Geometric Constraints and Covariant Phase Space: A Hamiltonian Approach to de Sitter Relativity

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Contents

Abstract

This thesis unifies two extensions of the Hamiltonian formalism: the extended phase space, treating time as a dynamical variable, and Dirac's theory of constrained systems. We first analyze the extended phase space, showing that the Poincaré group forms a subgroup of the canonical group in Minkowski spacetime and that the extended Hamiltonian is Lorentz-invariant. Next, Dirac's formalism is applied to phase space on curved surfaces, revealing an elegant relation between the hypersurface metric and the Poisson structure via the Dirac bracket. Combining these, we develop a consistent phase space description for relativistic systems on curved spacetimes. This framework is tested on de Sitter spacetime, which models our expanding universe's asymptotic structure. We derive a Poisson structure respecting curvature and time for a free massive particle, from which we demonstrate that the equations of motion reproduce geodesics and that de Sitter's isometry group is a subgroup of its canonical group. This work illustrates how the extended Dirac formalism provides, to a certain degree, a consistent Hamiltonian framework for particles in curved spacetimes.

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1 Introduction

In many areas of physics, the central goal is to describe the state of a system at any given moment and to predict how it evolves over time. For example, predicting the motion of celestial bodies has occupied scientists for centuries, while understanding the behavior of particles after detection has become especially important over the past century.

The idea of phase space, which provides the natural framework for this description, first appeared implicitly in Boltzmann's 1878 work on statistical mechanics and was later formalized by Gibbs and others [1]. Since then, it has become a cornerstone of classical mechanics, statistical physics, and even quantum theory. Phase space is defined as the space of all possible states of a system, where each state is uniquely specified by a complete set of generalized coordinates (such as positions or angles) and their conjugate momenta, which describe how the system responds to changes in those coordinates [2]. Within this framework, Hamiltonian mechanics provides a systematic method to derive the equations governing the system's evolution and to study how physical quantities change over time. However, the standard phase space formalism treats time merely as an external parameter, which becomes inadequate for describing systems that are relativistic or constrained. In Einstein's 1905 theory of special relativity, time and space are unified into a single spacetime framework [3], suggesting that time should be treated on the same footing as the spatial coordinates, that is, as a dynamical variable. This motivated the development of an extended phase space formalism, in which time is promoted to a coordinate and a separate evolution parameter is introduced [4].

Meanwhile, many physical systems come with intrinsic constraints that the standard Hamiltonian approach cannot handle consistently, for example curvature. To deal with these, Dirac developed a generalized Hamiltonian formalism that systematically treats these constraints [5]. These two extensions, the extended phase space and Dirac's constrained Hamiltonian mechanics, each address different limitations of the standard formalism and are usually discussed separately. However, in certain settings, such as relativistic particles in curved spacetimes like de Sitter space, both features become essential: the dynamics involve constraints that naturally mix spatial and temporal components, requiring a framework that combines both approaches.

We will discuss these formalism and their unification with our central focus towards the Poisson structure, which encodes the geometry of phase space and governs the evolution of observables through the Poisson algebra. We will discuss promoting time to a coordinate extending the standard Poisson brackets, while constraints modify this structure via the Dirac bracket. The main challenge in bringing both ideas together is to construct a consistent Poisson bracket that incorporates the extended geometry while remaining compatible with the constraints. The goal of this thesis is to motivate and present these two extensions in Chapters 3 and 4, building on the foundations laid in Chapter 2, and then to explore their unification in Chapter 5. Finally, in Chapter 6, the resulting framework is applied to the phase space of de Sitter spacetime.

2 Standard Lagrangian and Hamiltonian Formalism

In this chapter, the standard Lagrangian and Hamiltonian formalism will be discussed to establish the basic tools for describing classical dynamical systems. This includes how phase space and the Poisson bracket arise and why they are central to describing time evolution. Understanding these ideas is necessary because they form the basis for the extensions and modifications that follow later in this thesis.

2.1 Lagrangian Mechanics

Within Lagrangian mechanics, systems are described by a set of generalized coordinates $(q_1, q_2, \dots, q_n)$. The space spanned by the generalized coordinates is called the configuration space. As the system evolves in time, it traces out a path in configuration space, which is referred to as the motion of the system. For a given system, the Lagrangian $L : TQ \to \mathbb{R}$ is generally defined as

$$L(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n; t) \equiv T - V, \tag{1}$$

where T and V are the kinetic and potential energy of the system, respectively [6]. The Lagrangian is naturally defined on the tangent bundle of the configuration space TQ, as this provides the appropriate framework for specifying positions and their corresponding velocities.

The fundamental principle of Lagrangian mechanics is Hamilton's principle. It describes the motion of monogenic systems, which are physical systems in which the acting forces can be derived from scalar potentials. It states that the motion of a system from time t_1 to time t_2 is such that the action functional

$$S[q(t)] = \int_{t_1}^{t_2} L(q(t), \dot{q}(t), t)\, dt \tag{2}$$

is stationary with respect to variations of the path $q(t)$ that leave the endpoints fixed. That is, of all possible paths of motion that the system could take from its position at t_1 to its position at t_2, it takes the path for which

$$\delta S = 0. \tag{3}$$

Using methods from the calculus of variations , this condition leads to the Euler-Lagrange equations

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0, \tag{4}$$

where $\dot{q}_i = \frac{dq_i}{dt}$. The actual motion of the system satisfies these second-order differential equations.

2.2 Hamiltonian Mechanics

In the Lagrangian formalism, the motion of a system is determined by n independent generalized coordinates with the governing n Euler–Lagrange equations. Specifying $2n$ initial conditions fully determines the motion of the system for all times, because the Euler-Lagrange equations are second-order differential equations. The Hamiltonian formalism seeks to recast the dynamics in terms of $2n$ first-order differential equations. Because the number of degrees of freedom remains the same, the system must now be described by $2n$ independent variables. These variables form a new space called phase space, where each point represents a complete state of the system. The evolution of the system corresponds to a trajectory in phase space, which are determined by the first-order differential equations.

To start, it is natural to take the first n coordinates as the generalized positions $\dot{q}_i$. The remaining n coordinates are defined as the generalized (or conjugate) momenta

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i}, \tag{5}$$

which together form the momentum space. The transition from Lagrangian to Hamiltonian mechanics is made via the Legendre transformation. One defines an intermediate function to be

$$F(q,\dot{q},t) \equiv \sum_{i=1}^{n} \dot{q}_i p_i(q,\dot{q}) - L(q,\dot{q}(q,p),t), \tag{6}$$

where p_i are still functions of the coordinates $(q,\dot{q})$. Then one does the Legendre transformation by changing the variables from $(q,\dot{q},t)$ to the new set of coordinates (q,p,t). Assuming that this transformation is non-singular, i.e.

$$\det\left(\frac{\partial p_i}{\partial \dot{q}_j}\right) = \det\left(\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\right) \neq 0, \tag{7}$$

the relation $p_i = \frac{\partial L}{\partial \dot{q}_i}$ can be inverted to express $\dot{q}_i$ in terms of p_i. In that case (q,p) forms an independent coordinate system on phase space. These variables (q_i,p_i) are called canonical coordinates, since as we will see they give Hamilton's equations and their Poisson brackets a canonical structure. In this case, the Hamiltonian is obtained by expressing F in terms of the new variables (q,p)

$$H(q,p,t) \equiv \sum_{i=1}^{n} \dot{q}_i(q,p)\, p_i - L(q,\dot{q},t), \tag{8}$$

where $\dot{q}_i$ are now functions of the new coordinates (q,p). The action functional (2) will take on the form

$$S[q(t),p(t),t] = \int_{t_1}^{t_2} dt \left(\sum_{i=1}^{n} p_i \dot{q}^i - H(q,p,t)\right) \tag{9}$$

and Hamilton's principle becomes that (9) is stationary under the independent variations of $q(t)$ and $p(t)$.

The Hamiltonian, defined in this way, has the differential

$$\begin{aligned} dH &= \sum_{i=1}^{n} \left(d\left(\dot{q}_i p_i\right)\right) - dL \\ &= \sum_{i=1}^{n} \left(\dot{q}_i dp_i + p_i d\dot{q}_i - \frac{\partial L}{\partial q_i} dq_i - \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i\right) - \frac{\partial L}{\partial t} dt \\ &= \sum_{i=1}^{n} \left(\dot{q}_i dp_i - \frac{\partial L}{\partial q_i} dq_i\right) - \frac{\partial L}{\partial t} dt \\ &= \sum_{i=1}^{n} \left(\dot{q}_i dp_i - \dot{p}_i dq_i\right) - \frac{\partial L}{\partial t} dt, \end{aligned} \tag{10}$$

where the last step follows from the fact that the Euler Lagrange equations $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}$ imply $\dot{p}_i = \frac{\partial L}{\partial q_i}$. On the other hand, by the chain rule,

$$dH = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i \right) + \frac{\partial H}{\partial t} dt. \tag{11}$$

Comparing both expressions gives

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i} \qquad \frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t}. \tag{12}$$

The first two expressions are the $2n$ first-order differential equations, which govern the time evolution of the system and are called the canonical Hamilton's equations of motion, which me mentioned earlier.

2.3 Poisson Brackets and Canonical Transformations

In Hamiltonian mechanics, the Poisson bracket provides a fundamental tool for describing the time evolution of observables, identifying conserved quantities and bridging classical and quantum mechanics through canonical quantization. The Poisson bracket originates from the language of differential geometry. Readers unfamiliar with the foundations of differential geometry are encouraged to consult [7] for the necessary definitions and background.

The dynamics involved in phase space can be described by the so-called Poisson bracket. The definition of a Poisson bracket is the following [8]:

Definition 2.1 (Poisson Bracket)**.** *A Poisson bracket on the space $C^\infty(M)$ of smooth functions is a Lie bracket*

$$\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M),$$

satisfying the Leibniz identity

$$\{f, gh\} = \{f, g\}h + g\{f, h\} \qquad \forall f, g, h \in C^\infty(M). \tag{13}$$

Because it is a Lie bracket, Poisson brackets satisfy $\forall f, g, h \in C^\infty(M)$ and $a, b \in \mathbb{R}$

1. **$\mathbb{R}$-bilinearity:**

$$\{af + bg, h\} = a\{f, h\} + b\{g, h\}, \quad \text{and} \quad \{h, af + bg\} = a\{h, f\} + b\{h, g\}$$

2. **Skew-symmetry:**

$$\{f, g\} = -\{g, f\}$$

3. **Jacobi identity:**

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0.$$

Due to the Leibniz identity, the Poisson brackets can be restricted to local coordinate charts on the smooth manifold M. In a local chart $(U, x^1, \ldots, x^n)$ on an n-dimensional manifold the Poisson bracket can be written as

$$\{f, g\}|_U = \sum_{i,j}^{n} \pi^{ij} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j}, \tag{14}$$

where π^{ij} are the Poisson brackets of the local coordinates and are called the structure functions with respect to the chart

$$\pi^{ij} = \{x^i, x^j\}|_U. \tag{15}$$

This abstract definition of a Poisson bracket finds a concrete realization in Hamiltonian mechanics. Mathematically, phase space is modeled as the cotangent bundle of the configuration space T^*Q, which is a symplectic manifold[8]. Without going into details, this means that it possesses the geometric structure necessary for formulating Hamiltonian dynamics. Namely, a symplectic manifold has a symplectic form ω written as

$$\omega = \sum_{i=1}^{n} dx^i \wedge dy_i$$

on the local chart $(U, x^1, \ldots, x^n, y_1, \ldots, y_n)$ of T^*Q. The Poisson bracket is related to the symplectic form ω via the contraction formula

$$\{f, g\} = \omega(X_f, X_g), \tag{16}$$

where X_f, X_g are the Hamiltonian vector fields generated by f and g. In local coordinates, this symplectic form defines a Poisson bracket for which the bracket of two smooth functions f and g takes the form [7]

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x^i} \frac{\partial g}{\partial y_i} - \frac{\partial g}{\partial x^i} \frac{\partial f}{\partial y_i} \right). \tag{17}$$

The necessity of differentiating between upper and lower indices arises when the manifold possesses a metric tensor, as seen in geometric manifolds. Under coordinate frame transformations, upper and lower indices exhibit contra-variant and covariant behaviors, respectively [7].

Now we can relate this to Hamiltonian mechanics by the property that the symplectic form on the phase space T^*Q is now written in terms of the canonical coordinates

$$\omega = \sum_{i=1}^{n} dq^i \wedge dp_i. \tag{18}$$

So we get a Poisson bracket on phase space for which the bracket of two smooth functions $f, g : T^*Q \to \mathbb{R}$ takes the form

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q^i} \frac{\partial f}{\partial p_i} \right). \tag{19}$$

This gives the canonical Poisson bracket relations

$$\{q^i, q^j\} = 0, \qquad \{q^i, p_j\} = \delta^i_j \qquad \{p_i, p_j\} = 0. \tag{20}$$

and the canonical Poisson structure

$$\boldsymbol{\pi} = \begin{pmatrix} O_{n\times n} & I_{n\times n} \\ -I_{n\times n} & O_{n\times n} \end{pmatrix}, \tag{21}$$

where $O_{n\times n}$ is a $n\times n$ matrix with entries zero and $I_{n\times n}$ is the is a $n\times n$ identity matrix. So the canonical coordinates, for which the Hamiltonian can be written in the form (81), naturally define this local Poisson structure. For a function on the local chart $f: U \subset T^*Q \to \mathbb{R}$

$$\begin{aligned}\frac{df}{dt} &= \frac{\partial f}{\partial t} + \sum_{i=1}^{n} \frac{\partial f}{\partial q^i}\frac{dq^i}{dt} + \frac{\partial f}{\partial p_i}\frac{dp_i}{dt} \\ &= \frac{\partial f}{\partial t} + \sum_{i=1}^{n} \frac{\partial f}{\partial q^i}\frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial q^i}\frac{\partial f}{\partial p_i} \\ &= \frac{\partial f}{\partial t} + \{f,H\},\end{aligned} \tag{22}$$

where the Hamilton's equations $\dot{q}_i = \frac{\partial H}{\partial p_i}$ $\dot{p}_i = -\frac{\partial H}{\partial q^i}$ are used to obtain the final expression. From this it becomes apparent that functions with no explicit time-dependence are conserved if $\{f,H\}=0$. Specifically, if the Hamiltonian has no time dependence then the energy is conserved since $\{H,H\} = 0$.

Now most of the times we are interested in keeping everything canonical. So when we do transformations we are interested in the ones that keep the formalism canonical. To this extent we say a canonical transformation is a coordinate transformations $(q,p) \mapsto (Q(p,q),P(q,p))$ that satisfies the Poisson bracket relations

$$\{Q^i,Q^j\} = 0, \qquad \{Q^i,P_j\} = \delta^i_j \qquad \{P_i,P_j\} = 0. \tag{23}$$

This will be true when Hamilton's principle is satisfied for the transformed action functional

$$S[Q(t),P(t),t] = \int_{t_1}^{t_2} dt \left(\sum_{i=1}^{n} P_i\dot{Q}^i - H'\right), \tag{24}$$

where $H' = H'(Q,P,t)$ denotes the original Hamiltonian expressed in the new coordinates. Together with the fact that Hamilton's principle still holds for the original action functional (9), the two integrands differ at most by the time derivative of a function on phase space $F: T^*Q \to \mathbb{R}$ with continuous second-order derivatives [9]. So if the transformed set of variables (Q,P) is canonical, then we must have that

$$\sum_{i=1}^{n} p_i\dot{q}^i - H = \sum_{i=1}^{n} P_i\dot{Q}^i - H' + \frac{dF}{dt}. \tag{25}$$

This relation helps us to establish a bridge between the new and old coordinates using F. When working with two sets of canonical variables related by a canonical transformation, i.e. (q,p) and (Q,P), only $2n$ variables out of the $4n$ $\{q^i, p_i, Q^i, P_i\}$ can be independent. This is because the canonical transformation imposes $2n$ constraints. Now suppose we consider $\{q^i,Q^i\}$ to be the independent variables. We can write the function F as $F = F_1(q,Q,t)$, which is called a type 1 function. For such functions we get

$$\frac{dF_1}{dt} = \frac{\partial F_1}{\partial t} + \sum_{i=1}^{n} \frac{\partial F_1}{\partial q^i}\dot{q}^i + \frac{\partial F_1}{\partial Q^i}\dot{Q}^i. \tag{26}$$

Equation (25) then takes the form

$$\sum_{i=1}^{n} p_i\dot{q}^i - H = \sum_{i=1}^{n} P_i\dot{Q}^i - H' + \frac{\partial F_1}{\partial t} + \sum_{i=1}^{n} \frac{\partial F_1}{\partial q^i}\dot{q}^i + \frac{\partial F_1}{\partial Q^i}\dot{Q}^i \tag{27}$$

which can be rewritten into

$$\sum_{i=1}^{n}\left(p_i-\frac{\partial F_1}{\partial q^i}\right)\dot{q}^i-H=\sum_{i=1}^{n}\left(P_i+\frac{\partial F_1}{\partial Q^i}\right)\dot{Q}^i-H'+\frac{\partial F_1}{\partial t}. \tag{28}$$

Now, since the set $\{q^i, Q^i\}$ is taken to be independent and hence is linearly independent, the equation (28) can only hold if the coefficients vanish

$$p_i=\frac{\partial F_1(q,Q,t)}{\partial q^i}, \qquad P_i=-\frac{\partial F_1(q,Q,t)}{\partial Q^i}. \tag{29}$$

The remaining equation then is

$$H'(Q,P,t)=H(q,p,t)+\frac{\partial F_1(q,Q,t)}{\partial t}, \tag{30}$$

where H' is obtained by substituting (q,p) with their expressions in terms of (Q,P). Given a certain F_1, one can now find the associated canonical transformation using the two expressions in (29). The first expression defines p_i as functions of q_j, Q_j and t. Assuming the Hessian matrix is nonsingular, that is $\det\left(\frac{\partial^2 F}{\partial q^i \partial Q^j}\right) \neq 0$, then Q_i can be solved for in terms q_j, p_j and t. Using the second expression, one can find P_i as functions of q_j, p_j and t. Putting together, one finds the full canonical transformation.

It is possible that we would encounter a transformation such that p_i is a function of q_j, P_j and t instead of q_j, p_j and t, then $F_1(q,Q,t)$ would not be the most appropriate choice. In this case we write F with type 2 functions, where we take q and P to be the independent coordinates. We can do this by performing the Legendre transformation $(q,Q) \mapsto (q,P)$ such that

$$F=F_2(q,P,t)-\sum_{i=1}^{n}Q^iP_i. \tag{31}$$

Substituting this into equation (25) we find that

$$\sum_{i=1}^{n}\left(p_i-\frac{\partial F_2}{\partial q^i}\right)\dot{q}^i-H=\sum_{i=1}^{n}\left(\frac{\partial F_2}{\partial P_i}-Q^i\right)\dot{P}_i-H'+\frac{\partial F_2}{\partial t}. \tag{32}$$

By the same reasoning as before we get the expressions

$$p_i=\frac{\partial F_2(q,P,t)}{\partial q^i}, \qquad Q^i=\frac{\partial F_2(q,Q,t)}{\partial P_i}, \qquad H'(Q,P,t)=H(q,p,t)+\frac{\partial F_2(q,P,t)}{\partial t}. \tag{33}$$

Again, using the first expression one can find P_i written as a function of q_j, p_j and t if $\det\left(\frac{\partial^2 F}{\partial q^i \partial P_j}\right) \neq 0$. Then the functions Q^i in terms q_j, p_j and t can be found by the second expression.

In a similar manner one can find the expression relating the coordinates for which the independent variables are taken to be either Q and p or p and P. This can be summarized in the following table
To summarize, if one has a canonical coordinate system (q,p) and a coordinate transformation $(q,p) \mapsto (Q,P)$ for which the transformation equations are satisfied for some particular $F_{1,2,3,4}$, then by reverse reasoning, equation (25) holds and hence (Q,P) and the transformation are canonical. The

Type	Generating Function	Independent Variables	Transformation Equations
1	$F_1(q,Q,t)$	q,Q	$p_i = \frac{\partial F_1}{\partial q^i}, \quad P_i = -\frac{\partial F_1}{\partial Q^i}$
2	$F_2(q,P,t)$	q,P	$p_i = \frac{\partial F_2}{\partial q^i}, \quad Q^i = \frac{\partial F_2}{\partial P_i}$
3	$F_3(p,Q,t)$	p,Q	$q^i = -\frac{\partial F_3}{\partial p_i}, \quad P_i = -\frac{\partial F_3}{\partial Q^i}$
4	$F_4(p,P,t)$	p,P	$q^i = -\frac{\partial F_4}{\partial p_i}, \quad Q^i = \frac{\partial F_4}{\partial P_i}$

Table 1: Types of generating functions, their independent variables, and the resulting canonical transformation equations.

converse, however, is not entirely true. It is possible that a transformation cannot be derived from one of the generating functions $F_{1,2,3,4}$, but rather from a mixture of the four types. In such cases there is another way to check if a transformation is canonical:

A transformation is canonical if it satifies the symplectic condition with respect to the Poisson structure

$$\boldsymbol{A}^T \boldsymbol{\pi} \boldsymbol{A} = \boldsymbol{\pi}, \tag{34}$$

where $\boldsymbol{A}$ *is the jacobian matrix* $\frac{\partial(Q,P)}{\partial(q,p)}$ *of the transformation.*

Indeed only for a canonical transformation we see that

$$\begin{aligned}
\boldsymbol{A}^T \boldsymbol{\pi} \boldsymbol{A} &= \begin{pmatrix} \frac{\partial Q}{\partial q} & \frac{\partial P}{\partial q} \\ \frac{\partial Q}{\partial p} & \frac{\partial P}{\partial p} \end{pmatrix} \begin{pmatrix} O_{n\times n} & I_{n\times n} \\ -I_{n\times n} & O_{n\times n} \end{pmatrix} \begin{pmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{pmatrix} \\
&= \begin{pmatrix} \frac{\partial Q}{\partial q}\frac{\partial P}{\partial q} - \frac{\partial Q}{\partial q}\frac{\partial P}{\partial q} & \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} \\ \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} - \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} & \frac{\partial Q}{\partial p}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial p} \end{pmatrix} \\
&= \begin{pmatrix} 0 & \{Q,P\} \\ -\{Q,P\} & 0 \end{pmatrix} = \begin{pmatrix} O_{n\times n} & I_{n\times n} \\ -I_{n\times n} & O_{n\times n} \end{pmatrix} = \boldsymbol{\pi}.
\end{aligned} \tag{35}$$

Finally, the phase space we are primarily interested in is the phase space of $\mathbb{R}^n$, as it serves as the foundational setting for the theories discussed in this thesis. As we said, this is described by the cotangent bundle $T^*\mathbb{R}^n$, which is isomorphic to $\mathbb{R}^{2n}$ [8]. In this case, the canonical set of local coordinates $(q^1,\ldots,q^n,p_1,\ldots,p_n)$ become global because of the flat structure of $\mathbb{R}^{2n}$ is flat, and the Poisson bracket retains the form

$$\{f,g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q^i}\frac{\partial f}{\partial p_i} \right) \qquad \forall (q,p) \in T^*\mathbb{R}^n. \tag{36}$$

Therefore, all results that follow will hold globally, without requiring restriction to local coordinate charts. This will be useful when we consider manifolds embedded in a flat ambient space.

3 Time as a Dynamical Variable: Extended Phase Space

In this chapter, we will discuss a phase space formulation in which time is treated as a dynamical variable on equal footing with spatial coordinates. We first show that the standard Legendre transform approach fails due to the extended Lagrangian's homogeneity and the resulting constraints. To overcome this, we introduce an alternative construction of an extended Hamiltonian defined directly on an extended phase space including time and its conjugate momentum. We show the corresponding extended Poisson brackets and extended canonical transformations. Finally, we apply this framework to Minkowski spacetime, demonstrating that Lorentz transformations act as canonical transformations on the extended phase space and that the extended Hamiltonian is Lorentz-invariant. This sets the stage for a fully covariant Hamiltonian formalism suitable for relativistic systems. This chapter is based on [4], [10] and [8].

3.1 Failure of the Standard Legendre Procedure

As seen in the preceding chapter, the conventional way to obtain the Hamiltonian formalism is through Lagrangian mechanics. Therefore it seems only natural to start by promoting time to a dynamical variable and try to obtain the Legendre transform of the Lagrangian. To treat time as a dynamical variable, we introduce a reparameterization of the trajectory by promoting the configuration space coordinates and the time parameter to functions of a new parameter s. Consider the standard configuration space Q of a dynamical system with local coordinates q^i ($i = 1,\ldots,n$). We construct the extended configuration space $Q_e = Q \times \mathbb{R}$, where the additional factor $\mathbb{R}$ corresponds to the time t. The reparameterization is defined by introducing an arbitrary monotonically varying parameter s, such that time becomes a function $t = t(s)$, and the configuration variables become $q^i(s) = q^i(t(s))$. A trajectory is then described by the map

$$\gamma : \mathbb{R} \to Q_e, \quad s \mapsto \left(q^i(s), t(s)\right). \tag{37}$$

The requirement of s to be monotonically varying is such that $\frac{dt}{ds}$ is always finite and nonzero and hence trajectories become smooth.

Now to formulate the extended Lagrangian, we start by denoting derivatives with respect to the new parameter as $q'^i \equiv \frac{dq^i}{ds}$. By the chain rule we find that

$$q'^i = \frac{\partial q^i}{\partial t}\frac{dt}{ds} = \dot{q}^i t' \tag{38}$$

such that q'^i can be written as

$$\dot{q}^i = \frac{q'^i}{t'} \tag{39}$$

using the fact that t' is always nonzero. After reparameterization, we have that $dt = t'ds$ and so the action functional (2) becomes

$$S[q(s),t(s)] = \int_{s_0}^{s_1} ds\, L\left[q_i(s), \frac{q_i'(s)}{t'(s)}, t(s)\right] t'(s). \tag{40}$$

Defining the extended Lagrangian where time is treated among the dynamical variables as

$$L_e\left[q(s), q'(s), t(s), t'(s)\right] \equiv L\left[q(s), \frac{q_i'(s)}{t'(s)}, t(s)\right] t'(s), \tag{41}$$

results in the action functional becoming

$$S[q(s),t(s)] = \int_{s_0}^{s_1} ds\, L_e \left[q(s), q'(s), t(s), t'(s)\right]. \tag{42}$$

This action has the important property that it remains the same under any arbitrary monotonically varying reparameterization of the trajectory. We call this reparameterization invariance. One can see this by noting that for any nonzero scalar λ,

$$L_e\left[q(s), \lambda q'(s), t(s), \lambda t'(s)\right] = L\Big[q(s), \frac{\lambda q'(s)}{\lambda t'(s)}, t(s)\Big]\,(\lambda t'(s)) = \lambda L_e\left[q(s), q'(s), t(s), t'(s)\right]. \tag{43}$$

Thus, the extended Lagrangian is homogeneous of degree one in the generalized velocities. It is precisely this property that implies that the choice of a monotonic parameterization is arbitrary. More generally, it can be shown that any Lagrangian that is homogeneous of degree one in the velocities generates dynamics that are invariant under reparameterizations of the curve parameter [4]. Although the explicit form of the extended Lagrangian changes with different parameterizations, the resulting action is form-invariant and yields equivalent equations of motion.

To continue, we notice that equation (42) is analogous to the standard form and hence we extend on this formalism by now defining the extended conjugate momenta as

$$p_k \equiv \frac{\partial L_e}{\partial q'_k}, \tag{44}$$

where $k = 0, 1, ..., n$ and $q^0 \equiv t$. For the original generalized coordinates the momenta become

$$\begin{aligned} p_i = \frac{\partial L_e}{\partial q'^i} &= \frac{\partial}{\partial q'^i}\left(L\left[q^j, \frac{q'^i}{t'}, t\right] t'\right) \\ &= \sum_{j=0}^{n}\left(\frac{\partial L}{\partial \dot{q}^j}\cdot\frac{\partial \dot{q}^j}{\partial q'^i}\right) t' \\ &= \sum_{j=0}^{n}\left(\frac{\partial L}{\partial \dot{q}^j}\cdot\frac{\delta_i^j}{t'}\right) t' \\ &= \frac{\partial L}{\partial q'^i} \qquad i = 1, \ldots, k, \end{aligned} \tag{45}$$

which is exactly the same as the original conjugate momenta. The conjugate momenta for the time variable is given by

$$\begin{aligned} p_0 = \frac{\partial L_e}{\partial q'^0} &= \frac{\partial}{\partial t'}\left(L\left[q^j, \frac{q'^j}{t'}, t\right] t'\right) \\ &= L + \sum_{j=1}^{n}\left(\frac{\partial L}{\partial \dot{q}^j}\cdot\frac{\partial \dot{q}^j}{\partial t'}\right)t' \\ &= L + \sum_{j=1}^{n}\left(p_j \cdot \frac{\partial}{\partial t'}\left(\frac{q'^j}{t'}\right)\right)t' \\ &= L + \sum_{j=1}^{n}\left(-\frac{p_j q'^j}{t'^2}\right)t' \\ &= L - \sum_{j=1}^{n}\dot{q}^j p_j = -H, \end{aligned} \tag{46}$$

where H is the standard Hamiltonian. This means that p_0 is not an independent variable. It is algebraically constrained by q_i, p_i and t through the Hamiltonian H. Besides the fact that this shows that one cannot obtain a complete set of independent coordinates for phase space through the standard Lagrangian method, another problems arises. When we try to follow the procedure as shown in section 2.2 we will get a Hamiltonian that is identically zero:
Defining the intermediate function from the extended Lagrangian, we get that this is

$$F(q,\dot{q},t) = \sum_{k=0}^{n} q'_k p_k(q,\dot{q}) - L_e(q,\dot{q},t) = t'\sum_{i=1}^{n}\dot{q}_i p_i - t'H - t'L = t'\left(\sum_{i=1}^{n}\dot{q}_i p_i - L\right) - t'H = 0. \tag{47}$$

This function vanishes identically for all values of the variables involved. As a result, there is no well-defined Legendre transformation that can be performed, and thus no nontrivial Hamiltonian can be obtained through this method. It is clear that due to the relation (46) there is no way to obtain the standard Hamiltonian formalism through the Lagrangian formalism. Consequently, we must seek an alternative approach to formulate a consistent Hamiltonian framework where time is a dynamical variable.

3.2 Extended Hamiltonian

Our alternative approach starts similarly by extending the configuration space Q to the extended configuration space $Q_e = Q \times \mathbb{R}$ by including the time coordinate $q^0 \equiv t$. The extended phase space is the phase space of Q_e, namely T^*Q_e, with local coordinates

$$(q^0, q^1, \ldots, q^n, p_0, p_1, \ldots, p_n), \tag{48}$$

called the extended phase space coordinates, where p_0 is the conjugate momentum associated with time. Now it is important to emphasize that while $(q^1, \ldots, q^n, p_1 \ldots, p_n)$ are defined through the standard phase space with equation (5), we don't know yet the expression for p_0. Now instead of deriving the extended Hamiltonian from an extended Lagrangian, we define it directly through the extended action functional. That is, we will rewrite the extended action in terms of extended phase space coordinates such that it is completely analogous to the standard action

$$S[q(t),p(t)] = \int_{t_1}^{t_2} dt \left(\sum_{i=1}^{n} p_i \dot{q}_i - H \right). \tag{49}$$

From there we will define the extended Hamiltonian as the expression in the extended action principle which takes the same place of the H in the standard action. This goes as follows:

Again, we introduce a monotonically varying the evolution parameter s, and treat time $t = q^0(s)$ as a dynamical variable. The action functional can be written to form the extended action functional

$$\begin{aligned} S[q(s),p(s),t(s)] &= \int_{t_1}^{t_2} dt \left(\sum_{i=1}^{n} p_i \dot{q}_i - H \right) \\ &= \int_{s_1}^{s_2} ds\, t' \left(\sum_{i=1}^{n} p_i \dot{q}_i - H \right) \\ &= \int_{s_1}^{s_2} ds \left(\sum_{i=1}^{n} p_i q'^i - t'H \right) \\ &= \int_{s_1}^{s_2} ds \left(\sum_{k=0}^{n} p_k q'^k - (H + p_0)t' \right), \end{aligned} \tag{50}$$

where the prime still denotes differentiation with respect to s. Now we will define the extended Hamiltonian $H_e : T^*Q_e \to \mathbb{R}$ as

$$H_e(q^k, p_k) \equiv \alpha(H(q^i, p_i, t) + p_0), \qquad \alpha \equiv \frac{dt}{ds}. \tag{51}$$

We have written $\alpha \equiv \frac{dt}{ds}$ to emphasize that it must not be considered as a function on the extended phase space. Mathematically, $\alpha = \frac{dt}{ds}$ is a function on the tangent bundle TQ_e and hence merely a scaling factor external to the extended phase space T^*Q_e [10]. With the extended Hamiltonian simply being a function on the extended phase space, the action functional will be completely analogous to the standard action functional, as can be seen by substituting the expression for H_e into (50):

$$S[q(s),p(s),t(s)] = \int_{s_1}^{s_2} ds \left(\sum_{k=0}^{n} p_k q'^k - H_e \right). \tag{52}$$

So the extended Hamiltonian takes on the role of the standard Hamiltonian while the extended phase space coordinates $(q^0, q^1, \ldots, q^n, p_0, p_1 \ldots, p_n)$ become the canonical coordinates. It can be shown that this action is invariant under reparameterizations just like (42) [4]. Using Hamilton's principle in the form of (42), we get the following Euler-Lagrange equations with respect to the independent coordinates q^k and p_k:

$$\frac{\partial}{\partial q^k} \left(\sum_{j=0}^{n} p_j q'^j - H_e(q^k, p_k, t) \right) - \frac{d}{ds} \left(\frac{\partial}{\partial q'^j} \left(\sum_{j=0}^{n} p_j q'^j - H_e(q^k, p_k, t) \right) \right) = 0 \tag{53}$$

and

$$\frac{\partial}{\partial p_k} \left(\sum_{j=0}^{n} p_j q'^j - H_e(q^k, p_k, t) \right) - \frac{d}{ds} \left(\frac{\partial}{\partial p'_k} \left(\sum_{j=0}^{n} p_j q'^j - H_e(q^k, p_k, t) \right) \right) = 0. \tag{54}$$

Now from the fact that $\frac{\partial H_e}{\partial q'^k} = 0$ and $\frac{\partial H_e}{\partial p'_k} = 0$, one can easily verify that this reduces to the extended canonical equations of motion

$$\frac{dq^k}{ds} = \frac{\partial H_e}{\partial p_k}, \qquad \frac{dp_k}{ds} = -\frac{\partial H_e}{\partial q^k} \qquad\qquad k = 0, 1, \ldots, n. \tag{55}$$

We see now why it is important that we emphasized that $\frac{dt}{ds}$ is merely a scaling factor instead of a function on T^*Q_e, otherwise $\frac{\partial H_e}{\partial q^{0'}}$ would have been nonzero. Now from the equations of motion we see that for a physical system, where there is no explicit dependence on s, that

$$\frac{dH_e}{ds} = \sum_{k=0}^{n} \frac{\partial H_e}{\partial q^k}\frac{dq^k}{ds} + \frac{\partial H_e}{\partial p_k}\frac{dp_k}{ds} = \sum_{k=0}^{n} -p'_k q'^k + q'^k p'_k = 0.$$

So we see that due to Hamilton's equations of motion, the extended Hamiltonian is a constant. This means that $p_0(s)$ equals the value of $-H(q,p,t)$ up to a constant, because $H_e = \alpha(H(q^i, p_i, t) + p_0)$ and s is monotonically varying such that α is always nonzero. For simplicity the constant is set to zero, because selecting a nonzero constant for H_e would merely correspond to a trivial redefinition of the conventional Hamiltonian [11]. It seems we again recover the same issue as before, namely

$$p_0(s) + H(q^i, p_i, t) = 0. \tag{56}$$

However, the key difference with before is that the extended Hamiltonian does not vanish identically such as the Hamiltonian obtained through the Legendre transformation. What we have now is that the condition (56) is merely a consequence of the fact that due to Hamilton's principle, $H_e = 0$ constitutes an implicit function that defines a $(2n+1)$-submanifold $M \subset T^*Q_e$ in the extended phase space. This submanifold is the physical phase space describing the system. This is analogous to systems where the standard Hamiltonian equals to the energy giving the implicit function $E - H(q,p,t) = 0$ constituting an $(2n-1)$-hypersurface in the standard phase space. To isolate the physically meaningful phase space, one constructs coordinates that respect the constraint, providing a parametrization of the submanifold M.

3.3 Extended Poisson Brackets

We are now in a position to write the corresponding Poisson brackets for our extended phase space. As we saw in the preceding section the canonical coordinates are the extended phase space coordinates $(q^0, q^1, \ldots, q^n, p_0, p_1 \ldots, p_n)$. As shown in Section 2.3, this leads to a local Poisson bracket on T^*Q_e such that, for two smooth functions $f, g : T^*Q_e \to \mathbb{R}$, we have

$$\{f,g\}_e = \sum_{k=0}^{n}\left(\frac{\partial f}{\partial q^k}\frac{\partial g}{\partial p_k} - \frac{\partial g}{\partial q^k}\frac{\partial f}{\partial p_k}\right) = \{f,g\} + \left(\frac{\partial f}{\partial t}\frac{\partial g}{\partial p_0} - \frac{\partial g}{\partial t}\frac{\partial f}{\partial p_0}\right). \tag{57}$$

This is exactly of the same form as the standard Poisson bracket with just an additional independent coordinate and its conjugate momentum. From this extended bracket we get the extended form of the canonical Poisson bracket relations

$$\{q^k, q^l\} = 0, \qquad \{q^k, p_l\} = \delta^k_l \qquad \{p_k, p_l\} = 0, \qquad\qquad k, l = 0, 1, \ldots, n. \tag{58}$$

and the extended Poisson structure

$$\pi_e = \begin{pmatrix} O_{(n+1)\times(n+1)} & I_{(n+1)\times(n+1)} \\ -I_{(n+1)\times(n+1)} & O_{(n+1)\times(n+1)} \end{pmatrix}. \tag{59}$$

For a function $f : T^*Q_e \to \mathbb{R}$, with no explicit s-dependence, we have

$$\frac{df}{ds} = \sum_{k=0}^{n} \frac{\partial f}{\partial q^k}\frac{dq^k}{ds} + \frac{\partial f}{\partial p_k}\frac{dp_k}{ds} = \sum_{k=0}^{n} \frac{\partial f}{\partial q^k}\frac{\partial H_e}{\partial p_k} - \frac{\partial H_e}{\partial q^k}\frac{\partial f}{\partial p_k} = \{f, H_e\}_e. \tag{60}$$

Analogously to the standard phase space, in the extended phase space a function is a conserved if $\{f, H_e\}_e = 0$ and we see that Hamilton's equations of motion also follow from this bracket as

$$\frac{dq^k}{ds} = \{q^k, H_e\}_e = \sum_{j=0}^{n} \frac{\partial q^k}{\partial q^j}\frac{\partial H_e}{\partial p_j} - \frac{\partial H_e}{\partial q^j}\frac{\partial q^k}{\partial p_j} = \sum_{j=0}^{n} \delta_k^j \frac{\partial H_e}{\partial p_j} = \frac{\partial H_e}{\partial p_k} \tag{61}$$

and similarly

$$\frac{dp_k}{ds} = \{p_k, H_e\}_e = \sum_{j=0}^{n} \frac{\partial p_k}{\partial q^j}\frac{\partial H_e}{\partial p_j} - \frac{\partial H_e}{\partial q^j}\frac{\partial p_k}{\partial p_j} = -\sum_{j=0}^{n} \delta_k^j \frac{\partial H_e}{\partial q^j} = -\frac{\partial H_e}{\partial q^k}. \tag{62}$$

3.4 Extended Canonical Transformations

To find properties about the canonical transformations in the extended phase space, we will use the same approach as shown in section 2.4 using generating functions. The only difference is that we now start with the extended action principle

$$S[q(s), p(s), t(s)] = \int_{s_1}^{s_2} ds \left(\sum_{k=0}^{n} p_k {q'}^k - H_e \right) \tag{63}$$

and the modified extended action principle

$$S[Q(s), P(s), t(s)] = \int_{s_1}^{s_2} ds \left(\sum_{k=0}^{n} Q_k {P^k}' - H'_e \right), \tag{64}$$

where $H'_e = H'_e(Q, P, s)$ denotes the extended Hamiltonian expressed in the new coordinates and the coordinate transformations $(q, p) \mapsto (Q(p, q), P(q, p))$ now include t and p_0. As before, the transformation is canonical if the two integrands differ by a total derivative with respect to s of a function $F : T^*Q_e \to \mathbb{R}$, which is at least twice continuously differentiable. That is

$$\sum_{k=0}^{n} p_k {q'}^k - H_e = \sum_{k=0}^{n} P_k {q'}^k - H'_e + \frac{dF}{ds}. \tag{65}$$

If we consider a type 1 function, i.e. $F = F_1(q, Q, s)$, with q and Q as the independent variables, we find that

$$\frac{dF_1}{ds} = \frac{\partial F_1}{\partial s} + \sum_{k=0}^{n} \frac{\partial F_1}{\partial q^k}{q'}^k + \frac{\partial F_1}{\partial Q^k}{Q'}^k. \tag{66}$$

With this we can write (65) in the form

$$\sum_{k=0}^{n}\left(p_k - \frac{\partial F_1}{\partial q^k}\right) q'^k - H_e = \sum_{k=0}^{n}\left(P_k + \frac{\partial F_1}{\partial Q^k}\right) Q'^k - H'_e + \frac{\partial F_1}{\partial s}. \tag{67}$$

Since q and Q are taken as independent variables, they form a linear independent set and hence we find that

$$p_k = \frac{\partial F_1(q,Q,t)}{\partial q^k}, \qquad P_k = -\frac{\partial F_1(q,Q,t)}{\partial Q^k} \tag{68}$$

and the remaining equation becomes

$$H'_e(Q,P,s) = H_e(q,p,s) + \frac{\partial F_1(q,Q,s)}{\partial s} \tag{69}$$

similar to the standard results as in section 2.3. Given that the Hessian condition $\det\left(\frac{\partial^2 F_1}{\partial q^k \partial Q^l}\right) \neq 0$ is satisfied we can express the new set of coordinates (Q,P) in terms of (q,p,s). For the other types of function, it is good to note that given expressions (68) and (69) t and p_0 behave exactly the same as an extra coordinate would in the standard approach as in 2.4. It is therefore that the results for the other types of functions will also be identical. Therefore we omit the computations and we show the results in table 2. Finally, if we were to be dealing with a transformation which has no connected generating function, we can check it with the symplectic condition with the extended Poisson structure (59).

Type	**Generating Function**	**Independent Variables**	**Transformation Equations**
1	$F_1(q^i,Q^i,t,T)$	q^i,Q^i,t,T	$p_i = \frac{\partial F_1}{\partial q^i}, \quad P_i = -\frac{\partial F_1}{\partial Q^i},$ $p_0 = \frac{\partial F_1}{\partial t}, \quad P_0 = -\frac{\partial F_1}{\partial T}$
2	$F_2(q^i,P_i,t,P_0)$	q^i,P_i,t,P_0	$p_i = \frac{\partial F_2}{\partial q^i}, \quad Q^i = \frac{\partial F_2}{\partial P_i},$ $p_0 = \frac{\partial F_2}{\partial t}, \quad T = -\frac{\partial F_2}{\partial P_0}$
3	$F_3(p_i,Q^i,p_0,T)$	p_i,Q^i,p_0,T	$q^i = -\frac{\partial F_3}{\partial p_i}, \quad P_i = -\frac{\partial F_3}{\partial Q^i},$ $t = -\frac{\partial F_3}{\partial p_0}, \quad P_0 = -\frac{\partial F_3}{\partial T}$
4	$F_4(p_i,P_i,p_0,P_0)$	p_i,P_i,p_0,P_0	$q^i = -\frac{\partial F_4}{\partial p_i}, \quad Q^i = \frac{\partial F_4}{\partial P_i},$ $t = -\frac{\partial F_4}{\partial p_0}, \quad T = \frac{\partial F_4}{\partial P_0}$

Table 2: Types of generating functions, their independent variables, and the resulting canonical transformation equations.

3.5 Phase Space on Minkowski Spacetime $\mathbb{M}^4$

The main point of introducing time as a dynamical variable is to treat it on the same footing as the spatial variables. In this way, we can deal with systems where time is treated as a coordinate. The most noteworthy example of this is Minkowski spacetime. Minkowski spacetime is defined as the four-dimensional flat geometric manifold $\mathbb{M}^4$ equipped with a metric of signature $(-+++)$ [3]. This metric determines the spacetime interval as

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu, \tag{70}$$

where x^μ is the four-vector with components (ct, x, y, z) and $\eta_{\mu\nu}$ is the Minkowski metric, given by the diagonal matrix $\mathrm{diag}(-1, 1, 1, 1)$. The isometry group of $\mathbb{M}^4$ is the Poincaré group, which is the semidirect product of the Lorentz group $O(1,3)$ and the set of translations in $\mathbb{M}^4$. The Lorentz group, consisting of transformations called Lorentz transformations, has a specific representation on four-vectors q^μ and p_μ which is given by

$$x^\mu \to x'^\mu = \Lambda^\mu_\nu x^\nu, \qquad p_\mu \to p'_\mu = (\Lambda^{-1})^\mu{}_\nu p_\nu \tag{71}$$

where $\Lambda^\mu{}_\nu$ satisfies

$$\eta_{\mu\nu} = \eta_{\rho\sigma} \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu. \tag{72}$$

Now the Lorentz transformations consist of rotations in the spatial coordinates, boosts and compositions of these transformations. For a four vector with components (ct, x, y, z) a general boost has the form

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta_x & -\gamma\beta_y & -\gamma\beta_z \\ -\gamma\beta_x & 1+(\gamma-1)\frac{\beta_x^2}{\beta^2} & (\gamma-1)\frac{\beta_x\beta_y}{\beta^2} & (\gamma-1)\frac{\beta_x\beta_z}{\beta^2} \\ -\gamma\beta_y & (\gamma-1)\frac{\beta_y\beta_x}{\beta^2} & 1+(\gamma-1)\frac{\beta_y^2}{\beta^2} & (\gamma-1)\frac{\beta_y\beta_z}{\beta^2} \\ -\gamma\beta_z & (\gamma-1)\frac{\beta_z\beta_x}{\beta^2} & (\gamma-1)\frac{\beta_z\beta_y}{\beta^2} & 1+(\gamma-1)\frac{\beta_z^2}{\beta^2} \end{pmatrix}, \tag{73}$$

where $\gamma = 1/\sqrt{1 - v^2/c^2}$ and $\beta_i = \frac{v_i}{c}$ with $i = x, y, z$. For rotations the Lorentz matrix is block-diagonal

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix}, \tag{74}$$

where R is a 3×3 rotation matrix satisfying $R^{\mathrm{T}} R = I$ and $\det R = 1$, i.e. an element of SO(3).

To analyze phase space transformations in Minkowski spacetime, we start with the dynamical variables (t, x, y, z) and parameterize the system by the proper time τ. For a system with fixed energy E, we get $p_0 = -E$ and the canonical coordinates take the form $(t, x, y, z, -E, p_x, p_y, p_z)$. By performing the canonical transformation $(t, E) \mapsto (ct, E/c)$ we recast these variables into the standard relativistic four-vector notation, yielding canonical coordinates (x^μ, p_μ). This transformation aligns the phase space coordinates with the Lorentz covariant framework, where x^μ and p_μ transform respectively as contravariant and covariant four-vectors under Lorentz transformations. Now let us consider the generating function of type 2

$$F_2(x^\mu, p'_\mu) = \Lambda^\nu{}_\mu x^\mu p'_\nu, \tag{75}$$

where p'_ν is the transformed p_μ. First we observe that the corresponding Hessian matrix is nonsingular because

$$\frac{\partial^2 F_2}{\partial x^\mu \partial p'_\nu} = \frac{\partial}{\partial x^\mu} \left(\frac{\partial \left(\Lambda^\rho{}_\sigma x^\sigma p'_\rho \right)}{\partial p'_\nu} \right) = \frac{\partial}{\partial x^\mu} \left(\Lambda^\nu{}_\sigma x^\sigma \right) = \Lambda^\nu{}_\mu \tag{76}$$

and $\det\Lambda = \pm 1$. Secondly, we observe that the transformations equations are

$$p_\mu = \frac{\partial F_2}{\partial x^\mu} = \Lambda^\nu{}_\mu p'_\nu, \qquad x'^\mu = \frac{\partial F_2}{\partial p'_\mu} = \Lambda^\mu{}_\nu x^\nu, \tag{77}$$

which are precisely the Lorentz transformations on four vectors. Therefore, the transformation equations are satisfied, and the Lorentz transformations are canonical. The set of translations in Minkowski spacetime, given by

$$x^\mu \to x'^\mu = x^\mu + \varepsilon^\mu, \tag{78}$$

is trivially canonical since constant shifts vanish in the extended Poisson brackets. This shows that the Poincaré group forms a subgroup of the canonical group on Minkowski spacetime, i.e.the group of all canonical transformations on $\mathbb{M}^4$. Finally, it is quite easy to show that the value of the extended Hamiltonian does not depend on the reference frame, unlike the standard Hamiltonian. Looking at equations (73) and (74), we see that in both rotations and boosts, there is no explicit dependence on the parameter τ. By equation (69) we see that

$$H'_e(x'^\mu, p'_\mu, \tau) = H_e(x^\mu, p_\mu, \tau), \tag{79}$$

which means that the extended Hamiltonian is indeed Lorentz-invariant (here Lorentz-invariance refers to the fact that the transformed extended Hamiltonian H'_e takes on the same numerical value as H_e for every point $\left(q^\mu, p_\mu\right) \in T^*\mathbb{M}^4$).

4 Constrained phase space

In this chapter, we discuss the Hamiltonian formalism for systems with constraints, which naturally arise when motion is restricted to a surface or when redundant degrees of freedom are present, as in gauge theories. We first explain how constraints modify the canonical structure and distinguish between first-class and second-class constraints, showing how the former generate gauge transformations. To consistently handle second-class constraints, we introduce the Dirac bracket, which projects the dynamics onto the constraint surface. To illustrate how these ideas work in practice, we derive the Dirac structure for a particle on a sphere and generalize this to an arbitrary curved hypersurface, revealing how the algebra encodes the induced geometry. These results lay the foundation for the treatment of more complex systems, such as curved spacetimes, in later chapters. This chapter is based on references [5], [12], [13], [14] and [15].

4.1 The Legendre Transformation and The Equations of Motion

Like said before, normally one obtains the Hamiltonian of a system by performing a Legendre transformation on the Lagrangian, given that $\det\left(\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\right) \neq 0$. However, it is possible for systems to not satisfy this property. Given that we have a system for which $\det\left(\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\right) = 0$, the momenta $p_i = \frac{\partial L}{\partial \dot{q}_i}$ cannot be inverted to express velocities $\dot{q}_i$ as functions of the coordinates and momenta. Instead, there will be some $M \in \mathbb{N}$ constraints

$$\phi_m(q,p) = 0, \qquad m = 1,\ldots,M, \tag{80}$$

where the p's are replaced with their definition in terms of q's and $\dot{q}$'s. These are called primary constraints to emphasize that they arise directly from the singularity of the Legendre transformation, independently of the equations of motion, and that they do not necessarily correspond to any constraints present in the Lagrangian formalism [16]. Assuming that the rank of the Legendre transformation is a constant $n-k$ throughout TQ, there will be k independent constraints among the ϕ_m. The Legendre transformation then maps from the $2n$-dimensional TQ to a $(2n-k)$-dimensional submanifold $\Gamma \subset T^*Q$ defined by the constraints. We call this submanifold the primary constraint surface and it is on this surface that the system's evolution takes place. Since the transformation has rank $n-k$, for any point $(q,p) \in T^*Q$ there exist a k-dimensional manifold in TQ that maps to this point [16]. Now if we want to have a well-defined time evolution within the Hamiltonian formalism every point in phase space should have only one corresponding point in TQ, because otherwise different time evolutions will stem from the same initial conditions [15]. To make the transformation nonsingular we need to introduce at least k parameters, which will act as extra coordinates. We will later see that these parameters come in the form of Lagrange multipliers.

Now before we move on, we must first specify some details considering the primary constraints and the constrained surface. Whenever we are dealing with a set of dependent constraints, one can always locally reduce this set to a new set of independent constraints [16]. So from now on we assume that any set of constraints we are dealing with has been reduced to a set of k independent constraints. Also it will be convenient to use the sign $\approx$ whenever we equalities hold on the primary constrains surface, e.g. $\phi_k \approx 0$. With this in mind, for a set of independent constraints we introduce a useful theorem, for which the proof can be found in [16]:

Theorem 4.1. *If* $\sum_{i=1}^{n}\left(\lambda_i\delta q^i+\mu^i\delta p_i\right)=0$ *for arbitrary variations* δq^n, δp_n *tangent to the constraint surface, then*

$$\lambda_i\approx\sum_{j=1}^{k}u^j\frac{\partial\phi_j}{\partial q^i},\qquad \mu^i\approx\sum_{j=1}^{k}u^j\frac{\partial\phi_j}{\partial p_i}$$

for some functions u^k.

Now let us consider the function of the form

$$F(q,\dot q)\equiv\sum_{i=1}^{n}\dot q_i p_i(q,\dot q)-L(q,\dot q), \tag{81}$$

where there is no explicit time-dependence in L. If we were dealing with a Legendre transformation that is singular, it seems that we are unable to define the Hamiltonian by writing F as a function of q and p, because we cannot write $\dot q$ as a function of q and p. However, Legendre transformations have the property that $\dot q$ only appears through the combination of $p_i(q,\dot q)$. This can be shown by the fact that

$$\begin{aligned}\delta H&=\sum_{i=1}^{n}\left(\dot q^i\delta p_i+\delta\dot q^i p_i-\frac{\partial L}{\partial q^i}\delta q^i-\frac{\partial L}{\partial\dot q^i}\delta\dot q^i\right)\\&=\sum_{i=1}^{n}\left(\dot q^i\delta p_i+\delta\dot q^i p_i-\frac{\partial L}{\partial q^i}\delta q^i-p_i\delta\dot q^i\right)\\&=\sum_{i=1}^{n}\left(\dot q^i\delta p_i-\frac{\partial L}{\partial q^i}\delta q^i\right),\end{aligned} \tag{82}$$

which shows that H only changes through variations from q and p. Strictly speaking, this does allow us to consider F as a function of q and p and define the Hamiltonian. However, the problem is that it won't be uniquely determined. This can be understood by the fact that δp_i are restricted to move along the constraints (80), which are identities with the p's expressed as functions of $(q,\dot q)$ [16]. Also, the Hamiltonian is only well-defined on the primary constraint surface, because of the same idea that δp_i are restricted. So to actually find a useful Hamiltonian we first still have to consider F as a function of $(q,\dot q)$. Using equation (82) and the fact that

$$\delta F=\sum_{i=1}^{n}\left(\frac{\partial F}{\partial q^i}\delta q^i+\frac{\partial F}{\partial p_i}\delta p_i\right) \tag{83}$$

we get the following

$$\sum_{i=1}^{n}\left(\left(\frac{\partial F}{\partial q^i}+\frac{\partial L}{\partial q^i}\right)\delta q^i+\left(\frac{\partial F}{\partial p_i}-\dot q^i\right)\delta p_i\right)=0. \tag{84}$$

We can apply theorem 4.1 to get that

$$\frac{\partial F}{\partial q^i}+\frac{\partial L}{\partial q^i}\approx-\sum_{j=1}^{k}u^j\frac{\partial\phi_j}{\partial q^i},\qquad \frac{\partial F}{\partial p_i}-\dot q^i\approx-\sum_{j=1}^{k}u^j\frac{\partial\phi_j}{\partial p_i}, \tag{85}$$

which can be rewritten into the form

$$p_i=-\frac{\partial L}{\partial q^i}\approx\frac{\partial F}{\partial q^i}+\sum_{j=1}^{k}u^j\frac{\partial\phi_j}{\partial q^i},\qquad \dot q^i\approx\frac{\partial F}{\partial p_i}+\sum_{j=1}^{k}u^j\frac{\partial\phi_j}{\partial p_i}. \tag{86}$$

Now it turns out that the set $\{\frac{\partial \phi_j}{\partial p_i}\}$ is linearly independent [16]. So given a F, no two different sets of u^k can give the same expressions for $\dot{q}^i$. This allows us to express u^k uniquely as functions of the coordinates q^i and the velocities $\dot{q}^i$. We can think of the parameters u^k as the parameters which specify a point with the one corresponding point on the submanifold of TQ, we mentioned earlier [12]. With this we can form an invertible Legendre transformation from TQ to the space $\Gamma \times \{u^k\}$ by defining

$$q^i = q^i, \qquad p_i = \frac{\partial L}{\partial \dot{q}^i}(q,\dot{q}), \qquad u^k = u^k(q,\dot{q}). \tag{87}$$

The invertibility of the transformation becomes apparent from the fact that

$$q^i = q^i, \qquad \dot{q}^i = \frac{\partial F}{\partial p_i} + \sum_{j=1}^{k} u^j \frac{\partial \phi_j}{\partial p_i}, \qquad \phi_k(q,p) = 0. \tag{88}$$

The invertible Legendre transformation now allows for a unique time evolution as well as a Hamiltonian by replacing $\dot{q}$ with the expressions in terms of p in F. Using equation (4) and (86) we find the equations of motion for the i components

$$\dot{q}^i \approx \frac{\partial H}{\partial p_i} + \sum_{j=1}^{k} u^j \frac{\partial \phi_j}{\partial p_i}, \qquad \dot{p}_i \approx -\frac{\partial H}{\partial q^i} - \sum_{j=1}^{k} u^j \frac{\partial \phi_j}{\partial q^i} \tag{89}$$

together with the k constraints $\phi_k(q,p) \approx 0$.

To relate this to the actual phase space T^*Q, notice that the equations of motion could also be obtained by applying the variational principle on the action functional

$$S = \int_{t_1}^{t_2} dt \left(\sum_{i=1}^{n} p_i \dot{q}^i - H - \sum_{j=1}^{k} u^j \phi_j \right) \tag{90}$$

for arbitrary variations δq^i, δp_i and δu^k. The variables u^k of the space $\Gamma \times \{u^k\}$ have now become Lagrange multipliers enforcing the constraints in T^*Q. From the properties of Lagrange multipliers, the action functional reduces to the standard one

$$S = \int_{t_1}^{t_2} dt \left(\sum_{i=1}^{n} p_i \dot{q}^i - H. \right) \tag{91}$$

for variations subject to the conditions $\phi_k = 0$ and $\delta\phi_k = 0$ [16]. So where as we had defined u^k as coordinates in $\Gamma \times \{u^k\}$ to make the Legendre transforomation invertible, we see that when considering u^k as multipliers, this correspond precisely to the dynamics which happens on the primary constrained surface $\Gamma \subset T^*Q$. From the methods from chapter 2 and the equations (90) and (93) we get that for a function $F : T^*Q \to \mathbb{R}$

$$\dot{F} \approx \{F,H\} + \sum_{j=1}^{k} \{F, u^j \phi_j\} \approx \{F,H\} + \sum_{j=1}^{k} u^j \{F, \phi_j\}, \tag{92}$$

giving us the equations in Poisson brackets again.

Now there is a subtlety here: the Lagrange multipliers u^k, as before, are seen as functions on TQ

rather than on T^*Q. Recalling that the Poisson bracket is only defined on phase space, i.e. its domain is $C^\infty(T^*Q) \times C^\infty(T^*Q)$, this means that whenever u^k appears within the Poisson bracket, it is formally undefined. Strictly speaking, if we want to write the formalism using Poisson brackets, we would need to modify the standard bracket to include such functions. However considering any Poisson bracket, u^k only appears inside the bracket multiplying the constraints ϕ_k, and because from Leibniz identity

$$\{F, u^j \phi_j\} = \{F, u^j\}\phi_j + u^j\{F, \phi_j\} \approx u^j\{F, \phi_j\}, \tag{93}$$

the ill-defined term $\{F, u^j\}\phi_j$ weakly vanishes on the constraint surface. In other words, although the bracket is not formally defined for u^k, its only effect would be to produce terms proportional to the constraints, which vanish when restricted to the physical subspace. So considering our only intention is to formulate the dynamics of phase space, we neglect this issue.

4.2 Consistency Conditions

Assuming we have found some system subject to k primary constraints, it has to be subject to the constraints at all times. This means that the constraints cannot change over time, i.e. $\dot{\phi}_k \approx 0$. Using equation (92) we get the consistency conditions

$$\{\phi_i, H\} + \sum_{j=1}^{k} u^j \{\phi_i, \phi_j\} \approx 0 \qquad i = 1, \ldots, k. \tag{94}$$

After defining the matrix C with entries $C_{ij} \equiv \{\phi_i, \phi_j\}$ and $h_i \equiv \{\phi_i, H\}$, we get the system of equations

$$h_i + \sum_{j=1}^{k} C_{ij} u^j \approx 0 \qquad i = 1, \ldots, k. \tag{95}$$

There are two options. the first being, $\det C \not\approx 0$, in which case equation (95) unique defines $u^k \approx \sum_{j=1}^{k} C^{kj} h_j$, where C^{ij} is the inverse of C_{ij}. Then equation (92) becomes

$$\dot{F} \approx \{F, H\} - \sum_{j,k} \{F, \phi_j\} C^{jk} \{\phi_k, H\}. \tag{96}$$

This is generally not the case, as we will see at the end of this section. The other case is that $\det C \approx 0$, in which we either get a certain number l of new constraints $\phi_l(p,q) \approx 0$,called secondary constraints or constraints among the multipliers u^k [15]. Assuming we get secondary constraints, we combine them with the primary constraints and check the consistency condition on the secondary constraints

$$\{\phi_i, H\} + \sum_{j=1}^{k} u^j \{\phi_i, \phi_j\} \approx 0 \qquad i = k+1, \ldots, k+l \tag{97}$$

(notice that we still only sum over the primary constraints and not the new secondary constraints). This process is repeated until the consistency conditions yield only relations among the multipliers u^k and no further secondary constraints. Assuming one has found a finite set of constraints, one can reduce to a set of r constraints which give a new submanifold $\Gamma' \subset \Gamma \subset T^*Q$ defined by

$$\phi_i(q,p) \approx 0, \qquad i = 1, \ldots, r. \tag{98}$$

We can now look at the restrictions of the Lagrange multipliers u^k. Viewing the consistency conditions

$$\{\phi_i, H\} + \sum_{j=1}^{k} u^j \{\phi_i, \phi_j\} \approx 0, \qquad i = 1, \dots, r \tag{99}$$

as a set of r nonhomogenous linear equations with respect to the k unknowns, i.e. u^k, the general solution will be of the form

$$u^k(q,p) = U^k(q,p) + V^k(q,p), \tag{100}$$

where U^k is a particular solution of the system (99) and V^k is the general solution to the associated homogeneous system

$$\sum_{j=1}^{k} V^j \{\phi_i, \phi_j\} \approx 0 \qquad i = 1, \dots, r \tag{101}$$

[17]. We see that the last expression can be written in matrix form as

$$CV \approx O, \tag{102}$$

where $O = (0, \dots, 0)^T$, $V = (V^1, \dots, V^k)^T$ is the solution vector and C is the $k \times r$ matrix with again entries $C_j = \{\phi_i, \phi_j\}$. This shows that V is an element of the kernel of C, i.e. $V \in \ker C$, and therefore the components V^k can be expressed as a linear combination

$$V^k = \sum_{a=1}^{A} v^a V_a^k, \tag{103}$$

where $\{V_a^k\}$ forms a basis of $\ker C$ and $A = \dim \ker C$. This gives u^k the form

$$u^k \approx U^k + \sum_a v^a V_a^k, \tag{104}$$

where U^k was the particular solution of the system while V_a^k is an linearly independent set satisfying $\sum_{j=1}^{k} V_a^j \{\phi_i, \phi_j\} \approx 0$ each. We emphasize this again to show that these depend on the system and are not arbitrary. The coefficients v^a, however, are arbitrary and we can even take them to be arbitrary function of time while still satisfying all the requirements of our dynamical theory [5]. These arbitrary functions also appear in the equations of motion. By defining $\phi_a \equiv \sum_k V_a^k \phi_k$ we can write

$$\dot{F} \approx \{F, H\} + \sum_{j=1}^{k} \left(U^j + \sum_a v^a V_a^j \right) \{F, \phi_j\} \approx \{F, H\} + \sum_{j=1}^{k} U^j \{F, \phi_j\} + \sum_a v^a \{F, \phi_a\} \tag{105}$$

using the fact that

$$\sum_{a,k} v^a V_a^k \{F, \phi_k\} \approx \sum_{a,k} v^a \{F, V_a^k \phi_k\} = \sum_a v^a \{F, \phi_a\} \tag{106}$$

This suggests we are dealing with a formalism containing arbitrary features, e.g. gauge theories. We will study this arbitrariness and when it appears in section 4.6, but first we take a deeper look into what the restrictions on the u^k imply.

4.3 First and second-class Functions and Constraints

To fully determine the multipliers u^k, up until the arbitrariness, we classify constraints and, more generally, functions into two categories. A function F is said to be first-class if the Poisson bracket with all the constraints (primary and secondary) weakly vanishes, i.e.

$$\{F,\phi_i\} \approx 0, \quad \forall r \in \{1,\dots,r\}. \tag{107}$$

The function is called second-class if it is not first-class. To seperate these categories by their properties, we introduce another useful theorem, for which again the proof can be found in [16]:

Theorem 4.2. *If a smooth phase space function $F : T^*Q \to \mathbb{R}$ vanishes on the surface defined by $\phi_i = 0$ with $i = 1,\dots,r$, then there exist functions $f^r : T^*Q \to \mathbb{R}$ such that*

$$F = \sum_{i=1}^{r} f^i \phi_i.$$

Using this we can easily show that the set of first-class functions forms a closed set under the Poisson brackets:
Let $F,G : T^*Q \to \mathbb{R}$ be first-class. By theorem (4.2)

$$\{F,\phi_j\} = \sum_{i=1}^{r} f^i \phi_i, \qquad \{G,\phi_j\} = \sum_{i=1}^{r} g^j \phi_i \qquad \forall j \in \{1,\dots,r\}. \tag{108}$$

Then using the Jacobi identity we get our desired result

$$\begin{aligned}\{\{F,G\},\phi_j\} &= \{F,\{F,\phi_j\}\} - \{F,\{F,\phi_j\}\} \\ &= \sum_i \{F, f^i\phi_i\} - \sum_i \{G, g^i\phi_i\} \\ &= \sum_i \left(f^i\{F,\phi_i\} + \{F,f^i\}\phi_i\right) + \sum_i \left(g^i\{F,\phi_i\} + \{F,g^i\}\phi_i\right) \\ &\approx \sum_i f^i\{F,\phi_i\} + \sum_i g^i\{F,\phi_i\} \approx 0.\end{aligned} \tag{109}$$

From this we can see that the constraints ϕ_a form a complete set of first-class primary constraints. This is because $\sum_j V^j \phi_j = \sum_a v^a \phi_a$ is the most general solution to equation (101). So any first-class primary constraint is a linear combination of the ϕ_a. Another property of first-class constraints can be seen by noticing that whenever there is at least one first-class constraint among the constraints to define $C_{ab} = \{\phi_a, \phi_b\}$, the determinant of C will be zero because at least one row and column of zeros. This is the reason why we rarely have case one ($\det C \not\approx 0$), because we only need one first-class constraint for this not to hold.

Regarding second-class constraints, we denote these constraints with χ. The only property of the second-class constraints we need is that the matrix D consisting of entries

$$D_{ab} \equiv \{\chi_a, \chi_b\}$$

is non-singular. Because if it were singular, then its determinant would vanish and there would exist a nontrivial linear combination of the χ_a that has vanishing Poisson brackets with all the others.
To be precise, there would exist a function $\psi = \sum_i a^i \chi_i$ such that $\forall \chi$

$$\{\chi, \psi\} = 0. \tag{110}$$

That would mean that this combination is actually first-class, contradicting the assumption that all χ are second-class constraints.
Also, we must have that the number of second-class constraints must be even, because the matrix D is antisymmetric:

$$D_{ab} = -D_{ba}.$$

An antisymmetric matrix of odd dimension always has zero determinant and is therefore singular. So the matrix can only be invertible if its dimension (the number of second-class constraints) is even.

Now regarding the cases mentioned in section 4.2, one can only have $\det C \not\approx 0$ if all primary constraints are second-class. If that were the case the equations of motion were given by equation (96) with the matrix C replaced by D:

$$\dot{F} \approx \{F,H\} - \sum_{j,k} \{F,\phi_j\} D^{jk} \{\phi_k,H\}. \tag{111}$$

4.4 The Dirac Bracket

So we have found what the equations of motion look like when $\det C \not\approx 0$. We will Now look at the equations of motion when $\det C \approx 0$, meaning there are some first-class constraints. Consider the consistency conditions (99) again, but now focus on the part of the particular solution U^k, i.e.

$$\{\phi_i,H\} + \sum_{j=1}^{k} U^j \{\phi_i,\phi_j\} \approx 0, \qquad i = 1,\ldots,r. \tag{112}$$

Considering that among the total α second-class constraints there are some number β that are primary, we have:

$$\begin{aligned} \{\chi_i,H\} + \sum_{j=1}^{\alpha} D_{ij} U^j &= \{\chi_i,H\} + \sum_{j=1}^{\alpha} U^j \{\chi_i,\chi_j\} \\ &= \{\chi_i,H\} + \sum_{j=1}^{\beta} U^j \{\chi_i,\chi_j\} \\ &= \{\chi_i,H\} + \sum_{j=1}^{k} U^j \{\chi_i,\phi_j\} \approx 0. \end{aligned} \tag{113}$$

The second line follows from the fact that the coefficients $U^j = 0$ whenever χ_j is a secondary constraint, because only primary constraints appear explicitly in the total Hamiltonian and thus determine the Lagrange multipliers. We reach the final line since any additional first-class constraints vanish weakly and hence can be included in the sum. Now the reason for rewriting the system into this form is to make use of the property that the matrix D has a nonzero determinant so this can be solved for the multipliers by multiplying the inverse of D to both sides. What we get is that

$$\sum_{i=1}^{\alpha} D^{ji} \{\chi_i,H\} \approx U^j \tag{114}$$

such that we get for the primary and secondary second class constraints that

$$\sum_{i=1}^{\alpha} D^{ji} \{\chi_i,H\} \approx \begin{cases} U^j & \chi_j \text{ primary} \\ 0 & \chi_j \text{ secondary,} \end{cases} \tag{115}$$

where D^{ij} is the inverse of D_{ij}. So we see that the part of Lagrange multipliers corresponding to the particular solution U^k are fully determined by the consistency condintions with the second-class constraints. To use this, we rewrite (105) into the form

$$\dot{F} \approx \{F,H\} + \sum_a v^a \{F,\phi_a\} + \sum_\beta U^\beta \{F,\chi_\beta\}. \tag{116}$$

We can substitute the expression (115) into this to get

$$\dot{F} \approx \{F,H\} + \sum_a v^a \{F,\phi_a\} - \sum_{i,j} \{F,\chi_i\} D^{ij} \{\chi_j,H\}. \tag{117}$$

This is similar to the equations of motion for systems with $\det C \not\approx 0$, given by (111), except for the additional primary first-class constraints.

Motivated by the form of the equations of motion (both for $\det C \approx 0$ and $\det C \not\approx 0$), Paul Dirac introduced the following new Poisson bracket, which we now call the Dirac bracket [5]:
For $F,G : T^*Q \to \mathbb{R}$ the Dirac bracket is defined as

$$\{F,G\}_D \equiv \{F,G\} - \sum_{\alpha,\beta} \{F,\chi_\alpha\} D^{\alpha\beta} \{\chi_\beta,G\}. \tag{118}$$

Now it does not take a lot of work to show that this bracket is indeed a Poisson bracket as can be seen in [12]. The main motivation for introducing this bracket lies in its properties. The first is that, for systems involving only second-class constraints, the equations of motion take the form

$$\dot{F} \approx \{F,H\}_D. \tag{119}$$

as shown in equation (111). In addition, for any second-class constraint χ_i, we have

$$\begin{aligned} \{F,\chi_i\}_D &= \{F,\chi_i\} - \sum_{\alpha,\beta} \{F,\chi_\alpha\} D^{\alpha\beta} \{\chi_\beta,\chi_i\} \\ &= \{F,\chi_i\} - \sum_{\alpha,\beta} \{F,\chi_\alpha\} D^{\alpha\beta} D_{\beta i} \\ &= \{F,\chi_i\} - \sum_{\alpha} \{F,\chi_\alpha\} \delta_i^\alpha \\ &= \{F,\chi_i\} - \{F,\chi_i\} = 0. \end{aligned} \tag{120}$$

The Dirac bracket automatically eliminates any second-class constraint. In other words, for any function F and any second-class constraint χ_i, the Dirac bracket is identically zero which means that the condition $\chi_i = 0$ can be imposed either before or after evaluating the Dirac bracket, since the bracket enforces the constraint strongly. This avoids the tedious task of applying the constraints after one has evaluated the brackets. As we will see, there is a way to convert first-class constraints into second-class constraints without changing the physical content of the system. In this case, the Dirac bracket becomes particularly useful, because once all constraints are second-class, the Dirac bracket automatically enforces them strongly. As a result, we no longer need to distinguish between weak and strong equalities in the equations of motion.

4.5 The Total Hamiltonian

For the sake of compactness of the formalism we introduce a new function. Consider the following function $H_T : T^*Q \to \mathbb{R}$, called the total Hamiltonian,

$$H_T = H' + \sum_a v^a \phi_a, \tag{121}$$

where H' and ϕ_a are given by

$$H' = H + \sum_k U^k \phi_k, \qquad \phi_a = \sum_k V_a^k \phi_k. \tag{122}$$

With this we can write the equations of motion in a more compact form:
For a function $F : T^*Q \to \mathbb{R}$

$$\begin{aligned}\{F,H_T\} &= \{F,H\} + \sum_k \{F,U^k\phi_k\} + \sum_{a,k} v^a \{F,V_a^k \phi_k\} \\ &= \{F,H\} + \sum_k \Big(\{F,U^k\}\phi_k + U^k\{F,\phi_k\}\Big) + \sum_{a,k} v^a \Big(\{F,V_a^k\}\phi_k + V_a^k\{F,\phi_k\}\Big) \\ &\approx \{F,H\} + \sum_k U^k\{F,\phi_k\} + \sum_{a,k} v^a V_a^k \{F,\phi_k\} \\ &\approx \{F,H\} + \sum_k u^k \{F,\phi_k\} \\ &\approx \dot{F}.\end{aligned} \tag{123}$$

We can also see that H_T is a first-class function from the fact that both H' and ϕ_a are first-class:

For ϕ_a

$$\{\phi_a,\phi_j\} = \sum_k \{V_a^k\phi_k,\phi_r\} = \sum_k \Big(V_a^k\{\phi_k,\phi_r\} + \{V_a^k,\phi_r\}\phi_k\Big) \approx \sum_k V_a^k \{\phi_k,\phi_r\} \approx 0 \tag{124}$$

and for H'

$$\begin{aligned}\{H',\phi_r\} &= \{H,\phi_r\} + \sum_k \{U^k\phi_k,\phi_r\} \\ &= \{H,\phi_r\} + \sum_k \Big(U^k\{\phi_k,\phi_r\} + \{U^k,\phi_r\}\phi_k\Big) \\ &\approx \{H,\phi_r\} + \sum_k U^k\{\phi_k,\phi_r\} \approx 0,\end{aligned} \tag{125}$$

because U^k *is a particular solution.*
Using this, and the property that for any arbitrary first-class function G, the Dirac bracket satisfies

$$\{F,G\}_D = \{F,G\} - \sum_{\alpha,\beta} \{F,\chi_\alpha\} D^{\alpha\beta} \{\chi_\beta, G\} \;\approx\; \{F,G\}, \tag{126}$$

we conclude that

$$\{F,H_T\}_D \;\approx\; \{F,H_T\}. \tag{127}$$

Therefore, for any function $F : T^*Q \to \mathbb{R}$, the time evolution is given by

$$\dot{F} \;\approx\; \{F,H_T\}_D. \tag{128}$$

4.6 Gauge theory

Now we finally are in a point to talk about the primary first-class constraints. The presence of the arbitrary functions v^a in the total Hamiltonian, originating from the equations of motion, suggest some type of arbitrariness. The first thing to notice is that the number of arbitrary functions is the same as the number of primary first-class constraints. Now to get an idea of this arbitrariness, consider a physical state, that is a state that is determined only by (q,p). Now for physical systems, the initial state determines the state at any time. But as we have seen the time evolution is not uniquely determined due to the fact that v^a appears in the equations of motion. So the (q,p), corresponding to the state at a later time, is also not uniquely determined by the initial states. There are multiple values of (q,p) that correspond to the same state. It is now logical to search for the set of all the (q,p) that correspond to the same state. To find such sets we consider the fact that the all the (q,p) at a certain time, which evolve from one initial state, must correspond to the same physical state at that time [5]. For a general dynamical variable F, e.g. a function or the coordinates, with initial value F_0, its value at time $t+\delta t$ is

$$F(t+\delta t) = F_0 + \dot{F}\delta t = F_0 + \{F\, H_T\}_D \delta t = F_0 + \left(\{F, H'\} + \sum_a v^a \{F, \phi_a\} \right) \delta t. \tag{129}$$

Taking different values v and v', we see that the difference in the values of F at time $t+\delta t$ is

$$\Delta F = \sum_a \varepsilon^a \{F, \phi_a\}, \tag{130}$$

where $\varepsilon^a \equiv (v^a - v'^a)\delta t$. This shows that changing the arbitrary multipliers v^a generates a transformation of the dynamical variables along the directions defined by the first-class constraints. To emphasize, the important idea is that, regardless of the choice of v, the physical state remains the same. Changing all our dynamical variables (q,p) according to this rule gives new variables that describe the same physical state. So the transformations defined by (130) do not change the physical state and are called a gauge transformation and we see that these transfomations are generated by first-class constraints. The set of all phase-space points (q,p) related by gauge transformations forms a gauge orbit. Each orbit corresponds to exactly one physical state. Given there are multiple independent first-class constraints, we will have multiple gauge orbits.

To emphasize, first-class constraints and the resulting gauge freedom mean that multiple sets of canonical variables can describe the same physical state. In practice, it is often useful to remove this ambiguity by imposing additional conditions. This ensures a one-to-one correspondence between physical states and the remaining independent canonical variables. Such additional conditions are called gauge-fixing constraints. The purpose of a gauge-fixing condition is to eliminate the freedom generated by a first-class constraint by selecting a unique representative from each gauge orbit. This is achieved by imposing an extra constraint that pairs with the first-class constraint to form a second-class constraint pair. Once the gauge orbit is removed in this way, the Dirac bracket is defined to consistently take the second-class constraints into account, ensuring that the resulting reduced phase space describes only the true physical degrees of freedom. We will see an example of this in section 6.2.

4.7 Geometric Picture of the Dirac Bracket and Constraints

The Dirac bracket can be understood intuitively as a way of modifying the ordinary Poisson bracket such that the second-class constraints hold identically and cannot be violated by the dynamics.

Given a set of second-class constraints $\chi_\alpha \approx 0$, the Poisson bracket by itself does not guarantee that the evolution of a function F stays within the surface defined by these constraints. In particular, the Poisson brackets $\{F, \chi_\alpha\}$ generally do not vanish, so the constraints can be violated under time evolution. The Dirac bracket corrects this by subtracting off exactly the part of the Poisson bracket that would push the system away from the constraint surface

$$\{F,G\}_D = \{F,G\} - \sum_{\alpha,\beta} \{F,\chi_\alpha\} D^{\alpha\beta} \{\chi_\beta, G\}$$

and as a result

$$\{F,\chi_\alpha\}_D = 0.$$

This means that, when using the Dirac bracket, any operation using the Dirac Bracket automatically respects the constraints. All functions are essentially projected onto functions satisfying the second-class constraints. In this sense, the Dirac bracket acts as a projection of the Poisson bracket. It removes the components that would violate the second-class constraints and keeps only the part that is consistent with staying entirely within the allowed subspace.

After imposing the second-class constraints in this way, any remaining arbitrariness in the evolution comes from first-class constraints, whose freedom reflects gauge transformations. These do not need to be projected out because they do not restrict the physical degrees of freedom but instead generate different equivalent descriptions of the same physical state.

4.8 Dirac structure and D-Transformations

Given that most of the relevant theory is formulated, we are interested in transformations that do not change the canonical form. To discuss this we will write the Dirac bracket of the form

$$\{f,g\}_D = \sum_{ij}^{n} \omega^{ij} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j}, \tag{131}$$

just like we did with the Poisson bracket. We call ω^{ij} the Dirac structure and it can be easily seen that

$$\omega^{ij} = \pi^{ij} - \sum_{kl} \pi^{ik} \theta_{kl} \pi^{lj}, \tag{132}$$

follows from equation (118), where π^{ij} is the standard Poisson structure of the phase space and θ_{ij} is defined by

$$\theta_{ij} \equiv \sum_{ab} D^{ab} \frac{\partial \chi_i}{\partial x^a} \frac{\partial \chi_j}{\partial x^b}. \tag{133}$$

We can write this in an elegant form by writing π^{ij}, ω^{ij} and θ_{ij} as matrices to get

$$\omega = \pi - \pi\theta\pi. \tag{134}$$

Now, a canonical transformation with respect to the Poisson bracket is not necessarily canonical in the constrained formalism. Recall that canonical transformations were originally defined as those that leave the Poisson bracket relations invariant, since this guarantees that the equations of motion remain unchanged. However, once constraints are present, the true dynamics are governed by the Dirac brackets. Therefore, we now regard a transformation as canonical only if it preserves the

Dirac bracket relations, which depend on the specific system. We will call such transformations D-transformations, as first introduced in [14]. From (134) it follows that for a transformation to be a D-transformation, it must still be canonical with respect to the Poisson brackets, as well as the additional term

$$A^T(\pi - \pi\theta\pi)A = \pi - \pi\theta\pi. \tag{135}$$

This makes sense. A transformation that leaves the constraints unchanged must still respect the Poisson brackets in order to be truly canonical. So the set of all D-transformations is a subset of the canonical group. Even further, it forms a subgroup after seeing that:

- The identity is in the set:
$$I^T\omega I = \omega.$$
- Closure under composition: if $A^T\omega A = \omega$ and $B^T\omega B = \omega$, then
$$(AB)^T\omega(AB) = B^TA^T\omega AB = B^T\omega B = \omega.$$
- Closure under inversion: if $A^T\omega A = \omega$, then
$$\omega = A^T\omega A \implies A^{-T}\omega A^{-1} = \omega.$$

We will see an example of a D-transformation in section 6.2.

4.9 Constrained Phase Space Geometry

Let us finally put the machinery to the test. We do this by examining the Dirac structure for curved configuration spaces embedded in $\mathbb{R}^n$. Before we proceed, however, we must point out an important subtlety in how we interpret these curved surfaces. Namely, do we regard the curved surface itself as the physical space, or do we think of the ambient space as the true physical space, with the surface condition acting as a constraint on the motion, e.g. an ant forced to walk on a sphere? This distinction matters because the Hamiltonian must be defined on the space we take to be physically meaningful. One can define the Hamiltonian intrinsically on the curved surface and then rewrite it in ambient coordinates for convenience, treating the surface as the true configuration space and the embedding only as a tool. Or one can define the Hamiltonian directly in the ambient space and impose constraints that restrict the motion to the surface, viewing the system as living in flat space but confined to the submanifold. The first case will be more relevant when we look at the actual dynamics on de Sitter spacetime, which we will see in Chapter Six. But for now we stick to the simpler option by defining the Hamiltonian in the ambient space and just looking at the Dirac structure. This is fine because the Dirac bracket structure does not depend on the specific form of the Hamiltonian, unless the Hamiltonian itself is a constraint, as we saw in chapter 3 and will see again in chapters 5 and 6.

4.9.1 Dirac Structure on S^{n-1}

Let us consider the Lagrangian of a free particle of mass m

$$L = \frac{1}{2}m\sum_{i=1}^{n}(\dot{x}^i)^2 - \lambda\phi_1 \tag{136}$$

constrained by

$$\phi_1 = \frac{1}{2}\left(\sum_{i=1}^{n} x^i x_i - R^2\right). \tag{137}$$

This primary constraint makes sure the dynamics is constrained to the surface of the $(N-1)$-dimensional hypersphere S^{n-1}. For this Lagrangian, we get the total Hamiltonian

$$H_T = H + u\phi_1 = \frac{1}{2m}\sum_{i=1}^{n} p^i p_i + u\phi_1. \tag{138}$$

Now from the consistency condition $\dot{\phi}_1 \approx 0$ we get

$$\begin{aligned}
\dot{\phi}_1 &= \{\phi_1, H_T\} \\
&= \sum_{i=1}^{n} \frac{\partial\phi_1}{\partial x^i}\frac{\partial H_T}{\partial p_i} - \frac{\partial H_T}{\partial x^i}\frac{\partial\phi_1}{\partial p_i} \\
&= \sum_{i=1}^{n} \frac{\partial\phi_1}{\partial x^i}\frac{\partial H_T}{\partial p_i} \\
&= \sum_{i=1}^{n} x^i p_i \approx 0.
\end{aligned} \tag{139}$$

Note that this new constraint, $\phi_2 \equiv \sum_{i=1}^{n} x^i p_i \approx 0$ demands that p is simply tangent to the hypersphere. Now checking the consistency condition on ϕ_2 we find that

$$\begin{aligned}
\dot{\phi}_2 &= \{\phi_2, H_T\} \\
&= \sum_{i=1}^{n} \frac{\partial\phi_2}{\partial x^i}\frac{\partial H_T}{\partial p_i} - \frac{\partial H_T}{\partial x^i}\frac{\partial\phi_2}{\partial p_i} \\
&= \frac{1}{2}\sum_{i=1}^{n} p_i\frac{\partial H_T}{\partial p_i} - x^i\frac{\partial H_T}{\partial x^i} \\
&= \frac{1}{2}\sum_{i=1}^{n} \left(\frac{p^i p_i}{m} - u x^i x_i\right) \\
&\approx \frac{1}{2}\sum_{i=1}^{n} \frac{p^i p_i}{m} - uR^2 \approx 0
\end{aligned} \tag{140}$$

(the partial derivatives on u do not vanish because of property (93)). Now this does not give us a new constraint but a restriction on u. Namely we must have that $u \approx U$ for

$$U = \frac{1}{mR^2}\sum_{i=1}^{n} p^i p_i. \tag{141}$$

This means we only have the constraints ϕ_1 and ϕ_2. To obtain the Dirac brackets, we compute

$$\{\phi_1, \phi_2\} = \sum_{i=1}^{n} \frac{\partial\phi_1}{\partial x^i}\frac{\partial\phi_2}{\partial p_i} - \frac{\partial\phi_2}{\partial x^i}\frac{\partial\phi_1}{\partial p_i} = \sum_{i=1}^{n} x_i\frac{\partial\phi_2}{\partial p_i} = \sum_{i=1}^{n} x^i x_i \equiv x^2. \tag{142}$$

So we see that ϕ_1 and ϕ_2 are second class and that the D matrix and its inverse become

$$D = x^2\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad D^{-1} = \frac{1}{x^2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then, the Dirac bracket as defined in (118), gives

$$\{F,G\}_D = \{F,G\} - \frac{1}{x^2}\begin{pmatrix} -\sum_i x_i \frac{\partial F}{\partial p_i} \\ -\sum_i p_i \frac{\partial F}{\partial p_i} + x^i \frac{\partial F}{\partial x^i} \end{pmatrix}^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \sum_i x_i \frac{\partial G}{\partial p_i} \\ \sum_i p_i \frac{\partial G}{\partial p_i} - x^i \frac{\partial G}{\partial x^i} \end{pmatrix}.$$

In particular we get the Dirac algebra

$$\{x^a, x^b\}_D = 0, \tag{143}$$

$$\{x^a, p_b\}_D = \delta^a_b - \frac{x^a x_b}{x^2} \approx \delta^a_b - \frac{x^a x_b}{R^2}, \tag{144}$$

$$\{p_a, p_b\}_D = \frac{1}{x^2}(x_a p_b - p_a x_b) \approx \frac{1}{R^2}(x_a p_b - p_a x_b). \tag{145}$$

This algebra determines the time evolution while respecting the constraints. It can be shown that the second constraint ϕ_2 is the same for the more general canonical Hamiltonian $H = \frac{1}{2m}\sum_{i=1}^n p^i p_i + V(x)$ as well as the fact that $\dot{\phi}_2 \approx 0$ does not imply a new constraint [13]. So the Dirac algebra would be the same for a Hamiltonian including a potential.

4.9.2 Dirac Structure on a Arbitrary Curved Surface

Following the same procedure as in 4.9.1 we get that for an arbitrary curved surface $M \subset \mathbb{R}^n$ and a arbitrary Hamiltonian given by

$$\phi_1(q,p) = f(q^1,\ldots,q^n) \approx 0, \qquad H = \frac{1}{2m}\sum_{i=1}^n p^i p_i + V(q) \tag{146}$$

the total Hamiltonian becomes

$$H_T = \frac{1}{2m}\sum_{i=1}^n p^i p_i + V(q) - uf(q). \tag{147}$$

For the consistency condition on f, we find that

$$\begin{aligned} \dot{f}_1 &= \{f, H_T\} \\ &= \sum_{i=1}^n \frac{\partial f}{\partial q^i}\frac{\partial H_T}{\partial p_i} - \frac{\partial H_T}{\partial q^i}\frac{\partial f}{\partial p_i} \\ &= \sum_{i=1}^n \partial_i f \frac{\partial H_T}{\partial p_i} \\ &= \frac{1}{2m}\sum_{i=1}^n p^i \partial_i f \approx 0, \end{aligned} \tag{148}$$

where $\partial_i \equiv \frac{\partial}{\partial q^i}$. This gives us the new constraint $\phi_2 \equiv \sum_{i=1}^n p^i \partial_i f \approx 0$. The consistency condition on this constraint is then given by

$$\begin{aligned}
\dot{\phi}_2 &= \{\phi_2, H_T\} \\
&= \sum_{j=1}^{n} \left(\{p^j, H_T\}\partial_j f + p^j\{\partial_j f, H_T\}\right) \\
&= \sum_{j=1}^{n} \left((-\partial^j V - u\partial^j f)\partial_j f + p^j \sum_i \frac{1}{m} p^i \partial_j \partial_i f\right) \\
&= \frac{1}{m}\sum_{ij} p^i p^j \partial_i \partial_j f - \sum_j \left(\partial^j \partial_j V + u\partial^j f \partial_j f\right) \approx 0.
\end{aligned} \tag{149}$$

We find that this condition is again a condition on the multiplier:

$$u \approx U = \frac{1}{|\nabla f|^2}\left(\frac{1}{m}\sum_{ij} p^i p^j \partial_i \partial_j f - \sum_j \partial^j \partial_j V\right), \tag{150}$$

where $|\nabla f|^2 = \sum_j \partial^j f \partial_j f$. The commutator becomes $\{\phi_1, \phi_2\}$

$$\begin{aligned}
\{\phi_1, \phi_2\} &= \sum_{i=1}^{n} \{f, p^i \partial_i f\} \\
&= \sum_{i=1}^{n} \{f, p^i\}\partial_i f + p^i\{f, p^i \partial_i f\} \\
&\approx \sum_{i=1}^{n} \partial^i \partial_i f = |\nabla f|^2.
\end{aligned} \tag{151}$$

So we see that ϕ_1 and ϕ_2 are second class and that the D matrix and its inverse become

$$D = |\nabla f|^2 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad D^{-1} = \frac{1}{|\nabla f|^2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then, the Dirac bracket as defined in (118), gives

$$\{F,G\}_D = \{F,G\} - \frac{1}{|\nabla f|^2}\begin{pmatrix} -\sum_i \partial_i f \frac{\partial F}{\partial p_i} \\ -\sum_{ij} p^i(\partial_i \partial_j f)\frac{\partial F}{\partial p_i} + \sum_i \partial^i f \frac{\partial F}{\partial x^i} \end{pmatrix}^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \sum_i \partial_i f \frac{\partial G}{\partial p_i} \\ \sum_{ij} p^i(\partial_i \partial_j f)\frac{\partial G}{\partial p_i} - \sum_i \partial^i f \frac{\partial G}{\partial x^i} \end{pmatrix}.$$

In particular we get the Dirac algebra

$$\{q^i, q^j\}_D = 0, \tag{152}$$

$$\{q^i, p_j\}_D = \delta^i_j - \frac{\partial^i f \partial_j f}{|\nabla f|^2}, \tag{153}$$

$$\{p_i, p_j\}_D = \frac{1}{|\nabla f|^2}\sum_a p_a \left(\partial_i f(\partial_j \partial^a f) - \partial_j f(\partial_i \partial^a f)\right). \tag{154}$$

Now, there is something interesting about the Dirac bracket $\{q^i, p_j\}_D$. Given that our hypersurface is embedded in $\mathbb{E}^n$, we find:

$$\{q_i, p_j\}_D = \delta_{ij} - \frac{\partial_i f \partial_j f}{|\nabla f|^2}.$$

As it turns out, this is exactly the induced metric on the hypersurface embedded in $\mathbb{E}^n$ [13]:

$$g^*_{ij} = \delta_{ij} - \frac{\partial_i f \partial_j f}{|\nabla f|^2}.$$

Thus, the Dirac bracket of the position and momentum coordinates directly reproduces the induced metric in ambient coordinates:

$$\{q_i, p_j\}_D = g^*_{ij}.$$

The unit normal vector $\boldsymbol{n}$ to the hypersurface has components

$$n_i = \frac{\partial_i f}{|\nabla f|}.$$

One also finds for the momentum-momentum bracket:

$$\{p_i, p_j\}_D = n_i \partial_j(\boldsymbol{n}\cdot\boldsymbol{p}) \;-\; n_j \partial_i(\boldsymbol{n}\cdot\boldsymbol{p}), \quad \text{where} \quad \boldsymbol{p} = (p_1, \ldots, p_n).$$

As our final result in this section, we can write the Dirac structure for an arbitrary hypersurface embedded in $\mathbb{E}^n$ in block form:

$$\omega = \begin{pmatrix} 0 & g^*_{ij} \\ -g^*_{ij} & n_i \partial_j(\boldsymbol{n}\cdot\boldsymbol{p}) \;-\; n_j \partial_i(\boldsymbol{n}\cdot\boldsymbol{p}) \end{pmatrix}.$$

5 Reformulation of Dirac Theory with Time as a Dynamical Variable

Here we show how Dirac's theory of constraints naturally includes time as a canonical variable via reparameterization. This leads to a primary Hamiltonian constraint and an extended phase space where time and its momentum appear on equal footing with other coordinates. The extended Hamiltonian and action then arise directly from Dirac's formalism as a special case. We show that reparameterization invariance can be realized as a gauge symmetry of the total Hamiltonian.We conclude this section by illustrating the formalism through the example of a free massive particle in Minkowski spacetime.

5.1 Recovering the Extended Formalism from Dirac's Theory of Constraints

Let us recall our first attempt of obtaining a Hamiltonian formalism with time as a dynamical variable. In section 3.1, we started with obtaining the Lagrangian (41) where time is treated among the spatial coordinates. From there we tried to obtain the Hamiltonian through a Legendre transformation, from which the dynamics would follow. Instead we found that this transformation left us with a constraint $p_0 + H = 0$ from which we derived that the Hamiltonian was identically equal to zero, i.e. $H = 0$. Logically, we concluded that in the standard formulation of Hamiltonian mechanics there would be no dynamics. However if we consider Dirac's formalism, there is no reason to make this conclusion: Starting from the reparameterized action (42), we follow the standard procedure until we have obtained the relation $p_0 + H = 0$. As shown by [4], the Legendre transformation we performed is singular and $p_0 + H = 0$ is the only constraint among the coordinates that arises from this singularity. Assuming there are no other constraints imposed beforehand, the total Hamiltonian, which generates the dynamics, is not identically zero but

$$H_T = H + u(p_0 + H) = u(p_0 + H). \tag{155}$$

So unlike in chapter 3, where we introduced the extended phase space by explicitly adding time as an additional coordinate, here we see that Dirac's theory of constraints naturally includes time in the canonical framework through reparameterization of the action. Once time is treated as an ordinary configuration variable in the Lagrangian, its conjugate momentum arises automatically from the singular Legendre transformation. As a result, after reparameterization both the Poisson bracket and the Dirac bracket naturally include time and its conjugate momentum as canonical variables, without needing any separate extension prescription.

Upon further notice, we see that

$$\frac{dt}{ds} = (q^0)' = \{p_0, H_T\} = u. \tag{156}$$

and hence the Hamiltonian that is responsible for the dynamics is given by

$$H_T = \frac{dt}{ds}(p_0 + H). \tag{157}$$

The corresponding action is

$$\begin{aligned} S &= \int_{t_1}^{t_2} dt \left(\sum_{i=1}^{n} p_i \dot{q}_i - H \right) \\ &= \int_{s_1}^{s_2} ds \left(\sum_{k=0}^{n} p_k q^{k\prime} - (H + p_0) t' \right) \\ &= \int_{s_1}^{s_2} ds \left(\sum_{k=0}^{n} p_k q^{k\prime} - H_T \right). \end{aligned} \tag{158}$$

These expressions (157) and (158) are exactly identical to the extended Hamiltonian (51) and the corresponding extended action (52). With time now being included as an ordinary coordinate through the Lagrangian formalism, and the resulting action and Hamiltonian matching precisely, every result derived in Chapter 3 now follows directly from Dirac's general theory of constraints. We can conclude that the extended phase space formalism works not because it is fundamentally different, but because it is simply a special case of the general theory of singular systems.

5.2 Extended Dirac's formalism

Now that we have seen that the extended formalism is a special case of Dirac's formalism with time as a coordinate. Let us consider an overview of the general theory, where the action is reparameterized beforehand.

Now that time has become a dynamical variable and s the parameter, we will call this the extended Dirac formalism to distinguish it from the standard one where time is not a dynamical variable. Using the notation from chapter 3, we have that the total phase space is T^*Q_e. The extended Poisson bracket $\{\cdot,\cdot\}_e$ now replaces the standard Poisson bracket in Dirac's formalism. So first-class constraints γ and second-class χ are now with respect to the extended Poisson bracket. For systems in the extended Dirac's formalism, we will always have the primary constraint

$$\mathcal{H} \equiv p_0 + H \approx 0, \tag{159}$$

which we will call the Hamiltonian constraint and denote with $\mathcal{H}$. Potentially, there is some set of k other independent primary constraints ϕ_i. This set of constraints $\{\phi_i\}_{i=1}^{k}$ with $\mathcal{H}$ defines the primary constraint surface $M \subset T^*Q_e$. With this the total Hamiltonian becomes

$$H_T = \sum_{i=0}^{k} u^i \phi_i, \tag{160}$$

where $u^0 = \frac{dt}{ds}$. Given that s has become the parameter, we find that for a function $F : T^*Q_e \to \mathbb{R}$

$$F' \approx \{F, H_T\}_{ED}, \tag{161}$$

where $\{\cdot,\cdot\}_{ED} : C^\infty(T^*Q_e) \times C^\infty(T^*Q_e) \to C^\infty(T^*Q_e)$ is the extended Dirac bracket

$$\{F,G\}_{ED} \equiv \{F,G\}_e - \sum_{\alpha\beta} \{F,\chi_\alpha\}_e D^{\alpha\beta} \{\chi_\beta, G\}_e \tag{162}$$

with $D^{\alpha\beta}$ being the inverse of $D_{\alpha\beta} \equiv \{\chi_\alpha, \chi_\beta\}_e$. Regarding gauge transformations we have that they are now defined by

$$\Delta F = \sum_a \varepsilon^a \{F, \phi_a\}_e, \tag{163}$$

where $\varepsilon^a \equiv (v^a - v'^a)\,\delta s$. Finally, our Dirac structure becomes the extended Dirac structure

$$\omega_e = \pi_e - \pi_e \theta_e \pi_e, \tag{164}$$

where π_e is given by (59) and θ_e by (133) with t and p_0 included in the sum.

5.3 Reparameterization Invariance as a Gauge Symmetry

Normally, when dealing with relativistic systems, one starts with an action defined directly on spacetime, where time is treated as a dynamical coordinate. The formalism discussed here, however, is built on the fact that we reparameterize the action after a (potentially non-relativistic) Lagrangian is determined, from which we then find the Hamiltonian and hence the constraint

$$\mathcal{H} = p_0 + H \approx 0. \tag{165}$$

So when starting from a relativistic action, we need a way to correctly identify the Hamiltonian constraint. Now, in Chapter 3, it was shown that the action (52) is invariant under reparameterizations due to the appearance of $H_e = p_0 + H \approx 0$. Similarly, for constrained systems the action (158) is also invariant under reparameterizations [16], meaning the action stays the same under the infinitesimal transformation

$$s \to s' = s + \varepsilon(s). \tag{166}$$

This is equivalent to the gauge transformation generated by the first-class total Hamiltonian H_T,

$$\delta F = \varepsilon(s)\{F, H_T\}_e \approx \varepsilon(s)\frac{dF}{ds}, \tag{167}$$

which shows explicitly that reparameterization invariance appears as a gauge symmetry in the Hamiltonian framework.

5.4 Free Massive Particle in Minkowski Spacetime $\mathbb{M}^4$

Now, the action for a free massive spinless particle in flat Minkowski spacetime can be given by

$$S = -mc\int_\gamma ds = -mc\int_\gamma \sqrt{-\eta_{\mu\nu}dx^\mu dx^\nu} = -mc\int_{\gamma(\tau)} d\tau\,\sqrt{-\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}, \tag{168}$$

where the dot denotes differentiation with respect to the chosen parameter τ, which can be identified with the proper time.

In this case, there is no additional embedding or hyperboloid constraint since the particle moves in flat spacetime, so the Minkowski metric $\eta_{\mu\nu}$ directly defines the invariant line element. The parameterized action for the free spinless particle is therefore

$$S = -mc\int_\gamma ds = \int_{\gamma(\tau)} d\tau\left[-mc\,\sqrt{-\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}\right]. \tag{169}$$

The corresponding Lagrangian is

$$L = -mc\,\sqrt{-\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}, \tag{170}$$

and the canonical momenta are given by

$$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = -mc\,\frac{\eta_{\mu\nu}\dot{x}^\nu}{\sqrt{-\eta_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta}} = -mc\,\frac{dx_\mu}{ds} = m\,\frac{dx_\mu}{d\tau}, \tag{171}$$

where we used that $ds = -c\,d\tau$ for the metric signature $(-+++)$.
It then follows that the Legendre transform of L vanishes identically on-shell,

$$\dot{x}^\mu p_\mu - L = -mc\,\frac{\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}{\sqrt{-\eta_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta}} + mc\,\sqrt{-\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} = 0, \tag{172}$$

which is the expected result for a reparameterized free particle action in flat Minkowski spacetime. Instead we get the dynamics from the total Hamiltonian H_T. Contracting p^μ with itself gives

$$p_\mu p^\mu = m^2\,\frac{\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}{-\eta_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta} = -m^2. \tag{173}$$

Therefore, we obtain the primary constraint

$$p_\mu p^\mu + m^2 \approx 0, \tag{174}$$

which is the familiar mass-shell condition for a free relativistic particle in Minkowski spacetime. Given that there are no additional constraints, (174) becomes the Hamiltonian constraint $\mathcal{H}$ and the total Hamiltonian is given by

$$H_T = \alpha(p_\mu p^\mu + m^2) \approx 0. \tag{175}$$

With the canonical variables (x^μ, p^ν), the extended Poisson brackets are

$$\{x^\mu, x^\nu\}_e = 0, \quad \{x^\mu, p^\nu\}_e = \eta^{\mu\nu}, \quad \{p^\mu, p^\nu\}_e = 0. \tag{176}$$

As there are no second class constraints, the Dirac algebra is equal to the one above and we find that the equations of motion for a free particle are

$$\dot{x}^\mu = \{x^\mu, H_T\}_e = 2\alpha(\tau)p^\mu, \quad \dot{p}_\mu = \{p_\mu, H_T\}_e = 0, \tag{177}$$

6 Phase space on de Sitter spacetime

De Sitter spacetime plays a central role in modern cosmology as it describes the large-scale geometry of our universe when dominated by a positive cosmological constant. Because it captures the asymptotic behavior of our expanding universe, understanding particle dynamics and symmetries on de Sitter space is essential. In this section, we formulate the extended Dirac formalism on de Sitter spacetime, study its constraint structure, and demonstrate how free massive particles move along geodesics on the de Sitter hyperboloid. This sets the stage for a Hamiltonian description of physics in realistic cosmological settings.

6.1 de Sitter Spacetime $dS(4,1)$

De Sitter spacetime is the maximally symmetric Lorentzian manifold with positive scalar curvature that solves the Einstein field equations. This manifold $dS(4,1)$ can be defined as a hyperboloid in the pseudo-Euclidean space $\mathbb{E}^{4,1}$

$$\eta_{ab}\mathcal{X}^a\mathcal{X}^b = -\left(\mathcal{X}^0\right)^2 + \left(\mathcal{X}^1\right)^2 + \left(\mathcal{X}^2\right)^2 + \left(\mathcal{X}^3\right)^2 + \left(\mathcal{X}^4\right)^2 = \mathcal{R}^2, \tag{178}$$

where $(\mathcal{X}^a) = \left(\mathcal{X}^0, \mathcal{X}^1, \mathcal{X}^2, \mathcal{X}^3, \mathcal{X}^4\right)$ are the cartesian coordinates of $\mathbb{E}^{4,1}$. The isometry group of de Sitter space is $SO(4,1)$ [18], which are the transformations Λ satisfying

$$\Lambda^T\eta\Lambda = \eta \qquad \Longrightarrow \qquad \eta_{cd}\Lambda^c{}_a\Lambda^d{}_b = \eta_{ab}. \tag{179}$$

De Sitter space can also described using stereographic coordinates. We perform a stereographic projection of the hyperboloid from the point $(\mathcal{X}^0, \mathcal{X}^1, \mathcal{X}^2, \mathcal{X}^3, \mathcal{X}^4) = (0,0,0,0,\mathcal{R})$, i.e. the lowest point of the upper sheet of the hyperboloid, into the hyperplane $\mathbb{E}^{3,1}$ tangent at the point $\mathcal{X}^4 = -\mathcal{R}$. In this case every point on the hyperboloid, except from the point we project it from, can be written in the stereographic coordinates $(x^\mu) = (x^0, x^1, x^2, x^3)$ [19]:

$$\mathcal{X}^a = \frac{\delta^a_\mu x^\mu}{1+\frac{\sigma^2}{4\mathcal{R}^2}} = n(x)\delta^a_\mu x^\mu \quad \text{and} \quad \mathcal{X}^4 = -\mathcal{R}n(x)\left(1+\frac{\sigma^2}{4\mathcal{R}^2}\right) \tag{180}$$

where

$$n(x) \equiv \frac{1}{1+\frac{\sigma^2}{4\mathcal{R}^2}} \quad \text{and} \quad \sigma^2 = \eta_{\mu\nu}x^\mu x^\nu \tag{181}$$

where the Lorentz metric η having signature $(-+++)$. With this the line element $ds^2 = \eta_{ab}\mathcal{X}^a\mathcal{X}^b$ constrained to the hyperboloid becomes $ds^2 = g_{\mu\nu}x^\mu x^\nu$, with the metric tensor

$$g_{\mu\nu} = n^2(x)\eta_{\mu\nu}, \tag{182}$$

which shows that one recovers the Minkowski metric $\eta_{\mu\nu}$ when $\mathcal{R} \to \infty$.

6.2 Free Massive Spinless Particle on $dS(4,1)$

The action for a free massive spinless particle on de Sitter spacetime can be given by

$$S = -mc\int_\gamma ds = -mc\int_\gamma \sqrt{-g_{\mu\nu}x^\mu x^\nu} = -mc\int_{\gamma(\tau)} d\tau\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}, \tag{183}$$

where the dot represents dertivatives with respect to the parameter proper time τ. However we saw that the line segment $ds^2 = \eta_{ab}\mathcal{X}^a\mathcal{X}^b$ is equivalent to $g_{\mu\nu}x^\mu x^\nu$ as long as its constrained to be on the hyperboloid surface. So equivalently, we can write the parameterized action for a free spinless particle on de Sitter spacetime as

$$S = -mc\int_\gamma ds = \int_{\gamma(\tau)} d\tau \left[-mc\sqrt{-\eta_{ab}\dot{\mathcal{X}}^a\dot{\mathcal{X}}^b} + \lambda\left(\eta_{ab}\mathcal{X}^a\mathcal{X}^b - \mathcal{R}^2\right)\right], \tag{184}$$

where $\eta_{ab}\mathcal{X}^a\mathcal{X}^b - \mathcal{R}^2$ together with the Lagrange multipliers are added such that the equations of motion are constrained to the hyperboloid. The Lagrangian, parameterized with τ, is given by

$$\mathcal{L} = -mc\sqrt{-\eta_{ab}\dot{\mathcal{X}}^a\dot{\mathcal{X}}^b} + \lambda\left(\eta_{ab}\mathcal{X}^a\mathcal{X}^b - \mathcal{R}^2\right) \tag{185}$$

with the corresponding canonical momenta

$$\mathcal{P}_a = \frac{\partial\mathcal{L}}{\partial(\dot{\mathcal{X}}^a)} = -mc\frac{\eta_{ab}\dot{\mathcal{X}}^b}{\sqrt{-\eta_{ab}\dot{\mathcal{X}}^a\dot{\mathcal{X}}^b}} = -mc\frac{d\mathcal{X}_a}{ds} = m\frac{d\mathcal{X}_a}{d\tau}, \tag{186}$$

where the last equality follows from the fact that $ds = -cd\tau$ for a metric signature $(-++++)$. As a result, we see that the Legendre transform of $\mathcal{L}$ vanishes on the constrained surface

$$\dot{\mathcal{X}}^a\mathcal{P}_a - \mathcal{L} = -mc\frac{\eta_{ab}\dot{\mathcal{X}}^a\dot{\mathcal{X}}^b}{\sqrt{-\eta_{ab}\dot{\mathcal{X}}^a\dot{\mathcal{X}}^b}} + mc\sqrt{-\eta_{ab}\dot{\mathcal{X}}^a\dot{\mathcal{X}}^b} + \lambda\left(\eta_{ab}\mathcal{X}^a\mathcal{X}^b - \mathcal{R}^2\right) = \lambda\left(\eta_{ab}\mathcal{X}^a\mathcal{X}^b - \mathcal{R}^2\right) \approx 0, \tag{187}$$

which is what we would expect from a reparameterized action. Now, shown by [19], the Casimir invariant of the group $SO(4,1)$ is given by

$$\mathcal{C} = -\frac{1}{2\mathcal{R}^2}\Lambda_{ab}\Lambda^{ab} \quad \text{with} \quad \mathcal{C} = m^2c^4 \tag{188}$$

where

$$\Lambda_{ab} \equiv mc\left(\mathcal{X}^a\dot{\mathcal{X}}^b - \dot{\mathcal{X}}^a\mathcal{X}^b\right). \tag{189}$$

We can rewrite this in the form of a constraint on the phase space of the pseudo-euclidean space $T^*\mathbb{E}^{4,1}$. Namely, this can be rewritten in the form

$$\phi(\mathcal{X},\mathcal{P}) = \mathcal{K}_{ab}\mathcal{K}^{ab} + 2m^2c^2 \approx 0, \tag{190}$$

where

$$\mathcal{K}_{ab} \equiv \frac{1}{\mathcal{R}}\left(\mathcal{X}_a\mathcal{P}_b - \mathcal{P}_a\mathcal{X}_b\right) \tag{191}$$

This constraint can be interpreted as the mass-shell condition on de Sitter spacetime as it can be shown that when $\mathcal{R}\to\infty$ this correctly reduces to the mass-shell condition of Minkowski space [20]. Since the mass-shell condition of Minkowski space was the Hamitlonian constraint, we consider this to be the Hamiltonian constraint $\mathcal{H}$ for a free spinless particle on de Sitter space. So we have two primary constraints which vanish on the de Sitter hyperboloid

$$\mathcal{H} = \mathcal{K}_{ab}\mathcal{K}^{ab} + 2m^2c^2 \approx 0 \quad \text{and} \quad \phi_1 = \eta_{ab}\mathcal{X}^a\mathcal{X}^b - \mathcal{R}^2 \approx 0. \tag{192}$$

With this we can look at it from the extended Dirac's formalism. The total Hamiltonian is given by

$$H_T = \alpha\mathcal{H} + \lambda\phi_1 = \alpha\left(\mathcal{K}_{ab}\mathcal{K}^{ab} + 2m^2c^2\right) + \lambda\left(\eta_{ab}\mathcal{X}^a\mathcal{X}^b - \mathcal{R}^2\right) \approx 0 \tag{193}$$

and the canonical coordinates in $T^*\mathbb{E}^{4,1}$ are $(\mathcal{X},\mathcal{P})$ with the extended Poisson brackets

$$\{\mathcal{X}^a,\mathcal{X}^b\}_e = 0, \qquad \{\mathcal{X}^a,\mathcal{P}^b\}_e = \eta^{ab}, \qquad \{\mathcal{P}^a,\mathcal{P}^b\}_e = 0. \tag{194}$$

With this algebra we see that the constraints are first-class (Appendix. (A.1.5))

$$\{\phi_1,\mathcal{H}\}_e = \{\eta_{ab}\mathcal{X}^a\mathcal{X}^b - \mathcal{R}^2, \mathcal{K}_{ab}\mathcal{K}^{ab} + 2m^2c^2\}_e = 0. \tag{195}$$

Now that both constraints are first-class they both generate gauge symmetries. The gauge transformations generated by ϕ_1 are given by

$$\delta\mathcal{X}^a = \varepsilon(\tau)\{\mathcal{X}^a,\phi_1\}_e = 0, \quad \delta\mathcal{P}_a = \varepsilon(\tau)\{\mathcal{P}_a,\phi_1\}_e = 2\,\varepsilon(\tau)\,\eta_{ab}\mathcal{X}^b. \tag{196}$$

This can be interpreted as the fact that any shift of the momentum, parallel to the position vector, leaves the system physically unchanged. We impose a gauge condition for ϕ_1 so that it becomes second-class, ensuring that the Dirac bracket automatically enforces the hyperboloid constraint $\eta_{ab}\mathcal{X}^a\mathcal{X}^b = \mathcal{R}^2$ at all times. This is consistent with the idea that the physical momentum must always be tangent to the constraint surface. The gauge-fixing condition is

$$\phi_2 \equiv \eta_{ab}\mathcal{X}^a\mathcal{P}^b \approx 0, \tag{197}$$

which ensures that the momentum is tangent to the hyperboloid at all times and hence cannot change parallel to the position vector. With this we get (Appendix. (A.1.6))

$$\begin{aligned}\{\phi_1,\phi_2\}_e &= \{\eta_{ab}\mathcal{X}^a\mathcal{X}^b - \mathcal{R}^2, \eta_{cd}\mathcal{X}^c\mathcal{P}^d\}_e \\ &= 2\eta_{ab}\mathcal{X}^a\mathcal{X}^b \approx 2\mathcal{R}^2,\end{aligned} \tag{198}$$

and hence we see that it converts ϕ_1 into a second-class constraint. For the Hamiltonian constraint $\mathcal{H}$, we find that the Hamiltonian constraint remains first-class

$$\{\mathcal{H},\phi_2\}_e = 0. \tag{199}$$

So our general situation is now that we one first-class constraint $\mathcal{H} = \mathcal{K}_{ab}\mathcal{K}^{ab} + 2m^2c^2$ and two second-class constraints $\phi_1 = \mathcal{X}^2 - \mathcal{R}^2$ and $\phi_2 = \mathcal{X}_a\mathcal{P}^a$. Given that $\mathcal{H} = \mathcal{K}_{ab}K^{ab} = 2m^2c^2$ is now the only first-class constraint its gauge symmetry is the reparameterization invariance.

We can now look at the extended Dirac algebra, the canonical transformations and the equations of motion. Given that $\{\phi_1,\phi_2\}_e = 2\mathcal{X}^2$, we get that

$$D = 2\mathcal{X}^2\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \quad D^{-1} = \frac{1}{2\mathcal{X}^2}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \tag{200}$$

where $\mathcal{X}^2 \equiv \eta_{ab}\mathcal{X}^a\mathcal{X}^b$. Now in the canonical coordinates we have

$$\{F,G\}_e = \left(\frac{\partial F}{\partial\mathcal{X}^a}\frac{\partial G}{\partial\mathcal{P}_a} - \frac{\partial G}{\partial\mathcal{X}^a}\frac{\partial F}{\partial\mathcal{P}_a}\right) \tag{201}$$

and hence, the extended Dirac bracket as defined in (162), is given by

$$\{F,G\}_{ED} = \{F,G\}_e - \frac{1}{\mathcal{X}^2}\begin{pmatrix} -\mathcal{X}^a \frac{\partial F}{\partial \mathcal{P}^b} \\ -\mathcal{P}^a \frac{\partial F}{\partial \mathcal{P}^a} + \mathcal{X}^a \frac{\partial F}{\partial \mathcal{X}^a} \end{pmatrix}^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mathcal{X}^a \frac{\partial G}{\partial \mathcal{P}^a} \\ \mathcal{P}^a \frac{\partial G}{\partial \mathcal{P}^a} - \mathcal{X}^a \frac{\partial G}{\partial \mathcal{X}^a} \end{pmatrix}. \tag{202}$$

In particular we get the Dirac algebra

$$\{\mathcal{X}^a, \mathcal{X}^b\}_{ED} = 0, \tag{203}$$

$$\{\mathcal{X}^a, \mathcal{P}^b\}_{ED} = \eta^{ab} - \frac{\mathcal{X}^a \mathcal{X}^b}{\mathcal{X}^2} \approx \eta^{ab} - \frac{\mathcal{X}^a \mathcal{X}^b}{\mathcal{R}^2}, \tag{204}$$

$$\{\mathcal{P}^a, \mathcal{P}^b\}_{ED} = \frac{1}{\mathcal{X}^2}(\mathcal{X}^a \mathcal{P}^b - \mathcal{P}^a \mathcal{X}^b) \approx \frac{\mathcal{K}^{ab}}{R}. \tag{205}$$

Now it is not too hard to show that the isometry group $SO(4,1)$ is canonical. Given that $SO(4,1)$ is the group of transformations Λ satisfying

$$\Lambda^T \eta \Lambda = \eta \qquad \Longrightarrow \qquad \eta_{cd} \Lambda^c{}_a \Lambda^d{}_b = \eta_{ab} \tag{206}$$

we find that for $(\mathcal{X}^a, \mathcal{P}^b) \to (\mathcal{X}'^a, \mathcal{P}'^b) = (\Lambda^a{}_c \mathcal{X}^c, \Lambda^b{}_d \mathcal{P}^d)$

$$\{\mathcal{X}'^a, \mathcal{X}'^b\}_{ED} = \Lambda^a{}_c \Lambda^b{}_d \{\mathcal{X}^c, \mathcal{X}^d\}_{ED} = 0, \tag{207}$$

$$\{\mathcal{X}'^a, \mathcal{P}'^b\}_{ED} = \Lambda^a{}_c \Lambda^b{}_d \{\mathcal{X}^c, \mathcal{P}^d\}_{ED} = \Lambda^a{}_c \Lambda^b{}_d \left(\eta^{cd} - \frac{\mathcal{X}^a \mathcal{X}^b}{\mathcal{X}^2}\right) = \eta^{ab} - \frac{\mathcal{X}'^a \mathcal{X}'^b}{\mathcal{X}^2}, \tag{208}$$

$$\{\mathcal{P}'^a, \mathcal{P}'^b\}_{ED} = \Lambda^a{}_c \Lambda^b{}_d \{\mathcal{P}^c, \mathcal{P}^d\}_{ED} = \Lambda^a{}_c \Lambda^b{}_d \left(\frac{1}{\mathcal{X}^2}(\mathcal{X}^a \mathcal{P}^b - \mathcal{P}^a \mathcal{X}^b)\right) = \frac{1}{\mathcal{X}^2}(\mathcal{X}'^a \mathcal{P}'^b - \mathcal{P}'^a \mathcal{X}'^b), \tag{209}$$

which means the transformations are canonical. This is as expected because both constraints are invariant under any transformation of $SO(4,1)$.

For the motion of the free particle on de Sitter hyperboloid, we have that

$$\frac{d\mathcal{X}^a}{d\tau} = \{\mathcal{X}, H_T\}_{ED}. \tag{210}$$

It is important to remember that with Dirac brackets, second-class constraints become strong equalities. We can induce the second-class constraints beforehand to make the computation easier. We find that the total Hamiltonian reduces to (Appendix. (A.1.7))

$$\begin{aligned} H_T &= \alpha \mathcal{H} + \lambda \phi_1 \\ &\approx 2\alpha\left(\mathcal{P}^2 + m^2 c^2\right) \end{aligned} \tag{211}$$

With this, we find that (Appendix. (A.1.8))

$$\begin{aligned} \frac{d\mathcal{X}^a}{d\tau} &= \{\mathcal{X}, H_T\}_{ED} \\ &\approx 4\alpha \mathcal{P}^a \end{aligned} \tag{212}$$

and for the second derivative we get (Appendix. (A.1.9))

$$\begin{aligned}\frac{d^2\mathcal{X}^a}{d\tau^2} &= \left\{\frac{d\mathcal{X}^a}{d\tau}, H_T\right\}_{ED} \\ &\approx \frac{16\alpha^2\mathcal{P}^2}{\mathcal{R}^2}\mathcal{X}^a.\end{aligned} \tag{213}$$

Now using the fact that the velocity must always be tangent to the surface, we get the identity

$$\begin{aligned}0 &= \frac{d}{d\tau}(\eta_{ab}\mathcal{X}^a\mathcal{X}'^b) \\ &= \eta_{ab}\dot{\mathcal{X}}^a\dot{\mathcal{X}}^b + \eta_{ab}\mathcal{X}^a\ddot{\mathcal{X}}^b \\ &\approx \dot{\mathcal{X}}^2 + \eta_{ab}\mathcal{X}^a\left(\frac{16\alpha^2\mathcal{P}^2}{\mathcal{R}^2}\mathcal{X}^b\right) \\ &\approx \dot{\mathcal{X}}^2 + 16\alpha^2\mathcal{P}^2.\end{aligned} \tag{214}$$

Combining equations (213) and (214), we get the differential equation

$$\ddot{\mathcal{X}}^a + \frac{\dot{\mathcal{X}}^2}{\mathcal{R}^2}\mathcal{X}^a \approx 0. \tag{215}$$

This is the differential equation describing the motion of a free massive spinless particle on de Sitter spacetime. With the stereographic coordinates, as defined by (180), it is shown by [21] that this precisely corresponds to the geodesic equation

$$\ddot{x}^\lambda + \Gamma^\lambda_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0, \tag{216}$$

where (Appendix. (A.1.10))

$$\Gamma^\rho{}_{\mu\nu}(x) = -\frac{n(x)}{2\mathcal{R}^2}\left(\eta_{\mu\alpha}\,\delta^\rho_\nu + \eta_{\nu\alpha}\,\delta^\rho_\mu - \eta_{\mu\nu}\,\delta^\rho_\alpha\right)x^\alpha \tag{217}$$

are the Christoffel symbols for the metric $g_{\mu\nu}$. This means that the extended Dirac's formalism correctly predicts that free particles follow the geodesics on $dS(4,1)$. On this note we end our discussion.

7 Conclusion

In this thesis, we have investigated the unification of two natural extensions of the conventional phase space framework. We began by examining the extended phase space formalism, focusing on how time can be incorporated as a dynamical variable within phase space itself. It was demonstrated that applying the standard Lagrangian procedure in this context leads to a Hamiltonian that vanishes identically, $H = 0$, reflecting the inherent reparameterization invariance of the system. To overcome this, an alternative approach, as introduced in previous works [4] and [10], was represented. In this extended phase space formalism, time and its conjugate momentum are explicitly defined as dynamical variables from the outset, thereby extending the phase space. By reformulating the action functional accordingly, we derived a reparameterized action in which time is treated on equal footing with the other configuration variables. This naturally led to the emergence of an extended Hamiltonian, expressed implicitly as $H_e = p_0 + H$, which serves to define a submanifold $H_e = 0$ within the extended phase space. This submanifold encapsulates the physical dynamics of the system. In the context of the extended formalism, standard phase space constructions were discussed such as canonical transformations and the Poisson structure. From there we applied this description of phase space on Minkowski spacetime. We showed that the Poincaré group forms a subgroup of the canonical group as well as the fact that the extended Hamiltonian is Lorentz invariant.

After that, as introduced in previous works [5], [12], [13], [14] and [15], we examined Dirac's formalism on phase space, focusing on singular Lagrangians where the Legendre transformation fails to invert, leading to primary constraints. The constraints restrict the dynamics to a submanifold of phase space, with first-class constraints generating gauge transformations (redundancies) and second-class constraints reducing physical degrees of freedom. To enforce the constraints rigorously, the Dirac bracket was introduced, modifying the Poisson bracket to ensure constraints are preserved. The Dirac bracket was then used to determine dynamics that respect the constraints in the process. We applied this formalism to curved spaces as viewed as embeddings in a euclidean space, where the Dirac bracket can be used to naturally encode the geometry of the constraint surface. We showed that the metric of the curved space, written in the euclidean coordinates, is encoded in the Dirac structure giving us a nontrivial Poisson structure.

From there, we considered both formalisms simultaneously. Using Dirac's formalism we were able to continue the standard Lagrangian procedure, which was first rendered moot. Instead with the canonical Hamiltonian being identically zero, the constraint take over the dynamics given in the total Hamiltonian. It was shown that the extended phase space formalism was merely a special case of Dirac's formalism after reparameterizatizing the action. The Dirac formalism gives correctly the same constraint gives $p_0 + H \approx 0$ and the reparameterization invariance was shown to be the gauge symmetry of the total Hamiltonian. It was therefore concluded that time reparameterization and Dirac's formalism were compatible, forming the extended Dirac formalism.

This set the stepping stage for discussing phase space on Minkowski and de Sitter spacetime. Using the extended Dirac formalism, we showed a, similar looking to the sphere, Dirac algebra which represented a nontrivial Poisson structure incorporating the curvature and time on spacetime. From this we showed that the isometry group of de Sitter spacetime $SO(4,1)$ forms a subgroup of the canonical group on de Sitter. At last, we showed that for a free massive spineless particle on de Sitter spacetime, the equations of motion correctly describe the geodesic equation.

Together, this shows that the extended phase space formalism and Dirac's method fit naturally together into a consistent framework for describing reparameterization-invariant systems, with direct applications to curved spacetimes like de Sitter. This unification clarifies the link between gauge symmetries, constraints, and the geometry of phase space, showing how time and curvature can be treated in a single coherent structure. While the work stays within the classical picture, it opens the way to look at quantizing this extended Dirac formalism, especially in the context of quantum cosmology and constrained systems in general relativity.
$\Lambda^{\mu}{}_{\nu}$

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8 Appendix

A.1 Derivations

A.1.1 $\{\mathcal{X}^a, \mathcal{K}^{bc}\}_e$

$$\begin{aligned}
\{\mathcal{X}^a, \mathcal{K}^{bc}\}_e &= \frac{1}{\mathcal{R}}\left(\{\mathcal{X}^a, \mathcal{X}^b\mathcal{P}^c\}_e - \{\mathcal{X}^a, \mathcal{X}^c\mathcal{P}^b\}_e\right) \\
&= \frac{1}{\mathcal{R}}\left(\left(\{\mathcal{X}^a, \mathcal{X}^b\}_e\mathcal{P}^c + \mathcal{X}^b\{\mathcal{X}^a, \mathcal{P}^c\}_e\right) - \left(\{\mathcal{X}^a, \mathcal{X}^c\}_e\mathcal{P}^b + \mathcal{X}^c\{\mathcal{X}^a, \mathcal{P}^b\}\right)\right) \\
&= \frac{1}{\mathcal{R}}\left(\mathcal{X}^b\{\mathcal{X}^a, \mathcal{P}^c\}_e - \mathcal{X}^c\{\mathcal{X}^a, \mathcal{P}^b\}_e\right) \\
&= \frac{1}{\mathcal{R}}\left(\eta^{ac}\mathcal{X}^b - \eta^{ab}\mathcal{X}^c\right)
\end{aligned} \tag{218}$$

A.1.2 $\{\mathcal{P}^a, \mathcal{K}^{bc}\}_e$

$$\begin{aligned}
\{\mathcal{P}^a, \mathcal{K}^{bc}\}_e &= \frac{1}{\mathcal{R}}\left(\{\mathcal{P}^a, \mathcal{X}^b\mathcal{P}^c\}_e - \{\mathcal{P}^a, \mathcal{X}^c\mathcal{P}^b\}_e\right) \\
&= \frac{1}{\mathcal{R}}\left(\left(\{\mathcal{P}^a, \mathcal{X}^b\}_e\mathcal{P}^c + \mathcal{X}^b\{\mathcal{P}^a, \mathcal{P}^c\}_e\right) - \left(\{\mathcal{P}^a, \mathcal{X}^c\}_e\mathcal{P}^b + \mathcal{X}^c\{\mathcal{P}^a, \mathcal{P}^b\}\right)\right) \\
&= \frac{1}{\mathcal{R}}\left(\{\mathcal{P}^a, \mathcal{X}^b\}_e\mathcal{P}^c - \{\mathcal{P}^a, \mathcal{X}^c\}_e\mathcal{P}^b\right) \\
&= \frac{1}{\mathcal{R}}\left(\eta^{ab}\mathcal{P}^c - \eta^{ac}\mathcal{P}^b\right).
\end{aligned} \tag{219}$$

A.1.3 $\{\mathcal{X}^a, \mathcal{K}_{bc}\mathcal{K}^{bc}\}_e$

$$\begin{aligned}
\{\mathcal{X}^a, \mathcal{K}_{bc}\mathcal{K}^{bc}\}_e &= \{\mathcal{X}^a, \mathcal{K}_{bc}\}_e\mathcal{K}^{bc} + \mathcal{K}_{bc}\{\mathcal{X}^a, \mathcal{K}^{bc}\}_e \\
&= 2\mathcal{K}_{bc}\{\mathcal{X}^a, \mathcal{K}^{bc}\}_e \\
&= \frac{2}{\mathcal{R}}\mathcal{K}_{bc}\left(\eta^{ac}\mathcal{X}^b - \eta^{ab}\mathcal{X}^c\right) \\
&= \frac{2}{\mathcal{R}}\left(\eta^{ac}\mathcal{X}^b\mathcal{K}_{bc} - \eta^{ab}\mathcal{X}^c\mathcal{K}_{bc}\right) \\
&= \frac{4}{\mathcal{R}}\eta^{ac}\mathcal{X}^b\mathcal{K}_{bc}
\end{aligned} \tag{220}$$

A.1.4 $\{\mathcal{P}^a, \mathcal{K}_{bc}\mathcal{K}^{bc}\}_e$

$$\begin{aligned}
\{\mathcal{P}^a, \mathcal{K}_{bc}\mathcal{K}^{bc}\}_e &= \{\mathcal{P}^a, \mathcal{K}_{bc}\}_e\mathcal{K}^{bc} + \mathcal{K}_{bc}\{\mathcal{P}^a, \mathcal{K}^{bc}\}_e \\
&= 2\mathcal{K}_{bc}\{\mathcal{P}^a, \mathcal{K}^{bc}\}_e \\
&= \frac{2}{\mathcal{R}}\mathcal{K}_{bc}\left(\eta^{ab}\mathcal{P}^c - \eta^{ac}\mathcal{P}^b\right) \\
&= \frac{2}{\mathcal{R}}\left(\eta^{ab}\mathcal{P}^c\mathcal{K}_{bc} - \eta^{ac}\mathcal{P}^b\mathcal{K}_{bc}\right) \\
&= -\frac{4}{\mathcal{R}}\eta^{ac}\mathcal{P}^b\mathcal{K}_{bc}.
\end{aligned} \tag{221}$$

A.1.5 $\{\phi_1, \mathcal{H}\}_e$

$$\begin{aligned}
\{\phi_1, \mathcal{H}\}_e &= \{\eta_{ab}\mathcal{X}^a\mathcal{X}^b - \mathcal{R}^2, \mathcal{K}_{ab}\mathcal{K}^{ab} + 2m^2c^2\}_e \\
&= \{\mathcal{X}^a\mathcal{X}_a, \mathcal{K}_{bc}\mathcal{K}^{bc}\}_e \\
&= 2\mathcal{X}_a\{\mathcal{X}^a, \mathcal{K}_{bc}\mathcal{K}^{bc}\}_e \\
&= \frac{8}{\mathcal{R}}\mathcal{X}_a\eta^{ac}\mathcal{X}^b\mathcal{K}_{bc} \\
&= \frac{8}{\mathcal{R}^2}\mathcal{X}^a\mathcal{X}^b(\mathcal{X}_a\mathcal{P}_b - \mathcal{P}_a\mathcal{X}_b) \\
&= \frac{8}{\mathcal{R}^2}\mathcal{X}^a\mathcal{X}_a\mathcal{X}^b\mathcal{P}_b - \mathcal{X}^a\mathcal{P}_a\mathcal{X}^b\mathcal{X}_b \\
&= 0,
\end{aligned} \tag{222}$$

A.1.6 $\{\phi_1, \phi_2\}_e$

$$\begin{aligned}
\{\phi_1, \phi_2\}_e &= \{\eta_{ab}\mathcal{X}^a\mathcal{X}^b - \mathcal{R}^2, \eta_{cd}\mathcal{X}^c\mathcal{P}^d\}_e \\
&= \eta_{ab}\eta_{cd}\left(\{\mathcal{X}^a\mathcal{X}^b, \mathcal{X}^c\}_e\mathcal{P}^d + \mathcal{X}^c\{\mathcal{X}^a\mathcal{X}^b, \mathcal{P}^d\}_e\right) \\
&= \eta_{ab}\eta_{cd}\mathcal{X}^c\left(\{\mathcal{X}^a, \mathcal{P}^d\}_e\mathcal{X}^b + \mathcal{X}^a\{\mathcal{X}^b, \mathcal{P}^d\}_e\right) \\
&= \eta_{ab}\eta_{cd}\mathcal{X}^c\left(\eta^{ad}\mathcal{X}^b + \eta^{bd}\mathcal{X}^a\right) \\
&= 2\eta_{ab}\mathcal{X}^a\mathcal{X}^b \approx 2\mathcal{R}^2
\end{aligned} \tag{223}$$

A.1.7 H_T

$$\begin{aligned}
H_T &= \alpha\mathcal{H} + \lambda\phi_1 \\
&\approx \alpha\mathcal{H} \\
&\approx \alpha\left(\mathcal{K}_{ab}\mathcal{K}^{ab} + 2m^2c^2\right) \\
&\approx 2\alpha\left(\frac{1}{\mathcal{R}^2}\left((\eta_{ab}\mathcal{X}^a\mathcal{X}^b)(\eta_{cd}\mathcal{P}^c\mathcal{P}^d) - (\eta_{ab}\mathcal{X}^a\mathcal{P}^b)^2\right) + m^2c^2\right) \\
&\approx 2\alpha\left((\eta_{cd}\mathcal{P}^c\mathcal{P}^d) + m^2c^2\right) \\
&\approx 2\alpha\left(\mathcal{P}^2 + m^2c^2\right)
\end{aligned} \tag{224}$$

A.1.8 $\frac{d\mathcal{X}^a}{d\tau}$

$$\begin{aligned}
\frac{d\mathcal{X}^a}{d\tau} &= \{\mathcal{X}, H_T\}_{ED} \\
&\approx 2\alpha\{\mathcal{X}^a, \mathcal{P}^2 + m^2c^2\}_{ED} \\
&\approx 4\alpha\eta_{bc}\mathcal{P}^b\{\mathcal{X}^a, \mathcal{P}^c\}_{ED} \\
&\approx 4\alpha\eta_{bc}\mathcal{P}^b\left(\eta^{ac} - \frac{\mathcal{X}^a\mathcal{X}^c}{\mathcal{R}^2}\right) \\
&\approx 4\alpha\left(\mathcal{P}^a - \frac{\mathcal{X}^a(\eta_{bc}\mathcal{P}^b\mathcal{X}^c)}{\mathcal{R}^2}\right) \\
&\approx 4\alpha\mathcal{P}^a.
\end{aligned} \tag{225}$$

A.1.9 $\frac{d^2\mathcal{X}^a}{d\tau^2}$

$$\begin{aligned}
\frac{d^2\mathcal{X}^a}{d\tau^2} &= \left\{\frac{d\mathcal{X}^a}{d\tau}, H_T\right\}_{ED} \\
&\approx 8\alpha^2\{\mathcal{P}^a, \mathcal{P}^2 + m^2c^2\}_{ED} \\
&\approx 16\alpha^2\eta_{bc}\mathcal{P}^b\{\mathcal{P}^a, \mathcal{P}^c\}_{ED} \\
&\approx 16\alpha^2\eta_{bc}\mathcal{P}^b\left(\frac{\mathcal{K}^{\dashv|}}{\mathcal{R}}\right) \\
&\approx \frac{16\alpha^2}{\mathcal{R}^2}\mathcal{X}^a\left(\eta_{bc}\mathcal{P}^b\mathcal{P}^c\right) \\
&\approx \frac{16\alpha^2\mathcal{P}^2}{\mathcal{R}^2}\mathcal{X}^a
\end{aligned} \tag{226}$$

A.1.10 $\Gamma^{\lambda}{}_{\mu\nu}$

$$\begin{aligned}
\Gamma^{\lambda}{}_{\mu\nu} &= \frac{1}{2}g^{\lambda\rho}(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}) \\
&= \frac{1}{2}n^{-2}\eta^{\lambda\rho}\left(2n\,\partial_\mu n\,\eta_{\rho\nu} + 2n\,\partial_\nu n\,\eta_{\rho\mu} - 2n\,\partial_\rho n\,\eta_{\mu\nu}\right) \\
&= n^{-1}\left(\partial_\mu n\,\delta^\lambda_\nu + \partial_\nu n\,\delta^\lambda_\mu - \eta^{\lambda\rho}\partial_\rho n\,\eta_{\mu\nu}\right) \\
&\text{where}\quad \partial_\mu n = -\frac{1}{2\mathcal{R}^2}n^2 x_\mu \\
&= -\frac{n}{2\mathcal{R}^2}\left(x_\mu\,\delta^\lambda_\nu + x_\nu\,\delta^\lambda_\mu - \eta^{\lambda\rho}x_\rho\,\eta_{\mu\nu}\right).
\end{aligned} \tag{227}$$

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