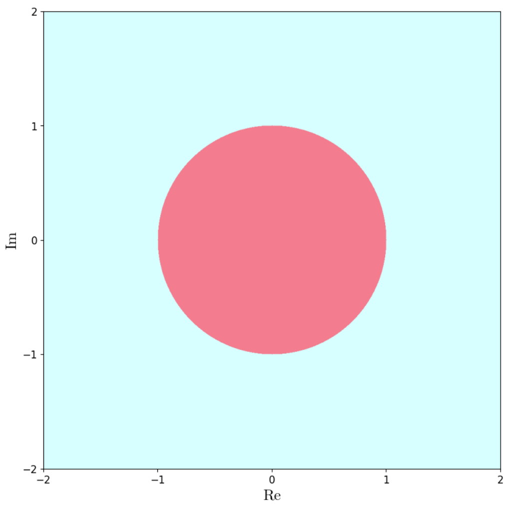

Figure 1: The Julia set of the square map, with convergence to 0 in pink and divergence to ∞ in light blue.

Lastly for $|a_0| = 1$, we get that

$$|a_k| = |S^k(a_0)| = |a_0|^{2^k} = 1$$

for all integers $k \geq 0$. If we let $a_0 = e^{\theta i}$, then the exponent gets doubled with each step. Therefore the points on the unit circle will have the same behaviours as the doubling map. This means there are periodic, preperiodic and chaotic points.

The behaviours we will classify in our main recurrence appear in the complex plane in a similar way. We will have an inside and an outside with two fairly trivial behaviours, and a bordering set, like the unit circle in the example, on which non-trivial behaviours exist.

The sequences (z_n) which we will analyse in this paper satisfy the following recurrence relation:

$$z_{n+1} = \frac{z_{n-1}^2}{z_n} + z_{n-1} = (z_n + z_{n-1}) \cdot \frac{z_{n-1}}{z_n}, \qquad n \geq 1, \tag{1}$$

with initial values $z_0, z_1 \in \mathbb{C}$. This recurrence is a variation on the Fibonacci recurrence. The sequence behaviours this recurrence generates are rather interesting, having both converging and diverging subsequences. Even though this is a recurrence in two variables, we can simplify the space of initial values. This is because for any $c \in \mathbb{C}\backslash\{0\}$, we have

$$\frac{(c \cdot z_{n-1})^2}{c \cdot z_n} + c \cdot z_{n-1} = c \cdot \frac{z_{n-1}^2}{z_n} + c \cdot z_{n-1} = c \cdot z_{n+1}, \qquad n \geq 1.$$

Hence if a sequence (z_n) satisfies (1), then so does $(c \cdot z_n)$. Therefore we can multiply the initial values with $\frac{1}{z_0}$ to get a sequence with the same behaviour, up to multiplying with a constant.

This only fails for $z_0 = 0$ as this causes division by 0. The sequences that $z_0 = 0$ generates are not very interesting however, so we will opt to ignore this case. For the rest of this paper we will fix $z_0 = 1$ and for each sequence generated by a choice of initial values we will only mention the choice for $z_1 \in \mathbb{C}$. We will encounter other initial values that cause division by zero, which we will not ignore. How we deal with division by zero in these sequences will be covered in section 2.

In section 3 we will plot and analyse the different sequence behaviours in the complex plane, which will give us a fun shape that we will call the 'Kirby fractal', because it looks a bit like the video game character Kirby. In section 4 we will look more closely at a subset of initial values in the boundary of the Kirby fractal.

2 Division by zero

To avoid problems with division by zero later on, we will clear up the way we handle these cases now. We will consider the function on the Riemann sphere $\mathbb{S} \cong \mathbb{C} \cup \{\infty\}$ so we can use ∞ in a reasonable way to denote when a number is without bound or equal to the so called "point at infinity"[7]. Let $h : \mathbb{S} \times \mathbb{S} \to \mathbb{S}$ be such that

$$h(w, z) := \frac{w^2}{z} + w.$$

Then we can write our main recursion as

$$z_{n+1} = h(z_{n-1}, z_n)$$

for all $n \geq 1$. This function h is well defined for $(w, z) \in \mathbb{C} \times \mathbb{C}\backslash\{0\}$, however for the other cases we have to do more work. In summary, for $w, z \in \mathbb{C}\backslash\{0\}$ the special cases for the recurrence function h will be defined as follows

$$\begin{aligned} h(\infty, z) &:= \infty, & h(w, \infty) &:= w, \\ h(\infty, 0) &:= \infty, & h(0, \infty) &:= 0, \\ h(w, 0) &:= \infty. \end{aligned}$$

For all of these definitions we just equate the function to the evaluation of the limit on the Riemann sphere at the given point:

$$\begin{aligned} |h(\infty, z)| &= \lim_{|w|\to\infty} \left| \frac{w^2}{z} + w \right| = \infty, \\ |h(\infty, 0)| &= \lim_{|w|\to\infty} \lim_{z\to 0} \left| \frac{w^2}{z} + w \right| = \infty, \\ h(w, \infty) &= \lim_{|z|\to\infty} \frac{w^2}{z} + w = 0 + w = w, \\ |h(w, 0)| &= \lim_{z\to 0} \left| \frac{w^2}{z} + w \right| = \infty, \\ h(0, \infty) &= \lim_{w\to 0} \lim_{|z|\to\infty} \frac{w^2}{z} + w = 0. \end{aligned}$$

There are two cases, namely $h(0, 0)$ and $h(\infty, \infty)$, that we will leave undefined as these do not have unique limit evaluations. Fortunately our choice of initial values $z_0 = 1, z_1 \in \mathbb{C}$, together with the definitions of the special cases that we were able to define, prevent these cases from occurring in a sequence. For instance, if $z_n = 0$ and $z_{n+1} = 0$, then this can only be preceded by $z_{n-1} = 0$ as the main recursion implies the inverse recursions

$$z_{n-1}^2 + z_n \cdot z_{n-1} - z_{n+1} z_n = 0 \implies z_{n-1} = \frac{-z_n \pm \sqrt{z_n^2 + 4 z_{n+1} z_n}}{2}.$$

Thus by induction this can only happen if $z_0 = z_1 = 0$. However we have fixed $z_0 = 1 \neq 0$, so this does not happen. On the other hand $z_n = \infty, z_{n+1} = \infty$ cannot be preceded by $z_{n-1} \in \mathbb{C}$, as then per our definition

$$z_{n+1} = h(z_{n-1}, \infty) = z_{n-1} \neq \infty.$$

Thus this case also cannot happen.

Now that our recursion is properly defined we can finally start examining the sequence behaviours this recursion generates.

3 Sequence behaviours

3.1 Convergent subsequences

We want to analyse the qualitative behaviours of the sequences that are generated by different choices of z_1 in the complex plane. Since our recursion is a variation on Fibonacci, let us first have a look at the sequence for initial value $z_1 = 1$ shown in the left-most column of table 1. The subsequence of the odd indices of the sequence generated by $z_1 = 1$ converges to an unknown

n	z_n	z_n	z_n	z_n	
0	1	1	1	1	
1	1	2	−3	1	−2i
2	2.0000	1.5000	0.6667	1.2000	+0.4000i
3	1.5000	4.6667	1.0500e+01	−2.2500	−4.2500i
4	4.6667	1.9821	0.7090	0.8990	+0.5418i
5	1.9821	1.5654e+01	1.6600e+02	−3.4523	+1.7748e+01i
6	1.5654e+01	2.2331	0.7120	0.9465	+0.5036i
7	2.2331	1.2538e+02	3.8868e+04	−3.0669e+02	+4.9625e+01i
8	1.2538e+02	2.2729	0.7120	0.9449	+0.5002i
9	2.2729	7.0419e+03	2.1217e+09	6.2085e+04	−6.5193e+04i
10	7.0419e+03	2.2736	0.7120	0.9449	+0.5003i
11	2.2736	2.1817e+07	6.3223e+18	−3.8693e+09	−6.5182e+09i
12	2.1817e+07	2.2736	0.7120	0.9449	+0.5003i
13	2.2736	2.0935e+14	5.6137e+37	−6.6988e+17	+5.3737e+19i
14	2.0935e+14	2.2736	0.7120	0.9449	+0.5003i

Table 1: The sequences generated by $z_1 = 1$, $z_1 = 2$, $z_1 = -3$, and $z_1 = 1 - 2i$.

number $2.27363902874548\ldots$, while the subsequence of the even indices diverges. This type of sequence behaviour is very common for our recurrence, even more so when the parities are swapped, some examples of which are given in the other three columns. Because of the frequency with which we encounter these sequence behaviours, we will define them as follows.

Definition 3.1. If a sequence converges/diverges on the even/odd index subsequence, we will call the sequence **even/odd convergent/divergent** respectively. If a sequence is *even* convergent in $\mathbb{C}$ and odd divergent to ∞, then we call the sequence **solely even convergent**. If a sequence is *odd* convergent in $\mathbb{C}$ and even divergent to ∞, then we call the sequence **solely odd convergent**. For convenience we will also refer to initial values as their corresponding sequence behaviour.

A natural question to ask is where these sequence behaviours occur in the complex plane. Numerical computations give us Figure 2. We see that there is an island of initial values that generate sequences with solely odd convergence, coloured pink, surrounded by a sea of initial values that generate sequences with solely even convergence, coloured light blue. The boundary of this island is fractal-like. Since looking at it sideways this shape looks a bit like the video game character Kirby, seen in Figure 3, we will call this the **Kirby fractal**. This funny name is also slightly useful. One can remember which type of convergence is on the inside of the Kirby fractal by remembering the mnemonic "Kirby is an oddball".

We also see that the Kirby fractal is symmetric across the real axis, which is because $z_0 = 1 = \bar{z}_0$ and

$$\bar{z}_{n+1} = \overline{\frac{z_{n-1}^2}{z_n} + z_{n-1}} = \frac{\bar{z}_{n-1}^2}{\bar{z}_n} + \bar{z}_{n-1}.$$

Hence taking the complex conjugate of z_1 only conjugates the whole sequence it generates and does not change whether the sequence has solely even or solely odd convergence.

As we noted already the Kirby fractal seems to be surrounded by a sea of solely even convergence. In section 3.2 we will show that all initial values with distance greater than 2 to the origin have solely even convergence. Hence everything beyond the confines of Figure 2 is part of the sea. However, before we can show this we need some tools which we will develop in this section. We will first need some definitions and theorems from the book "Complex Analysis" by Elias M. Stein and Rami Shakarchi [7].

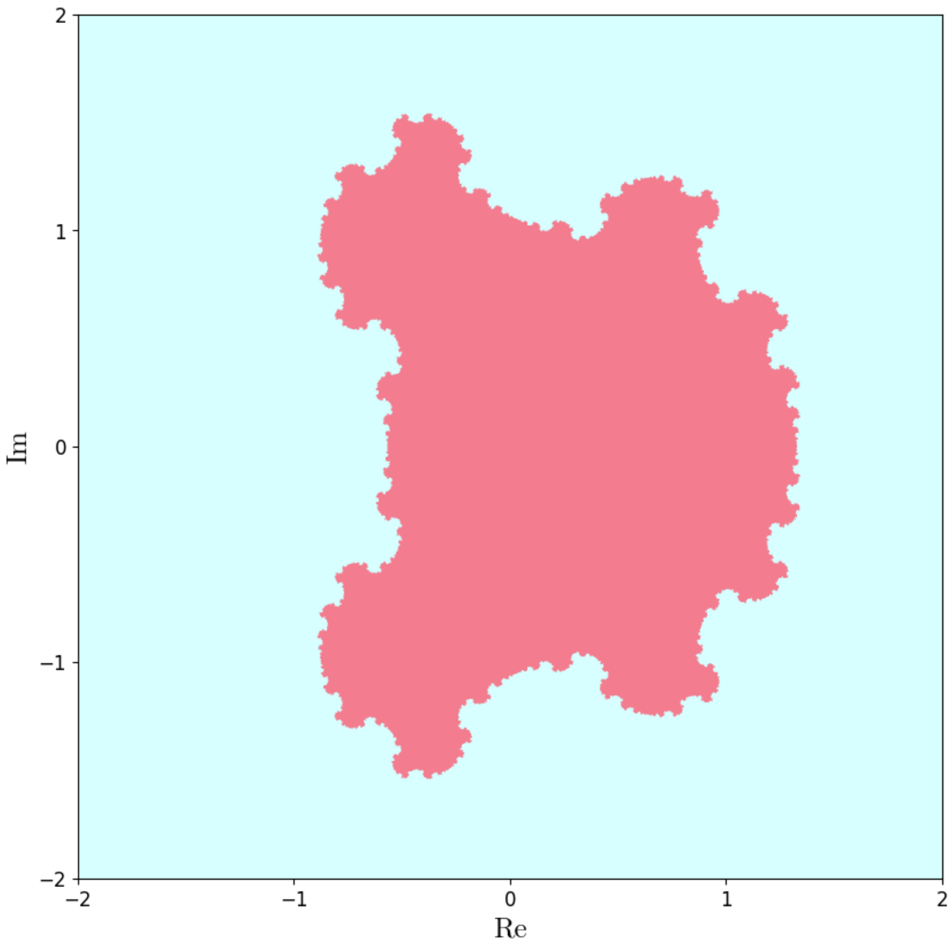

Figure 2: Solely even or solely odd convergence, respectively light blue and pink.

Definition 3.2. Let Ω be an open set in $\mathbb{C}$ and F a complex-valued function on Ω. The function F is said to be **holomorphic on** Ω if for every point $z \in \Omega$ the quotient

$$\frac{F(z+t)-F(z)}{t}$$

converges to a limit when $t \to 0$.

We will use holomorphicity to create bounds on the absolute value of rational functions in parts of the complex plane. The first proposition we will need is the following.

Proposition 3.3. *Every rational function is holomorphic on open sets that do not contain the poles of the function.*

The second proposition we need will make our bounds easier to compute.

Proposition 3.4 (Maximum modulus principle)**.** *If F is a non-constant holomorphic function in a connected open set $\Omega \subset \mathbb{C}$, then F cannot attain a maximum in Ω.*

Lastly to show convergence we will use Cauchy sequences.

Definition 3.5. A sequence (a_n) is said to be a **Cauchy sequence** (or simply **Cauchy**) if

$$|a_n - a_m| \to 0 \text{ as } n, m \to \infty.$$

In other words, given $\epsilon > 0$ there exists an integer $N > 0$ so that $|a_n - a_m| < \epsilon$ whenever $n, m > N$.

Proposition 3.6. *Every Cauchy sequence in $\mathbb{C}$ has a limit in $\mathbb{C}$.*

With these preliminaries out of the way we can now start proving our own lemmas. One of the facts that we will use in our proofs is that in our sequences the ratio of concurrent values is a function of the previous ratio of concurrent values. Since we will be using these functions a lot, we will give them formal definitions.

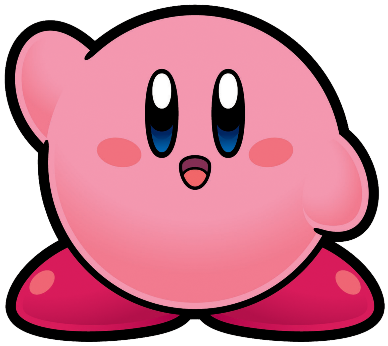

Figure 3: Videogame character Kirby. (Image taken from Wikipedia)

Definition 3.7. For any natural number k, the ratio $\frac{z_{n+k}}{z_n}$ is called the ***k*-step-ratio**. We define f to be the recurrence of 1-step-ratios of our sequences such that for all $n \geq 1$

$$\frac{z_{n+1}}{z_n} = f\left(\frac{z_n}{z_{n-1}}\right).$$

We define g to be the recurrence of 2-step-ratios of our sequences such that for all $n \geq 2$

$$\frac{z_{n+2}}{z_n} = g\left(\frac{z_{n+1}}{z_{n-1}}\right).$$

Lemma 3.8.

$$f(z) = \frac{z+1}{z^2} \text{ and } g(z) = 1 + \frac{1}{z^2 - z}.$$

The recurrence of 1-step-ratios can simply be found by dividing both sides of the main recurrence by z_n. Finding the recurrence of 2-step-ratios is a bit more involved, but only consists of some algebra, which is shown in Appendix A. These recurrence formulas are incredibly useful, because the behaviour of the sequence of 1-step-ratios

$$(\frac{z_1}{z_0}, \frac{z_2}{z_1}, \frac{z_3}{z_2}, \dots)$$

and the behaviour of the sequence of 2-step-ratios

$$(\frac{z_2}{z_0}, \frac{z_3}{z_1}, \frac{z_4}{z_2}, \dots)$$

individually determine the behaviour of the original sequence (z_n). In fact for these ratio sequences the behaviour of just the odd or just the even index subsequence of either is already enough to determine what the behaviour is of all three sequences. This gives us Theorem 3.9.

Theorem 3.9. *The following are equivalent:*

1. (z_n) *is solely even convergent*
2. (z_{n+1}/z_n) *is solely odd convergent to* 0
3. (z_{n+1}/z_n) *is odd convergent to* 0
4. (z_{n+1}/z_n) *is even divergent to* ∞
5. (z_{n+2}/z_n) *is solely even convergent to* 1
6. (z_{n+2}/z_n) *is even convergent to* 1
7. (z_{n+2}/z_n) *is odd divergent to* ∞

For simplicity we will divide the proof of this up into three parts. First the equivalences of the 1-step-ratio sequence, second the equivalences of the 2-step-ratio sequence, and lastly the equivalence of the three sequences. The first two Lemmas 3.10 and 3.11 will be quite trivial to prove. The third Lemma 3.19 however has a final step that needs a lot of justification, namely Lemmas 3.12, 3.13, 3.14, 3.15, 3.16, and 3.17. Because of this that final step will be its own Theorem 3.18.

Lemma 3.10.

$$(2) \iff (3) \iff (4)$$

Proof. By Definition 3.1 we have that (3) together with (4) is equivalent to (2). Thus we only need to show that (3) $\iff$ (4).

(3) $\implies$ (4): Let (z_{n+1}/z_n) be odd convergent to 0. Then

$$\lim_{k\to\infty} \frac{z_{2k+3}}{z_{2k+2}} = \lim_{k\to\infty} f\left(\frac{z_{2k+2}}{z_{2k+1}}\right) = \lim_{z\to 0} f(z) = \lim_{z\to 0} \frac{z+1}{z^2} = \infty.$$

(4) $\implies$ (3): Let (z_{n+1}/z_n) be even divergent to ∞. Then

$$\lim_{k\to\infty} \frac{z_{2k+2}}{z_{2k+1}} = \lim_{k\to\infty} f\left(\frac{z_{2k+1}}{z_{2k}}\right) = \lim_{z\to \infty} f(z) = \lim_{z\to \infty} \frac{z+1}{z^2} = 0.$$

This completes the proof. □

For the 2-step-ratio sequence equivalences we use the same argument, but with the 2-step-ratio recurrence formula.

Lemma 3.11.

$$(5) \iff (6) \iff (7)$$

Proof. By Definition 3.1 we have that (6) together with (7) is equivalent to (5). Thus we only need to show that (6) $\iff$ (7).

(6) $\implies$ (7): Let (z_{n+2}/z_n) be even convergent to 1. Then

$$\lim_{k\to\infty} \frac{z_{2k+3}}{z_{2k+1}} = \lim_{k\to\infty} g\left(\frac{z_{2k+2}}{z_{2k}}\right) = \lim_{z\to 1} g(z) = \lim_{z\to 1} 1 + \frac{1}{z^2-z} = \infty.$$

(7) $\implies$ (6): Let (z_{n+2}/z_n) be odd divergent to ∞. Then

$$\lim_{k\to\infty} \frac{z_{2k+4}}{z_{2k+2}} = \lim_{k\to\infty} g\left(\frac{z_{2k+3}}{z_{2k+1}}\right) = \lim_{z\to \infty} g(z) = \lim_{z\to \infty} 1 + \frac{1}{z^2-z} = 1.$$

This completes the proof. □

In Lemma 3.19, which states that (1), (2), and (5) are equivalent, we will be using an important equality that relates the 2-step-ratio to the 1-step-ratio:

$$\frac{z_{n+1}}{z_{n-1}} = \frac{1}{\frac{z_n}{z_{n-1}}} + 1.$$

It can be found by simply dividing our main recurrence by z_{n-1}. Since the behaviour of our sequences depends so much on the behaviour of the even or odd subsequence, we will be using the following equations a lot as well.

$$f^2(z) = \frac{z^4+z^3+z^2}{z^2+2z+1} \text{ and } g^2(z) = z^2 - z + \frac{1}{z^2-z+1}.$$

The next six lemmas are all needed to prove Theorem 3.18 that is the final step of Lemma 3.19. For the proof of Theorem 3.18 we will use the following lemma to create multiple bounds that are only slightly different, however each bound we create is useful in their own right.

Lemma 3.12. *If* $|z| < \frac{1}{2}$, *then*

$$\left|\frac{z}{z^2+z+1}\right| \le \sqrt{\frac{16}{27}}.$$

Proof. The poles of the rational function $\frac{z}{z^2+z+1}$ are

$$z^2+z+1 = 0 \implies z_\pm = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i.$$

These poles have absolute value $|z_\pm| = 1$, therefore they do not lie in the connected open set $|z| < \frac{1}{2}$. By Proposition 3.3 we have that $\frac{z}{z^2+z+1}$ is holomorphic in this region. Thus by Proposition 3.4 we have that for $|z| < \frac{1}{2}$

$$\left|\frac{z}{z^2+z+1}\right| \leq \sup_{|z|=\frac{1}{2}} \left|\frac{z}{z^2+z+1}\right| = \sup_{|z|=\frac{1}{2}} \frac{1}{\left|z+1+\frac{1}{z}\right|} = \frac{1}{\inf\limits_{|z|=\frac{1}{2}} \left|z+1+\dfrac{1}{z}\right|}.$$

Plugging $z = \frac{1}{2}e^{\theta i}$ into $\left|z+1+\frac{1}{z}\right|$ we get

$$\begin{aligned}\left|\frac{1}{2}e^{\theta i} + 1 + 2e^{-\theta i}\right| &= \left|\frac{5}{2}\cos(\theta) + 1 - \frac{3}{2}\sin(\theta) i\right| = \sqrt{\left(\frac{5}{2}\cos(\theta)+1\right)^2 + \frac{9}{4}\sin^2(\theta)} \\ &= \sqrt{4\cos^2(\theta) + 5\cos(\theta) + \frac{13}{4}}.\end{aligned}$$

The extremal points of $p(\theta) := 4\cos^2(\theta) + 5\cos(\theta) + \frac{13}{4}$ are

$$p'(\theta) = -8\sin(\theta)\cos(\theta) - 5\sin(\theta) = 0 \implies \sin(\theta) = 0 \text{ or } \cos(\theta) = -\frac{5}{8}.$$

Hence either $\cos(\theta) = 1$, $\cos(\theta) = -1$, or $\cos(\theta) = -\frac{5}{8}$. These extremal points evaluate to $p(\theta) = \frac{49}{4}$, $p(\theta) = \frac{9}{4}$, and $p(\theta) = \frac{27}{16}$, hence the latter is the minimal value. Therefore

$$\left|\frac{z}{z^2+z+1}\right| \leq \frac{1}{\inf_{|z|=\frac{1}{2}} \left|z+1+\frac{1}{z}\right|} = \sqrt{\frac{16}{27}}.$$

□

The next two lemmas are bounds that will be useful to show convergence once we apply them to the 2-step-ratio sequence. Their proofs are very similar.

Lemma 3.13. *If $|z-1| < \frac{1}{13}$, then there exists $\alpha \in (0,1)$ such that*

$$|g^2(z) - 1| \leq \alpha \left|\frac{1}{z} - 1\right|.$$

Proof. If $|z-1| = 0$, then $g^2(z) = g^2(1) = 1$ and hence

$$|g^2(1) - 1| = 0 = \alpha \left|\frac{1}{1} - 1\right|$$

for some $\alpha \in (0,1)$.

Let $0 < |z-1| < \frac{1}{13}$. We first note the following equivalences

$$|g^2(z) - 1| \leq \alpha \left|\frac{1}{z} - 1\right| \iff \left|\frac{z^2(z-1)^2}{z^2-z+1}\right| \leq \alpha \left|\frac{z-1}{z}\right| \iff \left|\frac{z^3(z-1)}{z^2-z+1}\right| \leq \alpha.$$

We will show the right most inequality. By the triangle inequality we have

$$|z| = |z-1+1| \leq |z-1| + 1 \leq \frac{1}{13} + 1 = \frac{14}{13} \implies |z|^3 \leq \left(\frac{14}{13}\right)^3. \tag{2}$$

Let $w := z - 1$, then $|w| < \frac{1}{13} < \frac{1}{2}$ and

$$\left|\frac{z-1}{z^2-z+1}\right| = \left|\frac{w}{w^2+w+1}\right|.$$

Thus by Lemma 3.12, we have

$$\left|\frac{z-1}{z^2-z+1}\right| \leq \sqrt{\frac{16}{27}} \leq \frac{4}{5}. \tag{3}$$

Applying inequalities (2) and (3), we get

$$\left|\frac{z^3(z-1)}{z^2-z+1}\right| \leq \left(\frac{14}{13}\right)^3 \cdot \frac{4}{5} = \frac{10976}{10985}.$$

Taking $\alpha := \frac{10976}{10985}$ finishes the proof. □

Lemma 3.14. *If $|z-1| < \frac{1}{2}$, then*

$$|g^2(z) - 1| \le \frac{9}{5} |z-1|^2 .$$

Proof. If $|z-1| = 0$, then $|g^2(1) - 1| = |1-1| = 0 = \frac{9}{5}|1-1|^2$.

Let $0 < |z-1| < \frac{1}{2}$. We first note the following equivalences

$$|g^2(z) - 1| \le \frac{9}{5}|z-1|^2 \iff \left|\frac{z^2(z-1)^2}{z^2-z+1}\right| \le \frac{9}{5}|z-1|^2 \iff \left|\frac{z^2}{z^2-z+1}\right| \le \frac{9}{5}$$
$$\iff \left|1 + \frac{z-1}{z^2-z+1}\right| \le \frac{9}{5}.$$

Let $w := z-1$, then $|w| < \frac{1}{2}$ and

$$\left|\frac{z-1}{z^2-z+1}\right| = \left|\frac{w}{w^2+w+1}\right|.$$

Thus by Lemma 3.12, we have

$$\left|\frac{z-1}{z^2-z+1}\right| \le \sqrt{\frac{16}{27}} \le \frac{4}{5}. \tag{4}$$

Applying the triangle inequality and inequality (4), we get

$$\left|1 + \frac{z-1}{z^2-z+1}\right| \le 1 + \left|\frac{z-1}{z^2-z+1}\right| \le 1 + \frac{4}{5} = \frac{9}{5}.$$

This finishes the proof. □

We will use the following two lemmas together with the Lemma 3.14 to show that, if the 2-step-ratio converges to 1, then its corresponding converging subsequence of (z_n) can only converge to 0 if it reaches 0 in a finite amount of steps.

Lemma 3.15. *For all integers $k \ge 6$, we have*

$$1 - \left(\frac{9}{10}\right)^{2^k} > e^{-2^{-k}}.$$

Proof. For $k = 6$, we have

$$1 - \left(\frac{9}{10}\right)^{2^6} = 0.9988\cdots > 0.9844\cdots = e^{-2^{-6}}.$$

Let $k \ge 6$ be such that

$$1 - \left(\frac{9}{10}\right)^{2^k} > e^{-2^{-k}}.$$

Then

$$e^{-2^{-(k+1)}} = e^{-\frac{1}{2}\cdot 2^{-k}} = \sqrt{e^{-2^{-k}}} < \sqrt{1 - \left(\frac{9}{10}\right)^{2^k}}.$$

We want to show that

$$\sqrt{1 - \left(\frac{9}{10}\right)^{2^k}} < 1 - \left(\frac{9}{10}\right)^{2^{k+1}}.$$

Define $a_k := \left(\frac{9}{10}\right)^{2^k}$. Then this is equivalent to

$$\begin{aligned} & 1 - a_k < \left(1 - a_k^2\right)^2 = 1 - 2a_k^2 + a_k^4 \\ \iff & 2a_k^2 < a_k + a_k^4 \\ \iff & a_k^3 - 2a_k + 1 > 0. \end{aligned}$$

Let $h(x) := x^3 - 2x + 1$, then

$$h(x) = (x^3 - x^2) + (x^2 - x) - (x - 1) = (x-1)\left(x^2 + x - 1\right) = (x-1)\left(x + \frac{1+\sqrt{5}}{2}\right)\left(x + \frac{1-\sqrt{5}}{2}\right)$$

and $h(0) = 1 > 0$. Since the roots of h are ordered as $-\frac{1+\sqrt{5}}{2} < \frac{\sqrt{5}-1}{2} < 1$ and $0 \in \left(-\frac{1+\sqrt{5}}{2}, \frac{\sqrt{5}-1}{2}\right)$, we therefore have $h(x) > 0$ for all $x \in \left(-\frac{1+\sqrt{5}}{2}, \frac{\sqrt{5}-1}{2}\right)$. For all integers $k \geq 6$, we have

$$\frac{\sqrt{5}-1}{2} = 0.618\dots > 0.001\dots = \left(\frac{9}{10}\right)^{2^6} \geq a_k > 0 \implies a_k \in \left(-\frac{1+\sqrt{5}}{2}, \frac{\sqrt{5}-1}{2}\right).$$

Thus

$$h(a_k) = a_k^3 - 2a_k + 1 > 0.$$

Hence

$$1 - \left(\frac{9}{10}\right)^{2^{k+1}} > \sqrt{1 - \left(\frac{9}{10}\right)^{2^k}} > e^{-2^{-(k+1)}}.$$

Therefore by mathematical induction the inequality holds for all $k \geq 6$. □

We can now bound the following infinite product from below, which will also bound our sequences, with 2-step-ratios convergent to 1, away from 0.

Lemma 3.16.

$$\prod_{k=0}^{\infty}\left(1 - \frac{5}{9}\left(\frac{9}{10}\right)^{2^k}\right) = r$$

for some real number $r > 0$.

Proof. First note that this product decreases monotonically. Hence to prove the statement we will create a positive lower bound. Since $-\frac{5}{9} > -1$ and by applying Lemma 3.15 we have

$$\begin{aligned}\prod_{k=0}^{\infty}\left(1 - \frac{5}{9}\left(\frac{9}{10}\right)^{2^k}\right) &> \prod_{k=0}^{5}\left(1 - \frac{5}{9}\left(\frac{9}{10}\right)^{2^k}\right) \cdot \prod_{k=6}^{\infty}\left(1 - \left(\frac{9}{10}\right)^{2^k}\right) \\ &> \prod_{k=0}^{5}\left(1 - \frac{5}{9}\left(\frac{9}{10}\right)^{2^k}\right) \cdot \prod_{k=6}^{\infty} e^{-2^{-k}}\end{aligned}$$

The exponent of the left most infinite product is a geometric series and therefore

$$\prod_{k=6}^{\infty} e^{-2^{-k}} = \exp\left(-\sum_{k=6}^{\infty}\frac{1}{2^k}\right) = \exp\left(-\frac{(\frac{1}{2})^6}{1 - \frac{1}{2}}\right) = \exp\left(-\frac{1}{32}\right).$$

We also have

$$1 - \frac{5}{9}\left(\frac{9}{10}\right)^{2^k} > 0 \iff 1 > \frac{5}{9}\left(\frac{9}{10}\right)^{2^k},$$

which is true for all integers $k \geq 0$. Thus

$$\prod_{k=0}^{\infty}\left(1 - \frac{5}{9}\left(\frac{9}{10}\right)^{2^k}\right) > \prod_{k=0}^{5}\left(1 - \frac{5}{9}\left(\frac{9}{10}\right)^{2^k}\right) \cdot \exp\left(-\frac{1}{32}\right) > 0.$$

This finishes the proof. □

We can now show that a sequence generated by our main recurrence is solely even convergent to 0 only if it reaches 0 in finite steps.

Lemma 3.17. *If (z_{n+2}/z_n) is even convergent to 1, then (z_n) is even convergent to 0 only if there exists $N \in \mathbb{N}$ such that for all integers $k \geq N$, we have $z_{2k} = 0$.*

Proof. Let (z_{n+2}/z_n) converge to 1 on the even indices. Let $w_n := z_{2n}$. Then there exists an integer N such that for all $n \geq N$, we have

$$\left|\frac{w_{n+1}}{w_n} - 1\right| < \frac{1}{2}.$$

Therefore by Lemma 3.14, we have

$$\left|\frac{w_{n+2}}{w_{n+1}} - 1\right| = \left|g^2\left(\frac{w_{n+1}}{w_n}\right) - 1\right| \leq \frac{9}{5}\left|\frac{w_{n+1}}{w_n} - 1\right|^2$$

for all integers $n \geq N$. By induction we get

$$\left|\frac{w_{N+m+1}}{w_{N+m}} - 1\right| \leq \left(\frac{9}{5}\right)^{2^m-1} \cdot \frac{1}{2^{2^m}} = \frac{5}{9} \cdot \left(\frac{9}{5}\right)^{2^m} \cdot \frac{1}{2^{2^m}} = \frac{5}{9} \cdot \left(\frac{9}{10}\right)^{2^m}.$$

Therefore a lower bound for the 2-step-ratio is

$$\left|\frac{w_{N+m+1}}{w_{N+m}}\right| \geq 1 - \frac{5}{9} \cdot \left(\frac{9}{10}\right)^{2^m}.$$

Hence

$$\lim_{k\to\infty} |w_k| = \lim_{k\to\infty} |w_N| \cdot \prod_{m=0}^{k} \frac{|w_{N+m+1}|}{|w_{N+m}|} \geq |w_N| \cdot \prod_{m=0}^{\infty} \left(1 - \frac{5}{9} \cdot \left(\frac{9}{10}\right)^{2^m}\right).$$

Thus by Lemma 3.16, if $w_n \neq 0$ for all $n \in \mathbb{N}$, then

$$\lim_{k\to\infty} |w_k| > 0.$$

If $w_N = 0$ for some $N \in \mathbb{N}$, then $w_n = 0$ for all integers $n \geq N$. This finishes the proof. □

With all this build up we have finally arrived at our ever so important Theorem 3.18, which provides the connection from k-step-ratio sequences back to the original sequence.

Theorem 3.18. *If (z_{n+2}/z_n) is even convergent to 1, then (z_n) has solely even convergence.*

Proof. Let the sequence (z_{n+2}/z_n) converge to 1 on the even indices. Then there must exist $N \in \mathbb{N}$ such that for all integers $k > N$, we have

$$\left|\frac{z_{2k+2}}{z_{2k}} - 1\right| < \frac{1}{13}.$$

By Lemma 3.13, we therefore have for some $\alpha \in (0, 1)$ that

$$\left|g^2\left(\frac{z_{2k+2}}{z_{2k}}\right) - 1\right| \leq \alpha \left|\frac{1}{\frac{z_{2k+2}}{z_{2k}}} - 1\right|$$

for all integers $k > N$. This implies

$$\begin{aligned}\left|\frac{z_{2k+4}}{z_{2k+2}} - 1\right| \leq \alpha \left|\frac{z_{2k}}{z_{2k+2}} - 1\right| &\implies |z_{2k+4} - z_{2k+2}| \leq \alpha |z_{2k} - z_{2k+2}| \\ &\implies |z_{2k+4} - z_{2k+2}| \leq \alpha |z_{2k+2} - z_{2k}|\end{aligned}$$

for all integers $k > N$ for some $\alpha \in (0, 1)$. Therefore the even index subsequence of (z_n) is a Cauchy sequence. We conclude that the even index subsequence converges. By Lemma 3.17 either the even index subsequence converges to a nonzero number, or the even index subsequence becomes zero after a finite amount of steps. In the case that the even index subsequence becomes zero after finite steps the tail of the main sequence becomes

$$(\ldots, 0, \infty, 0, \infty, 0, \infty, \ldots),$$

where even indices are zero and odd indices explode to infinity. Hence in this case the sequence (z_n) has solely even convergence. Since the even index 2-step-ratios converge to 1, we have

$$\lim_{k\to\infty} \left|\frac{z_{2k+1}}{z_{2k}}\right| = \lim_{k\to\infty} \left|\frac{1}{\frac{z_{2k+2}}{z_{2k}} - 1}\right| \to \infty.$$

Thus if the even index subsequence converges to a nonzero number, then the odd indices explode without bound. Therefore the sequence (z_n) has solely even convergence. This finishes the proof. □

The proof of Lemma 3.19 is now quite simple.

Lemma 3.19.

$$(1) \iff (2) \iff (5)$$

Proof. (1) $\implies$ (2): Let (z_n) be solely even convergent. Then

$$\lim_{k\to\infty} z_{2k+1} = \infty \text{ and } \lim_{k\to\infty} z_{2k} = z$$

for some $z \in \mathbb{C}$. Therefore

$$\lim_{k\to\infty} \frac{z_{2k+1}}{z_{2k}} = \lim_{k\to\infty} \frac{z_{2k+1}}{z} = \infty.$$

Thus the 1-step-ratio (z_{n+1}/z_n) is even divergent to ∞. By Lemma 3.10 we therefore have (2).

(2) $\implies$ (5): Let (z_{n+1}/z_n) be solely odd convergent to 0. Then (z_{n+1}/z_n) is even divergent to ∞

$$\lim_{k\to\infty} \frac{z_{2k+2}}{z_{2k}} = \lim_{k\to\infty} \frac{1}{\frac{z_{2k+1}}{z_{2k}}} + 1 = \lim_{z\to\infty} \frac{1}{z} + 1 = 1.$$

This implies that the 2-step-ratio (z_{n+2}/z_n) is even convergent to 1. Hence by Lemma 3.11 we have (5).

(5) $\implies$ (1): Let (z_{n+2}/z_n) be solely even convergent to 1. Then (z_{n+2}/z_n) is even convergent to 1. Therefore by Theorem 3.18 we have (1).

This concludes the proof. □

Lemmas 3.10, 3.11, and 3.19 together prove Theorem 3.9.

3.2 Bounding the Kirby fractal

Now that we have this more general tool for proving the behaviour of our sequences, we will need more specific lemmas to prove that the initial values with magnitude greater than 2 are all solely even convergent. The next lemma will be used to show even convergence of the 2-step-ratio sequence.

Lemma 3.20. *If $z \in \mathbb{C}$ such that $|z-1| < \frac{1}{2}$, then there exists $\alpha \in (0,1)$ such that*

$$|g^2(z) - 1| \leq \alpha |z-1|.$$

Proof. If $|z-1| = 0$, then $|g^2(z)-1| = |1-1| = 0 = \alpha|1-1| = \alpha|z-1|$ with $\alpha \in (0,1)$.

Let $0 < |z-1| < \frac{1}{2}$. We first note the following

$$|g^2(z) - 1| = \left| z^2 - z + \frac{1}{z^2-z+1} - 1 \right| = \left| \frac{(z^2-z)^2}{z^2-z+1} \right|.$$

Hence

$$\frac{|z|^2}{|z^2-z+1|} \leq 2\alpha \implies \frac{|z|^2 \cdot |z-1|}{|z^2-z+1|} < \frac{|z|^2}{2|z^2-z+1|} \leq \alpha \implies |g^2(z)-1| \leq \alpha|z-1|.$$

Let $w := z-1$, then $|w| < \frac{1}{2}$ and

$$\frac{|z|^2}{|z^2-z+1|} = \frac{|w+1|^2}{|w^2+w+1|} = \left| 1 + \frac{w}{w^2+w+1} \right|.$$

By the triangle inequality and Lemma 3.12, we have

$$\left| 1 + \frac{w}{w^2+w+1} \right| \leq 1 + \sqrt{\frac{16}{27}} < 1 + \frac{4}{5} = \frac{9}{5}.$$

Taking $\alpha := \frac{9}{10}$ finishes the proof. □

The lemma below helps show that ratios with magnitude greater than two stay greater than two after an even amount of steps.

Lemma 3.21. *If $|z| > 2$, then $|f^2(z)| > 2$.*

Proof. Let $z \in \mathbb{C}$ be such that $|z| > 2$. Then

$$|f^2(z)| = \left|\frac{z^4+z^3+z^2}{z^2+2z+1}\right| = |z|^2 \cdot \left|\frac{z^2+z+1}{z^2+2z+1}\right| > 4\left|\frac{z^2+z+1}{z^2+2z+1}\right|.$$

Let $w := \frac{1}{z}$, then $0 < w < \frac{1}{2}$ and hence by Lemma 3.12 we have

$$\left|\frac{z}{z^2+z+1}\right| = \left|\frac{\frac{1}{w}}{\frac{1}{w^2}+\frac{1}{w}+1}\right| = \left|\frac{w}{1+w+w^2}\right| \leq \sqrt{\frac{16}{27}} < 1.$$

The triangle inequality implies

$$\left|\frac{z}{z^2+z+1}\right| < 1 \implies \left|\frac{z^2+2z+1}{z^2+z+1}\right| \leq 1 + \left|\frac{z}{z^2+z+1}\right| < 2 \implies \left|\frac{z^2+z+1}{z^2+2z+1}\right| > \frac{1}{2}$$
$$\implies |f^2(z)| > 2.$$

This finishes the proof □

We can now conclude by proving that the Kirby fractal we see in Figure 2 contains all initial values that generate sequences with solely odd convergence.

Theorem 3.22. *For all $|z| > 2$, the sequence (z_n) has solely even convergence.*

Proof. Let $z \in \mathbb{C}$ be such that $|z| > 2$. Then by inductively applying Lemma 3.21, we have that

$$|f^{2k}(z)| > \ 2$$

for all $k \in \mathbb{N}$. Note that

$$\left|\frac{z_{2k+2}}{z_{2k}} - 1\right| = \left|\frac{z_{2k}}{z_{2k+1}}\right| = \frac{1}{|f^{2k}(z)|} < \frac{1}{2}.$$

Therefore by Lemma 3.20, we get for some $\alpha \in (0,1)$

$$\left|\frac{z_{2k+4}}{z_{2k+2}} - 1\right| = \left|g^2\left(\frac{z_{2k+2}}{z_{2k}}\right) - 1\right| \leq \alpha\left|\frac{z_{2k+2}}{z_{2k}} - 1\right|$$

for all $k \in \mathbb{N}$. This implies

$$\lim_{k\to\infty} \frac{z_{2k+2}}{z_{2k}} = 1.$$

Therefore by Theorem 3.9 the sequence (z_n) has solely even convergence. □

4 Periodic points of the 1-step-ratio recurrence

We now want to have a closer look at the 1-step-ratio recurrence $f(z) = \frac{z+1}{z^2}$. Let us define

$$r_n := \frac{z_n}{z_{n-1}}.$$

One of the first things to note is that we can telescope $\frac{z_n}{z_0}$ to get

$$\begin{aligned}\frac{z_n}{z_0} &= \frac{z_n}{z_{n-1}}\frac{z_{n-1}}{z_{n-2}}\cdots\frac{z_1}{z_0}\\ &= r_n r_{n-1}\cdots r_1\\ &= f^{n-1}(r_1)f^{n-2}(r_1)\cdots f(r_1)r_1.\end{aligned}$$

Since we fixed $z_0 = 1$ and $r_1 = \frac{z_1}{z_0}$, we get the formula

$$z_n = \prod_{k=0}^{n-1} f^k(z_1), \qquad n \geq 1. \tag{5}$$

We want to find the periodic points of f, because for a k-periodic point $f^k(z_1) = z_1$ we can determine the behaviour of (z_n) from the magnitude of $m(z_1) := |z_1 \cdot f(z_1) \cdots f^{k-1}(z_1)|$. From equation (5) we can see that the sequence (z_n) diverges to ∞ if $m(z_1) > 1$ and converges to 0 if $m(z_1) < 1$. Lastly if $m(z_1) = 1$ the modulus of the sequence (z_n) remains bounded between two positive numbers. Since all of these sequence behaviours are not solely odd nor solely even convergent, all these points z_1 must lie on the boundary of the fractal.

With our interest in periodic points of f justified, we will define the following terms.

Definition 4.1. A point z is a **fixed point** of the function F if and only if $F(z) = z$. A point z is called **k-periodic** if and only if $F^k(z) = z$. A point z **has period** k if and only if $F^k(z) = z$ and $F^p(z) \neq z$ for all $p < k$.

The fixed points of f are solutions of the polynomial

$$r = f(r) = \frac{r+1}{r^2} \iff r^3 - r - 1 = 0.$$

This means that the 'Plastic number' $\rho = 1.32471795724474602596\ldots$ and its algebraic conjugates

$$\frac{x^3 - x - 1}{x - \rho} = x^2 + \rho x + \frac{1}{\rho} = 0 \implies x_{1,2} = -\frac{1}{2}\rho \pm \frac{1}{2}\sqrt{3\rho^2 - 4} \cdot i$$

are the fixed points of f[5]. Note that

$$\left| -\frac{1}{2}\rho \pm \frac{1}{2}\sqrt{3\rho^2 - 4} \cdot i \right| = \sqrt{\frac{1}{4}\rho^2 + \frac{1}{4}(3\rho^2 - 4)} = \sqrt{\rho^2 - 1} = \sqrt{\frac{1}{\rho}} < 1.$$

Hence $|\rho| > 1$, while $|x_{1,2}| < 1$. Plugging these fixed points z_1 into equation (5), we then get the analytic solution $z_n = z_1^n$. Hence we see that $z_1 = \rho$ generates a sequence that diverges to infinity, while $z_1 = x_{1,2}$ generate sequences that converge to 0.

The 2-periodic points of f are

$$\begin{aligned} r = f^2(r) = \frac{\frac{r+1}{r^2} + 1}{\left(\frac{r+1}{r^2}\right)^2} = \frac{r^2 \cdot (r+1) + r^4}{(r+1)^2} &\iff r \cdot (r+1)^2 = r^4 + r^3 + r^2 \\ &\iff r^3 + 2r^2 + r = r^4 + r^3 + r^2 \\ &\iff r^4 - r^2 - r = 0 \\ &\iff r = 0 \text{ or } r^3 - r - 1 = 0. \end{aligned}$$

However, since $r = 0$ causes division by zero, we exclude this option and are left with only the fixed points.

We would like to have a general formula for $f^k(r)$. Let P_k, Q_k be polynomials in r such that

$$f^k(r) = \frac{P_k}{Q_k}.$$

Then we have

$$f^{k+1}(r) = f\left(\frac{P_k}{Q_k}\right) = \frac{\frac{P_k}{Q_k} + 1}{\left(\frac{P_k}{Q_k}\right)^2} = \frac{P_k Q_k + Q_k^2}{P_k^2} = \frac{P_{k+1}}{Q_{k+1}}.$$

We can define

$$\begin{cases} P_{k+1} := P_k Q_k + Q_k^2, \\ Q_{k+1} := P_k^2, \end{cases}$$

for all $k \geq 1$, with $P_0 = r, Q_0 = 1$.

Note that any k-periodic point of f must satisfy

$$f^k(r) = \frac{P_k}{Q_k} = r \iff P_k - Q_k \cdot r = 0.$$

Let us define $R_k := P_k - Q_k \cdot r$. If P_k and Q_k share a factor $(r - t)$, then t is a root of R_k. However evaluating $f^k(t)$ causes division by zero, hence this would not be an actual periodic point of f. We will show that this kind of error does not happen.

Lemma 4.2. *We have that* $\gcd(P_k, Q_k) = 1$ *for all* $k \geq 0$.

Proof. For $k = 0$ it holds, since $\gcd(P_0, Q_0) = \gcd(r, 1) = 1$.

Let $\gcd(P_k, Q_k) = 1$ for some $k \geq 0$. Assume for contradiction that $\gcd(P_{k+1}, Q_{k+1}) = g$, where g is a polynomial in r of positive degree. Then there exists a degree 1 polynomial d in r, such that $d \,|\, g$. Therefore

$$\begin{aligned} d \,|\, Q_{k+1} = P_k^2 &\implies d \,|\, P_k, \\ d \,|\, P_{k+1} = P_k Q_k + Q_k^2 &\implies d \,|\, P_k + Q_k \text{ or } d \,|\, Q_k \\ &\implies d \,|\, Q_k. \end{aligned}$$

Hence $d \,|\, \gcd(P_k, Q_k) = 1$, which is not possible for a degree 1 polynomial, thus we have reached a contradiction. We conclude that no such polynomial g exists and hence $\gcd(P_{k+1}, Q_{k+1}) = 1$. Therefore by mathematical induction we have that $\gcd(P_k, Q_k) = 1$ for all $k \geq 0$. □

The degree of R_k will help us understand the amount of k-periodic points, by providing an upper bound. To that end, let us first calculate the degree of P_k.

Lemma 4.3. *The degree of P_k is of the form*

$$\deg(P_k) = \begin{cases} 2^k & \text{if } k \text{ is even,} \\ 2^k - 2^{\frac{k-1}{2}} & \text{if } k \text{ is odd.} \end{cases}$$

Proof. The statement holds for $k = 0$, as

$$\deg(P_0) = \deg(r) = 1 = 2^0, \;\; \deg(P_1) = \deg(r + 1) = 1 = 2^1 - 2^{\frac{1-1}{2}}.$$

Assume the statement holds for all $k \in \{0, 1, \ldots, n\}$, then

$$\begin{aligned} \deg(P_{n+1}) &= \deg(P_n P_{n-1}^2 + P_{n-1}^4) \\ &= \deg(P_{n-1}^2) + \deg(P_n + P_{n-1}^2). \end{aligned}$$

If n is even, then $\deg(P_n) = 2^n$ and $\deg(P_{n-1}^2) = 2\deg(P_{n-1}) = 2 \cdot (2^{n-1} - 2^{\frac{n-1-1}{2}}) = 2^n - 2^{\frac{n}{2}}$, hence $\deg(P_n) \neq \deg(P_{n-1}^2)$.

If n is odd, then $\deg(P_n) = 2^n - 2^{\frac{n-1}{2}}$ and $\deg(P_{n-1}^2) = 2\deg(P_{n-1}) = 2 \cdot 2^{n-1} = 2^n$, hence $\deg(P_n) \neq \deg(P_{n-1}^2)$.

Hence $\deg(P_n) \neq \deg(P_{n-1}^2)$ in both cases, which implies

$$\begin{aligned} \deg(P_{n+1}) &= \deg(P_{n-1}^2) + \max(\deg(P_n), \deg(P_{n-1}^2)) \\ &= 2\deg(P_{n-1}) + \max(\deg(P_n), 2\deg(P_{n-1})). \end{aligned}$$

If $n + 1$ is even, then

$$\begin{aligned} \deg(P_{n+1}) &= 2 \cdot 2^{n-1} + \max(2^n - 2^{\frac{n-1}{2}}, 2 \cdot 2^{n-1}) \\ &= 2^n + \max(2^n - 2^{\frac{n-1}{2}}, 2^n) \\ &= 2^n + 2^n \\ &= 2^{n+1}. \end{aligned}$$

If $n + 1$ is odd, then

$$\begin{aligned} \deg(P_{n+1}) &= 2 \cdot (2^{n-1} - 2^{\frac{n-2}{2}}) + \max(2^n, 2 \cdot (2^{n-1} - 2^{\frac{n-2}{2}})) \\ &= 2^n - 2^{\frac{n}{2}} + \max(2^n, 2^n - 2^{\frac{n}{2}}) \\ &= 2^n - 2^{\frac{n}{2}} + 2^n \\ &= 2^{n+1} - 2^{\frac{n+1-1}{2}}. \end{aligned}$$

This finishes the proof. □

We can now calculate the degree of R_k.

Theorem 4.4. *For $k > 0$, the degree of R_k is of the form*

$$\deg(R_k) = \begin{cases} 2^k & \text{if } k \text{ is even,} \\ 2^k + 1 & \text{if } k \text{ is odd.} \end{cases}$$

Proof. We apply Lemma 4.3 multiple times to prove this statement. Since $Q_k = P_{k-1}^2$, we have that

$$\begin{aligned} \deg(Q_k) &= 2 \cdot \deg(P_{k-1}) \\ &= \begin{cases} 2 \cdot 2^{k-1} & \text{if } k-1 \text{ is even,} \\ 2 \cdot (2^{k-1} - 2^{\frac{k-1-1}{2}}) & \text{if } k-1 \text{ is odd,} \end{cases} \\ &= \begin{cases} 2^k - 2^{\frac{k}{2}} & \text{if } k \text{ is even,} \\ 2^k & \text{if } k \text{ is odd.} \end{cases} \end{aligned}$$

If k is even, then

$$\deg(P_k) > \deg(Q_k \cdot r) \iff 2^k > 2^k - 2^{\frac{k}{2}} + 1 \iff 2^{\frac{k}{2}} > 1 \iff k \geq 2.$$

Therefore $\deg(P_k) > \deg(Q_k \cdot r)$ for all even $k > 0$.

If k is odd, then

$$\deg(P_k) < \deg(Q_k \cdot r) \iff 2^k - 2^{\frac{k-1}{2}} < 2^k + 1 \iff -2^{\frac{k-1}{2}} < 1.$$

Hence $\deg(P_k) < \deg(Q_k \cdot r)$ for all odd k.

Since $k > 0$, we therefore have

$$\begin{aligned} \deg(R_k) &= \deg(P_k - Q_k \cdot r) \\ &= \max(\deg(P_k), \deg(Q_k \cdot r)) \\ &= \begin{cases} \deg(P_k) & \text{if } k \text{ is even,} \\ \deg(Q_k \cdot r) & \text{if } k \text{ is odd,} \end{cases} \\ &= \begin{cases} 2^k & \text{if } k \text{ is even,} \\ 2^k + 1 & \text{if } k \text{ is odd.} \end{cases} \end{aligned}$$

This finishes the proof. □

For the next theorem we will need the Möbius inversion formula, which uses the Möbius function[2].

Definition 4.5. For $k \in \mathbb{N}$ the **Möbius function** $\mu(k)$ is defined to be

$$\mu(k) = \begin{cases} 1 & \text{if } k = 1, \\ (-1)^n & \text{if } k \text{ is the product of } n \text{ distinct primes,} \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 4.6. *For all positive integers k, we have*

$$\sum_{d|k} \mu\left(\frac{k}{d}\right) = \mathbb{1}_{\{x=1\}}(k).$$

Proof. The formula holds for $k = 1$:

$$\sum_{d|1} \mu\left(\frac{1}{d}\right) = \mu(1) = 1 = \mathbb{1}_{\{x=1\}}(1).$$

Let $k > 1$ and write it as its unique prime decomposition $k = p_1^{e_1} \cdots p_n^{e_n}$. In the sum $\sum_{d|k} \mu(d)$ the only nonzero terms come from $d = 1$ and from those divisors of k which are products of distinct primes. Thus

$$\begin{aligned} \sum_{d|k} \mu(d) &= \mu(1) + \mu(p_1) + \cdots + \mu(p_1 p_2) + \cdots + \mu(p_{n-1} p_n) + \cdots + \mu(p_1 p_2 \cdots p_n) \\ &= 1 + \binom{n}{1}(-1) + \binom{n}{2}(-1)^2 + \cdots + \binom{n}{n}(-1)^n = (1-1)^n = 0 = \mathbb{1}_{\{x=1\}}(k). \end{aligned}$$

Note that we can replace d with $\frac{k}{d}$ as this will also iterate through all divisors of k exactly once. This finishes the proof. $\square$

Lemma 4.7 (Möbius inversion formula)**.** *For functions $F, G : \mathbb{N} \to \mathbb{N}$ such that*

$$F(k) = \sum_{d|k} G(d) \text{ for all } k \in \mathbb{N},$$

we have that

$$G(k) = \sum_{d|k} \mu\left(\frac{k}{d}\right) \cdot F(d) \text{ for all } k \in \mathbb{N}.$$

Proof. Plug the formula for F into the formula for G and apply Lemma 4.6. $\square$

Lemma 4.8. *For all positive integers k, we have*

$$\sum_{d|k} \mu\left(\frac{k}{d}\right)(-1)^d = -\mathbb{1}_{\{x=1\}}(k) + 2 \cdot \mathbb{1}_{\{x=2\}}(k).$$

Proof. Let k be a positive integer. Then we see $(-1)^k = 1$ if k is even and $(-1)^k - 1$ if k is odd. Because every integer is divisible by 1 and only even numbers are divisible by 2, we see that

$$\sum_{d|k} -\mathbb{1}_{\{x=1\}}(k) + 2 \cdot \mathbb{1}_{\{x=2\}}(k) = -1 + 2 \cdot \mathbb{1}_{\{2|x\}}(k) = (-1)^k.$$

Applying the Möbius inversion formula gives our result. $\square$

It is now possible to calculate an upper bound for the number of k-periodic points of f.

Theorem 4.9.

$$\#\text{points of } f \text{ with period } k \leq \begin{cases} 3 & \text{if } k = 1, \\ 1 & \text{if } k = 2, \\ \sum_{d|k} \mu(d) \cdot 2^{\frac{k}{d}} & \text{if } k \geq 3. \end{cases}$$

Proof. Let

$$T_k := \frac{R_k}{\prod_{d|k, d \neq k} T_d}, \qquad k \geq 1,$$
$$t_k := \deg(T_k), \qquad k \geq 1,$$

such that t_k counts the roots of R_k that have not appeared in previous R_j with $j \in \{1, 2, \ldots k-1\}$. Then

$$t_k = \deg(R_k) - \sum_{d|k, d \neq k} t_d \implies \deg(R_k) = \sum_{d|k} t_d.$$

By applying the Möbius inversion formula, we get

$$t_k = \sum_{d|k} \mu\left(\frac{k}{d}\right) \deg(R_d).$$

Note that we can rewrite

$$\deg(R_d) = \begin{cases} 2^d & \text{if } d \text{ is even}, \\ 2^d + 1 & \text{if } d \text{ is odd}, \end{cases}$$
$$= 2^d + \frac{1 - (-1)^d}{2}.$$

Using Lemmas 4.6 and 4.8, we get

$$\begin{aligned}
t_k &= \sum_{d|k} \mu\left(\frac{k}{d}\right) 2^d + \frac{1}{2} \sum_{d|k} \mu\left(\frac{k}{d}\right) - \frac{1}{2} \sum_{d|k} \mu\left(\frac{k}{d}\right) (-1)^d \\
&= \sum_{d|k} \mu\left(\frac{k}{d}\right) 2^d + \frac{1}{2} \cdot \mathbb{1}_{\{x=1\}}(k) - \frac{1}{2}(-\mathbb{1}_{\{x=1\}}(k) + 2 \cdot \mathbb{1}_{\{x=2\}}(k)) \\
&= \sum_{d|k} \mu\left(\frac{k}{d}\right) 2^d + \mathbb{1}_{\{x=1\}}(k) - \mathbb{1}_{\{x=2\}}(k) \\
&= \begin{cases} 3 & \text{if } k = 1, \\ 1 & \text{if } k = 2, \\ \sum_{d|k} \mu(d) \cdot 2^{\frac{k}{d}} & \text{if } k \geq 3. \end{cases}
\end{aligned}$$

Since

$$\#\text{points of } f \text{ with period } k \leq t_k,$$

this finishes the proof. □

We conclude that the number of points of f with period k is bounded above by the sequence

$$(3, 1, 6, 12, 30, 54, 126, 240, 504, 990, 2046, 4020, 8190, 16254, 32730, 65280, 131070, 261576, \dots).$$

5 Discussion

For our main recurrence there are two very common behaviours. When we plot these behaviours in the complex plane we obtain the Kirby fractal. We created some tools to analyse the behaviour of our sequence by looking at the behaviours of the sequences of k-step-ratios. We used these behaviours to confirm that the Kirby fractal is contained in the disk of radius 2. We noticed that points on the boundary of this fractal are very closely connected to periodic points of our 1-step-ratio recurrence. We then proved a bound on the number of points with period k that our 1-step-ratio recurrence can have.

There are a lot of things left out of this paper that warrant further research. For instance, the area of the Kirby fractal is approximately 4.4493. Is there an exact formula for this number? What is the dimension of the boundary of the Kirby fractal?

Another observation is that the bound we showed for the number of points of period k is probably the optimal bound. This is because consulting the Online Encyclopedia of Integer Sequences givfig:evenoddconves us a connection between the number of periodic points and the sequence A027375: Number of aperiodic binary strings of length n[1]. The function f has two inverses, say inverse zero and inverse one. Numerical testing has shown that an aperiodic composition of n of these inverses will be a function that, when iterated on the top half of the complex plane, converges to a unique period n point of f, if the amount of zeros and ones are even. If the amount of zeros or the amount of ones is odd, then iterating the composition an even amount of times converges to a unique period $2n$ point of f.

Here is list of some other conjectures.

Conjecture 5.1. *The number* 2.27363902874548… *is transcendental.*

This is only a hunch based on the fact that the number is approached rapidly by our sequence, where an accuracy of n digits in the current step results in an accuracy of $2n$ digits in the next step.

Conjecture 5.2. *The set of solely odd convergence and the set of solely even convergence are open. The boundaries of these sets are equal and each set is connected.*

We have a basic framework of a proof for the first part, but some details still need to be worked out before this can be released. The second part seems reasonable from Figure 2, but remains unproven.

We showed that all initial values outside of the bound of distance 2 to the origin are solely even convergent. We conjecture the following is the best bound.

Conjecture 5.3. *For all $|z| > \phi$, the golden ratio, the sequence (z_n) has solely even convergence.*

If this is true, then the outer most points on the boundary of the Kirby fractal are

$$-\frac{1}{2} \pm \frac{1}{2}\sqrt{5 + 2\sqrt{5}}i,$$

which are period 4 points of both the 1-step-ratio and the main recurrence itself.

Lastly, there could be chaotic behaviour of the iteration of f on the boundary fractal, just like there is chaotic behaviour on the boundary circle of the Julia set of the square map. Numerically speaking this conjecture is difficult to show an example of. At least, using the recurrence in the forward direction. This is because, however close you start to the boundary fractal, each iteration of f seems to push the points further and further away from the boundary fractal. We still reason that there is chaos analytically speaking as we can start anywhere in the complex plane and just randomly applying the inverses of f. This seems to always get closer and closer to the boundary with each iteration, while never converging to a single point. So it is not a stretch to believe that points on the boundary can behave just as randomly in the forward direction.

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A Expressions of the 2-step-ratio recurrence

Note that

$$z_{n+1} = \frac{z_{n-1}^2}{z_n} + z_{n-1} \implies \frac{z_{n+1}}{z_{n-1}} = \frac{1}{\frac{z_n}{z_{n-1}}} + 1 = \frac{1}{f\left(\frac{z_{n-1}}{z_{n-2}}\right)} + 1 = \frac{1}{f\left(\frac{1}{\frac{z_n}{z_{n-2}} - 1}\right)} + 1.$$

This implies by definition of g that

$$g(z) := \frac{1}{f\left(\frac{1}{z-1}\right)} + 1 = \frac{1}{\frac{\frac{1}{z-1}+1}{(\frac{1}{z-1})^2}} + 1 = \frac{1}{z-1+(z-1)^2} + 1 = 1 + \frac{1}{z^2 - z}.$$

$$\begin{aligned} g^2(z) &= 1 + \frac{1}{\left(1+\frac{1}{z^2-z}\right)^2 - \left(1+\frac{1}{z^2-z}\right)} = 1 + \frac{(z^2-z)^2}{(z^2-z+1)^2 - ((z^2-z)^2 + z^2 - z)} \\ &= 1 + \frac{(z^2-z)^2}{z^4 - 2z^3 + 3z^2 - 2z + 1 - z^4 + 2z^3 - z^2 - z^2 + z} \\ &= 1 + \frac{(z^2-z)^2}{z^2-z+1} = 1 + \frac{(z^2-z)^2 + (z^2-z) - (z^2-z)}{z^2-z+1} = 1 + z^2 - z + \frac{-(z^2-z)}{z^2-z+1} \\ &= z^2 - z + \frac{z^2-z+1-(z^2-z)}{z^2-z+1} = z^2 - z + \frac{1}{z^2-z+1}. \end{aligned}$$

B Polynomials

The following polynomials were computed with a program.

$$
\begin{aligned}
&P_0 = r\\
&P_1 = 1 + r\\
&P_2 = r^2 + r^3 + r^4\\
&P_3 = 1 + 4r + 7r^2 + 7r^3 + 5r^4 + 3r^5 + r^6\\
&P_4 = r^4 + 6r^5 + 18r^6 + 35r^7 + 50r^8 + 56r^9 + 53r^{10} + 44r^{11} + 33r^{12} + 21r^{13} + 11r^{14} + 4r^{15} + r^{16}\\
&P_5 = 1 + 16r + 124r^2 + 620r^3 + 2251r^4 + 6342r^5 + 14490r^6 + 27729r^7 + 45572r^8 + 65604r^9\\
&\quad + 84018r^{10} + 96911r^{11} + 101709r^{12} + 98004r^{13} + 87431r^{14} + 72783r^{15} + 56927r^{16} + 42036r^{17}\\
&\quad + 29343r^{18} + 19298r^{19} + 11859r^{20} + 6723r^{21} + 3458r^{22} + 1581r^{23} + 626r^{24} + 207r^{25} + 54r^{26}\\
&\quad + 10r^{27} + r^{28}
\end{aligned}
$$

$$
\begin{aligned}
&Q_0 = 1\\
&Q_1 = r^2\\
&Q_2 = 1 + 2r + r^2\\
&Q_3 = r^4 + 2r^5 + 3r^6 + 2r^7 + r^8\\
&Q_4 = 1 + 8r + 30r^2 + 70r^3 + 115r^4 + 144r^5 + 145r^6 + 120r^7 + 81r^8 + 44r^9 + 19r^{10} + 6r^{11} + r^{12}\\
&Q_5 = r^8 + 12r^9 + 72r^{10} + 286r^{11} + 844r^{12} + 1972r^{13} + 3803r^{14} + 6240r^{15} + 8922r^{16} + 11332r^{17}\\
&\quad + 12978r^{18} + 13542r^{19} + 12953r^{20} + 11386r^{21} + 9202r^{22} + 6832r^{23} + 4651r^{24} + 2890r^{25}\\
&\quad + 1625r^{26} + 814r^{27} + 355r^{28} + 130r^{29} + 38r^{30} + 8r^{31} + r^{32}
\end{aligned}
$$

$$
\begin{aligned}
&R_0 = 0\\
&R_1 = 1 + r - r^3\\
&R_2 = -r - r^2 + r^4\\
&R_3 = 1 + 4r + 7r^2 + 7r^3 + 5r^4 + 2r^5 - r^6 - 3r^7 - 2r^8 - r^9\\
&R_4 = -r - 8r^2 - 30r^3 - 69r^4 - 109r^5 - 126r^6 - 110r^7 - 70r^8 - 25r^9 + 9r^{10} + 25r^{11}\\
&\quad + 27r^{12} + 20r^{13} + 11r^{14} + 4r^{15} + r^{16}\\
&R_5 = 1 + 16r + 124r^2 + 620r^3 + 2251r^4 + 6342r^5 + 14490r^6 + 27729r^7 + 45572r^8 + 65603r^9\\
&\quad + 84006r^{10} + 96839r^{11} + 101423r^{12} + 97160r^{13} + 85459r^{14} + 68980r^{15} + 50687r^{16} + 33114r^{17}\\
&\quad + 18011r^{18} + 6320r^{19} - 1683r^{20} - 6230r^{21} - 7928r^{22} - 7621r^{23} - 6206r^{24} - 4444r^{25} - 2836r^{26}\\
&\quad - 1615r^{27} - 813r^{28} - 355r^{29} - 130r^{30} - 38r^{31} - 8r^{32} - r^{33}
\end{aligned}
$$

C Algorithms

Algorithm used to generate Figure 2:

```
function h(w, z)
    if w = 0 or w = ∞ or z = ∞ then
        return w
    if z = 0 then
        return ∞
    return w^2/z + w

image ← width × height matrix
for x ← 1 to width do
    for y ← 1 to height do
        a ← 1
        b ← (−2 + 4 · x/width) + (2 − 4 · y/height) · i
        n ← 2
        while True do
            c ← h(a, b)
            if |a − c| < 10^−10 then
                if n is even then
                    image[x, y] ← light blue
                else
                    image[x, y] ← pink
                break
            a ← b
            b ← c
            n ← n + 1
```