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Linear algebraic crowds

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Abstract

Crowds are generalizations of groups, and bands are generalizations of commutative rings. In this thesis, we continue the study of these crowds, with a focus on some linear algebraic crowds, namely the special linear group and general linear group over bands. We start the thesis by introducing bands and crowds and giving some general results. We then study the special and general linear groups over a band B, $SL_n(B)$ and $GL_n(B)$, and their comparison, with case studies for the Krasner hyperfield and the sign hyperfield. We follow this by defining the semidirect product of crowds, and the crowd version of the short exact sequence, which we use to learn more about $SL_n(B)$ and $GL_n(B)$.

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Introduction

A linear algebraic group is a structure that, using the elements of a ring, forms a subgroup of the special linear group of dimension n. An example is the general linear group over the real numbers, $GL_n(\mathbb{R})$. These groups have already been widely studied. However, when they are taken over weaker structures, the group law may fail. Then the algebraic group is not a group anymore, but it can still be a crowd.
Crowds were introduced by Lorscheid and Thas in 2023 in their paper [7], to study algebraic groups over the field of one element $\mathbb{F}_1$. They are generalizations of groups, and therefore can be used to study objects that are not groups, but do seem to have some structure to them. In his bachelor thesis [9], Maxson continued the study of crowds, in particular the special linear group over the Krasner hyperfield. This thesis aims to build on his research, by defining the general linear group and the symplectic group over bands, and studying their relation with the previously defined special linear group ([7, Chapter 5.4]). We include case studies where the band is the Krasner hyperfield, the sign hyperfield and the regular partial field. We also do further research on general crowds.

As mentioned, crowds are generalizations of groups. Their precise definition is:

Definition 0.1. A *crowd* is a triple $(G, 1, R)$, where G is a set, $1 \in G$ is the unit element and $R \subset G^3$ is the crowd law, that satisfies the following:

1. $(1, 1, 1) \in R$;
2. If $(a, 1, 1) \in R$ for some $a \in G$, then $a = 1$;
3. If $(a, b, 1) \in R$ for some $a, b \in G$, then $(b, a, 1) \in R$;
4. If $(a, b, c) \in R$ for some $a, b, c \in G$, then $(c, a, b) \in R$.

For all $a, b \in G$, the *inverse* of a is the set $a^{-1} = \{b \in G \mid (a, b, 1) \in R\}$, and the *product* of a and b is given by $a \cdot b = \{c \in G \mid \exists d \in G \text{ such that } c \in d^{-1} \text{ and } (a, b, d) \in R\}$ ([7, Chapter 5.1]) (see Definition 1.10).

This definition allows for a lot of variation, for example, both

$$(\{1, a\}, 1, \{(1, 1, 1)\}) \text{ and } (\{1, a\}, 1, \{(1, 1, 1), (a, a, 1), (1, a, a), (a, 1, a)\})$$

are crowds. Given a crowd $(G, 1_G, R_G)$, we can construct a *subcrowd* by taking subsets of G and R_G (see Definition 1.17), and a crowd morphism from G to another crowd by preserving the unit element and the triples in the crowd law (see Definition 1.13).
In this thesis, we use crowds to study linear algebraic groups over bands. *Bands* B are generalizations of commutative rings, where the addition operator is replaced by a set N_B, called the null set, that contains all sums we want to treat as 'zero' (see Definition 1.2). For example, the Krasner hyperfield $\mathbb{K} = \{0, 1\}$ forms a band with normal multiplication and null set $N_{\mathbb{K}} = \{n \cdot 1 | n \in \mathbb{N}_{\geq 0}, n \neq 1\}$(see Definition 1.5). Other bands are the sign hyperfield $\mathbb{S} = \{0, 1, -1\}$, with normal multiplication and null set $N_{\mathbb{S}} = \{p \cdot 1 + m \cdot (-1) | p = m = 0 \text{ or } p, m \in \mathbb{N}_{\geq 0}\}$ (see Definition 1.6) and the regular partial field $\mathbb{F}_1^{\pm} = \{0, 1, -1\}$, also with normal mutliplication, but with null set $N_{\mathbb{F}} = \{m \cdot 1 + m \cdot (-1) | m \in \mathbb{N}\}$ (see Definition 1.7).
The *special linear group* over a band, $SL_n(B)$, is not a group, but it is still a crowd. Because the null set of the band contains all sums we want to treat as zero, we can define the set as all n by n matrices A such that $\det(A) - 1 \in N_B$. The crowd law is given by the set of triples (A, C, D), such that the coefficients of $A \cdot C \cdot D - I_n$ are in N_B (see Definition 2.2). The crowd of *the general linear group*

over B, $GL_n(B)$, has a similar definition, except it contains all $n+1$ by $n+1$ matrices A such that $A = \left[\begin{smallmatrix} A_{11} & \mathbf{0} \\ \mathbf{0} & \delta \end{smallmatrix}\right]$ and $\det(A) - 1 \in N_B$ (see Definition 2.5).

In this thesis, we develop the following new results about linear algebraic crowds, and crowds in general.
We define the general linear group and the symplectic group over a band, $Sp_{2n}(B)$, and find that $Sp_2(B) = SL_2(B)$ for all bands B (see Definition 2.5, 2.16 and Theorem 2.18). Furthermore, we find that crowd isomorphisms map inverses to inverses and products to products (see Theorem 1.16). This is useful, because of the following theorem.

Theorem 0.2. *Let B be a band. Then the map $\varphi : SL_n(B) \to GL_n(B)$, $A \mapsto \left[\begin{smallmatrix} A & \mathbf{0} \\ \mathbf{0} & 1 \end{smallmatrix}\right]$ is an injective, full crowd morphism (see Theorem 3.1).*

This implies, as stated in Theorem 3.3, that $SL_n(B)$ is isomorphic to the image of φ, which is a full subcrowd of $GL_n(B)$. When we apply this morphism to the linear groups over the Krasner hyperfield, we find that $SL_n(\mathbb{K}) \cong GL_n(\mathbb{K})$ (see Theorem 3.4). Therefore, everything that was already proven for $SL_n(\mathbb{K})$ by Maxson in his thesis ([9]), also holds for $GL_n(\mathbb{K})$ (see Proposition 3.4.1 and 3.4.2). While there is no such isomorphism between $SL_n(\mathbb{S})$ and $GL_n(\mathbb{S})$, it is possible to construct all elements in $GL_n(\mathbb{S})$ and its crowd law, using only the elements in $SL_n(\mathbb{K})$ and its crowd law (see Theorems 3.5, 3.6, 3.7 and Corollaries 3.6.1 and 3.7.1).
We also compare the special and general linear groups taken over $\mathbb{F}_1^{\pm}$, $\mathbb{S}$ and $\mathbb{K}$ with each other, and find injective crowd morphisms f_n^S, f_n^G, surjective crowd morphism g_n^S, g_n^G and crowd isomorphisms θ, ψ, φ such that the following diagram commutes:

$$\begin{array}{ccccc} SL_n(\mathbb{F}_1^{\pm}) & \xrightarrow{f_n^S} & SL_n(\mathbb{S}) & \xrightarrow{g_n^S} & SL_n(\mathbb{K}) \\ \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\varphi} \\ GL_n(\mathbb{F}_1^{\pm}) & \xrightarrow{f_n^G} & GL_n(\mathbb{S}) & \xrightarrow{g_n^G} & GL_n(\mathbb{K}) \end{array}$$

Returning to general crowds, we find that, where groups always correspond to a crowd ([7, Chapter 5.2]), hypergroups always correspond to a specific type of crowd, namely a saturated crowd (see Theorem 5.7). Additionally, we define the semidirect product for crowds (see Definition 4.1):

Definition 0.3. Given crowds $(H, 1_H, R_H)$ and $(Q, 1_Q, R_Q)$ and crowd morphism $\theta : Q \to \mathrm{Aut}(H)$, the *semi-direct product* $H \rtimes_\theta Q$ is the crowd $(H \times Q, (1_H, 1_Q), R_{H \rtimes_\theta Q})$, with crowd law

$$R_{H \rtimes_\theta Q} = \left\{ ((h_1, q_1), (h_2, q_2), (h_3, q_3)) \in (H \times Q)^3 \mid (q_1, q_2, q_3) \in R_Q \text{ and } (h_1, \theta_{q_1}(h_2), \theta_{q_1}(\theta_{q_2}(h_3))) \in R_H \right\}.$$

Another notion from group theory that can be applied to crowds is that of the split short exact sequence. We define 2 types of sequences, a weakly split short exact sequence (see Definition 4.4) and a strongly split short exact sequence (see Definition 4.6).

Theorem 0.4. *Suppose the sequence*

$$1 \longrightarrow (H, 1_H, R_H) \xrightarrow{\alpha} (G, 1_G, R_G) \xrightarrow{\beta} (Q, 1_Q, R_Q) \longrightarrow 1,$$

where H, G and Q are crowds and α, β are crowd morphisms, is a weakly split short exact sequence. Then there is a bijection between the sets $H \times Q \to G$, and if G is finite, then so are H and Q and $\#G = \#H \cdot \#Q$ (see Theorem 4.5).
If it is a strongly split short exact sequence, then there is an additional bijection between the crow laws $R_Q \times R_H \to R_G$, and if R_G is finite, then so are R_H and R_Q and $\#R_G = \#R_Q \cdot \#R_H$. Furthermore, there is a crowd morphism $\theta : Q \to Aut(H)$ such that the crowd $G \cong H \rtimes_\theta Q$(see Theorems 4.7 and 4.8).

For example, there are crowd morphisms φ and ψ such that

$$1 \longrightarrow SL_n(\mathbb{K}) \xrightarrow{\varphi} GL_n(\mathbb{K}) \xrightarrow{\psi} \mathbb{K}^\times \longrightarrow 1$$

is a strongly split short exact sequence (see Example 4.9). For general bands B, we find that if $n \in \mathbb{N}$ such that $a_{n+1,n+1} \in B^\times$ for all $A \in GL_n(B)$, then there are crowd morphisms φ and ψ such that

$$1 \longrightarrow SL_n(B) \xrightarrow{\varphi} GL_n(B) \xrightarrow{\psi} B^\times \longrightarrow 1$$

is a strongly split short exact sequence (see Proposition 4.10). For example, this assumption holds for the bands $\mathbb{S}$ and $\mathbb{F}_1^\pm$ for all $n \in \mathbb{N}$. Since $\mathbb{S}^\times = \{-1, 1\}$ and $(\mathbb{F}_1^\pm)^\times = \{1, -1\}$, it follows that, if the sets are finite,

$$\#GL_n(\mathbb{S}) = 2 \cdot \#SL_n(\mathbb{S}) \text{ and } \#R_{GL_n(\mathbb{S})} = 4 \cdot \#R_{SL_n(\mathbb{S})} \text{ and}$$
$$\#GL_n(\mathbb{F}_1^\pm) = 2 \cdot \#SL_n(\mathbb{F}_1^\pm) \text{ and } \#R_{GL_n(\mathbb{F}_1^\pm)} = 4 \cdot \#R_{SL_n(\mathbb{F}_1^\pm)}.$$

(see Example 4.11).

In section 1, we give an introduction into bands and crowds, and some preliminary results. This includes the definition of bands, the Krasner hyperfield, the sign hyperfield, the regular partial field, crowds and subcrowds, and band and crowd morphisms and Theorem 1.16.
Section 2 introduces the special linear group, the general linear group and the symplectic group taken over bands. It contains Theorem 2.18, and these crowds when they are taken over $\mathbb{K}$, $\mathbb{S}$ and $\mathbb{F}_1^\pm$ (Example 2.7, 2.8 and 2.14).
In section 3, we study crowd morphisms between linear algebraic crowds. We look at the map $\varphi : SL_n(B) \to GL_n(B)$, give the Theorems 3.1 and 3.3, and study the situations when $B = \mathbb{K}$ and $B = \mathbb{S}$. This includes Theorems 3.4, 3.5, 3.6, 3.7 and Corollaries 3.6.1 and 3.7.1. We also compare the special linear groups and general linear groups over $\mathbb{F}_1^\pm$, $\mathbb{S}$ and $\mathbb{K}$ with each other, leading to Theorem 3.9 and more.
In section 4 we return to general crowds, and define the semidirect product of crowds, weakly split short exact sequence and strongly split short exact sequence. This section includes Theorems 4.5, 4.7, 4.8 and Proposition 4.10.1.
In the final section, we give the definition of saturated crowds, as defined by Maxson in [9], and show that $GL_n(\mathbb{K})$ and hypergroups are saturated crowds.

1 Bands and Crowds

In this section, we introduce bands and crowds. These structures are generalizations of commutative rings ([1]) and groups ([7, Introduction]), respectively, and can therefore be used to study objects with a weaker structure. We also introduce the Krasner hyperfield, the sign hyperfield and the regular partial fields as bands.

1.1 Bands

To define a band, we need the definitions of a *pointed monoid*, an *ambient semiring* and the *ideal of an ambient semiring*, as given by Baker, Jin and Lorscheid in [1, Chapter 1.1] and Lorscheid and Thas in [7, Introduction].

Definition 1.1. A *pointed monoid* is a set B, together with a commutative and associative operator $\cdot : B \times B \to B$, such that there is a unit element $1 \in B$ and absorbing element $0 \in B$ such that $a \cdot 1 = a$ and $a \cdot 0 = 0$ for all $a \in B$.
Its *ambient semiring* B^+ is the semiring $B^+ = \mathbb{N}[B]/\sim$, where $\sim$ is the equivalence relation

$$0 + \sum n_a \cdot a \sim \sum n_a \cdot a.$$

In other words:

$$B^+ = \{\sum_{a\in B-\{0\}} n_a \cdot a \mid n_a \in \mathbb{N}, n_a = 0 \text{ for all but finitely many } n_a\}.$$

An *ideal* of B^+ is a subset I such that

- $0 \in I$,
- For all $a, b \in I$, we have that $a + b \in I$ and
- For all $a \in I, b \in B$, we have that $b \cdot a \in I$.

Definition 1.2. A *band* is a pointed monoid $(B, 1, 0, \cdot)$ together with a set N_B, called the null set, such that N_B is an ideal of the ambient semiring B^+ and for every $a \in B$, there exists a unique $b \in B$ such that $a + b \in N_B$. We call this the *additive inverse* of a and write $-a = b$ ([1, Chapter 1.1]).

Example 1.3. As Maxson shows in [9, Chapter 2], every commutative ring $(R, +, \cdot, 0, 1)$ gives way to a band $(R, , \cdot, 1)$ with null set $N_R = \{\sum_{a\neq 0} n_a \cdot a = 0 | n_a \in \mathbb{N}, n_a = 0$ for all but finitely many $n_a\}$. Therefore bands are indeed a generalization of commutative rings.

So a band generalizes a commutative ring by replacing the addition operator with the null set. This null set can be seen as the set of sums we want to treat as 'zero'.

Lemma 1.4. *Let B be a band with zero element 0 and unit element 1. Then $B \cap N_B = \{0\}$. Furthermore $(-1)^2 = 1$ and for all $a \in B$, $-a = (-1) \cdot a$.*

Proof: This is proven in [1, Lemma 1.2]

Example 1.5. The Krasner hyperfield $\mathbb{K}$ is the band $(\mathbb{K}, 1, 0, \cdot)$, with the set $\mathbb{K} = \{0, 1\}$ and normal multiplication $\cdot : \mathbb{K} \to \mathbb{K}$, together with the null set $N_{\mathbb{K}} = \{m \cdot 1 \mid m \in \mathbb{N}_{\geq 0}, m \neq 1\}$. Note that $-1 = 1$ and $-0 = 0$.

Example 1.6. The sign hyperfield $\mathbb{S}$ is the band $(\mathbb{S}, 1, 0, \cdot)$, where $\mathbb{S} = \{0, 1, -1\}$ and $\cdot : \mathbb{S} \to \mathbb{S}$ is normal mutliplication, together with the null set $N_{\mathbb{S}} = \{p \cdot 1 + m \cdot (-1) | p = m = 0 \text{ or } p, m \in \mathbb{N}_{\geq 0}\}$. In this band, 1 and -1 are each others additive inverse, and the inverse of 0 is itself.

Example 1.7. The regular partial field $\mathbb{F}_1^{\pm}$ is similar to the sign hyperfield. It is the band $(\mathbb{F}_1^{\pm}, 1, 0, \cdot)$, where $\mathbb{F}_1^{\pm} = \{0, 1, -1\}$ and $\cdot : \mathbb{F}_1^{\pm} \to \mathbb{F}_1^{\pm}$ is again normal multiplication, but the null set is given by $N_{\mathbb{F}} = \{m \cdot 1 + m \cdot (-1) | m \in \mathbb{N}\}$. Similarly to $\mathbb{S}$, the inverse of 1 is -1 and the inverse of 0 is 0.

These examples are from [1, Chapter 1.2].

One can travel between bands by using *band morphisms*:

Definition 1.8. Given the bands $(B, 1_B, N_B)$ and $(C, 1_C, N_C)$, a *band morphism* $f : B \to C$ is a multiplicative map such that $f(1_B) = 1_C$ and $\sum_{a\in B-\{0\}} n_a f(a) \in N_C$ whenever $\sum_{a\in B-\{0\}} n_a a \in N_B$ ([1, Definition 1.3]).

Example 1.9. The maps $f : \mathbb{F}_1^{\pm} \to \mathbb{S}, b \mapsto b$ and $g : \mathbb{S} \to \mathbb{K}, 0 \mapsto 0, \pm 1 \mapsto 1$ are band morphisms. We prove this in Theorem 3.8

1.2 Crowds

A crowd generalizes a group by replacing the *group* law with a *crowd* law. For a group G with group law $\cdot : G \times G \to G$, all elements of G have a unique inverse ([11, Chapter III.1]). Thus for all $a, b, d \in G$, we can describe $a \cdot b = d$ as $a \cdot b \cdot c = 1$, where $c \in G$ such that $d \cdot c \cdot 1 = 1$. Therefore we can replace the operator with the set of all triples a, b, c such that $abc = 1$. This set is the crowd law ([7, Chapter 5]).

Definition 1.10. A *crowd* is a triple $(G, 1, R)$, where G is a set, $1 \in G$ is the unit element and $R \subset G^3$ is the crowd law, that satisfies the following:

1. $(1, 1, 1) \in R$;
2. If $(a, 1, 1) \in R$ for some $a \in G$, then $a = 1$;
3. If $(a, b, 1) \in R$ for some $a, b \in G$, then $(b, a, 1) \in R$;
4. If $(a, b, c) \in R$ for some $a, b, c \in G$, then $(c, a, b) \in R$.

For all $a, b \in G$, the *inverse* of a is the set $a^{-1} = \{b \in G \mid (a, b, 1) \in R\}$, and the *product* of a and b is given by $a \cdot b = \{c \in G \mid \exists d \in G \text{ such that } c \in d^{-1} \text{ and } (a, b, d) \in R\}$ ([7, Chapter 5.1]).

This definition also works when it is difficult to find a good group law, by making the crowd law from the triples of elements whose product we would like to be equivalent to 1. However, it is possible in a crowd that some inverses or products are empty sets, or contain more then 1 element. Therefore, what holds for groups does not have to hold for crowds.

Example 1.11. Let $(G, \cdot, 1)$ be a group. We can construct the crowd $(G, 1, R)$ by setting the crowd law as $R = \{(a, b, c) \in G^3 \mid abc = 1\}$. This is indeed a crowd, since: $1 \cdot 1 \cdot 1 = 1$, so $(1, 1, 1) \in R$; if $(a, 1, 1) \in R$, then $a \cdot 1 \cdot 1 = a = 1$; if $(a, b, 1) \in R$ for some $a, b, \in G$, then $a \cdot b = 1$ and since G is a group, this implies $b \cdot a = 1$, so $(b, a, 1) \in R$; if $(a, b, c) \in R$ for some $a, b, c \in G$, then $abc = 1$, so $ab = c^{-1}$, which implies $cab = 1$ and thus $(c, a, b) \in R$. Therefore all of the crowd axioms are satisfied, so $(G, 1, R)$ is indeed a crowd ([7, Chapter 5.2]). In this text, we say that G is both a group and a crowd. It is also possible to say that G is a crowd that comes from a group.

It is not necessary for crowd laws to be based on group operators. They can be quite arbitrary, and sets usually have multiple possible crowd laws.

Example 1.12. Let $G = \{1_G, a, b\}$ with unit element 1_G. Then possible crowd laws are

$$R_1 = \{(1_G, 1_G, 1_G), (a, b, 1_G), (1_G, a, b), (b, 1_G, a), (b, a, 1_G), (1_G, b, a), (a, 1_G, b)\} \text{ and}$$
$$R_2 = \{(1_G, 1_G, 1_G), (a, a, b), (b, a, a), (a, b, a)\}.$$

Similar to band and group morphisms, Lorscheid and Thas have defined *crowd morphisms* in [7, Chapter 5.1].

Definition 1.13. A *crowd morphism* $\varphi : G \to H$ is a map from the crowd $(G, 1_G, R_G)$ to the crowd $(H, 1_H, R_H)$ such that $\varphi(1_G) = 1_H$ and for all $(a, b, c) \in R_G$, it holds that $(\varphi(a), \varphi(b), \varphi(c)) \in R_H$.
If there exist a crowd morphism $\psi : H \to G$ such that $\varphi \circ \psi = \mathrm{id}_H$ and $\psi \circ \varphi = \mathrm{id}_G$, then φ is a *crowd isomorphism.*
If for all $a, b, c \in G$, $((\varphi(a), \varphi(b), \varphi(c)) \in R_H$ implies $(a, b, c) \in R_G$, then φ is a *full crowd morphism.*
The *kernel* of a crowd morphism φ is $\ker(\varphi) = \{a \in G | \varphi(a) = 1_H\}$.

Example 1.14. Let $(G, \odot_G, 1_G)$ and $(H, \odot_H, 1_H)$ be groups and $(G, 1_G, R_G)$ and $(H, 1_H, R_H)$ be the corresponding crowds. Suppose $f : G \to H$ is a group homomorphism, and let f' be the map from the crowd G to the crowd H such that $f'(g) = f(g)$. Then we can check that f' is a crowd morphism. Since $f'(1_G) = f(1_G) = 1_H$, the first condition is satisfied. Additionally, suppose that $a, b, c \in G$ such that $(a, b, c) \in R_G$. Then $a \odot_G \odot_G b \odot c = 1_G$, so

$$f(a) \odot_H f(b) \odot_H f(c) = f(a \odot_G b \odot_G c) = f(1_G) = 1_H,$$

which implies $(f(a), f(b), f(c)) = (f'(a), f'(b), f'(c)) \in R_H$. Therefore the second condition is satisfied as well, so f' is indeed a crowd morphism ([7, Proposition 5.7]). In fact, if f' is injective and $(f'(a), f'(b), f'(c)) \in R_H$ for some $a, b, c \in G$, then

$$\begin{aligned} f'(a) \odot_H f'(b) \odot_H f'(c) &= f(a) \odot_H f(b) \odot_H f(c) \\ &= f(a \odot_G b \odot c) = 1_H. \end{aligned}$$

Since f' is injective we must have that $a \odot_G b \odot_G c = 1_G$, which means $(a, b, c) \in R_G$. Therefore if f' is injective, then f' is a full crowd morphism.

In general, a crowd morphism is not full if and only if it is injective. This does not mean we cannot infer anything results from a full crowd morphism.

Lemma 1.15. *Let $(G, 1_G, R_G)$ and $(H, 1_H, R_H)$ be crowds, and $\varphi : G \to H$ be a full crowd morphism. Then ker$(\varphi) = \{1_G\}$.*

Proof: To proof the first statement, let $a \in \ker(\varphi)$ be arbitrary. Then

$$(\varphi(a), \varphi(1_G), \varphi(1_G)) = (1_H, 1_H, 1_H) \in R_H.$$

Since φ is full, this means $(a, 1_G, 1_G) \in R_H$, which implies that $a = 1_G$. Since a was chosen arbitrarily, this holds for all elements of $\ker(\varphi)$. Therefore $\ker(\varphi) = \{1_G\}$. □

Corollary 1.15.1. *Let G and H be both crowds and groups and $f' : G \to H$ a crowd morphism that corresponds to a group homomorphism $f : G \to H$. Then f' is a full crowd morphism if and only if it is injective.*

Proof: $(\Rightarrow)$: Suppose that f' is a full crowd morphism. Since it corresponds to a group homomorphism f, we know that $\ker(f') = \ker(f)$. Therefore it follows from Lemma 1.15 that $\ker(f) = \ker(f') = \{1_G\}$. This implies that the group homomorphism f, and therefore also the crowd morphism f', is injective.
$(\Leftarrow)$: This direction is proven in Example 1.14.
Therefore f' is a full crowd morphism if and only if it is injective. □

Similar to groups, crowd isomorphisms map inverses to inverses and products to products.

Theorem 1.16. *Let $(G, 1_G, R_G)$ and $(H, 1_H, R_H)$ be crowds such that there exists a crowd isomorphism $\varphi : G \to H$. Then for all $a, b, c \in G$, the following holds:*

1. *$b \in a^{-1}$ if and only if $\varphi(b) \in \varphi(a)^{-1}$;*
2. *$c \in a \cdot b$ if and only if $\varphi(c) \in \varphi(a) \cdot \varphi(b)$.*

Proof of 1: $(\Rightarrow)$: Suppose $a, b \in G$ such that $b \in a^{-1}$. Then $(a, b, 1_G) \in R_G$, which implies $(\varphi(a), \varphi(b), \varphi(1_G)) = (\varphi(a), \varphi(b), 1_H) \in R_H$, and thus $\varphi(b) \in \varphi(a)^{-1}$.
$(\Leftarrow)$: Suppose $a, b \in G$ such that $\varphi(b) \in \varphi(a)^{-1}$. Then $(\varphi(a), \varphi(b), 1_H) \in R_H$, which implies $(\varphi^{-1}(\varphi(a)), \varphi^{-1}(\varphi(b)), \varphi^{-1}(1_H)) = (a, b, 1_G) \in R_G$, so $b \in a^{-1}$. Therefore, for all $a, b \in G$, $b \in a^{-1}$ if and only if $\varphi(b) \in \varphi(a)^{-1}$.
Proof of 2.: $(\Rightarrow)$: Suppose $a, b, c \in G$ such that $c \in a \cdot b$. Then there is a $d \in G$ such that $c \in d^{-1}$ and $(a, b, d) \in R_G$. It follows from **(1)** that $\varphi(c) \in \varphi(d)^{-1}$. Furthermore, it follows that $(\varphi(a), \varphi(b), \varphi(d)) \in R_H$. Therefore $\varphi(c) \in \varphi(a) \cdot \varphi(b)$.
$(\Leftarrow)$: Suppose $a, b, c \in G$ such that $\varphi(c) \in \varphi(a) \cdot \varphi(b)$. Then there is an $e \in H$ such that $\varphi(c) \in e^{-1}$ and $(\varphi(a), \varphi(b), e) \in R_H$. Since φ is a crowd isomorphism, there is a $d \in G$ such that $\varphi(d) = e$, and therefore it follows from **(1)** that $c \in d^{-1}$. Furthermore, $(\varphi^{-1}(\varphi(a)), \varphi^{-1}(\varphi(b)), \varphi^{-1}(e)) = (a, b, d) \in R_G$. Therefore $c \in a \cdot b$, and so for all $a, b, c \in G$, $c \in a \cdot b$ if and only if $\varphi(c) \in \varphi(a) \cdot \varphi(b)$. □

Another similarity to groups is that, where groups can have subgroups, crowds can have *subcrowds*:

Definition 1.17. Let $(G, 1_G, R_G)$ and $(H, 1_H, R_H)$ be a crowds. Then H is a *subcrowd* of G if $1_H = 1_G \in H \subset G$ and $R_H \subset R_G \cap H^3$. If $R_H = R_G \cap H^3$, then H is a *full subcrowd* of G ([9, Chapter 1.4]).

Lemma 1.18. *Let $(G, 1, R_G)$ be a crowd and suppose $H \subset G$ such that $1 \in H$. Then $(H, 1, R_H)$, where $R_H = R_G \cap H^3$, is a full subcrowd of G.*

Proof: We simply need to check that H satisfies the 4 crowd axioms:

1. $(1, 1, 1) \in H^3$ and $(1, 1, 1) \in R_G$, so $(1, 1, 1) \in R_H$;
2. If $(a, 1, 1) \in R_H$, then $(a, 1, 1) \in R_G$, which implies $a = 1$, since G is a crowd;

3. If $(a,b,1) \in R_H$, then $(a,b,1) \in R_G$. It follows that $(b,a,1) \in R_G$, and since $(b,a,1) \in H^3$ as well, which implies $(b,a,1) \in R_H$;

4. If $(a,b,c) \in R_H$, then $(a,b,c) \in R_G$, which implies $(c,a,b) \in R_G$. Since $(c,a,b) \in H^3$ as well, this means $(c,a,b) \in R_H$.

Therefore the axioms are satisfied, and H is indeed a crowd. Since $1 \in H \subset G$ and $R_H = R_G \cap H^3$, it is a full subcrowd of G. □

Example 1.19. Let $(G, \cdot, 1)$ be a group with subgroup $(H, \cdot, 1)$. Then the corresponding crowd of H is a full subcrowd of G: By definition of subgroups, $1 \in H \subset G$, so the first axiom is satisfied. Furthermore, $(a,b,c) \in R_H$ if and only if $(a,b,c) \in H^3$ and $a \cdot b \cdot c = 1$, which holds if and only if $(a,b,c) \in R_G$. Therefore $R_H = R_G \cap H^3$, so H is indeed a full subcrowd.

2 Matrix crowds

In this section, we study the special linear group, the general linear group and the symplectic group when they are taken over bands. We find that these groups become crowds, and give some preliminary results. We specifically study the cases when the band is the Krasner hyperfield, the sign hyperfield and the regular partial field

2.1 Linear groups

Notation. $M_{n\times n}(B)$ refers to the set of n by n matrices with coefficients in B. In particular, the coefficients of the n by n identity matrix I_n are written as $\delta_{i,j}$.

Definition 2.1. Let B be a band. Then for all $A \in M_{n\times n}(B)$, the determinant of A is given by $\det(A) = \sum_{\sigma\in S_n} \operatorname{sign}(\sigma) \prod_{i=1}^n a_{i,\sigma(i)}$ ([7, Chapter 5.2]).

The special linear group over a band was defined by Lorscheid and Thas in their paper ([7, Chapter 5.4]) as follows.

Definition 2.2. Let B be a band. Then the *special linear group* over B is the crowd $(SL_n(B), I_n, R_{SL_n(B)})$, where the set is given by

$$SL_n(B) = \{A \in M_{n\times n}(B) |\ \det(A) - 1 \in N_B\},$$

and crowd law is given by

$$R_{SL_n(B)} = \left\{ \left(A^{(1)},\ A^{(2)},\ A^{(3)}\right) \in SL_n(B)^3 \ \middle|\ \forall \sigma \in A_3, \forall i,j = 1,...,n, \sum_{k,l=1,\dots,n} a_{i,k}^{(\sigma(1))} a_{k,l}^{(\sigma(2))} a_{l,j}^{(\sigma(3))} - \delta_{i,j} \in N_B \right\}.$$

Note that for all $A, C \in SL_n(B)$, the coefficients of $A \cdot C$ will be sums, and therefore not necessarily in B. Therefore the group structure fails, which is why the special linear group is defined as a crowd instead. The crowd law is taken as the set of all triples $\left(A^{(1)},\ A^{(2)},\ A^{(3)}\right) \in SL_n(B)^3$ such that $A^{(\sigma(1))} \cdot A^{(\sigma(2))} \cdot A^{(\sigma(3))} - I_n$ is coefficient wise in the null set. Thus, if B is a band that was constructed from a field F, then the crowd $SL_n(B)$ corresponds to the crowd constructed from the group $SL_n(F)$, as explained in Example 1.11.
The following result is given in Maxson's thesis [9, Chapter 3.1]. For completeness, we include a proof.

Theorem 2.3. *Let B be a band. Then the triple $(SL_n(B), I_n, R_{SL_n(B)})$ is indeed a crowd.*

Proof: Since $I_n \in SL_n(B)$ is a unit element, we simply need to check that all 4 crowd axioms are satisfied:

1. $(I_n, I_n, I_n) \in R_{SL_n(B)}$, since for all $\sigma \in A_3$ and for all $i, j = 1, ..., n$,

$$\begin{aligned}\sum_{k,l=1,\dots,n} \delta_{i,k}^{(\sigma(1))}\delta_{k,l}^{(\sigma(2))}\delta_{l,j}^{(\sigma(3))} - \delta_{i,j} &= \sum_{k,l=1,\dots,n} \delta_{i,k}\delta_{k,l}\delta_{l,j} - \delta_{i,j} \\ &= \sum_{k=1,\dots,n} \delta_{i,k}\delta_{k,j} - \delta_{i,j} = \delta_{i,j} - \delta_{i,j} \in N_B.\end{aligned}$$

   Therefore (I_n, I_n, I_n) satisfies the conditions to be in $R_{SL_n(B)}$, so the first axiom holds.

2. Suppose that there is an $A \in SL_n(B)$ such that $(A, I_n, I_n) \in R_{SL_n(B)}$. Then for all $i, j = 1, ..., n$, we have that

$$\begin{aligned}\sum_{k,l=1,\dots,n} a_{i,k}\delta_{k,l}\delta_{l,j} - \delta_{i,j} \in N_B &\Rightarrow \sum_{k=1,\dots,n} a_{i,k}\delta_{k,j} - \delta_{i,j} \in N_B \\ &\Rightarrow a_{i,j} - \delta_{i,j} \in N_B.\end{aligned}$$

   Since additive inverses in bands are unique, it follows that for all $i, j = 1, ..., n$, $a_{i,j} = \delta_{i,j}$. Therefore $(A, I_n, I_n) \in R_{SL_n(B)}$ implies that $A = I_n$, so the second axiom is satisfied.

3. Suppose $(A, B, I_n) \in R_{SL_n(B)}$ for some $A, B \in SL_n(B)$. Then for all $i, j = 1, ..., n$,

$$\begin{aligned}\sum_{k,l=1,\dots,n} a_{i,k}b_{k,l}\delta_{l,j} - \delta_{i,j} \in N_B &\Rightarrow \sum_{k=1,\dots,n} a_{i,k}b_{k,j} - \delta_{i,j} \in N_B \text{ and} \\ \sum_{k,l=1,\dots,n} b_{i,k}\delta_{k,l}a_{l,j} - \delta_{i,j} \in N_B &\Rightarrow \sum_{k=1,\dots,n} b_{i,k}a_{k,j} - \delta_{i,j} \in N_B.\end{aligned}$$

   This implies that for all $i, j = 1, ..., n$,

$$\begin{aligned}\sum_{k=1,\dots,n} a_{i,k}b_{k,j} - \delta_{i,j} &= \sum_{k,l=1,\dots,n} a_{i,k}\delta_{k,l}b_{l,j} - \delta_{i,j} \in N_B, \\ \sum_{k=1,\dots,n} b_{i,k}a_{k,j} - \delta_{i,j} &= \sum_{k,l=1,\dots,n} \delta_{i,k}b_{k,l}a_{l,j} - \delta_{i,j} \in N_B \text{ and} \\ \sum_{k=1,\dots,n} b_{i,k}a_{k,j} - \delta_{i,j} &= \sum_{k,l=1,\dots,n} b_{i,k}a_{k,l}\delta_{l,j} - \delta_{i,j} \in N_B.\end{aligned}$$

   Therefore $(B, A, I_n) \in R_{SL_n(B)}$ as well, so the third axiom is satisfied.

4. Suppose that $(A, B, C) \in R_{SL_n(B)}$ for some $A, B, C \in SL_n(B)$. Then for all $i, j = 1, ..., n$,

$$\begin{aligned}&\sum_{k,l=1,\dots,n} a_{i,k}b_{k,l}c_{l,j} - \delta_{i,j} \in N_B, \\ &\sum_{k,l=1,\dots,n} b_{i,k}c_{k,l}a_{l,j} - \delta_{i,j} \in N_B \text{ and} \\ &\sum_{k,l=1,\dots,n} c_{i,k}a_{k,l}b_{l,j} - \delta_{i,j} \in N_B.\end{aligned}$$

   These sums also imply that $(C, A, B) \in R_{SL_n(B)}$. Therefore the final axiom holds.

Since all axioms are satisfied, $(SL_n(B), I_n, R)$ is indeed a crowd. □

For fields F, the general linear group $GL_n(F)$ contains all n by n matrices whose determinant is invertible. An equivalent definition is stated in [2, Example I.1.6]:

Definition 2.4. The general linear group over a field F of dimension n is

$$\begin{aligned} GL_n(F) &= \left\{A \in M_{n+1\times n+1}(F) \mid A = [a_{11}, a_{12}, ..., a_{nn}, D^{-1}], \text{ where } D = \det(A)\right\} \\ &= \{A \in SL_{n+1}(F) \mid \forall i = 1, ..., n,\ a_{i,n+1} = a_{n+1,i} = 0\}. \end{aligned}$$

This definition allows us to define the general linear group over bands.

Definition 2.5. Let B be a band. Then the *general linear group* over B is the crowd $(GL_n(B), I_{n+1}, R_{GL_n(B)})$, where the set is given by

$$GL_n(B) = \{A \in SL_{n+1}(B) | \forall i = 1, ..., n,\ a_{i,n+1} = a_{n+1,i} = 0\},$$

and crowd law is given by

$$R_{GL_n(B)} = \left\{ \left(A^{(1)},\ A^{(2)},\ A^{(3)}\right) \in GL_n(B)^3 \ \middle| \ \begin{array}{c} \forall \sigma \in A_3, \forall i, j = 1, ..., n+1, \\ \sum_{k,l=1,...,n+1} a_{i,k}^{(\sigma(1))} a_{k,l}^{(\sigma(2))} a_{l,j}^{(\sigma(3))} - \delta_{i,j} \in N_B \end{array} \right\}.$$

Theorem 2.6. *Let B be a band. Then the triple $(GL_n(B), I_{n+1}, R_{GL_n(B)})$ is a crowd. Furthermore, it is a full subcrowd of the crowd $SL_{n+1}(B)$.*

Proof: It follows from Definition 2.5 that $I_{n+1} \in GL_n(B)$ and that $GL_n(B) \subset SL_{n+1}(B)$. Furthermore, for all $A, C, D \in M_{(n+1)\times(n+1)}(B)$, we have that $(A, C, D) \in R_{GL_n(B)}$ if and only if $(A, C, D) \in GL_n(B)^3$ and for all $i, j = 1, ..., n + 1$,

$$\begin{aligned} \sum_{k,l=1,...,n+1} a_{i,k} c_{k,l} d_{l,j} - \delta_{i,j} &\in N_B, \\ \sum_{k,l=1,...,n+1} c_{i,k} d_{k,l} a_{l,j} - \delta_{i,j} &\in N_B \text{ and} \\ \sum_{k,l=1,...,n+1} d_{i,k} a_{k,l} c_{l,j} - \delta_{i,j} &\in N_B. \end{aligned}$$

Since $(A, C, D) \in GL_n(B)^3 \subset SL_{n+1}(B)^3$, these sums hold if and only if $(A, C, D) \in R_{SL_{n+1}(B)}$. Therefore $R_{GL_n(B)} = R_{SL_{n+1}(B)} \cap GL_n(B)^3$, $I_{n+1} \in GL_n(B)$ and $GL_n(B) \subset SL_{n+1}(B)$, so it follows from Lemma 1.18 that $GL_n(B)$ is a crowd and a full subcrowd of $SL_{n+1}(B)$. □

In Section 1, the bands of the Krasner hyperfield, sign hyperfield and regular partial field were given. We can study their special linear and general linear groups:

Example 2.7 (The Krasner hyperfield). Since in $\mathbb{K}$, $-1 = 1$, the special linear group has the set

$$SL_n(\mathbb{K}) = \{A \in M_{n\times n}(B) |\ \det(A) + 1 \in N_{\mathbb{K}}\}.$$

Since $N_{\mathbb{K}} = \{m \cdot 1\ |m \in \mathbb{N} \text{ and } m \neq 1\}$, it follows that $A \in SL_n(\mathbb{K})$ if and only if $\det(A) \neq 0 \cdot 1$. Therefore, we can rewrite the set as follows:

$$SL_n(\mathbb{K}) = \{A \in M_{n\times n}(B) |\ \det(A) \neq 0 \cdot 1\}.$$

For example, $SL_1(\mathbb{K}) = \{[1]\}$ and, as Maxson shows in [9, Chapter 3.3],

$$SL_2(\mathbb{K}) = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}.$$

To find $GL_n(\mathbb{K})$, note that $GL_n(\mathbb{K}) \subset SL_{n+1}(\mathbb{K})$. Therefore, for all $A \in GL_n(\mathbb{K})$, it follows that $\det(A) \neq 0 \cdot 1$. This implies $a_{n+1,n+1} \neq 0$, meaning

$$GL_n(\mathbb{K}) = \left\{A \in SL_{n+1}(\mathbb{K}) \mid A = \left[\begin{smallmatrix} A_{11} & \mathbf{0} \\ \mathbf{0} & 1 \end{smallmatrix}\right] \text{ for some } A_{11} \in SL_n(\mathbb{K})\right\}.$$

In Section 3, we see that this observation leads to a crowd isomorphism between $GL_n(\mathbb{K})$ and $SL_n(\mathbb{K})$ (see Theorem 3.4). With this map, we can find all matrices in $GL_n(\mathbb{K})$. For example, when $n = 1$ the set is given by $GL_1(\mathbb{K}) = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ and when $n = 2$, we find the set

$$GL_2(\mathbb{K}) = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}.$$

Both in $SL_n(\mathbb{K})$ and $GL_n(\mathbb{K})$, all elements have nonempty inverses. Maxson shows this in [9, Chapter 3.2] for $SL_n(\mathbb{K})$, and since $SL_n(\mathbb{K}) \cong GL_n(\mathbb{K})$ (Theorem 3.4), it follows from Theorem 1.16 that the same must hold $GL_n(\mathbb{K})$. Maxson also shows that, for all $A \in SL_2(\mathbb{K})$,

$$A^{-1} = \left\{ \begin{bmatrix} a_{22} & a_{12} \\ a_{21} & a_{11} \end{bmatrix} \right\}.$$

Similarly, for all $A \in GL_2(\mathbb{K})$,

$$A^{-1} = \left\{ \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}.$$

Thus for $n = 2$, in both linear groups all elements have singeleton inverse sets.

Example 2.8 (The sign hyperfield). For the sign hyperfield, the crowds $SL_n(\mathbb{S})$ and $GL_n(\mathbb{S})$ are not isomorphic. When $n = 1$, the special linear group is a singleton set, $SL_1(\mathbb{S}) = \{[1]\}$, while $GL_1(\mathbb{S})$ contains 2 elements: $GL_1\{\mathbb{S}) = \left\{ \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right] \right\}$. When $n = 2$, the special linear crowd contains 32 matrices and the general linear crowd 64. We can find these matrices by looking at all the possible determinants. If $A \in M_{2\times 2}(\mathbb{S})$, then $\det(A) = ad - bc$ for some $a, b, c, d \in \mathbb{S}$. Thus $ad, bc \in \{0, 1, -1\}$, which implies

$$\begin{aligned} \det(A) &\in \{0-0, 0-1, 0-(-1), 1-0, 1-1, 1-(-1), (-1)-0, (-1)-1, (-1)-(-1)\} \\ &= \{0\cdot 1 + 0\cdot(-1), 0\cdot 1 + 1\cdot(-1), 1\cdot 1 + 0\cdot(-1), 1\cdot 1 + 1\cdot(-1), 2\cdot 1 + 0\cdot(-1), 0\cdot 1 + 2\cdot(-1)\}. \end{aligned}$$

Since $A \in SL_2(\mathbb{S})$ if and only if $\det(A) - 1 \in N_{\mathbb{S}} = \{p\cdot 1 + m\cdot(-1) | p = m = 0 \text{ or } p, m \in \mathbb{N}_{\geq 0}\}$, it follows that $A \in SL_2(\mathbb{S})$ if and only if

$$\det(A) \in \{1\cdot 1 + 0\cdot(-1), 2\cdot 1 + 0\cdot(-1), 1\cdot 1 + 1\cdot(-1)\}.$$

Therefore the set of the special linear group for $n = 2$ is given by

$$SL_2(\mathbb{S}) = \{A \in M_{2\times 2}(\mathbb{S}) \mid \det(A) \in \{1\cdot 1 + 0\cdot(-1), 2\cdot 1 + 0\cdot(-1), 1\cdot 1 + 1\cdot(-1)\}\}.$$

To find $GL_2(\mathbb{S})$, note that all $A \in GL_2(\mathbb{S})$ are of the form $A = \left[\begin{smallmatrix} A_{11} & \mathbf{0} \\ \mathbf{0} & \pm 1 \end{smallmatrix}\right]$ for some $A_{11} \in M_{2\times 2}(\mathbb{S})$. If $a_{33} = 1$, then $A \in GL_2(\mathbb{S})$ if and only if $\det(A) - 1 = \det(A_{11}) \cdot 1 - 1 \in N_{\mathbb{S}}$, so if and only if $A_{11} \in SL_2(\mathbb{S})$. If $a_{33} = -1$ however, then $\det(A) = -\det(A_{11})$, so $\det(A) - 1 \in N_{\mathbb{S}}$ if and only if $-\det(A_{11}) \in \{1\cdot 1+0\cdot(-1), 2\cdot 1+0\cdot(-1), 1\cdot 1+1\cdot(-1)\}$. Note that the only if $\det(A_{11}) = 0\cdot 1+0\cdot(-1)$, is it impossible for A to be in $GL_2(\mathbb{K})$. Therefore

$$GL_2(\mathbb{S}) = \{A = \left[\begin{smallmatrix} A_{11} & \mathbf{0} \\ \mathbf{0} & \pm 1 \end{smallmatrix}\right] \in SL_3(\mathbb{S}) \ |\det(A) \neq 0\cdot 1 + 0\cdot(-1)\}.$$

In fact, we can generalize these observations to general n.

Theorem 2.9. *Let $A \in M_{n\times n}(\mathbb{S})$. Then the determinant of A is of the form $p\cdot 1 + m\cdot(-1)$ for some $p, m \in \mathbb{N}$. Furthermore, $A \in SL_n(\mathbb{S})$ if and only if and $p > 0$.*

Proof: Let $A \in M_{n\times n}(\mathbb{S})$ be arbitrary. Then $\det(A) = \sum_{\sigma\in S_n} \operatorname{sign}(\sigma) \prod_{k=1,\ldots,n} a_{k,\sigma(k)}$, where $a_{k,\sigma(k)} \in \{0, 1, -1\}$ for all $k = 1, ..., n$ and $\sigma \in S_n$. Therefore $\operatorname{sign}(\sigma) \prod_{k=1,..,n} a_{k,\sigma(k)} \in \{0, 1, -1\}$

for all $\sigma \in S_n$, so the determinant of A is always a linear combination of $0, 1$ and -1. It follows that $\det(A) = p \cdot 1 + m \cdot (-1)$, for some $p, m \in \mathbb{N}$. Furthermore,

$$A \in SL_n(\mathbb{S}) \iff \det(A) - 1 \in N_{\mathbb{S}} \iff p \cdot 1 + m \cdot (-1) - 1 = p \cdot 1 + (m+1) \cdot (-1) \in N_{\mathbb{S}}.$$

Since $m \in \mathbb{N}$, $m + 1 > 0$. Therefore $p \cdot 1 + (m+1) \cdot (-1) \in N_{\mathbb{S}}$ if and only if $p > 0$. Since A was chosen arbitrarily, we have that for all $A \in M_{n \times n}(\mathbb{S})$, $\det(A) = p \cdot 1 + m \cdot (-1)$ for some $p, m \in \mathbb{N}$ and $A \in SL_n(\mathbb{S})$ if and only if and $p > 0$. □.

Theorem 2.10. *Suppose $A \in M_{(n+1)\times(n+1)}(\mathbb{S})$. Then $A \in GL_n(\mathbb{S})$ if and only if A is of the form $\left[\begin{smallmatrix} A_{11} & 0 \\ 0 & \delta \end{smallmatrix}\right]$, where $\delta = 1$ or $\delta = -1$ and $A_{11} \in M_{n\times n}(\mathbb{S})$ such that det$(A_{11}) = p \cdot 1 + m \cdot (-1)$ for some $p, m \in \mathbb{N}$ such that $p > 0$ or $m > 0$.*

Proof: Take an arbitrary $A \in M_{(n+1)\times(n+1)}(\mathbb{S})$. By definition, $A \in GL_n(\mathbb{S})$ if and only if A is of the form $\left[\begin{smallmatrix} A_{11} & 0 \\ 0 & \delta \end{smallmatrix}\right]$ and $\det(A) - 1 \in N_{\mathbb{S}}$. If $\delta = 0$, then $\det(A) - 1 = 0 - 1 \notin N_{\mathbb{S}}$, so δ must be equal to ± 1. Furthermore, if follows from Theorem 2.9 that $\det(A_{11}) = p \cdot 1 + m \cdot (-1)$ for some $p, m \in \mathbb{N}$. Therefore

$$\begin{aligned} \det(A) - 1 \in N_{\mathbb{S}} &\iff 1 \cdot (p \cdot 1 + m \cdot (-1)) - 1 \in N_{\mathbb{S}} \text{ or } (-1) \cdot (p \cdot 1 + m \cdot (-1)) - 1 \in N_{\mathbb{S}} \\ &\iff p \cdot 1 + (m+1) \cdot (-1) \in N_{\mathbb{S}} \text{ or } m \cdot 1 + (p+1) \cdot (-1) \in N_{\mathbb{S}} \\ &\iff p > 0 \text{ or } m > 0. \end{aligned}$$

Therefore $A \in GL_n(\mathbb{S})$ if and only if A is of the form $\left[\begin{smallmatrix} A_{11} & 0 \\ 0 & \pm 1 \end{smallmatrix}\right]$, where $\det(A_{11}) = p \cdot 1 + m \cdot (-1)$ for some $p, m \in \mathbb{N}$ such that $p > 0$ or $m > 0$. □

Example 2.11 (The sign hyperfield (continued)). Similar to the linear groups taken over $\mathbb{K}$, the elements of $SL_2(\mathbb{S})$ and $GL_2(\mathbb{S})$ all have nonempty inverses, since for $A \in SL_2(\mathbb{S})$ and $B \in GL_2(\mathbb{S})$, we have that

$$\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \in A^{-1} \text{ and}$$

$$\begin{bmatrix} b_{22} & -b_{12} & 0 \\ -b_{21} & b_{11} & 0 \\ 0 & 0 & b_{33} \end{bmatrix} \in B^{-1}.$$

However, the inverses are not necessarily singletons. For example, in $SL_2(\mathbb{S})$,

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}^{-1} = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \right\},$$

and in $GL_2(\mathbb{S})$,

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}^{-1} = \left\{ \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right\}.$$

The band of the regular partial field $\mathbb{F}_1^{\pm}$ is very similar to $\mathbb{S}$, except the null set is smaller. We see this reflected in the similar but stricter conditions for the matrices in the sets $SL_n(\mathbb{F}_1^{\pm})$ and $GL_n(\mathbb{F}_1^{\pm})$.

Theorem 2.12. *Let $A \in M_{n\times n}(\mathbb{F}_1^{\pm})$. Then there exist $p, m \in \mathbb{N}$ such that det$(A) = p \cdot 1 + m \cdot (-1)$. Furthermore, $A \in SL_n(\mathbb{F}_1^{\pm})$ if and only if $p = m + 1$.*

Proof: Let $A \in M_{n\times n}(\mathbb{F}_1^{\pm})$. Since all coefficients of A are in $\{0, 1, -1\}$, it follows that the determinant of A is a linear combination of $0, 1$ and -1 (just like in the proof of Theorem 2.9). Thus there exist $p, m \in \mathbb{N}$ such that $\det(A) = p \cdot 1 + m \cdot (-1)$. Furthermore,

$$\begin{aligned} A \in SL_n(\mathbb{F}_1^{\pm}) &\iff \det(A) - 1 \in N_{\mathbb{F}_1^{\pm}} \\ &\iff p \cdot 1 + (m+1) \cdot (-1) - 1 \in N_{\mathbb{F}_1^{\pm}}, \end{aligned}$$

which holds if and only if $p = m + 1$. □

Theorem 2.13. *Let $A \in M_{(n+1)\times(n+1)}(\mathbb{F}_1^{\pm})$. Then $A \in GL_n(\mathbb{F}_1^{\pm})$ if and only if A is of the form $\left[\begin{smallmatrix} A_{11} & 0 \\ 0 & \delta \end{smallmatrix}\right]$, where $\delta = 1$ or $\delta = -1$ and $A_{11} \in M_{n\times n}(\mathbb{F}_1^{\pm})$ such that $det(A_{11}) = p \cdot 1 + m \cdot (-1)$ for some $p, m \in \mathbb{N}$ such that $p = m+1$ or $m = p+1$.*

Proof: Let $A \in M_{(n+1)\times(n+1)}(\mathbb{F}_1^{\pm})$. Then by definition, $A \in GL_n(\mathbb{F}_1^{\pm})$ if and only if $A = \left[\begin{smallmatrix} A_{11} & 0 \\ 0 & \delta \end{smallmatrix}\right]$ for some $\delta \in \mathbb{F}_1^{\pm}$ and $A_{11} \in M_{n\times n}(\mathbb{F}_1^{\pm})$ such that $\det(A) - 1 \in N_{\mathbb{F}_1^{\pm}}$. Note that if $\delta = 0$, then $\det(A) - 1 = 0 - 1 \notin N_{\mathbb{F}_1^{\pm}}$, so $\delta = \pm 1$. Suppose, $p, m \in \mathbb{N}$ such that $\det(A_{11}) = p \cdot 1 + m \cdot (-1)$. Then

$$\begin{aligned} \det(A) - 1 \in N_{\mathbb{F}_1^{\pm}} &\iff 1 \cdot (p \cdot 1 + m \cdot (-1)) - 1 \in N_{\mathbb{F}_1^{\pm}} \text{ or } (-1) \cdot (p \cdot 1 + m \cdot (-1)) \in N_{\mathbb{F}_1^{\pm}} \\ &\iff p \cdot 1 + (m+1) \cdot (-1) \in N_{\mathbb{F}_1^{\pm}} \text{ or } m \cdot 1 + (p+1) \cdot (-1) \in N_{\mathbb{F}_1^{\pm}}, \end{aligned}$$

which holds if and only if $p = m+1$ or $m = p+1$. □

Example 2.14 (The regular partial field). Using Theorems 2.12 and 2.13, we can find the crowds $SL_n(\mathbb{F}_1^{\pm})$ and $GL_n(\mathbb{F}_1^{\pm})$. In particular, $SL_1(\mathbb{F}_1^{\pm}) = \{[1]\}$ and $GL_1(\mathbb{F}_1^{\pm}) = \left\{\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right]\right\}$. Furthermore,

$$\begin{aligned} SL_2(\mathbb{F}_1^{\pm}) &= \{A \in M_{2\times 2}(\mathbb{F}_1^{\pm}) \mid \det(A) \in 1 \cdot 1 + 0 \cdot (-1)\} \\ GL_2(\mathbb{F}_1^{\pm}) &= \{A \in SL_3(\mathbb{F}_1^{\pm}) \mid A = \left[\begin{smallmatrix} A_{11} & 0 \\ 0 & \pm 1 \end{smallmatrix}\right] \text{ and } \det(A_{11}) \in \{1 \cdot 1 + 0 \cdot (-1), 0 \cdot 1 + 1 \cdot (-1)\}\}, \end{aligned}$$

where $SL_2(\mathbb{F}_1^{\pm})$ contains 20 matrices and $GL_2(\mathbb{F}_1^{\pm})$ contains 40. Unlike in $SL_2(\mathbb{S})$ and $GL_2(\mathbb{S})$, inverse sets in $SL_2(\mathbb{F}_1^{\pm})$ and $GL_2(\mathbb{F}_1^{\pm})$ are singleton sets, and for $A \in SL_2(\mathbb{F}_1^{\pm})$ and $B \in GL_2(\mathbb{F}_1^{\pm})$ we have that

$$A^{-1} = \left\{\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}\right\} \text{ and}$$
$$B^{-1} = \left\{\begin{bmatrix} b_{22} & -b_{12} & 0 \\ -b_{21} & b_{11} & 0 \\ 0 & 0 & b_{33} \end{bmatrix}\right\}.$$

For a general band B, it can be difficult to find the entire crowd law of its special or general linear group. However, for some triples we can easily show that they cannot be in $R_{GL_n(B)}$. If $\left(\left[\begin{smallmatrix} A_{11} & \mathbf{0} \\ \mathbf{0} & \delta_A \end{smallmatrix}\right], \left[\begin{smallmatrix} C_{11} & \mathbf{0} \\ \mathbf{0} & \delta_C \end{smallmatrix}\right], \left[\begin{smallmatrix} D_{11} & \mathbf{0} \\ \mathbf{0} & \delta_D \end{smallmatrix}\right]\right) \in R_{GL_n(B)}$, then $\delta_A \cdot \delta_C \cdot \delta_D - 1 \in N_B$, $\delta_D \cdot \delta_A \cdot \delta_C - 1 \in N_B$ and $\delta_C \cdot \delta_D \cdot \delta_A - 1 \in N_B$. Because in bands all additive inverses are unique, this implies the products are equal to 1. In particular, if $\left[\begin{smallmatrix} A_{11} & \mathbf{0} \\ \mathbf{0} & \delta_A \end{smallmatrix}\right] \in \left[\begin{smallmatrix} C_{11} & \mathbf{0} \\ \mathbf{0} & \delta_C \end{smallmatrix}\right]^{-1}$, then δ_A is the multiplicative inverse of δ_C.

Example 2.15. In $GL_n(\mathbb{S})$ and $GL_n(\mathbb{F}_1^{\pm})$, $\left(\left[\begin{smallmatrix} A_{11} & \mathbf{0} \\ \mathbf{0} & \delta_A \end{smallmatrix}\right], \left[\begin{smallmatrix} B_{11} & \mathbf{0} \\ \mathbf{0} & \delta_B \end{smallmatrix}\right], \left[\begin{smallmatrix} C_{11} & \mathbf{0} \\ \mathbf{0} & \delta_C \end{smallmatrix}\right]\right)$ is in the crowd law if and only if $\delta_A = \delta_B = \delta_C = 1$, $\delta_A = 1$ and $\delta_B = \delta_C = -1$, $\delta_B = 1$ and $\delta_A = \delta_C = -1$ or $\delta_C = 1$ and $\delta_A = \delta_B = -1$. Furthermore, $\left[\begin{smallmatrix} A_{11} & \mathbf{0} \\ \mathbf{0} & \delta_A \end{smallmatrix}\right] \in \left[\begin{smallmatrix} B_{11} & \mathbf{0} \\ \mathbf{0} & \delta_B \end{smallmatrix}\right]^{-1}$ if and only if $\delta_A = \delta_B$.

2.2 Symplectic crowd

Suppose F is a field and let $\Omega = \left[\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right]$. Then the group of symplectic group over F, denoted as $Sp_{2n}(F)$, is defined to be the group of all matrices in $M \in M_{2n\times 2n}(F)$ such that $M^T \Omega M = \Omega$. It can be proven that this is a subgroup of $SL_{2n}(F)$ ([5, Definition 1.1]). When we take this group over a band B, the group structure fails again, similarly to how it did for the special and general linear group. Therefore we define it to be a crowd instead, with the same crowd law as for the linear groups. It is uncertain if this condition implies $Sp_{2n}(B) \subset SL_{2n}(B)$, as it does when these groups are taken over fields. Since we want the definition to correspond nicely to the symplectic group over a field, we immediately define $Sp_{2n}(B)$ to be a subset of $SL_{2n}(B)$.

Definition 2.16. The *symplectic group* of a band B is the crowd $(Sp_{2n}(B), I_{2n}, R_{Sp_{2n}(B)})$, where the set is given by

$$Sp_{2n}(B) = \left\{ M \in SL_{2n}(B) \mid \forall i, j = 1, ..., 2n, \sum_{k,l=1,...,2n} m_{k,i}\omega_{k,l}m_{l,j} - \omega_{i,j} \in N_B \right\},$$

for

$$\Omega = (\omega_{i,j}) = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix},$$

and the crowd law is given by

$$R = \left\{ \left(M^{(1)}, M^{(2)}, M^{(3)}\right) \in Sp_{2n}(B)^3 \mid \forall \sigma \in A_n, \forall i, j = 1, ..., 2n, \sum_{k,l=1,..,2n} m_{i,k}^{(\sigma(1))} m_{k,l}^{(\sigma(2))} m_{l,j}^{(\sigma(3))} - \delta_{ij} \in N_B \right\}.$$

Theorem 2.17. *The triple $(Sp_{2n}(B), I_{2n}, R_{Sp_{2n}(B)})$ is a full subcrowd of $SL_{2n}(B)$.*

Proof: Per definition, $I_{2n} \in Sp_{2n}(B) \subset SL_{2n}(B)$. Furthermore, $(A, C, D) \in R_{Sp_{2n}(B)}$ if and only if $(A, C, D) \in Sp_{2n}(B)^3$ and for all $i, j = 1, ..., 2n$,

$$\sum_{k,l=1,....,2n} a_{i,k}c_{k,l}d_{l,j} - \delta_{i,j} \in N_B, \quad \sum_{k,l=1,....,2n} d_{i,k}a_{k,l}c_{l,j} - \delta_{i,j} \in N_B \text{ and } \sum_{k,l=1,....,2n} c_{i,k}d_{k,l}a_{l,j} - \delta_{i,j} \in N_B,$$

which holds if and only if $(A, C, D) \in R_{SL_{2n}(B)}$. It follows that $R_{Sp_{2n}(B)} = R_{SL_{2n}(B)} \cap Sp_{2n}(B)^3$ and since $I_{2n} \in Sp_{2n}(B) \subset SL_{2n}(B)$, Lemma 1.18 implies that $Sp_{2n}(B)$ is a full subcrowd of $SL_{2n}(B)$. □

Theorem 2.18. *Let B be a band. Then $Sp_2(B) = SL_2(B)$.*

Proof: Let B be a band, then $Sp_2(B)$ is a full subcrowd of $SL_2(B)$, as is proven in Theorem 2.17. Let $M = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in SL_2(B)$ be arbitrary. Then $\det(M) - 1 = ad - bc - 1 \in N_B$, so

$$\sum_{k,l=1,2} m_{k,1}\omega_{k,l}m_{l,1} - \omega_{1,1} = ac - ca - 0 \in N_B,$$
$$\sum_{k,l=1,2} m_{k,2}\omega_{k,l}m_{l,2} - \omega_{i,j} = ad - cb - 1 \in N_B,$$
$$\sum_{k,l=1,2} m_{k,2}\omega_{k,l}m_{l,1} - \omega_{i,j} = bc - da + 1 \in N_B \text{ and}$$
$$\sum_{k,l=1,2} m_{k,2}\omega_{k,l}m_{l,2} - \omega_{i,j} = bd - db - 0 \in N_B.$$

Therefore $M \in Sp_2(B)$ as well, and since M was taken arbitrarily, this implies that the set of $Sp_2(B)$ is equivalent to the set of $SL_2(B)$. Since $Sp_2(B)$ is a full subcrowd of $SL_2(B)$, we can conclude that the crowds are equivalent. □

Example 2.19. It follows from Theorem 2.17 that $Sp_{2n}(\mathbb{K}), Sp_{2n}(\mathbb{S}), Sp_{2n}(\mathbb{F}_1^{\pm})$ are crowds. For $n = 1$, we can use Theorem 2.18 to find that

$$Sp_2(\mathbb{K}) = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\},$$
$$Sp_2(\mathbb{S}) = \{A \in M_{2\times 2}(\mathbb{S}) | \det(A) \in \{1 \cdot 1 + 0 \cdot (-1), 2 \cdot 1 + 0 \cdot (-1), 1 \cdot 1 + 1 \cdot (-1)\}\} \text{ and}$$
$$Sp_2(\mathbb{F}_1^{\pm}) = \{A \in M_{2\times 2}(\mathbb{F}_1^{\pm}) \mid \det(A) \in 1 \cdot 1 + 0 \cdot (-1)\}.$$

3 Comparison maps

In this section, we study the relationship between the special linear group and the general linear group over a band B. In particular, we look into this relation for the Krasner hyperfield and the sign hyperfield, where we find that $SL_n(\mathbb{K})$ is in fact isomorphic to $GL_n(\mathbb{K})$. Additionally, every element and triple $SL_n(\mathbb{S})$ and its crowd law is connected to an element or triple in $GL_n(\mathbb{S})$ and its crowd law. In the third subsection, we compare the special and general linear groups over $\mathbb{F}_1^{\pm}$, $\mathbb{K}$ and $\mathbb{S}$ with each other.

Theorem 3.1. *Let B be a band. Then the map $\varphi : SL_n(B) \to GL_n(B), A \mapsto \left[\begin{smallmatrix} A & 0 \\ 0 & 1 \end{smallmatrix}\right]$, is an injective, full crowd morphism.*

Proof: We start the proof by showing φ is well-defined and satisfies the axioms of full crowd morphisms. We follow this by showing φ is injective as well.
For all $A \in SL_n(B)$, we have that

$$\det(\varphi(A)) - 1 = \det(A) \cdot 1 - 1 = \det(A) - 1 \in N_B.$$

Therefore $\varphi(A) \in SL_{n+1}(B)$. Furthermore, $\varphi(a)_{i,n+1} = \varphi(a)_{n+1,i} = 0$ for all $i = 1, ..., n$, so we can conclude that $\varphi(A) \in GL_n(B)$, and therefore that φ is well-defined. Additionally,

$$\varphi(I_n) = I_{n+1},$$

which means φ satisfies the first crowd morphism axiom.
Thus φ is a full crowd morphism if for all $A, C, D \in SL_n(B)$, $(\varphi(A), \varphi(C), \varphi(D)) \in R_{GL_n(B)}$ if and only if $(A, C, D) \in R_{SL_n(B)}$. So let $A, C, D \in SL_n(B)$ be arbitrary. Then
$(\varphi(A), \varphi(C), \varphi(D)) \in R_{GL_n(B)}$ if and only if for all $i, j = 1, ..., n+1$,

$$\sum_{k,l=1,...,n} \varphi(a)_{i,k}\varphi(c)_{k,l}\varphi(d)_{l,j} - \delta_{i,j} \in N_B, \tag{1}$$

$$\sum_{k,l=1,...,n} \varphi(c)_{i,k}\varphi(d)_{k,l}\varphi(a)_{l,j} - \delta_{i,j} \in N_B \text{ and} \tag{2}$$

$$\sum_{k,l=1,...,n} \varphi(d)_{i,k}\varphi(a)_{k,l}\varphi(c)_{l,j} - \delta_{i,j} \in N_B. \tag{3}$$

When looking at condition **(1)**, note that, for all $i, j = 1, ..., n+1$,

$$\begin{aligned}\sum_{k,l=1,...,n+1} \varphi(a)_{i,k}\varphi(c)_{k,l}\varphi(d)_{l,j} - \delta_{i,j} &= \sum_{k,l=1,..,n} \varphi(a)_{i,k}\varphi(c)_{k,l}\varphi(d)_{l,j} + \sum_{k=1,...,n} \varphi(a)_{i,k}\varphi(c)_{k,n+1}\varphi(d)_{n+1,j} \\ &\quad + \sum_{l=1,...,n} \varphi(a)_{i,n+1}\varphi(c)_{n+1,l}\varphi(d)_{l,j} + \varphi(a)_{i,n+1}\varphi(c)_{n+1,n+1}\varphi(d)_{n+1,j} - \delta_{i,j} \\ &= \sum_{k,l=1,..,n} a_{i,k}c_{k,l}d_{l,j} + \varphi(a)_{i,n+1}\varphi(c)_{n+1,n+1}\varphi(d)_{n+1,j} - \delta_{i,j}.\end{aligned}$$

Therefore, if $i = n+1, j = 1, ..., n$ or $i = 1, ..., n, j = n+1$, then

$$\sum_{k,l=1,...,n+1} \varphi(a)_{i,k}\varphi(c)_{k,l}\varphi(d)_{l,j} - \delta_{i,j} = 0 + 0 - 0 \in N_B,$$

and if $i = j = n+1$, then

$$\sum_{k,l=1,...,n+1} \varphi(a)_{i,k}\varphi(c)_{k,l}\varphi(d)_{l,j} - \delta_{i,j} = 0 + 1 \cdot 1 \cdot 1 - 1 \in N_B,$$

so condition (1) is satisfied whenever $i = n+1$ or $j = n+1$, independent on the choice for A, C and D. However, if $i, j = 1, ..., n$, then

$$\sum_{k,l=1,...,n+1} \varphi(a)_{i,k}\varphi(c)_{k,l}\varphi(d)_{l,j} - \delta_{i,j} = \sum_{k,l=1,..,n} a_{i,k}c_{k,l}d_{l,j} + 0 - \delta_{i,j}.$$

So, the condition is satisfied if and only if for all $i, j = 1, ..., n$,

$$\sum_{k,l=1,..,n} a_{i,k}c_{k,l}d_{l,j} - \delta_{i,j} \in N_B.$$

Similarly, condition (2) and (3) are satisfied if and only if, for all $i, j = 1, ..., n$,

$$\sum_{k,l=1,..,n} c_{i,k}d_{k,l}a_{l,j} - \delta_{i,j} \in N_B$$

$$\sum_{k,l=1,..,n} d_{i,k}a_{k,l}c_{l,j} - \delta_{i,j} \in N_B$$

Therefore $(\varphi(A), \varphi(C), \varphi(D)) \in R_{GL_n(B)}$ if and only if $(A, C, D) \in R_{SL_n(B)}$, which implies φ is a full crowd morphism. Finally, note that for all $A, C \in SL_n(B)$, $\varphi(A) = \varphi(C)$ if and only if $\left[\begin{smallmatrix} A & \mathbf{0} \\ \mathbf{0} & 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} C & \mathbf{0} \\ \mathbf{0} & 1 \end{smallmatrix}\right]$, which holds if and only if $A = C$. Therefore $\varphi : SL_n(B) \to GL_n(B)$ is an injective, full crowd morphism. □

Notation. In the rest of this thesis, φ will always denote the map $\varphi : SL_n(B) \to GL_n(B), A \mapsto \left[\begin{smallmatrix} A & \mathbf{0} \\ \mathbf{0} & 1 \end{smallmatrix}\right]$, either for a general band B or for a particular band if specified.

The existence of an injective, full crowd morphism from $SL_n(B)$ to $GL_n(B)$ implies that $SL_n(B)$ is isomorphic to a full subcrowd of $GL_n(B)$.

Definition 3.2. Let B be a band. Then we define the subcrowd $GL_n^*(B)$ of $GL_n(B)$ as

$$GL_n^*(B) = \left\{ A \in GL_n(B) \mid A = \left[\begin{smallmatrix} A_{11} & \mathbf{0} \\ \mathbf{0} & 1 \end{smallmatrix}\right] \text{ for some } A_{11} \in M_{n\times n}(B) \right\},$$

with unit element I_{n+1} and crowd law

$$R_{GL_n^*(B)} = \left\{ \left(A^{(1)},\ A^{(2)},\ A^{(3)}\right) \in GL_n^*(B)^3 \;\middle|\; \begin{array}{c} \forall \sigma \in A_3, \forall i, j = 1, ..., n+1, \\ \sum_{k,l=1,...,n+1} a_{i,k}^{(\sigma(1))} a_{k,l}^{(\sigma(2))} a_{l,j}^{(\sigma(3))} - \delta_{i,j} \in N_B \end{array} \right\}.$$

Theorem 3.3. *Let B be a band. Then $GL_n^*(B)$ is a crowd. Furthermore, it is a full subcrowd of $GL_n(B)$, and $GL_n^*(B) \cong SL_n(B)$.*

Proof: It follows from the definition of $GL_n^*(B)$ that $I_{n+1} \in GL_n^*(B)$, $GL_n^*(B) \subset GL_n(B)$. Furthermore, it follows from the similarity of the crowd laws that $R_{GL_n^*(B)} = R_{GL_n(B)} \cap GL_n^*(B)^3$. Therefore Lemma 1.18 implies that $GL_n^*(B)$ is not only a crowd, but a full subcrowd of $GL_n(B)$.
To proof that $GL_n^*(B) \cong SL_n(B)$, note that $\text{im}(\varphi) \subset GL_n^*(B)$ and $\varphi(R_{SL_n(B)}) \subset R_{GL_n^*(B)}$. This implies the map $\varphi^* : SL_n(B) \to GL_n^*(B), A \mapsto \varphi(A)$ is well-defined and an injective, full crowd morphism as well. let $A \in GL_n^*(B)$ be arbitrary. Then A is of the form $\left[\begin{smallmatrix} A_{11} & \mathbf{0} \\ \mathbf{0} & 1 \end{smallmatrix}\right]$, where A_{11} is a n by n matrix. This implies

$$\det(A) - 1 \in N_B \iff \det(A_{11}) \cdot 1 - 1 = \det(A_{11}) - 1 \in N_B.$$

Therefore $A_{11} \in SL_n(B)$ and $\varphi^*(A_{11}) = A$, so $A \in \text{im}(\varphi)$. Since A was taken arbitrarily, it follows that φ^* is bijective, full crowd morphism. Therefore the inverse of φ^* is well defined, and $((\varphi^*)^{-1}(A), (\varphi^*)^{-1}(C), (\varphi^*)^{-1}(D)) \in R_{SL_n(B)}$ whenever $(A, C, D) \in R_{GL_n^*(B)}$, which means $(\varphi^*)^{-1}$ is a crowd morphism. Therefore φ^* is a crowd isomorphism, so $SL_n(B) \cong GL_n^*(B)$. □

The crowd morphism φ leads to interesting results. For example, it follows from Theorem 3.3 and Theorem 2.18, which states $Sp_2(B) = SL_2(B)$, that $Sp_2(B) \cong GL_2^*(B)$. In the following subsections, we study this morphism for the bands $\mathbb{K}$ and $\mathbb{S}$.

3.1 The Krasner Hyperfield

When B is the Krasner hyperfield, the crowd morphism φ becomes a crowd isomorphism.

Theorem 3.4. *The crowd $SL_n(\mathbb{K})$ is isomorphic to the crowd $GL_n(\mathbb{K})$*

Proof: As was noted in Example 2.7, if $A \in GL_n(\mathbb{K})$, then A is of the form $\left[\begin{smallmatrix} A_{11} & \mathbf{0} \\ \mathbf{0} & 1 \end{smallmatrix}\right]$, where $A_{11} \in SL_n(\mathbb{K})$. Therefore $GL_n(\mathbb{K}) = GL_n^*(\mathbb{K})$, so it follows from Theorem 3.3 that $GL_n(\mathbb{K}) \cong SL_n(\mathbb{K})$. □

Maxson found in his thesis ([9, Chapter 3]) many results for $SL_n(\mathbb{K})$. Using Theorem 1.16 and the crowd isomorphism $\varphi : SL_n(\mathbb{K}) \to GL_n(\mathbb{K})$, one can proof these also hold for $GL_n(\mathbb{K})$. For example, take proposition 3.5:

Proposition 3.4.1. *Let $A, B \in SL_n(\mathbb{K})$, then the following are equivalent.*

1. $B \in A^{-1}$.
2. $A \in B^{-1}$.
3. $\delta_{i,j} \in \sum_{k=1,\ldots,n}^{\boxplus} a_{i,k}b_{k,j}$ *and* $\delta_{i,j} \in \sum_{k=1,\ldots,n}^{\boxplus} b_{i,k}a_{k,j}$.

Here $\boxplus$ stands for the Krasner hyperaddition, where $0 \boxplus 0 = \{0\}, 0 \boxplus 1 = 1 \boxplus 0 = \{1\}$, $1 \boxplus 1 = \{0, 1\}$ and if $A \subset \mathbb{K}, b \in \mathbb{K}$, then $A \boxplus b = \bigcup_{a \in A} a \boxplus b$ ([3, Definition 2.2]). This proposition also holds for $GL_n(\mathbb{K})$:

Proposition 3.4.2. *Let $A, B \in GL_n(\mathbb{K})$, then following statements are equivalent:*

1. $B \in A^{-1}$.
2. $A \in B^{-1}$.
3. $\delta_{i,j} \in \sum_{k=1,\ldots,n+1}^{\boxplus} a_{i,k}b_{k,j}$ *and* $\delta_{i,j} \in \sum_{k=1,\ldots,n+1}^{\boxplus} b_{i,k}a_{k,j}$ *for all* $i, j = 1, ..., n+1$.

Proof: We proof that **(1)** is equivalent to **(2)**, and **(2)** is equivalent to **(3)**.
$(1 \iff 2)$: For all $A, B \in GL_n(\mathbb{K})$, we have that

$$\begin{aligned} B \in A^{-1} &\iff \varphi^{-1}(B) \in (\varphi^{-1}(A))^{-1} \\ &\iff \varphi^{-1}(A) \in (\varphi^{-1}(B))^{-1} \iff A \in B^{-1}, \end{aligned}$$

where the first and third equivalence follow from Theorem 1.16 and the second equivalence follows from Proposition 3.4.1. Therefore $B \in A^{-1}$ if and only if $A \in B^{-1}$.
$(2 \iff 3)$: Let $A, B \in GL_n(\mathbb{K})$ be arbitrary and take $C, D \in SL_n(\mathbb{K})$ such that $\varphi(C) = A$ and $\varphi(D) = B$. Then it follows from Proposition 3.4.1 that

$$\begin{aligned} A \in B^{-1} &\iff C \in D^{-1} \\ &\iff \forall i, j = 1, ..., n,\ \delta_{i,j} \in \sum_{k=1,\ldots,n}^{\boxplus} c_{i,k}d_{k,j} \text{ and } \delta_{i,j} \in \sum_{l=1,\ldots,n}^{\boxplus} d_{i,l}c_{l,j} \\ &\iff \forall i, j = 1, ..., n,\ \delta_{i,j} \in \sum_{k=1,\ldots,n}^{\boxplus} a_{i,k}b_{k,j} \text{ and } \delta_{i,j} \in \sum_{l=1,\ldots,n}^{\boxplus} b_{i,l}a_{l,j}, \end{aligned}$$

where the third equivalence holds because $\varphi(C) = A$ and $\varphi(D) = B$, so it must hold that $c_{i,j} = a_{i,j}$ and $d_{i,j} = b_{i,j}$ for all $i, j = 1, ..., n$. Furthermore, adding 0 does not change the outcome of a hypersum, which implies

$$A \in B^{-1} \iff \forall i, j = 1, ..., n,\ \delta_{i,j} \in \sum_{k=1,\ldots,n+1}^{\boxplus} a_{i,k}b_{k,j} \text{ and } \delta_{i,j} \in \sum_{l=1,\ldots,n+1}^{\boxplus} b_{i,l}a_{l,j}.$$

Additionally, note that for $i = n+1, j = 1, ..., n$ and $i = 1, .., n, j = n+1$ we have that

$$\sum_{k=1,..,n+1}^{\boxplus} a_{i,k}b_{k,j} = \sum_{k=1,..,n+1}^{\boxplus} b_{i,k}a_{k,j} = \sum_{k=1,...,n+1}^{\boxplus} 0 = \{0\},$$

and when $i = j = n+1$,

$$\sum_{k=1,..,n+1}^{\boxplus} a_{i,k}b_{k,j} = \sum_{k=1,..,n+1}^{\boxplus} b_{i,k}a_{k,j} = \left(\sum_{k=1,...,n}^{\boxplus} 0\right) \boxplus (1 \cdot 1) = \{1\}.$$

Therefore, if $i = n+1$ or $j = n+1$, it always holds that

$$\delta_{i,j} \in \sum_{k=1,...,n+1}^{\boxplus} a_{i,k}b_{k,j} \text{ and } \delta_{i,j} \in \sum_{k=1,...,n+1}^{\boxplus} b_{i,k}a_{k,j}.$$

It follows that

$$A \in B^{-1} \iff \forall i,j = 1, ..., n+1,\ \delta_{i,j} \in \sum_{k=1,...,n+1}^{\boxplus} a_{i,k}b_{k,j} \text{ and } \delta_{i,j} \in \sum_{l=1,...,n}^{\boxplus} b_{i,l}a_{l,j}.$$

Therefore **(2)** is equivalent to **(3)**. □

Statement **(3)** can be rewritten such that the Krasner hyperaddition is replaced by the null set $N_{\mathbb{K}}$.

Corollary 3.4.1. *Let $a_i \in \mathbb{K}$, where $i = 1, .., n$ for some $n \in \mathbb{N}$. Then*

1. $0 \in \sum_{i=1,...,n}^{\boxplus} a_i$ *if and only if* $\sum_{i=1,...,n} a_i - 0 \in N_{\mathbb{K}}$ *and*
2. $1 \in \sum_{i=1,...,n}^{\boxplus} a_i$ *if and only if* $\sum_{i=1,...,n} a_i - 1 \in N_{\mathbb{K}}$.

Proof of 1: Note that

$$\sum_{i=1,...,n} a_i - 0 \in N_{\mathbb{K}} \iff \sum_{i=1,...,n} a_i = 0 \text{ or } \sum_{i=1,...,n} a_i \geq 2,$$

which holds if and only if $a_i = 0$ for all $i = 1, ..., n$ or there are at least two distinct $i, j \in \{1, ..., n\}$ such that $a_i = a_j = 1$. If the latter is true, then $\sum_{i=1,...,n}^{\boxplus} a_i = \{0, 1\}$. Therefore, it follows that

$$\sum_{i=1,...,n} a_i - 0 \in N_{\mathbb{K}} \iff \sum_{i=1,...,n}^{\boxplus} a_i = \{0\} \text{ or } \sum_{i=1,...,n}^{\boxplus} a_i = \{0, 1\}$$
$$\iff 0 \in \sum_{i=1,...,n}^{\boxplus} a_i,$$

where the second equivalence holds because a hypersum is equal to $\{0\}$, $\{1\}$ or $\{0, 1\}$. Therefore $0 \in \sum_{i=1,...,n}^{\boxplus} a_i$ if and only if $\sum_{i=1,...,n} a_i - 0 \in N_{\mathbb{K}}$.

Proof of 2: Similarly,

$$\sum_{i=1,...,n} a_i - 1 \in N_{\mathbb{K}} \iff \sum_{i=1,...,n} a_i \geq 1,$$

which holds if and only if there is at least one $i \in \{1, .., n\}$ such that $a_i = 1$. This holds if and only if $\sum_{i=1,...,n}^{\boxplus} a_i = \{1\}$ or $\sum_{i=1,...,n}^{\boxplus} a_i = \{0, 1\}$, so if and only if $1 \in \sum_{i=1,...,n}^{\boxplus} a_i$. Therefore $1 \in \sum_{i=1,...,n}^{\boxplus} a_i$ if and only if $\sum_{i=1,...,n} a_i - 1 \in N_{\mathbb{K}}$. □

3.2 The sign hyperfield

The special linear group over the sign hyperfield is not isomorphic to the general linear group. However, we can describe all elements in $GL_n(\mathbb{S})$ and its crowd law, using the crowd $SL_n(\mathbb{S})$. To do this, we need the following notation.

Notation. Suppose $A \in M_{n\times n}(B)$ for some band B. Then:

- $A^{-(i,0)}$ is obtained by multiplying the ith row in A by -1;
- $A^{-(0,j)}$ is obtained by multiplying the jth column in A by -1;
- $\hat{A}^{i,j}$ is obtained by exchanging the ith row in A with the jth row. So $\hat{a}^{i,j}_{j,k} = a_{i,k}$ and $\hat{a}^{i,j}_{i,l} = a_{j,l}$;
- $\dot{A}^{i,j}$ is obtained by exchanging the ith column in A with the jth column. So $\dot{a}^{i,j}_{k,j} = a_{k,i}$ and $\dot{a}^{i,j}_{l,i} = a_{l,j}$.

Furthermore, in this subsection, $\varphi : SL_n(\mathbb{S}) \to GL_n(\mathbb{S})$.

Theorem 3.5. *Let $A \in M_{n\times n}(\mathbb{S})$, then the following statements are equivalent:*

1. *$A \in SL_n(\mathbb{S})$;*
2. *For all $i = 1, ..., n$ $\begin{bmatrix} A^{-(i,0)} & \mathbf{0} \\ \mathbf{0} & -1 \end{bmatrix} \in GL_n(\mathbb{S})$;*
3. *For all $j = 1, ..., n$ $\begin{bmatrix} A^{-(0,j)} & \mathbf{0} \\ \mathbf{0} & -1 \end{bmatrix} \in GL_n(\mathbb{S})$;*
4. *For all $i, j = 1, ..., n$, $\begin{bmatrix} \hat{A}^{ij} & \mathbf{0} \\ \mathbf{0} & -1 \end{bmatrix} \in GL_n(\mathbb{S})$;*
5. *For all $i, j = 1, ..., n$, $\begin{bmatrix} \dot{A}^{ij} & \mathbf{0} \\ \mathbf{0} & -1 \end{bmatrix} \in GL_n(\mathbb{S})$;*

Proof: Let $A \in M_{n\times n}(\mathbb{S})$ be arbitrary. Note that $\begin{bmatrix} A^{-(i,0)} & \mathbf{0} \\ \mathbf{0} & -1 \end{bmatrix}, \begin{bmatrix} A^{-(0,j)} & \mathbf{0} \\ \mathbf{0} & -1 \end{bmatrix}, \begin{bmatrix} \hat{A}^{ij} & \mathbf{0} \\ \mathbf{0} & -1 \end{bmatrix}, \begin{bmatrix} \dot{A}^{ij} & \mathbf{0} \\ \mathbf{0} & -1 \end{bmatrix}$ have the right form to be in $GL_n(\mathbb{S})$. Therefore we can proof the theorem by showing that they are in $SL_{n+1}(\mathbb{S})$ if and only if $A \in SL_n(\mathbb{S})$. When looking at their determinant, we find that, for all $i, j = 1, ..., n$,

$$\begin{aligned}
\det(\hat{A}^{i,j}) &= \sum_{\sigma\in S_n} \operatorname{sign}(\sigma) \prod_{k=1,\dots,n} \hat{a}^{i,j}_{k,\sigma(k)} \\
&= \sum_{\sigma\in S_n} \operatorname{sign}(\sigma) \cdot \hat{a}^{i,j}_{j,\sigma(j)} \cdot \hat{a}^{i,j}_{i,\sigma(i)} \cdot \prod_{k\in\{1,\dots,n\}-\{i,j\}} \hat{a}^{i,j}_{k,\sigma(k)} \\
&= \sum_{\sigma\in S_n} \operatorname{sign}(\sigma) \cdot a_{i,\sigma(j)} \cdot a_{j,\sigma(i)} \cdot \prod_{k\in\{1,\dots,n\}-\{i,j\}} a_{k,\sigma(k)}.
\end{aligned}$$

Let σ' be the permutation such that $\sigma'(i) = \sigma(j), \sigma'(j) = \sigma(i)$ and $\sigma'(k) = \sigma(k)$ for all $k \neq i, j$. Since σ' is σ with one extra permutation, $\operatorname{sign}(\sigma) = -\operatorname{sign}(\sigma')$. Therefore, for all $i, j = 1, ..., n$,

$$\begin{aligned}
\det(\hat{A}^{i,j}) &= \sum_{\sigma'\in S_n} (-1) \cdot \operatorname{sign}(\sigma') \cdot a_{i,\sigma'(i)} \cdot a_{j,\sigma'(j)} \cdot \prod_{k\in\{1,\dots,n\}-\{i,j\}} a_{k,\sigma'(k)} \\
&= (-1) \cdot \sum_{\sigma'\in S_n} \operatorname{sign}(\sigma') \prod_{k\in\{1,\dots,n\}} a_{k,\sigma'(k)} \\
&= -\det(A).
\end{aligned}$$

Using a similar argument, we find that for all $i, j = 1, ..., n$,

$$\det(\dot{A}^{i,j}) = -\det(A).$$

Similarly, for all $i,j=1,...,n$,

$$\det(A^{-(i,0)}) = \sum_{\sigma\in S_n} \mathrm{sign}(\sigma)(-a_{i,\sigma(i)})\cdot \prod_{k=1,..,\hat{i},..n}(a_{k,\sigma(k)}) = (-1)\cdot \sum_{\sigma\in S_n}\mathrm{sign}(\sigma)\prod_{k=1,...,n} a_{k,\sigma(k)} = -\det(A),$$

$$\det(A^{-(0,j)}) = \sum_{\sigma\in S_n} \mathrm{sign}(\sigma)(-a_{l,\sigma(l)})\cdot \prod_{k=1,..,\hat{l},..n}(a_{k,\sigma(k)}),\ \text{where } l\in\{1,...,n\} \text{ such that } \sigma(l)=j,$$

$$= (-1)\cdot\sum_{\sigma\in S_n}\mathrm{sign}(\sigma)\prod_{k=1,...,n} a_{k,\sigma(k)} = -\det(A).$$

Therefore, for all $i,j=1,...,n$

$$\begin{aligned} A\in SL_n(\mathbb{S}) &\iff \det(A)-1\in N_{\mathbb{S}} \\ &\iff -\det(A)\cdot(-1)-1\in N_{\mathbb{S}} \\ &\iff \det\left(\begin{bmatrix} A^{(i,0)} & \mathbf{0} \\ \mathbf{0} & -1\end{bmatrix}\right)-1\in N_{\mathbb{S}},\ \det\left(\begin{bmatrix} A^{(0,j)} & \mathbf{0} \\ \mathbf{0} & -1\end{bmatrix}\right)-1\in N_{\mathbb{S}},\ \det\left(\begin{bmatrix} \hat{A}^{ij} & \mathbf{0} \\ \mathbf{0} & -1\end{bmatrix}\right)-1\in N_{\mathbb{S}} \text{ and} \\ &\quad\ \det\left(\begin{bmatrix} \dot{A}^{ij} & \mathbf{0} \\ \mathbf{0} & -1\end{bmatrix}\right)-1\in N_{\mathbb{S}} \\ &\iff \begin{bmatrix} A^{(i,0)} & \mathbf{0} \\ \mathbf{0} & -1\end{bmatrix}, \begin{bmatrix} A^{(i,0)} & \mathbf{0} \\ \mathbf{0} & -1\end{bmatrix}, \begin{bmatrix} \hat{A}^{ij} & \mathbf{0} \\ \mathbf{0} & -1\end{bmatrix}, \begin{bmatrix} \dot{A}^{ij} & \mathbf{0} \\ \mathbf{0} & -1\end{bmatrix} \in GL_n(\mathbb{S}). \end{aligned}$$

Therefore statement **(1)** is equivalent to statement **(2)**, **(3)**, and **(4)**. □

This theorem allows us to find the set $GL_n(\mathbb{S})$ using the set $SL_n(\mathbb{S})$, and the other way around. The same result holds for their crowd laws.

Theorem 3.6. *Let $A,B,C\in M_{n\times n}(\mathbb{S})$, then $(A,B,C)\in R_{SL_n(\mathbb{S})}$ if and only if, for all $s=1,...,n$, $\left(\begin{bmatrix} A^{-(0,s)} & \mathbf{0} \\ \mathbf{0} & -1\end{bmatrix}, \begin{bmatrix} B^{-(s,0)} & \mathbf{0} \\ \mathbf{0} & -1\end{bmatrix}, \begin{bmatrix} C & \mathbf{0} \\ \mathbf{0} & 1\end{bmatrix}\right)\in R_{GL_n(\mathbb{S})}$.*

Proof: Let $A,B,C\in M_{n\times n}(\mathbb{S})$ be arbitrary. Then it follows from Theorem 3.5 that $A,B,C\in SL_n(\mathbb{S})$ if and only if, for all $s=1,...,n$, $\begin{bmatrix} A^{-(0,s)} & \mathbf{0} \\ \mathbf{0} & -1\end{bmatrix}, \begin{bmatrix} B^{-(s,0)} & \mathbf{0} \\ \mathbf{0} & -1\end{bmatrix}, \begin{bmatrix} C & \mathbf{0} \\ \mathbf{0} & a1\end{bmatrix}\in GL_n(\mathbb{S})$. Furthermore, since φ is a full crowd isomorphism, $(A,B,C)\in R_{GL_n(\mathbb{S})}$ if and only if $(\varphi(A),\varphi(B),\varphi(C))\in R_{GL_n(\mathbb{S})}$. Let $\varphi(A)=D, \varphi(B)=E$ and $\varphi(C)=F$. Then $(D,E,F)\in R_{GL_n(\mathbb{S})}$ so if and only if, for all $i,j=1,...,n+1$,

$$\sum_{k,l=1,...,n+1} d_{i,k}e_{k,l}f_{l,j}-\delta_{i,j}\in N_{\mathbb{S}},\quad \sum_{k,l=1,...,n+1} e_{i,k}f_{k,l}d_{l,j}-\delta_{i,j}\in N_{\mathbb{S}} \text{ and } \sum_{k,l=1,...,n+1} f_{i,k}d_{k,l}e_{l,j}-\delta_{i,j}\in N_{\mathbb{S}}.$$

However, for all $i,j=1,...,n+1$ and $s=1,...,n$,

$$\begin{aligned} \sum_{k,l=1,...,n+1} d_{i,k}e_{k,l}f_{l,j}-\delta_{i,j} &= \sum_{k,l=1,...,n+1,k\neq s} d_{i,k}e_{k,l}f_{l,j} + \sum_{l=1,...,n+1} d_{i,s}e_{s,l}f_{l,j}-\delta_{i,j} \\ &= \sum_{k,l=1,...,n+1,k\neq s} d_{i,k}e_{k,l}f_{l,j} + \sum_{l=1,...,n+1} (-d_{i,s})(-e_{s,l})f_{l,j}-\delta_{i,j}, \\ \sum_{k,l=1,...,n+1} f_{i,k}d_{k,l}e_{l,j}-\delta_{i,j} &= \sum_{k,l=1,...,n+1,l\neq s} f_{i,k}d_{k,l}e_{l,j} + \sum_{k=1,...,n+1} f_{i,k}d_{k,s}e_{s,j}-\delta_{i,j} \\ &= \sum_{k,l=1,...,n+1,l\neq s} f_{i,k}d_{k,l}e_{l,j} + \sum_{k=1,...,n+1} f_{i,k}(-d_{k,s})(-e_{s,j})-\delta_{i,j} \text{ and} \end{aligned}$$

$$\sum_{k,l=1,...,n+1} e_{i,k}f_{k,l}d_{l,j}-\delta_{i,j} = \begin{cases} (-1)\cdot\sum_{k,l=1,...,n+1}(-e_{s,k})f_{k,l}d_{l,j} & \text{if } i=s\neq j \\ (-1)\cdot\sum_{k,l=1,...,n+1} e_{i,k}f_{k,l}(-d_{l,s}) & \text{if } i\neq s=j \\ \sum_{k,l=1,...,n+1} e_{i,k}f_{k,l}d_{l,j}-\delta_{i,j} & \text{if } i,j\neq s \\ \sum_{k,l=1,...,n+1}(-e_{i,k})f_{k,l}(-d_{l,j})-\delta_{i,j} & \text{if } i=j=s \end{cases}$$

Recall that if $x \in N_{\mathbb{S}}$, then $(-1) \cdot x \in N_{\mathbb{S}}$ as well. Therefore, it follows that for all $i, j = 1, ..., n+1$,

$$\sum_{k,l=1,\dots,n+1} d_{i,k}e_{k,l}f_{l,j} - \delta_{i,j} \in N_{\mathbb{S}} \iff \sum_{k,l=1,\dots,n+1,k\neq s} d_{i,k}e_{k,l}f_{l,j} + \sum_{l=1,\dots,n+1} (-d_{i,s})(-e_{s,l})f_{l,j} - \delta_{i,j} \in N_{\mathbb{S}},$$

$$\sum_{k,l=1,\dots,n+1} f_{i,k}d_{k,l}e_{l,j} - \delta_{i,j} \iff \sum_{k,l=1,\dots,n+1,l\neq s} f_{i,k}d_{k,l}e_{l,j} + \sum_{k=1,\dots,n+1} f_{i,k}(-d_{k,s})(-e_{s,j}) - \delta_{i,j} \in N_{\mathbb{S}} \text{ and}$$

$$\sum_{k,l=1,\dots,n+1} e_{i,k}f_{k,l}d_{l,j} - \delta_{i,j} \in N_{\mathbb{S}} \iff \begin{cases} \text{if } i = s \neq j & \text{then } \sum_{k,l=1,\dots,n+1}(-e_{s,k})f_{k,l}d_{l,j} \in N_{\mathbb{S}} \\ \text{if } i \neq s = j & \text{then } \sum_{k,l=1,\dots,n+1} e_{i,k}f_{k,l}(-d_{l,s}) \in N_{\mathbb{S}} \\ \text{if } i, j \neq s & \text{then } \sum_{k,l=1,\dots,n+1} e_{i,k}f_{k,l}d_{l,j} - \delta_{i,j} \in N_{\mathbb{S}} \\ \text{if } i = j = s & \text{then } \sum_{k,l=1,\dots,n+1}(-e_{i,k})f_{k,l}(-d_{l,j}) \in N_{\mathbb{S}} \end{cases}$$

Therefore $(D, E, F) \in R_{GL_n(\mathbb{S})}$ if and only if, for all $s = 1, ..., n$, $(D^{-(0,s)}, E^{-(s,0)}, F) \in R_{GL_n(\mathbb{S})}$. Since $D^{-(0,s)} = \begin{bmatrix} A^{-(0,s)} & \mathbf{0} \\ \mathbf{0} & -1 \end{bmatrix}$ and $E^{-(s,0)} = \begin{bmatrix} B^{-(s,0)} & \mathbf{0} \\ \mathbf{0} & -1 \end{bmatrix}$, and (A, B, C) were chosen arbitrarily, it follows that $(A, B, C) \in R_{SL_n(\mathbb{S})}$ if and only if $\left(\begin{bmatrix} A^{-(0,s)} & \mathbf{0} \\ \mathbf{0} & -1 \end{bmatrix}, \begin{bmatrix} B^{-(s,0)} & \mathbf{0} \\ \mathbf{0} & -1 \end{bmatrix}, \begin{bmatrix} C & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}\right) \in R_{GL_n(\mathbb{S})}$. □

Corollary 3.6.1. *Let $A, B, C \in M_{n\times n}(\mathbb{S})$, then the following are equivalent:*

1. *$(A, B, C) \in R_{SL_n(\mathbb{S})}$;*
2. *For all $s = 1, ..., n$, $\left(\begin{bmatrix} A^{-(0,s)} & \mathbf{0} \\ \mathbf{0} & -1 \end{bmatrix}, \begin{bmatrix} B^{-(s,0)} & \mathbf{0} \\ \mathbf{0} & -1 \end{bmatrix}, \begin{bmatrix} C & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}\right) \in R_{GL_n(\mathbb{S})}$;*
3. *For all $s = 1, ..., n$, $\left(\begin{bmatrix} A & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}, \begin{bmatrix} B^{-(0,s)} & \mathbf{0} \\ \mathbf{0} & -1 \end{bmatrix}, \begin{bmatrix} C^{-(s,0)} & \mathbf{0} \\ \mathbf{0} & -1 \end{bmatrix}\right) \in R_{GL_n(\mathbb{S})}$;*
4. *For all $s = 1, ..., n$, $\left(\begin{bmatrix} A^{-(s,0)} & \mathbf{0} \\ \mathbf{0} & -1 \end{bmatrix}, \begin{bmatrix} B & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}, \begin{bmatrix} C^{-(0,s)} & \mathbf{0} \\ \mathbf{0} & -1 \end{bmatrix}\right) \in R_{GL_n(\mathbb{S})}$.*

Proof: It follows from Theorem 3.6 that **(1)** is equivalent to **(2)**. Furthermore, for all $s = 1, ..., n$,

$$\begin{aligned} (A, B, C) \in R_{SL_n(\mathbb{S})} &\iff (B, C, A) \in R_{SL_n(\mathbb{S})} \\ &\iff \left(\begin{bmatrix} B^{-(0,s)} & \mathbf{0} \\ \mathbf{0} & -1 \end{bmatrix}, \begin{bmatrix} C^{-(s,0)} & \mathbf{0} \\ \mathbf{0} & -1 \end{bmatrix}, \begin{bmatrix} A & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}\right) \in R_{GL_n(\mathbb{S})} \\ &\iff \left(\begin{bmatrix} A & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}, \begin{bmatrix} B^{-(0,s)} & \mathbf{0} \\ \mathbf{0} & -1 \end{bmatrix}, \begin{bmatrix} C^{-(s,0)} & \mathbf{0} \\ \mathbf{0} & -1 \end{bmatrix}\right) \in R_{GL_n(\mathbb{S})}, \end{aligned}$$

where the second equivalence follows from Theorem 3.6. Therefore **(1)** is equivalent to **(3)**. One can similarly proof that **(1)** is equivalent to **(4)**, so the theorem holds. □

We know from Example 2.15 that in every triple in $R_{GL_n(\mathbb{S})}$, zero or two matrices have a -1 in the right bottom corner. It follows that we can use $R_{SL_n(\mathbb{S})}$ and Corollary 3.6.1 to describe every triple in $R_{GL_n(\mathbb{S})}$. Similar results hold when we exchange rows and columns.

Theorem 3.7. *Let $A, B, C \in M_{n\times n}(\mathbb{S})$. Then for all $r, s = 1, ..., n$, $(A, B, C) \in R_{SL_n(\mathbb{S})}$ if and only if $\left(\begin{bmatrix} \dot{A}^{r,s} & \mathbf{0} \\ \mathbf{0} & -1 \end{bmatrix}, \begin{bmatrix} \hat{B}^{r,s} & \mathbf{0} \\ \mathbf{0} & -1 \end{bmatrix}, \begin{bmatrix} C & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}\right) \in R_{GL_n(\mathbb{S})}$.*

Proof: Suppose $A, B, C \in M_{n\times n}(\mathbb{S})$ and fix some $r, s \in \{1, ..., n\}$. Then $A, B \in SL_n(\mathbb{S})$ if and only if $\begin{bmatrix} \dot{A}^{r,s} & \mathbf{0} \\ \mathbf{0} & -1 \end{bmatrix}, \begin{bmatrix} \hat{B}^{r,s} & \mathbf{0} \\ \mathbf{0} & -1 \end{bmatrix} \in GL_n(\mathbb{S})$. For ease of notation, let $\dot{a}_{i,j}$ be the coefficients of $\begin{bmatrix} \dot{A}^{r,s} & \mathbf{0} \\ \mathbf{0} & -1 \end{bmatrix}$ and $\hat{b}_{i,j}$ be the coefficients of $\begin{bmatrix} \hat{B}^{r,s} & \mathbf{0} \\ \mathbf{0} & -1 \end{bmatrix}$. Then $\left(\begin{bmatrix} \dot{A}^{r,s} & \mathbf{0} \\ \mathbf{0} & -1 \end{bmatrix}, \begin{bmatrix} \hat{B}^{r,s} & \mathbf{0} \\ \mathbf{0} & -1 \end{bmatrix}, \begin{bmatrix} C & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}\right) \in R_{GL_n(\mathbb{S})}$ if and only if, for all $i, j = 1, ..., n+1$,

$$\sum_{k,l=1,\dots,n+1} \dot{a}_{i,k}\hat{b}_{k,l}c_{l,j} - \delta_{i,j} \in N_{\mathbb{S}}, \quad \sum_{k,l=1,\dots,n+1} \hat{b}_{i,k}c_{k,l}\dot{a}_{l,j} - \delta_{i,j} \in N_{\mathbb{S}} \text{ and } \sum_{k,l=1,\dots,n+1} c_{i,k}\dot{a}_{k,l}\hat{b}_{l,j} - \delta_{i,j} \in N_{\mathbb{S}}.$$

When $i = n+1$ and $j = 1, .., n$ or $j = n+1$ and $i = 1, ..., n$,

$$\sum_{k,l=1,...,n+1} \dot{a}_{i,k}\hat{b}_{k,l}c_{l,j} - \delta_{i,j} = \sum_{k,l=1,...,n+1} \hat{b}_{i,k}c_{k,l}\dot{a}_{l,j} - \delta_{i,j} = \sum_{k,l=1,...,n+1} c_{i,k}\dot{a}_{k,l}\hat{b}_{l,j} - \delta_{i,j} = 0 - 0 \in N_{\mathbb{S}},$$

and when $i = j = n+1$,

$$\sum_{k,l=1,...,n+1} \dot{a}_{i,k}\hat{b}_{k,l}c_{l,j} - \delta_{i,j} = \sum_{k,l=1,...,n+1} \hat{b}_{i,k}c_{k,l}\dot{a}_{l,j} - \delta_{i,j} = \sum_{k,l=1,...,n+1} c_{i,k}\dot{a}_{k,l}\hat{b}_{l,j} - \delta_{i,j} = 1 - 1 \in N_{\mathbb{S}},$$

so the conditions always hold when $i = n+1$ or $j = n+1$. Additionally, for all $i, j = 1, ..., n$,

$$\begin{aligned}
\sum_{k,l=1,...,n+1} \dot{a}_{i,k}\hat{b}_{k,l}c_{l,j} - \delta_{i,j} &= \sum_{k,l=1,...,n} \dot{a}_{i,k}\hat{b}_{k,l}c_{l,j} - \delta_{i,j} \\
&= \sum_{k,l=1,...,n,k\neq r,s} \dot{a}_{i,k}\hat{b}_{k,l}c_{l,j} + \sum_{l=1,...,n} \dot{a}_{i,s}\hat{b}_{s,l}c_{l,j} + \sum_{l=1,...,n} \dot{a}_{i,r}\hat{b}_{r,l}c_{l,j} - \delta_{i,j} \\
&= \sum_{k,l=1,...,n,k\neq r,s} a_{i,k}b_{k,l}c_{l,j} + \sum_{l=1,...,n} a_{i,r}b_{r,l}c_{l,j} + \sum_{l=1,...,n} a_{i,s}b_{s,l}c_{l,j} - \delta_{i,j} \\
&= \sum_{k,l=1,...,n} a_{i,k}b_{k,l}c_{l,j} - \delta_{i,j}
\end{aligned}$$

Similarly, for all $i, j = 1, ..., n$,

$$\sum_{k,l=1,...,n+1} c_{i,k}\dot{a}_{k,l}\hat{b}_{l,j} - \delta_{i,j} = \sum_{k,l=1,...,n} c_{i,k}a_{k,l}b_{l,j} - \delta_{i,j}$$

Finally, note that the sum $\sum_{k,l=1,...,n+1} \hat{b}_{i,k}c_{k,l}\dot{a}_{l,j} - \delta_{i,j}$ is in the null set of $\mathbb{S}$ for all $i, j = 1, ..., n$ if and only if $\sum_{k,l=1,...,n} b_{i,k}c_{k,l}a_{l,j} - \delta_{i,j}$ is in $N_{\mathbb{S}}$ for all $i, j = 1, ..., n$. Therefore, we can conclude that $\left(\begin{bmatrix} \dot{A}^{r,s} & \mathbf{0} \\ \mathbf{0} & -1 \end{bmatrix}, \begin{bmatrix} \hat{B}^{r,s} & \mathbf{0} \\ \mathbf{0} & -1 \end{bmatrix}, \begin{bmatrix} C & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}\right) \in R_{GL_n(\mathbb{S})}$ if and only if, for all $i, j = 1, ..., n$,

$$\sum_{k,l=1,...,n} a_{i,k}b_{k,l}c_{l,j} - \delta_{i,j} \in N_{\mathbb{S}}, \sum_{k,l=1,...,n} b_{i,k}c_{k,l}a_{l,j} - \delta_{i,j} \in N_{\mathbb{S}} \text{ and } \sum_{k,l=1,...,n} c_{i,k}a_{k,l}b_{l,j} - \delta_{i,j} \in N_{\mathbb{S}},$$

so if and only if $(A, B, C) \in R_{SL_n(\mathbb{S})}$. □

Corollary 3.7.1. *Let $A, B, C \in M_{n\times n}(\mathbb{S})$. Then the following are equivalent:*

1. $(A, B, C) \in R_{SL_n(\mathbb{S})}$;
2. *For all $r, s = 1, ..., n$,* $\left(\begin{bmatrix} \dot{A}^{r,s} & \mathbf{0} \\ \mathbf{0} & -1 \end{bmatrix}, \begin{bmatrix} \hat{B}^{r,s} & \mathbf{0} \\ \mathbf{0} & -1 \end{bmatrix}, \begin{bmatrix} C & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}\right) \in R_{GL_n(\mathbb{S})}$;
3. *For all $r, s = 1, ..., n$,* $\left(\begin{bmatrix} A & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}, \begin{bmatrix} \dot{B}^{r,s} & \mathbf{0} \\ \mathbf{0} & -1 \end{bmatrix}, \begin{bmatrix} \hat{C}^{r,s} & \mathbf{0} \\ \mathbf{0} & -1 \end{bmatrix}\right) \in R_{GL_n(\mathbb{S})}$;
4. *For all $r, s = 1, ..., n$,* $\left(\begin{bmatrix} \hat{A}^{r,s} & \mathbf{0} \\ \mathbf{0} & -1 \end{bmatrix}, \begin{bmatrix} B & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}, \begin{bmatrix} \dot{C}^{r,s} & \mathbf{0} \\ \mathbf{0} & -1 \end{bmatrix}\right) \in R_{GL_n(\mathbb{S})}$.

Proof: It follows from Theorem 3.7 that **(1)** is equivalent to **(2)**. Furthermore, for all $r, s = 1, ..., n$, we have that

$$\begin{aligned}
(A, B, C) \in R_{SL_n(\mathbb{S})} &\iff (B, C, A) \in R_{SL_n(\mathbb{S})} \\
&\iff \left(\begin{bmatrix} \dot{B}^{r,s} & \mathbf{0} \\ \mathbf{0} & -1 \end{bmatrix}, \begin{bmatrix} \hat{C}^{r,s} & \mathbf{0} \\ \mathbf{0} & -1 \end{bmatrix}, \begin{bmatrix} A & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}\right) \in R_{GL_n(\mathbb{S})} \\
&\iff \left(\begin{bmatrix} A & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}, \begin{bmatrix} \dot{B}^{r,s} & \mathbf{0} \\ \mathbf{0} & -1 \end{bmatrix}, \begin{bmatrix} \hat{C}^{r,s} & \mathbf{0} \\ \mathbf{0} & -1 \end{bmatrix}\right) \in R_{GL_n(\mathbb{S})},
\end{aligned}$$

where the second equivalence follows from Theorem 3.7. Therefore **(1)** is equivalent to **(3)**. A similar argument shows that **(1)** is equivalent to **(4)**, which implies that the theorem holds. □

3.3 Change of coefficients

We can also study the morphisms between bands and their linear groups. For example, for every band B, there is a unique band morphism $f : \mathbb{F}_1^{\pm} \to B$ ([1, Chapter 1.2.2/Lemma 1.39]). In the following, we study the sequence $\mathbb{F}_1^{\pm} \to \mathbb{S} \to \mathbb{K}$, and how it can be used to compare their special and general linear groups.

Theorem 3.8. *The maps* $f : \mathbb{F}_1^{\pm} \to \mathbb{S}, 0 \mapsto 0, \pm 1 \mapsto \pm 1$ *and* $g : \mathbb{S} \to \mathbb{K}, 0 \mapsto 0, \pm 1 \mapsto 1$ *are band morphisms. Furthermore, they induce the following crowd morphisms* $f_n^S : Sl_n(\mathbb{F}_1^{\pm}) \to SL_n(\mathbb{S})$, $f_n^G : GL_n(\mathbb{F}_1^{\pm}) \to GL_n(\mathbb{S})$, $g_n^S : SL_n(\mathbb{S}) \to Sl_n(\mathbb{K})$ *and* $g_n^G : GL_n(\mathbb{S}) \to GL_n(\mathbb{K})$.

Proof: Recall from Definition 1.8, that a map $h : B \to C$, for some bands B and C, is a band morphism if it is multiplicative map such that $h(1_B) = 1_C$ and $\sum_{a \in B-\{0\}} n_a h(a) \in N_C$ whenever $\sum_{a \in B-\{0\}} n_a a \in N_B$. It follows from the definitions of f and g that $f(1) = 1$ and $f(a \cdot b) = f(a) \cdot f(b)$ for all $a, b \in \mathbb{F}_1^{\pm}$, and similarly $g(1) = 1$ and $g(c \cdot d) = g(c) \cdot g(d)$ for all $c, d \in \mathbb{S}$. So they satisfy the first 2 conditions. To check the third condition, let $\sum_{a \in \mathbb{F}_1^{\pm}-\{0\}} n_a a \in N_{\mathbb{F}_1^{\pm}}$ be arbitrary. Then $\sum_{a \in \mathbb{F}_1^{\pm}-\{0\}} n_a a = m \cdot 1 + m \cdot (-1)$ for some $m \in \mathbb{N}$, so

$$\sum_{a \in \mathbb{F}_1^{\pm}-\{0\}} n_a f(a) = m \cdot f(1) + m \cdot f(-1) = m \cdot 1 + m \cdot (-1) \in N_{\mathbb{S}}.$$

Therefore f satisfies the third condition. For all $\sum_{a \in \mathbb{S}-\{0\}} n_a a \in N_{\mathbb{S}}$, we have that

$$\sum_{a \in \mathbb{S}-\{0\}} n_a g(a) = p \cdot g(1) + m \cdot g(-1) = (p + m) \cdot 1,$$

for some $p, m \in \mathbb{N}$ such that $p = m = 0$ or $m, p > 0$. Then $(p + m) \cdot 1 \neq 1 \cdot 1$, which implies $\sum_{a \in \mathbb{S}-\{0\}} n_a g(a) \in N_{\mathbb{S}}$, so g satisfies the third condition as well. Therefore, both f and g are band morphisms.

These band morphisms induce the maps on the special and general linear groups over $\mathbb{F}_1^{\pm}$ and $\mathbb{S}$, that map the coefficients $a_{i,j}$ of a matrix A to $f(a_{i,j})$ or $g(a_{i,j})$. Let f_n^S be the map on $SL_n(\mathbb{F}_1^{\pm})$ that is induced by f. Then for all $A \in SL_n(\mathbb{F}_1^{\pm})$, we have that

$$\begin{aligned}
\det(A) - 1 &= \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) \prod_{i=1,\dots,n} a_{i,\sigma(i)} - 1 \in N_{\mathbb{F}_1^{\pm}} \Rightarrow \\
\det(f_n^S(A)) - f(1) &= \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) \prod_{i=1,\dots,n} f(a_{i,\sigma(i)}) - 1 \in N_{\mathbb{S}}.
\end{aligned}$$

This implies $f_n^S(A) \in SL_n(\mathbb{S})$, so $f_n^S : Sl_n(\mathbb{F}_1^{\pm}) \to SL_n(\mathbb{S})$ is well-defined. Furthermore, $f_n^S(I_n) = I_n$ and for all $A, B, C \in SL_n(\mathbb{F}_1^{\pm})$ such that $(A, B, C) \in R_{SL_n(\mathbb{F}_1^{\pm})}$, we have that for all $i, j = 1, ..., n$,

$$\begin{aligned}
\sum_{k,l=1,\dots,n} a_{i,k} b_{k,l} c_{l,j} - \delta_{i,j} \in N_{\mathbb{F}_1^{\pm}} &\Rightarrow \sum_{k,l=1,\dots,n} f(a)_{i,k} f(b)_{k,l} f(c)_{l,j} - f(\delta_{i,j}) \in N_{\mathbb{S}}, \\
\sum_{k,l=1,\dots,n} b_{i,k} c_{k,l} a_{l,j} - \delta_{i,j} \in N_{\mathbb{S}} &\Rightarrow \sum_{k,l=1,\dots,n} f(b)_{i,k} f(c)_{k,l} f(a)_{l,j} - f(\delta_{i,j}) \in N_{\mathbb{S}} \text{ and} \\
\sum_{k,l=1,\dots,n} c_{i,k} a_{k,l} b_{l,j} - \delta_{i,j} \in N_{\mathbb{S}} &\Rightarrow \sum_{k,l=1,\dots,n} f(c)_{i,k} f(a)_{k,l} f(b)_{l,j} - f(\delta_{i,j}) \in N_{\mathbb{S}},
\end{aligned}$$

which implies $(f_n^S(A), f_n^S(B), f_n^S(C)) \in R_{SL_n(\mathbb{S})}$. Therefore f_n^S is a crowd morphism. It follows from symmetry that the induced maps $f_n^G : GL_n(\mathbb{F}_1^{\pm}) \to GL_n(\mathbb{S})$, $g_n^S : SL_n(\mathbb{S}) \to Sl_n(\mathbb{K})$ and $g_n^G : GL_n(\mathbb{S}) \to GL_n(\mathbb{K})$ are well-defined crowd morphisms as well. □

Theorem 3.9. *There exist crowd morphisms*

$$\begin{aligned}
&\theta : SL_n(\mathbb{F}_1^{\pm}) \to GL_n(\mathbb{F}_1^{\pm}), && f_n^S : Sl_n(\mathbb{F}_1^{\pm}) \to SL_n(\mathbb{S}), && g_n^S : SL_n(\mathbb{S}) \to Sl_n(\mathbb{K}),\\
&\psi : SL_n(\mathbb{S}) \to GL_n(\mathbb{S}), && f_n^G : GL_n(\mathbb{F}_1^{\pm}) \to GL_n(\mathbb{S}), && g_n^G : GL_n(\mathbb{S}) \to GL_n(\mathbb{K}),\\
&\varphi : SL_n(\mathbb{K}) \to GL_n(\mathbb{K}),
\end{aligned}$$

such that $\psi \circ f_n^S = f_n^G \circ \theta$ and $\varphi \circ g_n^S = g_n^G \circ \psi$. In other words, such that the following diagram commutes:

$$\begin{array}{ccccc}
SL_n(\mathbb{F}_1^{\pm}) & \xrightarrow{f_n^S} & SL_n(\mathbb{S}) & \xrightarrow{g_n^S} & SL_n(\mathbb{K})\\
\Big\downarrow \theta & & \Big\downarrow \psi & & \Big\downarrow \varphi\\
GL_n(\mathbb{F}_1^{\pm}) & \xrightarrow{f_n^G} & GL_n(\mathbb{S}) & \xrightarrow{g_n^G} & GL_n(\mathbb{K})
\end{array}$$

Proof: Let $f_n^S, g_n^S, f_n^G, g_n^G$ be the crowd morphisms defined in Theorem 3.8, and θ, ψ, φ the crowd morphisms defined in Theorem 3.1, where the general band B is taken to be $\mathbb{F}_1^{\pm}, \mathbb{S}$ and $\mathbb{K}$, respectively. Then, for all $A \in SL_n(\mathbb{F}_1^{\pm})$, we have that

$$\begin{aligned}
(\psi \circ f_n^S)(A) &= \psi(f_n^S(A)) = \psi(A)\\
&= \begin{bmatrix} A & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} = f_n^G\left(\begin{bmatrix} A & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}\right) = f_n^G(\theta(A)) = (f_n^G \circ \theta)(A).
\end{aligned}$$

Therefore $\psi \circ f_n^S = f_n^G \circ \theta$. To proof $\varphi \circ g_n^S = g_n^G \circ \psi$, let $C \in Sl_n(\mathbb{S})$ be arbitrary. Then

$$\begin{aligned}
(\varphi \circ g_n^S(C)) &= \varphi(g_n^S(C)) = \varphi\left((g(c_{i,j}))\right)\\
&= \begin{bmatrix} (g(c_{i,j})) & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} = g_n^G\left(\begin{bmatrix} C & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}\right) = g_n^G(\psi(C)) = (g_n^G \circ \psi)(C).
\end{aligned}$$

Since C was taken arbitrarily, this holds for all matrices in $SL_n(\mathbb{S})$. Therefore $\varphi \circ g_n^S = g_n^G \circ \psi$, and the diagram commutes. □

Remark. The special and general linear group are also called *algebraic crowds*, which means they are functors, a type of map, from the category of Bands to the category of Crowds. Theorem 3.9 implies that the maps $f_n^S, g_n^S, f_n^G, g_n^G$ are natural transformations (also known as morphisms of functors). For more information, algebraic crowds are defined and studied by Lorscheid and Thas in their paper [7]. An explanation of categories and functors can be found in Lorscheid's lecture notes [6].

Notation. In the rest of this section, f, f_n^S, f_n^G, g_n^S and g_n^G are the band and crowd morphisms as defined in Theorem 3.8.

With the diagram of Theorem 3.9, we can compare the different special and general linear groups.

Corollary 3.9.1. *f, f_n^S, f_n^G are injective maps and g, g_n^S, g_n^G are surjective maps.*

Proof: It follows from how f was defined in Theorem 3.8, that the map is injective. Furthermore, for all $A, B, \in SL_n(\mathbb{F}_1^{\pm})$ such that $f_n^S(A) = f_n^S(B)$, it follows that $f(a_{i,j}) = f(b_{i,j})$ for all $i, j = 1, ..., n$, which implies $A = B$. Similarly, for all $C, D \in GL_n(\mathbb{F}_1^{\pm})$ such that $f_n^G(C) = f_n^G(D)$, it holds that $f(c_{i,j}) = f(d_{i,j})$ for all $i, j = 1, ..., n+1$, which implies $C = D$. Therefore the crowd morphisms f_n^S and f_n^G are injective as well.
It also follows from its definition in Theorem 3.8 that g is surjective. To proof that g_n^S is surjective as well, let $A \in SL_n(\mathbb{K})$ be arbitrary and take $A' \in M_{n\times n}(\mathbb{S})$ such that, for all $i, j = 1, ..., n$,

$$a'_{i,j} = \begin{cases} 0 & \text{if } a_{i,j} = 0 \\ 1 & \text{if } a_{i,j} = 1. \end{cases}$$

Since $A \in SL_n(\mathbb{K})$ and $-1 = 1$, it follows that $\det(A) - 1 \neq 1 \cdot 1$, so $\det(A) \neq 0 \cdot 1$. This and Theorem 2.9 imply that $\det(A') = p \cdot 1 + m \cdot (-1)$ for some $p > 0$ or $m > 0$. If $p > 0$, then it follows from the same theorem that $A' \in SL_n(\mathbb{S})$, which would imply $A \in \text{im}(g_n^S)$. If $m > 0$, then we multiply the first row of A' by -1 to obtain the matrix $A'' \in M_{n\times n}(\mathbb{S})$. As shown in the proof of Theorem 3.5, $\det(A'') = -\det(A') = m \cdot 1 + p \cdot (-1)$, which means $A'' \in SL_n(\mathbb{S})$. Furthermore, $g(A'') = A$, so we again have that $A \in \text{im}(g_n^S)$. Therefore A must be in the image of g_n^S, and since A was chosen arbitrarily, this holds for all $A \in SL_n(\mathbb{K})$, which means g_n^S is surjective.
To show g_n^G is surjective, let $A = \left[\begin{smallmatrix} A_{11} & \mathbf{0} \\ \mathbf{0} & 1 \end{smallmatrix}\right] \in GL_n(\mathbb{K})$ be arbitrary and take $A'_{11} \in M_{n\times n}(\mathbb{S})$ such that, for all $i, j = 1, ..., n$,

$$a'_{i,j} = \begin{cases} 0 & \text{if } a_{i,j} = 0 \\ 1 & \text{if } a_{i,j} = 1. \end{cases}$$

Then we again have that $\det(A) = \det(A_{11}) \neq 0 \cdot 1$, which implies $\det(A'_{11}) = p \cdot 1 + m \cdot (-1)$, for some $p > 0$ or $m > 0$. It follows from Theorem 2.10 that $A' = \left[\begin{smallmatrix} A'_{11} & \mathbf{0} \\ \mathbf{0} & \delta \end{smallmatrix}\right] \in GL_n(\mathbb{S})$, where $\delta = 1$ or $\delta = -1$. Furthermore, $g_n^G(A') = A$, and since A was chosen arbitrarily, we can conclude that g_n^G is surjective as well.
Therefore f, f_n^S, f_n^G are injective maps and g, g_n^S, g_n^G are surjective maps. □

Corollary 3.9.2. *Let $A \in SL_n(\mathbb{K})$ and $B \in GL_n(\mathbb{K})$. If $A \in g_n^S(im(f_n^G))$, then there is an odd $x \in \mathbb{N}$ such that $det(A) = x \cdot 1$. Similarly, if $B \in g_n^G(im(f_n^S))$, then there is an odd $y \in \mathbb{N}$ such that $det(B) = y \cdot 1$.*

Proof: Suppose $A \in SL_n(\mathbb{K})$ such that $A \in g_n^S(\text{im}(f_n^S))$, then there is a $C \in \text{im}(f_n^S)$ such that $g_n^S(C) = A$. It follows from the definition of f_n^S, and Theorem 2.12, that $\det(C) = (m+1)\cdot 1 + m\cdot(-1)$ for some $m \in \mathbb{N}$. Therefore

$$\det(A) = (m + 1 + m) \cdot 1 = (2m + 1) \cdot 1 = x \cdot 1 \text{ for some odd } x \in \mathbb{N}.$$

So the first half of the statement holds.
To proof the second half, let $B \in GL_n(\mathbb{K})$ and suppose $B \in g_n^G(\text{im}(f_n^S))$. Then there is a matrix $D = \left[\begin{smallmatrix} D_{11} & \mathbf{0} \\ \mathbf{0} & \delta \end{smallmatrix}\right] \in \text{im}(f_n^G)$ such that $g_n^G(D) = B$. It follows from the definition of f_n^G and Theorem 2.13 that $\det(D_{11}) = p \cdot 1 + m \cdot (-1)$, where $p = m + 1$ or $m = p + 1$. Therefore

$$\det(B) = 1 \cdot \det(B_{11}) = (p + m) \cdot 1 = \begin{cases} (2m + 1) \cdot 1 & \text{if } p = m + 1 \\ (2p + 1) \cdot 1 & \text{if } m = p + 1, \end{cases}$$

which implies $\det(B) = y \cdot 1$ for some odd $y \in \mathbb{N}$. Therefore the second half of the theorem holds as well. □

When we study these maps for $n = 2$, we find that

$$\text{im}(g_2^S \circ f_2^S) = SL_2(\mathbb{K}) - \left\{\left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]\right\} \text{ and}$$
$$\text{im}(g_2^G \circ f_2^G) = GL_2(\mathbb{K}) - \left\{\left[\begin{smallmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right]\right\}.$$

Furthermore, we can observe that if $A \in g_2^S(\text{im}(f_2^S))$, then A has at least 1 zero coefficient and $\#(g_2^S)^{-1}(A) = 2^{x-1}$, where x is the number of ones in A. Similarly, if $B = \left[\begin{smallmatrix} B_{11} & \mathbf{0} \\ \mathbf{0} & 1 \end{smallmatrix}\right] \in g_2^G(\text{im}(f_2^G))$, then B_{11} has at least 1 zero coefficient and $\#(g_2^G)^{-1}(B) = 2^y$, where y is the number of ones in B_{11}. Another interesting comparison is between the inverse sets in these crowds. We state in the Examples 2.7, 2.11 and 2.14 that all the inverse sets in the special linear group and general linear groups over $\mathbb{K}$ and $\mathbb{F}_1^{\pm}$ are singleton sets, but that this does not hold for $SL_2(\mathbb{S})$ and $GL_2(\mathbb{S})$. In fact, the matrices whose inverse sets are not singletons, are those that not in the image of f_2^S or f_2^G, and are therefore the ones that are mapped to $\left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ or $\left[\begin{smallmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right]$ by g_2^S and g_2^G, respectively .

4 Semidirect products and short exact sequences

In this section, we define the semidirect product and weakly split and strongly split short exact sequence for crowds, and find that these sequences imply the existence of certain bijective maps. We follow this by finding conditions under which there is a strongly split short exact sequence that contains the special and general linear group over a band.

4.1 Semidirect product

Let $(H, 1_H, \odot^H)$ and $(Q, 1_Q, \odot^Q)$ be groups and $\theta : Q \to \mathrm{Aut}(H)$ be a group homomorphism. Then the *semidirect product* $H \rtimes_\theta Q$ is the group $(H \times Q, (1_H, 1_Q), \cdot)$, where $\cdot : (H \times Q) \times (H \times Q) \to H \times Q$ is the group law such that $(g, p) \cdot (h, q) = (g \odot^H \theta_p(h), p \odot^Q q)$. To generalize this to crowds, recall that crowd laws are supposed to contain all triples whose product is equal to the unit element, if such a product is defined. In the semidirect product of 2 groups, the product of three arbitrary elements $(h_1, q_1), (h_2, q_2), (h_3, q_3) \in H \times Q$ is given by

$$\begin{aligned}(h_1, q_1) \cdot (h_2, q_2) \cdot (h_3, q_3) &= (h_1, q_1) \cdot (h_2 \odot^H \theta_{q_2}(h_3), q_2 \odot^Q q_3)\\ &= (h_1 \odot^H \theta_{q_1}(h_2 \odot^H \theta_{q_2}(h_3)), q_1 \odot^Q q_2 \odot^Q q_3)\\ &= (h_1 \odot^H \theta_{q_1}(h_2) \odot^H \theta_{q_1}(\theta_{q_2}(h_3)), q_1 \odot^Q q_2 \odot^Q q_3),\end{aligned}$$

where the last equality holds because θ_{q_1} is a automorphism. This pair is equal to $(1_H, 1_Q)$ if and only if $h_1 \odot^H \theta_{q_1}(h_2) \odot^H \theta_{q_1}(\theta_{q_2}(h_3) = 1_H$ and $q_1 \odot^Q q_2 \odot^Q q_3 = 1_Q$. This gives us a logical definition for the semidirect product of crowds.

Definition 4.1. Given crowds $(H, 1_H, R_H)$ and $(Q, 1_Q, R_Q)$ and crowd morphism $\theta : Q \to \mathrm{Aut}(H)$, the *semi-direct product* $H \rtimes_\theta Q$ is the crowd $(H \times Q, (1_H, 1_Q), R_{H \rtimes_\theta Q})$, with crowd law

$$R_{H \rtimes_\theta Q} = \left\{((h_1, q_1), (h_2, q_2), (h_3, q_3)) \in (H \times Q)^3 \mid (q_1, q_2, q_3) \in R_Q \text{ and } (h_1, \theta_{q_1}(h_2), \theta_{q_1}(\theta_{q_2}(h_3))) \in R_H\right\}.$$

Theorem 4.2. *Given crowds $(H, 1_H, R_H)$ and $(Q, 1_Q, R_Q)$ and crowd morphism $\theta : Q \to Aut(H)$, the semi-direct product $H \rtimes_\theta Q$ is indeed a crowd.*

Proof: Let $(H, 1_H, R_H)$ and $(Q, 1_Q, R_Q)$ be crowds and $\theta : Q \to \mathrm{Aut}(H)$ be a crowd morphism. Note that it is possible for θ to be a crowd morphism, since $\mathrm{Aut}(H)$ is a group. To show that the semi-direct product $(H \times Q, (1_H, 1_Q), R_{H \rtimes_\theta Q})$ is a crowd, we need to check the 4 crowd axioms:

1. We know that $(1_Q, 1_Q, 1_Q) \in R_Q$ and $\theta(1_Q) = \theta_{1_Q} = id_H$. Therefore,

   $$(1_H, \theta_{1_Q}(1_H), \theta_{1_Q}(\theta_{1_Q}(1_H))) = (1_H, 1_H, 1_H) \in R_H,$$

   which implies $((1_H, 1_Q), (1_H, 1_Q), (1_H, 1_Q)) \in R_{H \rtimes_\theta Q}$ and the first axiom is satisfied.

2. Suppose $((h, q), (1_H, 1_Q), (1_H, 1_Q)) \in R_{H \rtimes_\theta Q}$ for some $h \in H$ and $q \in Q$. Then $(q, 1_Q, 1_Q) \in R_Q$, which implies $q = 1_Q$, and $(h, \theta_{1_Q}(1_H), \theta_{1_Q}(\theta_{1_Q}(1_H))) \in R_H$. It follows that

   $$(h, \theta_{1_Q}(1_H), \theta_{1_Q}(\theta_{1_Q}(1_H))) = (h, 1_H, 1_H) \in R_H,$$

   which means $h = 1_H$. Therefore $((h, q), (1_H, 1_Q), (1_H, 1_Q)) \in R_{H \rtimes_\theta Q}$ implies that the pair $(h, q) = (1_H, 1_Q)$, so the second axiom is satisfied as well.

3. Let $h, i \in H$ and $p, q \in Q$ be arbitrary and suppose that $((h, p), (i, q), (1_H, 1_Q)) \in R_{H \rtimes_\theta Q}$. Then $(h, \theta_p(i), \theta_p(\theta_q(1_H)) \in R_H$ and $(p, q, 1_Q) \in R_Q$, which implies $(q, p, 1_Q) \in R_Q$. It follows that $(\theta_q, \theta_p, id_H) \in R_{\mathrm{Aut}(H)}$, which means $\theta_q \circ \theta_p = id_H$, since $\mathrm{Aut}(H)$ is a group. Therefore,

   $$\begin{aligned}(h, \theta_p(i), \theta_p(\theta_q(1_H)) \in R_H &\Rightarrow (h, \theta_p(i), 1_H) \in R_H\\ &\Rightarrow (\theta_p(i), h, 1_H) \in R_H\\ &\Rightarrow (\theta_q(\theta_p(i)), \theta_q(h), 1_H) = (i, \theta_q(h), \theta_q(\theta_p(1_H))) \in R_H.\end{aligned}$$

   So $((h, p), (i, q), (1_H, 1_Q)) \in R_{H \rtimes_\theta Q}$ implies $(q, p, 1_Q) \in R_Q$ and $(i, \theta_q(h), \theta_q(\theta_p(1_H))) \in R_H$, which implies $((i, q), (h, p), (1_H, 1_Q)) \in R_{H \rtimes_\theta Q}$. Therefore the third axiom is satisfied.

4. Suppose $((h,p),(i,q),(j,r)) \in R_{H\rtimes_\theta Q}$ for some $h,i,j \in H$ and $p,q,r \in Q$. Then, per definition, $(h,\theta_p(i),\theta_p(\theta_q(j))) \in R_H$ and $(p,q,r) \in R_Q$, which implies $(r,p,q) \in R_Q$. It follows that $\theta_r \circ \theta_p \circ \theta_q = id_H$, which means,

$$
\begin{aligned}
(h,\theta_p(i),\theta_p(\theta_q(j))) \in R_H &\Rightarrow (\theta_p(\theta_q(j)),h,\theta_p(i)) \in R_H \\
&\Rightarrow (\theta_r(\theta_p(\theta_q(j))),\theta_r(h),\theta_r(\theta_p(i))) = (j,\theta_r(h),\theta_r(\theta_p(i))) \in R_H.
\end{aligned}
$$

Since $(j,\theta_r(h),\theta_r(\theta_p(i))) \in R_H$ and $(r,p,q) \in R_Q$, we have that $((j,r),(h,p),(i,q)) \in R_H$. Therefore the fourth axiom is satisfied.

Since all axioms are satisfied, $(H \times Q,(1_H,1_Q),R_{H\rtimes_\theta Q})$ is indeed a crowd. □

Example 4.3. Suppose $(H,1_H,\odot^H)$ and $(Q,1_Q,\odot^Q)$ are groups, and $\theta : Q \to \mathrm{Aut}(H), q \mapsto \theta_q$ is a group homomorphism. Then their semidirect product is a group, which means the corresponding crowd is given by $(H \times Q,(1_H,1_Q),R)$, with crowd law

$$R = \{((a,p),(b,q),(c,r)) \in (H \times Q)^3 \mid (a,p)\cdot(b,q)\cdot(c,r) = (1_H,1_Q)\}$$

We can also find the crowds corresponding do H and Q, and the crowd morphism corresponding to θ, to construct the semidirect product $(H \times Q,(1_H,1_Q),R_{H\rtimes_\theta Q})$. Then

$$
\begin{aligned}
R_{H\rtimes_\theta Q} &= \big\{((h_1,q_1),(h_2,q_2),(h_3,q_3)) \in (H \times Q)^3 \mid (q_1,q_2,q_3) \in R_Q \text{ and } (h_1,\theta_{q_1}(h_2),\theta_{q_1}(\theta_{q_2}(h_3))) \in R_H\big\} \\
&= \{((a,p),(b,q),(c,r)) \in (H \times Q)^3 \mid p \odot^Q q \odot^Q r = 1_Q \text{ and } a \odot^H \theta_p(b) \odot^H \theta_p(\theta_q(c)) = 1_H\} \\
&= \{((a,p),(b,q),(c,r)) \in (H \times Q)^3 \mid (a \odot^H \theta_p(b) \odot^H \theta_p(\theta_q(c)), p \odot^Q q \odot^Q r) = (1_H,1_Q)\} \\
&= \{((a,p),(b,q),(c,r)) \in (H \times Q)^3 \mid (a,p)\cdot(b,q)\cdot(c,r) = (1_H,1_Q)\} \\
&= R.
\end{aligned}
$$

Therefore the semidirect product of the groups H and Q is equivalent as to the semidirect product of the crowds H and Q.

4.2 Short exact sequences

Another useful concept from group theory is the short exact sequence. When 'translating' this to crowds, we see that, in order to get similar results, we have to base the definition on *split* short exact sequences instead. Furthermore, we define 2 types of short exact sequences of crowds: a weakly split short exact sequence, which focuses on the sets of the crowds, and a strongly split short exact sequence, which also includes the crowd laws.

Definition 4.4. A *weakly split short exact sequence of crowds* is a sequence of crowd morphisms

$$1 \longrightarrow (H,1_H,R_H) \xrightarrow{\alpha} (G,1_G,R_G) \xrightarrow{\beta} (Q,1_Q,R_Q) \longrightarrow 1,$$

where Q is also a group, together with a left group action $\lambda : Q \times G \to G$ or a right group action $\rho : G \times Q \to G$ such that

- α is injective,
- $\mathrm{im}(\alpha) = \ker(\beta)$ and
- $\beta \circ \lambda = \sigma$ or $\beta \circ \rho = \tau$, where σ is the map $\sigma : Q \times G \to Q, (q,g) \mapsto q \cdot \beta(g)$ and τ is the map $\tau : G \times Q \to Q, (g,q) \mapsto \beta(g) \cdot q$.

Remark. Since Q is both a crowd and a group, it is possible to have a group action of Q on a set, for example the set of a crowd.

Theorem 4.5. *Suppose the sequence*

$$1 \longrightarrow (H, 1_H, R_H) \xrightarrow{\alpha} (G, 1_G, R_G) \xrightarrow{\beta} (Q, 1_Q, R_Q) \longrightarrow 1,$$

together with a left group action $\lambda : Q \times G \to G$ or a right group action $\rho : G \times Q \to G$, is a weakly split short exact sequence. Then β is surjective and there exists a bijection $Q \times H \to G$. In particular, if G is finite, then so are H and Q and $\#G = \#Q \cdot \#H$.

Proof: We only proof the theorem for a sequence with a left group action $\lambda : Q \times G \to G$, as the proof for a sequence with a right group action follows by symmetry, and the notation $\lambda(q, g) = q.g$. So let

$$1 \longrightarrow (H, 1_H, R_H) \xrightarrow{\alpha} (G, 1_G, R_G) \xrightarrow{\beta} (Q, 1_Q, R_Q) \longrightarrow 1,$$

together with the group action $\lambda : Q \times G \to G, (q, g) \mapsto q.g$, be a weakly split short exact sequence. To show that β is surjective, note that for all $q \in Q$, $\lambda(q, 1_G) \in G$ and

$$\beta(\lambda(q, 1_G)) = \sigma(q, 1_G) = q \cdot \beta(1_G) = q \cdot 1_Q = q.$$

Therefore, the image of β is Q, which implies that β is surjective.
To proof the second half of the theorem, we show that the map $s : Q \times H \to G, (q, a) \mapsto q.\alpha(a)$ is a bijection. Suppose $a, b \in H$ and $p, q \in Q$ such that $s(p, a) = s(q, b)$, so $p.\alpha(a) = q.\alpha(b)$. Since Q is a group, there is a $p^{-1} \in Q$, which means

$$\begin{aligned} p.\alpha(a) = q.\alpha(b) &\Rightarrow p^{-1}.(p.\alpha(a)) = p.(q.\alpha(b)) \\ &\Rightarrow (p^{-1} \cdot p).\alpha(a) = \alpha(a) = (p \cdot q).\alpha(b). \end{aligned}$$

Therefore

$$\beta(\alpha(a)) = \beta((p^{-1} \cdot q).\alpha(b)) \Rightarrow 1_Q = (p^{-1} \cdot q) \cdot \beta(\alpha(b)) = p^{-1} \cdot q,$$

where $\beta((p^{-1} \cdot q).\alpha(b)) = (p^{-1} \cdot q) \cdot \beta(\alpha(b))$ because of the third axiom of weakly split short exact sequences. This implies that $p = q$, which means

$$\alpha(a) = (p^{-1} \cdot q).\alpha(b) = 1_Q.\alpha(b) = \alpha(b).$$

Since α is injective, if follows that $a = b$. Therefore $s(p, a) = s(q, b)$ implies $p = q$ and $a = b$, so s is injective.
To show s is surjective, let $g \in G$ be arbitrary. Take $q \in Q$ such that $\beta(g) = q$. Then $q^{-1}.g \in G$ and $\beta(q^{-1}.g)) = q^{-1} \cdot \beta(g) = q^{-1} \cdot q = 1_Q$, which implies $q^{-1}.g \in \ker(\beta)$. Since the kernel of β is equal to the image of α, it follows that there exists an $a \in H$ such that $\alpha(a) = q^{-1}.g$, which implies

$$g = 1_Q.g = (q \cdot q^{-1}).g = q.(q^{-1}.g) = q.\alpha(a) \in \operatorname{im}(s).$$

Since g was taken arbitrarily, this holds for all elements in G, so $\operatorname{im}(s) = G$ and s is surjective.
Since $s : Q \times H \to G$ is both injective and surjective, there exists a bijection $Q \times H \to G$. Furthermore, since G, H, Q are crowds, their sets are nonempty. It follows that if G is finite, so are Q and H and $\#G = \#Q \cdot \#H$. □

A weakly split short exact sequence tells us about the sets of its crowds. To gain similar information on their crowd laws, we need a few extra conditions.

Definition 4.6. A *strongly split short exact sequence of crowds* is a sequence of crowd morphisms

$$1 \longrightarrow (H, 1_H, R_H) \xrightarrow{\alpha} (G, 1_G, R_G) \xrightarrow{\beta} (Q, 1_Q, R_Q) \longrightarrow 1,$$

where Q is also a group, together with a left group action $\lambda : Q \times G \to G$ and a right group action $\rho : G \times Q \to G$ such that:

- the sequence is a weakly split short exact sequence when taken together with λ or ρ,
- α is a full crowd morphism,
- λ and ρ commute and
- the map $l : R_Q \times R_G \to R_G,\ ((p,q,r),(a,b,c)) \mapsto (\rho(a,p), \lambda(p^{-1}, \rho(b, r^{-1})), \lambda(r,c))$ is well-defined.

Theorem 4.7. *Suppose the sequence*

$$1 \longrightarrow (H, 1_H, R_H) \xrightarrow{\alpha} (G, 1_G, R_G) \xrightarrow{\beta} (Q, 1_Q, R_Q) \longrightarrow 1$$

together with a left group action $\lambda : Q \times G \to G$ and a right group action $\rho : G \times Q \to G$ is a strongly split short exact sequence. Then the set $\{(\beta(a), \beta(b), \beta(c)) \mid (a,b,c) \in R_G\} = R_Q$ and there exists a bijection $R_Q \times R_H \to R_G$. In particular, if R_G is finite, then so are R_H and R_Q and $\#R_G = \#R_Q \cdot \#R_H$.

Proof: In this proof, we use the following notation:

$$p.g.q = \lambda(p, \rho(g,q)),$$

so the map l can be rewritten as $l : R_Q \times R_G \to R_G, ((p,q,r),(a,b,c)) \mapsto (1_Q.a.p, p^{-1}.b.r^{-1}, r.c.1_Q)$. Note that because λ and ρ commute, the equalities

$$\begin{aligned} r.(p.g.q).s &= r.(\lambda(p, \rho(g,q))).s \\ &= \lambda(r, \rho(\lambda(p, \rho(g,q)), s)) \\ &= \lambda(r, \lambda(p, \rho(\rho(g,q), s))) \\ &= \lambda(r, \lambda(p, \rho(g, q \cdot s))) \\ &= \lambda(r \cdot p, \rho(g, q \cdot s)) \\ &= (r \cdot p).g.(q \cdot s) \end{aligned}$$

hold. Additionally, we let σ and τ be the maps defined in Definition 4.4. To show the first half of the statement, note that for all $p, q \in Q$ and $g \in G$,

$$\begin{aligned} \beta(p.g.q) = \beta(\lambda(p, \rho(g,q))) &= \sigma(p, \rho(g,q)) \\ = p \cdot \beta(\rho(g,q)) &= p \cdot \tau(g,q) \\ &= p \cdot \beta(g) \cdot q. \end{aligned}$$

Therefore, for all $(p,q,r) \in R_Q$, we have that

$$\begin{aligned} l((p,q,r),(1_G,1_G,1_G)) &= (1_Q.1_G.p, p^{-1}.1_G.r^{-1}, r.1_G.1_Q) \in R_G \text{ and} \\ (\beta(1_Q.1_G.p), \beta(p^{-1}.1_G.r^{-1}), \beta(r.1_G.1_Q)) &= (1_Q \cdot \beta(1_G) \cdot p, p^{-1} \cdot \beta(1_G) \cdot r^{-1}, r \cdot \beta(1_G) \cdot 1_Q) \\ &= (p, p^{-1} \cdot r^{-1}, r) = (p,q,r), \end{aligned}$$

where the fourth equality follows from the equation $p \cdot q \cdot r = 1_Q$. It follows that R_Q is a subset of $\{(\beta(a), \beta(b), \beta(c)) \mid (a,b,c) \in R_G\}$, and since β is a crowd morphism, we can conclude that $\{(\beta(a), \beta(b), \beta(c)) \mid (a,b,c) \in R_G\} = R_Q$.
To proof that there is a bijection $R_Q \times R_H \to R_H$, let $l' : R_Q \times R_H \to R_G$ be the map induced by l, so $l'((p,q,r),(a,b,c)) = (1_Q\alpha(a).p, p^{-1}.\alpha(b).r^{-1}, r.\alpha(c)1_Q)$ for all $(p,q,r) \in R_Q$ and $(a,b,c) \in R_G$. We want to show that this is a bijection. Let $p_i, q_i \in Q$ and $a_i, b_i \in H$, for $i = 1,2,3$, and suppose that $(p_1,p_2,p_3),(q_1,q_2,q_3) \in R_Q$, $(a_1,a_2,a_3),(b_1,b_2,b_3) \in R_H$ and

$$l'((p_1,p_2,p_3),(a_1,a_2,a_3)) = l'((q_1,q_2,q_3),(b_1,b_2,b_3)).$$

This implies

$$1_Q.\alpha(a_1).p_1 = 1_Q.\alpha(b_1).q_1,\ p_1^{-1}.\alpha(a_2).p_3^{-1} = q_1^{-1}.\alpha(b_2).q_3^{-1} \text{ and } p_3.\alpha(a_3).1_Q = q_3.\alpha(b_3).1_Q.$$

Recall from the proof of Theorem 4.5 that the map $s : Q \times H \to G, (p, a) \mapsto \lambda(p, \alpha(a))$ and the map $s' : H \times Q \to G, (b, q) \mapsto \rho(\alpha(b), q)$ are injective. Furthermore,

$$\begin{aligned} 1_Q.\alpha(a_1).p_1 = 1_Q.\alpha(b_1).q_1 &\Rightarrow \rho(\alpha(a_1), p_1) = \rho(\alpha(b_1), q_1) \\ &\Rightarrow s'(a_1, p_1) = s'(b_1, q_1) \text{ and} \\ p_3.\alpha(a_3).1_Q = q_3.\alpha(b_3).1_Q &\Rightarrow \rho(\alpha(a_3), 1_Q) = \rho(\alpha(b_3), 1_Q) \\ &\Rightarrow s(p_3, a_3) = s(q_3, b_3), \end{aligned}$$

which implies $a_1 = b_1,\ p_1 = q_1,\ a_3 = b_3$ and $p_3 = q_3$. It follows that

$$\begin{aligned} p_1^{-1}.\alpha(a_2).p_3^{-1} = q_1^{-1}.\alpha(b_2).q_3^{-1} &\Rightarrow p_1^{-1}.\alpha(a_2).p_3^{-1} = p_1^{-1}.\alpha(b_2).p_3^{-1} \\ &\Rightarrow (p_1 \cdot p_1^{-1}).\alpha(a_2).(p_3^{-1} \cdot p_3) = (p_1 \cdot p_1^{-1}).\alpha(b_2).(p_3^{-1} \cdot p_3) \\ &\Rightarrow \alpha(a_2) = \alpha(b_2) \Rightarrow a_2 = b_2, \end{aligned}$$

where the last implication holds because α is injective. Additionally, since $p_1 \cdot p_2 \cdot p_3 = 1_Q = q_1 \cdot q_2 \cdot q_3$, we can infer that

$$p_2 = p_1^{-1} \cdot p_3^{-1} = q_1^{-1} \cdot q_3^{-1} = q_2.$$

Therefore $l'((p_1, p_2, p_3), (a_1, a_2, a_3)) = l'((q_1, q_2, q_3), (b_1, b_2, b_3))$ implies that $p_i = q_i$ and $a_i = b_i$ for $i = 1, 2, 3$, so l' is injective.
To show that l' is surjective, let $(d, e, f) \in R_G$ be arbitrary and let p, q, r be the elements in Q such that $\beta(d) = p,\ \beta(e) = q$ and $\beta(f) = r$. Since β is a crowd morphism, this implies that $(p, q, r) \in R_Q$. Recall that Q is also a group, so its crowd law consists of all triples whose product is equal to 1_Q. It follows that $q = p^{-1} \cdot r^{-1}$ and

$$l((p^{-1}, p \cdot r, r^{-1}), (d, e, f)) = (1_Q.d.p^{-1}, p.e.r, r^{-1}.f.1_Q) \in R_G.$$

But

$$\begin{aligned} \beta(1_Q.d.p^{-1}) &= 1_Q \cdot \beta(d) \cdot p^{-1} = p \cdot p^{-1} = 1_Q \\ \beta(p.e.r) &= p \cdot \beta(e) \cdot r = p \cdot q \cdot r = 1_Q \text{ and} \\ \beta(r^{-1}.f.1_Q) &= r^{-1} \cdot \beta(f) \cdot 1_Q = r^{-1} \cdot r = 1_Q. \end{aligned}$$

Since $\ker(\beta) = \operatorname{im}(\alpha)$, it follows that there exist $a, b, c \in H$ such that $1_Q.d.p^{-1} = \alpha(a),\ p.e.r = \alpha(b)$ and $r^{-1}.f.1_Q = \alpha(c)$. Therefore $(\alpha(a), \alpha(b), \alpha(c)) \in R_G$, which implies $(a, b, c) \in R_H$, since α is a full crowd morphism, and

$$\begin{aligned} l'((p, p^{-1} \cdot r^{-1}, r), (a, b, c)) &= ((1_Q \cdot 1_Q).d.(p^{-1} \cdot p), (p^{-1} \cdot p).e.(r \cdot r^{-1}), (r \cdot r^{-1}).f.(1_Q \cdot 1_Q)) \\ &= (1_Q.d.1_Q, 1_Q.e.1_Q, 1_Q.f.1_Q) = (d, e, f), \end{aligned}$$

So (d, e, f) is in the image of l'. Since the triple was chosen arbitrarily, it follows that l' is surjective. Therefore l' is bijective, which means there exists a bijection $R_Q \times R_H \to R_G$. Since crowd laws are always nonempty, it holds that if R_G is finite, then so are R_Q and R_H and $\#R_G = \#R_Q \cdot \#R_H$. □

Given a split short exact sequence of groups, we can find a semidirect product that is isomorphic to 1 of the groups in the sequence, as explained in the notes of Conrad ([4, Definition 3.4]). An analogous theorem holds for crowds.

Theorem 4.8. *Suppose the sequence*

$$1 \to H \to^{\alpha} G \to^{\beta} Q \to 1,$$

together with a left group action $\lambda : Q \times G \to G$ and right group action $\rho : G \times Q \to Q$, is a strongly split short exact sequence of crowds. Then there is a crowd morphism $\theta : Q \to Aut(H)$ such that $G \cong H \rtimes_\theta Q$.

Proof: This proof consists of 2 steps, the first is to find a crowd morphism θ such that $H \rtimes_\theta Q$ is a semidirect product, and the second is to find a crowd isomorphism $\psi : H \rtimes_\theta Q \to G$. Just as in the proof of Theorem 4.7, we use the notation $p.g.q = \lambda(p, \rho(g, q))$.
Inspired by the proof of theorem 3.3 in [4], we let $\theta : Q \to \mathrm{Aut}(H), q \to \theta_q$ be the map such that $\theta_q(h) = h'$, where $h' \in H$ such that $\alpha(h') = q.\alpha(h).q^{-1}$ (so $\alpha(\theta_q(h)) = q.\alpha(h).q^{-1}$). To proof that this map is well-defined, let $q \in Q$ be arbitrary. Then we need to show that $\theta_q : H \to H$ is a well-defined crowd isomorphism:

1. For all $h \in H$, we have that

$$\beta(q.\alpha(h).q^{-1}) = q \cdot \beta(\alpha(h)) \cdot q^{-1} = q \cdot 1_Q \cdot q^{-1} = 1_Q,$$

   so $q.\alpha(h).q^{-1} \in \ker(\beta) = \mathrm{im}(\alpha)$. Therefore there is a $h' \in H$ such that $\alpha(h') = q.\alpha(h).q^{-1}$ and, since α is injective, this h' is unique. Therefore $\theta_q(h) \in H$ exists and $\theta_q : H \to H$ is well-defined.

2. Suppose $\theta_q(1_H) = h$ for some $h \in H$. Then $\alpha(h) = q.\alpha(1_H).q^{-1} = q.1_G.q^{-1}$. Furthermore,

$$\begin{aligned} l((q, q^{-1}, 1_Q), (1_G, 1_G, 1_G) &= (1_Q.1_G.q,\ q^{-1}.1_G.1_Q,\ 1_Q.1_G.1_Q) = (1_Q.1_G.q,\ q^{-1}.1_G.1_Q,\ 1_G) \in R_G, \\ &\Rightarrow (q^{-1}.1_G.1_Q,\ 1_Q.1_G.q,\ 1_G) \in R_G \\ &\Rightarrow (1_Q.1_G.q,\ 1_G,\ q^{-1}.1_G.1_Q) \in R_G \\ &\Rightarrow l((q^{-1},\ q \cdot q^{-1},\ q), (1_Q.1_G.q,\ 1_G,\ q^{-1}.1_G.1_Q)) \in R_G \\ &\Rightarrow (1_Q.1_G.1_Q,\ q.1_G.q^{-1},\ 1_Q.1_G.1_Q) \in R_G \\ &\Rightarrow (q.1_G.q^{-1},\ 1_Q.1_G.1_Q,\ 1_Q.1_G.1_Q) = (q.1_G.q^{-1},\ 1_G,\ 1_G) \in R_G, \end{aligned}$$

   where $(q, q^{-1}, 1_Q), (q^{-1}, q \cdot q^{-1}, q) \in R_Q$ because their triple products are equal to 1_Q. This implies $q.1_Q.q^{-1} = 1_G$, which means $\alpha(h') = q.1_G.q^{-1} = 1_G$. Therefore $\theta_q(1_H) = h = 1_H$, so the first crowd morphism axiom is satisfied.

3. To proof the second crowd morphism axiom, note that $(q^{-1}, q \cdot q^{-1}, q)$, $(q^{-1}, q, 1_Q) \in R_Q$. Therefore, whenever $h_1, h_2, h_3 \in H$ such that $(h_1, h_2, h_3) \in R_H$, it follows that

$$\begin{aligned} (\alpha(h_1), \alpha(h_2), \alpha(h_3)) \in R_G &\Rightarrow l((q^{-1}, q \cdot q^{-1}, q), (\alpha(h_1), \alpha(h_2), \alpha(h_3))) \in R_G \\ &\Rightarrow (1_Q.\alpha(h_1).q^{-1}, q.\alpha(h_2).q^{-1}, q.\alpha(h_3).1_Q) \in R_G \\ &\Rightarrow (q.\alpha(h_3).1_Q, 1_Q.\alpha(h_1).q^{-1}, q.\alpha(h_2).q^{-1}) \in R_G \\ &\Rightarrow l((q^{-1}, q, 1_Q), (q.\alpha(h_3).1_Q, 1_Q.\alpha(h_1).q^{-1}, q.\alpha(h_2).q^{-1})) \in R_G \\ &\Rightarrow (q.\alpha(h_3).q^{-1}, q.\alpha(h_1).q^{-1}, q.\alpha(h_2).q^{-1}) \in R_G \\ &\Rightarrow (q.\alpha(h_1).q^{-1}, q.\alpha(h_2).q^{-1}, q.\alpha(h_3).q^{-1}) \in R_G \\ &\Rightarrow (\alpha(\theta_q(h_1)), \alpha(\theta_q(h_2)), \alpha(\theta_q(h_3))) \in R_G \\ &\Rightarrow (\theta_q(h_1), \theta_q(h_2), \theta_q(h_3)) \in R_H, \end{aligned}$$

   where the last implication holds because α is a full crowd morphism. Therefore, if $(h_1, h_2, h_3) \in R_H$, then $(\theta_q(h_1), \theta_q(h_2), \theta_q(h_3)) \in R_H$ as well, so θ_q is indeed a crowd morphism.

4. To show that θ_q is an isomorphism, note that, since $q \in Q$ was chosen arbitrarily, $\theta_{q^{-1}}$ is a crowd morphism as well. Furthermore, for all $h \in H$,

$$\begin{aligned} \alpha((\theta_q \circ \theta_{q^{-1}})(h)) &= \alpha(\theta_q(\theta_{q^{-1}}(h))) = q.\alpha(\theta_{q^{-1}}(h)).q^{-1} \\ &= q.(q^{-1}.\alpha(h).q).q^{-1} = (q \cdot q^{-1}).\alpha(h).(q \cdot q^{-1}) = 1_Q.\alpha(h).1_Q \\ &= \alpha(h). \end{aligned}$$

   Since α is injective, it follows that $\theta_q \circ \theta_{q^{-1}} = id_H$ and since q was chosen arbitrarily, we can conclude that $\theta_{q^{-1}} \circ \theta_q = id_H$ as well. Therefore θ_q must be a crowd isomorphism.

Therefore $\theta_q \in \text{Aut}(H)$ for all $q \in Q$, which means the map $\theta : Q \to \text{Aut}(H)$ is well-defined. Additionally, for all $h \in H$, we have that $\alpha(\theta_{1_Q}(h)) = 1_Q.\alpha(h).1_Q = \alpha(h)$, which implies $\theta(1_Q) = id_H$. Furthermore, for all $p, q \in Q$ and $h \in H$, we have that

$$\begin{aligned}\alpha((\theta_p \circ \theta_q)(h)) &= p.\alpha(\theta_q(h)).p^{-1} = p.(q.\alpha(h).q^{-1}).p^{-1} \\ &= (p \cdot q).\alpha(h).(q^{-1} \cdot p^{-1}) = (p \cdot q).\alpha(h).(p \cdot q)^{-1} = \alpha(\theta_{p \cdot q}(h)).\end{aligned}$$

This implies $\theta(p) \circ \theta(q) = \theta(p \cdot q)$, which means θ is a group homomorphism. As shown in Example 1.14, this implies that θ is a crowd morphism as well. Furthermore, it follows from Definition 4.1 that $H \rtimes_\theta Q$ is a semidirect product.
To find the isomorphism ψ, recall from the proof of Theorem 4.7 that the map $s' : H \times Q \to G$, where the pair (h, q) is mapped to $\rho(\alpha(h), q)$, is a bijection. So let $\psi : H \rtimes_\theta Q \to G$ be the map such that $\psi(h, q) = s'(h, q)$. Then

$$\psi(1_H, 1_Q) = \rho(\alpha(1_H), 1_Q) = \alpha(1_H) = 1_G.$$

To show that ψ is a crowd isomorphism, we need to proof that for all $(h, p), (i, q), (j, r) \in H \rtimes_\theta Q$, it holds that $(\psi(h, p),\ \psi(i, q),\ \psi(j, r)) \in R_G$ if and only if $((h, p),\ (i, q),\ (j, r)) \in R_{H \rtimes_\theta Q}$. We will prove this by using the bijection

$$l' : R_Q \times R_H \to R_G, ((p, q, r), (a, b, c)) \mapsto (1_Q.a.p, p^{-1}.b.r^{-1}, r.c.1_Q)$$

from the proof of Theorem 4.7.
Note that, if $(p, q, r) \in R_Q$, then $p \cdot q \cdot r = 1_Q$ and the following equivalences hold:

$$\begin{aligned}(1_Q.\alpha(h).p, p^{-1}.\alpha(\theta_p(i)).r^{-1}, r.\alpha(\theta_p(\theta_q(j))).1_Q) \in R_G &\iff \\ (1_Q.\alpha(h).p, 1_Q.\alpha(i).(p^{-1} \cdot r^{-1}), (r \cdot p \cdot q).\alpha(j).(q^{-1} \cdot p^{-1})) \in R_G &\iff \\ (1_Q.\alpha(h).p, 1_Q.\alpha(i).q, 1_Q.\alpha(j).(q^{-1} \cdot p^{-1})) \in R_G &\iff \\ (1_Q.\alpha(h).p, 1_Q.\alpha(i).q, 1_Q.\alpha(j).r) \in R_G &\iff \\ (\rho(\alpha(h), p), \rho(\alpha(i), q), \rho(\alpha(j), r)) = (\psi(h, p), \psi(i, q), \psi(j, r)) \in R_G&.\end{aligned}$$

Let $(h, p), (i, q), (j, r) \in H \rtimes_\psi Q$ be arbitrary, and suppose that $((h, p), (i, q), (j, r)) \in R_{H \rtimes_\theta Q}$. Then, by definition, $(p, q, r) \in R_Q$ and $(h, \theta_p(i), \theta_p(\theta_q(j))) \in R_H$, which implies

$$l'((p, q, r), (h, \theta_p(i), \theta_p(\theta_q(j)))) = (1_Q.\alpha(h).p, p^{-1}.\alpha(\theta_p(i)).r^{-1}, r.\alpha(\theta_p(\theta_q(j))).1_Q) \in R_G.$$

It follows that $(\psi(h, p), \psi(i, q), \psi(j, r)) \in R_G$. Since $(h, p), (i, q), (j, r) \in H \rtimes_\psi Q$ were chosen arbitrarily, it holds for all $(h, p), (i, q), (j, r) \in H \rtimes_\psi Q$ that if $((h, p), (i, q), (j, r)) \in R_{H \rtimes_\theta Q}$, then $(\psi(h, p), \psi(i, q), \psi(j, r)) \in R_G$.
Suppose $(\psi(h, p), \psi(i, q), \psi(j, r)) \in R_G$ for some $(h, p), (i, q), (j, r) \in H \rtimes_\psi Q$. Then

$$\begin{aligned}(\beta(\psi(h, p)), \beta(\psi(i, q)), \beta(\psi(j, r))) &= (\beta(\rho(\alpha(h), p)), \beta(\rho(\alpha(i), q)), \beta(\rho(\alpha(j), r))) \\ &= (\beta(\alpha(h)) \cdot p, \beta(\alpha(i)) \cdot q, \beta(\alpha(j) \cdot r) = (p, q, r) \in R_Q.\end{aligned}$$

Since $(p, q, r) \in R_Q$ and $(\psi(h, p), \psi(i, q), \psi(j, r)) \in R_G$, it follows that $(1_Q.\alpha(h).p, p^{-1}.\alpha(\theta_p(i)).r^{-1}, r.\alpha(\theta_p(\theta_q(j))).1_Q)$ is in R_G. Therefore, we can take this triple together with $(p^{-1}, p \cdot r, r^{-1}) \in R_Q$ and apply l' to find that

$$(1_Q.\alpha(h).1_Q, 1_Q.\alpha(\theta_p(i)).1_Q, 1_Q.\alpha(\theta_p(\theta_q(j))).1_Q) \in R_G.$$

Since α is a full crowd morphism, this implies $(h, \theta_p(i), \theta_p(\theta_q(j))) \in R_H$. Therefore, it holds that if $(\psi(h, p), \psi(i, q), \psi(j, r)) \in R_G$, then $(h, \theta_p(i), \theta_p(\theta_q(j))) \in R_H$ and $(p, q, r) \in R_Q$, which means $((h, p), (i, q), (j, r)) \in R_{H \rtimes_\theta Q}$. Therefore ψ is indeed a crowd isomorphism.
Therefore there exists a crowd morphism $\theta : Q \to \text{Aut}(H)$ such that $G \cong H \rtimes_\theta Q$. □

Example 4.9 (The Krasner hyperfield). Take the sequence

$$1 \longrightarrow SL_n(\mathbb{K}) \xrightarrow{\varphi} GL_n(\mathbb{K}) \xrightarrow{\psi} \mathbb{K}^\times \longrightarrow 1,$$

where $\mathbb{K}^\times = \{1\}$ is the unit group of $\mathbb{K}$, $\varphi : SL_n(\mathbb{K}) \to GL_n(\mathbb{K}), A \mapsto \left[\begin{smallmatrix} A & \mathbf{0} \\ \mathbf{0} & 1 \end{smallmatrix}\right]$ is the crowd morphism from Theorem 3.1, and $\psi : GL_n(\mathbb{K}) \to \mathbb{K}^\times, A \mapsto 1$. It follows from Theorem 3.4 that $SL_n(\mathbb{K})$ is isomorphic to $GL_n(\mathbb{K})$ and since $\mathbb{K}^\times$ only contains 1 element, it makes sense that there is a bijection of sets from $SL_n(\mathbb{K}) \times \mathbb{K}^\times \to GL_n(\mathbb{K})$ and from $R_{SL_n(\mathbb{K})} \times R_{\mathbb{K}^\times} \to R_{GL_n(\mathbb{K})}$. However, we can also proof this with Theorem 4.7. Let $\lambda : \mathbb{K}^\times \times GL_n(\mathbb{K}) \to GL_n(\mathbb{K})$ and $\rho : GL_n(\mathbb{K}) \times \mathbb{K}^\times \to GL_n(\mathbb{K})$ be the unique left and right group actions. Then we can show that the sequence together with these group actions is a strongly split short exact sequence:

- It follows from Theorem 3.1 that φ is both injective and full. Furthermore, is follows from Theorem 3.3 that $\mathrm{im}(\varphi) = GL_n(B) = \ker(\psi)$;
- for all $A \in GL_n(\mathbb{K})$, it holds that

  $$\psi(\lambda(1, A)) = \psi(A) = 1 \cdot \psi(A) \text{ and } \psi(\rho(A, 1)) = \psi(A) = \psi(A) \cdot 1,$$

  so $\psi \circ \lambda = \sigma$ and $\psi \circ \rho = \tau$, where σ and τ are as in Definition 4.4;
- For all $A \in GL_n(\mathbb{K})$,

  $$\lambda(1, \rho(A, 1)) = \lambda(1, A) = A = \rho(A, 1) = \rho(\lambda(1, A), 1),$$

  which implies λ and ρ commute;
- For all $(A, B, C) \in R_{GL_n(\mathbb{K})}$,

  $$(\rho(A, 1), \lambda(1^{-1}, \rho(B, 1^{-1}), \lambda(1, C))) = (A, B, C) \in R_{GL_n(\mathbb{K})},$$

  and since $(1, 1, 1)$ is the only triple in $R_{\mathbb{K}^\times}$, it follows that the map

  $$l : R_{\mathbb{K}^\times} \times R_{GL_n(\mathbb{K})} \to R_{GL_n(\mathbb{K})}, ((p, q, r), (A, B, C)) \mapsto (\rho(A, p), \lambda(p^{-1}, \rho(B, r^{-1})), \lambda(r, C))$$

  is well-defined.

Therefore it is indeed a strongly split short exact sequence, which proof the bijections do indeed exist. This also implies that $GL_n(\mathbb{K}) \cong SL_n(\mathbb{K}) \rtimes_\theta \mathbb{K}^\times$, where $\theta : \mathbb{K}^\times \to \mathrm{Aut}(SL_n(\mathbb{K}))$ is the crowd morphism that sends 1 to the identity map on $SL_n(\mathbb{K})$. Note that

$$\begin{aligned} R_{SL_n(\mathbb{K}) \rtimes_\theta \mathbb{K}^\times} &= \left\{ ((A, p), (B, q), (C, r)) \in (SL_n(\mathbb{K}) \times \mathbb{K}^\times)^3 \mid (p, q, r) \in R_{\mathbb{K}^\times} \text{ and } (A, \theta_p(B), \theta_p(\theta_q(C))) \in R_{SL_n(\mathbb{K})} \right\} \\ &= \left\{ ((A, 1), (B, 1), (C, 1)) \in (SL_n(\mathbb{K}) \times \mathbb{K}^\times)^3 \mid (A, B, C) \in R_{SL_n(\mathbb{K})} \right\}. \end{aligned}$$

It is still uncertain if this sequence exists and is a strongly split short exact sequence for all bands B and $n \in \mathbb{N}$. However, we do know the following.

Theorem 4.10. *Suppose B is a band and $n \in \mathbb{N}$ such that $a_{n+1,n+1} \in B^\times$ for all $A \in GL_n(B)$, where $B^\times$ is the unit group of B. Then the map $\varphi : SL_n(B) \to GL_n(B)$, $A \mapsto \left[\begin{smallmatrix} A & \mathbf{0} \\ \mathbf{0} & 1 \end{smallmatrix}\right]$ and the map $\psi : GL_n(\mathbb{K}) \to \mathbb{K}^\times$, $A \mapsto a_{n+1,n+1}$ are crowd morphisms, and there exists a left group action $\lambda : B^\times \times GL_n(B) \to GL_n(B)$ and a right group action $\rho : GL_n(B) \times B^\times \to GL_n(B)$ such that the sequence*

$$1 \longrightarrow SL_n(B) \xrightarrow{\varphi} GL_n(B) \xrightarrow{\psi} B^\times \longrightarrow 1,$$

together with λ and ρ is a strongly split short exact sequence.

Proof: It follows from Theorem 3.1 that φ is a crowd morphism. Furthermore, $\psi(1) = 1$ and, if $(A, C, D) \in R_{GL_n(B)}$, then $a_{n+1,n+1}c_{n+1,n+1}d_{n+1,n+1} - 1 \in N_B$. It follows from the definition of a band that $a_{n+1,n+1}c_{n+1,n+1}d_{n+1,n+1} = 1$, which implies $(a_{n+1,n+1}, c_{n+1,n+1}, d_{n+1,n+1}) \in R_{B^\times}$, since $B^\times$ is both a group and a crowd. Therefore $(\psi(A), \psi(B), \psi(C)) \in R_{B^\times}$ whenever $(A, B, C) \in R_{GL_n(B)}$, which means ψ is a crowd morphism as well.
Suppose $\lambda : B^\times \times GL_n(B) \to GL_n(B)$ is the map that takes a $p \in B^\times$ and $A \in GL_n(B)$, and constructs

the matrix $A' \in GL_n(B)$ by multiplying the first *row* of A by p^{-1} and multiplying $a_{n+1,n+1}$ by p. Similarly, suppose $\rho : GL_n(B) \times B^\times \to GL_n(B)$ is the map that takes an $A \in GL_n(B)$ and $p \in B^\times$, and multiplies the first *column* of A with p^{-1} and $a_{n+1,n+1}$ with p to obtain the matrix $A'' \in GL_n(B)$. We want to proof that these maps are well-defined group actions, and that the sequence with these group actions is a strongly split short exact sequence.
We show that λ is a well-defined left group action, as it follows by symmetry that ρ is a well-defined right group action. Let $p \in B^\times$ and $A \in GL_n(B)$ be arbitrary. Note that $p \in B^\times$, so p^{-1} exists and $\lambda(p, A) = A' \in M_{(n+1)\times(n+1)}(B)$. Furthermore,

$$\begin{aligned}
\det(A') &= \sum_{\sigma \in S_{n+1}} \operatorname{sign}(\sigma) \prod_{k=1,\dots,n+1} a'_{k,\sigma(k)} \\
&= p \cdot a_{n+1,n+1} \cdot \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) \cdot p^{-1} \cdot a_{i,\sigma(i)} \prod_{k=1,..,\hat{i},..,n} a_{k,\sigma(k)}, \text{ where } i \text{ is such that } \sigma(i) = 1 \\
&= p \cdot p^{-1} \cdot a_{n+1,n+1} \cdot \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) \prod_{k=1,\dots,n+1} a_{k,\sigma(k)} \\
&= \det(A),
\end{aligned}$$

Therefore $\operatorname{im}(\lambda) \subset GL_n(B)$, so λ is well-defined. It is a left group action, since for all $A \in GL_n(B)$, $\lambda(1, A) = A$, and for all $p, q \in B^\times,\ A \in GL_n(B)$, we have that

$$\begin{aligned}
\lambda(p, \lambda(q, A)) &= \rho(A', p) \\
&= \begin{bmatrix} p^{-1}q^{-1}a_{11} & p^{-1}q^{-1}a_{12} & \dots & 0 \\ a_{21} & a_{22} & \dots & \dots \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & bca_{n+1,n+1} \end{bmatrix} \\
&= \begin{bmatrix} (pq)^{-1}a_{11} & (pq)^{-1}a_{12} & \dots & 0 \\ a_{21} & a_{22} & \dots & \dots \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & (pq)a_{n+1,n+1} \end{bmatrix} \\
&= \lambda(pq, A).
\end{aligned}$$

Therefore λ is indeed a left group action and, by symmetry, ρ is a right group action.
To proof that the sequence together with the group actions is a strongly split short exact sequence, we need to check that it satisfies all the axioms. We know that $B^\times$ is a group. Additionally, φ is injective and, as shown in the proof of Theorem 3.3,

$$\operatorname{im}(\varphi) = GL_n^*(B) = \{A \in GL_n(B) | \psi(A) = 1\} = \ker(\psi).$$

Let $\sigma : B^\times \times GL_n(B) \to B^\times, (p, A) \mapsto p \cdot \psi(A)$ and $\tau : GL_n(B) \times B^\times \to B^\times, (A, p) \mapsto \psi(A) \cdot p$ be maps. Then, for all $p \in B^\times$ and $A \in GL_n(B)$, we have that

$$\begin{aligned}
&\psi(\lambda(p, A)) = p \cdot a_{n+1,n+1} = p \cdot \psi(A) \text{ and} \\
&\psi(\rho(A, p)) = p \cdot a_{n+1,n+1} = a_{n+1,n+1} \cdot a = \psi(A) \cdot p,
\end{aligned}$$

so $\psi \circ \lambda = \sigma$ and $\psi \circ \rho = \tau$. Therefore the axioms of weakly split short exact sequences are satisfied, so the sequence together with λ and the sequence together with ρ is a weakly split short exact sequence. This means the first axiom of stronly split short exact sequences is satisfied. The second axiom is satisfied as well, since φ is a full crowd morphism. To show that λ and ρ commute, let $p, q \in B^\times$ and

$A \in GL_n(B)$ be arbitrary. Then

$$\begin{aligned}
\lambda(p, \rho(A, q)) &= \lambda(p, A'') \\
&= \begin{bmatrix} p^{-1}q^{-1}a_{11} & p^{-1}a_{12} & ... & 0 \\ q^{-1}a_{21} & a_{22} & ... & ... \\ ... & ... & ... & ... \\ 0 & 0 & ... & pqa_{n+1,n+1} \end{bmatrix} \\
&= \rho(A', q) \\
&= \rho(\lambda(p, A), q),
\end{aligned}$$

so the group actions commute, and therefore the third axiom is satisfied.
The fourth axiom states the map $l : R_{B^\times} \times R_{GL_n(B)} \to R_{GL_n(B)}$, where the pair $((p, q, r), (A, C, D))$ is mapped to $(\rho(A, p), \lambda(p^{-1}, \rho(C, r^{-1})), \lambda(r, C))$, must be well-defined. So let $(p, q, r) \in R_{B^\times}$ and $(A, B, C) \in R_{GL_n(B)}$ be arbitrary, and define $X = \rho(A, p)$, $Y = \lambda(p^{-1}, \rho(C, r^{-1}))$ and $Z = \lambda(r, C)$. Then we want to check that (X, Y, Z) satisfy the three conditions to be in $R_{GL_n(B)}$. Note that $x_{i,n+1} = pa_{i,n+1}$ for all $i = 1, ..., n+1$, because if $i \neq n+1$, then $a_{i,n+1} = 0$. This also holds for the $n+1$th row, and similar equalities hold the $n+1$th rows and columns of Y and Z.

1. For $i, j = 1, ..., n+1$, we have that

$$\begin{aligned}
\sum_{k,l=1,\ldots,n+1} x_{i,k}y_{k,l}z_{l,j} - \delta_{i,j} &= p^{-1}a_{i,1}prb_{1,1}r^{-1}c_{1,j} + \sum_{l=2,\ldots,n+1} p^{-1}a_{i,1}pb_{1,l}c_{l,j} + \sum_{k=2,\ldots,n+1} a_{i,k}rb_{k,1}r^{-1}c_{l,j} \\
&+ \sum_{k,l=2,\ldots,n} a_{i,k}b_{k,l}c_{l,j} + pa_{i,n+1}p^{-1}r^{-1}b_{n+1,n+1}rc_{n+1,j} - \delta_{i,j} \\
&= \sum_{k,l=1,\ldots,n+1} a_{i,k}b_{k,l}c_{l,j} - \delta_{i,j} \in N_B,
\end{aligned}$$

since $(A, B, C) \in R_{GL_n(B)}$.

2. We know that, for all $i, j = 1, ..., n+1$,

$$\sum_{k,l=1,\ldots,n+1} b_{i,k}c_{k,l}a_{l,j} - \delta_{i,j} \in N_B.$$

This implies that, whenever $i \neq j$,

$$\sum_{k,l=1,\ldots,n+1} b_{i,k}c_{k,l}a_{l,j} \in N_B \Rightarrow s \cdot \sum_{k,l=1,\ldots,n+1} b_{i,k}c_{k,l}a_{l,j} \in N_B \text{ for all } s \in B^\times.$$

Therefore, for all $i, j = 1, ..., n+1$, we have that

$$\begin{aligned}
\sum_{k,l=1,\ldots,n+1} y_{i,k}z_{k,l}x_{l,j} - \delta_{i,j} &= \sum_{l=1,\ldots,n+1} y_{i,1}r^{-1}c_{1,l}x_{l,j} + \sum_{l=1,\ldots,n} y_{i,n+1}c_{n+1,l}x_{l,j} \\
&+ \sum_{k,l=1,\ldots,n+1, k\neq 1,n+1} y_{i,k}c_{k,l}x_{l,j} + p^{-1}r^{-1}b_{i,n+1}rc_{n+1,n+1}pa_{n+1,j} - \delta_{i,j} \\
&= \begin{cases} \sum_{k,l=1,\ldots,n+1} b_{i,k}c_{k,l}a_{l,j} - \delta_{i,j} \in N_B & \text{if } i = j = 1 \\ p \cdot \sum_{k,l=1,\ldots,n+1} b_{i,k}c_{k,l}a_{l,j} \in N_B & \text{if } i = 1 \neq j \\ p^{-1} \cdot \sum_{k,l=1,\ldots,n+1} b_{i,k}c_{k,l}a_{l,j} \in N_B & \text{if } i \neq 1 = j \\ \sum_{k,l=1,\ldots,n+1} b_{i,k}c_{k,l}a_{l,j} - \delta_{i,j} \in N_B & \text{if } i, j \neq 1 \end{cases}
\end{aligned}$$

3. Similarly, for all $i, j = 1, ..., n+1$ such that $i \neq j$, we have that

$$\sum_{k,l=1,\ldots,n+1} c_{i,k}a_{k,l}b_{l,j} \in N_B \Rightarrow s \cdot \sum_{k,l=1,\ldots,n+1} c_{i,k}a_{k,l}b_{l,j} \in N_B \text{ for all } s \in B^\times.$$

Therefore, for all $i, j = 1, ..., n+1$,

$$\begin{aligned}\sum_{k,l=1,\dots,n+1} z_{i,k}x_{k,l}y_{l,j} - \delta_{i,j} &= \sum_{k=1,\dots,n+1} z_{i,k}p^{-1}a_{k,1}y_{1,j} + \sum_{l=1,\dots,n} z_{i,n+1}a_{n+1,l}y_{l,j} \\ &\quad + \sum_{k,l=1,\dots,n+1, l\neq 1,n+1} z_{i,k}a_{k,l}y_{l,j} + rc_{i,n+1}pa_{n+1,n+1}p^{-1}r^{-1}b_{n+1,j} - \delta_{i,j} \\ &= \begin{cases} \sum_{k,l=1,\dots,n+1} c_{i,k}a_{k,l}b_{l,j} - \delta_{i,j} \in N_B & \text{if } i=j=1 \\ r^{-1}\cdot \sum_{k,l=1,\dots,n+1} c_{i,k}a_{k,l}b_{l,j} \in N_B & \text{if } i=1\neq j \\ r\cdot \sum_{k,l=1,\dots,n+1} c_{i,k}a_{k,l}b_{l,j} \in N_B & \text{if } i\neq 1=j \\ \sum_{k,l=1,\dots,n+1} c_{i,k}a_{k,l}b_{l,j} - \delta_{i,j} \in N_B & \text{if } i,j\neq 1 \end{cases}\end{aligned}$$

This implies $(X, Y, Z) = (\rho(A, p), \lambda(p^{-1}, \rho(C, r^{-1})), \lambda(r, C)) \in R_{GL_n(B)}$, so l is well-defined. Therefore, the last axiom is satisfied, so there exist group action λ and ρ such that the sequence

$$1 \longrightarrow SL_n(B) \xrightarrow{\varphi} GL_n(B) \xrightarrow{\psi} B^\times \longrightarrow 1,$$

is a strongly split short exact sequence. □

Corollary 4.10.1. *Let B be a band. If, for all $A \in GL_n(B)$, it holds that $a_{n+1,n+1} \in B^\times$, then there is a bijection between the sets $GL_n(B) \to SL_n(B) \times B^\times$ and $R_{GL_n(B)} \to R_{SL_n(B)} \times R_{B^\times}$. Furthermore, if $GL_n(B)$ is finite, then so are $SL_n(B)$ and $B^\times$ and $\#GL_n(B) = \#SL_n(B) \cdot \#B^\times$. If $R_{GL_n(B)}$ is finite, then so are $R_{SL_n(B)}$ and $R_{B^\times}$ and $\#R_{GL_n(B)} = \#R_{SL_n(B)} \cdot R_{B^\times}$. There is also a crowd morphism $\theta : B^\times \to Aut(SL_n(B))$ such that $GL_n(B) \cong SL_n(B) \cong B^\times$.*

Proof: This follows directly from Theorems 4.10, 4.5, 4.7 and 4.10. □

Example 4.11 (The sign hyperfield and the regular partial field). Theorem 2.10 states that, for all $A \in GL_n(\mathbb{S})$, $a_{n+1,n+1} = \pm 1 \in \mathbb{S}^\times$. Therefore, it follows from Proposition 4.10 that there exist crowd morphisms φ and ψ, and a left group action λ and a right group action ρ such that

$$1 \longrightarrow SL_n(\mathbb{S}) \xrightarrow{\varphi} GL_n(\mathbb{S}) \xrightarrow{\psi} \mathbb{S}^\times \longrightarrow 1,$$

together with these group actions, is a strongly split short exact sequence. The proof shows that possible group actions are $\lambda : S^\times \times GL_n(\mathbb{S})$, that takes a matrix A and mutliplies the first row and $a_{n+1,n+1}$ by ± 1, and $\rho : GL_n(\mathbb{S}) \times \mathbb{S}^\times$, that takes a matrix A and multiplies the first column and $a_{n+1,n+1}$ by ± 1. Recall that it followed from Theorems 3.5 and 3.6, that we can find all elements of $GL_n(\mathbb{S})$ and $R_{GL_n(\mathbb{S})}$ from the crowd $SL_n(\mathbb{S})$, by multiplying a row or column by -1. This is very similar to these group actions. It turns out other possible groups actions λ and ρ to make the sequence strongly split short exact sequence, are those that multiply another row or column and $a_{n+1,n+1}$ by -1, or that exchange 2 rows or columns and multiply $a_{n+1,n+1}$ by -1. Note that they should affect the same rows or columns, so if λ affects the fifth row, then ρ should affect the fifth column.
Since the sequence is strongly split, there exist bijective maps on the sets $GL_n(\mathbb{S}) \to SL_n(\mathbb{S}) \times \mathbb{S}^\times$ and $R_{GL_n(\mathbb{S})} \to R_{SL_n(\mathbb{S})} \times R_{\mathbb{S}^\times}$. Since $\mathbb{S}^\times = \{1, -1\}$, and

$$R_{\mathbb{S}^\times} = \{(1, 1, 1), (-1, -1, 1), (1, -1, -1), (-1, 1, -1)\},$$

it follows that $GL_n(\mathbb{S})$ is finite if and only if $SL_n(\mathbb{S})$ is finite, in which case $\#GL_n(\mathbb{S}) = 2 \cdot SL_n(\mathbb{S})$, and $R_{GL_n(\mathbb{S})}$ is finite if and only if $R_{SL_n(\mathbb{S})}$ is finite, in which would imply $\#R_{GL_n(\mathbb{S})} = 4 \cdot R_{SL_n(\mathbb{S})}$.
Similarly, for all $A \in GL_n(\mathbb{F}_1^\pm)$, we have that $a_{n+1,n+1} = \pm 1 \in (\mathbb{F}_1^\pm)^\times$ (see Theorem 2.13). Additionally, $(\mathbb{F}_1^\pm)^\times = \{1, -1\}$ and $R_{(\mathbb{F}_1^\pm)^\times} = \{(1, 1, 1), (-1, -1, 1), (1, -1, -1), (-1, 1, -1)\}$. Therefore there is a bijection $GL_n(\mathbb{F}_1^\pm) \to SL_n(\mathbb{F}_1^\pm) \times (\mathbb{F}_1^\pm)^\times$, and $GL_n(\mathbb{F}_1^\pm)$ is finite if and only if $SL_n(\mathbb{F}_1^\pm)$ is finite, in which case $\#GL_n(\mathbb{F}_1^\pm) = 2 \cdot \#SL_n(\mathbb{F}_1^\pm)$. There is also a bijection for the crowd laws, $R_{GL_n(\mathbb{F}_1^\pm)} \to R_{SL_n(\mathbb{F}_1^\pm)} \times R_{(\mathbb{F}_1^\pm)^\times}$, which implies $R_{GL_n(\mathbb{F}_1^\pm)}$ is finite if and only if $R_{SL_n(\mathbb{F}_1^\pm)}$ is finite, in which case $\#R_{GL_n(\mathbb{F}_1^\pm)} = 4 \cdot \#R_{SL_n(\mathbb{F}_1^\pm)}$.

However, not every weakly split short exact sequence gives way to a strongly split short exact sequence. An example of this is a weakly split short exact sequence, where the first crowd morphism is injective, but not full.

5 Saturated crowds

In this section, we discuss some additional findings regarding a special type of crowd, namely saturated crowds. These were introduced by Maxson in his thesis ([9, Chapter 1.3]), and are defined as follows.

Definition 5.1. A *saturated crowd* is a crowd G, where $a^{-1} \subset b \cdot c$, $b^{-1} \subset c \cdot a$ and $c^{-1} \subset a \cdot b$, implies $(a, b, c) \in R_G$ for all $a, b, c \in G$.

Maxson follows this by showing that $SL_n(\mathbb{K})$ is in fact a saturated crowd ([9, Theorem 3.8]). Using Theorem 1.16 and the crowd isomorphism $\varphi : SL_n(\mathbb{K}) \to GL_n(\mathbb{K}),\ A \mapsto \left[\begin{smallmatrix} A & \mathbf{0} \\ \mathbf{0} & 1 \end{smallmatrix}\right]$ from Section 3.1, we can proof $GL_n(\mathbb{K})$ is saturated as well.

Theorem 5.2. *$GL_n(\mathbb{K})$ is a saturated crowd.*

Proof: Suppose $A, B, C \in GL_n(\mathbb{K})$ such that $A^{-1} \subset B \cdot C$, $B^{-1} \subset C \cdot A$ and $C^{-1} \subset A \cdot B$. Then, for all $D \in A^{-1}$, it follows that $\varphi^{-1}(D) \in (\varphi^{-1}(A))^{-1}$ and $\varphi^{-1}(D) \in \varphi^{-1}(B) \cdot \varphi^{-1}(C)$. We can conclude that $(\varphi^{-1}(A))^{-1} \subset \varphi^{-1}(B) \cdot \varphi^{-1}(C)$, and therefore it follows from symmetry that $(\varphi^{-1}(B))^{-1} \subset \varphi^{-1}(C) \cdot \varphi^{-1}(A)$ and $(\varphi^{-1}(C))^{-1} \subset \varphi^{-1}(A) \cdot \varphi^{-1}(B)$. Since $SL_n(\mathbb{K})$ is a saturated crowd, this implies $(\varphi^{-1}(A), \varphi^{-1}(B), \varphi^{-1}(C)) \in R_{SL_n(\mathbb{K})}$, which means $(A, B, C) \in R_{GL_n(\mathbb{K})}$. Therefore, for all $A, B, C \in GL_n(\mathbb{K})$, it holds that if $A^{-1} \subset B \cdot C, B^{-1} \subset C \cdot A$ and $C^{-1} \subset A \cdot B$, then $(A, B, C) \in R_{GL_n(\mathbb{K})}$, so $GL_n(\mathbb{K})$ is a saturated crowd. □

Other saturated crowds are hypergroups. These structures were originally introduced by Marty in 1934 ([8]). We use a slightly altered definition, which was given by Nakamura and Reyes ([10, Chapter 2]), that includes the existence of a unit element.

Definition 5.3. Let G be a set. A *hyperoperation* on G is a map $\odot : G \times G \to \mathcal{P}(G)$, where the $\mathcal{P}(G)$ is the set of all subsets of G. If $A, B \subset G$, then $A \odot B = \bigcup_{a \in A, b \in B} a \odot b$

Notation. Given an element $a \in G$ and a subset $B \subset G$, we write $a \odot B$ and $B \odot a$ instead of $\{a\} \odot B$ and $B \odot \{a\}$. So $a \odot B = \bigcup_{b \in B} a \odot b$ and $B \odot a = \bigcup_{b \in B} b \odot a$.

Definition 5.4. A *hypergroup* is a triple $(G, \odot, 1_G)$, where G is a set, $1_G \in G$ is the unit element and $\odot : G \times G \to \mathcal{P}(G)$ is a hyperoperation, such that

1. for all $x, y, z \in G$, $(x \odot y) \odot z = x \odot (y \odot z)$ (associativity);
2. for all $x, y \in G$, $x \odot y \neq \emptyset$;
3. for all $x \in G$, $\{x\} = x \odot e = e \odot x$ (unit element);
4. for all $x, y, z \in G$, $x \in y \odot z \Rightarrow y \in x \odot z^{-1}$ and $z \in y^{-1} \odot x$ (reversibility).

Lemma 5.5. *If $(G, \odot, 1_G)$ is a hypergroup, then*

1. *for all $x, y, z \in G$, $x \in y \odot z \iff y \in x \odot z^{-1} \iff z \in y^{-1} \odot x$ and*
2. *for all $x \in G$ there is a unique $y \in G$ such that $e \in x \odot y \cap y \odot x$. We write $y = x^{-1}$.*

Proof: This follows from the reversibility axiom, and the existence of a unit element. For a more detailed explanation, see [10, page 415]. □

A hypergroup has a corresponding crowd. Recall that the crowd law is supposed to be the set of all triples (a, b, c) such that abc is equal to the unit element. Therefore, for a hypergroup G, it makes sense to use the crowd law $R = \{(a, b, c) \in G \mid 1_G \in a \odot b \odot c\}$. Note that $a \odot (b \odot c) = (a \odot b) \odot c$, and since inverses are unique, $1_G \in (a \odot b) \odot c$ if and only if $c^{-1} \in a \odot b$. Therefore, the crowd of a hypergroup is defined as follows.

Definition 5.6. Let $(G, \odot, 1_G)$ be a hypergroup. Then the corresponding crowd is $(G, 1_G, R_G)$, with the crowd law

$$R_G = \{(a, b, c) \in G^3 \mid c^{-1} \in a \odot b\}.$$

However, Lemma 5.5 implies the following corollary.

Corollary 5.6.1. *Let G by a hypergroup. Then, for all $a, b, c \in G$, we have that*

$$c^{-1} \in a \odot b \iff b^{-1} \in c \odot a \iff a^{-1} \in b \odot c.$$

Proof: Let G be a hypergroup and $a, b, c \in G$ be arbitrary. Then

$$\begin{aligned} c^{-1} \in a \odot b &\iff b \in a^{-1} \odot c^{-1} \iff a^{-1} \in b \odot c \\ &\iff c \in b^{-1} \odot a^{-1} \iff b^{-1} \in c \odot a. \end{aligned}$$

Since $a, b, c \in G$ were chosen arbitrary, this holds for all $a, b, c \in G$. □

Therefore, equivalent definitions of the crowd law of a hypergroup G are

$$R_G = \{(a, b, c) \in G^3 \mid b^{-1} \in c \odot a\} \text{ and}$$
$$R_G = \{(a, b, c) \in G^3 \mid a^{-1} \in b \odot c\}.$$

Theorem 5.7. *Let $(G, \odot, 1_G)$ by a hypergroup. Then the corresponding crowd $(G, 1_G, R_G)$ is a saturated crowd.*

Proof: We need to check that G satisfies the 4 crowd axioms, and that it is saturated.

1. Since 1_G is the unit element, we know that $1_G \odot 1_G = \{1_G\}$. Therefore $1_G \in (1_G \odot 1_G) \cap (1_G \odot 1_G)$, which means $1_G^{-1} = 1_G$. This implies $1_G^{-1} \in 1_G \odot 1_G$, so $(1_G, 1_G, 1_G) \in R_G$ and the first axiom is satisfied.

2. Suppose $(a, 1_G, 1_G) \in R_G$ for some $a \in G$. Then $1_G^{-1} = 1_G \in a \odot 1_G = \{a\}$, so $a = 1_G$. Therefore the second axiom is satisfied.

3. Suppose $(a, b, 1_G) \in R_G$ for some $a, b, \in G$. Then $1_G^{-1} \in a \odot b$, so it follows from Corollary 5.6.1 that $b^{-1} \in 1_G \odot a = a \odot 1_G$. By applying the corollary again, we find that $1_G^{-1} \in b \odot a$, which means $(b, a, 1_G) \in R_G$. Therefore $(a, b, 1_G) \in R_G$ implies $(b, a, 1_G) \in R_G$, so the third axiom is satisfied.

4. Suppose $(a, b, c) \in R_G$ for some $a, b, c \in G$. Then $c^{-1} \in a \odot b$, so it follows from Corollary 5.6.1 that $b^{-1} \in c \odot a$, which implies $(c, a, b) \in R_G$. Therefore the fourth axiom is satisfied.

So $(G, 1_G, R_G)$ is indeed a crowd. To proof that it is saturated, let $a, b, c \in G$ and suppose that $a^{-1} \subset b \cdot c$, $b^{-1} \subset c \cdot a$ and $c^{-1} \subset a \cdot b$. Then, for all $f \in c^{-1}$, there is an $e \in G$ such that $(e, f, 1_G) \in R_G$ and $(a, b, e) \in R_G$. This implies

$$1_G \in c \odot f \Rightarrow f \in c^{-1} \odot 1_G = \{c^{-1}\} \Rightarrow f = c^{-1} \text{ and}$$
$$1_G \in e \odot f \Rightarrow f \in e^{-1} \odot 1_G = \{e^{-1}\} \Rightarrow f = e^{-1}.$$

It follows that $e^{-1} = c^{-1}$, so

$$\begin{aligned} (a, b, e) \in R_G &\Rightarrow e^{-1} \in a \odot b \\ &\Rightarrow c^{-1} \in a \odot b \\ &\Rightarrow (a, b, c) \in R_G. \end{aligned}$$

Therefore, if $a^{-1} \subset b \cdot c$, $b^{-1} \subset c \cdot a$ and $c^{-1} \subset a \cdot b$, then $(a, b, c) \in R_G$, which means G is a saturated crowd. □

Note that, in the proof above, the condition $c^{-1} \subset a \odot b$ was enough to proof that $(a, b, c) \in R_G$. Similarly, one can show that if $b^{-1} \subset c \cdot a$ or $c^{-1} \subset a \cdot b$, then $(a, b, c) \in R_G$.

6 Conclusion

With crowds, we can study linear algebraic groups, such as the special and general linear group, over bands. This leads to interesting results, including the theorem that the special linear group over a band is always isomorphic to a subcrowd of the general linear group over that band (Theorem 3.3). It turns out that there can be an even stronger connection between the linear groups. By defining and applying a crowd version of the split short exact sequence, we see that if B is a band and $n \in \mathbb{N}$ such that for all $A \in GL_n(B)$, the coefficient $a_{n+1,n+1} \in B^\times$, then there is a bijection between the sets $SL_n(B) \times B^\times \to GL_n(B)$ and the crowd laws $R_{SL_n(B)} \times R_{B^\times} \to R_{GL_n(B)}$ (Corollary 4.10.1).

An open question left by this thesis is whether this condition on the matrices in the general linear group holds for all bands and dimensions. If it does not, are there other sequences for these bands, that include $SL_n(B)$ and $GL_n(B)$, and for which there exist group actions such that it is a strongly split short exact sequence? Another unanswered question is whether it is necessary to define $Sp_{2n}(B)$ as a subset of $SL_{2n}(B)$, or if this follows from the rest of the definition (as it does for groups).
Further research could also include other (linear) algebraic crowds, such as the projective linear group $PGL_n(B)$ over a band B.

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