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Coherent States and Unitary Irreducible Representations of $SU(1, 1)$

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Student: Mert Yaroğlu
Physics supervisor: Prof. Dr. Daniël Boer
Mathematics supervisor: Prof. Dr. Holger Waalkens

Abstract: In this report, the transformation of coherent states under the group $SU(1,1)$ is investigated. The algebraic, geometric, and topological properties of $SU(1,1)$ are analyzed. These properties are used to construct the unitary irreducible representations of the group. For the discrete representations, the generalized coherent states are defined and examined. Finally, two methods are suggested to answer the question of the transformation of canonical coherent states of the quantum harmonic oscillator in $1+1$ dimensions.

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1 Introduction

The canonical coherent states (CCS) of the quantum harmonic oscillator are the most classical states of the system. Their dynamics and time evolution resemble those of a classical harmonic oscillator. They are also the minimal uncertainty states, meaning that the position and momentum uncertainties of these states assume the equality of Heisenberg's uncertainty principle. As such, they have many interesting properties and a wide range of applications, both in theoretical and experimental physics.

One of the most fruitful ways to investigate a physical system is to look at its symmetries. The symmetry of a system can tell us what is allowed and what is not allowed, what the conservation laws that govern the system are, and so on. Therefore, determining the symmetries of a physical system is a useful tool for gaining more knowledge about the system.

Once we determine the symmetry of a system, we can also ask how objects that are part of the system transform under the symmetry of the system. To make this point a little clearer, let us give some well-known examples from physics. For instance, in quantum field theory, we want our theory to be Lorentz invariant. Therefore, a natural symmetry of the system that we are working with (the Minkowski spacetime) is the Lorentz symmetry, which is denoted by the Lorentz group.

Consider a set of four-vectors that live in Minkowski space. In one inertial reference frame, they may look in a certain way, and in a different inertial reference frame, they may look different. Therefore, they transform between one inertial reference frame and the other, and since Lorentz transformations are the elements of the Lorentz group, we say that the four-vectors transform under a certain element of the group. This is what we mean when we say that x transforms as y under the group G.

Since the way certain objects transform under a group is a quite common and important theme in physics, we want to do the same thing for the coherent states. We want to investigate the transformation of the canonical coherent states under the symmetry group of the harmonic oscillator system in one space dimension. After introducing the coherent states and establishing that the symmetry group of the system is $SU(1,1)$, we have three goals in this report. Firstly, to examine the transformations of the coherent states, we need to come up with the unitary irreducible representations of the group $SU(1,1)$. We will classify the possible representations of this group and then realize them explicitly. Our second goal is to find a suitable generalization of the canonical coherent states of the harmonic oscillator for more general systems that share the same symmetry and apply it to our case for the discrete representations of $SU(1,1)$. Finally, by using the results of the first two goals, we want to find explicit descriptions of the transformation of the canonical coherent states under the discrete representations. The first two goals are achieved in this report. For the third goal, we also came up with an answer, but more work needs to be done in order to obtain more direct results.

Our plan for this report is as follows. After introducing the canonical coherent states in Chapter 2, we will first establish the symmetry group of our system and show that this is isomorphic to another group called $SU(1,1)$. Hence, we shall work towards the representations of the group. But some properties of the group directly affect the possible representations of the group, so in Chapter 3 we will look at the algebraic, geometric, and topological properties of the group $SU(1,1)$. Then, we want to find the irreducible representations of the group to see how the coherent states transform under the group. Moreover, we want to apply this to quantum systems, so we want a normalized vector to stay normalized under the transformation. Hence, we are interested in the unitary irreducible representations of the group. In Chapter 4, we will classify and realize all the unitary irreducible representations (irreps) of our group. In Chapter 5, we will introduce a generalization of the CCS called generalized coherent states (GCS) and investigate the GCS of our group $SU(1,1)$ under a certain unitary irrep of the group. In doing so, we will develop the necessary tools to answer the question of how the generalized coherent states transform under the representations of the group. Finally, in Chapter 6, we will return to the original question and ask how CCS transform under the group $SU(1,1)$. We will answer this question by establishing a correspondence between generalized and canonical coherent states. We will conclude the report by suggesting a different representation of the group that would be helpful to investigate the transformation of canonical coherent states in a more direct way.

Before starting the technical discussions, we would like to give some motivation for studying the subjects in this report, both from a mathematical and a physical point of view.

1.1 Motivation for Physics

In experimental and theoretical physics, canonical coherent states play an important role in quantum optics and laser physics [2, p. 107]. They bridge the gap between the quantum theory of light and the classical theory of light [10, p. 47]. Because they resemble the classical states, they have many nice properties that one can exploit.

In theoretical and mathematical physics, coherent states were used in the investigation of the following problems: condensation phenomena in a system of interacting bosons, the existence of a classical limit for quantum mechanical correlation functions in a class of field theories, proof of the virial theorem for liquid helium, obtaining a quasiclassical description of solitons in nonlinear field theories, and many other examples [14, p. 714].

As their name suggests, generalized coherent states are generalizations of the canonical coherent states of the harmonic oscillator. They naturally arise in physics problems with dynamical symmetries. Generalized coherent states allow for the simplification of the solution of a quantum problem by reducing it to a simpler, classical problem [14, p. 703]. For example, the generalized coherent states for the discrete series representation of the group $SU(1,1)$ have applications in the problems of parametric excitation of a quantum oscillator and the superfluidity of an almost-ideal Bose gas [14, pp. 717, 718].

The numerous examples above clearly demonstrate that both canonical and generalized coherent states have many applications in diverse areas of physics. Therefore, a better understanding of these states may lead to new insights in a variety of theoretical and experimental research areas. Transformation of coherent states is a very fundamental property that one can investigate to gain a better understanding of these states. However, their transformations are far from trivial, so it is quite desirable to obtain explicit results about their transformations.

1.2 Motivation for Mathematics

We introduce and extensively use Lie groups and Lie algebras. Lie groups have direct applications in a number of areas in mathematics, including but not limited to analysis, algebraic topology, algebraic geometry, differential geometry, number theory, and general topology. Another important application of Lie groups is mathematical physics, in particular topics like gauge theory, particle phenomenology, and general relativity. They act like a bridge between differential geometry and group theory, which is evident even from the definitions.

Due to their importance in modern mathematics, we wanted to study a subject that includes a significant amount of Lie theory. It is one of the best topics that one can study that is mathematically deep and rigorous with plenty of applications in physics.

Another topic on which this thesis is heavily based is representation theory. Representation theory has applications in algebra, number theory, topology, discrete mathematics, physics, and computer science. Especially the representation theory of non-compact Lie groups uses a lot of methods from different branches of mathematics and brings them together, which is also visible in this thesis: we use algebra, geometry, topology, complex analysis, and functional analysis to come up with certain representations of a non-compact Lie group.

One of the simplest non-compact, simple Lie groups is $SL(2,\mathbb{R})$. Due to this, any set of lecture notes written about the representation theory of Lie groups includes some discussion on the representations of $SL(2,\mathbb{R})$; for instance, see [9] in which there is an entire chapter on the unitary representations of $SL(2,\mathbb{R})$. This is essentially what we are doing in this report for the unitary irreducible representations of the group, after we identify its isomorphism with the group $SU(1,1)$. Because the group is non-compact, the unitary representations of the group except the trivial representations are all infinite dimensional, which makes the task of finding such representations highly nontrivial.

As a result, classifying unitary irreducible representations of the group $SU(1,1)$ is very applicable to a wide range of problems in representation theory.

2 Background Material

2.1 Prerequisites

In this thesis, we assume some background knowledge that is acquired by the physics and mathematics courses that are taught at the Bachelor's degree of the University of Groningen. Especially some familiarity with nonrelativistic quantum mechanics and basics of group theory (definition of groups, group homomorphisms, definition of representations, reducible/irreducible representations) is essential, and relevant definitions/theorems are used with very little or no explanation. Additional topics that are needed for the thesis include Hamiltonian mechanics, complex analysis, geometry, functional analysis, general topology, and theory of smooth manifolds. However, the later topics are used at a basic level and most of the report should still be accessible if some of the background knowledge is missing.

2.2 Quantum Harmonic Oscillator and Canonical Coherent States

Recall that the stationary states $|n\rangle$ of the quantum harmonic oscillator in $1+1$ dimensions have the position representations [11, p. 57]

$$|n\rangle = \langle x|n\rangle = \psi_n(x) = (\frac{m\omega}{\pi\hbar})^{1/4}\frac{1}{\sqrt{2^n n!}}H_n(\xi)e^{-\xi^2/2} \tag{2.1}$$

where m is the mass, ω is the angular frequency, $\hbar$ is Planck's constant, $\xi = \sqrt{\frac{m\omega}{\hbar}}x$ and $H_n(\xi)$ are the Hermite polynomials. Throughout this chapter, we will use $|n\rangle$ and ψ_n interchangeably to denote the n^{th} energy eigenstate of the harmonic oscillator.

When we calculate the position and momentum uncertainty of the stationary states, we find that their product depends on the state n: $\sigma_x\sigma_p = (2n+1)\hbar/2$. Hence, among the energy eigenstates of the system, only the ground state of $n = 0$ hits the uncertainty limit. One can ask if there are solutions to the harmonic oscillator system, other than the vacuum state, that also minimize the uncertainty product. In this section, we will define canonical coherent states (CCS) of the harmonic oscillator and show that they indeed minimize the uncertainty product. We will also investigate some of their properties.

Recall the raising and lowering operators of the harmonic oscillator: $a_\pm = \frac{1}{\sqrt{2\hbar m\omega}}(\mp ip + m\omega x)$ where x and p are the position and momentum operators, respectively. They have the property

$$a_-|n\rangle = \sqrt{n}|n-1\rangle, \quad a_+|n\rangle = \sqrt{n+1}|n+1\rangle \tag{2.2}$$

Furthermore, they are the Hermitian conjugates of each other: $(a_+)^* = a_-$. With these simple reminders, we are ready to give the definition of canonical coherent states.

Definition 2.1. The canonical coherent states of the quantum harmonic oscillator are the normalized eigenfunctions of the lowering operator: $a_-|\alpha\rangle = \alpha|\alpha\rangle$ with the eigenvalue α any complex number.

We now prove the most important property of CCS: they are minimum uncertainty states.

Theorem 2.2. *Let $|\alpha\rangle$ be a canonical coherent state. Then, $\sigma_x\sigma_p = \hbar/2$, where σ_o is the standard deviation of the operator o.*

Proof. We begin by calculating the expectation value of x, $\langle x\rangle = \langle\alpha|x|\alpha\rangle$. In terms of the raising and lowering operators, x is given by $x = \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-) := c(a_+ + a_-)$. Then, we get

$$\langle x\rangle = \langle\alpha|x|\alpha\rangle = \langle\alpha|ca_+ + ca_-|\alpha\rangle = c\langle a_-\alpha|\alpha\rangle + c\langle\alpha|\alpha|\alpha\rangle = c(\alpha + \overline{\alpha}) = \sqrt{\frac{\hbar}{2m\omega}}2\text{Re}(\alpha) \tag{2.3}$$

where $\overline{z}$ denotes the complex conjugate of z. With similar algebraic manipulations, we obtain

$$\langle x^2\rangle = \frac{\hbar}{2m\omega}(1 + 4\text{Re}^2(\alpha)), \quad \langle p\rangle = -i\sqrt{\frac{\hbar m\omega}{2}}2\text{Im}(\alpha), \quad \langle p^2\rangle = \frac{\hbar m\omega}{2}(1 - 4\text{Im}^2(\alpha)). \tag{2.4}$$

Using the expressions above, we get $\sigma_x^2 = \frac{\hbar}{2m\omega}$ and $\sigma_p^2 = \frac{\hbar m\omega}{2}$, so that $\sigma_x\sigma_p = \hbar/2$.

□

We would like to express the coherent states in the basis of the energy eigenstates. The next theorem provides an expression for this.

Theorem 2.3. *The canonical coherent state* $|\alpha\rangle$ *is given by* $|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle$ *in terms of the stationary states* $|n\rangle$ *of the harmonic oscillator.*

Proof. Because the energy eigenstates form a basis of the Hilbert space, we can write $|\alpha\rangle = \sum_{n=0}^{\infty} c_n|n\rangle$ for some coefficients c_n. For a fixed value of n, we can find this coefficient by calculating

$$c_n = \langle n|\alpha\rangle = \frac{1}{\sqrt{n!}}\langle(a_+)^n\psi_0|\alpha\rangle = \frac{1}{\sqrt{n!}}\langle\psi_0|(a_-)^n\alpha\rangle = \frac{a}{\sqrt{n!}}\alpha^n\langle\psi_0|\alpha\rangle = \frac{\alpha^n}{\sqrt{n!}}c_0. \tag{2.5}$$

We can determine c_0 by normalizing the coherent state. Observe that

$$\langle\alpha|\alpha\rangle = \sum_{n=0}|c_n|^2 = \sum_{n=0}\frac{|\alpha|^{2n}}{n!}|c_0|^2 = e^{|\alpha|^2}|c_0|^2 = 1. \tag{2.6}$$

Hence, we have $|c_0|^2 = e^{-|\alpha|^2}$, so we can choose $c_0 = e^{-|\alpha|^2/2}$. Putting everything together, we get

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle. \tag{2.7}$$

□

The canonical coherent states of the harmonic oscillator are usually said to be the most classical states, meaning that their dynamics resemble the oscillatory behavior of the classical harmonic oscillator the most. This is especially true in the field of quantum optics, where the canonical states play a crucial role because of their similarities to the classical states. For a great discussion of the similarities of the coherent states of light with the classical states of light, see Chapter 3 of [10]. On top of minimizing the uncertainty product, coherent states also stay coherent over time with their eigenvalues evolving in an oscillatory fashion, which is another reason why they resemble a set of classical states. The following theorem formalizes this [11, p. 129].

Theorem 2.4. *Adding the time dependence* $|n\rangle \to e^{-iE_nt/\hbar}|n\rangle$*, the time-dependent quantum state of the canonical coherent state* $|\alpha\rangle$*, denoted by* $|\alpha(t)\rangle$*, remains an eigenstate of* a_- *and the eigenvalue evolves as* $\alpha(t) = e^{-i\omega t}\alpha$ *over time.*

Proof. Recall that the energies of the stationary states are given by $E_n = (n + \frac{1}{2})\hbar\omega$. Hence, we obtain

$$|\alpha(t)\rangle = \sum_{n=0}\frac{\alpha^n}{\sqrt{n!}}e^{-|\alpha|^2/2}e^{-iE_nt/\hbar}|n\rangle = \sum_{n=0}\frac{\alpha^n}{\sqrt{n!}}e^{-|\alpha|^2/2}e^{-i(n+1/2)\omega t}|n\rangle = e^{-i\omega t/2}\sum_{n=0}\frac{(\alpha e^{-i\omega t})^n}{\sqrt{n!}}e^{-|\alpha|^2/2}|n\rangle. \tag{2.8}$$

Apart from the global phase factor, which is physically irrelevant, $|\alpha(t)\rangle$ is the same as $|\alpha\rangle$ with the eigenvalue $\alpha(t) = e^{-i\omega t}\alpha$.

□

Because their eigenvalues evolve over time, canonical coherent states are not energy eigenstates. However, their time evolution looks like one: $i\frac{d|\alpha(t)\rangle}{dt} = \omega|\alpha(t)\rangle$. This is one of the reasons why the time evolution of the canonical coherent states resembles the evolution of the classical harmonic states.

There is an alternative definition of the canonical coherent states of the harmonic oscillator in terms of the displacement operator. We start by defining this operator.

Definition 2.5. The displacement operator D is defined as the unitary operator $D(\alpha) = e^{\alpha a_+ - \overline{\alpha} a_-}$ with $\alpha \in \mathbb{C}$.

The displacement operator can also be expressed as follows.

Proposition 2.6. $D(\alpha) = e^{-|\alpha|^2/2} e^{\alpha a_+} e^{-\overline{\alpha} a_-}$

Proof. Let $A = \alpha a_+, B = -\overline{\alpha} a_-$. Then $[A, B] = |\alpha|^2$, which is a real number. Because real numbers commute with operators, we have $[A, [A, B]] = [B, [B, A]] = 0$. Using the Baker–Campbell–Hausdorff formula, we get

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\frac{1}{12}[A,[A,B]]+\frac{1}{12}[B,[B,A]]+\cdots} = e^{A+B+\frac{1}{2}|\alpha|^2} \tag{2.9}$$

so that $e^{A+B} = e^A e^B e^{-|\alpha|^2/2}$. This shows that $D(\alpha) = e^{-|\alpha|^2/2} e^{\alpha a_+} e^{-\overline{\alpha} a_-}$.

□

With this proposition at hand, we are ready to state the following theorem.

Theorem 2.7. *The coherent state $|\alpha\rangle$ is generated by the action of the displacement operator on the vacuum state: $|\alpha\rangle = D(\alpha)|0\rangle$.*

Proof. First, observe that we have $e^{-\overline{\alpha} a_-}|0\rangle = \sum_{l=0} \frac{(-\overline{\alpha} a_-)^l}{l!}|0\rangle = |0\rangle$ since $a_-^l|0\rangle = 0$ for all l except $l = 0$. We similarly compute

$$e^{\alpha a_+}|0\rangle = \sum_{n=0} \frac{\alpha^n}{n!}(a_+)^n|0\rangle = \sum_{n=0} \frac{\alpha^n}{\sqrt{n!}}|n\rangle \tag{2.10}$$

where we use the identity that $|n\rangle = \frac{1}{\sqrt{n!}}(a_+)^n|0\rangle$. Combining the equalities above, we obtain $|\alpha\rangle = D(\alpha)|0\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle$.

□

This theorem allows us to give an alternative definition: coherent states are generated by the action of $D(\alpha)$ on the vacuum state.

This theorem also shows why this operator is called the displacement operator. Eq. 2.4 shows that the expectation values for the operators x and p of the ground state $|0\rangle$ are both 0. Therefore, we can represent the ground state as the origin of a 2−dimensional phase space where the real axis represents the expectation value of x and the imaginary axis represents the expectation value of p with appropriate scaling factors. Again, the expressions in Eq. 2.4 show that the expectation values for x and p of the canonical coherent states with the eigenvalue α are also proportional to the real and imaginary parts of the complex number α. As a result, the displacement operator $D(\alpha)$ displaces the vacuum state from the origin of the phase space to the point α of the phase space, which corresponds to the canonical coherent state of $|\alpha\rangle$. For more details on the geometric interpretation of the canonical coherent states in phase space, see [10, Section 3.6].

Another interesting property of the coherent states is that they are not mutually orthogonal. In other words, if one takes two different coherent states, their inner product will be nonzero. We demonstrate this in the next proposition, where we see the value of the alternative definition in terms of the displacement operators [2, p. 114].

Proposition 2.8. *Coherent states do not form an orthogonal set. In particular, the inner product of two coherent states is given by $\langle\beta|\alpha\rangle = e^{-\frac{1}{2}(|\alpha|^2+|\beta|^2)+\alpha\overline{\beta}}$.*

Proof. Using Theorem 2.7 and Proposition 2.6, we calculate

$$\langle\beta|\alpha\rangle = \langle 0|D^\dagger(\beta)D(\alpha)|0\rangle = \langle 0|e^{-\beta a_+}e^{-\overline{\beta}a_-}e^{\alpha a_+}e^{-\overline{\alpha}a_-}|0\rangle e^{-\frac{1}{2}(|\alpha|^2+|\beta|^2)}. \tag{2.11}$$

Expanding the Taylor series for the exponential function, we can use our results from the proof of Theorem. 2.7 to get

$$\langle 0|(1+\overline{\beta}a_- + \cdots)(1+\alpha a_+ + \cdots)|0\rangle = (\cdots + \langle 1|\overline{\beta} + \langle 0|)(|0\rangle + \alpha|1\rangle + \cdots). \tag{2.12}$$

Finally, using the orthogonality of the stationary states, we get

$$\langle\beta|\alpha\rangle = e^{-\frac{1}{2}(|\alpha|^2+|\beta|^2)}(1+\alpha\overline{\beta} + \frac{1}{2!}(\alpha\overline{\beta})^2 + \cdots) = e^{-\frac{1}{2}(|\alpha|^2+|\beta|^2)+\alpha\overline{\beta}}. \tag{2.13}$$

□

Although coherent states are not orthogonal, they still have the following completeness relation, which is also called a resolution of the identity [2, p. 115].

Theorem 2.9. *The completeness relation of the coherent states are given by* $\frac{1}{\pi}\int d^2\alpha|\alpha\rangle\langle\alpha| = \mathbb{I}$, *where* $d^2\alpha$ *represents the area element* $dxdy$ *with* $z = x + ip$, *and the integration is evaluated over the entire complex plane* $\mathbb{C}$.

Proof. Using the expression of coherent states in terms of the energy eigenstates, we obtain

$$\frac{1}{\pi}\int d^2\alpha|\alpha\rangle\langle\alpha| = \frac{1}{\pi}\sum_{n,m\geq 0}\frac{1}{\sqrt{n!m!}}|n\rangle\langle m|\int d^2\alpha e^{-|\alpha|^2}\alpha^n(\overline{\alpha})^m. \tag{2.14}$$

Let $\alpha = re^{i\varphi}$. The integral on the right-hand side can be evaluated as

$$\int d^2\alpha e^{-|\alpha|^2}\alpha^n(\overline{\alpha})^m = \int_0^\infty rdre^{-r^2}r^{n+m}\int_0^{2\pi} d\varphi e^{i(n-m)\varphi} = 2\pi\int_0^\infty rdre^{-r^2}r^{n+m}\delta_{nm} \tag{2.15}$$

where δ_{nm} is the Kronecker delta. Moreover, with the change of variables of $r^2 = t, 2rdr = dt$, we can evaluate

$$2\int_0^\infty rdre^{-r^2}r^{2n} = \int_0^\infty e^{-t}t^n dt = \Gamma(n+1) = n! \tag{2.16}$$

to recover the definition of the gamma function. Combining everything, we obtain

$$\frac{1}{\pi}\int d^2\alpha|\alpha\rangle\langle\alpha| = \frac{1}{\pi}\sum_n\frac{1}{n!}|n\rangle\langle n|\pi n! = \sum_n|n\rangle\langle n| = \mathbb{I}. \tag{2.17}$$

□

Using the last two statements, we see that the coherent states form an overcomplete set. This means that any coherent state can be expanded in terms of the other coherent states, and hence they are not linearly independent. Combining Theorem 2.9 and Proposition 2.8, we get the following expression of a coherent state as a continuous linear combination of other coherent states:

$$|\beta\rangle = \frac{1}{\pi}\int d^2\alpha|\alpha\rangle\langle\alpha|\beta\rangle = \frac{1}{\pi}\int d^2\alpha|\alpha\rangle e^{-\frac{1}{2}(|\alpha|^2+|\beta|^2)+\alpha\overline{\beta}}. \tag{2.18}$$

2.3 Description of the Problem

In the last section, we defined the canonical coherent states of the harmonic oscillator and looked at some of their properties. A very fundamental question that one can ask is the following: How do the canonical coherent states transform under the symmetry of the harmonic oscillator system?

But to ask this question meaningfully, we should first answer a simpler yet crucial question: What is the symmetry of the harmonic oscillator system? In this section, we will answer this question, and once we find the answer, we will tackle the more challenging question about the transformation of the coherent states under this symmetry of the system.

The mechanical system of the quantum harmonic oscillator can be analyzed using Hamiltonian mechanics. Indeed, when we analyze the harmonic oscillator, we start with the Hamiltonian of the system and solve the Schrödinger equation using this Hamiltonian. Although the idea of generalized coordinates that is familiar in Hamiltonian mechanics does not seem to appear when we are working in quantum mechanics, this is not the fault of quantum mechanics but rather the picture that we usually work with. Indeed, in the Schrödinger picture of quantum mechanics, the wavefunctions are usually expressed in the position basis, and one can take the Fourier transform to move to the momentum basis. This hints at the interconnectedness of position and momentum while describing the system. In fact, there are alternative formulations of quantum mechanics where the quantum states depend both on position and momentum in a much more explicit way. An example of this formalism is called the Wigner functions, introduced by Eugene Wigner. For an introduction to Wigner functions, see [12] and [4]. As a simple example, the Wigner function of the ground state of the harmonic oscillator is given by

$$W_0(x,p) = \frac{1}{\pi\hbar}\exp(-a^2p^2/\hbar^2 - x^2a^2) \tag{2.19}$$

where $a^2 = \frac{\hbar}{m\omega}$.

In this view, the harmonic oscillator is also a Hamiltonian system, and we will look for the symmetry of the phase space. Although the following arguments are based on the Hamiltonian formalism of classical mechanics, for dynamical systems with quadratic potentials such as the quantum harmonic oscillator, the obtained results also correspond to the symmetries of quantum systems.

Recall the Hamilton's equations of motion in two-dimensional phase space:

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q} \tag{2.20}$$

where H is the Hamiltonian of the system, and q and p are the generalized position and momentum, respectively. Defining $\eta = (q,p)$, we can rewrite Eq. 2.20 as

$$\dot{\eta} = J\nabla_\eta H \tag{2.21}$$

where $\dot{\eta}$ is the time derivative of η, J is the matrix $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, and $\nabla_\eta = \left(\begin{smallmatrix} \frac{\partial}{\partial q} \\ \frac{\partial}{\partial p} \end{smallmatrix}\right)$ is the derivative with respect to η coordinates.

Consider now a differentiable function $F(\eta) = F\left(\begin{smallmatrix} q \\ p \end{smallmatrix}\right) = \left(\begin{smallmatrix} f(q,p) \\ g(q,p) \end{smallmatrix}\right) = \left(\begin{smallmatrix} q' \\ p' \end{smallmatrix}\right) = \varepsilon$ that preserves the form of Hamilton's equations. Such a coordinate transformation is called a canonical transformation. Denote the Jacobian of this function as $M := DF(\eta) = \left(\begin{smallmatrix} \frac{\partial f}{\partial q} & \frac{\partial f}{\partial p} \\ \frac{\partial g}{\partial q} & \frac{\partial g}{\partial p} \end{smallmatrix}\right)$. Under this change of coordinates, the partial derivatives transform as $\frac{\partial}{\partial q} = \frac{\partial f}{\partial q}\frac{\partial}{\partial q'} + \frac{\partial g}{\partial q}\frac{\partial}{\partial p'}$, $\frac{\partial}{\partial q} = \frac{\partial f}{\partial q}\frac{\partial}{\partial q'} + \frac{\partial g}{\partial q}\frac{\partial}{\partial p'}$. Observe that this can be written compactly as $\nabla_\eta H = M^T\nabla_\varepsilon H$ where we act on the Hamiltonian, so that

$$\dot{\eta} = J\nabla_\eta H = JM^T\nabla_\varepsilon H. \tag{2.22}$$

It also follows by the chain rule that $\dot{\varepsilon} = M\dot{\eta}$. Hence, we have the equation

$$\dot{\varepsilon} = J\nabla_\varepsilon H = M\dot{\eta} = MJM^T\nabla_\varepsilon H \tag{2.23}$$

which shows that $MJM^T = J$. As a result, the Jacobian matrices of all the canonical transformations satisfy the condition above, called the symplectic condition. These matrices form a group, called the symplectic group in dimension 2, denoted by $Sp(2,\mathbb{R})$. Of course, it is trivial to extend this to higher-dimensional phase spaces to obtain the symplectic group of $Sp(2N,\mathbb{R})$ where N is the dimension of the physical system.

Let us denote $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, then the symplectic condition gives

$$MJM^T = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & -bc+ad \\ -ad+bc & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) = J. \tag{2.24}$$

Hence, the group $Sp(2,\mathbb{R})$ is given by

$$Sp(2,\mathbb{R}) = \{M = \left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right) : \det M = 1, \quad a,b,c,d \in \mathbb{R}\}. \tag{2.25}$$

We can observe that this is also the definition of the group $SL(2,\mathbb{R})$, so that we automatically get $Sp(2,\mathbb{R}) \cong SL(2,\mathbb{R})$ as groups. In Chapter 3 we will define another group called $SU(1,1)$ that is also isomorphic to $Sp(2,\mathbb{R})$, and use this group for the rest of the report.

We are ready to define the problem that we will investigate in this report. Given the coherent states of the quantum harmonic oscillator, how do these states transform under the group $SU(1,1)$ which is a natural symmetry of the system? Is it possible to generalize the idea of coherent states and see how they transform under the group $SU(1,1)$? In the remainder of the report, we will develop tools to answer these questions.

In order to answer the question "How does x transform under the group G?" we need to use representation theory. Therefore, we need to find representations of the group $SU(1,1)$ that will act on the coherent states. Furthermore, the representations of finite or semi-simple Lie groups (see Definition 3.4) can be written as a direct sum of irreducible representations. Hence, to investigate the representations of the group, we should find the irreducible representations of $SU(1,1)$. Moreover, we want to act on the coherent states of the harmonic oscillator with these irreducible representations, which are special quantum states. In quantum mechanics, physical systems have a norm of one because of their statistical interpretation. Hence, after we transform a coherent state, we would like the transformed state to be also normalized. As a result, we would like to have a unitary representation that would preserve the inner product of the underlying Hilbert space. Putting everything together, we are after the unitary irreducible representations of the group $SU(1,1)$ to investigate the transformation of coherent states. Although this seems like a rather technical request, it naturally follows from all the conditions that we want to satisfy.

2.4 Lie Groups, Lie Algebras, Representations

In order to answer the questions we discussed above, we need tools from the branch of mathematics called Lie theory. It is a vast topic with many applications in mathematics and theoretical physics. In this section, we will very briefly introduce most of the notions that we need for the rest of the report related to Lie groups and Lie algebras. Some definitions will be introduced later when they are needed.

We start with the definition of a Lie group [16, p. 151].

Definition 2.10. A real Lie group G is a smooth manifold that is also an algebraic group, with the property that the multiplication map $\mu : G \times G \to G, \mu : (g,h) \mapsto gh$, and the inversion map $\iota : G \to G, \iota : g \mapsto g^{-1}$ are smooth.

In both physics and mathematics, Lie groups are analyzed mostly from an algebraic perspective. This motivates us to give an alternative working definition of a Lie group, which is not perfectly rigorous but works for most cases reasonably well: "A Lie group is a group whose elements depend on a set of parameters analytically, i.e. $g = g(x_1, \cdots, x_n)$ with the number of parameters denoting the dimension of the Lie group."

Next, we give the definition of a Lie algebra. Initially, it looks completely unrelated to the definition of a Lie group, but as we will see, they are intimately related to one another.

Definition 2.11. A real Lie algebra L of dimension n is a real n-dimensional vector space endowed with a product $[a, b]$ called the Lie product, with the properties

1. $[a, b] \in L, \quad \forall a, b \in L$
2. $[\alpha a + \beta b, c] = \alpha[a, c] + \beta[b, c], \quad \forall a, b, c \in L, \quad \forall \alpha, \beta \in \mathbb{R}$
3. $[a, b] = -[b, a]$
4. $[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0, \quad \forall a, b, c \in L$

where the last property is known as the Jacobi identity.

For the remainder of the report, we will be working with the commutator $[A, B] = AB - BA$ as the Lie product.

How is the Lie algebra related to the Lie group? The following theorem answers this question.

Theorem 2.12. *Let G be a Lie group. The tangent space to G at the identity element e is a Lie algebra in the sense of Definition 2.11. It is usually denoted by $T_eG := g$ (name of the group written in italic and lowercase letters, e.g. the group is $SU(1,1)$ and its Lie algebra is su(1,1)).*

We will not prove this theorem, but the interested reader can see a proof, for instance, in Appendix A of [16]. Since the Lie algebra of a Lie group is the tangent space at the identity, we can find the elements of the Lie algebra by defining smooth curves on the Lie group and looking at the vector space spanned by the derivatives of these smooth curves. Because we can see the Lie group as a group whose elements analytically depend on a set of parameters, we can generate the elements of the Lie algebra by differentiating the group with respect to different parameters at $t = 0$. Hence, one should choose a parameterization $g = g(x_1, \cdots, x_n)$ such that $g(x_1(t), \cdots, x_n(t))|_{t=0} = e$. Some examples will be given in the following chapter.

Because the Lie algebra L is a vector space of dimension n, we can fix a basis $\{a_1, \cdots, a_n\}$. Consider the Lie products $[a_p, a_q]$. Since they are also elements of the Lie algebra, they can be written as linear combinations of basis elements:

$$[a_p, a_q] = \sum_{r=1}^{n} f_{pq}^r a_r. \tag{2.26}$$

The terms f_{pq}^r are called the structure constants, and they fully specify the Lie algebra.

We saw that one can go from the Lie group to the Lie algebra by differentiating the curves that parameterize the group at the identity. To go from the Lie algebra to the Lie group, we use the exponential map exp, which is defined as the usual exponential map for matrices:

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}. \tag{2.27}$$

The following theorem justifies this [3, p. 26].

Theorem 2.13. *For all $a \in L$ there exists a one-parameter subgroup of G that can be defined by $T(t) = \exp(ta), t \in \mathbb{R}$.*

We now introduce a representation of the Lie algebra that will be useful for some of the upcoming theorems, called the adjoint representation [3, p. 31].

Definition 2.14. Let L be an n-dimensional Lie algebra with a basis $\{a_1, \cdots, a_n\}$. The adjoint representation ad is defined by

$$[a, a_j] = \sum_{k=1}^{n} a_k (ad(a))_{kj}. \tag{2.28}$$

One can check that the matrix elements of $ad(a)$ relate to the structure constants as follows:

$$(ad(a_p))_{kj} = f_{pj}^{k}. \tag{2.29}$$

We now introduce a definition that will be used a lot in the following chapters.

Definition 2.15. A Casimir operator is a quadratic operator that commutes with all elements of the Lie algebra.

Even though these terms may be new to physics students, we have been using them a lot without realizing it. Take, for instance, angular momentum, which is related to the rotations in space. The group of rotations in three dimensions is a Lie group called the special orthogonal group, denoted by $SO(3)$. In quantum mechanics, we define the angular momentum operators L_x, L_y, L_z, which are nothing but the basis elements of the Lie algebra *so(3)*. We say that they generate rotations about the $x, y, z-$ axes, respectively. We then define the operator $L^2 = L_x^2 + L_y^2 + L_z^2$ and note that it commutes with the operators L_x, L_y, L_z. L^2 is nothing but the Casimir operator, since it commutes with any vector of the form $a_x L_x + a_y L_y + a_z L_z$, so it commutes with every element of the Lie algebra. This example shows that Lie groups and Lie algebras show up everywhere in physics very naturally, and physics students are already using these concepts implicitly.

We continue with another definition.

Definition 2.16. The Killing form $B(a, b)$ of the two elements a, b of a Lie algebra L is defined by

$$B(a, b) = \mathrm{Tr}[ad(a)ad(b)] \tag{2.30}$$

where Tr is the trace function and ad is the adjoint representation.

If a_p is a basis element of L, then the adjoint representation of a_p is the matrix given by $ad(a_p)_{kj} = f_{pj}^{k}$, where f_{pj}^{k} are the structure constants of the Lie algebra. Now, take two arbitrary elements a, b in L and write them as

$$a = \sum_p \alpha_p a_p, \quad b = \sum_q \beta_q a_q. \tag{2.31}$$

Their matrix product in the adjoint representation is given by

$$(ad(a)ad(b))_{ij} = \sum_p \sum_q \sum_k \alpha_p (ad(a_p))_{ik} \beta_q (ad(a_q))_{kj} = \sum_p \sum_q \alpha_p \beta_q \sum_k f_{pk}^{i} f_{qj}^{k}. \tag{2.32}$$

We need to take the trace of this matrix:

$$\mathrm{Tr}[ad(a)ad(b)] = \sum_{p,q} (\sum_{k,j} f_{pk}^{i} f_{qj}^{k}) \alpha_p \beta_q = \sum_{p,q} g_{pq} \alpha_p \beta_q \tag{2.33}$$

where we defined

$$g_{mn} = g_{nm} = \sum_{k,l} f_{ml}^{k} f_{nk}^{l}. \tag{2.34}$$

We note that for a semi-simple Lie algebra (see Definition 3.7) one can view g_{mn} as a metric tensor where the inner product is given by the Killing form. Then g_{mn} is called the Cartan metric tensor.

There is much more to be said about Lie groups and Lie algebras, but we conclude this chapter with the following theorem [3, p. 100].

Theorem 2.17. *Let $\{a_1, \cdots, a_n\}$ be a basis of a semi-simple Lie algebra and V be the carrier space of some representation of L whose linear operators are denoted by $\Phi(a), a \in L$. Then the Casimir operator C_2 is given by*

$$C_2 = \sum_{p,q=1}^{n} g^{pq}\Phi(a_p)\Phi(a_q) \tag{2.35}$$

where g^{pq} is the inverse of the Cartan metric tensor.

3 Basics of $SU(1,1)$

As discussed in the previous chapter, we want to find the unitary irreducible representations of the group $SU(1,1)$. To achieve this, we need to take a closer look at the group itself. In both Chapter 4 and Chapter 5, we will make use of the geometric, algebraic, and topological properties of the group that we will investigate in this chapter.

3.1 Why $SU(1,1)$? The relation between $SU(1,1)$ and the Symplectic Group

In this section, we will define the group $SU(1,1)$ and show that it is isomorphic to $Sp(2,\mathbb{R})$ as a group. This makes it possible to work with $SU(1,1)$ as a symmetry group of the quantum harmonic oscillator.

Let U be a 2×2 matrix with complex entries, and define the matrix $g = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. The special unitary group of signature $(1,1)$, denoted by $SU(1,1)$ is the group of all matrices of unit determinant that satisfies the equation

$$U^\dagger g U = g \tag{3.1}$$

where $U^\dagger$ is the conjugate transpose of U. Noting that $g^{-1} = g$, this equation can equivalently be written as

$$U^\dagger = gU^{-1}g^{-1}. \tag{3.2}$$

Let $U = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$. Then, we have

$$gU^{-1}g^{-1} = \begin{pmatrix} \delta & \beta \\ \gamma & \alpha \end{pmatrix} = U^\dagger = \begin{pmatrix} \overline{\alpha} & \overline{\gamma} \\ \overline{\beta} & \overline{\delta}. \end{pmatrix} \tag{3.3}$$

This gives the relations $\delta = \overline{\alpha}$ and $\gamma = \overline{\beta}$. As a result, we have the following definition:

$$SU(1,1) = \{U \in M_2(\mathbb{C}) : \quad U = \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix}, \quad |\alpha|^2 - |\beta|^2 = 1\}. \tag{3.4}$$

As discussed before, the reason we are interested in the group $SU(1,1)$ is because it is isomorphic to the symmetry group of the 1-dimensional quantum harmonic oscillator, $Sp(2,\mathbb{R})$. We now establish this isomorphism [1, p. 591]. Define the map

$$f : SU(1,1) \to Sp(2,\mathbb{R}), \quad W \mapsto \tilde{W} = TWT^{-1} \tag{3.5}$$

where T is the unitary matrix $T = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 & -i \\ -i & 1 \end{smallmatrix}\right)$. We first check that the map is well-defined. Note that the matrix $\tilde{W}$ is given by $\tilde{W} = \begin{pmatrix} \alpha_1 - \beta_2 & -\alpha_2 + \beta_1 \\ \alpha_2 + \beta_1 & \alpha_1 - \beta_2 \end{pmatrix}$ where $\alpha = \alpha_1 + i\alpha_2, \beta = \beta_1 + i\beta_2$ and $W = \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix}, \quad |\alpha|^2 - |\beta|^2 = 1$. Clearly, the entries of $\tilde{W}$ are real. Furthermore, $\det \tilde{W} = (\alpha_1^2 + \alpha_2^2) - (\beta_1^2 + \beta_2^2) = 1$. As a result, $\tilde{W} \in Sp(2,\mathbb{R})$, so the map is well-defined.

We check that the map is a group homomorphism. This follows from

$$f(W_1W_2) = TW_1W_2T^{-1} = TW_1T^{-1}TW_2T^{-1} = f(W_1)f(W_2). \tag{3.6}$$

Finally, the map is bijective because the inverse of f is given by

$$f^{-1} : Sp(2,\mathbb{R}) \to SU(1,1), \quad \tilde{W} \mapsto= T^{-1}\tilde{W}T. \tag{3.7}$$

Therefore, we conclude that f is a group isomorphism and we have $SU(1,1) \cong Sp(2,\mathbb{R}) \cong SL(2,\mathbb{R})$. Establishing these isomorphisms is useful because we can go back and forth between the groups to determine their algebraic properties, based on convenience.

3.2 Parameterization, Lie Algebra, and Generators

Now that we have the definition of $SU(1,1)$, let us look at some of the algebraic properties of the group. We start with a simple parameterization of the group, from which we compute the corresponding Lie algebra of *su(1,1)*.

If $U \in SU(1,1)$, then we have $U = \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} = \begin{pmatrix} a-bi & c-di \\ c+di & a+bi \end{pmatrix}$ with $a^2+b^2-c^2-d^2=1$. If we pick $a>0$, then we can treat b,c,d as continuous parameters of the group and write $U(b,c,d)$ with $U(0,0,0)=\mathbb{I}$. From this, one can compute the derivatives of the parameters at the identity to compute the generators of the Lie algebra. We thus get

$$\left.\frac{\partial U}{\partial b}\right|_{\mathbb{I}} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \quad \left.\frac{\partial U}{\partial c}\right|_{\mathbb{I}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \left.\frac{\partial U}{\partial d}\right|_{\mathbb{I}} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{3.8}$$

We define the basis elements of the Lie algebra, in other words, the generators of *su(1,1)* as

$$b_0 = \frac{i}{2}\left.\frac{\partial U}{\partial b}\right|_{\mathbb{I}} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{1}{2}\sigma_3, \quad b_1 = \frac{i}{2}\left.\frac{\partial U}{\partial c}\right|_{\mathbb{I}} = \frac{i}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{i}{2}\sigma_1, \quad b_2 = \frac{i}{2}\left.\frac{\partial U}{\partial d}\right|_{\mathbb{I}} = \frac{i}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{i}{2}\sigma_2 \tag{3.9}$$

where σ_j are the Pauli matrices. They obey the commutation relations

$$[b_1,b_2] = -ib_0, \quad [b_2,b_0] = ib_1, \quad [b_0,b_1] = ib_2. \tag{3.10}$$

Therefore, the structure constants are given by

$$f_{12}^{0} = -i, \quad f_{20}^{1} = i, \quad f_{01}^{2} = i, \quad 0 \quad \text{otherwise}. \tag{3.11}$$

In Section 3.3.4 we will show that $SU(1,1)$ is simple, so we can construct the Cartan metric tensor with

$$g_{00} = \sum_{k,m} f_{0m}^{k} f_{0k}^{m} = 2, \quad g_{11} = -2, \quad g_{22} = -2, \quad 0 \quad \text{otherwise}. \tag{3.12}$$

This gives the Cartan metric tensor as

$$g = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{3.13}$$

with the inverse

$$g^{-1} = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & -1/2 \end{pmatrix}. \tag{3.14}$$

From this, we construct the second order Casimir operator by

$$C_2 = \sum_{p,q=0}^{2} g^{pq}\Phi(b_p)\Phi(b_q) = \frac{1}{2}\Phi^2(b_0) - \frac{1}{2}\Phi^2(b_1) - \frac{1}{2}\Phi^2(b_2) = \frac{1}{2}B_0^2 - \frac{1}{2}B_1^2 - \frac{1}{2}B_2^2 \tag{3.15}$$

where g^{pq} is the inverse of the Cartan metric tensor, and $\phi(b_j) = B_j$ are the operators in the carrier space of the Lie algebra. Of course, one can scale the structure constants without changing the structure of the Lie algebra, so that we can scale the Cartan metric tensor and the Casimir operator as well. It makes sense that if C_2 commutes with all elements of the Lie algebra, then λC_2 will commute with all elements as well, where λ is some scalar. Using this freedom, we define the Casimir operator of our group as

$$C_{as} = B_0^2 - B_1^2 - B_2^2. \tag{3.16}$$

3.3 Algebra and Topology

In this section, we will show some of the algebraic and topological properties of the group $SU(1,1)$.

3.3.1 Dimension

When we computed the generators of the Lie algebra in Eq. 3.9, we already found 3 linearly independent vectors in *su(1,1)* which span the whole space, so we know already that the dimension of the Lie algebra is 3. This was trivial to obtain because we had an explicit parameterization of the entire group, where we had 4 real parameters and 1 equation, giving us 3 free real parameters. Here, we present an alternative, more general way of determining the dimension of a Lie group from its Lie algebra.

To achieve this, we first state the following theorem [16, p. 39].

Theorem 3.1. *Let M be a smooth manifold of dimension n and $p \in M$. Then T_pM is a vector space of dimension n.*

A direct corollary of the above theorem is that we can find the dimension of a (connected) smooth manifold (i.e. the Lie group) by looking at the dimension of its tangent space at a point, which is precisely the Lie algebra if the point is the identity element of the group.

Let us use the group $SL(2,\mathbb{R})$ for this. Recall that $U \in SL(2,\mathbb{R})$ if $U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1$. Suppose U is close to the identity so that $U = \exp(ta)$ for some $a \in$ *sl(2,$\mathbb{R}$)* and for some sufficiently small $t \in \mathbb{R}$. Then, $\det(e^{ta}) = e^{\mathrm{Tr}(ta)} = 1$, so that $\mathrm{Tr}(a) = 0$. Therefore, *sl(2,$\mathbb{R}$)* consists of traceless 2×2 matrices. Therefore, *sl(2,$\mathbb{R}$)* has dimension 3. Since $SU(1,1)$ is isomorphic to $SL(2,\mathbb{R})$ as a group, *su(1,1)* is isomorphic to *sl(2,$\mathbb{R}$)* as a Lie algebra, and hence the dimension of *su(1,1)* is 3 as well. Using Theorem 3.1, we conclude that $SU(1,1)$ is three dimensional, verifying our initial finding.

3.3.2 Connectedness

In this section, we introduce another parameterization for $SU(1,1)$, which is used to show the connectedness of the group. Moreover, the subgroups generated by this parameterization will be used in the following chapter as well.

Let $g \in SU(1,1)$ such that $g = \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix}$, $|\alpha|^2 - |\beta|^2 = 1$. Without loss of generality, we can assume that $\alpha = \cosh\frac{t}{2} e^{i(\varphi+\psi)/2}$, $\beta = \sinh\frac{t}{2} e^{i(\varphi-\psi)/2}$ where (φ, t, ψ) runs through $0 \leq \varphi < 2\pi$, $0 \leq t < \infty$, $-2\pi \leq \psi < 2\pi$. This follows from the identity $\cosh^2 x - \sinh^2 x = 1$ and the fact that any phase can be obtained with combinations of $\frac{\varphi+\psi}{2}$, $\frac{\varphi-\psi}{2}$ in the given range of parameters. Essentially, we are parameterizing two complex numbers with magnitude squares differing by one, which is all we need to parameterize $SU(1,1)$.

Note, however, that this provides a very nice factorization of our group:

$$g(\varphi, t, \psi) = \begin{pmatrix} \cosh\frac{t}{2} e^{i(\varphi+\psi)/2} & \sinh\frac{t}{2} e^{i(\varphi-\psi)/2} \\ \sinh\frac{t}{2} e^{i(\psi-\varphi)/2} & \cosh\frac{t}{2} e^{-i(\varphi+\psi)/2} \end{pmatrix} = g(\varphi,0,0)g(0,t,0)g(0,0,\psi) \tag{3.17}$$

Based on the factorization in Eq. 3.17, we define the following one-parameter subgroups of $SU(1,1)$:

$$\omega_1(t) = g(0,t,0) = \begin{pmatrix} \cosh\frac{t}{2} & \sinh\frac{t}{2} \\ \sinh\frac{t}{2} & \cosh\frac{t}{2} \end{pmatrix}, \quad \omega_2(t) = \begin{pmatrix} \cosh\frac{t}{2} & i\sinh\frac{t}{2} \\ -i\sinh\frac{t}{2} & \cosh\frac{t}{2} \end{pmatrix}, \quad \omega_0(t) = g(t,0,0) = \begin{pmatrix} e^{it/2} & 0 \\ 0 & e^{-it/2} \end{pmatrix}. \tag{3.18}$$

Take an arbitrary matrix in $SU(1,1)$ and write it as above so that $g' = g(\varphi', t', \psi')$. Fix the parameters t', ψ' and continuously change the φ parameter to 0. Crucially observe that for any $0 \leq \varphi \leq \varphi'$, $g(\varphi, t', \psi')$ is still in $SU(1,1)$ as all the parameters are still in the given range. One can repeat the same argument with t' and ψ' to conclude that there is a path from any $g \in SU(1,1)$ to $g_0(0,0,0) = \mathbb{I}$. This shows that the group is path-connected. Furthermore, for a topological space, path-connectedness implies connectedness (we note that the converse also holds for a differentiable manifold, in particular for a Lie group). As a result, we find that $SU(1,1)$ is connected.

3.3.3 Non-compactness

Non-compactness of $SU(1,1)$ is the most straightforward topological property to verify. It is a fundamental result of the general topology that subsets of $\mathbb{R}^n$ are compact if and only if they are closed and bounded. In this view, take $U \in SU(1,1)$ and denote $U = \begin{pmatrix} a-bi & c-di \\ c+di & a+bi \end{pmatrix}$ with $a^2+b^2-c^2-d^2 = 1$. Clearly, the parameter space of our Lie group, which is also a smooth manifold, can be viewed as a subset of $\mathbb{R}^4$. Therefore, it suffices to show that $SU(1,1)$ is not bounded to conclude that $SU(1,1)$ is not compact.

Theorem 3.2. *$SU(1,1)$ is not bounded.*

Proof. We can show that $SU(1,1)$ is not bounded by equivalently showing that for any $k > 0 \in \mathbb{R}$, there exists a point $p = (a,b,c,d) \in SU(1,1) \subseteq \mathbb{R}^4$ with $a^2+b^2-c^2-d^2 = 1$ such that $||p|| > k$. For any $k > 0$, take $P = (k,1,k,0)$. Clearly, $a^2+b^2-c^2-d^2 = 1$ so that $p \in SU(1,1)$. Moreover, $||p|| = \sqrt{a^2+b^2+c^2+d^2} = \sqrt{2k^2+1} > k$. This shows that $SU(1,1)$ is not bounded. □

Of course, boundedness is not a topological property; it is not preserved under homeomorphism. However, compactness is, so showing that $SU(1,1)$ is not bounded for any parameterization is sufficient to conclude that it is not compact.

3.3.4 $SU(1,1)$ is Simple

In this section, we show a group-theoretic property of $SU(1,1)$, namely that it is a simple Lie group. We start with two definitions.

Definition 3.3. A Lie group is simple if it is a connected, non-Abelian Lie group which does not posses a proper normal Lie subgroup.

Definition 3.4. A Lie group is semi-simple if it is a connected, non-Abelian Lie group which does not posses an Abelian normal Lie subgroup.

As a common theme in Lie theory, we will look at the Lie algebra to investigate the group. More definitions are needed to achieve this, where the definitions are adapted from [3].

Definition 3.5. The vector subspace $L' \subseteq L$ is said to be a subalgebra of the Lie algebra L if it satisfies the following conditions:

1. L' is endowed with the same Lie product as L is a Lie algebra;
2. If $a', b' \in L'$, then $[a', b'] \in L'$.

Furthermore, if $[a', b] \in L'$ for all $a' \in L'$ and for all $b \in L$, then L' is an invariant subalgebra of L.

With this definition at hand, we are ready to define simple and semi-simple Lie algebras.

Definition 3.6. A Lie algebra is said to be simple if it is not Abelian and does not possess a proper invariant Lie subalgebra.

Definition 3.7. A Lie algebra is said to be semi-simple if it does not possess an Abelian invariant Lie subalgebra.

It is not clear how one can determine if a Lie algebra is simple or semi-simple from the above definitions. Instead, we use a criterion developed by the French mathematician Élie Cartan, using the the Killing form.

Theorem 3.8 (Cartan's criterion)**.** *The Lie algebra L is semi-simple if and only if its Killing form is nondegenerate, or equivalently, if and only if* $\det g \neq 0$.

Let us apply this theorem for the Lie algebra *sl(2,ℝ)*. We can check that the following three matrices form a basis for *sl(2,ℝ)*:

$$b_1 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad b_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad b_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.19}$$

The commutation relations are given by

$$[b_1, b_2] = b_3, \quad [b_1, b_3] = b_2, \quad [b_2, b_3] = b_1. \tag{3.20}$$

Using this, we obtain the structure constants

$$f_{12}^3 = 1, \quad f_{13}^2 = 1, \quad f_{23}^1 = 1, \quad 0 \quad \text{otherwise}. \tag{3.21}$$

It is now possible to calculate the metric tensor g using Eq. 2.34. From the structure constants, it is clear that non-diagonal entries of the matrix are zero. The diagonal entries are calculated as

$$g_{11} = \sum_{k,m} f_{1m}^k f_{1k}^m = 2, \quad g_{33} = 2, \quad g_{22} = -2. \tag{3.22}$$

Therefore, we get

$$g = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \tag{3.23}$$

and hence $\det g = -8 \neq 0$. By Cartan's criterion, we conclude that *sl(2,ℝ)* is semi-simple. In fact, we can do more by exploiting the following theorem.

Theorem 3.9. *Consider a semi-simple Lie algebra L with an invariant subalgebra L', and define its orthogonal complement with respect to the Killing form B as $L'^{\perp} := \{a \in L : B(a, c) = 0 \quad \forall c \in L'\}$. Then $L'^{\perp}$ is also an invariant subalgebra.*

Proof. For all $a, b, c \in L$, we have

$$B([a, b], c) = \text{Tr}(ad[a, b]ad(c)) = \text{Tr}([ad(a), ad(b)]ad(c)) = \text{Tr}([ad(b), ad(c)]ad(a)) = B([b, c], a) \tag{3.24}$$

where the third equality follows from the linearity and the cyclic property of trace. In particular, for $c \in L'$, $a \in L'^{\perp}$, and $b \in L$, we have $[b, c] \in L'$, and hence

$$B([b, c], a) = 0 \Rightarrow \quad B([a, b], c) = 0 \Rightarrow \quad [a, b] \in L'^{\perp} \tag{3.25}$$

which shows that $L'^{\perp}$ is an invariant subalgebra of L.

□

Theorem 3.10. *sl(2,ℝ) is a simple Lie algebra.*

Proof. In Section 3.3.1 we showed that the dimension of *sl(2,ℝ)* is 3. Now, suppose that *sl(2,ℝ)* possesses a proper invariant Lie subalgebra L'. Because it is a proper subalgebra, it is either one-dimensional, or its complement is. By Theorem 3.9, either way, there is a one-dimensional invariant subalgebra of *sl(2,ℝ)*. But one-dimensional algebras are Abelian, so *sl(2,ℝ)* possesses an Abelian invariant Lie subalgebra. This contradicts the semi-simplicity of *sl(2,ℝ)*, which shows that *sl(2,ℝ)* does not possess a proper invariant Lie subalgebra. Hence, by definition, *sl(2,ℝ)* is a simple Lie algebra.

□

Because they are isomorphic as Lie algebras, the above theorem shows that *su(1,1)* is a simple Lie algebra. We complete our argument with the following theorem:

Theorem 3.11. *A simple Lie group is a connected Lie group whose Lie algebra is simple.*

Using this theorem, as well as the fact that $SU(1,1)$ is connected and its Lie algebra is simple, we conclude that $SU(1,1)$ is a simple Lie group.

We use the fact that $SU(1,1)$ is a connected, non-compact simple Lie group in the following theorem [8, p. 566].

Theorem 3.12 (Nontrivial Unitary Reps of Noncompact Simple Groups)**.** *If G is a connected, non-compact, simple Lie group, then G possesses no finite-dimensional unitary representations apart from the trivial representation in which $\rho(g) = \mathbb{I}$ for all $g \in G$.*

This theorem has serious consequences for us. Namely, when we classify the nontrivial unitary irreps of $SU(1,1)$ in Chapter 4, they will all be infinite-dimensional.

3.4 Geometry of the Parameter Space

In this section, we will introduce some models of hyperbolic geometry. These constructions are taken from [5, Chapter 8.1]. There are two motivations for this. Firstly, the parameterization discussed in Section 3.3.2 is hyperbolic in nature, and we will take a closer look at this, which will lead to other geometric models that are useful for studying the group $SU(1,1)$. Secondly, in Chapter 5, we will introduce the notion of a generalized coherent state, and parameterizing the generalized coherent states will be possible thanks to the geometric models that we will consider in this section.

3.4.1 Pseudo-Sphere Model

We start with the vector space $\mathbb{R}^3$ and consider the Minkowski metric on this space defined by the symmetric bilinear form $\langle x, y\rangle = -x_0y_0 + x_1y_1 + x_2y_2$. We are interested in the surface defined by the equation $\{x \in \mathbb{R}^3 : \langle x, x\rangle = -1\}$. This gives a hyperboloid with two sheets. We choose the upper sheet to define the pseudo-sphere $P\mathbb{S}^2$:

$$P\mathbb{S}^2 = \{x = (x_0, x_1, x_2) \in \mathbb{R}^3 : -x_0^2 + x_1^1 + x_2^2 = -1, \quad x_0 > 0\}. \tag{3.26}$$

One possible parameterization for this surface is given by the so-called pseudopolar coordinates:

$$x_0 = \cosh\tau, \quad x_1 = \sinh\tau\cos\varphi, \quad x_2 = \sinh\tau\sin\varphi \tag{3.27}$$

where the parameters satisfy $0 \le \tau < \infty, 0 \le \varphi < 2\pi$.

The first fundamental form can be used to derive the line and area elements of the surface $P\mathbb{S}^2$ induced by the Minkowski metric. The following example illustrates this process. First, write a generic point on $P\mathbb{S}^2$ as $X(\tau,\varphi) = \begin{pmatrix}\cosh\tau \\ \sinh\tau\cos\varphi \\ \sinh\tau\sin\varphi\end{pmatrix}$, and take the partial derivatives respectively as $X_\tau = \begin{pmatrix}\sinh\tau \\ \cosh\tau\cos\varphi \\ \cosh\tau\sin\varphi\end{pmatrix}$;

$X_\varphi = \begin{pmatrix} 0 \\ -\sinh\tau\sin\varphi \\ \sinh\tau\cos\varphi \end{pmatrix}$. Then, define $E = \langle X_\tau, X_\tau\rangle; F = \langle X_\tau, X_\varphi\rangle; G = \langle X_\varphi, X_\varphi\rangle$ where the inner product is the one derived by the metric. Finally, we obtain the formulas $ds^2 = Ed\tau^2 + 2Fd\tau d\varphi + Gd\varphi^2$ for the line element and $dA = \sqrt{EG - F^2}d\tau d\varphi$ for the area element. In our examples, we calculate $E = 1, F = 0, G = \sinh^2\tau$ to get $ds^2 = d\tau^2 + \sinh^2\tau d\varphi^2$ and $d^2\boldsymbol{n} = \sinh\tau d\tau d\varphi$ where $\boldsymbol{n}$ is an arbitrary point on $P\mathbb{S}^2$. One can use the line and area elements to define lengths and areas on the surface that are derived from the underlying metric.

The invariance group of the symmetric bilinear form introduced at the beginning of this section is the Lorentz group $O(2,1)$ in three dimensions. This means $A \in O(2,1)$ if and only if $\langle Ax, Ax\rangle = \langle x, x\rangle$. Equivalently, $A^T LA = L$ where L is the Minkowski metric in three dimensions $L = \left(\begin{smallmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$. The subgroup of $O(2,1)$ consisting of matrices with unit determinant is denoted by $SO(2,1)$. This group has two connected components, so we introduce the connected component that contains $\mathbb{I}$ as $SO_0(2,1)$, which is also a group. We note that, by construction, the pseudo-sphere $P\mathbb{S}^2$ is invariant under $SO_0(2,1)$. One can check that the Lie algebra *so(2,1)* of the group $SO_0(2,1)$ is isomorphic with *su(1,1)* [5, p. 228]. This shows that the two Lie groups are locally isomorphic. Precisely because of this reason, studying hyperbolic geometry provides us more insight about the group $SU(1,1)$.

3.4.2 Poincare Disk Model

We will present an alternative way to parameterize $P\mathbb{S}^2$ on the unit disk $\mathbb{D}$ without the boundary. Pick a point (x_0, x_1, x_2) on $P\mathbb{S}^2$, and project it onto the plane $x_0 = 0$ by intersecting this point with the line drawn through $(-1, 0, 0)$. This gives a stereographic projection of $P\mathbb{S}^2$ onto the unit disk $\mathbb{D}$. The Cartesian coordinates (x_0, x_1, x_2) on the pseudo-sphere and (y_0, y_1) on the disk are given by the formulas

$$y_i = \frac{x_i}{1 + x_0}, \quad (x_0, x_i) = \frac{(1 + y_1^2 + y_2^2, 2y_i)}{1 - y_1^2 - y_2^2}. \tag{3.28}$$

Defining $r = \sqrt{y_1^2 + y_2^2}, \tan\varphi = y_2/y_1$ gives the parameterization

$$X(r, \varphi) = \begin{pmatrix} (1 + r^2)/(1 - r^2) \\ 2r\cos\varphi/(1 - r^2) \\ 2r\sin\varphi/(1 - r^2) \end{pmatrix}. \tag{3.29}$$

From this, one can compute the elements of the first fundamental form as $E = \frac{4}{(1-r^2)^2}, F = 0, G = \frac{4r^2}{(1-r^2)^2}$ to obtain

$$ds^2 = \frac{4(dr^2 + r^2 d\varphi^2)}{(1 - r^2)^2}, \quad \zeta = \frac{4rdrd\varphi}{(1 - r^2)^2} \tag{3.30}$$

for the line and area elements, respectively, where $\zeta = re^{-i\varphi}$ is a complex representation of $\mathbb{D}$.

The metric obtained is conformal, which means that the angles for the metric coincide with the angles in the Euclidean geometry [5, p. 230]. Therefore, the isometries of $\mathbb{D}$ are conformal transformations of $\mathbb{D}$. Such transformations f of the unit disk $\mathbb{D}$ are given by [5, p. 230]:

$$f(\zeta) = \frac{\alpha\zeta + \beta}{\overline{\beta}\zeta + \overline{\alpha}}, \quad \text{where} \quad |\alpha|^2 - |\beta|^2 = 1. \tag{3.31}$$

Therefore, the isometries of $\mathbb{D}$ are given by Möbius transformations where the coefficients of the transformations come from the group $SU(1,1)$. In other words, the symmetry group of $\mathbb{D}$ is given by the following representation of $SU(1,1)$: $g \mapsto H_g$, where $H_g(\zeta) = f(\zeta), g = \left(\begin{smallmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{smallmatrix}\right)$. The following proposition makes this statement more precise.

Proposition 3.13. $H_g = \mathbb{I}_{\mathbb{D}}$ *if and only if* $g = \pm\mathbb{I}$.

Proof. (⇐) Suppose $g = \pm\mathbb{I}$. Then, $g = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$. For any $\zeta \in \mathbb{D}$, $f(\zeta) = \frac{\alpha\zeta+\beta}{\overline{\beta}\zeta+\overline{\alpha}} = \frac{\zeta+0}{0+1} = \zeta$, and hence $H_g = \mathbb{I}_{\mathbb{D}}$.
(⇒) Suppose $f(\zeta) = \frac{\alpha\zeta+\beta}{\overline{\beta}\zeta+\overline{\alpha}} = \zeta$ for any $\zeta \in \mathbb{D}$. This holds in particular for $\zeta = 0$, so $f(0) = \frac{\beta}{\overline{\alpha}} = 0$, which holds only if $\beta = 0$. We then have $|\alpha|^2 = 1$, and hence $\alpha = e^{i\theta}$. This gives $f(\zeta) = e^{2i\theta}\zeta = \zeta$. Therefore, $\theta = 0$ or $\theta = \pi$, which gives $\alpha = \pm 1$ and $g = \pm\mathbb{I}$. □

Identifying the isometries of the disk with the group $SO_0(2,1)$, this shows that $SO_0(2,1) \cong SU(1,1)/\{\pm\mathbb{I}\}$. Of course, this does not come as a surprise, as we have already established that the groups were locally isomorphic. As a result, one group is expected to be a covering space of the other one; in this case, $SU(1,1)$ is the double cover of $SO_0(2,1)$. It is also nice that, whereas we previously used algebra to deduce local isomorphism, we recovered the same fact geometrically as well.

3.4.3 Alternative Model for $P\mathbb{S}^2$

We will present yet another way of looking at the pseudo-sphere $P\mathbb{S}^2$. It turns out, $P\mathbb{S}^2$ can be seen as a quotient of $SU(1,1)$ by its $U(1)$ subgroup [5, p. 230]. This model is important because it will be used to parameterize the generalized coherent states when we define them in Chapter 5.

Lemma 3.14. *For any $\boldsymbol{n} = (\cosh\tau, \sinh\tau\sin\varphi, \sinh\tau\cos\varphi)$, define the matrix*

$$g_{\boldsymbol{n}} = \begin{pmatrix} \cosh\frac{\tau}{2} & \sinh\frac{\tau}{2}e^{-i\varphi} \\ \sinh\frac{\tau}{2}e^{i\varphi} & \cosh\frac{\tau}{2} \end{pmatrix}. \tag{3.32}$$

Then the map $\boldsymbol{n} \mapsto \{g_{\boldsymbol{n}}\omega_0(t), t \in [-2\pi, 2\pi)\}$ is a bijection of $P\mathbb{S}^2$ onto the right cosets of $SU(1,1)$ modulo $U(1)$, where $U(1)$ is identified with the matrices $\begin{pmatrix} e^{it/2} & 0 \\ 0 & e^{-it/2} \end{pmatrix}$, $t \in [0, 4\pi)$ and $\omega_0(t)$ is the one-parameter subgroup defined in Eq. 3.18.

Proof. Denote $g(\alpha,\beta) = \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} \in SU(1,1)$. Observe the following decomposition: there exist $\alpha' > 0, \beta' \in \mathbb{C}, 0 \leq t < 4\pi$ unique such that

$$g(\alpha,\beta) = g(\alpha',\beta')\omega_0(t) \tag{3.33}$$

with $\alpha' = |\alpha|, t = \arg\alpha, \beta' = e^{it/2}\beta$. The uniqueness of the decomposition implies that the right cosets of $SU(1,1)$ modulo $U(1)$ are in bijection with the points on the pseudo-sphere.

□

We conclude this chapter with a summary of the spaces that we introduced in this chapter, their symmetry groups, and how they relate to each other.

2−dimensional phase space → $SU(1,1)$

$P\mathbb{S}^2 \to SO_0(2,1)$

Open unit disk $\mathbb{D} \to SO_0(2,1)$

$SO_0(2,1) \cong SU(1,1)/\mathbb{Z}_2$ as groups

$P\mathbb{S}^2 \iff SU(1,1)/U(1)$ as sets

4 Unitary Irreducible Representations of $SU(1,1)$

Throughout this chapter, we closely follow the treatment in [5, Chapter 8], with minor adjustments and slightly different explanations and derivations.

4.1 Classification of Possible Representations

In order to find the unitary irreducible representations of the group $SU(1,1)$, we will first look at some of the properties that they satisfy. This will naturally give us a way to classify different families of unitary irreps. In this regard, let ρ be a unitary irreducible representation of $SU(1,1)$ in some Hilbert space $\mathcal{H}$.

Recall Theorem 3.12 that for all nontrivial unitary irreps of $SU(1,1)$, our Hilbert space $\mathcal{H}$ will be infinite dimensional. We will define the operators on this space as follows:

$$B_j := i\frac{d}{dt}\rho(w_j(t))\Big|_{t=0} \tag{4.1}$$

where $j = 0,1,2$, b_j are the generators of the Lie algebra, and $\omega_j(t)$ are the one-parameter subgroups introduced in Eq. 3.18. They satisfy the familiar commutation relations

$$[B_1, B_2] = -iB_0, \quad [B_2, B_0] = iB_1, \quad [B_0, B_1] = iB_2. \tag{4.2}$$

In order to obtain ladder operators, we can form complex linear combinations of $B_\pm = B_2 \pm iB_1$. We then obtain the following:

$$[B_-, B_+] = 2B_0 \quad [B_0, B_\pm] = \pm B_\pm. \tag{4.3}$$

One can observe that $\rho(\omega_0(t)) = e^{-itB_0}$ due to the following computation:

$$i\frac{d}{dt}\rho(w_0(t))\Big|_{t=0} = i\frac{d}{dt}e^{-itB_0}\Big|_{t=0} = i(-iB_0)e^{-itB_0}\Big|_{t=0} = B_0 \cdot \mathbb{I} = B_0. \tag{4.4}$$

Because $\omega_0(4\pi) = \mathbb{I}$, by the properties of a group homomorphism, the representation ρ should map the identity element of the group to the identity operator, which gives us $e^{-i4\pi B_0} = \mathbb{I}$. Using this equality, we obtain the following.

Proposition 4.1. *The eigenvalues of the operator B_0 are a subset of $\{\frac{k}{2}, k \in \mathbb{Z}\}$.*

Proof. Suppose λ is an eigenvalue of the operator B_0 with the eigenvector ψ. Since $e^{-i4\pi B_0} = \mathbb{I}$, we have

$$e^{-i4\pi B_0}\psi = \mathbb{I}\psi - 4i\pi B_0\psi + \frac{(-4i\pi)^2 B_0^2}{2}\psi + \cdots = \mathbb{I}\psi - 4i\pi\lambda\psi + \frac{(-4i\pi)^2\lambda^2}{2}\psi + \cdots = e^{-i4\pi\lambda}\psi = \psi. \tag{4.5}$$

Therefore, λ must be a half-integer. But λ was an arbitrary eigenvalue of B_0, so any eigenvalue of B_0 must be a half-integer. Therefore, the set of eigenvalues of B_0 is a subset of $\{\frac{k}{2}, k \in \mathbb{Z}\}$. □

Take a vector ψ_0 in $\mathcal{H}$ such that $||\psi_0|| = 1$ and $B_0\psi_0 = \lambda\psi_0$, $\lambda = \frac{k_0}{2}$, $k_0 \in \mathbb{Z}$. Because $[B_0, B_+] = B_0B_+ - B_+B_0 = B_+$, we get that

$$B_0B_+\psi_0 = B_+B_0\psi_0 + B_+\psi_0 = B_+\lambda\psi_0 + B_0\psi_0 = (\lambda+1)B_+\psi_0. \tag{4.6}$$

With a similar reasoning, one can obtain the following equality:

$$B_0B_-\psi_0 = (\lambda-1)B_-\psi_0. \tag{4.7}$$

In fact, by the property of a commutator bracket, one has

$$[B_0, (B_+)^k] = [B_0, B_+](B_+)^{k-1} + B_+[B_0, B_+](B_+)^{k-2} + \cdots = k(B_+)^k. \tag{4.8}$$

This immediately gives $B_0(B_+)^k = (B_+)^kB_0 + k(B_+)^k$, so that

$$B_0(B_+)^k\psi_0 = (\lambda+k)(B_+)^k\psi_0 \tag{4.9}$$

$$B_0(B_-)^k\psi_0 = (\lambda-k)(B_-)^k\psi_0. \tag{4.10}$$

We now introduce the Casimir operator C_{as} of the representation. Due to Schur's lemma, because our representation is irreducible, the Casimir operator is proportional to the identity, $C_{as} = c_{as}\mathbb{I}$, where

$$C_{as} = B_0^2 - B_1^2 - B_2^2 = B_0^2 - \frac{1}{2}(B_-B_+ + B_+B_-). \tag{4.11}$$

Using this, we get

$$(B_-B_+ + B_+B_-)\psi_0 = (2B_0 - 2C_{as})\psi_0 = 2(\lambda^2 - c_{as})\psi_0 \tag{4.12}$$

$$(B_-B_+ - B_+B_-)\psi_0 = 2\lambda\psi_0. \tag{4.13}$$

Adding and subtracting the two equations above give the following equalities:

$$(B_-B_+)\psi_0 = (\lambda(\lambda+1) - c_{as})\psi_0 \tag{4.14}$$

$$(B_+B_-)\psi_0 = (\lambda(\lambda-1) - c_{as})\psi_0. \tag{4.15}$$

Our goal now is to produce an expression similar to Eq. 4.14, where we want to replace $B_+\psi_0$ with $(B_+)^k\psi_0$. We start with rewriting Eq. 4.11 as $B_-B_+ + B_+B_- = 2B_0^2 - 2C_{as}$. Multiplying from right by B_+^{k-1}, we get

$$B_-B_+^k + B_+B_-B_+^{k-1} - = 2B_0^2B_+^{k-1} - 2C_{as}B_+^{k-1}. \tag{4.16}$$

We also have the following equality:

$$B_-B_+^k - B_+B_-B_+^{k-1} = [B_-, B_+]B_+^{k-1} = 2B_0B_+^{k-1}. \tag{4.17}$$

By adding these two equations, using Eq. 4.9, and letting the operators act on ψ_0, we obtain the equation that we were after:

$$B_-B_+^k\psi_0 = B_0B_0B_+^{k-1}\psi_0 + B_0B_+^{k-1}\psi_0 - C_{as}B_+^{k-1}\psi_0 = ((\lambda+k-1)(\lambda+k) - c_{as})B_+^{k-1}\psi_0. \tag{4.18}$$

The corresponding equation for the lowering operator, which can be derived similarly, reads

$$B_+B_-^k\psi_0 = ((\lambda-k+1)(\lambda-k) - c_{as})B_+^{k-1}\psi_0. \tag{4.19}$$

For ease of notation, let us denote $((\lambda+k-1)(\lambda+k) - c_{as}) = \nu_k^+$ and $((\lambda-k+1)(\lambda-k) - c_{as}) = \nu_k^-$.

We are one step away from our goal of classifying unitary irreps. We need to establish that the raising and lowering operators that we defined above are Hermitian conjugates of each other. This becomes straightforward to show if we can establish that B_1 and B_2 are Hermitian operators. The following theorem precisely states this fact [7, p. 50].

Theorem 4.2. *Every unitary operator U on a Hilbert space can be written as $U = \exp(iA)$ for some Hermitian operator A.*

Let f, g be two elements in $\mathcal{H}$. Using the Hermicity of B_1 and B_2, we get

$$\langle f|B_\pm g\rangle = \langle f|(B_2 \pm iB_1)g\rangle = \langle f|B_2 g\rangle \pm i\langle f|B_1 g\rangle = \langle B_2 f|g\rangle \pm i\langle B_1 f|g\rangle = \langle (B_2 \mp iB_1)f|g\rangle = \langle B_\mp f|g\rangle. \tag{4.20}$$

This indeed shows that the raising and lowering operators are conjugates of each other. We can combine this fact with Eq. 4.18 and Eq. 4.19 to obtain the relation between the norms of different states that we reach by acting with the ladder operators:

$$||B_+^{k+1}\psi_0||^2 = \nu_{k+1}^+ ||B_+^k\psi_0||^2 \tag{4.21}$$

$$||B_-^{k+1}\psi_0||^2 = \nu_{k+1}^- ||B_-^k\psi_0||^2. \tag{4.22}$$

Using Eq. 4.21, we can start the classification [5, p. 234].

1. Suppose that for all $k \in \mathbb{N}$, $B_+^k\psi_0 \neq 0$ and $B_-^k\psi_0 \neq 0$. In other words, there is not a top or bottom rung; the ladder goes on forever in each direction. Then, for each $k \in \mathbb{N}$, $\lambda \pm k$ is an eigenvalue of the operator B_0.
   (a) If 0 is in this family, then the set of eigenvalues is the integers. Without loss of generality, we can assume $B_0\psi_0 = 0$, so that $\lambda = 0$. Eq. 4.21 now implies that $c_{as} < 0$.
   (b) If 0 is not in this family, then we can take $\lambda = 1/2$ and the set of eigenvalues is given by $\{\frac{1}{2} + k, k \in \mathbb{Z}\}$. Again using Eq. 4.21, we conclude that $c_{as} < -\frac{1}{4}$.
2. Suppose now instead that the chains terminate.
   (a) Assume there exists $k_0 \in \mathbb{N}$ such that $B_+^{k_0+1}\psi_0 = 0$ and $B_+^{k_0}\psi_0 \neq 0$. In other words, there is a top rung of the ladder. By scaling the eigenvector accordingly, we can assume that $B_0\psi_0 = \lambda\psi_0$, and $B_+\psi_0 = 0$. This gives $\nu_1^+ = 0$, which in turn gives $c_{as} = \lambda(\lambda + 1)$. Now, if $\lambda = 0$, one can check that $\nu_1^- = 0$ as well, which shows $B_-\psi_0 = 0$. Hence, the Hilbert space is one-dimensional. This case corresponds to the trivial representation. Of course, when we do things in full generality, it is nice to verify that the trivial representation also belongs to one of these families, as it should be. On the other hand, we are interested in the nontrivial representations, so we can take $\lambda \neq 0$. The set of eigenvalues of B_0 is given by $\{\lambda - k : k \in \mathbb{N}\}$.
   (b) If there is a bottom rung instead, we obtain analogous results with the set of eigenvalues $\{\lambda + k : k \in \mathbb{N}\}$.

For the first two infinite families of representations, the eigenvalues of the Casimir operator c_{as} change continuously and hence they are called continuous series. For the last two infinite families, the Casimir parameters vary discretely, and the representations belong to discrete series.

This is a good point to take a step back from all the calculations and see what we have done. Only using the Lie algebra, by fixing a nontrivial unitary irreducible representation ρ, we managed to classify the possible families of different irreps of the group. These derivations do not prove that such representations exist. What is said instead is, if we find some nontrivial unitary irrep of our group, it will belong to one of these four families. In fact, in the rest of this chapter, we will explicitly realize each of these families of unitary irreps, thereby classifying all of the unitary irreducible representations of our group $SU(1,1)$.

4.2 Discrete Series Representation

For the rest of this report, we will mainly focus on the discrete series representation. In this section, we will define a Hilbert space and define the discrete unitary irreps of $SU(1,1)$.

4.2.1 Hilbert Space $H_n(\mathbb{D})$

Let n be an integer such that $n \geq 2$. Define the Hilbert space $H_n(\mathbb{D})$ as the space of holomorphic functions f on the open unit disk $\mathbb{D}$ of the complex plane $\mathbb{C}$. We require that the norm of f be finite, where the norm is derived by the following inner product: if $f, g \in H_n(\mathbb{D})$, then

$$\langle f, g\rangle := \frac{n-1}{\pi} \int_{\mathbb{D}} \overline{g(z)} f(z)(1 - |z|^2)^{n-2} dxdy \tag{4.23}$$

where $\overline{f(z)}$ denotes the complex conjugate, and $z = x + iy$.

Without going into the details of measure theory, we can state that the function

$$d\nu_n(z) = \frac{n-1}{\pi}(1-|z|^2)^{n-2}dxdy \tag{4.24}$$

is a probability measure on $\mathbb{D}$. The fact that this is a measure follows from the standard theory of integration. One can observe that it is a probability measure, i.e. its integral over the unit disk is one, from the following computation, where we use the polar coordinates $z = re^{i\theta}$, and the change of variables $u = (1-r^2)$:

$$\int_{\mathbb{D}} d\nu_n(z) = \frac{n-1}{\pi}\int_{\mathbb{D}}(1-r^2)^{n-2}rdrd\theta = 2(n-1)\int_1^0 u^{n-2}\cdot -\frac{1}{2}du = 1. \tag{4.25}$$

It is a celebrated theorem in complex analysis that holomorphic functions are analytic, so we can represent f as $f(z) = \sum_{k\geq 0} c_k(f)z^k$, where $c_k(f)$ are the Taylor coefficients of f. Using the Taylor series expression, we can compute the norm of a function in the Hilbert space as

$$||f|| = 2(n-1)\sum_{k\geq 0}|c_k(f)|^2\int_0^1 r^{2k+1}(1-r^2)^{n-2}dr. \tag{4.26}$$

An explicit expression for this integral is provided in terms of gamma functions in [5, p. 235]:

$$||f|| = \sum_{k\geq 0}|c_k(f)|^2\frac{\Gamma(n)\Gamma(k+1)}{\Gamma(n+k)}. \tag{4.27}$$

Therefore, after defining $\gamma_{n,l} := \frac{\Gamma(n)\Gamma(l+1)}{\Gamma(n+l)} = \frac{1}{\binom{n+l-1}{l}}$, the inner product between any two functions in $H_n(\mathbb{D})$ can be expressed as

$$\langle f_1, f_2\rangle = \sum_{k\geq 0}\overline{c_k(f_1)}c_k(f_2)\gamma_{n,k}. \tag{4.28}$$

Using Eq. 4.27 and Eq. 4.28, we get that $e_l(z) := \{\frac{z^l}{\sqrt{\gamma_{n,l}}}\}_{l\geq 0}$ forms an orthonormal set. In addition, one can show that the span of this set is dense in $H_n(\mathbb{D})$. Hence, $\{e_l(z)\}$ forms an orthonormal basis for $H_n(\mathbb{D})$, and the Hilbert space is separable [17, p. 55]. This orthonormal basis will be useful in the upcoming sections.

4.2.2 Defining Discrete Representations

After we have defined the appropriate carrier space in the preceding section, we are ready to construct a unitary representation of our group, as is done in [5, Chapter 8.2.2.2]. The irreducibility of the representation will be discussed in Section 4.2.4.

Recall from Section 3.4 that the isometries of the Poincaré disk were given by Möbius transformations with the coefficients coming from the parameterization of $SU(1,1)$. This observation leads to the question of whether it is possible to define group actions of $SU(1,1)$ on the unit disk $\mathbb{D}$, which would be useful to construct representations. The answer is positive and, after reminding the reader of the definition of group actions, this is exactly what we will do.

Definition 4.3. Let G be a group with the identity element e and X a nonempty set. A (left) group action of G on X is a map $G\times X\to X$ which can be denoted by $(g,x)\mapsto gx$, satisfying

1. $ex = x$ for every $x\in X$;
2. $(gh)x = g(hx)$ for all $g,h\in G$ and all $x\in X$.

Let g be an arbitrary element of $SU(1,1)$, given by $g = \begin{pmatrix}\alpha & \beta\\ \overline{\beta} & \overline{\alpha}\end{pmatrix}$. We want to establish that the map defined by

$$M_-(g)(z) = \frac{\overline{\beta} + \overline{\alpha} z}{\alpha + \beta z} \tag{4.29}$$

is a left group action of $SU(1,1)$ on $\mathbb{D}$.

Lemma 4.4. *The map 4.29 is well-defined.*

Proof. We need to check that the denominator is never zero. First, note that $|\beta z| < |\beta|$ since z is in the open unit disk. This shows that $|\alpha| - |\beta z| > |\alpha| - |\beta|$. Furthermore, $|\alpha|^2 - |\beta|^2 = 1$ implies that $|\alpha|^2 > |\beta|^2$, and since $|\alpha|$, $|\beta|$ are nonnegative, we have $|\alpha| > |\beta|$, so that $|\alpha| - |\beta| > 0$.

Combining these inequalities, we get $||\alpha| - |\beta z|| > ||\alpha| - |\beta|| > 0$. Now using the reverse triangle inequality shows that $|\alpha + \beta z| \geq ||\alpha| - |\beta z|| > ||\alpha| - |\beta|| > 0$. Thus, $\alpha + \beta z$ is not zero, so the map is well defined for any $g \in SU(1,1)$. □

Lemma 4.5. *Eq. 4.29 maps the open unit disk to the open unit disk.*

Proof. We should check that $\frac{\overline{\beta} + \overline{\alpha} z}{\alpha + \beta z}$ is in the open unit disk. First, note that if $|z| = 1$, then $|M_-(g)(z)| = 1$. Recall from complex analysis that the maximum modulus principle states that $|M_-(g)(z)|$ assumes its maximum on the boundary of $\overline{\mathbb{D}}$, that is, on the unit circle. Therefore, $|M_-(g)(z)| < 1$ if $|z| < 1$. As a result, the image of $M_-(g)(z)$ is in $\mathbb{D}$. □

Using Lemma 4.4 and Lemma 4.5, we are ready to prove the group action property.

Theorem 4.6. *The map $M_- : SU(1,1) \times \mathbb{D} \to \mathbb{D}$ given by $M_-(g)(z) = \frac{\overline{\beta} + \overline{\alpha} z}{\alpha + \beta z}$ is a left group action.*

Proof. The map is well defined due to the previous two lemmas. We thus check the two criteria in the definition.

1. The identity element e is given by $e = \begin{pmatrix} \alpha = 1 & \beta = 0 \\ \overline{\beta} = 0 & \overline{\alpha} = 1 \end{pmatrix}$, so that $M_-(e)(z) = \frac{0+1z}{1+0z} = z$.

2. Let $g_1 = \begin{pmatrix} \alpha_1 & \beta_1 \\ \overline{\beta_1} & \overline{\alpha_1} \end{pmatrix}$, $g_2 = \begin{pmatrix} \alpha_2 & \beta_2 \\ \overline{\beta_2} & \overline{\alpha_2} \end{pmatrix}$. Then, we have $M_-(g_2)(z) = \frac{\overline{\beta_2} + \overline{\alpha_2} z}{\alpha_2 + \beta_2 z}$. Hence, we obtain $M_-(g_1)(M_-(g_2)(z)) = \frac{(\overline{\beta_1}\alpha_2 + \overline{\alpha_1}\overline{\beta_2}) + (\overline{\beta_1}\beta_2 + \overline{\alpha_1}\overline{\alpha_2})z}{(\alpha_1\alpha_2 + \beta_1\overline{\beta_2}) + (\alpha_1\beta_2 + \beta_1\overline{\alpha_2})z} = M_-(g_1 g_2)(z)$.

□

In the same spirit, we can also define $M_+(g)(z) = \frac{-\overline{\beta} + \alpha z}{\overline{\alpha} - \beta z}$ and show that this is also a group action.

We are now ready to define the unitary representation of the group acting on $H_n(\mathbb{D})$. The way we do this is as follows. For $f \in H_n(\mathbb{D}), z \in \mathbb{D}$ and $g \in SU(1,1)$, we define

$$\mathcal{D}_n^-(g)f(z) = c_g^-(z)(f(M_-(g^{-1})(z))) \tag{4.30}$$

where the scaling factor c_g^- is chosen to ensure unitarity. The following theorem realizes this explicitly.

Theorem 4.7. *For every integer $n \geq 2$, the following are unitary representations of $SU(1,1)$ in $H_n(\mathbb{D})$:*

$$\mathcal{D}_n^-(g)f(z) = (\overline{\alpha} - \beta z)^{-n} f(\frac{-\overline{\beta} + \alpha z}{\overline{\alpha} - \beta z}) \tag{4.31}$$

$$\mathcal{D}_n^+(g)f(z) = (\alpha + \overline{\beta} z)^{-n} f(\frac{\beta + \overline{\alpha} z}{\alpha + \overline{\beta} z}) \tag{4.32}$$

Proof. It follows from 4.6 that the maps $\mathcal{D}_n^{\pm}$ are group homomorphisms, so indeed they are group representations. For $\mathcal{D}_n^{-}$, we can prove that this representation is unitary. The proof for $\mathcal{D}_n^{+}$ follows analogously. Introduce the holomorphic change of variables $Z = \frac{-\overline{\beta}+\alpha z}{\overline{\alpha}-\beta z}$, $Z = X + iY$, $z = x + iy$. We get $z = \frac{\overline{\beta}+\overline{\alpha}Z}{\alpha+\beta Z}$ and $\frac{dz}{dZ} = (\alpha + \beta Z)^{-2}$ by the quotient rule. In order to apply the change of variables, we observe that $|\det \frac{\partial(x,y)}{\partial(X,Y)}| = |\frac{dz}{dZ}|^2 = |\alpha + \beta Z|^{-4}$, so that $dxdy = |\alpha + \beta Z|^{-4} dXdY$.

Moreover, we have $|\overline{\alpha}-\beta z|^{-2n} = |\alpha+\beta Z|^{2n}$, as well as $(1-|z|^2)^{n-2} = (1-|\frac{\overline{\beta}+\overline{\alpha}Z}{\alpha+\beta Z}|^2)^{n-2} = \frac{(|\alpha+\beta z|^2-|\overline{\beta}+\overline{\alpha}Z|^2)^{n-2}}{|\alpha+\beta Z|^{2n-4}}$. Putting everything together, we obtain

$$|\overline{\alpha} - \beta z|^{-2n}(1 - |z|^2)^{n-2} dxdy = (|\alpha + \beta z|^2 - |\overline{\beta} + \overline{\alpha}Z|^2)^{n-2} dXdY = (1 - |Z|^2)^{n-2} dXdY. \tag{4.33}$$

As a result, we get

$$\begin{aligned} \langle \mathcal{D}_n^-(g) f(z) | \mathcal{D}_n^-(g) f(z) \rangle &= \frac{n-1}{\pi} \int_{\mathbb{D}} |\overline{\alpha} - \beta z|^{-2n} |f(\frac{-\overline{\beta} + \alpha z}{\overline{\alpha} - \beta z})|^2 (1 - |z|^2)^{n-2} dxdy \\ &= \frac{n-1}{\pi} \int_{\mathbb{D}} |f(Z)|^2 (1 - |Z|^2)^{n-2} dXdY = \langle f(z) | f(z) \rangle. \end{aligned} \tag{4.34}$$

This proves that $\mathcal{D}_n$ is unitary.

□

4.2.3 Casimir Operator and Labeling Irreps

This is once again a good place to take a break from the derivations and focus on the big picture. In this chapter, we are trying to obtain all unitary irreducible representations of $SU(1,1)$. First, we established that there are four different possible families of unitary irreps of the group. Then, we defined the Hilbert spaces $H_n(\mathbb{D})$ indexed by integers greater than 1, and constructed two families of unitary representations of our group that act on these Hilbert spaces. We still need to do two things. Firstly, we should prove that these unitary representations are irreducible. Secondly, we need to find to which infinite families of unitary irreps these representations belong to. We will start with the second task, and as the title suggests, we will conclude that these representations are the discrete irreps.

Now that we have explicit unitary representations, we can construct the corresponding Lie algebra representations. First, we calculate the derivatives of the representations at the identity. Using the chain rule gives [5, p. 237]

$$\frac{d}{dt} \mathcal{D}_n^-(\omega_0(t)) f(z) \Big|_{t=0} = \frac{-i}{2}(n + 2z\frac{d}{dz}) f(z) \tag{4.35}$$

$$\frac{d}{dt} \mathcal{D}_n^-(\omega_1(t)) f(z) \Big|_{t=0} = \frac{1}{2}(nz + (z^2 - 1)\frac{d}{dz}) f(z) \tag{4.36}$$

$$\frac{d}{dt} \mathcal{D}_n^-(\omega_2(t)) f(z) \Big|_{t=0} = \frac{-i}{2}(nz + (z^2 + 1)\frac{d}{dz}) f(z). \tag{4.37}$$

From these computations, we get three Hermitian operators $B_j = id\mathcal{D}_n^-(b_j)$ where d denotes the differential of the group at the identity, and b_j are the generators of the Lie algebra:

$$B_0 = \frac{n}{2} + z\frac{d}{dz} \tag{4.38}$$

$$B_1 = \frac{i}{2}(nz + (z^2 - 1)\frac{d}{dz}) \tag{4.39}$$

$$B_2 = \frac{1}{2}(nz + (z^2+1)\frac{d}{dz}). \tag{4.40}$$

As expected, they satisfy the familiar commutation relations

$$[B_1, B_2] = -iB_0, \quad [B_2, B_0] = iB_1, \quad [B_0, B_1] = iB_2. \tag{4.41}$$

By acting on an arbitrary function $f(z)$ in $H_n(\mathbb{D})$, one can check the above commutation relations. Using the notation $B_\pm = B_2 \mp iB_1$, we get

$$B_- = \frac{d}{dz} \tag{4.42}$$

$$B_+ = nz + z^2\frac{d}{dz} \tag{4.43}$$

as well as

$$[B_-, B_+] = 2B_0 \quad [B_0, B_\pm] = \pm B_\pm. \tag{4.44}$$

We are ready to compute the Casimir operator C^n_{as} for the representation $\mathcal{D}^-_n$. From Eq. 4.11 we get:

$$C^n_{as} = B_0^2 - B_1^2 - B_2^2 = B_0^2 - \frac{1}{2}(B_-B_+ + B_+B_-). \tag{4.45}$$

By Schur's lemma, if the representation is irreducible, we can write the Casimir operator C^n_{as} as $C^n_{as} = c^n_{as}\mathbb{I}$. Hence, for now, let us assume $\mathcal{D}^-_n$ is irreducible and calculate the number c_{as}. We do this as follows: Take an arbitrary $f(z)$ in the Hilbert space and write $f = \sum_{k\geq 0} c_k z^k$. This gives

$$B_+B_-f = nz(\sum_{k\geq 1} kc_kz^{k-1}) + z^2\frac{d}{dz}(\sum_{k\geq 1} kc_kz^{k-1}) = \sum_{k\geq 0}(nk + k(k-1))c_kz^k. \tag{4.46}$$

We similarly compute

$$B_-B_+f = \sum_{k\geq 0}((n+k)(k+1))c_kz^k \tag{4.47}$$

$$B_0^2 f = \sum_{k\geq 0}(k+\frac{n}{2})^2 c_kz^k. \tag{4.48}$$

Plugging these expressions into Eq. 4.45 gives

$$C^n_{as}f = (B_0^2 - \frac{1}{2}(B_-B_+ + B_+B_-))f = (\frac{n^2}{4} - \frac{n}{2})\sum_{k\geq 0} c_kz^k = \frac{n}{2}(\frac{n}{2}-1)f \tag{4.49}$$

which shows that $c_{as} = \frac{n}{2}(\frac{n}{2}-1), n \geq 2$.

We indeed obtained a discrete representation of the group: the Casimir parameter changes discretely.

Let us compare our results with the classifications of the unitary irreps that we obtained at the beginning of the chapter. Take $f(z) \in H_n(\mathbb{D})$ the eigenstate of the operator B_0 with the eigenvector $\frac{n}{2}+k$. In other words, we have $B_0 f = (\frac{n}{2}+k)f$. Denote f by $f = |n,k\rangle$. Then, we have the eigenvector equations

$$C^n_{as}|n,k\rangle = \frac{n}{2}(\frac{n}{2}-1)|n,k\rangle, \quad B_0|n,k\rangle = (\frac{n}{2}+k)|n,k\rangle. \tag{4.50}$$

They are analogous equations for the theory of angular momentum in quantum mechanics: the L^2 operator is the Casimir operator, and the L_z operator does what the B_0 operator does in this case.

Comparing with the initial classification, we found that $c_{as} = \lambda(\lambda - 1)$. Hence, we obtained $\lambda := \frac{n}{2}$. Then, the set of eigenvalues of B_0 is given by $\{\lambda + k : k \in \mathbb{N}\} = \{\frac{n}{2} + k : k \in \mathbb{N}\} = \{\frac{n}{2}, \frac{n}{2} + 1, \frac{n}{2} + 2, \cdots\}$.

To make things more concrete, let us take a simple example and apply our results. Take $n = 3$. Then, $\lambda = \frac{3}{2}$ and $c_{as}^3 = \frac{3}{2}(\frac{3}{2} - 1) = \frac{3}{4}$. We have the eigenstates of B_0 $|3,0\rangle, |3,1\rangle, |3,2\rangle, \cdots, |3,k\rangle$ with $C_{as}^3|3,k\rangle = \frac{3}{4}|3,k\rangle$ and $B_0|3,k\rangle = (\frac{3}{2} + k)|3,k\rangle$, e.g. $B_0|3,1\rangle = \frac{5}{2}|3,1\rangle$.

We can do even more than this. Let $n = 2$ and observe that the constant function $\psi_0(z) = 1$ has the property

$$B_0\psi_0 = (\frac{2}{2} + z\frac{d}{dz})\psi_0 = 1 \cdot \psi_0 \tag{4.51}$$

where we used Eq. 4.38 for the computation. This shows that it is an eigenvector of the operator B_0 with the eigenvalue $1 = 1 + 0 = \frac{2}{2} + 0 = \frac{n}{2} + 0$. Furthermore, we can compute its magnitude:

$$||\psi_0||^2 = \frac{1}{\pi}\int_{\mathbb{D}}(1 - |z|^2)^{2-2}dxdy = 1 \tag{4.52}$$

which follows from the fact that $d\nu_n(z) = \frac{n-1}{\pi}(1-|z|^2)^{n-2}dxdy$ is a probability measure as we demonstrated in Eq. 4.25. Therefore, we have a normalized eigenvector of B_0 with the eigenvalue $\frac{2}{2} + 0$, so we can label $|2,0\rangle = \psi_0(z) = 1$. For the next step, we can act on this state with the raising operator:

$$B_+|2,0\rangle = (2z + z^2\frac{d}{dz})1 = 2z. \tag{4.53}$$

Acting with B_0 gives $B_0B_+|2,0\rangle = (1 + z\frac{d}{dz})2z = 4z = 2 \cdot 2z = (\frac{2}{2} + 1)2z = (\frac{2}{2} + 1)|2,0\rangle$, which is expected due to Eq. 4.6. Hence, we want to label $|2,1\rangle$ as $f(z) = 2z$. In principle, we can do this, but we would prefer to have the states $|2,k\rangle$ normalized. One can compute

$$||2z||^2 = \frac{1}{\pi}\int_{\mathbb{D}}4|z|^2dxdy = \frac{1}{\pi}2\pi\int_0^1 4r^2rdr = 2. \tag{4.54}$$

In fact we did not even need to make this computation explicitly, since Eq. 4.21 automatically gives the square of the norm of this state as 2. Hence, we define

$$|2,1\rangle = \sqrt{2}z. \tag{4.55}$$

Hence, recursively, one can produce the normalized eigenstates of the operator B_0 using the relations that we developed throughout this chapter. Therefore, in principle, we can do all the calculations very explicitly on the functions that live in the Hilbert space $H_n(\mathbb{D})$. This demonstrates the power of the method of finding the Casimir operator and labeling the irreps of our group.

As a result, we see that our discrete unitary irreps satisfy the necessary conditions that we figured out at the beginning of this chapter. This shows that we explicitly realized two of the four infinite families of the unitary irreps of $SU(1,1)$, namely the two discrete irreps of $\mathcal{D}_n^-$ and $\mathcal{D}_n^+$ for integers $n \geq 2$. The only detail missing from this derivation is that we still did not prove these unitary representations are indeed irreducible. At various points, we assumed this and proceeded with our calculations. We are now ready to prove the irreducibility of the representations and complete the classification of the discrete unitary irreps of $SU(1,1)$.

4.2.4 Irreducibility of the Series

The argument we present here is adapted from [5, p. 238].

Theorem 4.8. $\mathcal{D}_n^-$ *is an irreducible representation.*

Proof. In order to prove that the representation is irreducible, we need to show that there are no nontrivial invariant subspaces of $H_n(\mathbb{D})$. Suppose E is a closed invariant subspace of $H_n(\mathbb{D})$. We need to show that E is the entire Hilbert space of $H_n(\mathbb{D})$ if E is not the subspace with only the zero vector.

Note that the subgroup $g(\theta,0,0)$ is compact and commutative, so the restriction of $\mathcal{D}_n^-$ to this subgroup is a sum of one-dimensional unitary representations. Therefore, there exists $u \in E, \nu \in \mathbb{R}$ such that

$$\mathcal{D}_n^-(g(\theta,0,0))u = e^{i\nu\theta}u, \quad \forall\theta \in \mathbb{R}. \tag{4.56}$$

Writing $u(z) = \sum a_j z^j$, we calculate

$$\mathcal{D}_n^-(g)(\sum a_j z^j) = \sum a_j(\mathcal{D}_n^-(g)z^j) = \sum a_j e^{i\theta n/2}e^{i\theta j}z^j = e^{i\nu\theta}u = \sum a_j e^{i\nu\theta}z^j \tag{4.57}$$

and by comparing the coefficients, we get that $a_j e^{i\nu\theta} = a_j e^{i\theta(n/2+j)}$. If u is not the zero vector, there exists some j_0 such that $a_{j_0} \neq 0$, which implies that $e^{i\theta(n/2+j_0)=e^{i\nu\theta}}$. But this equality holds for all $\theta \in \mathbb{R}$ if and only if $\nu = \frac{n}{2} + j_0$. This implies that for $j \neq j_0$, $a_j e^{i\nu\theta} = a_j e^{i\theta(n/2+j_0)}$ holds for all $\theta \in \mathbb{R}$, which is possible only if $a_j = 0$ for $j \neq j_0$. This shows that $u = z^{j_0}$ is in E.

Now, playing around with the operators $B_\pm$, it can be shown that E contains all the monomials z^j with $j \in \mathbb{N}$. Therefore, the monomials are a total system in $H_n(\mathbb{D})$, so that $E = H_n(\mathbb{D})$. This proves that the representation $\mathcal{D}_n^-$ is irreducible.

□

4.3 Principal and Complementary Series

In this thesis, we will mostly be working with discrete series. However, for completeness, we also present the principal and complementary series unitary irreducible representations of the group $SU(1,1)$, mostly without proofs. For this section, we closely follow the treatment in [5, Chapter 8.2].

The representations for the principal series can be realized in the Hilbert space of functions on the unit circle, denoted by $L^2(\mathbb{S}^1)$, with the inner product

$$\langle f_1, f_2\rangle = \frac{1}{2\pi}\int_0^{2\pi} \overline{f_1}(\theta)f_2(\theta)d\theta. \tag{4.58}$$

First, note that the map $z \mapsto \frac{\alpha z+\overline{\beta}}{\beta z+\overline{\alpha}}$ maps the unit circle into itself. Using this, take a nonnegative number λ and a point z on the unit circle and define the map

$$\mathcal{P}_{i\lambda}(g)f(z) = |\beta z + \overline{\alpha}|^{-1+2i\lambda} f(\frac{\alpha z+\overline{\beta}}{\beta z+\overline{\alpha}}). \tag{4.59}$$

We are ready to state the main theorem.

Theorem 4.9. *For any $\lambda \in \mathbb{R}$, $\mathcal{P}_{i\lambda}$ is a unitary irreducible representation of $SU(1,1)$ in the Hilbert space $L^2(\mathbb{S}^1)$.*

The generators of the Lie algebra are given by

$$L_0 = \frac{d}{d\theta}, \quad L_1 = (-\frac{1}{2}+i\lambda)\cos\theta - \sin\theta\frac{d}{d\theta}, \quad L_2 = -(-\frac{1}{2}+i\lambda)\sin\theta - \cos\theta\frac{d}{d\theta}. \tag{4.60}$$

Forming the linear combinations of $B_\pm = \pm L_1 + iL_2$, one obtains

$$B_+ = (-\frac{1}{2}+i\lambda)e^{-i\theta} - ie^{-i\theta}\frac{d}{d\theta}, \quad B_- = -(-\frac{1}{2}+i\lambda)e^{i\theta} - ie^{i\theta}\frac{d}{d\theta}. \tag{4.61}$$

Upon defining $B_0 = iL_0 = i\frac{d}{d\theta}$, one can calculate the Casimir operator

$$C := B_0 - \frac{1}{2}(B_-B_+ + B_+B_-) = (\frac{1}{4} - \lambda^2)\mathbb{I}. \tag{4.62}$$

For the complementary series, we present the following theorem.

Theorem 4.10. *For all $0 < \sigma < 1/2$, $\mathcal{C}_\sigma$ is a unitary irreducible representation of $SU(1,1)$ in the Hilbert space H_σ where*

$$\mathcal{C}_\sigma = |\beta z + \overline{\alpha}|^{-1+2\sigma} f(\frac{\alpha z + \overline{\beta}}{\beta z + \overline{\alpha}}). \tag{4.63}$$

Furthermore, its Casimir operator is given by $C_\sigma = (\sigma^2 - \frac{1}{4})\mathbb{I}$.

For the construction of the Hilbert space H_σ, see [5, p. 241].

5 Generalized Coherent States for the Discrete Series

We defined canonical coherent states in Section 2.2 and looked at some of their properties. Most importantly, they were the "closest states to the classical harmonic oscillator." Of course, this construction only works for the quantum harmonic oscillator. Therefore, one can ask the following question: can we similarly define coherent states for more general systems where these states somehow represent the closest states to a classical system?

5.1 Defining Generalized Coherent States

The answer is yes, and there are different ways of constructing such generalized coherent states. In this report, we will use the Gilmore–Perelomov generalized coherent states (GCS). The following explanations and definitions are adapted from Perelomov's book titled "Generalized Coherent States and Their Applications" [15].

Consider an arbitrary Lie group G with a unitary irreducible representation $T(g)$, acting in the Hilbert space $\mathcal{H}$. Take a fixed vector $|\psi_0\rangle$ in $\mathcal{H}$ and consider the set $\{|\psi_g\rangle\}$ where $|\psi_g\rangle = T(g)|\psi_0\rangle$ and $g \in G$. In this construction, we declare two vectors $|\psi_a\rangle$ and $|\psi_b\rangle$ equivalent (corresponding to the same state) if they differ only by a global phase factor, i.e., if $|\psi_a\rangle = e^{i\alpha}|\psi_b\rangle$ where $0 \leq \phi < 2\pi$. Hence, $|\psi_{g_1}\rangle$ and $|\psi_{g_2}\rangle$ correspond to the same state if

$$|\psi_{g_1}\rangle = e^{i\alpha}|\psi_b\rangle \Rightarrow \quad T(g_1)|\psi_0\rangle = e^{i\alpha}T(g_2)|\psi_0\rangle \Rightarrow \quad T(g_2^{-1}g_1)|\psi_0\rangle = e^{i\alpha}|\psi_0\rangle. \tag{5.1}$$

Suppose $H \leq G$ is a subgroup with the property $h \in H$ if $T(h)|\psi_0\rangle = e^{i\alpha}|\psi_0\rangle$. When the subgroup H is maximal, it is called the isotropy subgroup of the state $|\psi_0\rangle$.

This construction shows that if g_1, g_2 are two group elements belonging to the same left coset of G modulo the subgroup H, then $|\psi_a\rangle$ and $|\psi_b\rangle$ will only differ by a phase factor and determine the same state. Hence, by choosing a representative $g(x)$ in any equivalence class x, one gets a set of states $\{|\psi_{g(x)}\rangle\}$, where $x \in G/H$. With this construction in mind, we are ready to define generalized coherent states [15, p. 41].

Definition 5.1 (Generalized coherent states)**.** Consider an arbitrary Lie group G with a unitary irreducible representation $T(g)$, acting in the Hilbert space $\mathcal{H}$. The system of states $\{|\psi_g\rangle\}, |\psi_g\rangle = T(g)|\psi_0\rangle$, where g are elements of the group G and $|\psi_0\rangle$ is a fixed vector in $\mathcal{H}$, is called the coherent state system $\{T, |\psi_0\rangle\}$. Let H be the isotropy subgroup for the state $|\psi_0\rangle\}$, i.e. $H = \{h \in G : T(h)\psi_0 = e^{i\alpha}\psi_0\}$. Then a coherent state $|\psi_g\rangle$ is determined by a point $x = x(g)$ in the coset space G/H, corresponding to the element $g : |\psi_g\rangle = \exp(i\alpha)|x\rangle, |\psi_0\rangle = |0\rangle$.

We shall not go into detail here, but the reason we can generalize the coherent states this way is because we can obtain the CCS as a set of generalized coherent states with the above definition, where the Lie group G is taken to be the Heisenberg-Weyl group. We refer the reader to [15] for further details.

5.2 Choosing the Fixed State

(For the rest of the report, when we say GCS, we mean GCS with the group $SU(1,1)$ and the representation $\mathcal{D}_n^-(g)$.)

The definition above clearly depends on the chosen fixed state $|\psi_0\rangle$, where there is no restriction on how to choose this state. At the same time, we want our coherent states to be the most classical states in some way. This procedure depends on the nature of the group G and on the type of representation T.

Perelomov gives the recipe for a discrete series representation $T^k(g)$ of a real semi-simple group G [15, p. 47]. We already established that $G = SU(1,1)$ is a real, simple Lie group with a discrete unitary irreducible representation $\mathcal{D}_n^-(g)$, so we can adapt this construction for our group.

Let *su(1,1)* be the Lie algebra of G with the basis vectors B_0, B_1, B_2. The representation $\mathcal{D}_n^-$ is characterized by an integer $n \geq 2$, and the basis vectors $|\psi_m\rangle$ are the eigenvectors of the operator B_0 with the eigenvalue m, i.e. $B_0|\psi_m\rangle = m|\psi_m\rangle$. Comparing this with Eq. 4.50, we see that $m = \frac{n}{2} + k$. Denote the Casimir operator by $C_{as}^n = C_2 = B_0^2 - B_1^2 - B_2^2$ and recall that $C_2|\psi\rangle = \frac{n}{2}(\frac{n}{2} - 1)|\psi\rangle$.

We define the dispersion of the operator C_2 as

$$\Delta C_2 = \langle B_0^2 - B_1^2 - B_2^2 \rangle - \langle B_0 \rangle^2 + \langle B_1 \rangle^2 + \langle B_2 \rangle^2 \tag{5.2}$$

where $\langle O \rangle$ denotes the expectation value of the operator O. According to Perelomov, the coherent state system constructed based on the vector $|\psi_k\rangle$ that maximizes the dispersion of C_2 is the closest to the classical system [15, p. 47] (note that in the original source, the author minimizes this equation because the Casimir operator defined there is the negative of the Casimir operator that we use in this report. When one does everything consistently, one reaches the same conclusion as expected). Hence, we try to maximize

$$\Delta C_2 = \langle B_0^2 - B_1^2 - B_2^2 \rangle - \langle B_0 \rangle^2 + \langle B_1 \rangle^2 + \langle B_2 \rangle^2 = (\frac{n^2}{4} - \frac{n}{2}) - (\frac{n}{2} + k)^2 = -(\frac{n}{2} + nk + k^2) \tag{5.3}$$

where we evaluate the expectation values based on the state $|\psi_k\rangle$. Hence, we maximize this by taking $k = 0$, which shows that we should choose our fixed state as the eigenvector of B_0 with the lowest eigenvalue, which is $\frac{n}{2}$. Note that choosing $\psi_0(z) = 1$ suffices, since

$$B_0\psi_0 = \frac{n}{2} = \frac{n}{2}\psi_0 \tag{5.4}$$

so that ψ_0 is the eigenvector of B_0 with the eigenvalue $n/2$.

We introduced many definitions in the previous paragraphs, so it is a good idea to take a step back and see what we have done. Firstly, we had the canonical coherent states of the quantum harmonic oscillator at the very beginning of this report with very nice properties and a lot of applications. The most important feature of these states was that they resembled a system of classical states. However, they were only defined for the harmonic oscillator system. Therefore, secondly, we asked the question of whether it is possible to generalize this idea of finding a system of most classical states in a broader context. The answer was positive, and we defined the generalized coherent states to achieve this task. However, the definition of the coherent states that we presented depends on an initial fixed state to generate the system. There seems to be no reason to choose one state over the other one a priori. Therefore, thirdly, we asked the question of how we should choose this initial state. Perelomov answered this question by defining the dispersion of the Casimir operator and stating that the initial state should maximize the dispersion. Hence, we calculated this to determine the fixed initial state ψ_0. After all of this work, we managed to generalize the idea of coherent states for the discrete unitary irreps of the Lie group $SU(1,1)$.

5.3 Parameterizing Generalized Coherent States

Now that we have chosen our fixed state, Definition 5.1 tells us that we need to determine the isotropy subgroup H for the state ψ_0 to parametrize the generalized coherent states.

The isotropy subgroup for our system is given by

$$H = \{h \in SU(1,1) : \mathcal{D}_n^-(h)\psi_0 = e^{i\phi}\psi_0, \quad 0 \leq \psi < 2\pi\} \tag{5.5}$$

where $\mathcal{D}_n^-(g)f(z) = (\overline{\alpha} - \beta z)^{-n} f(\frac{-\overline{\beta}+\alpha z}{\overline{\alpha}-\beta z})$ and $\psi_0(z) = 1$. Therefore, we need

$$\mathcal{D}_n^-(h)\psi_0(z) = (\overline{\alpha} - \beta z)^{-n}\psi_0(\frac{-\overline{\beta} + \alpha z}{\overline{\alpha} - \beta z}) = (\overline{\alpha} - \beta z)^{-n}\psi_0(z). \tag{5.6}$$

Therefore, we need all $h \in SU(1,1)$ such that $(\overline{\alpha} - \beta z)^{-n} = e^{i\phi}$. Furthermore, this should hold for all $z \in \mathbb{D}$, so that $\beta = 0$. Therefore, for any $h = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix}$ we have that $h \in H$. Therefore, $H \cong U(1)$ through the exact identification that we used in Lemma 3.14.

As a result, using Lemma 3.14, we can parametrize the coherent states for the representation $\mathcal{D}_n^-$ using the pseudo-sphere $P\mathbb{S}^2$, since $P\mathbb{S}^2$ is in bijection with the right cosets of $SU(1,1)$ modulo $U(1)$. In the following computation, we will give an explicit parameterization.

Take an arbitrary point $\boldsymbol{n} = (\cosh\tau, \sinh\tau\sin\varphi, \sinh\tau\cos\varphi) \in P\mathbb{S}^2$ and denote $g_{\boldsymbol{n}} = \begin{pmatrix} \cosh\frac{\tau}{2} & \sinh\frac{\tau}{2}e^{-i\varphi} \\ \sinh\frac{\tau}{2}e^{i\varphi} & \cosh\frac{\tau}{2} \end{pmatrix}$. Recall that, by Lemma 3.14, $g_{\boldsymbol{n}}$ is a representative for the right cosets of $SU(1,1)$ for each $\boldsymbol{n}$. Therefore, we should look at the orbits of the fixed state ψ_0 under the representation for each $\boldsymbol{n} \in P\mathbb{S}^2$. Hence, we define

$$\psi_{\boldsymbol{n}} = \mathcal{D}_n^-(g_{\boldsymbol{n}})\psi_0. \tag{5.7}$$

Because $P\mathbb{S}^2$ is in bijection with $\mathbb{D}$, we can equivalently parametrize the coherent states using the unit disk as follows. Simply plugging in gives

$$\psi_{\boldsymbol{n}} = \mathcal{D}_n^-(g_{\boldsymbol{n}})\psi_0 = (\cosh\frac{\tau}{2} - \sinh\frac{\tau}{2}e^{-i\varphi}z)^{-n}. \tag{5.8}$$

Let $\zeta = \tanh\frac{\tau}{2}e^{i\varphi}$. Clearly, for each $\boldsymbol{n}$, there is a unique ζ and vice versa. Then, note that $(1-|\zeta|^2)^{n/2} = (\operatorname{sech}^2\frac{\tau}{2})^{n/2} = (\cosh\frac{\tau}{2})^{-n}$. At the same time, $(1-\overline{\zeta}z)^{-n} = (1-\tanh(\frac{\tau}{2})e^{-i\varphi}z)^{-n}$. Combining the two, we get

$$(1-|\zeta|^2)^{n/2}(1-\overline{\zeta}z)^{-n} = (\cosh\frac{\tau}{2})^{-n}(1-\tanh(\frac{\tau}{2})e^{-i\varphi}z)^{-n} = (\cosh\frac{\tau}{2} - \sinh\frac{\tau}{2}e^{-i\varphi}z)^{-n} = \psi_{\boldsymbol{n}}. \tag{5.9}$$

Therefore, we can parameterize the generalized coherent states using the unit disk $\zeta \in \mathbb{D}$:

$$\psi_\zeta(z) = (1-|\zeta|^2)^{n/2}(1-\overline{\zeta}z)^{-n}. \tag{5.10}$$

5.4 Basic Properties of the Generalized Coherent States

In this section, we will take a look at some of the properties of the generalized coherent states. We will show that they form an overcomplete set in the underlying Hilbert space and demonstrate the resolution of the identity, which are some of the properties that they share with the canonical coherent states. We will also rewrite Eq. 5.10 in terms of the basis vectors of $H_n(\mathbb{D})$, which will be useful for the discussions in the following chapter.

We start by rewriting the coherent states in terms of the canonical basis elements of $H_n(\mathbb{D})$ [5, p. 247].

Proposition 5.2. *The coherent state* ψ_ζ *can be written as* $\psi_\zeta = (1-|\zeta|^2)^k \sum_{l\in\mathbb{N}}(\frac{\Gamma(2k+l)}{\Gamma(l+1)\Gamma(2k)})^{1/2}(\bar{\zeta})^l e_l$ *where* $n := 2k$.

Proof. We start by using the expansion $(1-az)^{-2k} = \sum_{n=0}^{\infty} a^n(-z)^n\binom{-2k}{n}$. Then, by using the identity $\binom{-m}{k} = (-1)^k\binom{m+k-1}{k}$, we get $\binom{-2k}{n} = (-1)^n\binom{2k+n-1}{n}$. As a result, we have the equality

$$(1-\bar{\zeta}z)^{-2k} = \sum_{l=0}^{\infty}(\bar{\zeta})^l(-z)^l\binom{-2k}{l} = \sum_{l=0}^{\infty}(\bar{\zeta})^l(-z)^l(-1)^l\binom{2k+l-1}{l} = \sum_{l=0}^{\infty}(\bar{\zeta}z)^l\frac{(2k+l-1)!}{(2k-1)!l!}. \tag{5.11}$$

This proves that

$$\psi_\zeta = (1-|\zeta|^2)^k \sum_{l\in\mathbb{N}}(\frac{\Gamma(2k+l)}{\Gamma(l+1)\Gamma(2k)})^{1/2}(\bar{\zeta})^l e_l. \tag{5.12}$$

□

Next, we show that two distinct coherent states are not orthogonal to each other.

Theorem 5.3. *The coherent state family* $\psi_{\boldsymbol{n}} := |\boldsymbol{n}\rangle$ *is not an orthogonal system in* $H_n(\mathbb{D})$.

Proof. We can prove this claim by explicitly computing the inner product of two coherent states $|\boldsymbol{n}\rangle$ and $|\boldsymbol{n'}\rangle$. We start by computing the Fourier coefficient of the state $|\zeta\rangle$ in the basis e_l, which is obtained by using Eq. 5.12: $\langle e_l|\zeta\rangle = (1-|\zeta|^2)^k(\frac{\Gamma(2k+l)}{\Gamma(l+1)\Gamma(2k)})^{1/2}(\bar{\zeta})^l$. Using the Parseval identity, we calculate the inner product:

$$\langle\boldsymbol{n'}|\boldsymbol{n}\rangle = \sum_{l\geq 0}(\frac{\Gamma(2k+l)}{\Gamma(l+1)\Gamma(2k)})^{1/2}(1-|\zeta|^2)^k\zeta^l(\frac{\Gamma(2k+l)}{\Gamma(l+1)\Gamma(2k)})^{1/2}(1-|\zeta'|^2)^k(\overline{\zeta'})^l. \tag{5.13}$$

Note that $(1-\overline{\zeta'}\zeta)^{-2k} = \sum_{l=0}^{\infty}(\overline{\zeta'}\zeta)^l\frac{(2k+l-1)!}{(2k-1)!l!}$, which gives

$$\langle\boldsymbol{n'}|\boldsymbol{n}\rangle = (1-|\zeta|^2)^k(1-|\zeta'|^2)^k(1-\overline{\zeta'}\zeta)^{-2k}. \tag{5.14}$$

Alternatively, introducing $\zeta = \tanh(\frac{\tau}{2})e^{-i\varphi}$ for returning to pseudopolar coordinates, this expression can also be expressed as

$$\langle\boldsymbol{n'}|\boldsymbol{n}\rangle = (\cosh\frac{\tau'}{2}\cosh\frac{\tau}{2} - \sinh\frac{\tau'}{2}\sinh\frac{\tau}{2}e^{-i(\varphi'-\varphi)})^{-2k}. \tag{5.15}$$

□

Finally, we state the following two equivalent resolutions of the identity [5, p. 249].

Proposition 5.4. *We have the formulas*

$$\frac{2k-1}{4}\int_{P\mathbb{S}^2} d\boldsymbol{n}|\boldsymbol{n}\rangle\langle\boldsymbol{n}| = \mathbb{I} \tag{5.16}$$

$$\int_{\mathbb{D}} d\nu_{2k}(\zeta)|\zeta\rangle\langle\zeta| = \mathbb{I} \tag{5.17}$$

with the measure $d\nu_n(\zeta) = \frac{n-1}{\pi}\frac{d^2\zeta}{(1-|\zeta|^2)^2}, d^2\zeta = \frac{|d\zeta\wedge d\bar{\zeta}|}{2}$.

Proof. Recall $\psi_\zeta(z) = |\zeta\rangle = (1-|\zeta|^2)^k\sum_{l\in\mathbb{N}}\gamma_{2k,l}^{-1/2}\bar{\zeta}^l e_l(z)$. Then, we have

$$|\zeta\rangle\langle\zeta| = (1-|\zeta|^2)^{2k}\sum_{k,l\in\mathbb{N}}\gamma_{2k,l}^{-1/2}\gamma_{2k,m}^{-1/2}|e_l\rangle\langle e_m|\bar{\zeta}^l\zeta^m. \tag{5.18}$$

Therefore, we have

$$\int_{\mathbb{D}} d\nu_{2k}|\zeta\rangle\langle\zeta| = \sum_{k,l\in\mathbb{N}} \gamma_{2k,l}^{-1/2}\gamma_{2k,m}^{-1/2}|e_l\rangle\langle e_m| \int_{\mathbb{D}} d\nu_{2k}(1-|\zeta|^2)^{2k}\overline{\zeta}^l\zeta^m. \tag{5.19}$$

Let $\zeta = re^{i\varphi}$. Then, the integral on the right-hand side is given by

$$\int_0^{2\pi}\int_0^1 r^{m+l+1}\frac{(2k-1)}{\pi}(1-r^2)^{2k-2}e^{i(m-l)\varphi}drd\varphi = \int_0^1 r^{m+l+1}2(2k-1)(1-r^2)^{2k-2}\delta_{ml}dr \tag{5.20}$$

where we used the integral expression for the Dirac delta function. Therefore, the initial integral is given by

$$\int_{\mathbb{D}} d\nu_{2k}|\zeta\rangle\langle\zeta| = \sum_{l\in\mathbb{N}} \gamma_{2k,l}^{-1}|e_l\rangle\langle e_l| \int_0^1 r^{2l+1}2(2k-1)(1-r^2)^{2k-2}dr = \sum_{l\in\mathbb{N}} \gamma_{2k,l}^{-1}\gamma_{2k,l}^{1}|e_l\rangle\langle e_l| = \sum_{l\in\mathbb{N}} |e_l\rangle\langle e_l| = \mathbb{I} \tag{5.21}$$

where we made use of Eq. 4.27 to solve the integral.

□

Because the coherent state family is not an orthogonal system and because we have the resolution of the identity, generalized coherent states are not linearly independent and they form an overcomplete set, just like the canonical coherent states:

$$|\boldsymbol{n}\rangle = \frac{n-1}{4}\int_{P\mathbb{S}^2} d\boldsymbol{n}'|\boldsymbol{n}'\rangle\langle\boldsymbol{n}'|\boldsymbol{n}\rangle = \frac{n-1}{4}\int_{P\mathbb{S}^2} d\boldsymbol{n}'|\boldsymbol{n}'\rangle(\cosh\frac{\tau'}{2}\cosh\frac{\tau}{2} - \sinh\frac{\tau'}{2}\sinh\frac{\tau}{2}e^{-i(\varphi'-\varphi)})^{-2n}. \tag{5.22}$$

6 Transformation of the Canonical Coherent States

We started the thesis with the following question: How do the canonical coherent states of the quantum harmonic oscillator transform under the group $SU(1,1)$? In order to answer this question, we presented the unitary irreducible representation of the group $SU(1,1)$.

For the discrete series, these representations act on the Hilbert space $H_n(\mathbb{D})$. The problem is that we do not know what the canonical coherent states look like in $H_n(\mathbb{D})$. However, we know the expressions for the generalized coherent states that we derived in the previous chapter. Therefore, to answer the question of "how do generalized coherent states transform under $SU(1,1)$," we have all the tools that we need: we have the unitary irreducible representation $\mathcal{D}_n^-$ acting on $H_n(\mathbb{D})$, and we have the coherent states $\psi_\zeta(z)$ living in $H_n(\mathbb{D})$. We can take certain elements or certain subgroups of $SU(1,1)$ and check the transformation of the coherent states under these elements/subgroups. In principle, we can answer all the questions related to the transformation of these states.

This was a success, but we still want to investigate the transformation of the canonical coherent states. In this chapter, we will do two things. First, we will attempt to find a correspondence between the generalized and canonical coherent states. Since we know the transformation of generalized coherent states, if we can link these states to the canonical coherent states, we can also investigate the transformation of the canonical coherent states. Secondly, we will present an idea to construct a unitary representation of the group $SL(2,\mathbb{R})$ acting on the space $L^2(\mathbb{R})$. This is very desirable because we know the expressions for the canonical coherent states in $L^2(\mathbb{R})$. Using these expressions, we can act on them with the group representation to answer the question of how canonical coherent states transform.

6.1 Identifying Canonical and Generalized Coherent States

To examine the transformation of canonical coherent states, the first idea is to try to identify the generalized and canonical coherent states with each other. If we know how generalized coherent states transform, and if we can find a bijective correspondence onto canonical coherent states, or at least some correspondence, then we could also identify how canonical coherent states transform.

We know the expressions for canonical coherent states in $L^2(\mathbb{R})$, whereas generalized coherent states live in $H_n(\mathbb{D})$. Therefore, we can try to match the vector spaces with each other. This is possible due to the following result from functional analysis [17, p. 55]:

Theorem 6.1. *All separable Hilbert spaces are isometrically isomorphic.*

We already established that $H_n(\mathbb{D})$ is separable at the end of Section 4.2.1. It is also true that $L^2(\mathbb{R})$ is separable, which can be seen by the fact that the energy eigenstates of the quantum harmonic oscillator $|n\rangle$ form an orthonormal basis for the space. Therefore, we can find an isomorphism between the two Hilbert spaces. This is done in [13, p. 15] as follows.

Take a normalized vector $|\psi\rangle$ in $L^2(\mathbb{R})$ and write it as $|\psi\rangle = \sum c_n|n\rangle$ where $|n\rangle$ are the energy eigenstates of the quantum harmonic oscillator in 1 space dimension. Then, define the function

$$\psi(\zeta) = \sum_{n=0}^{\infty} \sqrt{\frac{\Gamma(2k+n)}{\Gamma(n+1)\Gamma(2k)}} c_n \zeta^n. \tag{6.1}$$

Clearly, $\psi(\zeta)$ is holomorphic on the unit disk, and $||\psi||^2 = \int_{\mathbb{D}} |\psi(\zeta)^2| d\nu_n(z) dxdy < \infty$. Therefore, $\psi(\zeta) \in H_n(\mathbb{D})$, and Eq. 6.1 establishes the isomorphism between $H_n(\mathbb{D})$ and $L^2(\mathbb{R})$.

We can make this isomorphism more explicit. Note that an energy eigenstate $|l\rangle$ is mapped under this isomorphism to the function $|l\rangle \mapsto \sqrt{\binom{l+2k-1}{l}}\zeta^l$. But recall that

$$e_l(\zeta) = \frac{\zeta^l}{\sqrt{\gamma_{2k,l}}} = \frac{\zeta^l}{\sqrt{\frac{\Gamma(2k)\Gamma(l+1)}{\Gamma(2k+l)}}} = \sqrt{\binom{l+2k-1}{l}}\zeta^l. \tag{6.2}$$

As a result, the isomorphism established in Eq. 6.1 maps nothing but orthonormal basis elements to orthonormal basis elements. Hence, we can define it by extending the following map linearly:

$$Y : L^2(\mathbb{R}) \to H_n(\mathbb{D}), \qquad |l\rangle \mapsto e_l. \tag{6.3}$$

Now that we established the isomorphism between the Hilbert spaces, we can see how GCS look like in $L^2(\mathbb{R})$. Recall that, in the basis e_l, a coherent state is given by $\psi_\zeta(z) = (1-|\zeta|^2)^k \sum_{l\in\mathbb{N}} \gamma_{2k,l}^{-1/2} \overline{\zeta}^l e_l(z)$. Denote by $\tilde{\psi}_\zeta^k \in L^2(\mathbb{R})$ as the image of $\psi_\zeta(z)$ under the isomorphism Y. Then, we have

$$\tilde{\psi}_\zeta^k = (1-|\zeta|^2)^k \sum_{l\in\mathbb{N}} \gamma_{2k,l}^{-1/2} \overline{\zeta^l} |l\rangle. \tag{6.4}$$

We can rewrite this by noting that

$$\gamma_{2k,l}^{-1/2} = \sqrt{\frac{(l+2k-1)!}{l!(2k-1)!}} = \frac{1}{\sqrt{l!}}\sqrt{(l+2k-1)_l} \tag{6.5}$$

where $(x)_n$ is the falling factorial. As a result, we get

$$\tilde{\psi}_\zeta^k = (1-|\zeta|^2)^k \sum_{l\in\mathbb{N}} \sqrt{(l+2k-1)_l}\frac{\overline{\zeta^l}}{\sqrt{l!}}|l\rangle. \tag{6.6}$$

This expression is not in the form of a CCS that we derived in Eq. 2.7, so we cannot work with it. However, there is a limiting behavior between the generalized and canonical coherent states that relates the two systems, which is expressed in [5, p. 250].

Theorem 6.2 (Large k limit)**.** $\lim_{k\to\infty} \tilde{\psi}_{\zeta/\sqrt{2k}}^k = |\overline{\zeta}\rangle$

Proof. We have the first limit

$$\lim_{k\to\infty}(1-|\frac{\zeta}{\sqrt{2k}}|^2)^k = \exp(-|\zeta|^2/2) \tag{6.7}$$

and the second limit

$$\lim_{k\to\infty}\sum_{l\in\mathbb{N}}\sqrt{(l+2k-1)_l}\frac{(\overline{\zeta^l})/(\sqrt{2k})^l}{\sqrt{l!}}|l\rangle = \lim_{k\to\infty}\sum_{l\in\mathbb{N}}\sqrt{\frac{(l+2k-1)!}{l!(2k-1)!}}\frac{\overline{\zeta^l}}{(\sqrt{2k})^l}|l\rangle = \sum_{l\in\mathbb{N}}\frac{1}{\sqrt{l!}}\overline{\zeta}^l|l\rangle. \tag{6.8}$$

Combining the two, we obtain

$$\lim_{k\to\infty}\tilde{\psi}^k_{\zeta/\sqrt{2k}} = \exp(-|\zeta|^2/2)\sum_{l\in\mathbb{N}}\frac{\overline{\zeta^l}}{\sqrt{l!}}|l\rangle = |\overline{\zeta}\rangle. \tag{6.9}$$

□

This is an interesting result, for which we are not sure of the correct interpretation. When we take a generalized coherent state and map it to $L^2(\mathbb{R})$, we obtain the state $\tilde{\psi}^k_\zeta$, and when the number k goes to infinity as the parameter value of ζ goes to zero at a certain rate, we recover a canonical coherent state. This theorem suggests that there is a deeper connection between the generalized and canonical coherent states that the author of this report is not aware of, which may be helpful to better understand the transformation of canonical coherent states.

6.2 Changing the Carrier Space

Another approach we can take is as follows. When we constructed the irreducible representations of $SU(1,1)$ in Chapter 4, the Hilbert space on which the representations acted on was $H_n(\mathbb{D})$. Since $H_n(\mathbb{D})$ and $L^2(\mathbb{R})$ are isomorphic as Hilbert spaces, can we construct unitary irreps of $SU(1,1)$ that act on $L^2(\mathbb{R})$ instead? This would be very useful, as the representations could act directly on the canonical coherent states.

The first approach we can try is to realize the operator $\mathcal{D}_n^-(g)$ as an operator acting on $L^2(\mathbb{R})$. Since $\mathcal{D}_n^-$ is a linear operator, we should find out how it acts on the basis elements of $L^2(\mathbb{R})$, because if we can find this, then we know how it acts on any vector in $L^2(\mathbb{R})$ due to linearity.

Let us take an element from the orthonormal basis of $H_n(\mathbb{D})$ and see how $\mathcal{D}_n^-$ acts on it. We simply compute

$$\mathcal{D}_n^-(g)e_l(z) = (\overline{\alpha}-\beta z)^{-n}e_l(\frac{-\overline{\beta}+\alpha z}{\overline{\alpha}-\beta z}) = (\overline{\alpha}-\beta z)^{-(n+l)}(-\overline{\beta}+\alpha z)^l\cdot\frac{1}{\sqrt{\gamma_{n,l}}}. \tag{6.10}$$

This is a complex rational function, so we can write it as a Taylor series: $\mathcal{D}_n^-(g)e_l(z) = c_0 + c_1 z^1 + \cdots$. Unfortunately, after some algebraic manipulations and some literature search, we are not aware of simple, closed-form expressions for the Taylor coefficients c_j of the above function.

For a moment, suppose that we can find explicit expressions for the Taylor coefficients, so that we write $\mathcal{D}_n^-(g)e_l(z) = c_0 + c_1 z^1 + \cdots$. By scaling z^l terms, we would get an expression in terms of the orthonormal basis $\{e_l\}_{l\geq 0}$. Because Y maps $|l\rangle$ to e_l, this would give an expression for the action of $\mathcal{D}_n^-(g)$ on $|l\rangle$ in terms of other energy eigenstates, where $\mathcal{D}_n^-$ now acts on $L^2(\mathbb{R})$. From this, using linearity, it would be possible to construct a unitary irrep of $SU(1,1)$ acting on the relevant Hilbert space.

As a matter of fact, the matrix elements of the form $\langle e_l|\mathcal{D}_n^-(g)|e_m\rangle$ are derived by Bargmann in terms of hypergeometric functions in [1, Chapter 10]. Once again, in principle, this is all we need to define the representation in an alternative Hilbert space, as the matrix elements determine the action of the operator on any vector in $L^2(\mathbb{R})$.

6.3 Directions for Further Research

Although one can investigate the transformation of canonical coherent states as a limiting case of generalized coherent states, this approach has limitations. For instance, it is not clear how we can start with a canonical coherent state in $L^2(\mathbb{R})$ and see how it transforms under an element g of the group $SU(1,1)$. With our current approach, we necessarily start from a generalized coherent state and obtain a canonical coherent state as a limiting case, where we lose some information about the initial generalized coherent state. Therefore, although certain properties of the transformation of canonical coherent states can be obtained this way, we ideally would like to come up with an alternative approach to investigate their transformation that is more direct.

To overcome these limitations, we would like to come up with unitary irreducible representations in $L^2(\mathbb{R})$. If this can be realized, then these representations can directly act on the canonical coherent states, as we have the expressions for canonical coherent states in the space $L^2(\mathbb{R})$. This would provide a more direct way of investigating the transformation of canonical coherent states under arbitrary elements of the group $SU(1,1)$.

We end the report by suggesting a possible way to come up with such a representation. Without proofs, we will construct a group action of $SL(2,\mathbb{R})$ and conjecture a unitary irreducible representation of $SL(2,\mathbb{R})$ acting on $L^2(\mathbb{R})$. Since $SL(2,\mathbb{R})$ is isomorphic to $SU(1,1)$ as a group, the following constructions can also be adapted for $SU(1,1)$.

First, we define the 2×2 projective special linear group $PSL(2,\mathbb{R})$ over real numbers as $PSL(2,\mathbb{R}) = SL(2,\mathbb{R})/\{\pm\mathbb{I}\}$, which is nothing but the quotient of $SL(2,\mathbb{R})$ by its center. Earlier in the report, we used Eq. 3.31 and Proposition 3.13 to show that $SU(1,1)$ quotient out by its center, or equivalently $SL(2,\mathbb{R})$ quotient out by its center, is isomorphic to the group of conformal automorphisms of the unit disk. With our new definition, we can say that the group $PSL(2,\mathbb{R})$ is isomorphic to the group of conformal automorphisms of the unit disk. Using this automorphism, we defined a group action in Theorem 4.6, which was used to construct a unitary representation of $SU(1,1)$ in Theorem 4.7. Our goal is to follow a similar path to come up with a different representation.

Similarly to their relation with the open unit disk, the elements of $PSL(2,\mathbb{R})$ are also homographies on the real projective line $\mathbb{R}P^1$, where $\mathbb{R}P^1 = \mathbb{R} \cup \{\infty\}$. Using this, we have the following proposition.

Theorem 6.3. *$M_- : SL(2,\mathbb{R}) \times \mathbb{R}P^1 \to \mathbb{R}P^1$, $\quad M_-(g)(r) = \frac{ar+b}{cr+d}$ where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1$ and $r \in \mathbb{R}P^1$ is a left group action.*

Proof. The proof is identical to the proof of Theorem 4.6. $\square$

Next, we would like to use this group action to construct a representation. We can do this with an equation of the form $\tilde{\mathcal{D}}^-(g)f(x) = c_g^-(x)(f(M_-(g^{-1})(x))$ where $g \in SL(2,\mathbb{R})$, $f \in L^2(\mathbb{R})$, and $c_g^-(x)$ is chosen so that the representation is unitary. The expression $f(M_-(g^{-1})(x))$ is given by $f(\frac{dx-b}{-cx+a})$. One should be careful here, as for certain values of x, c, and a the argument of f may be infinite, whereas the domain of f is the real numbers. We propose to overcome this technicality as follows. In general, it is not true that an arbitrary function $f \in L^2(\mathbb{R})$ will go to 0 as $x \to \pm\infty$. However, for the wavefunctions f that we consider in quantum mechanics, this is the case (for instance, if a function f is in the domain of the momentum operator $\hat{p}$, then f indeed goes to 0 as $x \to \pm\infty$). Therefore, we define $f(\infty) = 0$ so that the function on the right-hand side should be well-defined for any wavefunction. Using this, we state the main conjecture of this report.

Conjecture 6.4. $\tilde{D}^-(g)f(x) = (xc+d)f(\frac{dx-b}{-cx+a})$ is a unitary representation of $SL(2,\mathbb{R})$ in $L^2(\mathbb{R})$.

We note that the factor $(xc+d)$ makes the representation unitary, and since $\int_{\mathbb{R}} |f(x)|dx < \infty$, the expression on the right-hand side also has finite L^2 norm, which makes it square integrable. If the way we tried to overcome the technical details explained above is mathematically justified, we believe that this is indeed a unitary representation of $SL(2,\mathbb{R})$ acting in $L^2(\mathbb{R})$. We underline two important questions for further research. The first is related to the well-definedness of this representation and modifying the arguments above to make this statement completely rigorous. Secondly, one can investigate whether the representation is irreducible or not. If this is a valid irreducible representation of $SL(2,\mathbb{R})$ in $L^2(\mathbb{R})$ and if all such representations can be found, one would have all the tools to investigate the transformation of canonical coherent states.

Assuming Conjecture 6.4 is a valid unitary representation of $SL(2,\mathbb{R})$ in $L^2(\mathbb{R})$, we can investigate the transformation of canonical coherent states under this representation. The canonical coherent states in the position basis are given by [12, p. 2]

$$|\alpha\rangle = \psi_\alpha(x) = (\frac{m\omega}{\pi\hbar})^{1/4}\exp(-(|\alpha|^2+\alpha^2)/2)\exp(\sqrt{\frac{2m\omega}{\hbar}}\alpha x)\exp(-\frac{m\omega}{2\hbar}x^2). \tag{6.11}$$

Denote $s = \sqrt{\frac{\hbar}{m\omega}}$. Then, we have

$$\tilde{D}^-(g)\psi_\alpha(x) = (xc+d)(\frac{m\omega}{\pi\hbar})^{1/4}\exp(-(|\alpha|^2+\alpha^2)/2)e^{\frac{\sqrt{2}\alpha}{s}(\frac{dx-b}{a-cx})}e^{\frac{-1}{2s^2}(\frac{dx-b}{a-cx})^2} := \varphi_{g,\alpha}(x). \tag{6.12}$$

We calculate

$$\hat{x}\varphi_{g,\alpha}(x) = x\varphi_{g,\alpha}(x) \tag{6.13}$$

$$\hat{p}\varphi_{g,\alpha}(x) = -i\hbar\frac{\partial}{\partial x}\varphi_{g,\alpha}(x) = -i\hbar\varphi_{g,\alpha}(x)[\frac{c}{(xc+d)} + \frac{\sqrt{2}\alpha}{s}\frac{1}{(a-cx)^2} - \frac{1}{s^2}\frac{(dx-b)}{(a-cx)^3}]. \tag{6.14}$$

Since $a_- = \frac{1}{\sqrt{2\hbar m\omega}}(i\hat{p}+m\omega\hat{x})$, we get

$$a_-\varphi_{g,\alpha}(x) = \frac{\varphi_{g,\alpha}(x)}{\sqrt{2\hbar m\omega}}[\frac{\hbar c}{xc+d} + \sqrt{2m\omega\hbar}\alpha\frac{1}{(a-cx)^2} - m\omega\frac{(dx-b)}{(a-cx)^3} + m\omega x]. \tag{6.15}$$

We now introduce a special decomposition of $SL(2,\mathbb{R})$ called the Iwasawa decomposition. Define the following subgroups of $SL(2,\mathbb{R})$:

$$K = \{\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\},\quad A = \{\begin{pmatrix}r & 0\\ 0 & 1/r\end{pmatrix} : r>0\},\quad N = \{\begin{pmatrix}1 & \gamma\\ 0 & 1\end{pmatrix}\}. \tag{6.16}$$

Then, we have the following theorem [6, p. 1].

Theorem 6.5. *We have a decomposition $SL(2,\mathbb{R}) = KAN$: every $g \in SL(2,\mathbb{R})$ has a unique representation as $g = kan$ where $k \in K, a \in A$, and $n \in N$.*

We can investigate the transformation of $|\alpha\rangle$ under these subgroups. Let g_n be an element of the subgroup N, also called the parabolic subgroup. We then have

$$a_-\varphi_{g_n,\alpha}(x) = \varphi_{g_n,\alpha}(x)[\alpha + \sqrt{\frac{m\omega}{2\hbar}}\gamma] = |\alpha + \sqrt{\frac{m\omega}{2\hbar}}\gamma\rangle. \tag{6.17}$$

This is a new canonical coherent state with eigenvalue $\alpha + \sqrt{\frac{m\omega}{2\hbar}}\gamma$, showing that we can add any real number $\tilde{\gamma} = \sqrt{\frac{m\omega}{2\hbar}}\gamma$ to α to obtain a new canonical coherent state. Under the parabolic subgroup, canonical coherent states transform into other canonical coherent states.

Similarly, under the hyperbolic subgroup A, we obtain a squeezed coherent state with any positive squeezing parameter ζ. For the definition and discussion of squeezed coherent states, see [10, Chapter 7]. Denote the squeezed coherent state with parameters α, ζ as $|\alpha,\zeta\rangle = \psi_{\alpha,\zeta}$. Hence, under the subgroup AN, the canonical coherent states first turn into another canonical coherent state with the real part scaled, and then they are squeezed to become a squeezed coherent state.

We could not realize the transformation of the canonial coherent states under the subgroup K as special states of the harmonic oscillator such as squeezed coherent states, but we have the closed-form expression of the transformed state due to Conjecture 6.4. Regardless, this is precisely what we are looking for. For a fixed canonical coherent state $|\alpha\rangle$, we can pick an arbitrary $g \in SL(2, \mathbb{R})$, and write it uniquely as $g = kan$. Since group representations are group homomorphisms, we have

$$\tilde{D}^{-}(g)\psi_{\alpha}(x) = \tilde{D}^{-}(kan)\psi_{\alpha}(x) = \tilde{D}^{-}(k)\tilde{D}^{-}(an)\psi_{\alpha}(x) = \tilde{D}^{-}(k)\psi_{\tilde{\alpha},\tilde{\zeta}} \tag{6.18}$$

for some $\tilde{\alpha} \in \mathbb{C}, \tilde{\zeta} > 0$ that depend on the group element g. Expressions for the squeezed coherent states in terms of the energy eigenstates are known. Therefore, we get the final expression for the transformed canonical coherent state as

$$\tilde{D}^{-}(g)\psi_{\alpha}(x) = \tilde{D}^{-}(k)\psi_{\tilde{\alpha},\tilde{\zeta}} = (\sin\theta x + \cos\theta)\psi_{\tilde{\alpha},\tilde{\zeta}}(\frac{\cos\theta x + \sin\theta}{-\sin\theta x + \cos\theta}). \tag{6.19}$$

If our approach is correct, under this unitary irreducible representation of $SL(2, \mathbb{R})$, this is how the canonical coherent states transform.

7 Conclusion

In this thesis, we investigated the transformation of coherent states under the group $SU(1, 1)$.

First, we introduced the canonical coherent states of the quantum harmonic oscillator, which are the normalized eigenstates of the lowering operator. We demonstrated that they have many nice properties, including minimizing the uncertainty product. After this, we established the symmetry group of the harmonic oscillator, called the symplectic group $Sp(2, \mathbb{R})$, which was later shown to be isomorphic to the group $SU(1, 1)$. Hence, we formulated one of the main questions in this report, which is about the transformations of the canonical coherent states under $SU(1, 1)$. To answer this question, we needed two related branches of mathematics, namely Lie theory and representation theory. Lie theory is introduced to analyze the properties of the group $SU(1, 1)$, and the representation theory is used to come up with the unitary irreducible representations of $SU(1, 1)$.

Later, we studied the algebraic and topological properties of the group $SU(1, 1)$, including several parameterizations, Lie algebra, Casimir operator, dimension, connectedness, compactness, and simplicity. We also looked at the geometry of the parameter space of $SU(1, 1)$ with different models. Each of these properties has an influence on the representations of the group. In particular, we concluded that the nontrivial unitary representations of $SU(1, 1)$ were all infinite-dimensional.

In Chapter 4, we started by classifying the possible unitary irreducible representations of the group $SU(1, 1)$ based on some conditions that they need to satisfy. Then, our goal was to explicitly realize these representations. The first family of representations is the discrete series. After defining the carrier space $H_n(\mathbb{D})$, we constructed a family of representations of the group. We showed that the representations were unitary, irreducible, and discrete, verifying that they belonged to the discrete series. Without proofs, we presented analogous results for the other families of representations.

In Chapter 5, we used discrete series to define generalized coherent states for the group $SU(1, 1)$. We showed that the generalized coherent states are, as their name suggests, a generalization of the canonical coherent states of the harmonic oscillator. We justified this generalization by comparing some of their properties with the canonical coherent states. Finally, we parameterized the generalized coherent states and found explicit equations for them. In doing so, we obtained the necessary tools to investigate the transformation of generalized coherent states under $SU(1, 1)$.

In the last chapter, we returned to the original question of the transformation of canonical coherent states. We tried to answer this question with two methods. In the first one, we tried to establish a connection between the generalized and canonical coherent states. We presented a way of identifying canonical coherent states as limiting cases of generalized coherent states, based on a theorem in [5, p. 250]. This makes it possible to investigate their transformation. Although this approach was successful, it is not the most suitable way to analyze the transformation of canonical coherent states. Therefore, we took a second approach in which we tried to construct representations in the space $L^2(\mathbb{R})$ since the expressions for the coherent states in this Hilbert space are better understood. After pointing out in the literature where the matrix elements of the operator are calculated, which makes it possible to construct representations in $L^2(\mathbb{R})$, we conjectured yet another unitary representation of the group $SL(2, \mathbb{R})$ acting on the space $L^2(\mathbb{R})$. If this representation is well-defined and irreducible, it would provide us with the tools to explicitly answer the question of the transformation of canonical coherent states.

Overall, we managed to answer some of the questions we asked ourselves at the beginning. In particular, we found in the literature the classification of the unitary irreducible representations of $SU(1,1)$ and presented them in an organized way based on [5, Chapter 8], which are essential for analyzing the transformation of coherent states. We also presented a method that is adapted from [15] to generalize the canonical coherent states and obtained the necessary results to investigate their transformation under the group $SU(1,1)$. The question of the transformation of the canonical coherent states under the discrete representations is also answered, but there is more work to be done to obtain a more direct answer to this question.

There are many open questions to be answered. We still wonder if there are more explicit ways of investigating the transformations of the canonical coherent states, which was the biggest goal of this report. Furthermore, there are more properties of the generalized coherent states that can be understood better. For instance, the Wigner functions of the canonical coherent states preserve an area in the complex plane, which is related to their minimization of the uncertainty product. Is it possible to find a similar geometric meaning for generalized coherent states? Do they preserve an area on the unit disk, or do they minimize a certain operator that is related to their behavior of being the most classical states? What is the physical interpretation of the large k limit that relates them to canonical coherent states? Clearly, more research can be done on the generalized coherent states to answer these and other related questions.

Moreover, in this report, we mostly worked with the discrete series. One can also construct the generalized coherent states for other unitary irreducible representations and ask the same questions that we examined. It is possible that there is a more clear connection between the generalized coherent states of different representations and the canonical coherent states.

Finally, one can analyze the transformation of coherent states in higher dimensions. Almost everything in this report should be done from the beginning. One should first define the canonical coherent states of the harmonic oscillator in N dimensions. Then, the symmetry group of the oscillator is $Sp(2N, \mathbb{R})$, so one should find the unitary irreducible representations of the group $Sp(2N, \mathbb{R})$. After this, one can investigate the transformation of coherent states, both generalized and canonical. It would be nice to generate a systematic way to obtain these results for an arbitrary dimension N.

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References

[1] V. Bargmann. "Irreducible Unitary Representations of the Lorentz Group". In: *Annals of Mathematics* (1946).

[2] R.A. Bertlmann. *Theoretical Physics T2 Quantum Mechanics*. Lecture Notes. 2008. URL: `https://homepage.univie.ac.at/reinhold.bertlmann/pdfs/T2_Skript_final.pdf`.

[3] D. Boer. *Lie Groups in Physics*. Lecture Notes. 2024.

[4] W.B. Case. "Wigner functions and Weyl transforms for pedestrians". In: *American Association of Physics Teachers* (2008).

[5] M. Combescure and D. Robert. *Coherent States and Applications in Mathematical Physics*. Springer, 2012. ISBN: 978-94-007-0196-0.

[6] K. Conrad. *DECOMPOSING SL2(R)*. Lecture Notes. URL: `https://homepage.univie.ac.at/reinhold.bertlmann/pdfs/T2_Skript_final.pdf`.

[7] J.B. Conway. *A Course in Operator Theory*. American Mathematical Society, 2000. ISBN: 978-1-4704-8061-5.

[8] J.F. Cornwell. *Group Theory in Physics (Volume 2)*. Academic Press, 1986. ISBN: 978-0121898045.

[9] M. Einsiedler and T. Ward. *Unitary Representations and Unitary Duals*. Lecture Notes. 2024. URL: `https://tbward0.wixsite.com/books/unitary`.

[10] C.C. Gerry and P.L. Knight. *Introductory Quantum Optics*. Cambridge University Press, 2005. ISBN: 978-0-511-22949-7.

[11] D.J. Griffiths. *Introduction to Quantum Mechanics*. Pearson Education Limited, 2014. ISBN: 978-1-292-02408-0.

[12] J. Mostowski and J. Pietraszewicz. "Wigner Function for Harmonic Oscillator and The Classical Limit". In: (2021).

[13] A.M. Perelomov. "Coherent States for Arbitrary Lie Group". In: *Communications in Mathematical Physics* (1972).

[14] A.M. Perelomov. "Generalized coherent states and some of their applications". In: *Soviet Physics Uspekhi* (1977).

[15] A.M. Perelomov. *Generalized Coherent States and Their Applications*. Springer-Verlag Berlin Heidelberg, 1986. ISBN: 978-3-642-61629-7.

[16] M. Seri. *Analysis on Manifolds*. Lecture Notes. 2025. URL: `https://www.mseri.me/analysis-on-manifolds/`.

[17] A. Sterk and H. Snoo. *Functional Analysis An Introduction*. Lecture Notes. 2023.