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Bachelor's thesis - Isotropic gauges in the relativistic Kepler problem

Bachelor's Project Physics and Mathematics

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Abstract: In this thesis the relativistic Kepler problem is discussed. The effective Hamiltonian of the Kepler problem is written in a post-Newtonian expansion. The transformations to get to the isotropic LRL, amplitude and PMOOSH gauges at first post-Newtonian order are given. It is shown that one can get to those isotropic gauges up to any arbitrary post-Newtonian order by using a canonical transformation.

Contents

1 Introduction

The Kepler problem is one of the classical problems of physics. In the Kepler problem (also known as the two-body problem) there are two bodies orbiting each other due to an attractive (gravitational) force. In classical Newtonian mechanics, gravity is seen as an attractive central force that is inversely proportional to the square distance between the two bodies and proportional to the masses of the bodies. In this theory the force of gravity is transmitted instantly: changes in the position of one body immediately affect the gravitational force experienced by the other body. This is not compatible with the concept of causality in (special) relativity. In order to fix this several relativistic theories of gravity have been proposed, such as the dilaton and general relativity. Another problem with Newton's law of gravitation is that it could not explain the anomalous precession of Mercury. The relativistic corrections of general relativity are able to explain this anomaly[4].

In general relativity massive bodies curve spacetime according to the Einstein field equations. These equations both describe the curvature of spacetime created by matter and the geodesic trajectories followed by bodies. Another option for a relativistic theory of gravity is to have gravity being carried by a scalar field φ. In order to preserve causality, the scalar field can only couple to the metric in two ways: conformally (i.e. $g_{\mu\nu} = \eta_{\mu\nu} e^{\varphi}$) and disformally (i.e. $g_{\mu\nu} = \eta_{\mu\nu} + \partial_\mu \varphi \partial_\nu \varphi$). Combining these two coupling into one effective metric results in the so-called Bekenstein metric [1, 6]. The disformal coupling results in a faster fall-off of the gravitational force than $1/r^2$. If the bodies are orbiting at a speed significantly lower than the speed of light, then the conformal coupling gives the same gravitational force as Newtonian gravity. The relativistic corrections given by these theories results in the orbits precessing. Conformal coupling and general relativity predict different directions for the precession[10]. The empirical observations of the precession of mercury are better described by GR than by conformal coupling: the precession predicted by conformal coupling has the opposite sign as the observed anomalous precession. Although general relativity is able to predict the precession of Mercury[4], at larger scales GR needs dark matter to explain the rotation curves of galaxies. The dilaton can be combined with GR to potentially explain the behaviour of the expansion of the universe [2].

For these theories of gravity an effective hamiltonian can be created. Due to the SO(3) symmetry of the Kepler problem the Hamiltonian can only depend on p^2, $p_r^2 = (\vec{p} \cdot \hat{q})^2$ and $1/r = 1/|\vec{q}|$. In this research project it will be shown that under certain conditions a canonical transformation can be applied to remove the p_r^2 dependence of the Hamiltonian, this transformation is referred to as 'changing gauge'.

This will be done using post-Newtonian expansions (which are expansion in terms of the small parameter $1/c^2$) of the Hamiltonian and the generator of the canonical transformation. Removing the p_r^2 dependence can simplify the Hamiltonian. The canonical transformations discussed in this research can also be used to check whether seemingly different Hamiltonians actually result in the same dynamics. Specifying that the Hamiltonian does not have a dependence on p_r^2 does not fully specify the gauge. Two gauges that will be considered in this thesis in particular are the LRL gauge and the amplitude gauge. In the LRL gauge the Hamiltonian is written in a form that makes it easier to determine how quickly the LRL vector changes. The LRL vector, named after Laplace, Runge and Lenz is a vector that is preserved in the classical nonrelativistic Kepler problem. The transformations required to get to the LRL, amplitude and PMOOSH gauges will be explicitly determined for the first post-Newtonian order. It will also be shown that there are canonical transformations to get to these isotropic gauges up to any specified post-Newtonian order, which answer the research question of whether there is a canonical transformation to get to these isotropic gauges.

2 Hamiltonian Formalism

We will be describing the Kepler problem using the framework of Hamiltonian mechanics, however we will first introduce Lagrangian mechanics. This section is mainly based upon the discussion of Hamiltonian and Lagrangian mechanics in [7, 10, 11]. One formulation of classical mechanics is Lagrangian mechanics. Unlike Newtonian mechanics, which deals with forces, Lagrangian mechanics works with a Lagrangian. The Lagrangian is given by

$$\mathcal{L} = T - U,$$

where T is the kinetic energy and U the potential energy. The Lagrangian is written in terms of generalized coordinates and their derivatives. One of the advantages of Lagrangian mechanics over Newtonian mechanics is that the Lagrangian can use generalized coordinates, while Newtonian mechanics only works well with Cartesian coordinates in an inertial reference frame.

The equations of motion in the Lagrangian formalism can be derived from the principle of least action, where the action is

$$S = \int \mathcal{L} dt$$

. The principle of least action states that the path followed by particles is a critical point (typically a minimum) of the action.

Applying the calculus of variations to find such a critical point gives the equations of motion:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{\partial \mathcal{L}}{\partial q_i} \tag{2.1}$$

Another formalism, the one that will be used to describe the Kepler problem, is the Hamiltonian formalism, which works with coordinates and momenta. This is different from the Lagrangian works with generalized coordinates and their time derivates. For this we will first define the generalized momenta:

The generalized momenta are given by

$$p_i = \frac{\partial L}{\partial q_i}.$$

The Hamiltonian can be written in terms of the Lagrangian using the Legendre transform [7]:

$$H = \left(\sum p_i q_i\right) - \mathcal{L} \tag{2.2}$$

In the Hamiltonian framework the equations of motion are given by

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \tag{2.3}$$

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \tag{2.4}$$

2.1 Symplectic Geometry

The phase space of the configuration of a physical system can mathematically be described using symplectic manifold. A symplectic manifold is a manifold equipped with a symplectic two-form, where a symplectic two-form is an anti-symmetric two-form that is nondegenerate and closed[3].

By Darboux's theorem the symplectic form ω on a $2m$-dimensional manifold M (the nondegeneracy of ω implies that M is even-dimensional) can locally be described using coordinates p_i and q_i such that $\omega = \sum_{i=1}^{m} dp_i \wedge dq_i$.

A Hamiltonian $H : M \to \mathbb{R}$ is a function that induces dynamics on M according to Hamilton's equations. In terms of the local Darboux coordinates these equation are given by equations 2.3 and 2.4.

For this thesis we will be considering the mechanics induced by the keplereian hamiltonian on the phase space given by the manifold $T^*\mathbb{R}^3 = \mathbb{R}^3 \times \mathbb{R}^3$.

2.2 Poisson Brackets

An important bilinear operator in classical mechanics is the Poisson bracket. It plays a similar role in classical mechanics as the commutator does in quantum mechanics. [11]

The Poisson bracket is given by

$$\{f, g\} = \sum_{i=1}^{N} \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}. \tag{2.5}$$

Where the q_i's are (local) coordinates and the p_i's their respective generalized momenta. The vale N is the number of degrees of freedom of the system (which will be 3 for the Kepler problem).

The Poisson bracket satisfies various identities, namely:

- linearity: $\{f + g, h\} = \{f, h\} + \{g, h\}$ and $\{f, g + h\} = \{f, g\} + \{f, h\}$
- antisymmetry: $\{f, g\} = -\{g, f\}$
- Leibniz rule: $\{fg, h\} = f\{g, h\} + \{f, h\}g$
- Jacobi identity: $\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0$

Another important property of the Poisson bracket is that the canonical Poisson bracket has the following value:

$$\{q_i, p_j\} = \delta_{ij} \tag{2.6}$$

The Poisson bracket between two coordinates is zero and so is the Poisson bracket between momenta:

$$\{q_i, q_j\} = 0, \{p_i, p_j\} = 0 \tag{2.7}$$

Hamilton's equations can also be written in terms of Poisson brackets:

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} = \{q_i, H\} \tag{2.8}$$

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} = \{p_i, H\} \tag{2.9}$$

The time evolution of a function f of p, q and t is given as follows:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \sum_{i=1}^{N} \frac{\partial f}{\partial q_i}\frac{dq_i}{dt} + \frac{\partial f}{\partial p_i}\frac{dp_i}{dt} \tag{2.10}$$

$$= \frac{\partial f}{\partial t} + \sum_{i=1}^{N} \frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q_i} \tag{2.11}$$

$$= \frac{\partial f}{\partial t} + \{f, H\} \tag{2.12}$$

Therefore, if a function of p and q has no explicit time dependence, then its time evolution is given by $\frac{df}{dt} = \{f, H\}$.

From equation 2.10 we can see that if some quantity f has no explicit time dependence and Poisson commutes with the Hamiltonian, that is $\{f, H\} = 0$, then the time derivative of that quantity is zero, hence that quantity is conserved. This is the statement of Noether's theorem in Hamiltonian mechanics.

2.3 Canonical Transformations

A canonical transformation is a invertible transformation of the phase space coordinates p_i and q_i that leaves the form of Hamilton's equations invariant (although the Hamiltonian might change).

That is, for the transformed coordinates Q and P, which are both functions of q, p and possibly time we want to have some new Hamiltonian $K(P, Q, t)$ such that the following set of equations hold

$$\frac{dQ_i}{dt} = \frac{\partial K}{\partial P_i} \tag{2.13}$$

$$\frac{dP_i}{dt} = -\frac{\partial K}{\partial Q_i} \tag{2.14}$$

We will be considering restricted canonical transformations, which only depend on the phase space coordinates, and not on time.

The demand of that Hamilton's equations remain invariant can also be formulated in a different way using Poisson brackets: the canonical Poisson brackets (equations 2.6 and 2.7) should remain invariant. That is

$$\{Q_i, P_j\} = \delta_{ij}, \{Q_i, Q_j\} = 0, \{P_i, P_j\} = 0 \tag{2.15}$$

where the Poisson brackets are evaluated with respect to the untransformed coordinates.

We will first consider an infinitesimal canonical transformation and then derive the non-infinitesimal version of it.

Suppose we have some infinitesimal transformation, with ϵ infinitesimal given by

$$q_i \to Q_i = q_i + \epsilon A_i \tag{2.16}$$

$$p_i \to P_i = p_i + \epsilon B_i \tag{2.17}$$

For this transformation to be canonical we need that $\{Q_i, P_j\} = \delta_{ij}$. Hence

$$\{q_i + \epsilon A_i + p_j \epsilon B_j\} = \{q_i, p_j\} + \epsilon\{q_i, B_j\} + \epsilon\{A_i, p_j\} \tag{2.18}$$
$$\delta_{ij} = \delta_{ij} + \epsilon\{q_i, B_j\} + \epsilon\{A_i, p_j\} \tag{2.19}$$
$$\{q_i, B_j\} = \{p_j, A_i\} \tag{2.20}$$
$$\frac{\partial B_j}{\partial p_i} = -\frac{\partial A_i}{\partial q_j} \tag{2.21}$$

This can be solved by the ansatz $A_i = \{X, q_i\}$ and $B_i = \{X, p_i\}$ for any X. This yields the following transformations:

$$q_i \to Q_i = q_i + \epsilon\{X, q_i\} \tag{2.22}$$
$$p_i \to P_i = p_i + \epsilon\{X, p_i\}, \tag{2.23}$$

where X is called the generating function of the infinitesimal transformation[7]. Besides the fact that $\{Q_i, P_j\} = \delta_{ij}$, which is solved by the ansatz, it is also needed that $\{Q_i, Q_j\} = \{P_i, P_j\} = 0$ in order for the transformation to be canonical. Indeed,

$$\{Q_i, Q_j\} = \{q_i + \epsilon\{X, q_i\}\, q_j + \epsilon\{X, q_j\}\} \tag{2.24}$$
$$= \{q_i, q_j\} + \epsilon\{q_i, \{X, q_j\}\} + \epsilon\{\{X, q_i\}, q_j\} \tag{2.25}$$
$$= 0 + \epsilon(-\{X, \{q_j, q_i\}\} + \{X, \{q_j, q_i\}\} + \{q_i, \{X, q_j\}\} + \{\{X, q_i\}, q_j\}) \tag{2.26}$$
$$= -\epsilon\{X, \{q_j, q_i\}\} \tag{2.27}$$
$$= 0. \tag{2.28}$$

A similar argument can be used to show that $\{P_i, P_j\} = 0$. Hence this infinitesimal transformation is canonical.

We will define $\{X, \cdot\}$ to be a linear operator given by $\{X, \cdot\}f = \{X, f\}$. From the infinitesimal transformation one could take an ansatz that the non-infinitesimal version is given by

$$q_i \to q_i + \{X, q_i\} + \frac{1}{2!}\{X, \{X, q_i\}\} + ... = e^{\{X,\cdot\}} q_i$$
$$p_i \to p_i + \{X, p_i\} + \frac{1}{2!}\{X, \{X, p_i\}\} + ... = e^{\{X,\cdot\}} p_i.$$

To show that it is in fact a canonical transformation we can parametrize it by the parameter $\alpha \in [0, 1]$, which gives

$$q_i(t) \to Q_i(t, \alpha) = e^{\alpha\{X,\cdot\}} q_i$$
$$p_i(t) \to P_i(t, \alpha) = e^{\alpha\{X,\cdot\}} p_i.$$

Differentiating with respect to α (while keeping t, p and q constant) yields

$$\left(\frac{\partial Q_i}{\partial \alpha}\right)_t = \{X, Q_i\}$$

and

$$\left(\frac{\partial P_i}{\partial \alpha}\right)_t = \{X, P_i\}.$$

X	map	name
$-\tau H$	$q_i(t) \to q_i(t+\tau), p_i \to p_i(t+\tau)$	time translation
ap_j	$q_i \to q_i - a\delta_{ij}, p_i \to p_i$	space translation
$\epsilon(\vec{p}\cdot\vec{q})$	$q_i \to e^{-\epsilon}q_i, p_i \to e^{\epsilon}p_i$	–
$\theta L_z = \theta(q_2p_1 - q_1p_2)$	$q_1 \to q_1\cos\theta - q_2\sin\theta, p_1 \to p_1\cos\theta - p_2\sin\theta$ $q_2 \to q_2\cos\theta + q_1\sin\theta, p_2 \to p_2\cos\theta + p_1\sin\theta$	rotation

Table 2.1: Examples for X and the canonical transformation generated by them

These equations are very similar to Hamilton's equations, therefore this transformation can be seen as flowing from the original coordinates in phase space to new coordinates following some Hamiltonian flow given by the Poisson bracket with X up until $\alpha = 1$. Hamiltonian flow is known to be a canonical transformation[7], but we will also show it be considering the canonical Poisson bracket:

$$\begin{aligned}
\frac{\partial}{\partial\alpha}\{Q_i, P_j\} &= \left\{\frac{\partial Q_i}{\partial\alpha}, P_j\right\} + \left\{Q_i, \frac{\partial P_j}{\alpha}\right\} \\
&= \{\{X, Q_i\}, P_j\} + \{Q_i, \{X, P_j\}\} \\
&= \{P_j, \{Q_i, X\}\} + \{Q_i, \{X, P_j\}\} \\
&= -\{X, \{P_j, Q_i\}\} + \{X, \{P_j, Q_i\}\} + \{P_j, \{Q_i, X\}\} + \{Q_i, \{X, P_j\}\} \\
&= -\{X, \{P_j, Q_i\}\} \\
&= \{X, \{Q_i, P_j\}\}
\end{aligned}$$

We obtained a differential equation for $\{Q_i, P_j\}$ with the initial condition $\{Q_i, P_j\} = \{q_i, p_j\} = \delta_{ij}$ for $\alpha = 0$. The unique solution to this equation is $\{Q_i, P_j\} = \delta_{ij}$. Using a similar argument it can be shown that $\{Q_i, Q_j\} = \{P_i, P_j\} = 0$. Since the Poisson bracket structure stays preserved the transformation is canonical.

In general any function depending on $\vec{p}$ and $\vec{q}$ will transform in the same way as the coordinates when transformed actively, namely:

$$f \to f + \{X, f\} + \frac{1}{2!}\{X, \{X, f\}\} + ... \tag{2.29}$$

The derivation of this identity is given in appendix A.1. Such an active transformation amounts to replacing all instances of $\vec{p}$ with $\vec{P}$ and all instances of $\vec{q}$ with $\vec{Q}$.

When the Hamiltonian is actively transformed it will transform in the following way:

$$H \to K = H + \{X, H\} + \frac{1}{2!}\{X, \{X, H\}\} + ... \tag{2.30}$$

Taking the time evolution under the original Hamiltonian and then actively transforming the coordinates is the same as first actively transforming the phase space coordinates and then taking the time evolution under the transformed Hamiltonian. In that sense the dynamics of this transformed Hamiltonian are the same as the original Hamiltonian.

Another way to see this transformation is in a more passive way, here we will consider the (transformed) Hamiltonian in terms of P and Q as the original Hamiltonian. Rewriting this Hamiltonian in terms of p and q is the same as replacing all instances of P and Q with p and q (i.e. doing the inverse active transformation) and then performing this active transformation given by equation 2.30.

3 Kepler Problem

3.1 Classical Mechanics

In the Kepler problem there are two bodies orbiting each other due to the attractive force of gravity. In classical mechanics the gravitational potential is given by $U = -\frac{Gm_1m_2}{r}$. Thus the Lagrangian is given by

$$\mathcal{L} = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \frac{Gm_1m_2}{r} \tag{3.1}$$

Where m_1 and m_2 are the masses of the bodies and v_1 and v_2 their respective velocities. The distance between the two bodies is given by r.

This two body problem can be reduced to a problem of one body orbiting some static point. To do this let $\vec{r_1}$ and $\vec{r_2}$ be the positions of the two masses. Then let $\vec{r} = \vec{r}_2 - \vec{r}_1$. We will work in the center of mass frame, that is $\vec{R} = \frac{m_1\vec{r_1}+m_2\vec{r_2}}{m_1+m_2} = 0$. Then $\vec{r}_1 = -\frac{m_2}{m_1+m_2}\vec{r}$ and $\vec{r}_2 = \frac{m_1}{m_1+m_2}\vec{r}$.

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}m_1\dot{\vec{r}}_1^2 + \frac{1}{2}m_2\dot{\vec{r}}_2^2 + \frac{Gm_1m_2}{r} && (3.2)\\
&= \frac{1}{2}m_1\left(-\frac{m_2\dot{\vec{r}}}{m_1+m_2}\right)^2 + \frac{1}{2}m_2\left(\frac{m_1\dot{\vec{r}}}{m_1+m_2}\right)^2 + \frac{Gm_1m_2}{r} && (3.3)\\
&= \frac{1}{2}\frac{m_1m_2^2+m_2m_1^2}{(m_1+m_2)^2}\dot{\vec{r}}^2 + \frac{Gm_1m_2}{r} && (3.4)\\
&= \frac{1}{2}\frac{m_1m_2}{m_1+m_2}\dot{\vec{r}}^2 + \frac{Gm_1m_2}{r} && (3.5)\\
&= \frac{1}{2}\mu\dot{\vec{r}}^2 + \frac{Gm_1m_2}{r} && (3.6)
\end{aligned}$$

Where r is the distance between the two bodies, m_1 and m_2 are the masses of the two bodies and $\mu = \frac{m_1m_2}{m_1+m_2}$ is the reduced mass. The Lagrangian now only depends of the distance (and its time derivative) between the bodies and not the positions of both bodies. This Lagrangian is equivalent to the case where we have one body of mass μ in a static potential given by $U = -\frac{Gm_1m_2}{r}$ where r is the distance from the body to the origin.

The Hamiltonian corresponding to this one body system is given by

$$H = \frac{p^2}{2\mu} - \frac{Gm_1m_2}{r},$$

where the canonical momentum $\vec{p} = \mu\dot{\vec{r}}$.

The solutions to the equations of motion given by this Lagrangian/Hamiltonian are conic sections [11]. The closed orbits are ellipses (which don't precess). From the rotational symmetry of the Hamiltonian it can be seen that the angular momentum is conserved. The energy is also conserved, as the Hamiltonian doesn't have explicit time dependence.

Another conserved quantity is the LRL vector, given by $\vec{A} = \frac{1}{\mu}\vec{v}\times\vec{L} - \hat{r}$. It is not immediately apparent from that Hamiltonian that this vector is conserved . Due to this extra conserved quantity the Hamiltonian is superintegrable. The energy, combined with the angular momentum and LRL vector gives $1+3+3 = 7$

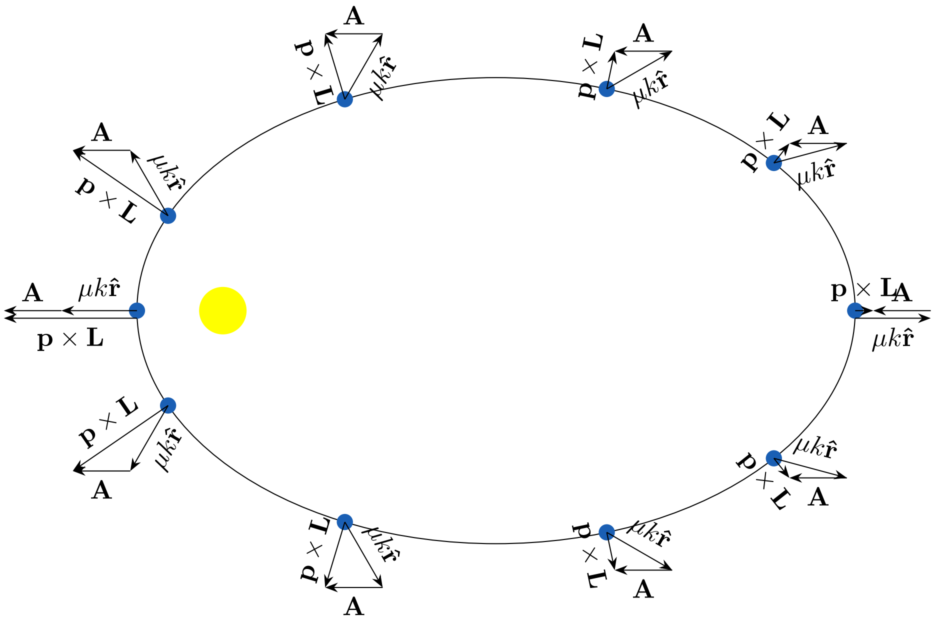

Figure 3.1: An illustration of the LRL vector of a small body (blue) as it orbits the origin (yellow)

conserved quantities. However these quantities are not fully independent: for example the LRL vector is always perpendicular to the angular momentum. It turns that there are a total of 5 independent conserved quantities in the Kepler problem[7]. As 5 conserved quantities is the maximal number of conserved quantities for a three dimensional system, the Kepler problem is maximally superintegrable.

3.2 Post-Newtonian and Post-Minkowskian Expansion

For studying the relativistic corrections of the Kepler problem, an important tool is the post-Newtonian expansion, which is a power series expansion in terms of the small parameter $1/c^2$. The post-Newtonian expansion up to order n, denoted by nPN, is an expansion with terms proportional to powers of $(1/c^2)$ from $(1/c^2)^0$ up to $(1/c^2)^n$. The 0PN order accounts for the non-relativistic terms, while higher orders account for the relativistic corrections. The post-Newtonian expansion physically amounts to an expansion both in terms of low velocity and a weak field.

Similar to the post-Newtonian expansion, the post-Minkowskian expansion is also a power series expansion. However, instead of being in terms of $1/c^2$ it is in terms of the coupling parameter κ. For the Kepler problem this parameter is given by $\kappa = Gm_1m_2/c^2$. The post-Minkowskian expansion is an asymptotic expansion for a weak field. Unlike the 0PN order, the 0PM order does contain relativistic corrections pertaining to high velocities.

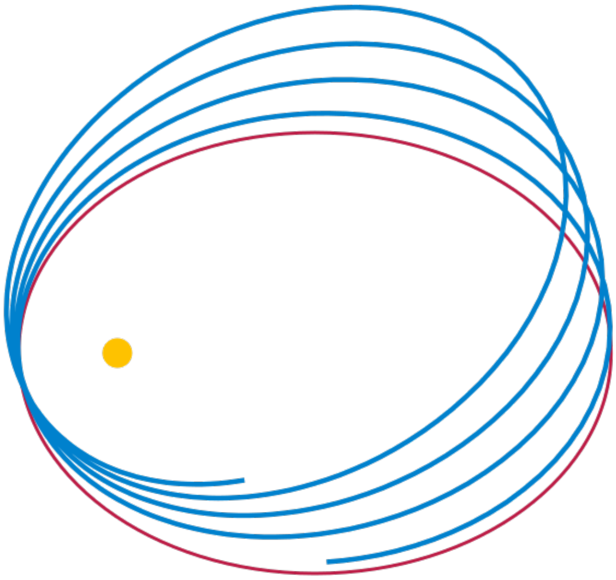

Figure 3.2: Relativistic precession of an elliptical orbit in the Kepler problem [9]

3.3 General Relativity

As was mentioned in the introduction, one of the problems with the classical Newtonian description of gravity is that it is not compatible with special relativity. According to the classical description of gravity the force is transmitted instantaneously. However, this transmission of information faster than the speed of light violates causality.

In special relativity one uses spacetime coordinates $x^\mu = (ct, x, y, z)$. The distance between two events x^μ and $x^\mu + dx^\mu$ is given by $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$, where $\eta_{\mu\nu} = \text{diag}(-1, +1, +1, +1)$ is the Minkowski metric.

In general relativity gravity is described using the geometry of spacetime. One of the important principles of general relativity is the (weak) equivalence principle, which states that gravitational and inertial mass are the same. Masses curve spacetime according to the Einstein field equations[12]. The curvature of spacetime is described by the metric $g_{\mu\nu}$, the role of $g_{\mu\nu}$ in general relativity is similar to the role of the Minkowski metric in special relativity. Sone of the main differences between $g_{\mu\nu}$ and $\eta_{\mu\nu}$ are that $g_{\mu\nu}$ depends on the spacetime coordinates and that $g_{\mu\nu}$ is not necessarily diagonal. The distance is given by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{3.7}$$

One solution to the Einstein field equations is the Schwarschild metric, given by

$$ds^2 = -\left(1 - \frac{2GM}{rc^2}\right) dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{3.8}$$

This metric has an asymptote at $r = \frac{2GM}{c^2}$, which is called the Schwarzschild radius. The Schwarschild metric describes the metric induced by a point particle of mass M at the origin. The Schwarzschild radius is related to (the event horizon of) black holes. When the distance between two masses is significantly larger than the Schwarshild radius the gravitational attraction is typically well-descrbied by Newtonian gravity. However, the closer the two bodies get to each other and the faster they move the more important the relativistic corrections become [12].

We will be using the metric signature $(-+++)$. In relativity the time is relative: different observers might not agree at which time an event occurs. Therefore instead of parametrizing the trajectory of a particle by time, the worldline (position and time) of the particle will be parametrized by some parameter σ. The dot indicates differentiation with respect to σ. Using this we can write the Lagrangian of the particle as

$$\mathcal{L} = \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}$$

and the action is given by

$$S = -mc \int L d\sigma.$$

It turns out that the equations of motion for such a particle are precisely the geodesic equations.

The effective Hamiltonian that describes the motion of two bodies of similar mass orbiting each other has also been derived (equation (3.1) of [5]). Letting $\nu = \frac{\mu}{m_1+m_2} = \frac{m_1 m_2}{(m_1+m_2)^2}$ the Hamiltonian is given as follows in the center of mass frame (up to 1PN order):

$$H = \frac{p^2}{2\mu} - \frac{\kappa}{r} - \frac{\mu}{2}(1-3\nu)\left(\frac{p^2}{2\mu}\right)^2 - (3+\nu)\frac{p^2}{2\mu r} - \nu\frac{p_r^2}{2\mu r} + \frac{1}{2}\frac{\kappa^2}{\mu r^2} + ... \quad (3.9)$$

Unlike in the non-relativistic Newtonian theory of gravity, the LRL vector is not conserved in GR due to these relativistic corrections. Hence the orbits can precess, as shown in figure 3.2.

3.4 Dilaton

Another approach to get a theory of gravity that is compatible with relativity is to have gravity being carried by a scalar field. In order to preserve causality, the scalar field can only couple to the metric in two ways: conformally and disformally[1]. Combining these two coupling into one effective metric yields the Bekenstein metric:

$$g_{\mu\nu} = e^{C\left(\frac{\varphi}{M_P}\right)}\eta_{\mu\nu} + D\left(\frac{\varphi}{M_P}\right)\frac{\partial_\mu\varphi\partial_\nu\varphi}{M_P^2 M_\partial^2} \quad (3.10)$$

Where $g_{\mu\nu}$ is the effective metric and $\eta_{\nu\mu}$ is the Minkowski metric. The constants M_P and M_∂ define the mass scales. The functions C and D determine the conformal and disformal coupling. For the dilaton we consider the conformal coupling for a field φ that induces an effective metric of

$$g_{\mu\nu} = e^{2a\varphi}\eta_{\nu\mu} \quad (3.11)$$

The Lagrangian of the field itself is given by $\mathcal{L} = -2(\partial\varphi)^2$, while the Lagrangian of a point mass is given by [10]

$$\mathcal{L} = \frac{1}{2} m e^{2a\varphi} \eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu - \frac{m}{2} \tag{3.12}$$

To solve this we consider the situation where there is one heavy mass that determines the metric and one light mass orbiting the heavy mass. The heavy mass remains stationary.

The equation of motion for the field φ is given by $\partial_\mu \partial^\mu \varphi = 0$ everywhere except at the origin. The field φ, which solves this equation of motion is given by

$$\varphi = -\frac{aM}{r}, \tag{3.13}$$

where M is the mass of the heavy particle located at the origin.

We will parametrize the worldline of the small massed particle using it's proper time, this gives $\dot{x}^0 = 1$. The Lagrangian becomes

$$\begin{aligned} \mathcal{L} &= \frac{1}{2} m e^{2a\varphi} \eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu - \frac{m}{2} \\ &= \frac{1}{2} m e^{2a\varphi} (-1 + \dot{x}^1 \dot{x}^1 + \dot{x}^2 \dot{x}^2 + \dot{x}^3 \dot{x}^3) - \frac{m}{2} \end{aligned}$$

We can then approximate this Lagrangian up to some post-Newtonian order and derive the corresponding Hamiltonian. A more general approach where the effective one body (EOB) Hamiltonian is derived for two orbiting bodies of a potentially similar mass and a more general scalar-tensor theory of gravity is given in [8].

Similar to the case of general relativity the orbits precess. Although the direction of precession given by the dilaton is opposite of the direction of precession predicted by GR.

3.5 General form of Hamiltonian

If one of the masses in the Kepler problem is significantly larger than the other, then the position of the larger mass can be considered to stationary at the origin. The smaller mass will orbit the larger mass. In the centre of mass frame, due to rotational symmetry[6, 10], the Hamiltonian of the two-body system can then be written in terms of p^2, p_r^2 and $1/r$, where p is the momentum, $p_r = \frac{\vec{p}\cdot\vec{q}}{r}$ is the radial component of momentum and r is the distance between the two bodies. The Hamiltonian of this system can thus be written as the following post-Newtonian expansion:

$$H = \mu c^2 \sum_{a,b,n \geq 0} H_n^{a,b} \left(\frac{p^2}{2\mu^2 c^2} \right)^a \left(\frac{p_r^2}{2\mu^2 c^2} \right)^b \left(\frac{\kappa}{\mu r c^2} \right)^n. \tag{3.14}$$

The post-Newtonian order of each term is given by $a+b+n-1$. We will define the quantity $a+b+n$ to be the degree of each term. The post-Minkowskian order of each term is given by n. Later on we will show that under certain conditions a transformation can be applied to remove the p_r dependence of the Hamiltonian.

For example for classical Newtonian gravity the coefficients are given by $\kappa = Gm_1m_2 2$, $H_0^{1,0} = 1$, $H_1^{0,0} = -1$ and all other coefficients of the Hamiltonian are zero.

4 Isotropic Gauges

In an isotropic gauge the Hamiltonian does not depend on p_r^2, i.e. $H_n^{a,b} = 0$ for $b > 0$. The name 'isotropic gauge' seems to be coined by [6]. One potential reasoning why these gauges are called isotropic is that in an isotropic gauge the Hamiltonian only depends on the norms of the vectors $\vec{p}$ and $\vec{q}$, while the p_r dependence also depends on the relative orientation of p and q. Thus there is no dependence on the directions of $\vec{p}$ and $\vec{q}$. One of the major advantages of isotropic gauges is that the number of terms in the PN expansion of the Hamiltonian is significantly reduced. In a general non-isotropic gauge the number of terms of the Hamiltonian up to xPN is of order x^3 while in an isotropic gauge the number is of order x^2. In general relativity the word gauge is often used to describe a choice of coordinates, these choices of coordinates can be related to the original coordinates via a canonical transformation.

As we have seen in section 3.5 the Hamiltonian has the following form when written as a PN expansion:

$$H = \mu \sum_{a,b,n \geq 0} H_n^{a,b} T_n^{a,b} \tag{4.1}$$

Where $T_n^{a,b} = \left(\frac{p^2}{2\mu^2}\right)^a \left(\frac{p_r^2}{2\mu^2}\right)^b \left(\frac{\kappa}{\mu r}\right)^n$ and we once again take $c = 1$.

It will be shown that a certain X can generate a transformation that kills the terms depending on p_r^2 up to any specified order, thus creating a gauge that is isotropic up to that order. This will be done by writing X as a power series, going order by order, setting the coefficients of X to kill the terms of H that have order that is one higher. It will turn out that there remains one free parameter left at each order. This parameter can be varied to get to different isotropic gauges, such as the amplitude and LRL gauges. But first, we will define these specific isotropic gauges.

4.1 LRL gauge

In the LRL gauge the Hamiltonian is written in the following form:

$$H = \sum_{0 \leq n \neq 1, a \geq 0} h_n^a (H_N)^a \left(\frac{\kappa}{\mu r}\right)^n$$

Where $H_N = \frac{p^2}{2\mu^2} - \frac{\kappa}{\mu r}$ is the classical Newtonian Hamiltonian for the Kepler problem and h_n^a are the coefficients of the LRL gauge. Note that the superscript a in h_n^a is a label, it is not the ath power of h_n.

In the classical Kepler problem the LRL vector is conserved. Therefore the Poisson bracket of the LRL vector with H_N is $\{\vec{A}, H_N\} = 0$. By the Leibniz rule we can derive that all higher powers of H_N also Poisson commute with the LRL vector.

In this gauge the lowest order terms with which the LRL vector doesn't Poisson commute have post-Minkowskian order 2 , unlike in the general case where one could have order 1 terms that don't Poisson commute with the LRL vector. This gauge is hence useful for analysing the conservation of the LRL vector, since the rate of change in the LRL vector, which is given by the Poisson bracket between the LRL vector and the Hamiltonian is of a larger post-Minkowskian order than it would be in other isotropic gauges.

Because this Hamiltonian in LRL gauge is written in terms of powers of H_N instead of powers of p^2 it is harder to compare to the general Hamiltonian given by equation 4.1. When we remove the $n \neq 1$ restriction from the sum we can write any isotropic gauge in such a form. For any PN order $x-1$ we pick $h_0^x = H_0^{x,0}$, then h_1^{x-1} is picked such that the coefficient of H_1^{x-1} matches. We repeat this process until we reach the coefficient h_x^0. This way all of the coefficients of the LRL gauge can be derived from an isotropic Hamiltonian.

However in the LRL gauge there is no h_1^{x-1} coefficient, hence we need that after setting $h_0^x = H_0^{x,0}$ there should be no remaining $H_1^{x-1,0}$ term. Expanding $h_0^x(H_N)^x$ we see that it is needed that the coefficient $H_1^{x-1,0} = -xh_0^x = -xH_0^{x,0}$. This is the extra condition of the coefficients of H that is required for the LRL gauge, besides $H_n^{a,b} = 0$ for $b > 0$.

4.2 Amplitude gauge

In the amplitude gauge the coefficients are chosen such that they correspond to the results given by amplitude analysis. In amplitude analysis Feynman diagrams are used to determine scattering amplitudes, from which the coefficients of the Hamiltonian can be deduced[6]. The terms involving only p^2, hence corresponding to the kinetic energy, should be the same as what is obtained through special relativity and amplitude analysis, that is,

$$H = \sqrt{p^2 + m_1^2} + \sqrt{p^2 + m_1^2} + \mu \sum_{n\geq 1, a\geq 0} v_n^{(a)} \left(\frac{p^2}{2\mu^2}\right)^a \left(\frac{\kappa}{\mu r}\right)^n \tag{4.2}$$

$$= \left(m_1 + m_2 + \frac{1}{2}(1/m_1 + 1/m_2)p^2 - \frac{1}{8}(1/m_1^3 + 1/m_2^3)p^4 + ...\right) + \mu \sum_{n\geq 1, a\geq 0} v_n^{(a)} \left(\frac{p^2}{2\mu^2}\right)^a \left(\frac{\kappa}{\mu r}\right)^n \tag{4.3}$$

$$= \left((m_1 + m_2) + \frac{p^2}{2\mu} - \frac{1}{2}\mu\left(1 - 3\frac{\mu}{m_1 + m_2}\right)\left(\frac{p^2}{2\mu^2}\right)^2 + ...\right) + \mu \sum_{n\geq 1, a\geq 0} v_n^{(a)} \left(\frac{p^2}{2\mu^2}\right)^a \left(\frac{\kappa}{\mu r}\right)^n \tag{4.4}$$

Comparing this to equation 4.1, we see that for the amplitude gauge the extra condition is that $H_0^{x,0}$ agrees with the power series expansion of the kinetic term.

4.3 PMOOSH gauge

Another possible choice of gauge is the PMOOSH (Post-Minkowskian Order One Seperable Hamiltonian) gauge. In this gauge there are no $(p^2)^n/r$ terms for $n > 0$. For this gauge in the limit of low coupling, that is at 1PM order, the Hamiltonian can be separated into a sum of a kinetic term $T(p^2)$ only depending on p^2 and a potential term depending on $1/r$. This causes the time derivative of momentum to only depend on position and the time derivative of position to only depend on momentum. In equation form the Hamiltonian looks like

$$H = T(p^2) + \sum_{0\leq n\neq 1, a\geq 0} h_n^a \left(\frac{p^2}{2\mu^2}\right)^a \left(\frac{\kappa}{\mu r}\right)^n \tag{4.5}$$

Comparing this equation with the general Hamiltonian in section 3.5 we see that the coefficients $H_0^{x,0}$ are accounted for in the kinetic term $T(p^2)$. The coefficients of the form $H_n^{a,0}$ are given by h_n^a. All of the other coefficients of the general Hamiltonian need to be zero when written in this gauge.

4.4 Transforming to Isotropic Gauges

We will assume that $H_0^{a,b} = 0$ if $b > 0$. Physically this corresponds to the Hamiltonian not depending on p_r^2 in the limit as $r \to \infty$ or $\kappa \to 0$. This is necessary to ensure that the Hamiltonian reduces to the special relativistic Hamiltonian when there is no gravitational influence. Without this condition particles won't move in a straight path in the limit of no gravitational coupling. Moreover the coefficients H_0^{10} and H_1^{00} should be non-zero, just like in the classical Kepler problem. Since the addition of a constant does not affect the dynamics induced by the Hamiltonian we can take H_0^{00} to be any value we want. Multiplying the Hamiltonian by a constant also doesn't change the nature of the dynamics, it only affects the speed, therefore we can multiply the Hamiltonian by a constant to set $H_0^{10} = 1$. The value of κ can then be chosen to make sure that $H_1^{00} = -1$. Using these three transformations we can ensure that the Hamiltonian is the same as the classical Hamiltonian up to 0PN order.

We will take X to be in a similar form as H, where in order to make the following calculations easier we take $\epsilon_0^{00} = 0$. This gives:

$$X = \sum_{a,b,n \geq 0} \epsilon_n^{a,b} T_n^{a,b} \vec{p} \cdot \vec{q} \tag{4.6}$$

To determine the transformed Hamiltonian given by equation 2.30 we need to evaluate $\{X, H\}$. Substituting the definition of X (equation 4.6) and the expansion of H given by equation 4.1 into $\{X, H\}$ and applying the linearity of the Poisson bracket we get that we need to evaluate the following family of Poisson brackets:

$$\begin{aligned}
&\{\epsilon_n^{a,b} \left(\frac{p^2}{2\mu^2}\right)^a \left(\frac{p_r^2}{2\mu^2}\right)^b \left(\frac{\kappa}{\mu r}\right)^n \vec{p} \cdot \vec{q}, H_m^{d,e} \left(\frac{p^2}{2\mu^2}\right)^d \left(\frac{p_r^2}{2\mu^2}\right)^e \left(\frac{\kappa}{\mu r}\right)^m\} = \\
=&\epsilon_n^{a,b} H_m^{d,e}((2b+1)(2d+2e+m) - 2e(2a+2b+n)) \left(\frac{p^2}{2\mu^2}\right)^{a+d} \left(\frac{p_r^2}{2\mu^2}\right)^{b+e} \left(\frac{\kappa}{\mu r}\right)^{n+m} \\
&- 2\epsilon_n^{a,b} H_m^{d,e}((2b+n)d - a(2e+m)) \left(\frac{p^2}{2\mu^2}\right)^{a+d-1} \left(\frac{p_r^2}{2\mu^2}\right)^{b+e+1} \left(\frac{\kappa}{\mu r}\right)^{n+m} \\
=&\epsilon_n^{a,b} H_m^{d,e} (A_{n,m}^{a,b,d,e} T_{n+m}^{a+d,b+e} + B_{n,m}^{a,b,d,e} T_{n+m}^{a+d-1,b+e+1})
\end{aligned}$$

Where $A_{n,m}^{a,b,d,e} = ((2b+1)(2d+2e+m) - 2e(2a+2b+n))$ and $B_{n,m}^{a,b,d,e} = -2((2b+n)d - a(2e+m))$

Note that the post-Minskowian order of this Poisson bracket is $n+m$, which is the sum of the post-Minskowskian orders of the two terms. The degree of the result of the Poisson bracket is equal to the sum of the degrees of the two degrees. This power counting property will become important later on, since it can be used to determine which terms could occur at each order in the transformed Hamiltonian.

4.5 Transformations at 1PN

In this subsection the transformation of the coefficients of H up to first post-Newtonian order will be considered. We will then use these transformations to transform the Hamiltonian into an isotropic form.

Since X only has terms of degree at least one and the degree of the result of the Poisson bracket of two terms described in equation 4.7 is the sum of the two degrees, it can be deduced that taking the Poisson bracket with X increasing the degree by (at least) one. Moreover since $H_0^{00}T_0^{00}$ Poisson commutes with X, we have that the degree of the term of smallest degree of $\{X, H\}$ is 2, $\{X, \{X, H\}\}$ only has terms of degree at least 3, $\{X, \{X, \{X, H\}\}\}$ has degree at least 4.

Because of these reasons, we have that to first order, which includes terms up to second degree, we only have to consider the first two terms of the transformation of the Hamiltonian, that is $H \to H + \{X, H\}$.

We can write the transformed Hamiltonian in a similar form as the original Hamiltonian, but with different coefficients that depend on the coefficients of the original Hamiltonian and X, which the generator of the transformation. Thus we are effectively redefining the coefficients of the Hamiltonian. These new (redefined) coefficients are the coefficients in the expansion of the transformed Hamiltonian.

Expanding the Hamiltonian to first order and X up to first degree and then evaluating the Poisson bracket using equation 4.7 we find the following transformations of the coefficients of the Hamiltonian, where the last 6 equations are the transformations of the 1PN coefficients:

$$H_0^{0,0} \to H_0^{0,0} \tag{4.7}$$
$$H_1^{0,0} \to H_1^{0,0} \tag{4.8}$$
$$H_0^{0,1} \to H_0^{0,1} \tag{4.9}$$
$$H_0^{1,0} \to H_0^{1,0} \tag{4.10}$$
$$H_2^{0,0} \to H_2^{0,0} + H_1^{0,0}\epsilon_1^{0,0} \tag{4.11}$$
$$H_1^{0,1} \to H_1^{0,1} - 2H_0^{1,0}\epsilon_1^{0,0} + 3H_1^{0,0}\epsilon_0^{0,1} + 2H_1^{0,0}\epsilon_0^{1,0} \tag{4.12}$$
$$H_0^{0,2} \to H_0^{0,2} + 2H_0^{0,1}\epsilon_0^{0,1} - 4H_0^{1,0}\epsilon_0^{0,1} + 4H_0^{0,1}\epsilon_0^{1,0} \tag{4.13}$$
$$H_1^{1,0} \to H_1^{1,0} + 2H_0^{1,0}\epsilon_1^{0,0} + H_1^{0,0}\epsilon_0^{1,0} \tag{4.14}$$
$$H_0^{1,1} \to H_0^{1,1} + 6H_0^{1,0}\epsilon_0^{0,1} - 2H_0^{0,1}\epsilon_0^{1,0} \tag{4.15}$$
$$H_0^{2,0} \to H_0^{2,0} + 2H_0^{1,0}\epsilon_0^{1,0} \tag{4.16}$$
$$\tag{4.17}$$

It can be seen that the terms of the Hamiltonian up to 0PN order remain invariant under this transformation. By taking into account the assumption that $H_0^{a,b} = 0$ if $b > 0$ we can simplify the other redefinitions to the following:

$$
\begin{aligned}
H_2^{0,0} &\to H_2^{0,0} + H_1^{0,0}\epsilon_1^{0,0} && (4.18)\\
H_1^{0,1} &\to H_1^{0,1} - 2H_0^{1,0}\epsilon_1^{0,0} + 3H_1^{0,0}\epsilon_0^{0,1} + 2H_1^{0,0}\epsilon_0^{1,0} && (4.19)\\
H_0^{0,2} &\to -4H_0^{1,0}\epsilon_0^{0,1} && (4.20)\\
H_1^{1,0} &\to H_1^{1,0} + 2H_0^{1,0}\epsilon_1^{0,0} + H_1^{0,0}\epsilon_0^{1,0} && (4.21)\\
H_0^{1,1} &\to +6H_0^{1,0}\epsilon_0^{0,1} && (4.22)\\
H_0^{2,0} &\to H_0^{2,0} + 2H_0^{1,0}\epsilon_0^{1,0} && (4.23)\\
& && (4.24)
\end{aligned}
$$

In order to not regenerate any p_r dependence, that is to make sure that the terms of the Hamiltonian that we assumed were zero remain zero, we need that $\epsilon_0^{01} = 0$, leaving us with the following redefinitions:

$$
\begin{aligned}
H_2^{0,0} &\to H_2^{0,0} + H_1^{0,0}\epsilon_1^{0,0} && (4.25)\\
H_1^{0,1} &\to H_1^{0,1} - 2H_0^{1,0}\epsilon_1^{0,0} + 2H_1^{0,0}\epsilon_0^{1,0} && (4.26)\\
H_1^{1,0} &\to H_1^{1,0} + 2H_0^{1,0}\epsilon_1^{0,0} + H_1^{0,0}\epsilon_0^{1,0} && (4.27)\\
H_0^{2,0} &\to H_0^{2,0} + 2H_0^{1,0}\epsilon_0^{1,0} && (4.28)
\end{aligned}
$$

To remove the p_r dependence the H_1^{01} coefficient should become 0, thus we need $H_1^{0,1} - 2H_0^{1,0}\epsilon_1^{0,0} + 2H_1^{0,0}\epsilon_0^{1,0}$ to be zero.

Since there are two free variables (ϵ_1^{00} and ϵ_0^{10}) and only one equation there is still some residual freedom; this freedom can be used to get to the specific isotropic gauges we discussed earlier.

4.5.1 LRL Gauge

At first order the Hamiltonian in the LRL gauge should be of the form

$$H/\mu = h_0^1 H_N + h_0^2 (H_N)^2 + h_2^0 \left(\frac{\kappa}{\mu r}\right)^2 \tag{4.29}$$

$$= h_0^1 \left(\frac{p^2}{2\mu}\right) - h_0^1 \left(\frac{\kappa}{\mu r}\right) + h_0^2 \left(\frac{p^2}{2\mu}\right)^2 - 2h_0^2 \left(\frac{p^2}{2\mu}\right)\left(\frac{\kappa}{\mu r}\right) + (h_0^2 + h_2^0)\left(\frac{\kappa}{\mu r}\right)^2 \tag{4.30}$$

To get to this form we need $H_0^{1,0} = -H_1^{0,0}$, $-2H_0^{2,0} = H_1^{1,0}$ and $H_1^{0,1} = 0$. The first equation $H_0^{1,0} = -H_1^{0,0}$ is satisfied, because in section 3.5 we took the Hamiltonian to be the same as the classical Kepler Hamiltonian up to 0PN order. The other two equations can be satisfied by setting the parameters of X accordingly.

To get to this form in first order $H_1^{10} + 2H_0^{20}$ needs to be to zero by using the residual freedom. Explicitly this means that

$$
\begin{aligned}
\begin{bmatrix} H_1^{0,1} \\ H_1^{1,0} + 2H_0^{2,0} \end{bmatrix} &\to \begin{bmatrix} H_1^{0,1} \\ H_1^{1,0} + 2H_0^{2,0} \end{bmatrix} + \begin{bmatrix} -2H_0^{1,0} & 2H_1^{0,0} \\ 2H_0^{1,0} + 2H_1^{0,0} & H_1^{0,0} \end{bmatrix} \begin{bmatrix} \epsilon_1^{0,0} \\ \epsilon_0^{1,0} \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

The solution to this linear system of equations gives the values of the parameters $\epsilon_1^{0,0}$ and $\epsilon_0^{1,0}$ that are required to get to the LRL gauge

$$\begin{aligned}\begin{bmatrix}\epsilon_1^{0,0}\\ \epsilon_0^{1,0}\end{bmatrix} &= -\begin{bmatrix}-2H_0^{1,0} & 2H_1^{0,0}\\ 2H_0^{1,0}+2H_1^{0,0} & H_1^{0,0}\end{bmatrix}^{-1}\begin{bmatrix}H_1^{0,1}\\ H_1^{1,0}+2H_0^{2,0}\end{bmatrix}\\ &= -\begin{bmatrix}-2 & -2\\ 0 & -1\end{bmatrix}^{-1}\begin{bmatrix}H_1^{0,1}\\ H_1^{1,0}+2H_0^{2,0}\end{bmatrix}\\ &= \frac{1}{2}\begin{bmatrix}H_1^{0,1}-2H_1^{1,0}-4H_0^{2,0}\\ 2H_1^{1,0}+4H_0^{2,0}\end{bmatrix}\end{aligned}$$

4.5.2 Amplitude Gauge

To get to this gauge in 1PN, we need to set $\epsilon_1^{0,0}$ to the value such that the $H_2^{0,0}$ coefficient attains the correct value of $-\frac{1}{2}(1-3\frac{\mu}{m_1+m_2})$. Then the other parameter needs to be $\epsilon_0^{1,0} = \frac{-H_1^{0,1}+2H_0^{1,0}\epsilon_1^{0,0}}{2H_1^{0,0}}$ in order to set $H_1^{0,1}$ to zero.

4.5.3 PMOOSH Gauge

Therefore to get to this gauge at first order it is needed that $H_1^{1,0} \to 0$. Hence,

$$\epsilon_1^{0,0} = \frac{H_1^{0,1}-2H_1^{1,0}}{6H_0^{1,0}}, \tag{4.31}$$

$$\epsilon_0^{1,0} = -\frac{H_1^{0,1}+H_1^{1,0}}{3H_1^{0,0}}, \tag{4.32}$$

$$H_2^{0,0} \to H_2^{0,0} + \frac{H_1^{0,0}(H_1^{0,1}-2H_1^{1,0})}{6H_0^{1,0}}, \tag{4.33}$$

$$H_0^{0,1} \to 0, \tag{4.34}$$

$$H_1^{1,0} \to 0, \tag{4.35}$$

$$H_0^{2,0} \to H_0^{2,0} - \frac{2(H_1^{0,1}+H_1^{1,0})H_0^{1,0}}{3H_1^{0,0}}. \tag{4.36}$$

4.6 Higher orders

In this subsection it will be shown that there exists a transformation given by X that can be used to go to an isotropic gauge at every order. In particular we will first consider how to get to the amplitude gauge at every PN order.

The approach we will take to remove the p_r^2 dependence is to work order by order. We will use terms of X of degree x to remove the p_r^2 terms of H that have degree $x+1$. Therefore for each degree we are only interested in the coefficients $\epsilon_n^{a,b}$ with $a+b+n=x$, this is because we have already used the epsilons corresponding to terms of lower degree to cancel the lower order terms of the Hamiltonian. The terms of X of degree $x+1$ ar higher do not affect the transformation of the degree x terms of the Hamiltonian due to the degree summing property of equation 4.7.

The transformed Hamiltonian, given by equation 2.30 contains infinitely many terms, starting with $H+\{X,H\}+\frac{1}{2!}\{X,\{X,H\}\}+....$ For each order we only care about the $\{X, H\}$ term, because it contains terms of X with parameters $\epsilon_n^{a,b}$ of the highest order. The other nested Poisson brackets will contain terms that are products of multiple ϵ's, whose sum is limited to the same value as the single ϵ occurring in terms of $\{X, H\}$, therefore the order of each epsilon is lower and thus already used to cancel terms of lower order. We can see the more nested Poisson brackets as a constant, since the parameters defining it have already been defined when working order by order.

By linearity of the Poisson bracket we can determine the $\{X, H\}$ term of the transformed Hamiltonian as follows

$$\begin{aligned}\{X,H\} &= \left\{\sum_{a,b,n\geq 0} \epsilon_n^{a,b} T_n^{a,b} \vec{p}\cdot\vec{q}, \sum_{d,e,m\geq 0} H_m^{d,e} T_m^{d,e}\right\} \\ &= \sum_{a,b,n\geq 0}\sum_{d,e,m\geq 0} \epsilon_n^{a,b} H_m^{d,e} \left\{T_n^{a,b}\vec{p}\cdot\vec{q}, T_m^{d,e}\right\}\end{aligned}$$

It is now possible to apply equation 4.7 to this equation. Then we can rearrange the sum. To decrease the amount of edge cases we will be defining $\epsilon_n^{a,b} = H_n^{a,b} = 0$ for negative indices, i.e. $a < 0 \vee b < 0 \vee n < 0$. This gives the following:

$$\begin{aligned}\{X,H\} = &\sum_{a,b,n\geq 0}\sum_{d,e,m\geq 0} \epsilon_n^{a,b} H_m^{d,e} A_{n,m}^{a,b,d,e} T_{n+m}^{a+d,b+e} \\ &+ \sum_{a,b,n\geq 0}\sum_{d,e,m\geq 0} \epsilon_n^{a,b} H_m^{d,e} B_{n,m}^{a,b,d,e} T_{n+m}^{a+d-1,b+e+1} \\ = &\sum_{g,h,l\geq 0}\sum_{a,b,n\geq 0} \epsilon_n^{a,b} H_{l-n}^{g-a,h-b} A_{n,l-n}^{a,b,g-a,h-b} T_l^{g,h} \\ &+ \sum_{g,h,l\geq 0}\sum_{a,b,n\geq 0} \epsilon_n^{a,b} H_{l-n}^{g-a+1,h-b-1} B_{n,l-n}^{a,b,g-a+1,h-b-1} T_l^{g,h} \\ = &\sum_{g,h,l\geq 0} T_l^{g,h}\left(\sum_{a,b,n\geq 0} \epsilon_n^{a,b}\left(H_{l-n}^{g-a,h-b} A_{n,l-n}^{a,b,g-a,h-b} + H_{l-n}^{g-a+1,h-b-1} B_{n,l-n}^{a,b,g-a+1,h-b-1}\right)\right) \\ = &\sum_{g,h,l\geq 0} T_l^{g,h} \begin{pmatrix}\epsilon_{l-1}^{g,h} H_1^{0,0} A_{l-1,1}^{g,h,0,0} + \epsilon_l^{g-1,h} H_0^{1,0} A_{l,0}^{g-1,h,0,0} \\ + \epsilon_{l-1}^{g+1,h-1} H_1^{0,0} B_{l-1,1}^{g+1,h-1,0,0} + \epsilon_l^{g,h-1} H_0^{1,0} B_{l,0}^{g,h-1,1,0} + \ldots\end{pmatrix}.\end{aligned}$$

For the last equality the highest order terms of ϵ were singled out. This $\epsilon_n^{a,b}$ of high order occurs when the Hamiltonian constant corresponds to the term of the lowest order, i.e. $H_0^{1,0}$ and $H_1^{0,0}$. There is no $H_0^{0,1}$ term, because we assumed $H_0^{0,1} = 0$. The reason for this is that when working order by order we will always use these high order epsilons to fix the coefficients of H, as we have already used the lower order terms of X to fix the lower order terms of H. Applying this equation to the equation of the transformed Hamiltonian we get that the coefficients of the Hamiltonian transform as follows:

$$
\begin{aligned}
H_n^{a,b} \to H_n^{a,b} &+ \text{lower order terms of } X \\
&+ A_{n-1,1}^{a,b,0,0}\epsilon_{n-1}^{a,b}H_1^{0,0} + A_{n,0}^{a-1,b,1,0}\epsilon_n^{a-1,b}H_0^{1,0} \qquad (4.37)\\
&+ B_{n-1,1}^{a+1,b-1,0,0}\epsilon_{n-1}^{a+1,b-1}H_1^{0,0} + B_{n,0}^{a,b-1,1,0}\epsilon_n^{a,b-1}H_0^{1,0} \\
= H_n^{a,b} &+ \text{lower order terms of } X \\
&+ (2b+1)\epsilon_{n-1}^{a,b}H_1^{0,0} + 2(2b+1)\epsilon_n^{a-1,b}H_0^{1,0} \qquad (4.38)\\
&+ 2(a+1)\epsilon_{n-1}^{a+1,b-1}H_1^{0,0} - 2(2(b-1)+n)\epsilon_n^{a,b-1}H_0^{1,0}
\end{aligned}
$$

To not regenerate p_r dependence that doesn't decay with $1/r$ it suffices to set $\epsilon_0^{a,b} = 0$ for $b > 0$. To see this we must once again consider equation 4.7. For this Poisson bracket to give rise to a term $T_0^{g,h}$ with $h > 0$ we need that $n + m = 0$, hence n and m are both zero. By assumption we also have that $H_0^{d,e} = 0$ for $e > 0$ and $\epsilon_0^{a,b} = 0$ for $b > 0$. Hence to have a non-zero coefficient we also need both b and e to be 0. If b and e are both zero then the $T_{n+m}^{a+d,b+e}$ term of equation 4.7 will not regenerate terms of the form $T_0^{g,h}$ with $h > 0$. The only possibility for such a term to arise would be due to the $T_{n+m}^{a+d-1,b+e+1}$ term. However its associated coefficient $B_{n,m}^{a,b,d,e} = 0$ when $b = n = e = m = 0$. Thus when $\epsilon_0^{a,b} = 0$ for $b > 0$ and $H_0^{d,e} = 0$ for $e > 0$, the Poisson bracket between X and H, given by $\{X, H\}$ will not contain terms of the form $T_0^{g,h}$ with $h > 0$.

We have made no specific assumptions about H, except that it can be written like equation 4.1 with $H_0^{d,e} = 0$ for $e > 0$, that is, it has no terms of the form $T_0^{d,e}$ for $e > 0$. Thus that argument will also work for anything of such a form, for example take $O = \sum_{a,b,n\geq 0} O_n^{a,b}T_n^{a,b}$, with $O_0^{a,b} = 0$ for $b > 0$. We can apply the same argument as in the previous paragraph to show that $\{X, O\}$ also does not contain terms of the form $T_0^{d,e}$ for $e > 0$.

We can use induction to show that the nth Poisson bracket of X with H does not contain terms of the form $T_0^{d,e}$ for $e > 0$. The base case ($n = 0$) is true by the assumption that $H_0^{d,e} = 0$ for $e > 0$. For the induction step we can let O be the nth Poisson bracket with X of H, that is $O = \{X, \cdot\}^n H$ and we assume that O does not contain terms of the form $T_0^{d,e}$ for $e > 0$. By the previous paragraph we see that $\{X, O\} = \{X, \cdot\}^{n+1}H$ also does not contain terms of the form $T_0^{d,e}$ for $e > 0$. Thus by induction all nested Poisson brackets of X with H do not contain such terms. Hence the transformed Hamiltonian also doesn't have those terms. This completes the proof that it suffices to have $\epsilon_0^{a,b} = 0$ to not regenerate p_r dependence.

4.6.1 Number of coefficients and parameters

We will now compare the number of available coefficients of X to the number of terms in the Hamiltonian that need to be killed.

The number of terms in H of order x is $\binom{x+2}{2}$; the number of ways to put x indistinguishable ball into 3 distinguishable bins, with each bin having a non-negative integer number of balls. This can be derived using a stars and bars argument.

There are $x + 1$ terms with $n = 0$, one of those only has p^2 dependence and the other terms all have p_r^2 dependence therefore there are x terms of H that are excluded due to having p_r^2 but not $1/r$. There are also $x+1$ terms with $b=0$.

There are hence $\binom{x+2}{2}-x-(x+1)=(x+2)(x+1)/2-2x-1=x^2/2-x/2$ terms in H that are relevant.

The number of terms X of order $x-1$ is $\binom{x+1}{2}$, and number of terms that are excluded (to not regenerate p_r^2 dependence) is $x-1$.

We have $x(x+1)/2-x+1=x^2/2-x/2+1$ available terms in X and $\binom{x+2}{2}-x-(x+1)=(x+2)(x+1)/2-2x-1=x^2/2-x/2$ terms to kill. There is one more available term in X than the number of terms to kill, thus at each order we have one residual degree of freedom assuming that the available terms in X are independent. This one residual degree of freedom at each order is precisely what is needed to get to the specific isotropic gauges. However, it still needs to be shown that the available parameters act independently, which we will do next.

4.6.2 Amplitude Gauge

We can write the linear part of the transformation as a matrix. The claim we want to prove is that if we order the coefficients of H that need to become zero in increasing order b primarily (and increasing order a is b-values are the same) and add the coefficient that needs to be set for the amplitude gauge ($H_0^{x,0}$) at the transition from $b=1$ to $b=2$ (since this coefficient is quite different from the other ones it is marked blue) then the resulting matrix becomes upper-triangular.

For example for the first few post-Newtonian orders we have that

1PN:

$$\begin{bmatrix} H_1^{0,1} \\ H_0^{2,0} \end{bmatrix} \to \begin{bmatrix} H_1^{0,1} \\ H_0^{2,0} \end{bmatrix} + \begin{bmatrix} -2H_0^{1,0} & 2H_1^{0,0} \\ 0 & 2H_0^{1,0} \end{bmatrix} \begin{bmatrix} \epsilon_1^{0,0} \\ \epsilon_0^{1,0} \end{bmatrix} \tag{4.39}$$

2PN:

$$\begin{bmatrix} H_2^{0,1} \\ H_1^{1,1} \\ H_0^{3,0} \\ H_1^{0,2} \end{bmatrix} \to \begin{bmatrix} H_2^{0,1} \\ H_1^{1,1} \\ H_0^{3,0} \\ H_1^{0,2} \end{bmatrix} + \text{lower order terms } X \tag{4.40}$$

$$+ \begin{bmatrix} -4H_0^{1,0} & 2H_1^{0,0} & 0 & 3H_1^{0,0} \\ 0 & -2H_0^{1,0} & 4H_1^{0,0} & 6H_0^{1,0} \\ 0 & 0 & 2H_0^{1,0} & 0 \\ 0 & 0 & 0 & -6H_0^{1,0} \end{bmatrix} \begin{bmatrix} \epsilon_2^{0,0} \\ \epsilon_1^{1,0} \\ \epsilon_0^{2,0} \\ \epsilon_1^{0,1} \end{bmatrix} \tag{4.41}$$

3PN:

$$\begin{bmatrix} H_3^{0,1} \\ H_2^{1,1} \\ H_1^{2,1} \\ H_0^{4,0} \\ H_2^{0,2} \\ H_1^{1,2} \\ H_1^{0,3} \end{bmatrix} \to \begin{bmatrix} H_3^{0,1} \\ H_2^{1,1} \\ H_1^{2,1} \\ H_0^{4,0} \\ H_2^{0,2} \\ H_1^{1,2} \\ H_1^{0,3} \end{bmatrix} + \text{lower order terms } X \tag{4.42}$$

$$+ \begin{bmatrix} -6H_0^{1,0} & 2H_1^{0,0} & 0 & 0 & 3H_1^{0,0} & 0 & 0 \\ 0 & -4H_0^{1,0} & 4H_1^{0,0} & 0 & 6H_0^{1,0} & 3H_1^{0,0} & 0 \\ 0 & 0 & -2H_0^{1,0} & 6H_1^{0,0} & 0 & 6H_0^{1,0} & 0 \\ 0 & 0 & 0 & 2H_0^{1,0} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -8H_0^{1,0} & 2H_1^{0,0} & 5H_1^{0,0} \\ 0 & 0 & 0 & 0 & 0 & -6H_0^{1,0} & 10H_0^{1,0} \\ 0 & 0 & 0 & 0 & 0 & 0 & -10H_0^{1,0} \end{bmatrix} \begin{bmatrix} \epsilon_3^{0,0} \\ \epsilon_2^{1,0} \\ \epsilon_1^{2,0} \\ \epsilon_0^{3,0} \\ \epsilon_2^{0,1} \\ \epsilon_1^{1,1} \\ \epsilon_1^{0,2} \end{bmatrix} \tag{4.43}$$

We will define the indicator function $1(P) = \begin{cases} 1 & \text{if } P \\ 0 & \text{otherwise} \end{cases}$ for any proposition P. The coefficients of the Hamiltonian $H_n^{a,b}$ (except $H_0^{x,0}$) in the column vector satisfy $a+b+n = x$, where $x-1$ is the post-Newtonian order, $b > 0$, and $n > 0$. The position of each coefficient $H_n^{a,b}$ (except $H_0^{x,0}$) in the column vector is given by the following function:

$$f : \{(a,b,n) \mid (a,b,n) \in \mathbb{Z}^3_{\geq 0}, a+b+n = x, b > 0, n > 0\} \to \mathbb{Z}_{\geq 0}$$

$$
\begin{aligned}
f(a,b,n) :&= 1(H_0^{x,0} \text{ comes before } H_n^{a,b}) \\
&\quad + \sum_{a'+b'+n'=x} 1(H_{n'}^{a',b'} \text{ needs to be set to zero and comes before } H_n^{a,b}) \\
&= 1(b>1) + \sum_{b'=0}^{x} \sum_{a'+n'=x-b'} 1(b'>0 \wedge n'>0 \wedge (b'<b \vee (b'=b \wedge a'<a))) \\
&= 1(b>1) + \sum_{b'=1}^{b} \sum_{a'=0}^{x-b'-1} 1(b'<b \vee (b'=b \wedge a'<a)) \\
&= 1(b>1) + \sum_{b'=1}^{b} \sum_{a'=0}^{x-b'-1} 1(b'<b) + 1(b'=b \wedge a'<a) \\
&= 1(b>1) + \sum_{b'=1}^{b} \sum_{a'=0}^{x-b'-1} 1(b'<b) + \sum_{b'=1}^{b} \sum_{a'=0}^{x-b-1} 1(b'=b \wedge a'<a) \\
&= 1(b>1) + \sum_{b'=1}^{b-1} \sum_{a'=0}^{x-b'-1} 1 + \sum_{a'=0}^{x-b-1} 1(a'<a) \\
&= 1(b>1) + \sum_{b'=1}^{b-1} (x-b') + \sum_{a'=0}^{a+n-1} 1(a'<a) \\
&= 1(b>1) + \sum_{b'=1}^{b-1} x - \sum_{b'=1}^{b-1} b' + \sum_{a'=0}^{a-1} 1 \\
&= (b-1)x - \frac{b(b-1)}{2} + a + 1(b>1)
\end{aligned}
$$

The position of $H_0^{x,0}$ is after $H_1^{x-2,1}$, hence given by $f(x-2,1,1)+1 = x-1$.

The coefficients $\epsilon_n^{a,b}$ satisfy $a+b+n = x-1$, and ($b=0$ or $n>0$). For the position of $\epsilon_n^{a,b}$ we have that it is given by the function g:

$$
g : \{(a,b,n) \mid (a,b,n) \in \mathbb{Z}_{\geq 0}^3 \wedge a+b+n = x-1 \wedge (b=0 \vee n>0)\} \to \mathbb{Z}_{\geq 0}
$$

$$
\begin{aligned}
g(a,b,n) :&= \sum_{a'+b'+n'=x-1} 1(\epsilon_{n'}^{a',b'} \text{ comes before } \epsilon_n^{a,b}) \\
&= \sum_{b'=0}^{x} \sum_{a'+n'=x-b'-1} 1((b'=0 \vee n'>0) \wedge (b'<b \vee (b'=b \wedge a'<a)))
\end{aligned}
$$

We will first consider the case $b = 0$, in this case we have that

$$\begin{aligned}
g(a, b = 0, n) &= \sum_{b'=0}^{x} \sum_{a'+n'=x-b'-1} 1((b' = 0 \vee n' > 0) \wedge (b' < 0 \vee (b' = 0 \wedge a' < a))) \\
&= \sum_{b'=0}^{x} \sum_{a'+n'=x-b'-1} 1(b' = 0 \wedge (n' > 0 \vee a' < a)) \\
&= \sum_{a'+n'=x-1} 1(n' > 0 \vee a' < a) \\
&= \sum_{a'=0}^{x-1} 1(x - 1 - a' > 0 \vee a' < a) \\
&= \sum_{a'=0}^{a-1} 1(x - 1 > a') \\
&= \sum_{a'=0}^{a-1} 1 \\
&= a
\end{aligned}$$

For the $b > 0$ case we have that

$$\begin{aligned}
g(a, b, n) &:= \sum_{a'+b'+n'=x-1} 1(\epsilon_{n'}^{a',b'} \text{ comes before } \epsilon_{n}^{a,b}) \\
&= \sum_{b'=0}^{x} \sum_{a'+n'=x-b'-1} 1((b' = 0 \vee n' > 0) \wedge (b' < b \vee (b' = b \wedge a' < a))) \\
&= \sum_{b'=1}^{x} \sum_{a'+n'=x-b'-1} 1((n' > 0) \wedge (b' < b \vee (b' = b \wedge a' < a))) \\
&+ \sum_{a'+n'=x-1} 1(0 < b \vee (0 = b \wedge a' < a)) \\
&= \sum_{b'=1}^{x} \sum_{a'=0}^{x-b'-2} 1(b' < b \vee (b' = b \wedge a' < a)) + \sum_{a'+n'=x-1} 1 \\
&= \sum_{b'=1}^{b-1} \sum_{a'=0}^{x-b'-2} 1 + \sum_{a'=0}^{x-b-2} 1(a' < a) + x \\
&= \sum_{b'=1}^{b-1} (x - b' - 1) + \sum_{a'=0}^{a+n-2} 1(a' < a) + x \\
&= (b - 1)(x - 1) - \sum_{b'=1}^{b-1} b' + a + x \\
&= x + (b - 1)(x - 1) - \frac{b(b - 1)}{2} + a \\
&= x + (b - 1)(x - 1) - \frac{b(b - 1)}{2} + a - 1 + 1(b > 0) \\
&= b(x - 1) - \frac{b(b - 1)}{2} + a + 1(b > 0)
\end{aligned}$$

Combining these two cases we get that for any b the function g is given by

$$g(a,b,n) = b(x-1) - \frac{b(b-1)}{2} + a + 1(b>0) \tag{4.44}$$

We want to show that the matrix is upper-triangular, for this we need that for all non-zero entries in the matrix the index of the column should be greater or equal to the index of the row. We will first show this for the rows corresponding to the 'anisotropic coefficients'. Considering equation 4.38 we see that we want the following four inequalities to hold whenever the arguments of the functions are in their domain:

$$f(a,b,n) \leq g(a,b,n-1) \tag{4.45}$$
$$f(a,b,n) \leq g(a,b-1,n) \tag{4.46}$$
$$f(a,b,n) \leq g(a+1,b-1,n-1) \tag{4.47}$$
$$f(a,b,n) \leq g(a-1,b,n) \tag{4.48}$$

We will now prove that those four inequalities hold:

$$\begin{aligned}
f(a,b,n) &= bx - x - \frac{b(b-1)}{2} + a + 1(b>1) && (4.49)\\
&\leq bx - (b-2) - \frac{b(b-1)}{2} + a + 1(b>1) && (4.50)\\
&< bx - b - \frac{b(b-1)}{2} + a && (4.51)\\
&\leq g(a,b,n-1) && (4.52)
\end{aligned}$$

$$\begin{aligned}
f(a,b,n) &= bx - x - \frac{b(b-1)}{2} + a + 1(b>1) && (4.53)\\
&= (b-1)x - \frac{(b-2)(b-1)+2(b-1)}{2} + a + 1(b>1) && (4.54)\\
&= (b-1)x - (b-1) - \frac{(b-2)(b-1)}{2} + a + 1(b-1>0) && (4.55)\\
&= g(a,b-1,n) && (4.56)\\
&< g(a+1,b-1,n-1) && (4.57)
\end{aligned}$$

$$\begin{aligned}
f(a,b,n) &= bx - x - \frac{b(b-1)}{2} + a + 1(b>1) && (4.58)\\
&= bx - (x-1) - \frac{b(b-1)}{2} + a - 1 + 1(b>1) && (4.59)\\
&= bx - (a+b+n-1) - \frac{b(b-1)}{2} + a - 1 + 1(b>1) && (4.60)\\
&= bx - b - \frac{b(b-1)}{2} + a - 1 + (1(b>1) - a - n - 1) && (4.61)\\
&< bx - b - \frac{b(b-1)}{2} + a - 1(b>0) && (4.62)\\
&= g(a-1,b,n) && (4.63)
\end{aligned}$$

The row corresponding to $H_0^{x,0}$ can be derived from its transformation given by equation 4.38:

$$H_0^{x,0} \to H_0^{x,0} + \text{lower order terms of } X + 2\epsilon_0^{x-1,0} H_0^{1,0}.$$

The non-zero entry on this row will have position $g(x-1,0,0) = x-1$ which is the same as the position of the row of $H_0^{x,0}$. Thus this row also follows the upper-triangular structure of the matrix.

The other entries along the diagonal, that are on the rows corresponding to the anisotropic coefficients, are also non-zero. Since $f(a,b,n) = g(a,b-1,n)$, the entries along the diagonal that are on the row of $H_n^{a,b}$ are at the position corresponding to the column of $\epsilon_n^{a,b-1}$. The value of these diagonal coefficients are given by $2(2(b-1)+n)H_0^{10}$, which is non-zero, because b and n are positive and H_0^{10} is non-zero. This completes the proof that the matrix is upper-triangular with non-zero diagonal entries. Therefore the matrix has full rank. Thus the parameters of X can be tuned to set the non-isotropic coefficients of the Hamiltonian to zero and get to the amplitude gauge.

4.6.3 LRL and PMOOSH gauges

A similar technique as what was used for the amplitude gauge can be used to show the existence of the LRL and PMOOSH gauges at higher orders. For the PMOOSH gauge we want to set $H_1^{x-1,0}$ to zero, while for the LRL gauge the linear combination $H_1^{x-1,0} + xH_0^{x,0}$ needs to be set to zero. Replacing the coefficient $H_0^{x,0}$ with one of these linear combinations of coefficients will result in a matrix that is almost upper-triangular: on the row corresponding to $H_1^{x-1,0}$ or $H_1^{x-1,0} + xH_0^{x,0}$ there is one entry one position to the left of the diagonal. By applying a row operation (adding a multiple of another row to this row) the matrix can be brought into upper-triangular form without affecting it's invertibility. If all of the entries on the diagonal are non-zero after this row addition (this will be shown later on), then the matrix is invertible and hence the LRL and PMOOSH gauges exist.

To be a bit more concrete for the 3PN PMOOSH gauge the following transformation is of importance:

$$\begin{bmatrix} H_3^{0,1} \\ H_2^{1,1} \\ H_1^{2,1} \\ H_1^{3,0} \\ H_2^{0,2} \\ H_1^{1,2} \\ H_1^{0,3} \end{bmatrix} \to \begin{bmatrix} H_3^{0,1} \\ H_2^{1,1} \\ H_1^{2,1} \\ H_1^{3,0} \\ H_2^{0,2} \\ H_1^{1,2} \\ H_1^{0,3} \end{bmatrix} + \text{lower order terms } X \tag{4.64}$$

$$+ \begin{bmatrix} -6H_0^{1,0} & 2H_1^{0,0} & 0 & 0 & 3H_1^{0,0} & 0 & 0 \\ 0 & -4H_0^{1,0} & 4H_1^{0,0} & 0 & 6H_0^{1,0} & 3H_1^{0,0} & 0 \\ 0 & 0 & -2H_0^{1,0} & 6H_1^{0,0} & 0 & 6H_0^{1,0} & 0 \\ 0 & 0 & 2H_0^{1,0} & 1H_1^{0,0} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -8H_0^{1,0} & 2H_1^{0,0} & 5H_1^{0,0} \\ 0 & 0 & 0 & 0 & 0 & -6H_0^{1,0} & 10H_0^{1,0} \\ 0 & 0 & 0 & 0 & 0 & 0 & -10H_0^{1,0} \end{bmatrix} \begin{bmatrix} \epsilon_3^{0,0} \\ \epsilon_2^{1,0} \\ \epsilon_1^{2,0} \\ \epsilon_0^{3,0} \\ \epsilon_2^{0,1} \\ \epsilon_1^{1,1} \\ \epsilon_1^{0,2} \end{bmatrix} \tag{4.65}$$

The entry that prevents the matrix from being upper-triangular is coloured red. By adding the third row to the fourth row the matrix can be made upper-triangular with a non-zero diagonal.

For the PMOOSH and LRL gauges we want to set some linear combination of $H_0^{x,0}$ and $H_1^{x-1,0}$ to zero at every PN order $x-1$. We have already shown that this possible up to 1PN order, now will will show it for 2PN and above, hence we consider $x \geq 3$. In general the rows corresponding to $H_1^{x-2,1}$ and $\lambda_1 H_0^{x,0} + \lambda_2 H_0^{x-1,1}$ are given by the transformations

$$\begin{aligned} H_1^{x-2,1} &\to H_1^{x-2,1} + \text{lower order terms of } X + 6\epsilon_1^{x-3,1} H_0^{1,0} + 2(x-1)\epsilon_0^{x-1,0} H_1^{0,0} - 2\epsilon_1^{x-2,0} H_0^{1,0} \\ H_0^{x,0} &\to H_0^{x,0} + \text{lower order terms of } X + 2\epsilon_0^{x-1,0} H_0^{1,0} \\ H_1^{x-1,0} &\to H_1^{x-1,0} + \text{lower order terms of } X + \epsilon_0^{x-1,0} H_1^{0,0} + 2\epsilon_1^{x-2,0} H_0^{1,0}. \end{aligned}$$

This results in to following two rows corresponding to $H_1^{x-2,1}$ and $\lambda_1 H_0^{x,0} + \lambda_2 H_0^{x-1,1}$:

$$\begin{bmatrix} H_1^{x-2,1} \\ \lambda_1 H_0^{x,0} + \lambda_2 H_0^{x-1,1} \end{bmatrix} \to \begin{bmatrix} H_1^{x-2,1} \\ \lambda_1 H_0^{x,0} + \lambda_2 H_0^{x-1,1} \end{bmatrix} + \text{lower order terms } X$$
$$+ \begin{bmatrix} \cdots & -2H_0^{1,0} & 2(x-1)H_1^{0,0} & \cdots & 6H_0^{1,0} & \cdots \\ \cdots & {\color{red}2\lambda_2 H_0^{1,0}} & 2\lambda_1 H_0^{1,0} + \lambda_2 H_1^{0,0} & \cdots & 0 & \cdots \end{bmatrix} \begin{bmatrix} \vdots \\ \epsilon_1^{x-2,0} \\ \epsilon_0^{x-1,0} \\ \vdots \\ \epsilon_1^{x-3,1} \\ \vdots \end{bmatrix}.$$

By adding a multiple of λ_2 of the row corresponding to $H_1^{x-2,1}$ to the row corresponding to $\lambda_1 H_0^{x,0} + \lambda_2 H_0^{x-1,1}$ the matrix (with all the rows, like in the proof for the amplitude gauge) can be brought to upper-triangular form. The diagonal element of the row corresponding to $\lambda_1 H_0^{x,0} + \lambda_2 H_0^{x-1,1}$ then becomes $2\lambda_1 H_0^{1,0} + (2x-1)\lambda_2 H_1^{0,0}$.

For the PMOOSH gauge we have $\lambda_1 = 0$ and $\lambda_2 = 1$. This results in the diagonal element $(2x-1)H_1^{0,0}$, which is non-zero for integer values of x. For the LRL gauge we have $\lambda_1 = 1$, $\lambda_2 = x$, $H_0^{1,0} = 1$, and $H_1^{0,0} = -1$. This results in a value of $2 - (2x-1)x = -2x^2 + x + 2$ on the diagonal. For integer values of x this is also non-zero. Thus for the LRL and PMOOSH gauges the matrix is also invertible at every order. Hence the parameters of X can be chosen to go to those gauges.

5 Conclusions

The Kepler problem can have relativistic corrections according to several theories, such as general relativity, but also via conformal coupling of a scalar field. The effective Hamiltonian for these theories can be written as a post-Newtonian expansion in terms of the parameters p^2, p_r^2 and $1/r$. A canonical transformation can then be used to transform this Hamiltonian into a form that does not depend on p_r^2.

There are several possibilities for these so-called isotropic gauges, where the Hamiltonian doesn't depend on p_r^2: for example one can take the amplitude gauge, the LRL gauge or the PMOOSH gauge. In the amplitude gauge the terms in the expansion of the Hamiltonian that only depend on p^2 correspond with the expression for kinetic energy in special relativity. When a Hamiltonian is written in this gauge it can easily be compared to a Hamiltonian derived using scattering amplitudes. In the LRL gauge the terms that cause the non-conservation of the LRL vector have a post-Minkowskian order of at least 2. This makes this gauge useful for analysing precession of orbits. When an orbit precesses the LRL vector also changes direction. If the coefficients in front of all those terms are then the LRL vector stays conserved. In the PMOOSH gauge at 1PM order the Hamiltonian is separable, that is, it can be written as a sum of a kinetic term depending on momentum and a potential terms depending of position.

The transformations to get to these gauges up to any post-Newtonian order have been shown to exist for the amplitude, LRL, and PMOOSH gauges. This was done by working order by order and showing that the coefficients of X affect the coefficients of H that need to be set to a specific value in a linearly independent way.

These isotropic gauges can be used to reduce the number of terms in the post-Newtonian expansion of the Hamiltonian. They can also be used to compare two seemingly different Hamiltonians by writing them in the same gauge. Further research could be done to come up with a more insightful reasoning behind the existence of these gauges and provide necessary conditions for the existence of certain isotropic gauges.

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A Appendix

A.1 Active Transformation of Function

This is a derivation of equation 2.29 using the chain rule

$$\begin{aligned}
\left(\frac{\partial f}{\partial \alpha}\right)_{t,q,p} &= \left(\frac{\partial f}{\partial \alpha}\right)_{t,Q,P} + \left(\frac{\partial f}{\partial Q}\right)_{t,\alpha,P}\left(\frac{\partial Q}{\partial \alpha}\right) + \left(\frac{\partial f}{\partial P}\right)_{t,\alpha,Q}\left(\frac{\partial P}{\partial \alpha}\right) && \text{(A.1)}\\
&= 0 + \left(\frac{\partial f}{\partial Q}\right)_{t,\alpha,P}\{X,Q\} + \left(\frac{\partial f}{\partial P}\right)_{t,\alpha,Q}\{X,P\} && \text{(A.2)}\\
&= \left(\frac{\partial f}{\partial Q}\right)_{t,\alpha,P}\left(\frac{\partial X}{\partial q}\frac{\partial Q}{\partial q} - \frac{\partial X}{\partial p}\frac{\partial Q}{\partial q}\right) + \left(\frac{\partial f}{\partial P}\right)_{t,\alpha,Q}\left(\frac{\partial X}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial X}{\partial p}\frac{\partial P}{\partial q}\right) && \text{(A.3)}\\
&= \frac{\partial X}{\partial q}\left(\left(\frac{\partial f}{\partial Q}\right)_{t,\alpha,P}\frac{\partial Q}{\partial q} + \left(\frac{\partial f}{\partial P}\right)_{t,\alpha,Q}\frac{\partial P}{\partial p}\right) - \frac{\partial X}{\partial p}\left(\left(\frac{\partial f}{\partial Q}\right)_{t,\alpha,P}\frac{\partial Q}{\partial q} + \left(\frac{\partial f}{\partial P}\right)_{t,\alpha,Q}\frac{\partial P}{\partial q}\right) && \text{(A.4)}\\
&= \frac{\partial X}{\partial q}\frac{\partial f}{\partial p} - \frac{\partial X}{\partial p}\frac{\partial f}{\partial q} && \text{(A.5)}\\
&= \{X,f\} && \text{(A.6)}
\end{aligned}$$

Thus $f \to e^{\{X,\cdot\}}f$

B Appendix 2

B.1 Derivation of Equation 4.7

We will first consider the following Poisson bracket:

$$\{(p^2)^a(p\cdot q)^b(q^2)^c,(p^2)^d(p\cdot q)^e(q^2)^f\}$$
$$=\sum_{i=1}^{N}\frac{\partial}{\partial q_i}((p^2)^a(p\cdot q)^b(q^2)^c)\frac{\partial}{\partial p_i}((p^2)^d(p\cdot q)^e(q^2)^f)-\frac{\partial}{\partial p_i}((p^2)^a(p\cdot q)^b(q^2)^c)\frac{\partial}{\partial q_i}((p^2)^d(p\cdot q)^e(q^2)^f)$$

$$\frac{\partial}{\partial q_i}((p^2)^a(p\cdot q)^b(q^2)^c)=(p^2)^a(bp_i(p\cdot q)^{b-1})(q^2)^c+(p^2)^a(p\cdot q)^b(2cq_i(q^2)^{c-1}) \tag{B.1}$$
$$=bp_i(p^2)^a(p\cdot q)^{b-1}(q^2)^c+2cq_i(p^2)^a(p\cdot q)^b(q^2)^{c-1} \tag{B.2}$$

Similarily

$$\frac{\partial}{\partial p_i}((p^2)^a(p\cdot q)^b(q^2)^c)=2ap_i(p^2)^{a-1}(p\cdot q)^b(q^2)^c+bq_i(p^2)^a(p\cdot q)^{b-1}(q^2)^c \tag{B.3}$$
$$\frac{\partial}{\partial q_i}((p^2)^d(p\cdot q)^e(q^2)^f)=ep_i(p^2)^d(p\cdot q)^{e-1}(q^2)^f+2fq_i(p^2)^d(p\cdot q)^e(q^2)^{f-1} \tag{B.4}$$
$$\frac{\partial}{\partial p_i}((p^2)^d(p\cdot q)^e(q^2)^f)=2dp_i(p^2)^{d-1}(p\cdot q)^e(q^2)^f+eq_i(p^2)^d(p\cdot q)^{e-1}(q^2)^f \tag{B.5}$$
$$\tag{B.6}$$

Now,

$$\frac{\partial}{\partial q_i}((p^2)^a(p\cdot q)^b(q^2)^c)\frac{\partial}{\partial p_i}((p^2)^d(p\cdot q)^e(q^2)^f) \tag{B.7}$$
$$=(bp_i(p^2)^a(p\cdot q)^{b-1}(q^2)^c+2cq_i(p^2)^a(p\cdot q)^b(q^2)^{c-1})(2dp_i(p^2)^{d-1}(p\cdot q)^e(q^2)^f+eq_i(p^2)^d(p\cdot q)^{e-1}(q^2)^f) \tag{B.8}$$
$$=2bdp_i^2(p^2)^{a+d-1}(p\cdot q)^{b+e-1}(q^2)^{c+f}+bep_iq_i(p^2)^{a+d}(p\cdot q)^{b+e-2}(q^2)^{c+f} \tag{B.9}$$
$$+4cdq_ip_i(p^2)^{a+d-1}(p\cdot q)^{b+e}(q^2)^{c+f-1}+2ceq_i^2(p^2)^{a+d}(p\cdot q)^{b+e-1}(q^2)^{c+f-1} \tag{B.10}$$

Summing over i yields

$$\sum_{i=1}^{N}\frac{\partial}{\partial q_i}((p^2)^a(p\cdot q)^b(q^2)^c)\frac{\partial}{\partial p_i}((p^2)^d(p\cdot q)^e(q^2)^f) \tag{B.11}$$
$$=2bd(p^2)^{a+d}(p\cdot q)^{b+e-1}(q^2)^{c+f}+be(p^2)^{a+d}(p\cdot q)^{b+e-1}(q^2)^{c+f} \tag{B.12}$$
$$+4cd(p^2)^{a+d-1}(p\cdot q)^{b+e+1}(q^2)^{c+f-1}+2ce(p^2)^{a+d}(p\cdot q)^{b+e-1}(q^2)^{c+f} \tag{B.13}$$

Thus the poisson bracket is given by

$$\{(p^2)^a(p\cdot q)^b(q^2)^c,(p^2)^d(p\cdot q)^e(q^2)^f\} \tag{B.14}$$
$$= 2bd(p^2)^{a+d}(p\cdot q)^{b+e-1}(q^2)^{c+f} + be(p^2)^{a+d}(p\cdot q)^{b+e-1}(q^2)^{c+f} \tag{B.15}$$
$$+ 4cd(p^2)^{a+d-1}(p\cdot q)^{b+e+1}(q^2)^{c+f-1} + 2ce(p^2)^{a+d}(p\cdot q)^{b+e-1}(q^2)^{c+f} \tag{B.16}$$
$$- 2ae(p^2)^{a+d}(p\cdot q)^{b+e-1}(q^2)^{c+f} - be(p^2)^{a+d}(p\cdot q)^{b+e-1}(q^2)^{c+f} \tag{B.17}$$
$$- 4af(p^2)^{a+d-1}(p\cdot q)^{b+e+1}(q^2)^{c+f-1} - 2bf(p^2)^{a+d}(p\cdot q)^{b+e-1}(q^2)^{c+f} \tag{B.18}$$
$$= 2(bd+ce-ae-bf)(p^2)^{a+d}(p\cdot q)^{b+e-1}(q^2)^{c+f} + 4(cd-af)(p^2)^{a+d-1}(p\cdot q)^{b+e+1}(q^2)^{c+f-1} \tag{B.19}$$

Using the fact that $p_r = \frac{p\cdot q}{\|q\|}$ we can derive that

$$\{\left(\frac{p^2}{2\mu^2}\right)^a\left(\frac{p_r^2}{2\mu^2}\right)^b\left(\frac{\kappa}{\mu r}\right)^n p\cdot q, \left(\frac{p^2}{2\mu^2}\right)^d\left(\frac{p_r^2}{2\mu^2}\right)^e\left(\frac{\kappa}{\mu r}\right)^m\} \tag{B.20}$$
$$= \left(\frac{1}{2\mu^2}\right)^{a+b+d+e}\left(\frac{\kappa}{\mu}\right)^{n+m}\{(p^2)^a(p\cdot q)^{1+2b}(q^2)^{-b-n/2},(p^2)^d(p\cdot q)^{2e}(q^2)^{-e-m/2}\} \tag{B.21}$$

Combining this with the previous result equation 4.7 can be found.

B.2 Transformation of Hamiltonian up to 2PN

$$H_0^{0,0} \to H_0^{0,0}$$
$$H_1^{0,0} \to H_1^{0,0}$$
$$H_0^{1,0} \to H_0^{1,0}$$
$$H_2^{0,0} \to H_2^{0,0} + H_1^{0,0}\epsilon_1^{0,0}$$
$$H_1^{0,1} \to H_1^{0,1} - 2H_0^{1,0}\epsilon_1^{0,0} + 2H_1^{0,0}\epsilon_0^{1,0}$$
$$H_1^{1,0} \to H_1^{1,0} + 2H_0^{1,0}\epsilon_1^{0,0} + H_1^{0,0}\epsilon_0^{1,0}$$
$$H_0^{2,0} \to H_0^{2,0} + 2H_0^{1,0}\epsilon_0^{1,0}$$
$$H_3^{0,0} \to H_3^{0,0} + 2H_2^{0,0}\epsilon_1^{0,0} + H_1^{0,0}\epsilon_2^{0,0} + \\ + \frac{1}{2!}(2H_1^{0,0}\epsilon_1^{0,0}\epsilon_1^{0,0})$$
$$H_2^{0,1} \to H_2^{0,1} + H_1^{0,1}\epsilon_1^{0,0} - 2H_1^{1,0}\epsilon_1^{0,0} + 4H_2^{0,0}\epsilon_0^{1,0} - 4H_0^{1,0}\epsilon_2^{0,0} + 3H_1^{0,0}\epsilon_1^{0,1} + 2H_1^{0,0}\epsilon_1^{1,0} + \\ + \frac{1}{2!}(-6H_0^{1,0}\epsilon_1^{0,0}\epsilon_1^{0,0} + 4H_1^{0,0}\epsilon_1^{0,0}\epsilon_0^{1,0})$$
$$H_1^{0,2} \to H_1^{0,2} + 6H_1^{0,1}\epsilon_0^{1,0} - 6H_0^{1,0}\epsilon_1^{0,1} + \\ + \frac{1}{2!}(-12H_0^{1,0}\epsilon_1^{0,0}\epsilon_0^{1,0} + 12H_1^{0,0}\epsilon_0^{1,0}\epsilon_0^{1,0})$$
$$H_2^{1,0} \to H_2^{1,0} + 3H_1^{1,0}\epsilon_1^{0,0} + 2H_2^{0,0}\epsilon_0^{1,0} + 2H_0^{1,0}\epsilon_2^{0,0} + H_1^{0,0}\epsilon_1^{1,0} + \\ + \frac{1}{2!}(6H_0^{1,0}\epsilon_1^{0,0}\epsilon_1^{0,0} + 5H_1^{0,0}\epsilon_1^{0,0}\epsilon_0^{1,0})$$
$$H_1^{1,1} \to H_1^{1,1} - 4H_0^{2,0}\epsilon_1^{0,0} - H_1^{0,1}\epsilon_0^{1,0} + 2H_1^{1,0}\epsilon_0^{1,0} + 6H_0^{1,0}\epsilon_1^{0,1} - 2H_0^{1,0}\epsilon_1^{1,0} + 4H_1^{0,0}\epsilon_0^{2,0} + \\ + \frac{1}{2!}(-2H_0^{1,0}\epsilon_1^{0,0}\epsilon_0^{1,0} + 0H_1^{0,0}\epsilon_0^{1,0}\epsilon_0^{1,0})$$
$$H_1^{2,0} \to H_1^{2,0} + 4H_0^{2,0}\epsilon_1^{0,0} + 3H_1^{1,0}\epsilon_0^{1,0} + 2H_0^{1,0}\epsilon_1^{1,0} + H_1^{0,0}\epsilon_0^{2,0} + \\ + \frac{1}{2!}(14H_0^{1,0}\epsilon_1^{0,0}\epsilon_0^{1,0} + 3H_1^{0,0}\epsilon_0^{1,0}\epsilon_0^{1,0})$$
$$H_0^{3,0} \to H_0^{3,0} + 4H_0^{2,0}\epsilon_0^{1,0} + 2H_0^{1,0}\epsilon_0^{2,0} + \\ + \frac{1}{2!}(8H_0^{1,0}\epsilon_0^{1,0}\epsilon_0^{1,0})$$