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Post-Newtonian Corrections to the Kepler Problem

Bachelor Research Project

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Abstract

To accurately describe motion in the Kepler two-body problem, such as the precessing orbit of Mercury, Kepler's Hamiltonian requires higher-order Post-Newtonian (PN) corrections from General Relativity. Alternative scalar field theories of gravity can also be studied to compare the resulting PN corrections and their impacts on two-body dynamics. While Nordström's theory of gravity with conformal coupling in the metric has been examined previously, the corrections arising from disformal coupling in Dirac-Born-Infeld (DBI) scalar field theory have not yet been investigated in detail. The aim of this research was to calculate the leading-order PN corrections to the Kepler problem Hamiltonian under disformal coupling and to understand how the resulting dynamics differs from the classical and relativistic cases. We find that the leading PN correction to the Hamiltonian with disformal coupling falls off as $\propto \frac{\dot{r}^2}{r^4}$ and does not support closed, bounded orbits, in contrast to General Relativity and Nordström's theory.

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Contents

Chapter 1

Introduction

Since antiquity, humans have sought to understand celestial motion and the fundamental forces governing the Universe. The initial scientific endeavours, based on astronomical observations, led to the development of one of the most significant physical theories: the universal law of gravitation, formulated by Isaac Newton. The prominent success of Newton's mechanics was largely contained in a correct prediction of the observed motion of Solar System objects, described empirically by Johannes Kepler. This 17th-century astronomer and mathematician had proposed the three laws of two-body dynamic, stating that "the orbits of bodies in the Solar System are ellipses that trace out equal areas in equal times, with periods inversely proportional to the 3/2 power of their diameters" [1]. By this day, the problem of determining the motion of two spherically symmetric objects subjected to mutual gravitational attraction is known as *the Kepler Problem.* [1].

Despite two centuries of overwhelming success, the failure of the classical law of gravity to predict Mercury's perihelion advancement (observed in the middle of the 19th century) resulted in a crisis in the scientific community. The discrepancy between the observation and Newton's theory prediction was firstly postulated to arise due to the gravitational influences of another planet between Mercury and the Sun, provisionally named Vulcan. Despite systematic astronomical searches, no credible evidence of its existence had ever been established [1].

The search for the explanation of enigmatic perihelion precession was finally concluded in 1915 with Albert Einstein's General Theory of Relativity [2]. The correct prediction of the advancement of 42.98 arcseconds per century of Mercury's perihelion constitutes one of the first observational confirmations of Einstein's theory [1].

Einstein's solution to the Kepler problem involved providing correction terms to the Hamiltonian describing the two-body dynamics. Hamiltonian and Lagrangian formalisms have proven to be a powerful framework for studying theoretical physics. Nevertheless, the complexity of General Relativity equations does not allow for an exact solution. Instead, the problem can be tackled by conducting a *Post-Newtonian expansion* - an expansion in inverse powers of the speed of light of a Lagrangian, a Hamiltonian, or a physical observable [3]. By this day, the relativistic corrections to the Kepler problem have been calculated up to the 4PN (Post-Newtonian) order [4]. Nonetheless, the Post-Newtonian expansion remains a potent tool for modern research on gravity [5], and it constitutes a centrepiece of this study.

Einstein's theory of General Relativity (GR) replaced the flat Minkowski spacetime metric, $\eta_{\mu\nu}$, with a dynamic metric, $g_{\mu\nu}$, that varies according to the Einstein's field equations based on the distribution of matter (and energy) [6]:

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu \qquad \to \qquad ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \tag{1.1}$$

In this context, Special Relativity (SR) with the Minkowski metric $\eta_{\mu\nu}$ describing a flat spacetime becomes a local approximation for the global curvature. This is often referred to as the *Einstein's Equivalence Principle.* As the general curved spacetime locally seems flat, in small enough regions, laws of physics reduce to those of SR and the gravitational field is impossible to detect [7].

However, it is worth noting that GR was not the only theory that aimed to extend Newtonian gravity to make it compatible with special relativity. An alternative approach was provided in 1913 by a Finnish physicist Gunnar Nordström, who introduced so-called *conformal coupling* in the space-time metric, given by an exponential of a scalar field φ:

$$ds^2 = e^{2a\varphi} \eta_{\mu\nu} dx^\mu dx^\nu. \tag{1.2}$$

The scalar field φ is known as the *dilaton field*, and a is a free parameter. Indeed, Nordström's theory was the first metric theory of gravitation where the effects of gravity stem from the geometry of spacetime [8]. The Equivalence Principle also applies for the conformally coupled metric, as in the expansion of the exponential factor identity remains a dominant term. Nonetheless, the theory's failure to predict Mercury's correct perihelion precession was one of the reasons it was ruled out as a competitor to General Relativity [8]. Although inconsistent with astronomical observations, predicting the precession in the opposite direction, Nordström's conformally coupled metric remains an interesting theoretical model, still investigated in a search for novel gravitational effects [9].

While General Relativity is considered the most accurate theory of gravity, other metric theories have appeared in theoretical studies. A model most closely approximating the GR effects is known as the Bekenstein metric [10]. The Bekenstein model, besides the conformally coupled part, also includes additional so-called *disformal coupling* perturbation [10, 11]. This model, introduced by Jacob D. Bekenstein in 1993, was aimed to be the most general description of the field theories of gravity that couple two geometries: the Riemannian geometry of spacetime and the physical geometry in which the matter plays out its dynamics. For instance, in Nordström's theory, the two geometries are conformally related via the exponential factor. Indeed, the way to generalize such theories of gravity was to introduce a disformal transformation to the metric equation, which changes the units of length with respect to the gradient (derivative) of the scalar field φ [10].

While different metric theories were considered, the disformally coupled metric on its own (with the conformal coupling constant set to zero) has not yet been investigated in much detail in the context of Post-Newtonian Hamiltonian correction. In particular, such a disformally coupled metric is given by [12]:

$$g_{\mu\nu} = \eta_{\mu\nu} + A\partial_\mu\varphi\partial_\nu\varphi, \tag{1.3}$$

where A is a free parameter, and φ is a scalar field. Such a metric is characteristic for the Dirac-Born-Infeld (DBI) scalar field theory [13]. The DBI theory was initially introduced in 1934 as a non-linear theory of electromagnetism modifying the Maxwellian Lagrangian [13]. While the model was eventually ruled out by quantum electrodynamics, the disformal coupling of gravitational scalar field to additional matter fields constitutes an active area of theoretical research [14].

Indeed, this study aims to establish what the Post-Newtonian corrections to Kepler problem's Hamiltonian resulting from the disformal coupling are, and deduce their effect on the two-body dynamics. In particular, the following research questions are considered:

- What are the corrections to the Kepler problem's Hamiltonian resulting from the disformal coupling?
- To what extent does the Kepler 2-body dynamics resulting from disformal coupling differs from classical and relativistic cases?

To investigate these issues in more detail, this thesis considers three theories of gravity: General Relativity, Nordström's theory, and the DBI theory. Re-derivation of the known Hamiltonian corrections and the resulting Kepler dynamics for the first two theories provides a suitable machinery for the study of disformal coupling. The calculations were conducted up to the first Post-Newtonian order. At the end, the leading order correction to Kepler's Hamiltonian resulting from disformal coupling were found to be $\propto \frac{\dot{r}^2}{r^4}$. This results in an alternative two-body dynamics, considered in the final part of this work. In particular, contrary to General Relativity with Schwarzschild's metric solution and the Nordström's theory with conformal coupling, the disformal coupling does not give rise to bounded circular or elliptical orbits. This different trajectory of motion is a consequence of the coupling in the metric which does not give any linear scalar field terms in the expansion.

Although this research explores gravitational effects in a "toy model" for a two-body problem that differs from their actual dynamics, best described by General Relativity, it contributes to our understanding of gravitational metric models. Such models have the potential to approximate complex GR calculations, contributing to more efficient research on gravity. On the other hand, the development of gravitational wave astronomy [15] enables to probe the modern theories of gravity and look for signs of any deviations from General Relativity [16]. Notably, the disformal coupling can affect stars in the vicinity of black holes or neutron stars in binary systems [12], making it potentially useful for further astrophysical applications. Moreover, the scalar field theories of gravity provide an interesting example of how the orbital motion could look in universes with alternative geometries.

Chapter 2

Principles of Classical and Relativistic Dynamics

2.1 Lagrangian and Hamiltonian Formalisms

The Lagrangian and Hamiltonian formalisms constitute a powerful formulation of classical mechanics. Furthermore, those formalisms had proven useful in the development of modern physical theories like General Relativity (GR), or Quantum Field Theory (QFT). This section presents an overview of the main concepts of Lagrangian and Hamiltonian mechanics, which can be found in multiple textbooks and lecture materials such as [17] or [18] . For more mathematical discussion of Hamiltonian Mechanics one can also refer to [19].

Lagrangian and Hamiltonian formalisms of classical mechanics are based on the methods of calculus of variations. Those techniques enable to find a function $y(x)$ that extremes a functional [17]:

$$J = \int_{x_1}^{x_2} f\{y(x), y'(x); x\} dx. \tag{2.1}$$

A dynamical system follows a path in a configuration space that minimizes the time integral of a quantity referred to as Lagrangian, which is know as the *Hamilton's Principle*. Therefore, in Lagrangian formalism, we are interested in the extreme value of the *action*, S, a functional expressed as:

$$S = \int_{t_1}^{t_2} \mathcal{L}\{q_i(t), \dot{q}_i(t); t\} dt, \tag{2.2}$$

where $\mathcal{L}$ denotes the Lagrangian depending on the generalized coordinates q_i (with $i = 1, 2, ...$ being an index), their time derivatives $\dot{q}_i$, and the time t itself [17]. Hamilton's Principle, expressed as $\delta S = 0$, is satisfied by the set of functions solving the *Euler-Lagrange Equations*[17]:

$$\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = 0, \qquad i = 1, 2, ... \tag{2.3}$$

This set of differential equations, together with constraints and initial conditions, completely describe the motion of a system.

In classical mechanics, the Lagrangian $\mathcal{L}$ is given by the difference of kinetic and potential energies of the system. In classical field theory, the Lagrangian $\mathcal{L}$ is a function of field φ (a quantity defined at every point in four-dimensional spacetime) and its time and spatial derivatives. The Lagrangian $\mathcal{L}$ governs the dynamics of the field itself and its interactions [20]. The action and Euler-Lagrange equations can then by generalized to:

$$S = \int \mathrm{d}^4 x \mathcal{L}, \tag{2.4}$$

and

$$\frac{\delta \mathcal{L}}{\delta \varphi} - \partial_\mu \left(\frac{\delta \mathcal{L}}{\delta(\partial_\mu \varphi)} \right) = 0, \tag{2.5}$$

where $\partial_\mu = (-\frac{1}{c}\frac{d}{dt}, \vec{\nabla})$ is the partial derivative operator, with $\mu = 0, 1, 2, 3$ being the index of the spacetime coordinate [20].

We note that the minus sign in front of the zeroth (time) component is an indicator of the mostly-positive signature metric convention, $(-, +, +, +)$. This convention has been utilized throughout the whole study. Moreover, in the context of the equation above, we also note the use of Einstein's notation convention, where $x_\mu y^\mu = \sum_\mu x_\mu y^\mu$. This notation has also been widely used throughout this paper.

Another closely related approach to classical mechanics is the Hamiltonian formalism. Hamiltonian is a quantity that can be expressed as [17]:

$$H = \sum_i \dot{q}_i p_i - \mathcal{L}, \tag{2.6}$$

where p_i is the conjugate momentum of the generalized coordinate q_i, that is given by[17]:

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}. \tag{2.7}$$

In classical mechanics, the Hamiltonian is often equal to the total energy of the system, i.e., the sum of kinetic and potential energies.

The set of equations of motion governing the dynamics of the system, know as the *canonical equations of motion*, is now given by [17]:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad -\dot{p}_i = \frac{\partial H}{\partial q_i}. \tag{2.8}$$

Similarly as before, the Hamiltonian formalism can also be generalized to classical field theory. Then, the momentum conjugate to the field $\varphi_a(x)$ is defined as [20]:

$$\pi^a(x) = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}_a(x)}, \tag{2.9}$$

where a is the index denoting a field. The canonical equations of motion are now given by[20]:

$$\dot{\varphi}_a = \frac{\partial H}{\partial \pi_a}, \qquad -\dot{\pi}_a = \frac{\partial H}{\partial \varphi_a}. \tag{2.10}$$

2.2 General Theory of Relativity

Einstein's Theory of General Relativity describes gravitation (one of the four fundamental interactions) in terms of an elegant mathematical structure - the differential geometry of curved spacetime. This one of the most influential physical theories of constitutes an indispensable part of modern theoretical physics [7]. This section presents an introduction to General Relativity, with a focus on the main concepts utilized in the derivation of the results of this thesis. More comprehensive overview of the subject has been covered in numerous textbooks and educational resources, such as [7] or [6].

2.2.1 Riemannian Geometry

General relativity is a theory of gravity based on the mathematical framework of differential geometry. A fundamental object of differential geometry is a *differentiable manifold*. Conceptually, a manifold can be thought of as an n-dimensional object which globally possesses a curvature, while locally appears to be flat. An example could be a sphere or a torus which both are curved objects, but their surfaces appear to be flat if zoomed in close enough. A formal introduction to the concept of differentiable manifolds and related definitions were included in Appendix A.

In particular, General Relativity is based on a subfield of differential geometry referred to as Riemannian geometry, whose central idea is the fact that manifolds admit Riemannian (or pseudo-Riemannian) *metrics*. That is, it is possible to define a way to evaluate a distance on a manifold, and an object referred to as metric specifics how to do that. Therefore, knowing the metric allows one to calculate the distances and motion in the curved spacetime, in relation to the strength of gravitational interactions. Conceptually, the metric can be understood as an object quantifying the spacetime curvature, which depends on mass and energy distribution [6]. Einstein's Theory of General Relativity provides the way to find such a metric based on the energy distribution.

The the (pseudo-)Riemannian metric is formally defined as follows.

Definition 2.2.1. ([6])

A **pseudo-Riemannian metric** g on a manifold M is a $(0,2)$ tensor field that is:

- Symmetric: $g(X,Y) = g(Y,X)$,
- Non-Degenerate: If for any $p \in M$, $g(X,Y)|_p = 0$ for all $Y \in T_pM$[1], then $w_p = 0$.

If g is also positive definite, then it is a **Riemannian metric** [21].

[1]The symbol T_pM indicates the tangent space of all tangent vectors at point p. Formal definitions of this and related concepts were presented in Appendix A.

With a choice of coordinates, the metric can be written as [6]:

$$g = g_{\mu\nu}(x) dx^\mu \otimes dx^\nu. \tag{2.11}$$

Here, the symbol $\otimes$ denotes a tensor product between the two 1-forms, and $g_{\mu\nu}$ denotes a an element of a symmetric matrix [6]. The metric g is often denoted as a line element ds^2, and abbreviated as [6]:

$$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu. \tag{2.12}$$

Moreover, the elements of the matrix can be extracted via evaluation of the metric on a pair of basis vectors $\mathbf{e}_i = \frac{\partial}{\partial x^i}$, $i = \mu, \nu$, of T_pM [6]:

$$g_{\mu\nu}(x) = g\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right). \tag{2.13}$$

A specific type of pseudo-Riemannian metric that is of interest in GR is the one that has one diagonal component with a minus sign (or can be transformed to such form). This is referred to as *Lorentzian geometry*. A well-known example of a Lorentzian manifold is $M = \mathbb{R}^4$ with the Minkowski metric $g = \eta_{\mu\nu}(x) dx^\mu \otimes dx^\nu$, where

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.14}$$

The (M, g) is a spacetime manifold of Special Relativity (SR), with $(x^0, x^1, x^2, x^3) = (ct, x, y, z)$ being the temporal and spatial components [21]. A number of negative entries is know as the *signature* of the metric, a concept that has already been referred to in Section 2.1.

A general Riemannian metric provides a way to measure a length of vector X, and the angle θ between any two vectors each point [6]:

$$|X| = \sqrt{g(X, X)}, \qquad g(X, Y) = |X||Y| \cos(\theta). \tag{2.15}$$

Moreover, g allows for measuring the distance between two points, p and q, along a curve on the manifold M. For a curve parametrized by $\sigma : [a, b] \to M$ with $\sigma(a) = p$ and $\sigma(b) = q$, the distance is given by [6]:

$$\text{distance} = \int_a^b dt \sqrt{g(X, X)|_{\sigma(t)}}, \tag{2.16}$$

where X is the tangent vector to the curve. For instance, for the curve with coordinates $x^\mu(t)$, the tangent vector is $X^\mu = \frac{dx^\mu}{dt}$, and the distance reads [6]:

$$\text{distance} = \int_a^b dt \sqrt{g_{\mu\nu}(x) \frac{dx^\mu}{dt} \frac{dx^\nu}{dt}}. \tag{2.17}$$

Additionally, we note that the distance is independent of the choice parametrization [6].

Considering now a Lorentzian four-dimensional manifold (manifold with one negative diagonal entry), we state that at any point p on M, it is always possible to find an orthonormal basis for T_pM, such that, locally, the metric looks like the Minkowski metric [6]:

$$g_{\mu\nu}|_p = \eta_{\mu\nu}. \tag{2.18}$$

This property is related to the *Einstein's Equivalence Principle* stating that: "In small enough regions of spacetime, the laws of physics reduce to those of Special Relativity; it is impossible to detect the existence of a gravitational field by means of local experiments" [7]. In other words, since the motion of freely falling (unaccelerated) particles is the same in a gravitational field and in a uniformly accelerated frames, gravity cannot be distinguished from acceleration. However, we emphasise that this is only the case in local inertial frames (small enough regions). In larger spacetime regions there will be inhomogeneities in the gravitational field leading to tidal forces, which can be detected [7]. This is related to the spacetime curvature, which cannot be detected on small enough regions, as the spacetime there is homeomorphic to the flat Minkowski spacetime (Equation 2.18).

Local resemblance to SR indicates that one can define a vector $X_p \in T_pM$, at any point $p \in M$ to be [6]:

- *timelike* if $g(X_p, X_p) < 0$,
- *null* if $g(X_p, X_p) = 0$,
- *spacelike* if $g(X_p, X_p) > 0$.

Then, we note that, analogously to SR, the timelike vectors are the ones inside the lightcone of point p on M, with the null vectors lying at the lightcone's boundary. Moreover, a curve on M is called *timelike* if its tangent vector is everywhere timelike. In this case, using the metric, one can measure a distance on a Lorentzian manifold in the following way [6]:

$$\tau = \int_a^b dt \sqrt{-g_{\mu\nu}(x) \frac{dx^\mu}{dt} \frac{dx^\nu}{dt}}, \tag{2.19}$$

where we denoted the parametrization and the endpoints of the curve analogously to Equation 2.17. The quantity τ is also referred to as the *proper time*. Moreover, τ also denotes the action for a point particle moving is spacetime with a metric $g_{\mu\nu}$ [6]. The Equation 2.19 or its equivalent action have been used through the further derivations of this paper.

Furthermore, let us note that to consider the action of the gravitational field itself, one should quantify its curvature, interpreted as a measure of deviations from flat Minkowski spacetime. The *Riemann tensor* provides a local description of the curvature of each point of the spacetime manifold. This is a $(1,3)$ tensor, that is calculated as follows [7]:

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}. \tag{2.20}$$

The Riemann tensor is expressed in terms of the *Christoffel symbols*, that are calculated from the metric as [7]:

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \right). \tag{2.21}$$

Moreover, the contraction of the Riemann tensor is know as the *Ricci tensor* [7]:

$$R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu}, \tag{2.22}$$

and the trace of the Ricci tensor is the *Ricci scalar* (or *curvature scalar*) [7]:

$$R = R^\mu{}_\mu = g^{\mu\nu} R_{\mu\nu}. \tag{2.23}$$

Let us emphasise that the action of gravity should be attributed to the curvature of spacetime itself, as it is a fundamental feature of a background on which all matter fields propagate. This refers to the Equivalence Principle arising from the idea that gravity is *universal* and affects all particles (forms of energy-momentum) in the same way [7]. Consequently, it should be possible to generalize the laws of physics into the curved-spacetime context. This is referred to as the *minimal-coupling principle*. The principle asserts that if a law of physics is valid in an inertial coordinates in flat spacetime, it remains true in curved spacetime when written in a coordinate-invariant (tensorial) form. Moreover, the Equivalence Principle attest that the interactions of matter fields to curvature are minimal. That is, they do not involve direct couplings to the Riemann tensor or its contractions [7]. This is indeed the case for all field theories of gravity considered in this study: General Relativity, Nordström's theory, and the DBI theory.

2.2.2 The Einstein-Hilbert Action

As discussed above, the information on the spacetime curvature is incorporated in the metric. Knowing the metric enables one to identify the gravitational field that governs the motion of bodies subjected to force of gravity. However, the metric is a dynamic quantity, varying for different points on the manifold. Therefore, it is of interest to specify the method to find the metric in terms of the spacetime coordinates.

Let us recall the metric can be identified with the gravitational field [6]. Therefore, the variational techniques from the classical field theory can be applied to find it. Indeed, the precise form of the spacetime metric can be found by varying action and solving the resulting Euler-Lagrange equations. The simplest action that one can consider for a gravitational field in a vacuum is the *Einstein-Hilbert action* given by [6]:

$$S_{EH} = \int \mathrm{d}^4 x \sqrt{g} \frac{c^3}{16\pi G} R(g), \tag{2.24}$$

where $g = -\det g_{\mu\nu}$, and $R(g)$ is the curvature scalar - the trace of the Ricci tensor, which on its own is a trace of the Riemann tensor. The minus sign comes from the negative determinant of metric on a Lorentzian manifold. The factor $\frac{c^3}{16\pi G}$, where c is the speed of light and G is the gravitational constant, is included for dimensional accuracy [6].

Noting that the Ricci scalar can be expressed as $R = g^{\mu\nu} R_{\mu\nu}$, a variation of the Einstein-Hilbert action gives [6]:

$$\delta S_{EH} = \frac{c^3}{16\pi G} \int \mathrm{d}^4 x \left((\delta\sqrt{-g}) g^{\mu\nu} R_{\mu\nu} + \sqrt{-g}(\delta g^{\mu\nu}) R_{\mu\nu} + \sqrt{-g} g^{\mu\nu} (\delta R_{\mu\nu}) \right), \tag{2.25}$$

which can be simplified to [6]:

$$\delta S_{EH} = \frac{c^3}{16\pi G} \int \mathrm{d}^4 x \sqrt{-g} \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) \delta g^{\mu\nu}. \tag{2.26}$$

Requiring action to be minimized, i.e., $\delta S_{EH} = 0$, we obtain the *Einstein field equations* in the absence of matter [6]:

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0, \tag{2.27}$$

where $G_{\mu\nu}$ is known as *the Einstein tensor*. Moreover, the contraction with inverse metric $g^{\mu\nu}$ gives:

$$R_{\mu\nu} g^{\mu\nu} - \frac{1}{2} R g_{\mu\nu} g^{\mu\nu} = R - \frac{1}{2} R = \frac{1}{2} R = 0 \implies R = 0, \tag{2.28}$$

which substituted back to Equation 2.27 gives a Ricci flat metric [6]:

$$R_{\mu\nu} = 0. \tag{2.29}$$

The solutions to Einstein field equations give the metric on the Lorentzian spacetime manifold. One of the simplest and most well-known solutions is the Schwarzschild's metric given by [6]:

$$ds^2 = -\left(1 - \frac{2MG}{rc^2}\right) dt^2 + \left(1 - \frac{2MG}{rc^2}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2\theta d\phi^2) \tag{2.30}$$

The coordinates r, θ, ϕ are the usual spherical coordinates with $\theta \in [0, \pi]$, $\phi \in [0, 2\pi)$ and M is a mass of a heavy body that is the source of gravitational field. Moreover, in the limit $r \to \infty$, we obtain a flat Minkowski metric (in spherical coordinates). Additionally,

$$g_{00} = -\left(1 + \frac{2\Phi}{c^2}\right) \quad \text{with} \quad \Phi(r) = -\frac{GM}{r}. \tag{2.31}$$

Therefore, we can identify $\Phi(r)$ as a classical Newtonian gravitational potential for an object of mass M [6].

Chapter 3

Classical Kepler Problem

Before addressing the main research questions of this study, let us consider the Kepler two-body problem in Newtonian mechanics. At first, the Kepler's Hamiltonian will be analysed, enabling to derive the equations of the orbital motion further on. This section discusses the well-know classical results, which can later on be compared with the relativistic Kepler problem, presented in Section 4.

3.1 Kepler's Hamiltonian

First, let us explicitly define the problem of our interest. The aim of this derivation is to establish what is the motion of a system consisting of two bodies affected by a force directed along the line connecting the centres of the two bodies (referred to as a *central force*) [17]. To describe the system, let us define the following variables:

- m_1, m_2 are the masses of the two bodies,
- $\mathbf{r}_1$, $\mathbf{r}_2$ denote the three-dimensional positions vectors of the two masses.

Let us specify that we consider a coordinate system, where the origin is given by the centre of mass of the two-body system. Now, denoting $r = |\mathbf{r}_1 - \mathbf{r}_2|$, the Lagrangian for the system is given by [17]:

$$\mathcal{L} = \frac{1}{2}m_1|\dot{\mathbf{r}}_1|^2 + \frac{1}{2}m_2|\dot{\mathbf{r}}_2|^2 - U(r), \tag{3.1}$$

where $U(r)$ is the potential energy. We specify that the Newton's gravitational potential energy is given by:

$$U(r) = -\frac{m_1 m_2 G}{r}, \tag{3.2}$$

where G is the gravitational constant. Considering $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, the Lagrangian can also be written as [18]:

$$\mathcal{L} = \frac{1}{2}\mu|\dot{\mathbf{r}}|^2 + \frac{m_1 m_2 G}{r}, \tag{3.3}$$

where is μ the *reduced mass* given by:

$$\mu = \frac{m_1 m_2}{m_1 + m_2}. \tag{3.4}$$

Using the methods outlined in Section 2.1, the Lagrangian form Equation 3.1 can then be transformed into the following Hamiltonian:

$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} - \frac{m_1 m_2 G}{r}, \tag{3.5}$$

where p_1, p_2 are the conjugate momenta of the coordinates $\mathbf{r}_1$, $\mathbf{r}_2$. Let us now denote the total momentum of the system as $P = p_1 + p_2$. Considering that the total momentum should be conserved, $P = c \in \mathbb{R}^3$, let us choose such a system, that $P = 0 \in \mathbb{R}^3$ [22]. Then, $p_1 = -p_2 \equiv p$, and we write

$$H = \frac{p^2}{2\mu} - \frac{m_1 m_2 G}{r}. \tag{3.6}$$

Now, in the limit where $m_2 >> m_1$, $\mu \approx m_1$, and denoting $m_2 = M$ and $m_1 = m$, we find that the Kepler Hamiltonian in the one centre system is given by:

$$H = \frac{p^2}{2m} - \frac{mMG}{r}. \tag{3.7}$$

3.2 Dynamics and Orbital Motion

With the Lagrangian (or Hamiltonian) specified, it is now possible to consider the resulting dynamics of the two-body system. Let us again consider the Lagrangian from Equation 3.9:

$$\mathcal{L} = \frac{1}{2}\mu|\dot{\mathbf{r}}|^2 - U(r). \tag{3.8}$$

Considering $\dot{\mathbf{r}}$ as the velocity vector, in the polar coordinates (r, θ), the Lagrangian above can be expressed as [17]:

$$\mathcal{L} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}) - U(r). \tag{3.9}$$

The Euler-Lagrange equation for θ now gives us the conserved quantity of angular momentum [17]:

$$l = \mu r^2 \dot{\theta}, \tag{3.10}$$

and the equation for r gives [17]:

$$\mu(\ddot{r} - r\dot{\theta}^2) = -\frac{dU(r)}{dr}. \tag{3.11}$$

Moreover, since the Lagrangian possesses a time translation symmetry (due to lack of explicit time dependence), another conserved quantity is the total energy of the system [17]:

$$E = T + U = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}) + U(r). \tag{3.12}$$

Finally, solving the Equation 3.11 for $U(r) = -\frac{k}{r}$, where we denoted $k = mMG$, gives the solution [17]:

$$\frac{\alpha}{r} = 1 + \varepsilon\cos\theta, \tag{3.13}$$

with the constants defined as [17]:

$$\alpha \equiv \frac{\ell^2}{\mu k}, \tag{3.14}$$

$$\varepsilon \equiv \sqrt{1 + \frac{2E\ell^2}{\mu k^2}}. \tag{3.15}$$

We consider that the Equation 3.13 is an equation of a conic section with one focus at the origin. The *eccentricity*, ε, is a measure of how much a conic section deviates from the ellipse. The quantity 2α is the *latus rectum* of the orbit - a line segment that passes through the focus of a conic section, perpendicular to its major axis [17]. Different types of conic section depend on the value of eccentricity and have been presented in Figure 3.1. Therefore, we conclude that, in the classical Kepler two-body problem, the resultant trajectories of motion have a shape of circular, elliptical, parabolic, or hyperbolic orbits.

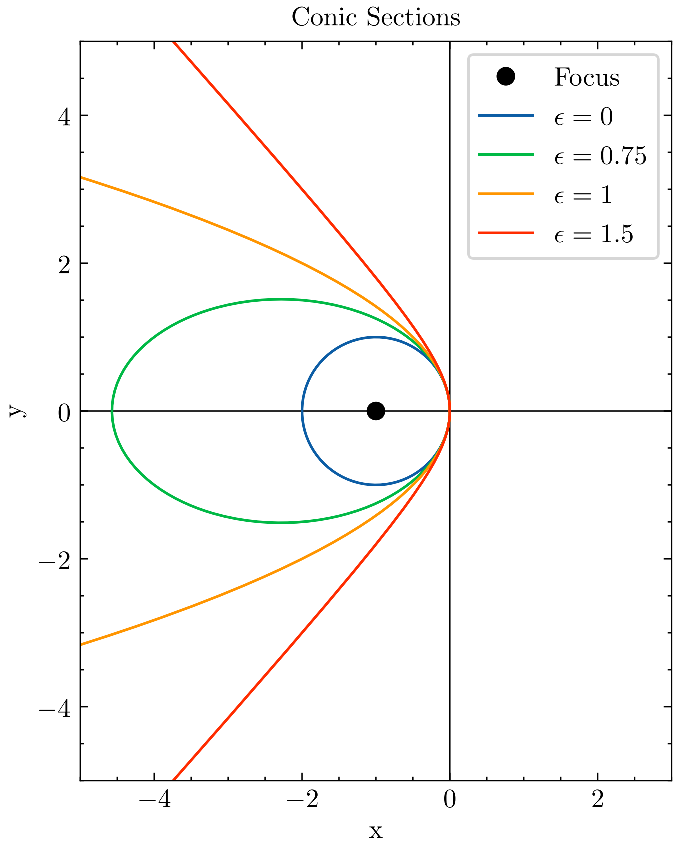


Figure 3.1: Examples of conic sections dependent on eccentricity ϵ, with $\epsilon = 0$ for a circle (in blue); $0 < \epsilon < 1$ for an ellipse (in green); $\epsilon = 1$ for a parabola (in yellow); and $\epsilon > 1$ for a hyperbola (in red).

Chapter 4

Relativistic Kepler Problems

The classical solution to the two-body problem does not give accurate trajectory for orbits of fast moving celestial bodies, such as Mercury. It was Einstein's Theory of General Relativity that gave a correct result of the precessing orbit of this planet [23]. However, this result was not calculated exactly. Instead, Einstein conducted what we now call the *Post-Newtonian expansion* and assumed a limit in which mass of the Sun is infinite compared to the mass of Mercury. Post-Newtonian (PN) expansion is an expansion in terms of $\frac{1}{c}$, where c is the speed of light, of Lagrangian, Hamiltonian, or an observable (e.g. as scattering angle). The formal definition of the Post-Newtonian expansion can be considered as follows.

Definition 4.0.1. ([3])

A Hamiltonian function $B(\epsilon; q, p)$ depending on a small parameter $\epsilon > 0$ is in **post-Newtonian expansion to Nth order** if it is of the form

$$B(\epsilon; q, p) = \sum_{j=0}^{N} \epsilon^j B_j(q, p) + \mathcal{O}(\epsilon^{N+1}), \tag{4.1}$$

for some regular enough Hamiltonian functions B_j. Here, q and p denote the positions and momenta.

We denote $\epsilon = \frac{1}{c^2}$. Consideration of the speed of light c as large with $\epsilon = 0$ yields the classical limit of General Relativity [3]. Nevertheless, the choice of the constant ϵ whose exponent j determines the order of the term is to some extend arbitrary. In this study, we define the PN order as the order of $\frac{1}{c^2}$. While this is a standard definition in General Relativity and Nordström's theory, it may be ambiguous when applied to DBI theory.

The limit in which the mass of one body is much larger than the mass of the other is known as the *one-centre system* [22]. Physical examples of relativistic one-centre systems include the relativistic Coulomb problem (spin-1 theory) and the relativistic gravitational case (spin-2 theory). The remaining of three minimally coupled theories (meaning they couple only to the lowest moments: charge or mass) is a dilaton gravity. This is a spin-0 field that couples through a conformal factor in the metric [22].

In this chapter, we present three approaches to the Kepler problem that consider the relativistic effects. Section 4.1 presents an overview of the main issues regarding the relativistic two-body problems. Subsequently, the relativistic corrections to the Hamiltonian are derived for three cases: the Schwarzschild's metric, the conformal coupling, and, finally, disformal coupling.

4.1 General Approach

In order to derive relativistic corrections to the Hamiltonian, one should consider how to obtain the Hamiltonian in the first place. As presented in Section 2.1, in classical mechanics, Hamiltonian can be derived from Lagrangian that minimizes the action. However, in the relativistic case, the action has a more complicated form, taking into account a general varying metric instead of the flat Minkowski metric. Using the principles of field theory, Lagrangian (and consequently Hamiltonian) no longer has a straightforward form of the difference of kinetic and potential energies. Instead, it is dependent on the symmetry that the system and the underlying fields posses. Furthermore, one needs to consider not only the two bodies, but also the interactions with the gravitational fields. In this section, we present a general approach to relativistic Lagrangian and Hamiltonian formalisms and how to apply it in the case of the two-body problem. Later on, this will enable us to derive the Hamiltonian corrections using the Post-Newtonian expansion for specific metrics.

Let us first consider the Hamiltonian for a relativistic free particle, H_{fp}, interpreted as a total relativistic energy [3]:

$$H_{\mathrm{fp}} = \sqrt{m^2c^4 + p^2c^2} = mc^2\sqrt{1 + \frac{p^2}{m^2c^c}} = mc^2\left(1 + \frac{p^2}{2m^2c^2} - \frac{p^4}{8m^4c^4} + \mathcal{O}\left(\frac{p^6}{m^6c^6}\right)\right), \tag{4.2}$$

where assuming $p << mc$ and using first terms of the Taylor series we constructed a PN expansion for H_{fp}. In the free particle case, the only concern is the kinetic energy, since any interactions are absent. In the Kepler problem, we are dealing with two massive particles, whose kinetic energy is now given by [22]:

$$\begin{aligned} K_{\mathrm{rel}} &= m_1c^2\sqrt{1 + \frac{p_1^2}{m_1^2c^c}} + m_2c^2\sqrt{1 + \frac{p_2^2}{m_2^2c^c}} \\ &= (m_1 + m_2)c^2 + \frac{p^2}{2\mu} - \frac{m_1^3 + m_2^3}{8(m_1m_2)^3}\frac{p^4}{c^2} + \ldots \end{aligned} \tag{4.3}$$

Here, we defined $\mu = \frac{m_1m_2}{m_1+m_2}$ to be the reduced mass and the relative momentum $p = p_1 = -p_2$, as in the Kepler problem reduced to one centre system.

Let us notice that the reduction of the system to a one-centre problem is not longer straightforward for the higher order corrections of the PN expansion. Moreover, now we should also consider the interaction potential of the two bodies. Because of the finite speed of light, the potential terms for each body will depend on the many past positions of the other body, significantly complicating the derivation of the expression for the Hamiltonian [22].

For comparison, let us consider a relativistic two body problem in the context of field theories instead. In this case, the total action of the system could be expressed as:

$$S = S_{\mathrm{fields}} + S_{\mathrm{matter}} \tag{4.4}$$

For instance, for GR, assuming the limit where both bodies are point particles and disregarding spin, the total action will have the form [22, 2]:

$$S = \int \mathrm{d}^4x\sqrt{g}\frac{c^3}{16\pi G}R(g) - \sum_a m_a c\int \mathrm{d}\tau_a\sqrt{-g_{\mu\nu}(x_a^\lambda)\dot{x}_a^\mu\dot{x}_a^\nu}, \tag{4.5}$$

where the used symbols indicate [22]:

- $a = 1, 2$ - the index of each particle,
- $g_{\mu\nu}(x_a^\lambda)$ - metric tensor depending on space-time coordinates $x^\lambda, \lambda = 0, 1, 2, 3$; $g = -\det g_{\mu\nu}$,
- $R(g)$ - the curvature scalar (trace of the Ricci tensor, which on its own is a trace of Riemann tensor (contracted Riemann tensor)).
- τ_a - the world line parameter of the world line of body a (proper time).

In the general relativistic action for the two-body problem given in Equation 4.5, one can distinguish two parts. The first integral corresponding to the term S_{fields} is the Einstein-Hilbert action, previously presented in Section 2.2.2. The Einstein-Hilbert action describes the dynamics of gravitational fields and is fully dependent on the metric. The second part (sum over two massive particles) is the matter action, S_{matter}, that minimizes the trajectory of each particle in curved space-time, similarly as given in Equation 2.19. For each particle, it is an integral of a form determined by the metric. It is parametrized by the world line of each particle (although parametrizations by other parameters are also possible). This term is analogous to the kinetic energy term of each individual particle.

Let us note that by inclusion of the Einstein-Hilbert action, we found a way to incorporate the interaction terms into the total action for the system. In fact the two massive particles interact with each other via the underlying gravitational field, and the field determining the space-time curvature influences their trajectory.

To work out the Hamiltonian directly from Equation 4.5, we need to express the action in Equation 4.5 as one integral, and find a minimizing Lagrangian. However, in the general relativistic case, varying the action becomes complicated, as the metric is a dynamic object changing as the particles move through space-time. Moreover, each particle's contribution is parametrized by particle's own world line, making it difficult to express the two contributions using the same coordinates. Although other parametrizations can be chosen, the calculation still remains non-trivial.

Fortunately, in case of the two-body problem, one can consider an exceptional situation when one body is infinitely heavier than the other: $m_2 >> m_1$. This is an analogous approach to the one applied for Equation

3.7 in Section 3.1. Despite being an idealized limiting case, this is a widely-used approximation for multiple astrophysical systems, where a mass of a central body (star or a black hole) is much larger than masses of their satellites. As a consequence, the metric the heavy body feels is independent of the coordinates of the lighter body - the mass of m_1 has a negligible effect on the spacetime curvature. In this case, the velocity of the heavy body is zero in the centre of mass frame. Consequently, the following integral becomes [22]:

$$m_2 \int \mathrm{d}\tau_2 \sqrt{-g_{\mu\nu}(x_2^\lambda)\dot{x}_2^\mu \dot{x}_2^\nu} = m_2. \tag{4.6}$$

The expression under the square root becomes 1, as we parametrized the integral by the proper time of m_2. This is because, in $\dot{x}_2^\mu$, only the derivative of a temporal component remains (velocity components are zero), which is a constant, and the equation can be parametrized suitably to give the answer of 1.

Let us specify the advantages of such one-centre approximation. One can now assume that the heavy body of mass m_2 is stationary and generates a constant gravitational background field in which the lighter body moves. Therefore, one can solve for the specific metric $g_{\mu\nu}$ describing this field by minimizing the Einstein-Hilbert action. The metric then quantifies the curvature of the space in which the lighter particle moves. Consequently, it is possible to determine the Lagrangian and Hamiltonian from the matter action S_{m_1} (where we consider only the motion of the mass m_1). Because of the complicated form of the expression and the metric, the Hamiltonian cannot be solved for exactly, but one can apply the Post-Newtonian expansion given by Definition 4.0.1, to find the first relativistic terms.

The approach described above is indeed the procedure that will be applied not only for GR, but also for alternative theories of gravity: dilaton field with the conformal coupling in the metric, and the DBI theory with disformal coupling. However, we note that for the latter two, another action has to be chosen than the Einstein-Hilbert action. This is due to different symmetries of the dilaton and DBI theories, discussed more in detail in Sections 4.3 and 4.4.

Additionally, we note that one can utilize scattering amplitudes and the framework of Effective Field Theory (EFT) to constrain the possible shapes the Kepler Hamiltonian can take. Indeed, the higher order Post-Newtonian correction terms have been calculated using EFT methods [24, 25]. Following the Definition 4.0.1 of the PN expansion, we obtain an order counting system for the terms occurring in relativistic Hamiltonian. Such Hamiltonian, based on the general form that the EFT provides, is given by:

$$H_{\text{rel}} = \sum_{j=0}^{\infty} \epsilon^j \Lambda_j(\alpha), \qquad \Lambda_j(\alpha) = \sum_{l+m+n=j} \alpha_{l,m,n} \frac{(p^2)^l (p_r^2)^n}{r^m}. \tag{4.7}$$

We note that $\epsilon = \frac{1}{c^2}$, $(l, m, n) \in \mathbb{N}^3$, $\sqrt{q^2} = r$, $p_r = \frac{q \cdot p}{r}$ is the radial momentum, and the Hamiltonian is written in the general gauge [22]. A particular form of the Hamiltonian, i.e., the power exponents: l, m, n, depend on a specific gauge choice. More detailed consideration of possible gauges in the relativistic Kepler problem has been considered in [16].

Furthermore, the number j gives us the PN orders of $\Lambda_j(\alpha)$. In this context, it is possible to note that the factors $\frac{1}{r}$, p^2, and p_r^2 contribute on equal footing to the PN order of a term. This is related to the virial theorem [22] that relates the average kinetic and potential energies of the system. The theorem provides an equivalence of $\frac{p^2}{m^2c^2} = \frac{v^2}{c^2} \sim \frac{GM}{rc^2}$, where m, M denote the lighter and heavier masses subsequently. Therefore, the order of a term in Post-Newtonian expansion is often determined by the sum of the exponents of $\frac{1}{r}$, p^2, and p_r^2, as in the Equation 4.7 above where $l + m + n = j$. However, while this definition of the PN order matches the one following the exponent of $\frac{1}{c^2}$ for GR and conformal coupling, it becomes more ambiguous for the disformal coupling. This issue is discussed to further extent in Section 4.4.1. For consistency, we define the PN order of a term in the Hamiltonian expansion as the power of $\frac{1}{c^2}$ factor in front of the terms, following the Definition 4.0.1.

While the EFT methods constitute a useful alternative calculation framework, they were not the centrepiece of this project. Nevertheless, they allow for useful comparison and verification of the obtained results, as discussed in the Section 4.4.3.

4.2 Schwarzschild's metric

As discussed in Section 2.2.2, by requiring the Einstein-Hilbert action to be minimized, i.e., $\delta S_{EH} = 0$, the Einstein field equations can be derived. One of their simplest and most well-known solutions is the Schwarzschild's metric given by [6]:

$$ds^2 = -\left(1 - \frac{2MG}{rc^2}\right) dt^2 + \left(1 - \frac{2MG}{rc^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{4.8}$$

The coordinates r, θ, ϕ are the usual spherical coordinates with $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi)$. We also note that in our case $M = m_2$.

This metric is a solution for the Kepler problem in one centre approximation, and is often used to describe the gravitational field of massive objects such as stars or black holes [6]. Therefore, it constitutes a suitable choice for the derivation of Hamiltonian corrections.

4.2.1 Hamiltonian Corrections

Knowing the metric, we can now use it in the equation for the particle action:

$$S_{\text{matter}} = -\sum_a m_a c \int \mathrm{d}\tau_a \sqrt{-g_{\mu\nu}(x_a^\lambda)\dot{x}_a^\mu \dot{x}_a^\nu} = -m_1 c \int \mathrm{d}\tau_1 \sqrt{-g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu} - m_2 c, \tag{4.9}$$

where we denoted $x_1 := x$, and we used the result from Equation 4.6. We also note that the $-m_2 c$ offset may now be ignored.

To indicate the difference in masses, let us denote $m_1 = m$ and $m_2 = M$. Therefore, we take the Lagrangian:

$$\mathcal{L} = -mc\sqrt{-g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu}. \tag{4.10}$$

Taking $g_{\mu\nu}$ to be the Schwarzschild's metric, and assuming a planar motion in the equatorial plane $\theta = \frac{\pi}{2}$, such that $\dot{\theta} = 0$, we obtain:

$$\mathcal{L} = -mc\sqrt{-\left[-\left(1 - \frac{2MG}{rc^2}\right)c^2 + \left(1 - \frac{2MG}{rc^2}\right)^{-1}\dot{r}^2 + r^2\dot{\phi}^2\right]}. \tag{4.11}$$

We first consider that assuming a weak field limit of $\frac{2MG}{r} << c^2$, we can expand the radial component of the metric as follows:

$$\left(1 - \frac{2MG}{rc^2}\right)^{-1} = \sum_{k=0}^{\infty}\left(\frac{2MG}{rc^2}\right)^k. \tag{4.12}$$

Taking the two terms of the lowest order, we obtain the following expression for $\mathcal{L}$:

$$\begin{aligned}\mathcal{L} &= -mc^2\sqrt{1 - \frac{2MG}{c^2 r} - \frac{\dot{r}^2 + r^2\dot{\phi}^2}{c^2} - \frac{2MG\dot{r}^2}{rc^4}} \\ &= -mc^2\sqrt{1 - \frac{1}{c^2}\left[\frac{2MG}{r} + \dot{r}^2 + r^2\dot{\phi}^2 + \frac{2MG\dot{r}^2}{rc^2}\right]}.\end{aligned} \tag{4.13}$$

Now, considering the term in bracket as much smaller than c^2, in the weak field and low velocity approximations of Post-Newtonian expansion, we expand the square root as:

$$\mathcal{L} = -mc^2 + \frac{mMG}{r} + \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{mMG\dot{r}^2}{rc^2} + \frac{m(\dot{r}^2 + r^2\dot{\phi}^2)^2}{8c^2} + \frac{m(\dot{r}^2 + r^2\dot{\phi}^2)MG}{2rc^2} + \frac{M^2G^2m}{2r^2}. \tag{4.14}$$

We now consider that the conjugate momenta are:

$$p_r = \frac{\delta\mathcal{L}}{\delta\dot{r}} = m\dot{r}\left(1 + \frac{3GM}{rc^2} + \frac{(\dot{r}^2 + r^2\dot{\phi}^2)}{2c^2}\right), \tag{4.15}$$

$$p_\phi = \frac{\delta\mathcal{L}}{\delta\dot{\phi}} = mr^2\dot{\phi}\left(1 + \frac{GM}{rc^2} + \frac{(\dot{r}^2 + r^2\dot{\phi}^2)}{2c^2}\right), \tag{4.16}$$

and recall that the Hamiltonian can be obtained via:

$$H = \sum_i \dot{q}_i p_i - \mathcal{L}. \tag{4.17}$$

Plugging in the results from above and considering the expansion up to 1PN order, we obtain the following Hamiltonian [2, 4]:

$$H = mc^2 + \frac{p^2}{2m} - \frac{mMG}{r} + \frac{1}{c^2}\left[-\frac{1}{8}\frac{p^4}{m^3} - \frac{3}{2}\frac{GMp^2}{rm} + \frac{G^2M^2m}{2r^2}\right]. \tag{4.18}$$

where we note that $p^2 = p_r^2 + \frac{p_\phi^2}{r^2}$. The 1PN order general relativistic corrections to the Kepler problem therefore are given by [2, 4]:

$$H_{1\text{PN}} = \frac{1}{c^2}\left[-\frac{1}{8}\frac{p^4}{m^3} - \frac{3}{2}\frac{GMp^2}{rm} + \frac{G^2M^2m}{2r^2}\right]. \tag{4.19}$$

4.2.2 Two-body Dynamics

To determine what the impact of the Hamiltonian corrections on the motion in the two body system is, let us first consider that instated of the action:

$$S_{\text{matter}} = -mc\int \mathrm{d}\tau \sqrt{-g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu}, \tag{4.20}$$

one can take [6]:

$$S_{\text{matter}} = \int \mathrm{d}\tau \frac{m}{2} g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu. \tag{4.21}$$

This simplification comes from parametrization of the integral with the proper time. Indeed, the proper time $\tau(\sigma)$ (where σ is the world line parameter) is defined by the following [6]:

$$c\tau(\sigma) = \int_0^\sigma d\sigma' L(\sigma') = \int_0^\sigma d\sigma' \sqrt{-g_{\mu\nu}(x)\frac{dx^\mu}{d\sigma'}\frac{dx^\nu}{d\sigma'}}. \tag{4.22}$$

Therefore,

$$c\frac{d\tau}{d\sigma} = L(\sigma), \tag{4.23}$$

and using the parametrization by τ:

$$L(\tau) = \sqrt{-g_{\mu\nu}(x)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}} = \sqrt{-g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu} = c. \tag{4.24}$$

If we will now a constraint that:

$$g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu = -c^2, \tag{4.25}$$

then we can now consider the simplified action S_{matter}. Therefore, we note that the particle's action is:

$$S_{\text{matter}} = \int \mathrm{d}\tau \frac{m}{2} g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu, \tag{4.26}$$

where we added the factor $\frac{1}{2}$ that to regain the classical Kepler Hamiltonian in the limiting case. Nevertheless, it should result with the same equations of motion, as we recall that action is invariant under rescaling. Therefore, the Lagrangian is given by:

$$\mathcal{L} = \frac{m}{2} g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu. \tag{4.27}$$

The Lagrangian is also invariant under addition of a constant. Thus, we can add a term $-\frac{mc^2}{2}$ to later on obtain a rest mass energy term in the particle's Lagrangian:

$$\mathcal{L} = \frac{m}{2} g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu - \frac{mc^2}{2}. \tag{4.28}$$

With the final form of the Lagrangian of our interest, we can substitute the Schwarzschild's metric in the equation above. Additionally, we can make an assumption of a planar motion in the equatorial plane $\theta = \frac{\pi}{2}$, so $\dot{\theta} = 0$. This gives:

$$\mathcal{L} = \frac{m}{2}\left[-\left(1-\frac{2MG}{rc^2}\right)c^2 + \left(1-\frac{2MG}{rc^2}\right)^{-1}\dot{r}^2 + r^2\dot{\phi}^2\right] - \frac{mc^2}{2}. \tag{4.29}$$

It follows that the solution to the equations of motion resulting from the Lagrangian given in the Equation 4.29 above slightly differs compared to the one presented in Section 3.2. While the closed orbits still resemble the ellipses, the position of perihelion (point on the orbit closest to the central heavy mass) shifts in time. The angular difference between two consecutive points of the closest approach is given by [26]:

$$\Delta\phi = 6\pi\frac{GM}{c^2a(1-\epsilon^2)}, \tag{4.30}$$

where a is the semi-major axis of the ellipse and ϵ is its eccentricity. For instance, for Mercury, the perihelion advance is $\Delta\phi = 0.103''$ per revolution, giving the shift of $43''$ per century [26]. An illustrative sketch of the perihelion advance of a planet orbiting the Sun was presented in Figure 4.1.

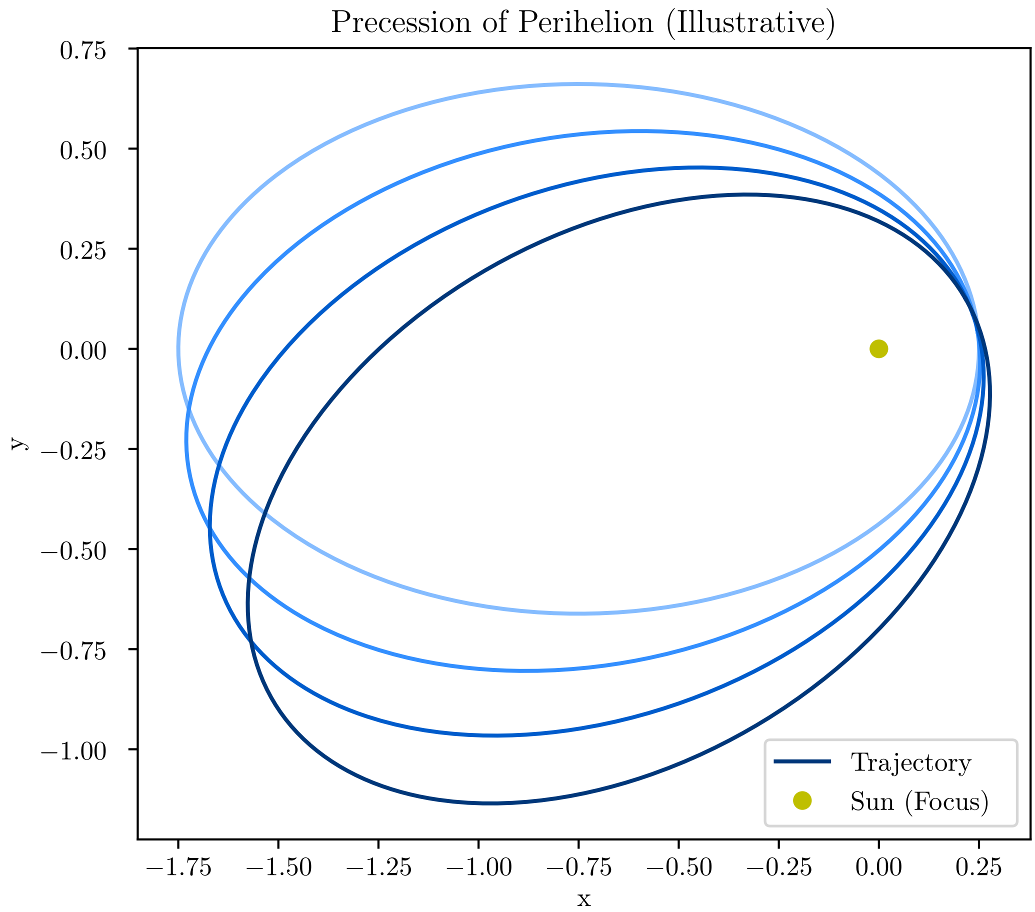


Figure 4.1: Illustrative presentation of the perihelion advance of a planet orbiting the Sun located at the focus of an ellipse. The plot shows four consecutive revolutions of a planet around the Sun, which represents the central heavy mass sourcing the gravitational field. The trajectory was plotted in increasingly darker shades of blue, starting from the lightest ellipse.

4.3 Conformal Coupling

We will now conduct a similar derivation as in the previous Section, but for the so-called dilaton field. In this case, a particle of mass $m_1 = m$ moves in spacetime whose curvature is determined by a background scalar field that changes (dilates) as a function of position [22]. Therefore, we consider that the metric on the spacetime manifold is flat, except for a rescaling given by the exponential of the scalar field [22].

The conformally coupled metric is given by:

$$g_{\mu\nu} = e^{2a\varphi}\eta_{\mu\nu}, \tag{4.31}$$

where a is a constant and φ is the background dilaton field. Here, the Kepler problem is considered in the same set-up as in the previous section. Thus, we take the following Lagrangian for a particle moving in dilaton field [22]:

$$\mathcal{L} = \frac{m}{2}e^{2a\varphi}\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu - \frac{mc^2}{2}, \tag{4.32}$$

where we use of the flat Minkowski metric with signature $(-+++)$.

However, to fully evaluate the Lagrangian above, a precise form of the metric $g_{\mu\nu}$ has to be established. Therefore, one has to solve for the scalar field φ that quantifies the conformal coupling in the metric. However, for conformal coupling (dilaton field), one needs to consider an alternative field Lagrangian than the one following from Einstein-Hilbert action, to account for different structural symmetry of the Nordström's scalar theory of gravity. In case of GR, the Lagrangian $\sqrt{-g}R$ possesses the diffeomorphisms invariance [14], while the dilaton field showcases a four-dimensional Poincaré symmetry. Additionally, the DBI theory of conformal coupling considered in Section 4.4 has a higher dimensional Poincaré symmetry (5-dimensional) [14].

The aforementioned field Lagrangian for the dilaton filed is given by[22]:

$$\mathcal{L}_{\text{field}} = -2(\partial\varphi)^2. \tag{4.33}$$

This leads to a vacuum equation where we pick out the solution corresponding to a single charge at the origin:

$$\varphi = -\frac{aM}{r}, \tag{4.34}$$

where M is the mass. This can be derived as follows. We recall that the general Euler-Lagrange equations read [20]:

$$\partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\varphi)}\right) - \frac{\delta\mathcal{L}}{\delta\varphi} = 0. \tag{4.35}$$

Considering the notation $(\partial\varphi)^2 = \partial_\mu\varphi\partial^\mu\varphi$, and substituting $\mathcal{L}_{\text{field}}$ into the equation above, we derive that:

$$-4\partial_\mu\partial^\mu\varphi = 0 \implies \Box\varphi = 0, \tag{4.36}$$

where $\Box$ represents the d'Alembertian operator.

Now, let us recall that in the one centre problem, it is assumed that the gravitational field is generated by a massive body $m_2 = M$, which is effectively stationary, and a smaller mass $m_1 = m$ is moving through the field, without influencing the spacetime curvature (its own gravitational field is negligible). Therefore, we can assume that the background scalar field φ is constant in time. Moreover, the masses can be assumed to be perfectly symmetric, which gives rise to a spherical symmetric scalar field φ (generated by M), meaning that it showcases a radial dependence only. Therefore, one obtains the following differential equation:

$$\begin{aligned}
&\nabla^2\varphi = \frac{1}{r^2}\frac{\delta}{\delta r}\left(r^2\frac{\delta\varphi}{\delta r}\right) = 0 \implies \\
&r^2\frac{\delta\varphi}{\delta r} = a \implies \frac{\delta\varphi}{\delta r} = \frac{a}{r^2} \implies \\
&\varphi = -\frac{a}{r}.
\end{aligned} \tag{4.37}$$

where a is a numerical constant. For dimensional consistency, we can consider that:

$$\varphi = -\frac{aM}{r}, \tag{4.38}$$

where M is the central mass.

4.3.1 Hamiltonian Corrections

Plugging in the scalar field obtained above, we obtain the following Lagrangian (given in spherical coordinates):

$$\mathcal{L} = \frac{m}{2} e^{\frac{-2a^2M}{r}} (-c^2 + \dot{r}^2 + r^2\dot{\phi}^2) - \frac{mc^2}{2}, \tag{4.39}$$

where the coordinates r, θ, ϕ are the usual spherical coordinates with $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi)$, and we made the assumption of a planar motion in the equatorial plane $\theta = \frac{\pi}{2}$, such that $\dot{\theta} = 0$.

We note that the associated conjugate momenta are:

$$p_r = m\dot{r}e^{\frac{-2a^2M}{r}}, \tag{4.40}$$

$$p_\phi = mr^2\dot{\phi}e^{\frac{-2a^2M}{r}}. \tag{4.41}$$

By Legendre transformation, we obtain the following Hamiltonian:

$$\begin{aligned} H &= m\dot{r}^2 e^{\frac{-2a^2M}{r}} + mr^2\dot{\phi}^2 e^{\frac{-2a^2M}{r}} - \frac{m}{2}e^{\frac{-2a^2M}{r}}(-c^2 + \dot{r}^2 + r^2\dot{\phi}) + \frac{mc^2}{2} \\ &= \frac{1}{2}\left(m\dot{r}^2 e^{\frac{-2a^2M}{r}} + mr^2\dot{\phi}^2 e^{\frac{-2a^2M}{r}}\right) + \frac{mc^2}{2}\left(1 + e^{\frac{-2a^2M}{r}}\right) \\ &= \frac{1}{2}\left(p_r\dot{r} + p_\phi\dot{\phi}\right) + \frac{mc^2}{2}\left(1 + e^{\frac{-2a^2M}{r}}\right). \end{aligned} \tag{4.42}$$

Furthermore, the calculation of conjugate momenta gives:

$$\dot{r} = \frac{p_r}{m}e^{\frac{2a^2M}{r}}, \tag{4.43}$$

$$\dot{\phi} = \frac{p_\phi}{mr^2}e^{\frac{2a^2M}{r}}, \tag{4.44}$$

which results in the following form of the Hamiltonian:

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\phi^2}{r^2}\right) \cdot e^{\frac{2a^2M}{r}} + \frac{mc^2}{2}\left(1 + e^{\frac{-2a^2M}{r}}\right). \tag{4.45}$$

Considering now the expansion with $2a^2M << r$, we obtain:

$$H = \frac{p^2}{2m}\left(1 + \frac{2a^2M}{r} + \frac{2a^4M^2}{r^2} + \frac{4}{3}\frac{a^6M^3}{r^3} + ...\right) + \frac{mc^2}{2}\left(1 + 1 - \frac{2a^2M}{r} + \frac{2a^4M^2}{r^2} - \frac{4}{3}\frac{a^6M^3}{r^3} + ...\right). \tag{4.46}$$

Considering that a is a free parameter, if we set $a = \frac{\sqrt{G}}{c}$, in the first order we obtain the following post-Newtonian expansion of the Kepler Hamiltonian (up to first order):

$$H = mc^2 + \frac{p^2}{2m} - \frac{mMG}{r} + \frac{1}{c^2}\left[\frac{GMp^2}{mr} + \frac{mG^2M^2}{r^2}\right]. \tag{4.47}$$

We note the opposite sign that we obtained for the 1PN correction with p^2 leading term compared to the Schwarzschild's metric derivation. This suggests the opposite direction of the perihelion precession, which can derived from Hamilton's equations of motion. Such a result is in agreement with historical derivations [22] and is discussed more in detail in the following section.

4.3.2 Two-body Dynamics and Disagreement with Observations

Similarly to the case of Schwarzschild's metric, let us know consider what is the effect of 1PN corrections to the Hamiltonian on the dynamics of the two-body problem. We recall the Lagrangian for the motion of a particle in a gravitational field is given by:

$$\mathcal{L} = \frac{m}{2}e^{2a\varphi}\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu - \frac{mc^2}{2}, \tag{4.48}$$

with the scalar field φ:

$$\varphi = -\frac{aM}{r}. \tag{4.49}$$

One can derive that the equations of motion resulting from this Lagrangian lead to a similar result as the Schwarzschild's metric in the case of General Relativity. The 0PN Hamiltonian is the same as in classical mechanics. Therefore, the conformal coupling should induce a small correction to the elliptical orbit.

Indeed, the Nordström's scalar theory of gravity also results in a perihelion precession. However, the dilaton field results in an opposite shift compared to GR, which can be expressed as [9]:

$$\Delta\phi = -\frac{1+b}{6}\Delta\phi_{GR}, \tag{4.50}$$

where b is a constant relating to the free parameter of the scalar field, and $\Delta\phi_{GR}$ is the perihelion precession in General Relativity, given by Equation 4.30. In Nordström's first theory, the parameter $b = 1$ [22]. Therefore, for a conformally coupled gravitational field, the elliptical orbit shifts in opposite direction to the case of Schwarzschild's metric. An illustrative comparison of the two shifts was presented in Figure 4.2.

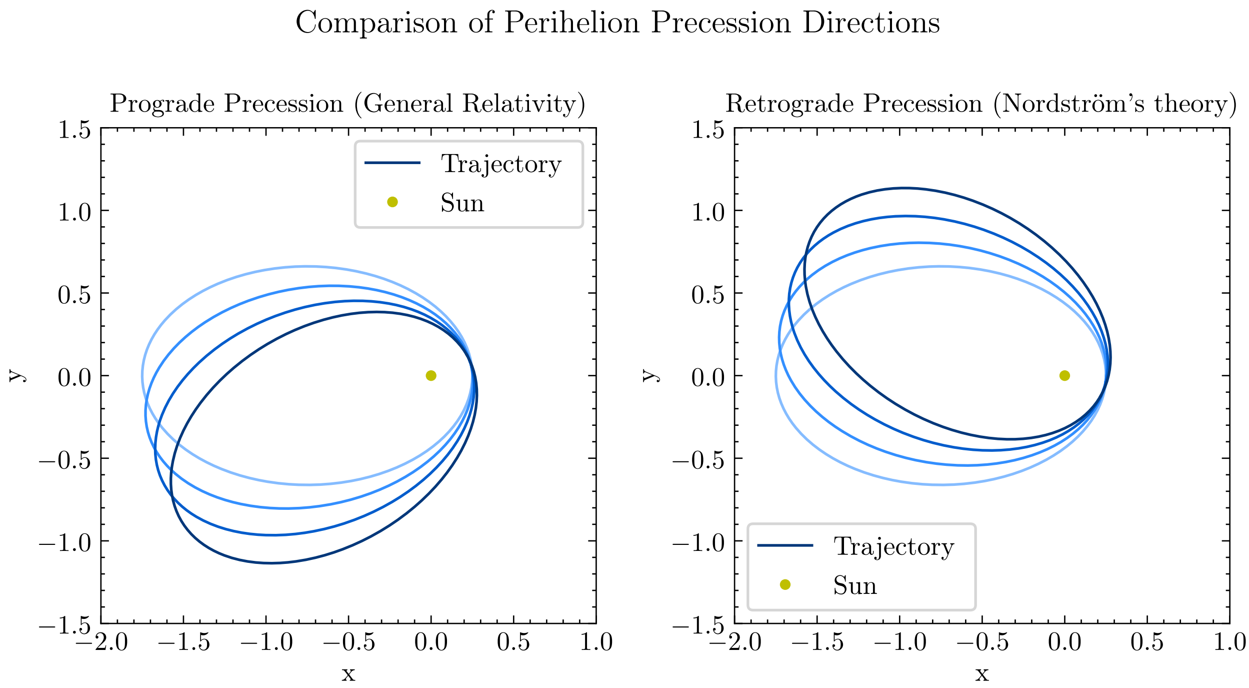


Figure 4.2: Illustrative comparison of the perihelion precession in opposite directions for General Relativity and Nordström's theory (conformal coupling). Each plot shows four consecutive revolutions of a planet around the Sun (focus of the ellipse), which represents the central heavy mass sourcing the gravitational field. The trajectory was plotted in increasingly darker shades of blue, starting from the lightest ellipse.

4.4 Disformal coupling

Finally, let us consider a relativistic two-body problem with the disformally coupled metric given by [14]:

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{\alpha}{\Lambda^4}\partial_\mu\varphi\partial_\nu\varphi, \tag{4.51}$$

where φ is a background scalar field, and the perturbation constant now consists of Λ representing the energy scale and the dimensionless parameter α [14]. This metric is characteristic for the DBI (Dirac-Born-Infeld) scalar field theory [13]. Once again, we start form the Lagrangian for the background field, in order to solve for φ and obtain particular form of the metric.

We recall that for each of the scalar theories of gravity, one need to apply a different field action (Lagrangian) for to account for different structural symmetries of the three theories. Now, the DBI theory (with disformally coupled metric) is described in four space-time dimensions by the action [14]:

$$S_{\text{DBI}} = -\frac{\Lambda^4}{\alpha}\int \mathrm{d}^4x\sqrt{1+\frac{\alpha}{\Lambda^4}(\partial\varphi)^2}, \tag{4.52}$$

with Λ and α defined in the same way as for the metric [14]. Expanding the square root in the integral above gives:

$$S_{\text{DBI}} = \int \mathrm{d}^4x\left(-\frac{\Lambda^4}{\alpha} - \frac{1}{2}(\partial\varphi)^2 + \frac{1}{8}\frac{\alpha}{\Lambda^4}(\partial\varphi)^4 - \frac{1}{16}\frac{\alpha^2}{\Lambda^8}(\partial\varphi)^6 + \frac{5}{128}\frac{\alpha^3}{\Lambda^{12}}(\partial\varphi)^8 + \ldots\right). \tag{4.53}$$

Considering the variation techniques, the Lagrangian minimizing of the background field φ the DBI action can be written as:

$$\mathcal{L}_{\text{field}} = -\frac{1}{2}(\partial\varphi)^2 + \frac{1}{8}\frac{\alpha}{\Lambda^4}(\partial\varphi)^4 - \frac{1}{16}\frac{\alpha^2}{\Lambda^8}(\partial\varphi)^6 + \frac{5}{128}\frac{\alpha^3}{\Lambda^{12}}(\partial\varphi)^8 + \ldots, \tag{4.54}$$

where we ignored the constant term $-\frac{\Lambda^4}{\alpha}$. Consequently, in the formula above, we obtained the expansion of the field Lagrangian.

Let us now consider the first expansion term only, giving the Lagrangian of the lowest order:

$$\mathcal{L}_{\text{field}_1} = -\frac{1}{2}(\partial\varphi)^2. \tag{4.55}$$

We recall that the general Euler-Lagrange equations read [20]:

$$\partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\varphi)}\right) - \frac{\delta\mathcal{L}}{\delta\varphi} = 0. \tag{4.56}$$

Considering the notation $(\partial\varphi)^2 = \partial_\mu\varphi\partial^\mu\varphi$, and substituting $\mathcal{L}_{\text{field}_1}$ into the equation above, we derive that:

$$-\partial_\mu\partial^\mu\varphi = 0 \implies \Box\varphi = 0, \tag{4.57}$$

where $\Box$ represents the d'Alembertian operator.

Now, let us recall that in the one centre problem it is assumed that the gravitational field is generated by a massive body $m_2 = M$, which is effectively stationary, and a smaller mass $m_1 = m$ is moving through the field, without influencing the spacetime curvature (its own gravitational field is negligible). Therefore, we can assume that the background scalar field φ (generated by M) is constant in time and spherically symmetric, showcasing only the radial dependence. Therefore, one obtains the following differential equation:

$$\begin{aligned}
&\nabla^2\varphi = \frac{1}{r^2}\frac{\delta}{\delta r}\left(r^2\frac{\delta\varphi}{\delta r}\right) = 0 \implies \\
&r^2\frac{\delta\varphi}{\delta r} = a \implies \frac{\delta\varphi}{\delta r} = \frac{a}{r^2} \implies \\
&\varphi = -\frac{a}{r}.
\end{aligned} \tag{4.58}$$

where a is a numerical constant. For consistency with previous derivations, we can consider that:

$$\varphi = -\frac{aM}{r}, \tag{4.59}$$

where M is the mass of the central body. Therefore, we obtain an analogous result as in the calculation conducted in Section 4.3. We note that the heavier mass is included as the source of the gravitational field. Moreover, heavier mass should result in a stronger gravitational coupling, which reflects on the proportionality in the equation above.

On the other hand, considering the two first terms of the Lagrangian $\mathcal{L}_{\text{field}}$:

$$\mathcal{L}_{\text{field}_{1,2}} = -\frac{1}{2}(\partial\varphi)^2 + \frac{1}{8}\frac{\alpha}{\Lambda^4}(\partial\varphi)^4, \tag{4.60}$$

and plugging them in the Euler-Lagrange equation, we obtain:

$$-\Box\varphi + \frac{\alpha}{\Lambda^4}\Box\varphi\left[\partial^\mu\varphi\partial_\mu\varphi\right] + \frac{1}{2}\frac{\alpha}{\Lambda^4}(\partial_\mu\partial_\mu\varphi)\partial^\mu\varphi\partial^\mu\varphi = 0. \tag{4.61}$$

We realize that the field solution for the lowest order Lagrangian, $\varphi = -\frac{aM}{r}$, no longer causes the equation above to vanish. This is because of the last no-vanishing term, which when evaluated gives:

$$\frac{1}{2}\frac{\alpha}{\Lambda^4}(\partial_\mu\partial_\mu\varphi)\partial^\mu\varphi\partial^\mu\varphi = \frac{1}{2}\frac{\alpha}{\Lambda^4}\cdot\left(-\frac{2aM}{r^3}\right)\cdot\left(\frac{aM}{r^2}\right)^2 = -\frac{\alpha}{\Lambda^4}\frac{a^3M^3}{r^7}. \tag{4.62}$$

As we only consider the two lowest order terms in the Lagrangian, the only two lowest order terms of the background scalar field are of our interest. Therefore, by expanding the scalar field as:

$$\varphi = -\frac{aM}{r} + \frac{\alpha}{\Lambda^4}\frac{b}{r^5} + ..., \tag{4.63}$$

where b is a numerical constant, and plugging the expression in Equation 4.61, we obtain:

$$-\frac{\alpha}{\Lambda^4}\frac{20b}{r^7} - \frac{\alpha}{\Lambda^4}\frac{a^3M^3}{r^7} + \mathcal{O}\left[\frac{\alpha^2}{\Lambda^8}\right] = 0, \tag{4.64}$$

with $\mathcal{O}\left[\frac{\alpha^2}{\Lambda^8}\right]$ indicating negligible higher order expansion terms. Therefore, we find that $b = -\frac{a^3M^3}{20}$, and the background scalar field is:

$$\varphi = -\frac{aM}{r} - \frac{\alpha}{\Lambda^4}\frac{a^3M^3}{20r^5}. \tag{4.65}$$

4.4.1 Hamiltonian Corrections

Considering the scalar field given by $\varphi = -\frac{aM}{r}$, let us now derive the resulting Kepler Hamiltonian at the lowest order. Taking an analogous Lagrangian for the motion in scalar field as in Section 4.3:

$$\mathcal{L} = \frac{m}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu - \frac{mc^2}{2}, \tag{4.66}$$

but with the disformal metric, we obtain:

$$\mathcal{L} = \frac{m}{2}\left[\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + \frac{\alpha}{\Lambda^4}\partial_\mu\varphi\partial_\nu\varphi\dot{x}^\mu\dot{x}^\nu\right] - \frac{mc^2}{2}. \tag{4.67}$$

As before, we consider that r, θ, ϕ are the usual spherical coordinates with $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi)$, and we make the assumption of a planar motion in the equatorial plane $\theta = \frac{\pi}{2}$, giving $\dot{\theta} = 0$. Then,

$$\partial_\mu\varphi = \left(0, \frac{\delta}{\delta r}\varphi, 0, 0\right) = \frac{aM}{r^2}\mathbf{e_r}, \tag{4.68}$$

where $\mathbf{e_r}$ is the basis radial vector in spherical coordinate system. Now,

$$\dot{x}^\mu = \partial_0 x^\mu = \partial_0 x^0\mathbf{e_0} + \partial_0 x^r\mathbf{e_r} + ..., \tag{4.69}$$

where subscript 0 denotes the time coordinate and r - the radial coordinate basis vectors. Considering that:

$$\mathbf{e_r}\cdot\mathbf{e_r} = 1 \tag{4.70}$$

we obtain:

$$\partial_\mu\varphi\dot{x}^\mu = \frac{aM}{r^2}\mathbf{e_r}\cdot\partial_0 x^r\mathbf{e_r} = \frac{aM}{r^2}\cdot\dot{r}. \tag{4.71}$$

Therefore,

$$\begin{aligned}\mathcal{L} &= \frac{m}{2}\left[(-c^2 + \dot{r}^2 + r^2\dot{\phi}^2) + \frac{\alpha}{\Lambda^4}\frac{a^2M^2}{r^4}\cdot\dot{r}^2\right] - \frac{mc^2}{2} \\ &= -mc^2 + \frac{m}{2}(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{\alpha}{\Lambda^4}\frac{m}{2}\frac{a^2M^2}{r^4}\cdot\dot{r}^2.\end{aligned} \tag{4.72}$$

Finally, the conjugate momenta are:

$$p_r = m\dot{r} + m\frac{\alpha}{\Lambda^4}\frac{a^2M^2}{r^4}\cdot\dot{r}, \tag{4.73}$$

$$p_\phi = mr^2\dot{\phi}, \tag{4.74}$$

and the Hamiltonian is:

$$H = m\dot{r}^2 + m\frac{\alpha}{\Lambda^4}\frac{a^2M^2}{r^4}\cdot\dot{r}^2 + mr^2\dot{\phi}^2 - \mathcal{L} \tag{4.75}$$

$$= mc^2 + \frac{m}{2}(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{m}{2}\frac{\alpha}{\Lambda^4}\frac{a^2M^2}{r^4}\cdot\dot{r}^2$$

$$= mc^2 + \frac{1}{2}\left(p_r\dot{r} + p_\phi\dot{\phi}\right) \tag{4.76}$$

$$= mc^2 + \frac{1}{2m}\left(p_r^2\left[1 + \frac{\alpha}{\Lambda^4}\frac{a^2M^2}{r^4}\right]^{-1} + \frac{p_\phi^2}{r^2}\right), \tag{4.77}$$

Considering that the total momentum is given by $p^2 = p_r^2 + \frac{p_\phi^2}{r^2}$, we obtain the following expansion of the Hamiltonian:

$$H = mc^2 + \frac{p^2}{2m} - \frac{\alpha}{\Lambda^4}\frac{a^2M^2}{2mr^4}p_r^2 + ..., \tag{4.78}$$

and evaluating the conjugate momentum, at the order of $\frac{\alpha}{\Lambda^4}$, we obtain:

$$H = mc^2 + \frac{p^2}{2m} - \frac{\alpha}{\Lambda^4}\frac{a^2M^2m}{2r^4}\dot{r}^2 + \mathcal{O}\left[\frac{\alpha^2}{\Lambda^8}\right]. \tag{4.79}$$

Let us note that the resulting Hamiltonian contains a potential term that falls as $\frac{\dot{r}^2}{r}$, which shows a discrepancy with the Newtonian gravitational potential (falling off as $\frac{1}{r}$). We also note the presence of the $\frac{\alpha}{\Lambda^4}$ in front on the effective potential term. This is a reminiscence of the disformal metric given by Equation 4.51, which already contains the $\frac{\alpha}{\Lambda^4}$ order perturbation to the flat Minkowski metric. Therefore, since the source of gravitational interaction in the metric is of the lower order, this is also the case for the effective gravitational potential in the Hamiltonian.

This result differs from the effective gravitational potential for the DBI theory presented in [14], where the leading order term is found to fall off as $\propto r^{-7}$. However, we note that the result above is also proportional to $\dot{r}^2$, which differs from the result in [14]. Nevertheless, let us note that the aforementioned work utilized Effective Field Theory and the method of scattering amplitudes to compute the gravitational potential, which was not considered in this study. Meanwhile, the calculated Hamiltonian is dimensionally justified ($\propto \frac{\dot{r}^2}{r^4}$) given the form of the metric perturbation: $\frac{\alpha}{\Lambda^4}\partial_\mu\varphi\partial_\nu\varphi$. This is because this form of the metric allows only for the even inverse powers of r in the expansion.

We also note that a is another free parameter determining the strength of the field. Setting $a = \frac{\sqrt{G}}{c}$ as in the case of conformal coupling, we find that the effective potential is of 1PN order in terms of the expansion in powers of $\frac{1}{c^2}$:

$$H_{\text{1PN}} = \frac{1}{c^2}\left[-\frac{\alpha}{\Lambda^4}\frac{GM^2m}{2r^4}\dot{r}^2\right]. \tag{4.80}$$

This suggests that 0PN order Hamiltonian does not contain the gravitational part, which could be justified by the fact that the disformal metric in the leading order follows the flat Minkowski metric structure. The $\eta_{\mu\nu}$ metric indeed represents the space-time without gravitational interaction.

Nevertheless, one should note that the parameter a can be arbitrary chosen, and in the derivation we consider the order of $\frac{\alpha}{\Lambda^4}$ as determining the leading and sub-leading expansion terms. Therefore, the constant a can be potentially set to 1 (or absorbed into M), leaving the parameter α or the the powers of r as determining the order of Post-Newtonian expansion.

To evaluate higher order Hamiltonian corrections, let us now include the second lowest order term in the scalar field, as given by Equation 4.81, restated below:

$$\varphi = -\frac{aM}{r} - \frac{\alpha}{\Lambda^4}\frac{a^3M^3}{20r^5}. \tag{4.81}$$

The Lagrangian of a massive particle moving in space-time with a disformally coupled metric is again given by:

$$\mathcal{L} = \frac{m}{2}\left[\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + \frac{\alpha}{\Lambda^4}\partial_\mu\varphi\partial_\nu\varphi\dot{x}^\mu\dot{x}^\nu\right] - \frac{mc^2}{2}. \tag{4.82}$$

As before, we consider r, θ, ϕ as spherical coordinates with $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi)$, and we make the assumption of the planar motion. Then,

$$\partial_\mu \varphi = \left(0, \frac{\delta}{\delta r}\varphi, 0, 0\right) = \left(\frac{aM}{r^2} + \frac{\alpha}{\Lambda^4}\frac{a^3M^3}{4r^6}\right)\mathbf{e_r}, \tag{4.83}$$

where $\mathbf{e_r}$ is the basis radial vector in spherical coordinate system. Then,

$$\partial_\mu \varphi \dot{x}^\mu = \left(\frac{aM}{r^2} + \frac{\alpha}{\Lambda^4}\frac{a^3M^3}{4r^6}\right)\mathbf{e_r}\cdot\partial_0 x^r \mathbf{e_r} = \left(\frac{aM}{r^2} + \frac{\alpha}{\Lambda^4}\frac{a^3M^3}{4r^6}\right)\cdot\dot{r}, \tag{4.84}$$

and the resulting Lagrangian is given by:

$$\begin{aligned}\mathcal{L} &= \frac{m}{2}\left[(-c^2 + \dot{r}^2 + r^2\dot{\phi}^2) + \frac{\alpha}{\Lambda^4}\left(\frac{aM}{r^2} + \frac{\alpha}{\Lambda^4}\frac{a^3M^3}{4r^6}\right)^2\cdot\dot{r}^2\right] - \frac{mc^2}{2} \\ &= -mc^2 + \frac{m}{2}(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{\alpha}{\Lambda^4}\frac{m}{2}\left(\frac{aM}{r^2} + \frac{\alpha}{\Lambda^4}\frac{a^3M^3}{4r^6}\right)^2\cdot\dot{r}^2.\end{aligned} \tag{4.85}$$

Finally, the conjugate momenta are:

$$p_r = m\dot{r} + \frac{\alpha}{\Lambda^4}m\left(\frac{aM}{r^2} + \frac{\alpha}{\Lambda^4}\frac{a^3M^3}{4r^6}\right)^2\cdot\dot{r}, \tag{4.86}$$

$$p_\phi = mr^2\dot{\phi}, \tag{4.87}$$

and the Hamiltonian is:

$$H = m\dot{r}^2 + m\frac{\alpha}{\Lambda^4}\left(\frac{aM}{r^2} + \frac{\alpha}{\Lambda^4}\frac{a^3M^3}{4r^6}\right)^2\cdot\dot{r}^2 + mr^2\dot{\phi}^2 - \mathcal{L} \tag{4.88}$$

$$= mc^2 + \frac{m}{2}(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{m}{2}\frac{\alpha}{\Lambda^4}\left(\frac{aM}{r^2} + \frac{\alpha}{\Lambda^4}\frac{a^3M^3}{4r^6}\right)^2\cdot\dot{r}^2$$

$$= mc^2 + \frac{1}{2}\left(p_r\dot{r} + p_\phi\dot{\phi}\right) \tag{4.89}$$

$$= mc^2 + \frac{1}{2m}\left(p_r^2\left[1 + \frac{\alpha}{\Lambda^4}\left(\frac{aM}{r^2} + \frac{\alpha}{\Lambda^4}\frac{a^3M^3}{4r^6}\right)^2\right]^{-1} + \frac{p_\phi^2}{r^2}\right), \tag{4.90}$$

Considering that the total momentum is given by $p^2 = p_r^2 + \frac{p_\phi^2}{r^2}$, we obtain the following expansion of the Hamiltonian:

$$H = mc^2 + \frac{p^2}{2m} - \frac{\alpha}{\Lambda^4}\frac{a^2M^2}{2mr^4}p_r^2 + \frac{\alpha^2}{\Lambda^8}\frac{a^4M^4}{4mr^8}p_r^2 + ... \tag{4.91}$$

Evaluating the conjugate momentum, up to the order of $\frac{\alpha^2}{\Lambda^8}$, we obtain:

$$H = mc^2 + \frac{p^2}{2m} - \frac{\alpha}{\Lambda^4}\frac{a^2M^2m}{2r^4}\dot{r}^2 + \frac{\alpha^2}{\Lambda^8}\frac{3a^4M^4m}{4r^8}\dot{r}^2 + \mathcal{O}\left[\frac{\alpha^3}{\Lambda^{12}}\right]. \tag{4.92}$$

Choosing the free parameter $a = \frac{\sqrt{G}}{c}$, we obtain the 2PN order Hamiltonian correction:

$$H_{\text{2PN}} = \frac{1}{c^4}\left[\frac{\alpha^2}{\Lambda^8}\frac{3G^2M^4m}{4r^8}\dot{r}^2\right]. \tag{4.93}$$

Nevertheless, we recall that the parameter a can also be set to 1, leaving the Hamiltonian expansion in terms of the powers of $\frac{\alpha}{\Lambda^8}$ in the derivation above.

In Equation 4.80 and 4.93, we applied the definition of Post-Newtonian order determined by the power of the factor $\frac{1}{c^2}$, following the Definition 4.0.1. This suggests that $\frac{1}{r}$ and p^2 (or p_r^2) are no longer on equal footing in the Kepler Hamiltonian. This is different then in the case of the Kepler Hamiltonian for GR and Nordström's theory. Therefore, the use of $\frac{1}{c^2}$ as a factor dictating the expansion order is justified. This is because it provides consistent framework for comparison of the relativistic corrections between different scalar field theories of gravity.

Furthermore, one should highlight that there are no momentum independent terms for neither 1PN nor 2PN correction, presented subsequently in Equation 4.80 and 4.93. This is because the $\dot{r}$ term is related to the conjugate radial momentum p_r, as considered in both derivations. This discrepancy with the results of General Relativity and Nordström's theory has interesting consequences for the two-body dynamics, considered in the following section.

4.4.2 Two-body Dynamics and Unbounded Orbits

With an aim to determine the dynamics of a two-body system with disformal coupling at 1PN order, let us consider again the Lagrangian given in equation 4.72:

$$\mathcal{L} = -mc^2 + \frac{m}{2}(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{\alpha}{\Lambda^4}\frac{m}{2}\frac{a^2M^2}{r^4}\cdot\dot{r}^2. \tag{4.94}$$

Now, solving the Euler-Lagrange equations, the resulting equations of motions read:

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\phi}}\right) = \frac{d}{dt}\left(mr^2\dot{\phi}\right) = 0, \tag{4.95}$$

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{r}}\right) - \frac{\partial\mathcal{L}}{\partial r} = m\left(\ddot{r}\left(1 + \frac{\alpha}{\Lambda^4}\frac{a^2M^2}{r^4}\right) + \dot{r}^2\left(1 + \frac{-2\alpha}{\Lambda^4}\frac{a^2M^2}{r^5}\right) - r\dot{\phi}^2\right) = 0. \tag{4.96}$$

Therefore, we obtain:

$$mr^2\dot{\phi} = l = \text{constant (conservation of angular momentum)}, \tag{4.97}$$

$$\ddot{r}\left(1 + \frac{k}{r^4}\right) = r\dot{\phi}^2 + \frac{2k}{r^5}\dot{r}^2, \tag{4.98}$$

where we denoted $k = \frac{\alpha a^2 M^2}{\Lambda^4}$. Then, the equation 4.98 can be expressed as:

$$\ddot{r}\left(1 + \frac{k}{r^4}\right) = \frac{l^2}{m^2r^3} + \frac{2k}{r^5}\dot{r}^2. \tag{4.99}$$

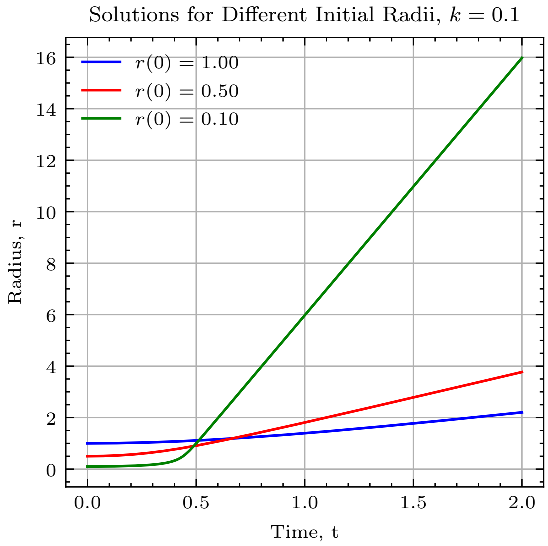
First of all, let us note that Equation 4.99 does not have any circular orbit solutions. That is, setting $\ddot{r} = \dot{r} = 0$, as required for a circular orbit with a constant radius, results in $\frac{l^2}{m^2r^3} = 0$. This would require the angular momentum l to be equal to zero, which is not a physically possible solution.

To consider other possible orbital trajectories, Equation 4.99[1] can be solved with a use of a numerical solver. For simplicity, it was assumed that $\frac{l^2}{m^2} = 1$ and the parameter k was set to 0.1. The solutions for different initial conditions were plotted in Figures 4.3 and 4.4. In particular, Figure 4.3 presents a comparison of radii evolution in time for different initial radii and zero initial radial velocity, while Figure 4.4 - the solutions for different initial velocities. Figure 4.3a presents the initial radial changes in time, and Figure 4.3b their evolution over a longer time period.

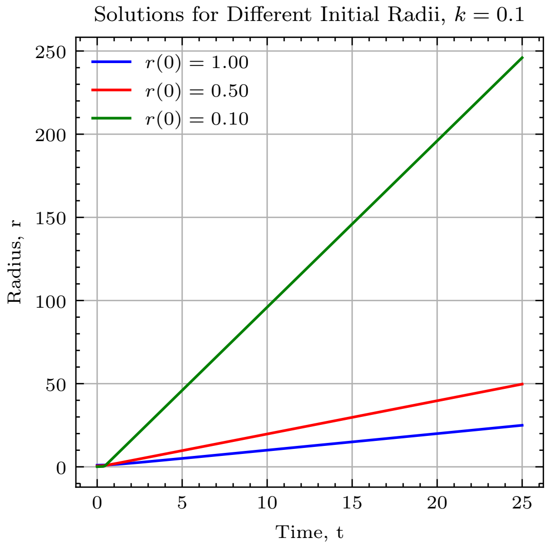
In Figures 4.3 and 4.4, we observe that temporal evolution of the orbital radius follows mostly stable linear increase. Only the initial radial trajectory can vary, depending on the initial conditions. This suggest that as time t increases, the radius r of the orbit of a mass moving in a disformally coupled gravitational field diverges, meaning that Equation 4.99 does not have any bounded orbit solutions.

This result in opposite to the cases of Newtonian gravity, General Relativity, or Nordström's theory, which give multiple circular and elliptical orbits (which might be precessing). The unbounded orbits of the DBI theory of gravity might stem from the disformal metric itself. As considered in Sections 4.2 and 4.3, in the case of GR and conformal coupling, while expanding the metric in terms of the scalar field, one always obtains a linear term in φ with gives rise to Newtonian-like potential proportional to $-\frac{1}{r}$. As a consequence, at the 0PN order, the Hamiltonian reduces to the classical Kepler Hamiltonian, as stated by Equations 4.18 and 4.47. As the trajectory of motion is largely determined by the lowest order terms, the bounded orbits have mostly circular or elliptical shapes, with the effect of precession arising with the higher order corrections. In contrast, the disformal perturbation to the metric has a form of a derivative of the scalar field. As a result, there are no linear φ terms and the 0PN Hamiltonian resembles the one for a free particle. Therefore, only the higher order correction terms are responsible for the gravitational interaction. Then, the two-body dynamics is not significantly influenced by the gravitational pull, and the final trajectory of motion remains unbounded at 1PN order correction.

[1] One can also consider an approximated equation of motion, by multiplication by the factor $\left(1 + \frac{k}{r^4}\right)^{-1}$ and its subsequent expansion up to 1PN order (considering the terms up to first order in k). The approximated equation has one circular orbit solution. However, this result requires to equal the terms of different orders in k, which is not a physically meaningful result. Therefore, not all of the solutions to the approximated equation will satisfy Equation 4.99. More detailed discussion on this topic was presented in Appendix B.



(a) Initial Temporal Evolution.



(b) Temporal Evolution over Longer Time Interval.

Figure 4.3: Numerical solution to the ODE given in Equation 4.99, plotted for different initial radii: $r(0) = 1.00$ (in blue); $r(0) = 0.50$ (in red); $r(0) = 0.10$ (in green), with the parameter k set to 0.1 and initial radial velocity $\dot{r}(0) = r'(0) = 0$. As visible in plot 4.3a, although at the beginning the radii do not change significantly (resulting in a mostly flat curve), at the end all three functions become more steep and diverge. This trend is confirmed by Figure 4.3b, which presents the same plots but over a longer time interval. This suggest that all of the three radii functions diverge in time.

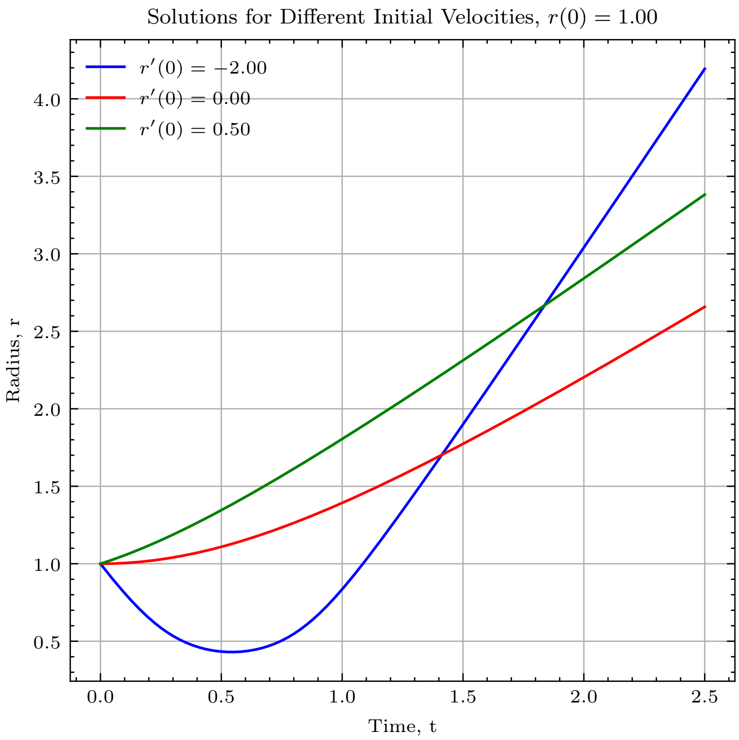


Figure 4.4: Numerical solution to the ODE given in Equation 4.99, plotted for different initial radial velocities: $r'(0) = -2.00$ (in blue); $r(0) = 0.00$ (in red); $r(0) = 0.50$ (in green), with the parameter k set to 0.1 and initial radius of $r(0) = 1.00$. Although at the beginning, the radii's temporal evolution is influenced by the initial radial velocity, at the end all of the trajectories diverge. This is visible the most in case of initial negative radial velocity (the blue curve), which at the beginning causes the radius to decrease, but afterwards trend reverses and the radius shows a stable increase later in time.

4.4.3 Comparison with Amplitude Calculations

An alternative approach to the scalar field theories of gravity is the framework of the Effective Field Theory. In particular, the gravitational interaction in Kepler problem can be considered as a scattering of two very heavy excitations of the matter field (particles) [22]. Calculation of the scattering amplitude of the graviton exchange between the two particles is then a method that allows us to describe their interaction. Indeed, the method of scattering amplitudes was used by Bonifacio et al. in the paper [14] to calculate the effective potential for the DBI theory to fall off $\propto r^{-7}$. This section aims to discuss more in depth the discrepancies and similarities between this result and the Hamiltonian obtained in Equation 4.92.

First of all, let us note that to consider an exact scattering amplitude, one would have to consider contributions from all possible Feynman diagrams. However, with increasing diagram complexity, the relevance of particular interactions decreases, allowing to consider only the lowest order diagrams. For the two-body problem, the simplest possible Feynman diagram is a *tree diagram*, that corresponds to an exchange of graviton between massive particles. In particular, scattering of this type of gives rise to a Newtonian gravitational potential between the two massive objects, $-\frac{GMm}{r}$ [14].

However, in the DBI scalar field theory, there are no three-point couplings, i.e., no tree diagrams, due to $\mathbb{Z}_2$ symmetry of the theory [14]. Therefore, one should consider a higher order *loop diagram* in the calculation of the scattering amplitudes. This types of diagrams involve an exchange of more than one graviton between the massive particles, effectively creating a loop. A schematic comparison of the tree and loop diagrams in EFT was presented in Figure 4.5.

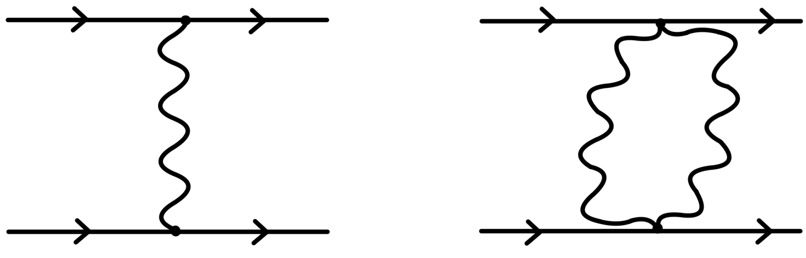


Figure 4.5: Comparison of scattering diagrams in Effective Field Theory. The thick lines represent the compact objects and the wiggly lines - the exchanged graviton.

To summarize, loop diagrams do not give a scattering amplitude that would result in a Newtonian gravitational potential. This is analogous to the Hamiltonian for disformal coupling obtained in the Equation 4.92. Although the leading correction term falls of as $\propto \frac{\dot{r}^2}{r^4}$ instead of $\propto r^{-7}$, the 0PN Hamiltonian does not include the Newtonian potential term $\propto r^{-1}$. This result is in agreement with [14], confirming that there are no tree diagrams in the disformal coupling in the DBI theory. Moreover, the lack of a Newtonian potential term also relates to lack of linear scalar field terms in the expansion of the metric, as discussed at the end of the previous section.

The discrepancy of the $\propto \frac{\dot{r}^2}{r^4}$ versus $\propto r^{-7}$ fall of in the Hamiltonian remains an open question of this study. The reason for this might be the different methods considered in the calculations. This project aimed to derive the Hamiltonian corrections straight from the gravitational metric plugged into the equation for the particle's action. Meanwhile, Bonifacio et al. first considered the scattering amplitude of the loop diagram, and applied the Fourier transform to obtain the interaction potential and not the full Hamiltonian [14]. This methods involves the calculation of complicated integral equations, in contrast to the relatively less complex expansion considered in this thesis.

Chapter 5

Conclusion

This thesis has explored the topic of the Post-Newtonian corrections to the Kepler problem. The aim of this study was to establish what the lowest order relativistic corrections to the Kepler Hamiltonian are, and how they influence the two-body dynamics. In particular, this project focused on three theories of gravity: General Relativity, Nordström's theory with conformal coupling in the metric, and the DBI theory with disformal coupling. While the corrections for GR and conformal coupling had been studied before, the derivation of Hamiltonian correction for disformal coupling constitutes a novel result.

First of all, this study managed to rederive the first Post-Newtonian order Hamiltonian corrections for the Einstein's theory of General Relativity, a result that has been presented in Section 4.2.1. Secondly, an analogous derivation was considered for conformal coupling given by Nordström's scalar theory. It resulted in one of the obtained 1PN correction terms, the leading momentum term $\propto p^2$, to be the same as for Schwarzschild's metric, but with an opposite sign. These key findings suggest that GR and Nordström's theory both result in a bounded elliptical orbits with precessing perihelion. Nevertheless, the perihelion shift precesses in opposite directions in the two theories, which was illustrated in Figure 4.2. Finally, the Hamiltonian for the DBI theory had been derived up to 2PN order. It was found that, at the 0PN order, the gravitational potential vanishes giving the lowest order Hamiltonian for a free particle. This is because of the disformal parameter in the metric, which already constitutes a first order perturbation. Indeed, the 1PN result gives the Hamiltonian correction $\propto \frac{\dot{r}^2}{r^4}$. This finding differs from the one given by Bonifacio et al. in [14], who found the effective gravitational potential to vary as $\propto r^{-7}$. Moreover, the corrections obtained for the disformal coupling suggest that the DBI theory does not give a bounded motion for the two-body dynamics at the 1PN order, as presented in Figures 4.3 and 4.4.

The findings for the Schwarzschild's and conformal metric present a confirmation of the previously-known results, which testifies to their significance. Moreover, they attest to the validity of the methods of Hamiltonian formalism applied to the theories of gravity. Despite the discrepancy of the result for disformal metric with the previously found effective potential for the DBI theory [14], it should be noted that alternative methods were applied for the two calculations. While this thesis derived the Hamiltonian corrections by considering the Lagrangian for a particle moving in a background gravitational field, Bonifacio et al. had used the methods of Effective Field Theory [14]. This might be the reason for the alternative final results. Nevertheless, the obtained unbounded orbit for the DBI theory suggests that the calculation findings of this study might still be significant.

All of the results had assumed a one centre approximation, in which the mass of one body is much heavier than the other. Additionally, calculations were conducted with an assumption of planar motion in the equatorial plane. Therefore, the findings of this study do not constitute the most general result for a two-body system. Moreover, the results are valid for systems in which $m_2 >> m_1$, which might be consider as a limitation. Consequently, the derived equations are not applicable for binary systems, where the two interacting objects have similar masses. However, one should note that without one centre approximation, the calculations could not be conducted in a straightforward way.

To expand on the findings of this thesis, future research could explore alternative scalar field theories of gravity. One of the possibilities could be the special galileon theory, considered for instance in [14]. Moreover, one could conduct the derivation of the Hamiltonian corrections resulting from a gravitational metric with both the disformal and conformal coupling, as it was conducted in the work by Brax and Davis [12]. Finally, alternative methods can be applied to confirm the validity of the obtained results, such as the techniques of Effective Field Theory [14].

Overall, this work contributes to a deeper understanding of alternative theories of gravitation. By presenting a range of results - both well-established and novel - it offers an overview of key techniques in scalar field theory and their implications for two-body dynamics. Although the Kepler problem has been studied for centuries, it continues to raise open questions in modern theoretical physics. Therefore, this thesis can be seen as an introduction to this rich and intriguing area of research.

Chapter 6

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Appendix A

Differentiable Manifolds

In order to define central concepts of General Relativity, it is necessary to introduce a fundamental object of differential geometry - a *differentiable manifold*. To formally introduce the concept of differentiable manifold, let us consider the definitions below, formulated as in [21].

Definition A.0.1. ([21])
Let X be some set and $\mathcal{T}$ a set of subsets of X. A pair $(X, \mathcal{T})$ is a **topological space** if

1. X and $\varnothing$ are open, i.e., $X \in \mathcal{T}$ and $\varnothing \in \mathcal{T}$;
2. arbitrary unions of families of open sets are open;
3. the intersection of finitely many open subsets is open.

The set $\mathcal{T}$ is know as a **topology** on X.

Definition A.0.2. ([21])
A map $f : X \to Y$ between two topological spaces $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ is called:

- **continuous** if $U \in \mathcal{U}$ implies that $f^{-1}(U) \in \mathcal{T}$, i.e., preimages of open sets under f are open;
- a **homeomorphism** if it is bijective and continuous with continuous inverse.

Definition A.0.3. ([21])
A topological space M is a **topological manifold** of dimension n, or topological n-manifold is it has the following properties:

1. M is a Hausdorff space, meaning that $\forall\ x, y \in X,\ x \neq y,\ \exists\ U_x, U_y \in \mathcal{T} : x \in U_x, y \in U_y$ and $U_x \cap U_y = \varnothing$. That is, every two distinct points admit disjoint open neighborhoods.
2. M is second countable, i.e., there exists a countable set $\mathcal{B} \subset \mathcal{T}$ such that any open set can be written as a union of set in $\mathcal{B}$.
3. M is **locally euclidean** of dimension n, i.e., for any point $p \in M$ there exists an open subset $U \subset M$ with $p \in U$, and an open subset $V \subset \mathbb{R}^n$ and a homeomorphism $\varphi : U \to V$.

The pair (U, φ) is called a **(coordinate) chart**.

Definition A.0.4. ([21])
A **differentiable manifold** of dimension n is a manifold M equipped with a **smooth structure**. That is, for any two charts (U_1, φ_1) and (U_2, φ_2) on M either $U_1 \cap U_2 = \varnothing$ or the map:

$$\varphi_1 \circ \varphi_2^{-1} : \varphi_2(U_1 \cap U_2) \to \varphi_1(U_1 \cap U_2) \tag{A.1}$$

is a smooth diffeomorphism (the function and its inverse are continuously differentiable infinitely many times). Such charts are said to be **compatible**, and a collection of such charts that cover whole M is called a **smooth atlas**.

Manifolds are omnipresent in physics. Examples include phase and configuration spaces in mechanics or state spaces in thermodynamics. In fact, the spacetime itself is a differentiable manifold with special properties, described in Section 2.2.1.

While the formal definition of the differentiable manifold presented above may seem abstract, charts that map neighborhoods on the manifold to euclidean spaces are the objects that enable to define operations like differentiation or integration on curved, non-euclidean spaces. In essence, constructing charts is a formal way of stating that a space is locally flat (euclidean).

As we can locally map a manifold to an euclidean space, the properties of functions on the manifold can be translated into properties of the functions on $\mathbb{R}^n$. For instance,

Definition A.0.5. ([21])
A function $f : M \to \mathbb{R}$ from a smooth n-manifold to $\mathbb{R}$ is said to be **smooth** if for any smooth chart (V, φ) for M the map $f \circ \varphi^{-1} : \varphi(V) \subset \mathbb{R}^n \to \mathbb{R}$ is smooth as an euclidean function on the open subset $\varphi(V) \subset \mathbb{R}^n$. We denote the space of smooth functions by $C^\infty(M)$.

Equipped with the definition above, we can now consider the derivations of C^∞ functions on specific points on a manifold.

Definition A.0.6. ([21])
Let $p \in W \subset M$, where M is a smooth manifold, and W is any neighborhood of p. A map $w_p : C^\infty(W) \to \mathbb{R}$ is called a **derivation of** $C^\infty(W)$ **at** p if it is linear over $\mathbb{R}$ and satisfies the Leibniz rule:

$$w_p(fg) = f(p)w_p(g) + g(p)w_p(f). \tag{A.2}$$

To make sense of the definition above, let us specify that a derivation is map that assigns a real number to each smooth function at point p on a manifold M. Analogously, derivation of a real value function $g : \mathbb{R} \to \mathbb{R}$ at a specific point in the functions domain is also an assignment of a real number, i.e., $\frac{dg(x)}{dx}|_{x=a} = g'(a) = c \in \mathbb{R}$.

A powerful feature of differential geometry is that an operation of differentiation can be defined in many equivalent ways. It turns out that a derivation w_p is equivalent to a tangent vector $v([f]_p)$ at a point $p \in M$ to a function $f \in C^\infty(W)$ [21]. Let us note that v has now two different interpretations. It is both a vector in $\mathbb{R}^n$ and a linear map. As a consequence, the set of all tangent vectors to point p has a vector space structure [21]. Therefore, every point p on a manifold has an associated vector space where all of the tangent vectors (derivations) of functions f on M live.

Definition A.0.7. ([21])
A **tangent space** T_pM is a vector space of all tangent vectors at point $p \in M$, where M is a smooth n-manifold.

Vector space is a fundamental concept of linear algebra. In fact, a vector space has an associated *dual space*, which is a space of linear maps from the vector space to $\mathbb{R}$. The dual space itself has a vector space structure. Therefore, having defined the tangent space, we can also consider its dual space.

Definition A.0.8. ([21])
The dual space T_p^*M of the tangent space T_pM is called the **cotangent space** of M at p (where M is a differentiable manifold, and $p \in M$). The elements of T_p^*M are called **cotangent vectors**, **covectors**, or **(differential) 1-forms** at p.

With the two previous definitions in mind, we can finally state the formal definition of a *tensor* on the manifold M.

Definition A.0.9. ([6])
A **tensor of rank** (r, s) at point $p \in M$ is defined to be a multi-linear map:

$$T : \underbrace{T_p^*(M) \times \cdots \times T_p^*(M)}_{r} \times \underbrace{T_p(M) \times \cdots \times T_p(M)}_{s} \to \mathbb{R}. \tag{A.3}$$

Such tensor is said to have a total rank of $r + s$. The space of tensors of type (r, s) is denoted by $T_s^r(T_pM)$. Moreover, a smooth map from M to $T_s^r(T_pM)$ is referred to as (r, s) **tensor field**.

Appendix B

Solutions to the Approximated Equation of Motion

The equation of motion describing the radius of the orbit in the DBI theory of gravity can be expressed as:

$$\ddot{r}\left(1+\frac{k}{r^4}\right)=\frac{l^2}{m^2r^3}+\frac{2k}{r^5}\dot{r}^2, \tag{B.1}$$

where $k=\frac{\alpha a^2M^2}{\Lambda^4}$, m is mass of a moving body and l is the conserved angular momentum. Transforming the equation by multiplication by the factor $\left(1+\frac{k}{r^4}\right)^{-1}$ and its subsequent expansion up to 1PN order (considering the terms up to first order in k), we obtain the following differential equation:

$$\ddot{r}=\frac{l^2}{m^2r^3}-\frac{l^2k}{m^2r^7}+\frac{2k}{r^5}\dot{r}^2. \tag{B.2}$$

We note that the only circular solution to the equation above is:

$$r(t)=k^{1/4}=\frac{\alpha^{1/4}(aM)^{1/2}}{\Lambda}, \tag{B.3}$$

obtain by setting $\ddot{r}=\dot{r}=0$. However, we note that this solution requires to consider terms of zeroth and first order in k at the same footing. Therefore, it does not correspond to a reliable solution for the orbit. Moreover, we note that this result does not satisfy the general equation of motion, restated here as Equation B.1. Therefore, this circular orbit solutions does not constitute a physically meaningful result.

To solve for other cases one can make use of a numerical solver. For simplicity, let us assume $\frac{l^2}{m^2}=1$. Then, the Equation B.2 becomes:

$$\ddot{r}=\frac{1}{r^3}-\frac{k}{r^7}+\frac{2k}{r^5}\dot{r}^2. \tag{B.4}$$

Solving the Equation B.4 numerically, we obtain the results plotted in Figures B.1 and B.2, dependent on the initial conditions, with the parameter k set to 0.1. Figure B.1 shows three plots for the temporal evolution of the radius depending on different initial radii with zero initial radial velocity. We observe that the orbiting body will either escape the system or crash into a central mass. This evolution of the trajectory depends on whether the initial radius is larger or smaller than the radius of the bounded circular orbit, $r=k^{1/4}$, presented on the central plot.

On the other hand, Figure B.2 presents how the radius changes in time depending on initial radial velocity. All of the plots consider the same initial radius of $r(0)=k^{1/4}$. The plot suggest that the circular orbit solution is unstable. Small initial velocity along or against the radial direction will result in either a diverging trajectory of motion, or the smaller body crashing into the central mass.

In contrast to the approximated Equation B.1, the general equation of motion discussed in Section 4.4.2 does not give the colliding trajectories, where the radius of trajectory is bounded by its initial value. This might be the result of treating the terms of first order in k on equal footing with the zeroth order terms, and overstating the significance of the gravitational interaction in the Hamiltonian. In general, the 0PN Hamiltonian for disformal coupling does not include the interaction terms, thus, the gravitational pull is not significant, as discusses in Section 4.4.2. Therefore, the exact Equation 4.99 will favour the diverging trajectories rather then the colliding ones or the circular bounded orbit.

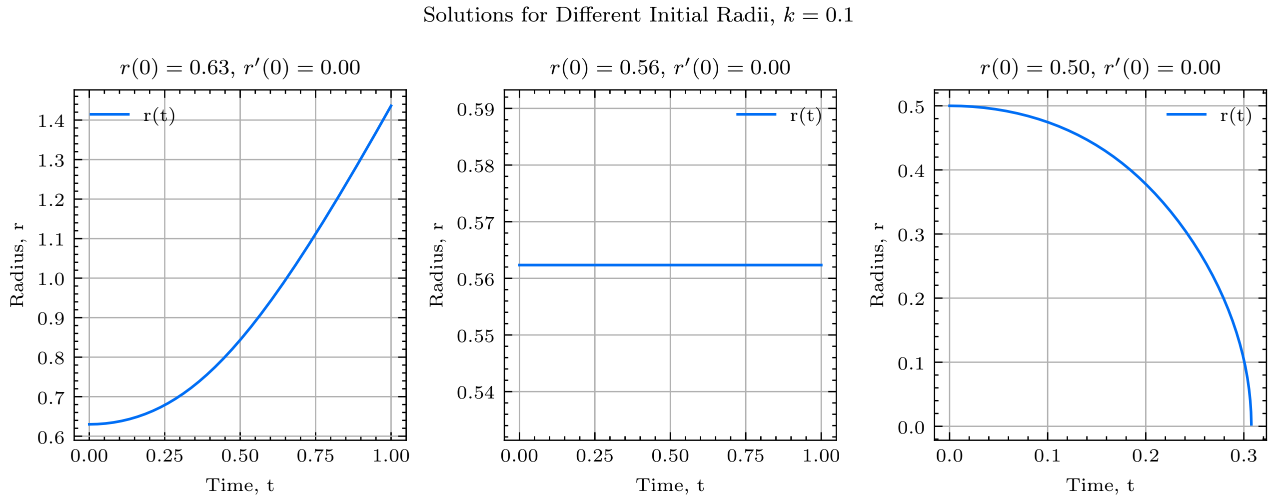

Figure B.1: Numerical solution to the ODE given in Equation B.4, plotted for different initial radii: $r(0) = 0.63; r(0) = k^{1/4} \approx 0.56; r(0) = 0.50$, with the parameter k was set to 0.1 and initial radial velocity $\dot{r}(0) = r'(0) = 0$. The initial condition determine the subsequent trajectory. If the initial radius is bigger than $k^{1/4}$, then the radius diverges in time and trajectory is not bounded, as suggested by the left plot. If $r(0)$ is less than $k^{1/4}$, then the radius decreases to 0 and the orbiting body crashes into the central mass, as shown on the right plot. The central plot shows the only circular solution for the trajectory of motion, with the constant radius of $r = k^{1/4}$.

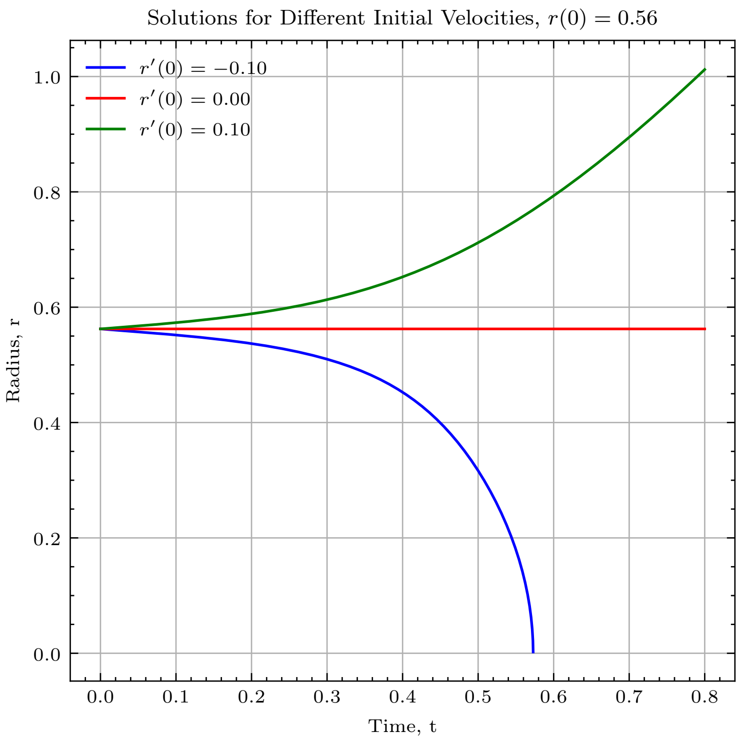

Figure B.2: Numerical solution to the ODE given in Equation B.4, plotted for different initial radial velocities: $r'(0) = -0.10$ (in blue); $r(0) = 0.00$ (in red); $r(0) = 0.10$ (in green), with the parameter k set to 0.1 and initial radius $r(0) = k^{1/4} \approx 0.56$. Initial outward or inward radial velocity causes the radius of trajectory to either diverge or decrease to zero due to collision, suggesting that the constant radius solution (in red) is unstable.