university of groningen / faculty of science and engineering / mathematics and applied mathematics

Exploration of the Banach-Tarski paradox

Bachelor's Project Mathematics

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Abstract

This thesis covers the proof of the Banach-Tarski paradox for the unit ball in $\mathbb{R}^3$, as well as extensions of this result to various objects and spaces. The Banach-Tarski paradox is generalized to the rational unit sphere in $\mathbb{R}^3$, the unit sphere in $\mathbb{R}^n$ for $n \geq 3$, and the unit square in $\mathbb{R}^2$. Notions such as free group, equidecomposability, and paradoxical decomposition are defined and applied to prove various results leading to the proof of the Banach-Tarski paradox. Based on the paradoxical decompositions that originate from the proofs of the Banach-Tarski paradox, Mathematica code that creates a visualisation of sets that are used in the construction of the Banach-Tarski is developed. This code is applied to create figures for the unit sphere and the unit square.

Contents

1 Introduction

Mathematics was developed to describe the world and its natural phenomena. This branch of science started as a way to provide tools to measure, count, and describe everyday objects. Even though the ideas studied by mathematicians soon became abstract, a confusion might still arise when mathematical results do not reflect what is expected to happen in the real world. Such results are called paradoxes. Their consequences on the existing theory can be staggering, as every paradoxical result makes mathematicians question the assumptions made to obtain such an unintuitive outcome. One of the most famous paradoxes in mathematics is the Banach-Tarski paradox, proved by Stefan Banach and Alfred Tarski in 1924.

Theorem 1 (The Banach-Tarski paradox)**.** *A solid ball in* $\mathbb{R}^3$ *can be taken apart into finitely many pieces that can be rearranged using isometries to form two solid balls, each of the same volume as the initial one [3].*

Claiming that this result is confusing is an understatement. Indeed, it turns out that it is possible to double the volume of a ball by using only volume preserving transformations. What is more, this result can be achieved by decomposing the ball into a minimum of five pairwise disjoint subsets that are later rearranged using suitable rotations to form two balls [12]. This procedure is far from what can be observed in real life, so it is not a surprise that this result soon obtained the label of a paradox. The Banach-Tarski paradox, and results inspired by it, are to these days a rich source of counterexamples in the area of mathematics called measure theory.

It is a well-known fact that working with uncountably infinite sets and the Axiom of Choice, which needs to be assumed to conclude the result obtained by Banach and Tarski, can lead to counter-intuitive results. The pivotal part of many arguments considered in this thesis plays the free group with two generators. Visual representation of the structure of this group, in form of a Cayley diagram, can be seen in Figure 1. The proof of the Banach-Tarski paradox for the unit ball, as well as for other sets, heavily relies on finding very particular decompositions of them. It turns out that finding such decompositions can be simplified by first considering the suitable decomposition of the free group, and a way in which it acts on a given set. Later this decomposition of a group can be translated onto the set.

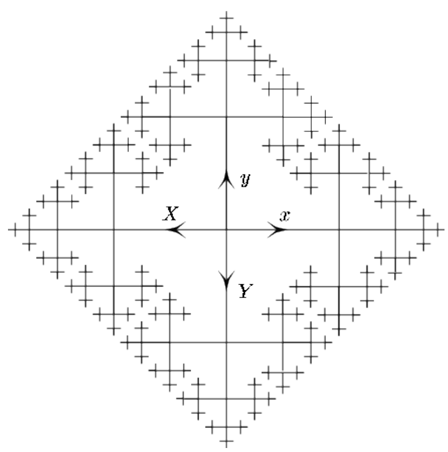

Figure 1: Cayley diagram of the free group with two generators, more information about this group can be found in Section 3.1 [5].

This thesis provides the introduction to the theory of paradoxical sets that is crucial to understand the proof of the Banach-Tarski Paradox. Notions such as paradoxical decomposition and equidecomposability are discussed. The concept of the free groups, with the emphasis on a free group generated by two rotations, is defined. An exhaustive proof of the Banach-Tarski paradox in $\mathbb{R}^3$ for the unit ball is presented, and the possible extensions of this result are explored. Extensions include generalization of the result to the rational unit sphere, which allows for the paradox to take place without assuming the Axiom of Choice. The Banach-Tarski paradox is also proved to hold in Euclidean spaces of dimension greater than three. For the case of the Banach-Tarski paradox in a plane, the result is proved for a square. This requires adjusting the standard set up of the paradox, which leads to the exploration of the area preserving transformations in $\mathbb{R}^2$.

Given the fact that the proof of the Banach-Tarski paradox requires assuming the Axiom of Choice, the proof is not constructive. However, an attempt can be made at visualizing the initial stages of the decomposition that the set undergoes during the Banach-Tarski paradox. To do that, Mathematica code is developed to generate a finite approximation of the construction of the Banach-Tarski paradox for the unit sphere. This code is then adjusted to provide a visualization of the paradox for a square.

2 Congruence by dissection, equidecomposability and paradoxical decompositions

The Banach-Tarski paradox describes an unintuitive relation between sets in $\mathbb{R}^3$. However, this is not the first result that puts in question the geometric intuition of mathematicians. In fact, the idea of trying to relate two different objects by decomposing one into finitely many pieces and then constructing another object out of them, has been present in geometry from the times of ancient Greeks. Over the years, many notions were developed as a way to formalize the idea of being able to build two different objects from the same pieces.

2.1 Congruence by dissection

The idea that a polygon can be decomposed into finitely many pieces that are later rearranged to create another polygon should be intuitively plausible. Elementary examples of this phenomena can be easily observed in the real life, for example when cutting a piece of paper into polygonal pieces and then rearranging them to create some other polygon of the same area. This observation inspired mathematicians, who turned the process of cutting shapes into pieces and rearranging them to form a different shapes into a rigorous mathematical operation called the congruence by dissection.

Definition 2.1. Two polygons are congruent by dissection if one of them can be decomposed into finitely many polygonal piece that are rearranged using isometries to form the second polygon (ignoring the boundaries).

It is a natural consequence of this definition that two polygons that are congruent by dissection must have the same area. What is more, a stronger statement that was proven by Bolyai and Gerwien holds.

Theorem 2 (Bolyai-Gerwien Theorem). *Two polygons are congruent by dissection if and only if they have the same area.*

Proof. A proof can be found in Chapter 3 of [17]. □

Figure 2 demonstrates an example of a triangle and a square being congruent by dissection. A triangle is cut into four polygonal pieces that are appropriately rotated and translated to form a square.

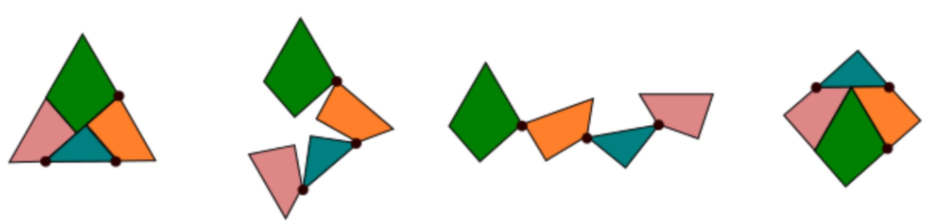

Figure 2: Example of a square and a triangle being congruent by dissection [8].

Theorem 2 describes a relation between shapes that have the same area; a polygon can be transformed by dissection into another polygon if and only if both have the same area. An important fact that needs to be reminded, is that this statement holds only if the pieces used to create a decomposition are polygonal, and there is a finite number of them. Now, it can be considered what would be the result of dropping the restriction on the shape of the pieces.

2.2 Equidecomposability

Leaving the idea of decomposing objects into polygonal pieces behind, with an intent to generalize the notion of congruence by dissection, leads to the idea of equidecomposability.

Definition 2.2. Consider two sets $A, B \subset X$ and a group G that acts on X. The sets A and B are equidecomposable (or G-equidecomposable) if it holds that

$$A = \bigcup_{i=1}^{n} A_i \qquad B = \bigcup_{i=1}^{n} B_i,$$

where the collections of sets A_i and B_i form partitions of A and B, respectively, and there exist some elements $g_1, \dots g_n \in G$ such that $B_i = g_i(A_i)$ for all $i \leq n$.

If sets A and B are G-equidecomposable, it is denoted by $A \sim_G B$. Sometimes subscript G is omitted, or the notation $\sim_n$ is used to underline the number of pieces used in the equidecomposition. Having introduced the definition of equidecomposability, the Banach-Tarski paradox, first presented in Theorem 1, can be reformulated.

Theorem 3 (The Banach-Tarski paradox, equidecomposability version)**.** *The unit ball B in $\mathbb{R}^3$ and two unit balls $B \sqcup B$ are $SO(3)$-equidecomposable.*

This reformulation of the Banach-Tarski paradox underlines the idea that the unit ball and two unit balls can be decomposed into the same number of subsets that are related by elements of the group SO(3). In particular, it was proved by Robinson that the minimum number of pieces into which a ball can be decomposed during the construction of the Banach-Tarski paradox is five [12]. It means that the first ball that is the outcome of the paradox can be created from two pieces of the original ball, and the second one from three pieces.

The notion of equidecomposability generalizes the idea of congruence by dissection, as it does not impose the condition on the shape of the subsets being polygonal. Thus, equidecomposability allows for the freedom of decomposing sets into more diverse subsets, for example ones that are not Lebesgue measurable. It is well known that Lebesgue measure is preserved under isometries, so the immediate conclusion is that the Banach-Tarski paradox cannot take place if the unit sphere is decomposed into measurable subsets. Otherwise, the volume of the unit ball would have to be doubled by applying isometries, which clearly cannot be achieved. Because of this, the Banach-Tarski paradox was one of the first evidence of the fact that there exist non Lebesgue measurable sets.

An important property of the equidecomposability is the fact that it is an equivalence relation.

Lemma 2.1. *Equidecomposability is an equivalence relation.*

Proof. Consider pairwise disjoint sets $A, B, C \subset X$ and a group G that acts on X. To show that equidecomposability is an equivalence relation, three properties need to be satisfied.

<u>*Reflexivity*</u>: Clearly $A \sim_G A$ holds as the set A can be decomposed twice in the same way.

<u>*Symmetry*</u>: If $A \sim_G B$ then it immediately follows that $B \sim_G A$ as order in which the sets are decomposed does not matter.

<u>*Transitivity*</u>: Assume that $A \sim_G B$ using n pieces and $B \sim_G C$ using m pieces. The sets A and B can be equidecomposed as $A = \bigcup_{i=1}^{n} A_i$ and $B = \bigcup_{i=1}^{n} B_i$ such that $A_i = g_i(B_i)$, where $g_i \in G$ for all $i \leq n$. Similarly, B and C can be equidecomposed as $B = \bigcup_{j=1}^{m} B'_j$ and $C = \bigcup_{j=1}^{m} C_j$ such that $h_j(B'_j) = C_j$, where $h_j \in G$ for all $j \leq m$.

To prove that $A \sim_G C$, define sets $X_{i,j} = B_i \cap B'_j$ for all i, j. Group elements $g_1, \dots, g_n$ and $h_1, \dots, h_m$ can be used to map sets $X_{i,j}$ to subsets of A and C, respectively. It follows that $g_i(X_{i,j}) \subset A$ and $h_j(X_{i,j}) \subset C$.

The set A can be decomposed using sets $g_i(X_{i,j})$ and the set C can be decomposed using sets $h_j(X_{i,j})$. The decompositions are

$$\bigcup_{i,j} g_i(X_{i,j}) = \bigcup_{i}^{n} \bigcup_{j}^{m} g_i(B_i \cap B'_j) = \bigcup_{i}^{n} g_i(B_i) = A, \tag{1}$$

and

$$\bigcup_{i,j} h_j(X_{i,j}) = \bigcup_{j}^{m} \bigcup_{i}^{n} h_j(B_i \cap B'_j) = \bigcup_{j}^{m} h_j(B'_j) = C. \tag{2}$$

Clearly all sets $X_{i,j}$ are pairwise disjoint and as $g_i(X_{i,j}) \subset g_i(B_i)$ and $h_j(X_{i,j}) \subset h_j(B'_j)$, so all $g_i(X_{i,j})$ and $h_j(X_{i,j})$ are pairwise disjoint too.

To conclude that Equations 1 and 2 form an equidecomposition of A with C there must exist some collection of elements $l_{i,j} \in G$ such that $l_{i,j}[g_i(X_{i,j})] = h_j(X_{i,j})$ for all i, j. Indeed, if $l_{i,j} = h_j g_i^{-1}$ are chosen then it holds that

$$l_{i,j}[g_i(X_{i,j})] = h_j g_i^{-1} g_i(X_{i,j}) = h_j(X_{i,j}),$$

so it follows that $A \sim_G C$. By this reasoning, it holds that equidecomposability satisfies the properties of the equivalence relation. It can be noted that the equidecomposition $A \sim_G C$ uses at most nm pieces, as some sets $X_{i,j}$ may be empty. □

2.3 Paradoxical decomposition

Equidecomposability is one of the ways in which, by considering decompositions, it is possible to capture similarities in the construction of sets. Another one, particularly useful while considering the Banach-Tarski paradox is the notion of a paradoxical decomposition.

Definition 2.3. Let G be a group and X a set on which G acts. Consider some subset $E \subset X$. The set E is G-paradoxical (or paradoxical) if there exist two disjoint, finite collections of pairwise disjoint subsets of E, $A_1, A_2, \dots, A_n$ and $B_1, B_2, \dots, B_m$ for some $n, m \in \mathbb{N}$ and elements $g_1, g_2, \dots, g_n$ and $h_1, h_2, \dots, h_m$ of G such that

$$E = \bigcup_{i=1}^{n} g_i(A_i) = \bigcup_{j=1}^{m} h_j(B_j).$$

The decomposition described in the Definition 2.3 is called a paradoxical decomposition of a set. The Banach-Tarski paradox can be again reformulated using the notion of the paradoxical decomposition.

Theorem 4 (The Banach-Tarski paradox, paradoxical version)**.** *The unit ball in $\mathbb{R}^3$ is $SO(3)$-paradoxical.*

This means that the unit ball can be covered by its subsets in two ways, as a union of two, or three of its pairwise disjoint subsets that are rearranged by some elements of the group SO(3). To explore the properties of paradoxical decompositions and their relation to equidecomposability, we consider some results that are useful while proving the Banach-Tarski paradox.

Lemma 2.2. *A union of some collection of pairwise disjoint, G-paradoxical sets is G-paradoxical.*

Proof. Let E_i be an element of some collection E of pairwise disjoint, G-paradoxical sets indexed by the set I. For each $i \in I$ it holds that $E_i = \bigcup_{k=1}^{n_i} g_k^i(A_k^i) = \bigcup_{j=1}^{m_i} h_j^i(B_j^i)$ is a paradoxical decompositions of E_i. The union of these sets can be expressed as

$$E = \bigcup_{i \in I} E_i = \bigcup_{i \in I} \bigcup_{k=1}^{n_i} g_k(A_k^i) = \bigcup_{i \in I} \bigcup_{j=1}^{m_i} h_j(B_j^i),$$

which is a paradoxical decomposition of E. □

What follows is one of the crucial results that defines the relation between paradoxical sets and equidecomposability.

Lemma 2.3. *Let X be some set and G a group that acts on it. A set X is G-paradoxical if and only if it has two disjoint subsets A and B such that $A \sim_G X$ and $B \sim_G X$ [17].*

Proof. First, assume that $A \sim_G X$ and $B \sim_G X$. The equidecomposition $A \sim_G X$ is $A = \bigcup_{i=1}^n A_i$ and $X = \bigcup_{i=1}^n X_i$, where $g_i(A_i) = X_i$. The equidecomposition $B \sim_G X$ is $B = \bigcup_{j=1}^m B_j$ and $X = \bigcup_{j=1}^m X_j'$, where $h_j(B_j) = X_j'$ holds. Combining these two equidecompositions it follows that

$$X = \bigcup_{j=1}^{m} h_j(B_j) = \bigcup_{i=1}^{n} g_i(A_i)$$

is a paradoxical decomposition of X.

Now, assume that X is paradoxical and its paradoxical decomposition is $X = \bigcup_{i=1}^n g_i(A_i) = \bigcup_{j=1}^m h_j(B_j)$. It can be noticed that the sets

$$X = \bigcup_{i=1}^{n} g_i(A_i) \quad \text{and} \quad A = \bigcup_{i=1}^{n} A_i$$

form an equidecomposition of X with some subset $A \subset X$. In a similar way it holds that

$$X = \bigcup_{j=1}^{m} h_j(B_j) \quad \text{and} \quad B = \bigcup_{j=1}^{m} B_j$$

is an equidecomposition of some subset $B \subset X$ with X. It follows that $X \sim_G A$ and $X \sim_G B$, which completes the proof. □

The following lemma describes the fact that equidecomposability preserves the paradoxicality of sets.

Lemma 2.4. *Consider two subsets $A, B \subset X$ and a group G acting on X. If A and B are G-equidecomposable and A is G-paradoxical then B is G-paradoxical [17].*

Proof. By Lemma 2.3 it holds that if A is G-paradoxical then there exist two disjoint subsets $C, D \subset A$ such that $C \sim_G A$ and $D \sim_G A$. Recall that by Lemma 2.1 equidecomposability is an equivalence relation. As it is assumed that $A \sim_G B$ then it holds that $B \sim_G C$ and $B \sim_G D$, so B is G-paradoxical. □

3 Free groups and Isometries

The proof of the Banach-Tarski paradox relies not only on finding suitable subsets of the unit ball, but also on a group that acts on them. As it turns out, groups that are particularly useful while proving the Banach-Tarski paradox are free groups.

3.1 Free groups

To define the notion of a free group we need to recall some ideas from group theory.

Definition 3.1. Consider some group G. A word is an element of G of form $g_1^{\epsilon_1} g_2^{\epsilon_2} \dots g_n^{\epsilon_n}$, where $\epsilon_i = \pm 1$ and elements $g_1, \dots, g_n$ are generators of G. The number n denotes the length of a word. Empty word represents a unit element of the group and the words are allowed to be of infinite length.

To simplify the notation and make sure that each element of the group is represented by a unique word, without the need to consider the equivalence classes, from now on, it is assumed that all words considered are reduced.

Definition 3.2. A reduced word is a word where no generator and its inverse appear next to each other. Thus all elements of the form $g_i g_i^{-1}$ and $g_i^{-1} g_i$ are simplified. Simplifying such pairs of elements is called reduction.

Finally, we can define a product of two words.

Definition 3.3. A product of two words $h_1 = g_1^{\epsilon_1} g_2^{\epsilon_2} \dots g_n^{\epsilon_n}$ and $h_2 = \overline{g}_1^{\overline{\epsilon_1}} \overline{g}_2^{\overline{\epsilon_2}} \dots \overline{g}_m^{\overline{\epsilon_m}}$ in some group G is defined by a concatenation of h_1 with h_2 of form $h_1 h_2 = g_1^{\epsilon_1} \dots g_n^{\epsilon_n} \overline{g}_1^{\overline{\epsilon_1}} \dots \overline{g}_m^{\overline{\epsilon_m}}$.

Note that even if words that are being concatenated are reduced, their product might not be. As of our interest are only reduced words, the product is defined as the concatenation that is immediately followed by reduction of the resulting word. With all of the notation recalled, a free group can be defined.

Definition 3.4. Consider some set M, called the generating set, where no relations except the notion of an inverse exist between the elements of M. A free group F generated by M is the set of all of the reduced words that can be created from the elements of M and their inverses. The identity element is an empty word and the group operation is concatenation.

Definition 3.5. The rank of a free group F is the cardinality of the generating set M. Free group of rank n is denoted by F_n.

Free groups are by definition non-abelian, although there exists a notion of an abelian free group, which is not considered in this paper. Of interest, will be groups that are isomorphic to some free group.

Definition 3.6. A group that is isomorphic to a free group is called free.

An example of a free group is $(\mathbb{Z}, +)$, which is a free group of rank 1 generated by $\{1\}$. It naturally follows that a non-trivial finite group cannot be free, as the set of generators of a free group must consists of elements of non-finite order. Thus, in search for free groups, the attention needs to be redirected to infinite groups and their subgroups. It turns out that when considering the Banach-Tarski paradox, free groups isomorphic to F_2, so a free group with two generators, suffice to create a paradoxical decomposition. A Cayley diagram, named after Arthur Cayley, is a visual tool by which the structure of a group can be described. It is particularly useful when considering groups that are isomorphic to F_2.

Definition 3.7. Let G by a group and M its generating set. The Cayley diagram is an edge-coloured, directed graph for which it holds that:

1. each element of G is represented by some vertex,
2. each element m of the generating set M (and their inverses) is assigned a colour c_m,
3. from any vertex representing some element $g \in G$ there exists a directed edge of colour c_m to a vertex representing the element mg.

In Figure 3, the Cayley diagram of a free group F_2 is presented.

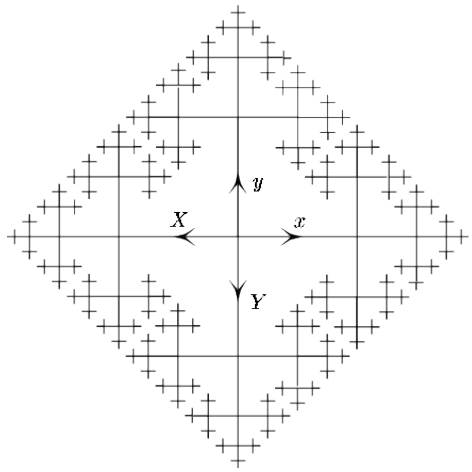

Figure 3: A Cayley diagram of F_2, where x, y are generators of the group and $X = x^{-1}$, $Y = y^{-1}$ [5].

From the definition of the Cayley diagram it follow that for every word in F_2 there exists a unique, directed path from the vertex representing the identity element, to a vertex assigned to the element that is considered. In Figure 4 there is presented an example of the directed path to the vertex representing the word Yxy.

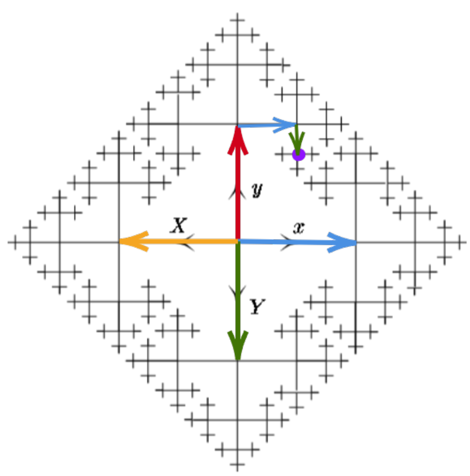

Figure 4: The Cayley diagram of F_2 with the directed path from the vertex representing identity, to the vertex representing the word Yxy (in purple).

Before the attention is redirected to free groups acting on sets, it is useful to further consider the structure of the group F_2. If a group is treated as a set of words that is acted on by the group itself, and it is possible to find a paradoxical decomposition of it, then such group is called paradoxical.

Theorem 5. *The group F_2 is F_2-paradoxical, where F_2 acts on itself by left multiplication [17].*

Proof. Let σ, τ be generators of F_2. Recall the construction of the Cayley diagram of F_2, and consider the partition of words in F_2 into 5 sets:

1. $W(\sigma)$: words in F_2 that end with σ,
2. $W(\sigma^{-1})$: words in F_2 that end with σ^{-1},
3. $W(\tau)$: words in F_2 that end with τ,
4. $W(\tau^{-1})$: words in F_2 that end with τ^{-1},
5. $\{e\}$: identity element of F_2.

To create a paradoxical decomposition of F_2, it can be noticed that if all of the elements of the set $W(\sigma^{-1})$ are multiplied from the left by σ, it holds that $\sigma W(\sigma^{-1}) = W(\sigma^{-1}) \cup W(\tau^{-1}) \cup W(\tau) \cup \{e\}$. So the set $\sigma W(\sigma^{-1})$ consists of all of the words in F_2 that do not end with σ. This is the case as all words in F_2 are reduced, so there cannot be a word in $W(\sigma^{-1})$ that has σ as its second to last entry, because it would simplify with σ^{-1}. It follows that the union of the set $\sigma W(\sigma^{-1})$ with $W(\sigma)$ covers the whole group F_2. This observation allows for finding an explicit paradoxical decomposition of F_2

$$W(\sigma) \cup \sigma W(\sigma^{-1}) = W(\tau) \cup \tau W(\tau^{-1}) = F_2. \tag{3}$$

This result concludes the proof. However, it can be noticed that the paradoxical decomposition in 3 does not use the identity element. It can be modified to include it. In order to do so, consider the set $W(\sigma) \cup \{e\}$ instead of $W(\sigma)$. Indeed, $F_2 = W(\sigma) \cup \{e\} \cup \sigma W(\sigma^{-1})$ holds, but this decomposition is not paradoxical as the sets are not disjoint. It follows from the fact that $\{e\} \in \sigma W(\sigma^{-1})$.

Excluding the element σ^{-1} from the set $W(\sigma^{-1})$ fixes the problem of the identity element being contained in $\sigma W(\sigma^{-1})$, as no element cancels out with σ. The element σ^{-1} must be included into the decomposition as an additional set. However, then it can be noticed that σ^{-1} is also contained in $\sigma W(\sigma^{-1})$ as $\sigma^{-2} \in W(\sigma^{-1})$. Thus σ^{-2} must also be excluded from the set $W(\sigma^{-1})$. By this reasoning, it can be noticed that all elements of the form σ^{-n} for any $n \in \mathbb{N}$ must be excluded from $W(\sigma^{-1})$, as otherwise the element σ^{-n+1} is repeated. To do this, define sets $K = \{\sigma^{-n} : n \in \mathbb{N}\}$ and $R = W(\sigma^{-1}) \setminus K$. Finally, the modified paradoxical decomposition of F_2 that uses all of the elements of F_2 is

$$F_2 = W(\sigma) \cup \{e\} \cup K \cup \sigma R = W(\tau) \cup \tau W(\tau^{-1}). \tag{4}$$

□

Visual representations of sets that are used to construct the paradoxical decomposition of F_2, defined as in Equation 3, are presented on Cayley diagrams of F_2 in Figures 5 and 6.

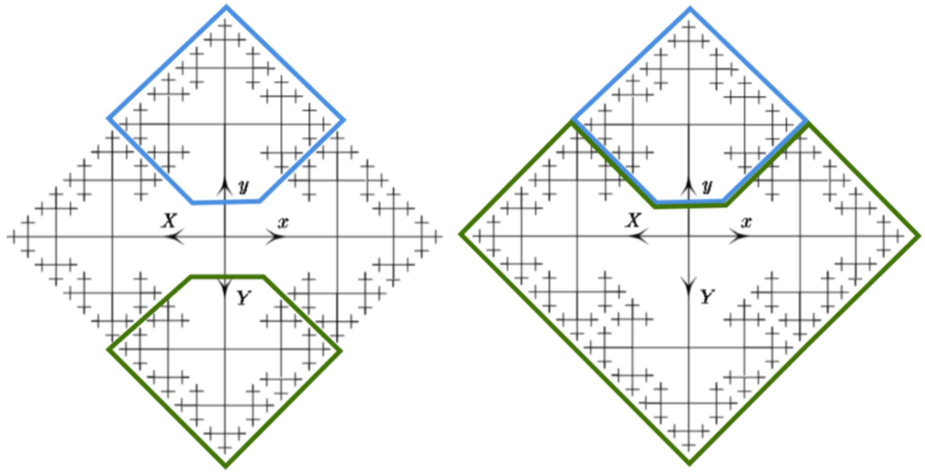

Figure 5: On the left: Cayley diagram of F_2 with subsets $W(y)$ (blue) and $W(Y)$ (green). On the right: Cayley diagram of F_2 with subsets $W(y)$ and $yW(Y)$ (green).

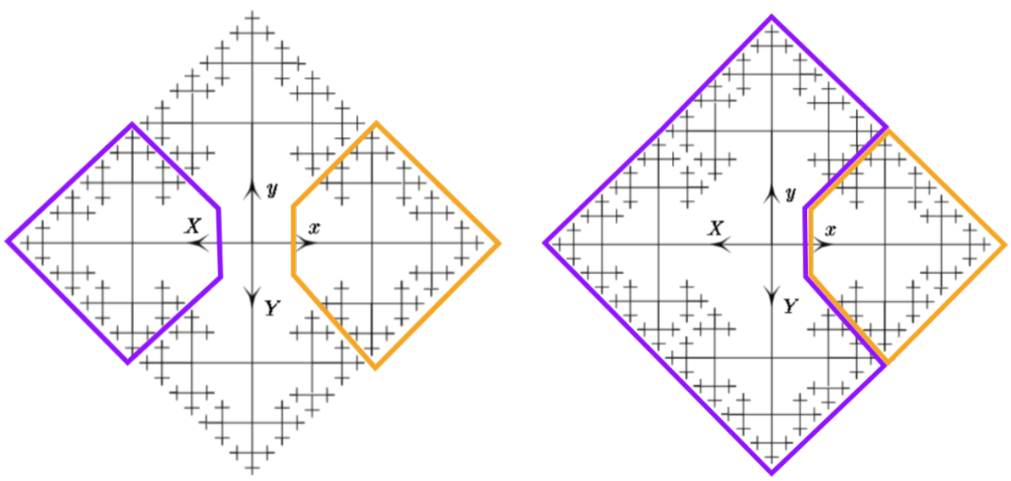

Figure 6: On the left: Cayley diagram of F_2 with subsets $W(x)$ (orange) and $W(X)$ (purple). On the right: Cayley diagram of F_2 with subsets $W(x)$ and $xW(X)$ (purple).

A similar visualisation of the refined decomposition described in Equation 4 is presented in Figures 6 and 7.

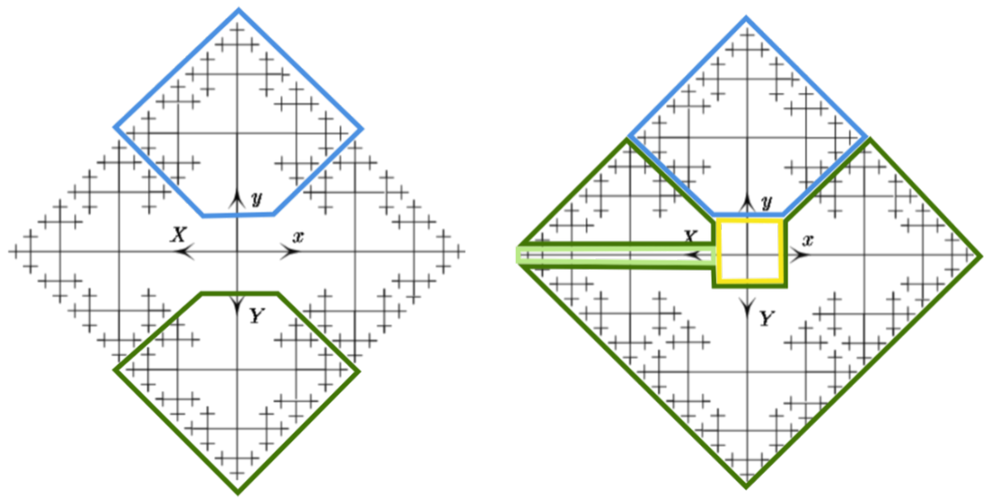

Figure 7: On the left: Cayley diagram of F_2 with subsets $W(y)$ (blue) and $W(Y)$ (green). On the right: Cayley diagram of F_2 with subsets $W(y)$, $\{e\}$ (yellow), K (light green) and yR (dark green).

Finally, the whole paradoxical decomposition of F_2 from Equation 3 can be seen in Figure 8.

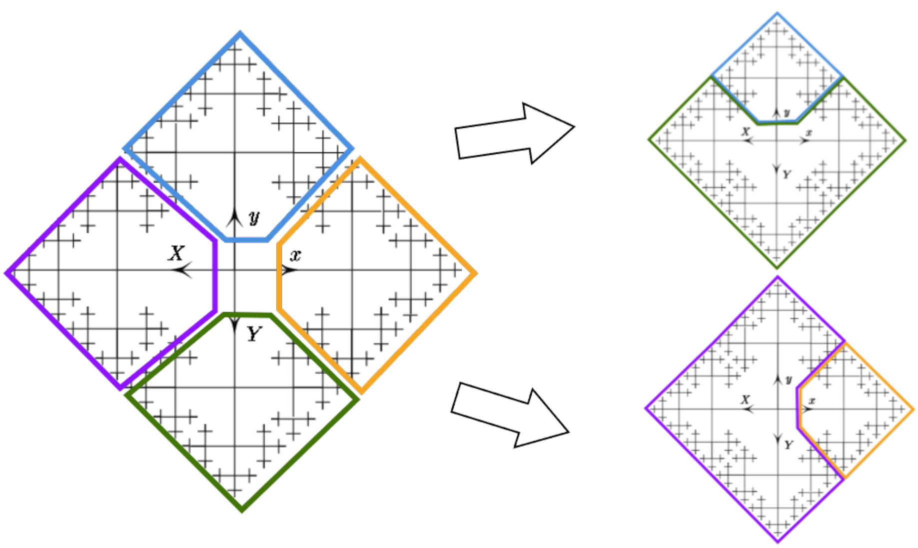

Figure 8: Visualization of the paradoxical decomposition of F_2 using Cayley diagrams.

In many instances, working with the definition of a paradoxical set may not be the most convenient. Concluding whether the set is paradoxical in this way requires explicitly constructing a paradoxical decomposition, which may be difficult. This issue can be resolved by translating a paradoxical decomposition of a group onto a set on which it acts. As it turns out, proving the fact that some group is paradoxical allows, under some assumptions, to conclude that a set on which it acts is paradoxical too. To do that first recall the definition of a fixed point.

Definition 3.8. Consider some group G that acts on a set X. A point $x \in X$ is called a fixed point if there exists some element $g \in G$ such that $g(x) = x$.

If some group acts without non-trivial fixed points on some set then this action is called free.

Lemma 3.1. *If a group F_2 acts without non-trivial fixed points on some set X then the set X is F_2-paradoxical [18].*

Proof. As the action of F_2 on X is free, orbits of X under F_2 form a partition of X. Now, by the Axiom of Choice, one representative of each orbit is chosen and the set containing all representatives is denoted by M. Axiom of Choice is used here as number of orbits might be uncountable. Rewrite the set X as $X = \bigcup\{g(M) : g \in F_2\}$, where the union is taken over all elements of F_2. All sets $\{g(M) : g \in F_2\}$ are pairwise disjoint and form a partition of X. Now, the paradoxical decomposition of F_2 can be transformed into a paradoxical decomposition of X. If a subset $S \subset F_2$ is chosen then the set $S' = \{g(M) : g \in S\}$ is a subset of X. If he group F_2 has a paradoxical decomposition $F_2 = \bigcup_{i=1}^{n} g_i(A_i) = \bigcup_{j=1}^{m} h_j(B_j)$ then define sets $A'_i = \{g(M) : g \in A_i\}$ for all $i \leq n$ and $B'_j = \{g(M) : g \in B_i\}$ for all $j \leq m$. It follows that $X = \bigcup_{i=1}^{n} g_i(A'_i) = \bigcup_{j=1}^{m} h_j(B'_j)$ form a paradoxical decomposition of X. □

Lemma 3.2. *Consider some group G and its subgroup $E < G$. If E is a paradoxical group then G is paradoxical.*

Proof. Let G be a group that has a paradoxical subgroup E. Recall that subgroups act without non-trivial fixed points on the group they belong to (as inverses of all elements of E are in E). Thus by Lemma 3.1 it holds that the group G is paradoxical. □

3.2 Isometries acting on spheres

Now begins the search for a free group used in the construction of a paradoxical decomposition in the Banach-Tarski paradox. The paradox takes place in the 3-dimensional Euclidean space denoted by $\mathbb{R}^3$ and isometries are used to construct it.

Definition 3.9. A map $f : \mathbb{R}^3 \to \mathbb{R}^3$ is an isometry if it holds that $d(a, b) = d(f(a), f(b))$ for any $a, b \in \mathbb{R}^3$, where d is a metric on $\mathbb{R}^3$. Thus isometry is a distance-preserving transformation.

In particular, we are interested in operations that preserve the euclidean distance between points which are called Euclidean isometries (also called rigid transformations). Isometry group is a set of all of the bijective isometries of $\mathbb{R}^3$ onto itself. Euclidean isometries in $\mathbb{R}^n$ form a group denoted by $E(n)$ called the Euclidean group. All elements of this group are affine transformations.

The most important subgroups of the Euclidean group include translational group $T(3)$ and orthogonal group $O(3)$. As a name suggests, translational group consists of all of the translations in $\mathbb{R}^3$. Orthogonal group is a group consisting of all of the distance preserving transformations of a Euclidean space that fix some point. Each element of $O(3)$ can be represented as a orthogonal matrix. Any element of $E(3)$ can be uniquely represented as the composition of a translation and an orthogonal transformation. The orthogonal group has a subgroup that is used in the construction of the Banach-Tarski paradox.

Definition 3.10. The special orthogonal group SO(3) in $\mathbb{R}^3$ is the group of all orthogonal matrices in $\mathbb{R}^{3\times 3}$ with determinant 1.

The group SO(3) is also called a rotation group, as it consists of all of the rotations about origin in $\mathbb{R}^3$. Every non-trivial rotation can be fully characterized by the axis of rotation an the angle of rotation, and uniquely expressed in form of a orthogonal matrix. Examples of matrices that represents a counter-clockwise rotation with respect to z-axis and x-axis, by an angle θ are:

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}.$$

The group $SO(3)$ is not finite nor abelian, as rotations in $\mathbb{R}^3$ do not commute. It is an indication that it might have a subgroup which is isomorphic to F_2. In fact, it turns out that such subgroups of SO(3) exist, and may be found explicitly.

Lemma 3.3. *The group $SO(3)$ has a subgroup generated by two independent rotations that is isomorphic to F_2 [15].*

Proof. An explicit example of two rotations that generate a group isomorphic to F_2 was found by Świerczkowski in [15]. He has shown that two rotations by the same angle θ, with respect to perpendicular axis (here z-axis and x-axis)

$$A = \begin{pmatrix} \frac{a}{b} & -\frac{\sqrt{c}}{b} & 0 \\ \frac{\sqrt{c}}{b} & \frac{a}{b} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{a}{b} & -\frac{\sqrt{c}}{b} \\ 0 & \frac{\sqrt{c}}{b} & \frac{a}{b} \end{pmatrix},$$

where $\cos\theta = \frac{a}{b}$ and $c = b^2 - a^2$, generate a free subgroup of SO(3) if and only if $\frac{a}{b} \notin \{0, \pm\frac{1}{2}, \pm 1\}$. □

The group defined in Lemma 3.3 can be used to provide a neat proof of the Banach-Tarski paradox. What is more, this result can be generalized to higher dimensions.

Proposition 3.1. *The group $SO(n)$ has a free subgroup for any $n \geq 3$.*

Proof. From Lemma 3.3 it is known that SO(3) has a subgroup isomorphic to F_2. Now, as SO(3) is a subgroup of SO(n) for any $n \geq 3$ then all groups SO(n) have a subgroup isomorphic to F_2. □

Corollary 3.1. *Group $SO(n)$ is paradoxical for any $n \geq 3$.*

Proof. This result follows from Lemma 3.2 and Proposition 3.1. □

Notice that in the statement of the Banach-Tarski paradox the group SO(3) acts on the unit ball. However, this does not mean that all of the elements of this group need to be used to construct the paradoxical decomposition. In fact, it is enough to consider its suitable subgroup, for example tone from Lemma 3.3. Clearly, if a group generated by rotations acts on a ball, or a sphere, then this action is not fixed point free. Every element of this group fixes the points of intersection of its axis of rotation with the unit ball. Even though we found a paradoxical group that acts on the unit ball, Lemma 3.1 cannot be used to conclude the fact that this set is paradoxical, as the group action is not free. Because of that it seems constructive to investigate the nature of fixed points appearing while some free group generated by rotations acts on a ball. Following results are proved for a sphere, but they can be immediately extended to a ball.

Lemma 3.4. *Consider a sphere S^2 and a free group G acting on it. Any point that is an element of the orbit of a fixed point is also a fixed point.*

Proof. Consider some point $p \in S^2$ and a rotation $\gamma \in G$ such that $\gamma p = p$, so γ fixes p. Now, any element x of the orbit of p is of the form $x = \phi p$ for some rotation $\phi \in G$. It follows that if

$$\gamma\phi^{-1}\phi p = p \quad \text{then} \quad \phi\gamma\phi^{-1}\phi p = \phi p, \quad \text{and it holds that} \quad \phi\gamma\phi^{-1}x = x.$$

Thus a point x is fixed by the rotation $\theta = \phi\gamma\phi^{-1}$. As x is arbitrary, it can be concluded that any element of the orbit that contains a fixed point is a fixed point. □

From this lemma it can be concluded that orbits fall under two categories, those consisting only of fixed points and those which are fixed point free. Now we may be wondering if a rotation that fixes some point is unique.

Theorem 6. *Consider some orbit of S^2, under the action of a free group G, that consisting of fixed point. Let θ be a rotation that has a fixed point $p \in S^2$. If there exists some other rotation σ such that $\sigma p = p$ then it follows that $\sigma = \theta^a$ for some integer a [12].*

Proof. Let a free group G be generated by rotations $\gamma_1, \gamma_2, \ldots$. Without loss of generality consider a rotation θ that has the shortest length of all of the rotations that have a fixed point in this orbit. Now, assume that there exists some other rotation $\sigma \neq \theta$ such that $\sigma p = p$, so it also fixes p. Rotations σ and θ rewritten as words in G are of form

$$\sigma = \gamma_{n_1}^{\pm 1}\gamma_{n_2}^{\pm 1}\ldots\gamma_{n_k}^{\pm 1}, \qquad \theta = \gamma_{m_1}^{\pm 1}\gamma_{m_2}^{\pm 1}\ldots\gamma_{m_l}^{\pm 1}.$$

Notice that θ cannot start and end with elements that are inverses of one another, for example τ and τ^{-1}, as then $\tau^{-1}\theta\tau$ is a shorter word that also fixes some point in the orbit. This follows from the construction in the proof of Lemma 3.4. Because of that, θ and θ^{-1} cannot start, nor end with the same elements. Now, if θ and σ have the same fixed point p then it follows that $\theta\sigma = \sigma\theta$, as if rotations fix the same point, they must have the same axis of rotation, so they commute.

First assume that the expression $\theta\sigma$ does not simplify, which means that neither $\sigma\theta$ does. It follows that σ must start with an element θ, as if $\theta\sigma = \sigma\theta$ is rewritten as

$$\theta\gamma_{n_1}^{\pm 1}\gamma_{n_2}^{\pm 1}\ldots\gamma_{n_k}^{\pm 1} = \gamma_{n_1}^{\pm 1}\gamma_{n_2}^{\pm 1}\ldots\gamma_{n_k}^{\pm 1}\theta,$$

then it is clear that $\gamma_{n_1}^{\pm 1}\ldots\gamma_{n_l}^{\pm 1} = \theta$. But it also means that

$$\theta\gamma_{n_{l+1}}^{\pm 1}\ldots\gamma_{n_k}^{\pm 1} = \gamma_{n_{l+1}}^{\pm 1}\ldots\gamma_{n_k}^{\pm 1}\theta,$$

so $\gamma_{n_{l+1}}^{\pm 1}\ldots\gamma_{n_{2l+1}}^{\pm 1} = \theta$ too. This process repeats until σ consists only of elements θ. In this way, it follows that σ is constructed from $r = \frac{k}{l}$ many elements θ, which shows that $\sigma = \theta^r$.

If $\theta\sigma$ simplifies then $\sigma\theta^{-1}$ does not, as it was assumed σ and σ^{-1} cannot start with the same element. The same procedure as in the case of the lack of simplification can be repeated, but now for $\sigma\theta^{-1} = \theta^{-1}\sigma$. From this it follows that $\sigma = \theta^{-r}$, which concludes the proof. □

4 The Banach-Tarski paradox in $\mathbb{R}^3$

4.1 Proof of the Banach-Tarski paradox

Recall that the unit ball B in $\mathbb{R}^3$ is a subset of $\mathbb{R}^3$ such that $B = \{x \in \mathbb{R}^3 : ||x||_3 \leq 1\}$. If the centre of the ball is excluded then the centreless ball can be treated as a union of concentric spheres with radiuses smaller or equal to the radius of the initial ball.

Lemma 4.1. *Every ball B_R of radius R, without a centre in $\mathbb{R}^3$ is equal to the union of concentric spheres S_r with positive radii r such that $r \leq R$. Thus it holds that $B_R \setminus \{0\} = \bigcup_{0<r\leq R} S_r$.*

Proof. Every point $y \in B_R$ is contained on a sphere S_r, where r is a distance from y to the centre of the ball. Additionally, every point $z \in S_r$ belongs to the ball B_R as $||z||_3 \leq R$, so $z \in B_R$. The only point for which this reasoning does not work is the centre, as there are no spheres of radius zero. □

Working with balls is significantly more difficult than working with spheres. Fortunately, it turns out that it is enough to prove the Banach-Tarski paradox for the unit sphere. As it holds that $S_r^2 = rS^2$ for a sphere of radius r, a paradoxical decomposition of the unit sphere can be scaled and adjusted to work for any sphere of arbitrary radius. Thus, if it can be proved that the unit sphere is paradoxical then it follows that any sphere is. By Lemma 4.1 and Lemma 2.2 it then follows that a centreless ball is paradoxical. The issue of including the centre into the decomposition is taken care of later.

Recall the statement of the Banach-Tarski paradox.

Theorem 7 (Banach-Tarski paradox)**.** *The unit ball in $\mathbb{R}^3$ is $SO(3)$-paradoxical.*

Proof. Consider the unit ball B in $\mathbb{R}^3$ and the group SO(3). To show that B satisfies the Definition 2.3 with SO(3), an explicit paradoxical decomposition can be constructed. Without loss of generality, it is enough to show that the unit sphere S^2 is paradoxical with some subgroup of SO(3). Consider a subgroup of SO(3) that is generated by two rotations σ, τ, both by the angle $\theta = \arccos \frac{1}{3}$, with respect to z-axis and x-axis, respectively. Here any rotations that satisfy Lemma 3.3 would work, so σ and τ generate a free subgroup of $SO(3)$, which is denoted by G_2.

As every rotation in G_2 fixes the points of intersection of its axis of rotation with the unit sphere, it follows that the action of the group G_2 on S^2 is not free. Define the set $D = \{p \in S^2 : g(p) = p \text{ where } g \in F_2, g \neq e\}$ which contains all of the fixed points of the action of G_2 on S^2. Now, by Lemma 3.1 it can be immediately concluded that the set $S^2 \setminus D$ is SO(3)-paradoxical. However, finding the explicit paradoxical decomposition of $S^2 \setminus D$ turns out to be useful to understand the construction of the paradox. To do this, decompose the set $S^2 \setminus D$ into orbits of the action of G_2 on it. Recall that all orbits are either pairwise disjoint or equal, as the action is fixed point free. Another observation that can be made is the fact that the number of elements in G_2 is countable, so each orbit must contain countably many points. As the number of points on the sphere is uncountable, it means that there exist uncountably many orbits of the action of G_2 on S^2. By the Axiom of Choice, select from each orbit one element m and call it a representative of the orbit. All of the representatives are collected in a set M.

Consider the orbit of some representative m and notice that all of its elements are of form $a(m)$, where a is some element of G_2, so a is constructed from the elements from the set $\{\tau, \tau^{-1}, \sigma, \sigma^{-1}\}$. Similarly to the proof of Theorem 5, divide this orbit into five classes:

1. m: initial point ,
2. $W_m(\tau)$: points obtained by acting on m with all words in G_2 ending with τ,
3. $W_m(\tau^{-1})$: points obtained by acting on m with all words in G_2 ending with τ^{-1},

4. $W_m(\sigma)$: points obtained by acting on m with all words in G_2 ending with σ,
5. $W_m(\sigma^{-1})$:points obtained by acting on m with all words in G_2 ending with σ^{-1}.

Notice that these sets form a partition of the orbit of m. Now, all of the orbits can be separated into these 5 classes, and can be combined to form sets

$$M = \bigcup_{m \in M} m, \quad W_\tau = \bigcup_{m \in M} W_m(\tau),$$
$$W_{\tau^{-1}} = \bigcup W(\tau^{-1}), \quad W_\sigma = \bigcup_{m \in M} W_m(\sigma), \quad W_{\sigma^{-1}} = \bigcup_{m \in M} W_m(\sigma^{-1}),$$

which form a partition of $S^2 \setminus D$. Important thing that needs to be noted here is the fact that if the set D was not excluded then these sets would not be pairwise disjoint. Consider some fixed point $z \in S^2$ that is a representative of some orbit, and without loss of generality assume that z is fixed by some word γ in G_2 that ends with τ. Then, it follows that $z \in W_z(\tau)$ as $z = \gamma z \in W_z(\tau)$. Thus the point z needs to be excluded from the decomposition. As z was arbitrary, and from Lemma 3.4 we know that if an orbit contains a fixed point then all of its points are fixed, in fact all of the orbits that contain fixed points need to be excluded. The set of orbits containing fixed points is equal to the set D. Note, that the set D is countable, as the number of orbits that contain fixed points is countable.

Unsurprisingly, the paradoxical decomposition of $S^2 \setminus D$ is found to be

$$S^2 \setminus D = W_\sigma \cup \sigma W_{\sigma^{-1}} = W_\tau \cup \tau W_{\tau^{-1}}.$$

This paradoxical decomposition is very similar to the one of F_2, which should not be a surprise. The set W_σ contains all of the points that are obtained by rotating elements of M by words in G_2 that end with σ, and the set $\sigma W_{\sigma^{-1}}$ contains all of the points that are obtained by rotating M by all of the elements in G_2 that do not end with σ. These sets are disjoint and their union is the whole of $S^2 \setminus D$. Similarly it holds for the second decomposition.

It can be noticed that the set M is not used to construct the paradoxical decomposition, but it can be included into it. Consider the union of M with one of the already existing sets, for example introduce the set $W_\sigma \cup M$, instead of W_σ. However, now $M \subset \sigma W_{\sigma^{-1}}$ and sets are not disjoint. Notice that the set $\sigma^{-1}M$ is causing the problem as σ^{-1} cancels out with σ. Excluding the set $\sigma^{-1}M$ from $W_{\sigma^{-1}}$ solves this issue, but now $\sigma^{-1}M$ must be included into the decomposition as a separate set. However, then also the set $\sigma^{-2}M$ must be excluded from $W_{\sigma^{-1}}$, as $\sigma^{-2}M \in W_{\sigma^{-1}}$ and again first σ^{-1} will cancel out with σ leaving the set $\sigma^{-1}M$. Inductively, it can be concluded that all sets of form $\sigma^{-n}M$ for any $n \in \mathbb{N}$ must be excluded from $W_{\sigma^{-1}}$, as otherwise the set $W_{\sigma^{-n+1}}$ is repeated.

Define new sets $K = \sigma^{-1}M \cup \sigma^{-2}M \cup \dots$ and $R = W_{\sigma^{-1}} \setminus K$. The decomposition of form

$$S^2 \setminus D = W_\sigma \cup \sigma R \cup K \cup M = W_(\tau) \cup \tau W(\tau^{-1})$$

is a paradoxical decomposition of $S^2 \setminus D$ that uses all of the elements of the set.

The only part left to show is the fact that $S^2 \setminus D$ being paradoxical is equivalent to S^2 being paradoxical. Recall that by Lemma 2.4, if a set is equidecomposable with a paradoxical set then it is also paradoxical. It we show that S^2 and $S^2 \setminus D$ are equidecomposable, it follows that S^2 is paradoxical. To find equidecomposition of S^2 and $S^2 \setminus D$ we use the fact that it is always possible (as set D is countable) to find some line l through the origin of the sphere that does not contain any points of D. What is more, the set D can be rotated with respect to this line l by some angle γ, call this rotation δ, in a way that none of the points of D are mapped back to D by δ^n for any $n \in \mathbb{N}$. It is possible to find such rotation as the number of rotations that map some point of D back to D is countable.

Define sets $K_n = \{\delta^n(d) : d \in D\}$ for all $n \in \mathbb{N}$ and a set $G = \bigcup_{n=0}^{\infty} K_n$. All sets K_n are pairwise disjoint, as δ was chosen in a way that it does not fix any points of D. Clearly, the sets G and $S^2 \setminus G$ are disjoint and cover the whole set S, thus

$$S^2 = G \cup S^2 \setminus G \tag{5}$$

is a partition of S. To form a partition of $S \setminus D$, define the set $\delta G = \bigcup_{n=1}^{\infty} K_n = G \setminus D$. It is the set of all of the points that are reachable by δ from D that is rotated once more by δ, so the whole orbit without the set D. From this it follows that

$$\delta G \cup S^2 \setminus G = S \setminus D \tag{6}$$

forms a partition of $S \setminus D$. It holds Equations 5 and 6 form an equidecomposition of S^2 with $S \setminus D$ using the group SO(3).

As S^2 and $S^2 \setminus D$ are equidecomposable and $S^2 \setminus D$ is $SO(3)$-paradoxical, it can be concluded by Lemma 2.4 that S^2 is SO(3)-paradoxical. This result can be extended to a sphere of arbitrary radius, so by Lemmas 4.1 and 2.2 it follows that $B \setminus \{c\}$, where $c = (0, 0, 0)$, is also SO(3)-paradoxical. What needs to be shown is the fact that B is paradoxical. This is done by showing that $B \setminus \{c\}$ and B are equidecomposable.

Notice that a ball without centre is trivially equidecomposable with a ball that is missing one point on the surface. Without loss of generality chose a point $k = (0, 1, 0)$. We want to show that $B \setminus \{k\} \sim B$. Consider the rotation τ, as it does not fix k, and define the set $T = \{\tau^n k : n \in \mathbb{N}\}$. It follows that a partition of the unit ball is

$$B = T \cup B \setminus T. \tag{7}$$

Now, for the decomposition of the unit ball without a point consider

$$B \setminus \{k\} = B \setminus T \cup \tau T. \tag{8}$$

Thus Equations 7 and 8 form an equidecomposition of B with $B \setminus \{k\}$. As $B \setminus \{k\} \sim B$ and $B \setminus \{c\} \sim B \setminus \{k\}$ then by transitivity also $B \setminus \{c\} \sim B$. As $B \setminus \{c\}$ is SO(3)-paradoxical, it follows that B is SO(3)-paradoxical, which finishes the proof. □

4.2 Notes on the proof

Lemma 4.1 allows for first finding a paradoxical decomposition of the unit spheres and then translating it onto the centreless unit ball. In hindsight, it can be noticed that the whole centreless ball can be decomposed at once. However, then the action of F_2 on $B \setminus \{c\}$ has orbits that are uncountably infinite, so orbits that contain fixed points are uncountable too. This could be an issue, as the method by which the fixed points are included into the decomposition is based on the fact that the set of fixed points can always be mapped to its complement. This is proven using the fact that the set of fixed points is countable. Fortunately, it turns out that finding an appropriate rotation is still possible.

The number of fixed points on each of the spherical layers of the sphere is still countable. What is more, if we consider all fixed points in a ball of some rotation, we can notice that they can be connected by a line from the centre of the ball to the fixed point located on the unit sphere. Examples of such lines can be seen in Figure 9. This procedure can be repeated for all of the rotations, so there is a countable number of such lines. Clearly, points contained on concentric spheres with different radii cannot be mapped to one another using rotations with the axis of rotation going through the centre of the ball. Because of that, it can be noticed that the same rotation used to create an equidecomposition of $S^2 \setminus D$ with S^2 can be used to find equidecomposition of the whole centreless ball with a ball missing the set of fixed points.

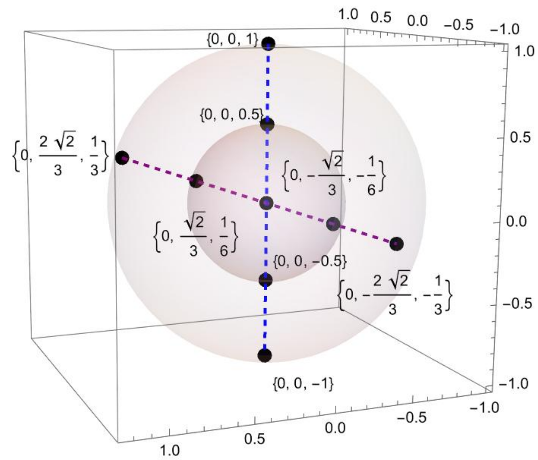

Figure 9: Fixed points of two rotations connected by lines.

Including the centre of the ball was shown by an argument involving finding an equidecomposition of a whole ball with a ball without the centre, and a ball missing some point on the surface. This problem can be solved in a more elementary way, but it requires allowing for rotations with respect to lines that do not pass through the origin. To include the centre we can find a circle C that contains the centre of the ball and some point on the unit sphere, an example of such circle can be found in Figure 10. Then consider a rotation $\omega \in \mathrm{SO}(2)$ by an irrational multiple of π, with respect to the centre of the circle, in the plane containing C.

Now, it is possible to find a equidecomposition of a unit circle C with a circle $C \setminus \{c\}$. To do this, define the set $A = \bigcup_{n\in\mathbb{N}} \omega^n c$. Then the equidecomposition of C with $C \setminus \{c\}$ is

$$C = C \setminus A \cup A, \qquad C \setminus \{c\} = \omega A \cup C \setminus A.$$

This decomposition can be performed both for the centre, and for any fixed point, see Figure 10 for a possible example of a circle used to include a fixed point $(0, 0, 1)$.

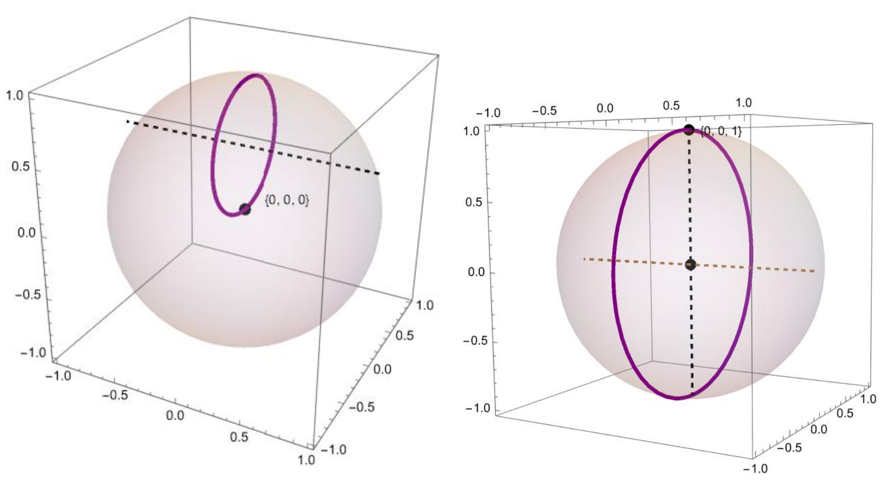

Figure 10: On the left: Circle containing the centre of the unit ball and some point on the unit sphere. On the right: Circle on the surface of the unit sphere that contains a fixed point.

5 Modifications of the Banach-Tarski paradox

5.1 Rational Sphere

We know that when a paradoxical group acts without non-trivial fixed points on some set then the set is paradoxical. However, so far there was no explicit example of such set presented, as the action of the group SO(3) on a sphere is not free. An example of an object where it is possible to conclude paradoxicality without an additional argument needed to include the fixed points, is the rational sphere.

Definition 5.1. The rational sphere $R = S^2 \cap \mathbb{Q}^3$ is a set of all points on the unit sphere with rational coordinates.

It may be wondered to what extent does the classic sphere differ from the rational one, as in the latter case all of the points with at least one irrational coordinate are excluded. Fortunately, it turns out that the rational sphere is a fairly decent approximation of the classical sphere.

Lemma 5.1. *The rational sphere is dense in S^2.*

Proof. The proof can be found in Chapter 2 of [17]. □

The main advantage of the rational unit sphere over the classic unit sphere is the fact that the number of points contained in it is countable. This property allows for creating a paradoxical decomposition of it without assuming the Axiom of Choice. In Lemma 3.3, a class of matrices that generate a free subgroup of SO(3) was presented, but not all of them generate a group that acts on R without fixed points. It can be suspected that if rotations with rotation axis intersecting the sphere at irrational points are chosen as generators, then no rational points are fixed by any of the elements of this group. This turns out to be true.

An example of a free group generated by two rotations whose axis of rotation intersect the sphere at irrational points was found by Satô. He showed in the article [14] that the matrices

$$\mu = \frac{1}{7}\begin{pmatrix} 6 & 2 & 3 \\ 2 & 3 & -6 \\ -3 & 6 & 2 \end{pmatrix}, \qquad \nu = \frac{1}{7}\begin{pmatrix} 2 & -6 & 3 \\ 6 & 3 & 2 \\ -3 & 2 & 6 \end{pmatrix}, \tag{9}$$

generate a free subgroup of SO(3) that acts without non-trivial fixed points on R.

The proof of this result highly relies on the fact that rotations in $\mathbb{R}^3$ can be represented in the form of quaternions [9]. Recall that the space of quaternions is defined as $\mathbb{H} = \mathbb{R} \oplus \mathbb{P}$, where $\mathbb{P}$ is a vector space over $\mathbb{R}$ with an orthonormal basis $\{i, j, k\}$, so all quaternions are of form $q = q_0 + q_1 i + q_2 j + q_3 k$, where $q_0, q_1, q_2, q_3 \in \mathbb{R}$. To denote quaternions in a compact form, the notation (c, s) is used, where $c \in \mathbb{R}$ and $s \in \mathbb{P}$. Quaternion multiplication is defined by the formula

$$(c', s') * (c, s) = (c'c - s' \cdot s, cs' + c's + s' \times s), \tag{10}$$

where $\cdot$ is a dot product and $\times$ is a cross product on $\mathbb{R}^3$. The norm on the space of quaternions is $||(c, s)|| = c^2 + |s|^2$, where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^3$.

Any unit quaternion $\gamma \in \mathbb{H}$, so a quaternion for which it holds that $||(c, s)|| = 1$, can be associated with an isometry operation of form $\psi_\gamma : \mathbb{P} \to \mathbb{P}$ defined for any $x \in \mathbb{P}$ as $\psi_\gamma(x) = \gamma x \gamma^{-1}$. This map describes a rotation γ, in a quaternion form, acting on some vector x. Recall that any rotation in $\mathbb{R}^3$ can be fully characterized by a vector that defines the axis of rotation, and by the angle representing the magnitude of a rotation. Any quaternion $\pm(c, s)$ such that $c \in \mathbb{R}$, $s \in \mathbb{R}^3$ and $||(c, s)|| = 1$ represents a rotation. Rotation is trivial if $s = 0$. The angle of a rotation is determined by the formula $c = \cos \frac{\theta}{2}$ and s is a vector parallel to the axis of rotation. The inverse of a rotation $\gamma = (c, s)$ is represented by $\gamma^{-1} = (c, -s)$.

Theorem 8. *The rotation axis of every non-trivial rotation generated by rotations $\mu, \nu, \mu^{-1}, \nu^{-1}$, defined as in 9, intersects the unit sphere at irrational points [14].*

Proof. First recall that a rotation matrix must have a real eigenvalue equal to 1, as it is an isometry. The eigenvector belonging to it is located on the axis of rotation. Eigenpairs of μ and ν that are of interest are

$$\lambda_\mu = 1, v_\mu = (2, 1, 0) \quad \text{and} \quad \lambda_\nu = 1, v_\nu = (0, 1, 2).$$

The points of intersection of these vectors with the unit sphere are $p_\mu = \pm\frac{1}{\sqrt{5}}(2, 1, 0)$ and $p_\nu = \pm\frac{1}{\sqrt{5}}(0, 1, 2)$, which are clearly irrational points.

Assume that there exists some rotation θ in the group generated by μ, ν such that its axis of rotation intersects the sphere at a rational point. This rotation clearly cannot be just a multiple of $\mu^{\pm 1}$ or $\nu^{\pm 1}$, as its rotation axis would still intersect the sphere at a irrational point. It means that θ must be a composition of both $\mu^{\pm 1}$ and $\nu^{\pm 1}$. What is more, as conjugation preserves the fact that a rotation has a fixed point, it can be assumed that θ does not start and end with elements that are inverses of one another. It can be concluded that that θ is either of form $\theta = \mu^{\pm 1} \dots \nu^{\pm 1}$ or $\theta = \nu^{\pm 1} \dots \mu^{\pm 1}$. Without loss of generality we can assume that such rotation is of form $\theta = \mu^{\pm 1} \dots \nu^{\pm 1}$ (otherwise consider its inverse).

Rotations $\mu^{\pm 1}$ and $\nu^{\pm 1}$ can be written in the quaternion form

$$q_{\nu^{\pm 1}} = \frac{1}{\sqrt{14}}(3, \pm(0, 1, 2)) \quad \text{and} \quad q_{\mu^{\pm 1}} = \frac{1}{\sqrt{14}}(3, \pm(2, 1, 0)).$$

Every word in the group is a composition of these rotations obtained by using formula 10, so if the word is of length n, then it will have a factor of $\frac{1}{\sqrt{14}^n}$ in front of it. As every unit quaternion represents a unique rotation, multiplication of the quaternion by a constant does not change the rotation represented. Because of that, the result of composing generators can be multiplied by $\sqrt{14}^n$, which makes the notation cleaner and allows for working with just integers.

Denote the rotation θ in the quaternion form by $q_\theta = (c_\theta, s_\theta) = (c_\theta, (x_\theta, y_\theta, z_\theta))$. The points of the intersection of s_θ with the unit sphere are $r_\theta = \pm\frac{(x_\theta, y_\theta, z_\theta)}{\sqrt{x_\theta^2 + y_\theta^2 + z_\theta^2}}$. If these points are irrational then $x_\theta^2 + y_\theta^2 + z_\theta^2$ cannot be a perfect square. To show that the denominator is not a perfect square, it is useful to recall that if a number is not perfect square modulo prime, then it is not a perfect square in $\mathbb{N}$ either. The goal then is to show that $x_\theta^2 + y_\theta^2 + z_\theta^2 \equiv 3, 5$ or $6 \mod 7$ for any rotation in the group, as 3, 5 and 6 are not perfect squares mod 7. Reduction of the quaternion mod 7 is performed by reducing all of its coefficients mod 7, so $\overline{q_\theta} = (\overline{c_\theta}, (\overline{x_\theta}, \overline{y_\theta}, \overline{z_\theta}))$. To make calculations less cumbersome, the quaternion $\overline{q}_\theta$, can be multiplied by $\overline{c_\theta}^{-1}$. This does not change the fact whether the sum is a perfect square, as it holds that if $x_\theta^2 + y_\theta^2 + z_\theta^2$ is a perfect square then $(cx_\theta)^2 + (cy_\theta)^2 + (cz_\theta)^2 = c^2(x_\theta^2 + y_\theta^2 + z_\theta^2)$ is also a perfect square. From now on, it may be assumed that $\overline{c_\theta} = 1$.

If all of the coefficients of two quaternions are equivalent mod 7, then these quaternions are called equivalent and this relation is denoted by $\sim$. It can be shown that the representations of $\mu^{\pm n}$ and $\nu^{\pm n}$ mod 7 are the same for all $n \in \mathbb{N}$. Thus the statements that can be shown is

$$\overline{q}_{\mu^n} \sim (1, (3, 5, 0)), \quad \overline{q}_{\mu^{-n}} \sim (1, (4, 2, 0)),$$
$$\overline{q}_{\nu^n} \sim (1, (0, 5, 3)), \quad \overline{q}_{\nu^{-n}} \sim (1, (0, 2, 4)),$$

for any $n \in \mathbb{N}$. This result is proven by induction.

<u>*Base case*</u>: For $n = 1$ recall that

$$\overline{q_\mu} = (3,(2,1,0)), \qquad \overline{q_{\mu^{-1}}} = (3,(-2,-1,0)),$$
$$\overline{q_\nu} = (3,(0,1,2)), \qquad \overline{q_{\nu^{-1}}} = (3,(0,-1,-2))$$

As $3^{-1} \equiv 5 \mod 7$ then multiplying all representations above by 5 gives the desired result.
Induction step: Assume that $\overline{q_\mu}^{n-1} \sim (1,(3,5,0))$. We want to show that $\overline{q_\mu}^{n} \sim (1,(3,5,0))$. This can be achieved by performing quaternion multiplication

$$\overline{q_\mu}^{n} = \overline{q_\mu}^{n-1} \cdot \overline{q_\mu} \sim (1,(3,5,0)) \cdot (3,(2,1,0)) \sim (1,(3,5,0)) \cdot (1,(3,5,0)) \sim (2,(6,3,0)) \sim (1,(3,5,0)).$$

Similar computation can be performed for μ^{-1}, ν and ν^{-1} to finish the proof.

From this step it follows that any power of a generator can be reduced to ± 1 without affecting the final result. Recall that any word that fixes some point on the rational sphere must contain at least one factor $\mu^{\pm 1}$ and $\nu^{\pm 1}$, so by the assumption on the structure of the words, this word must consist of blocks of form $\mu^{\pm 1}\nu^{\pm 1}$. All such blocks in mod 7 are equivalent to

$$\mu\nu \sim (1,(1,1,5)), \qquad \mu^{-1}\nu \sim (1,(5,1,1)), \tag{11}$$
$$\mu\nu^{-1} \sim (1,(4,3,4)), \qquad \mu^{-1}\nu^{-1} \sim (1,(6,5,6)). \tag{12}$$

Every word that fixes some point on R is constructed from such pieces. The rest of the proof follows by tedious calculations which lead to the fact that the sum of the squares of the coefficients of quaternions that are generated by elements in 11 are equivalent to 3, 5 or 6 in mod 7, so they are not perfect squares, thus the points of intersection of the axis of rotation with the unit sphere are in fact irrational. Details can be found in [14].

□

Corollary 5.1. *The rational sphere is paradoxical with a group generated by rotations ν, μ, defined as in Equation 9.*

Proof. This corollary holds by Lemma 2.3, as matrices in Equation 9 generate a paradoxical group that acts without non-trivial fixed points on the rational sphere. □

An even more general statement can be considered. The group found by by Satô was generalized by Agata, who has shown in [1] that all matrices of the form

$$A_x = \frac{1}{n+1}\begin{pmatrix} n & x & x+1 \\ x & x+1 & -n \\ -x-1 & n & x \end{pmatrix}, \qquad B_x = \frac{1}{n+1}\begin{pmatrix} x & -n & x+1 \\ n & x+1 & x \\ -x-1 & x & n \end{pmatrix} \tag{13}$$

for $n = x \cdot (x+1)$, where x is some positive integer, generate a free subgroup of SO(3) that acts without non-trivial fixed points on R.

Theorem 9. *A free subgroup of $SO(3)$ generated by A_x, B_x, defined as in 13, acts without non-trivial fixed points on R [1].*

Proof. The proof of this theorem follows in a similar way as for the result by Satô. It begins by representing rotations $A_x^{\pm 1}, , B_x^{\pm 1}$ in the quaternion form

$$q_{A_x} = \pm \frac{1}{\sqrt{2(1+x+x^2)}}(x+1+xi+j)$$
$$q_{A_x^{-1}} = \pm \frac{1}{\sqrt{2(1+x+x^2)}}(x+1-xi-j)$$
$$q_{B_x} = \pm \frac{1}{\sqrt{2(1+x+x^2)}}(x+1+j+xk)$$
$$q_{B_x^{-1}} = \pm \frac{1}{\sqrt{2(1+x+x^2)}}(x+1-j-xk).$$

Then, the procedure is equivalent to the proof of Theorem 8, details can be found in [1]. □

Corollary 5.2. *Rational sphere is paradoxical using the free group generated by rotations A_x, B_x for any positive integer x.*

Proof. This corollary follows immediately from Lemma 3.1 and Theorem 9. □

5.2 The Banach-Tarski paradox in higher dimensions

In Section 7, it is proven that the Banach-Tarski paradox holds in $\mathbb{R}^3$. However, Corollary 3.1 hints at the fact that this phenomena could also be observed in higher dimensions due to the fact that all groups $SO(n)$ for $n \geq 3$ are paradoxical. As it turns out, the answer is positive.

Theorem 10. *The unit sphere S^{n-1} in $\mathbb{R}^n$ is $SO(n)$-paradoxical for any $n \geq 3$.*

Proof. This theorem is proven by induction.

Base case: The case for $n = 3$ is the statement of Theorem 7.

Induction hypothesis: Assume that the claim holds for some $n \in \mathbb{N}$, so the unit sphere S^{n-1} in $\mathbb{R}^{n+1}$ is SO(n)-paradoxical.

Induction step: The goal is to show that if the induction hypothesis is assumed then the sphere S^n is also SO(n)-paradoxical. The paradoxical decomposition of the sphere S^{n-1} is $S^{n-1} = \bigcup_{r=1}^{a} g_r(A_r) = \bigcup_{q=1}^{b} f_q(B_q)$.

A partition of S^n can be constructed from the partition of S^{n-1}. To define new sets A'_r and B'_q that form a paradoxical decomposition of S^n, notice that the points on the n-dimensional sphere are of the form $(x_1, x_2, \dots, x_{n-1}, x_n)$, where the first $n-1$ coefficients can be identified with the points on the $(n-1)$-dimensional sphere. Now, using the fact that the paradoxical decomposition of S^{n-1} exists, the points on S^n can be classified into subsets using the rule that $(x_1, x_2, \dots, x_{n-1}, x_n) \in A'_r$ if $\frac{1}{|(x_1,\dots,x_{n-1})|}(x_1, x_2, \dots, x_{n-1}) \in A_r$. The sets B'_q are formed in a similar way.

This construction works for all points on the sphere S^n except for the points $(0, 0, \dots, 1)$ and $(0, 0, \dots, -1)$, as the point $(0, \dots, 0)$ does not belong to the sphere S^{n-1}. The sets A'_r and B'_q are clearly pairwise disjoint and cover the whole of $S^n \setminus \{(0, \dots, \pm 1)\}$, as their structure follows explicitly from the properties of sets A_r and B_q, thus they form a partition of the set $S^n \setminus \{(0, \dots, \pm 1)\}$.

To find elements of the group SO(n) that act on sets A'_r, B'_r, rotations f_q and g_r can be extended to act on S^n by

$$g'_r = \begin{pmatrix} & & 0 \\ & g_r & \vdots \\ 0 & \dots & 1 \end{pmatrix}, \quad f'_q = \begin{pmatrix} & & 0 \\ & f_q & \vdots \\ 0 & \dots & 1 \end{pmatrix}.$$

This rotations fix the n-th coordinate and generate a free subgroup of SO(n). In this way, the paradoxical decomposition

$$S^n \setminus \{(0,\dots,\pm 1)\} = \bigcup_{r=1}^{a} g'_r(A'_r) = \bigcup_{q=1}^{b} f'_q(B'_q)$$

of $S^n \setminus \{(0,\dots,\pm 1)$ in $\mathbb{R}^{n+1}$ can be created. To include the points $(0,\dots,\pm 1)$ it can be shown that S^n is equidecomposable with $S^n \setminus \{(0,\dots,\pm 1)$, which can be done using the same strategy as in the proof of Theorem 7. Thus by induction it holds that the n-dimensional unit sphere is SO(n)-paradoxical. □

This result can be extended to any sphere in $\mathbb{R}^n$, thus there is an immediately conclusion of this result.

Corollary 5.3. *Any n-dimensional solid ball in $\mathbb{R}^n$ is $SO(n)$-paradoxical.*

Proof. This result follows from Theorem 5.2 and Lemma 2.2. □

5.3 Amenable groups

So far the consideration of Banach-Tarski paradox was restricted to spaces $\mathbb{R}^n$ for $n \geq 3$. The question still remains, whether there is an equivalent result in $\mathbb{R}^2$. As it turns out, if the operations are restricted to the group SO(2), the answer is negative.

To understand why this is the case, we need to recall some information about transformations in a Euclidean plane. Orthogonal group $O(2)$ is a group containing all of the distance preserving transformations in $\mathbb{R}^2$ that fix some point and it is the symmetry group of a circle. Special orthogonal group SO(2) is the subgroup of $O(2)$ that is isomorphic to the multiplicative group of complex numbers with norm 1. What is more, the group $SO(2)$ is commutative and consists of all of the rotations in the plane that fix the origin and all of its elements can be presented in form of matrices

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

where θ is some angle.

To prove the result that the Banach-Tarski paradox does not take place in $\mathbb{R}^2$ using the special orthogonal groups, it is necessary to diverge a bit into the group theory and define the notion of an amenable group.

Definition 5.2. A group G is amenable if there exists some finitely additive measure μ on the power set $P(G)$ such that $\mu(G) = 1$ and for all $g \in G$ and $A \subset G$ it holds that $\mu(A) = \mu(gA)$.

This definition describes the fact that if there exists some finitely additive, probability measure on the group G that it is preserved under the action of the group G on itself, then such a group is called amenable. Measures that are preserved under the action of the group G are called G-invariant. The connection between the existence of a G-invariant measure on a set, and the paradoxicality of it was first described by Tarski.

Theorem 11 (Tarski's Theorem)**.** *Consider some group G that acts on a set X and some subset $E \subset X$. There exists a finitely additive G-invariant measure $\mu : P(X) \to [0, \infty]$ with $\mu(E) = 1$ if and only if E is not G-paradoxical [16].*

Proof. Proof can be found in Chapter 11 of [17]. □

Tarski's Theorem allows us to connect the notions of a group being amenable and paradoxical. In fact, it can be shown that a group cannot have both of these properties at the same time.

Theorem 12. *Consider some group G acting on a set X. The group G cannot be both paradoxical and amenable [2].*

Proof. First assume that G is amenable, so by definition there exists some G-invariant measure μ on $P(G)$ such that $\mu(G) = 1$. To show that G cannot be paradoxical, a measure on X, satisfying the properties mentioned in the Tarski Theorem, can be constructed. Consider some set $A \subset X$ and define a measure on $P(X)$ by $\nu : P(X) \to [0, 1]$, where $\nu(A) = \mu(\{g \in G : g(x) \in A \text{ for some } x \in X\})$. It needs to be checked that the measure ν is such that $\nu(X) = 1$, finitely additive, and G-invariant. Clearly ν is finitely additive, as μ is. What is more, $\nu(X) = \mu(g \in G : g(x) \in X\}) = \mu(G) = 1$. Finally, for some $k \in G$ it holds that

$$\nu(kA) = \mu(g \in G : g(x) \in kA) = \mu(k^{-1}g \in G : k^{-1}g(x) \in A) = \mu(A),$$

so by Tarski's Theorem G is not paradoxical.

Now, assume that G is paradoxical. If it was also amenable then there must exists a G-invariant measure μ on $P(G)$ such that $\mu(G) = 1$. However, as G is paradoxical, the Banach-Tarski paradox construction can be performed on it, and it must hold that $\mu(G) = 2\mu(G)$ which is clearly a contradiction. It can be concluded that G cannot be paradoxical if it is amenable. □

Corollary 5.4. *All groups $SO(n)$ are not amenable for $n \geq 3$.*

Proof. This result follows immediately from Theorem 12, as groups $\mathrm{SO}(n)$ for $n \geq 3$ are paradoxical. □

Going back to the case of the Banach-Tarski in two dimensions, to prove that it does not take place in $\mathbb{R}^2$, it can be shown that the group SO(2) is amenable, so it cannot be paradoxical. To do this another result needs to be considered.

Theorem 13. *Abelian groups are amenable.*

Proof. The proof of this theorem is out of the scope of this thesis and can be found in [2]. □

Corollary 5.5. *The Banach-Tarski paradox does not take place in $\mathbb{R}^2$ using the group $SO(2)$.*

Proof. Group SO(2) is abelian thus by Theorem 13 it is amenable, so by Theorem 12 not paradoxical and cannot be used to construct the Banach-Tarski paradox. □

5.4 The Banach-Tarski paradox in $\mathbb{R}^2$

In the previous section it is shown that the Banach-Tarski paradox cannot take place in $\mathbb{R}^2$ using the special orthogonal group SO(2). However, this does not contradict the fact that the paradox cannot happen at all. It was established that the group SO(2) is not paradoxical, but following the idea of Von Neumann [10], the Banach-Tarski paradox can still take place in the plane if the group SO(2) is expanded by allowing for more area-preserving transformations in the plane.

Affine group $A_2(\mathbb{R})$ is a group consisting of all affine transformations in $\mathbb{R}^2$. These are automorphisms of the Euclidean plane that preserve lines and parallelisms, but they may not preserve angles not Euclidean distances. Examples of affine transformations include translations, rotations, scaling, reflections, shear transformations and their compositions. Affine maps are of form $\sigma : x \to Mx + v$, where $M \in \mathbb{R}^{2\times 2}$ denotes some linear transformation and v is some vector in $\mathbb{R}^2$ representing a translation. These maps scale an object on which they act by the factor $|\det M|$.

Special affine group $\mathrm{SA}_2(\mathbb{R})$ is a subgroup of the affine group and consists of all affine transformations that preserve area and orientation. It consists of all maps of form $\gamma : x \to Mx + v$, where $M \in \mathbb{R}^{2\times 2}$ is a matrix such that $\det M = 1$ and v is some vector in $\mathbb{R}^2$ representing a translation. Special affine group has its linear analogue, called the special linear group $\mathrm{SL}_2(\mathbb{R})$. This group contains all of the matrices in $\mathbb{R}^{2\times 2}$ with determinant 1. If the Banach-Tarski paradox was to happen in the plane using elements of $\mathrm{SA}_2(\mathbb{R})$, then this result could be considered to have the same paradoxical nature as it is the case for its three dimensional equivalent. The Banach-Tarski paradox would double the area of a set using area preserving transformations.

Von Neumann has shown that a square $J^2 = [0, 1)^2$ is $\mathrm{SA}_2(\mathbb{R})$-paradoxical [10]. In this thesis, a simpler argument due to Wagon [18] is followed. The idea of the proof relies on the fact that a free subgroup of $\mathrm{SL}_2(\mathbb{R})$ can be generated by two independent elements of it. In our case, it is a free group generated by two shears. Before the proof of this claim is considered, it is useful to recall some information about shear transformations [6].

Definition 5.3. A shear mapping is an affine transformation which fixes some line in the plane and all of the points not contained in it are shifted parallel to it by a the amount proportional to their distance from the line.

Linear transformations induced by shears have determinant equal to 1, thus they fix the area. However, they are not isometries as they do not maintain the distance between points, nor angles. In fact, it holds that any element of the special linear group $\mathrm{SL}_2(\mathbb{R})$ can be written in terms of shears. Compositions of shears still preserve the area, but the may not be a shear anymore.

Two shears of our interest are of form $\sigma_1 : (x, y) \to (x + 2y, y)$, so it fixes the x-axis and stretches objects along it by a factor 2. And the second shear is $\sigma_2 : (x, y) \to (x, y + 2x)$, so it fixes the y-axis and stretches objects with respect to it by a factor 2. These shear transformations can be represented in the matrix form as

$$\sigma_1 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}. \tag{14}$$

The way in which σ_1 and σ_2 act on J^2 can be seen in Figure 11.

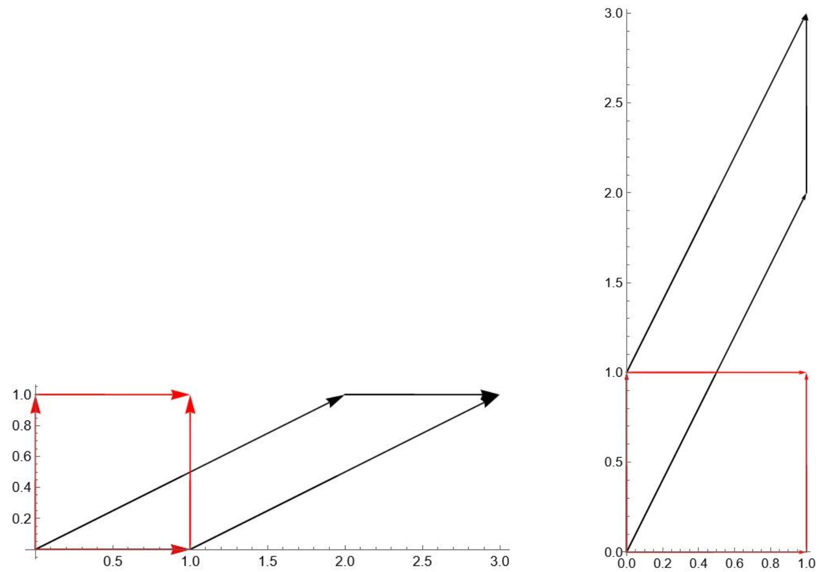

Figure 11: On the left: shear σ_1 acting on a square J^2. On the right: shear σ_2 acting on a square J^2.

Consider a group G_2^* generated by all of the elements of $G_2 = \mathrm{SO}(2)$ and a shear σ_1. Notice that it holds that

$$\sigma_2 = \rho^{-1}\sigma_1^{-1}\rho, \quad \text{for} \quad \rho = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

so in fact $\sigma_2 \in G_2^*$. It was first proven by Sanov in [13] that the group F generated by shears σ_1 and σ_2 is a free subgroup of G_2^*. This result was later improved on by Brenner [4], who showed that any pair of matrices

$$A_m = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}, \qquad A_m^T = \begin{pmatrix} 1 & 0 \\ m & 0 \end{pmatrix}$$

generate a free group for any $m \in \mathbb{R}$ such that $m \geq 1$. From this we know that the group G_2^* has a subgroup F isomorphic to F_2.

Define another subgroup $H < G_2^*$ which is generated by F and the group T of all translations in the plane. Any element of H is of form $h = \tau M$, where τ is a translation and $M \in F$. To prove the Banach-Tarski paradox, we need to find a paradoxical group that on a square without non-trivial fixed points. However, it can be easily noticed that even the shear σ_1 does not map all points of the square back to itself. To fix this issue, consider an equivalence relation $\sim$ on $\mathbb{R}^2$

$$x \sim y \text{ if and only if } x - y \in \mathbb{Z}^2. \tag{15}$$

Thus two points $x, y \in \mathbb{R}^2$ are equivalent if and only if it holds that $x = y + (m, n)$, where $(m, n) \in \mathbb{Z}^2$. Denote by $\hat{x}$ a unique point in J^2 that is equivalent to x under this equivalence relation.

Theorem 14. *The unit square $J^2 = [0, 1)^2$ is G_2^*-paradoxical.*

Proof. The group $F < H < G_2^*$ generated by shears σ_1, σ_2 was already proven to be isomorphic to F_2 by Sanov [13], so it is also paradoxical. However, Lemma 3.1 cannot be explicitly used to conclude that J^2 is paradoxical, as the elements of F, nor H, map J^2 to itself and these groups do not act without non-trivial fixed points on the square. To adjust the elements of H to map a square to itself, aforementioned equivalence relation in 15 can be used. Consider the function

$$\hat{h} : J^2 \to J^2 \text{ such that } \hat{h}(x) = \widehat{h(x)}.$$

This function maps a point $x \in J^2$ to $h(x)$ by some element $h \in H$, and then to the unique point in J^2 that is equivalent to $h(x)$. All elements of groups H, F and T can modified using the equivalence relation and groups containing them are denoted by $\hat{H}, \hat{F}, \hat{T}$, respectively.

It was shown that F is free, but what needs to be proven is whether $\hat{F}$ is free and if it is generated by $\hat{\sigma_1}$ and $\hat{\sigma_2}$. This can be done by showing that F is isomorphic to $\hat{F}$. To do this, define a map $I : H \to \hat{H}$, that maps elements of the group H to their equivalent elements in $\hat{H}$, and show that it is an isomorphism between F and $\hat{F}$.

It needs to be shown is that I is a bijective homomorphism. First, to show that it is a homeomorphism consider two elements $h_1, h_2 \in H$ and it needs to be proved that $\widehat{h_1 h_2} = \hat{h}_1 \hat{h}_2$ holds. Begin by noticing that if $x \sim y$ then it follows that $h(x) \sim h(y)$ as if $(x - y) \in \mathbb{Z}^2$ then $h(x) - h(y) = h(x - y) \in \mathbb{Z}^2$. This holds as σ_1, σ_2 and τ, where τ is some translation, are linear transformations. It follows that

$$\hat{h}_1 \hat{h}_2(x) = \hat{h}_1 \widehat{h_2(x)} = \widehat{h_1 \widehat{h_2(x)}} = h_1 \widehat{(h_2(x))} = \widehat{h_1 h_2(x)},$$

where second to last inequality follows from the observation above. It can be concluded that the map I is a homomorphism. Clearly, the map I is surjective, so the only thing left to show is injectivity. Aiming for contradiction assume that there exists some element $h \in H$ such that $I(h) = \hat{e}$ where $\hat{e}$ is the identity element of $\hat{H}$, but $h \neq e$.

Now, the structure of elements in $\hat{H}$ needs to be considered. Clearly not all elements $\hat{h} \in \hat{H}$ can be expressed in the form $\tau \circ h$, for all points in J^2, as some points would be mapped outside of the square. It follows that elements of $\hat{H}$ are defined piecewise as compositions of elements of H and translations that map points $h(x)$ back to J^2. In fact, it can be concluded that elements of $\hat{H}$ are of form $\hat{h}(a, b) = h(a, b) - \lfloor h(a, b) \rfloor$ for $(a, b) \in \mathbb{R}^2$.

Consider some subset of J^2 on which $\hat{h} = 1$ is defined as $1 = \hat{h} = \tau \circ h$ for some translation τ. This means that $\tau \circ h$ is equal to identity for at least three non-collinear points. Which means that sing the fact that if two affine maps have the same image for three non-collinear points then the maps are equal, it can be concluded that in fact $\tau \circ h = 1$. It means that $h = \tau^{-1}$ which is only possible id $\tau = h = 1$.From this it can be concluded that the map I is injective. What follows is the fact that I is an isomorphism so F and $\hat{F}$ and $\hat{F}$ is a free group generated by $\hat{\sigma}_1$ and $\hat{\sigma}_2$.

As the issue of the group H not mapping J^2 to itself is resolved by considering the group $\hat{H}$. What is left to show is that the action of $\hat{F}$ on J^2 is without fixed points, or more precisely, that fixed points can be included in the decomposition. Define the set of all non-trivial fixed points of J^2 under $\hat{F}$ by $D = \{x \in J^2 : \hat{f}(x) = x \text{ for some } \hat{f} \in \hat{F}, \hat{f} \neq 1\}$. Lemma 3.1 can be applied to the set $J^2 \setminus D$ to conclude that $J^2 \setminus D$ is $\hat{F}$-paradoxical. What is more, as all elements of $\hat{F}$ are defined piecewise by elements of H, so the set $J^2 \setminus D$ is also H-paradoxical and as $H < G_2^*$, then it is G_2^*-paradoxical. Now, we need to show that J^2 is paradoxical. This can be done by proving that $J^2 \setminus D$ is T-equidecomposable with J^2 and as $T < H$, then it can be concluded that they are H-equidecomposable.

Notice that if a point is fixed by some element of $\hat{F}$, then it must mean that it holds that $\hat{f}(x, y) = (x, y) + (m, n)$ for some $(m, n) \in \mathbb{Z}^2$. Linear transformations in the plane that are generated by shears fix either points, as in the case of rotations, or line segments (shears). The number of elements in $\hat{F}$ is countable and the action of $\hat{F}$, so the set D can be separated into two subsets $D = D_0 \cup D_1$ where D_0 is a set of points and D_1 is a set of line segments; both of these sets are countable. However, the whole set D contains uncountably many points, so to find the desired equidecomposition, first an equidecomposition of $J^2 \setminus D$ with a set $J^2 \setminus P$, where P is some countable set, is found.

To construct the set P we want to find some translation $\tau \in T$ such that the set $D \cap \hat{\tau}^n(D)$ is countable for all $n \in \mathbb{N}$. Number of points in D_0 is countable, so they do not need to be taken into consideration while finding the translation. As there is a countable number of line segments in D_1, they can be numbered by $S_0, S_1, \ldots$. Consider two line segments S_i, S_j such that $i \neq j$, either S_i and S_j are parallel, or not. If they are not parallel then for any $n \in \mathbb{N}$ it holds that $S_i \cap \tau^n(S_j)$ contains at most one point, so no translation causes the problem of uncountable intersection.

The issue arises when line segments S_i and S_j are parallel. If any point $p \in S_j$ is mapped by some translation τ^n to some point $q + (a, b)$, where $(a, b) \in \mathbb{N}$ and $q \in S_i$, then it means that $\hat{\tau}^n(S_j) \cap S_i$ is non-empty, possibly uncountable. This is the case as the element $\hat{\tau^n}$ maps S_j back to the square in a way that $\hat{\tau}(S_j)$ and S_i share a point, so if this is not an endpoint then their intersection may contain uncountably many points.

To make sure that the intersection is countable, all such translations need to be discarded. We need to make sure that there will be at least one translation leftover. For each of the countably many combinations of sets S_i, S_j and n, at most countably many translations map some point of S_i to some point $q + (a, b)$, so in total there is a countable number of translations discarded. As number of translations in the plane is countable, there must exist at least one τ translation leftover for which $S_i \cap \hat{\tau^n}(S_j) = \emptyset$ for all $n \in \mathbb{N}$ and all pairs of line segments that are parallel. So the set $D \cap \hat{\tau}^n(D)$ is countable for all $n \in \mathbb{N}$.

Now define a set $P = \{D \cap \hat{\tau^n}(D) : n \in \mathbb{Z} \setminus \{0\}\}$, which is a countable subset of D. We want to show that $J^2 \setminus P$ and $J^2 \setminus D$ are $\hat{T}$-equidecomposable. To do this consider sets $\hat{\tau}^n(D \setminus P)$ for all $n \in \mathbb{N}$. These are the points that are not in the image of D under $\hat{\tau}^n$ for any non-zero $n \in \mathbb{Z}$ that are translated by $\hat{\tau}^n$. These sets are pairwise disjoint, and disjoint from P. To show this assume that there exist some $n, m \in \mathbb{N}$ such that for some $d_1, d_2 \in D \setminus P$ it holds that $\hat{\tau}^n(d_1) = \hat{\tau}^m(d_2)$. Then $d_1 = \hat{\tau}^{m-n} d_2$, which is a contradiction as it follows that d_2 can be mapped to d_1 by $\hat{\tau}^{m-n}$, so these points must belong to P, which is a contradiction.

Consider the set $A = \bigcup_{n=0}^{\infty} \hat{\tau}^n(D \setminus P)$. A trivial decomposition of $J^2 \setminus P$ is $J^2 \setminus P = A \cup [(J^2 \setminus P) \setminus A]$. To find a decomposition of $J^2 \setminus D$ consider the set $\hat{\tau}(A) = \bigcup_{n=1}^{\infty} \hat{\tau}^n(D \setminus P) = (J^2 \setminus D) \setminus [(J^2 \setminus P) \setminus A]$, so we get that the equidecomposition of $J^2 \setminus D$ and $J^2 \setminus P$ is

$$J^2 \setminus P = A \cup (J^2 \setminus P) \setminus A,$$
$$J^2 \setminus D = [(J^2 \setminus P) \setminus A] \cup \hat{\tau}(A).$$

This reasoning concludes the proof that $J^2 \setminus P$ and $J^2 \setminus D$ are $\hat{T}$-equidecomposable. Now, as $J^2 \setminus D$ is paradoxical then $J^2 \setminus P$ must be too. The thing left to show is that $J^2 \setminus P$ and J^2 are equidecomposable. This follows in exactly the same way as the proof for $J^2 \setminus D$ and $J^2 \setminus P$ being $\hat{T}$-equidecomposable, but the process is easier as now the set P is countable, so there is no need to worry about uncountable intersections. We can find a translation τ such that $\hat{\tau}^n(P) \cap P = \emptyset$ and follow the procedure as before. This shows that J and $J \setminus C$ are $\hat{T}$-equidecomposable from which it can be concluded that $J \sim J \setminus P \sim J \setminus C$, so as $J \setminus P$ is paradoxical then by translation J is too. □

6 Visualisation of the Banach-Tarski paradox

6.1 Visualisation for the unit sphere

The Banach-Tarski paradox is in principle a result that takes advantage of the fact that the subsets into which a sphere is decomposed are uncountable and dense. If these subsets were to be presented accurately, it would mean that picture would need to include uncountably many points that are dense in the sphere, so they would not visually differ from the whole sphere. Thus, to maintain the visual soundness of the graphical representation, the process that a sphere undergoes during the Banach-Tarski paradox is presented only for one orbit and there is an upper bound put on the length of the words in the free group that acts on it.

Consider a point $p = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$ located on the unit sphere. The orbit of this point under the action of F_2 is used to demonstrate the construction of the paradoxical decomposition of the sphere. The location of the point p on the sphere is presented in Figure 12.

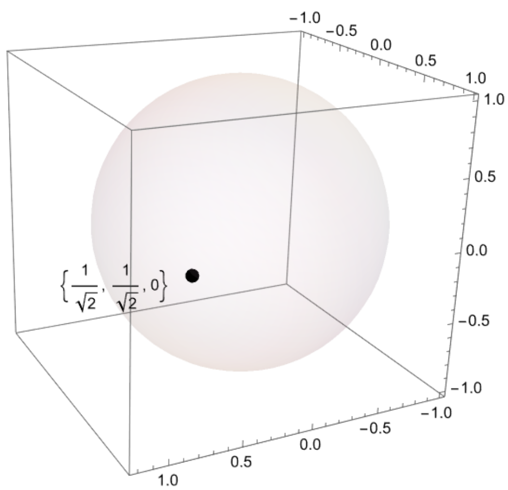

Figure 12: Point $p = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$ on a unit sphere.

The generators of the group that acts on S^2 are rotations with respect to z-axis and x-axis by the angle $\theta = \arccos \frac{1}{3}$:

$$\sigma = \begin{pmatrix} \frac{1}{3} & -\frac{2\sqrt{2}}{3} & 0 \\ \frac{2\sqrt{2}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \sigma^{-1} = \begin{pmatrix} \frac{1}{3} & \frac{2\sqrt{2}}{3} & 0 \\ -\frac{2\sqrt{2}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & -\frac{2\sqrt{2}}{3} \\ 0 & \frac{2\sqrt{2}}{3} & \frac{1}{3} \end{pmatrix}, \quad \tau^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{2\sqrt{2}}{3} \\ 0 & -\frac{2\sqrt{2}}{3} & \frac{1}{3} \end{pmatrix}.$$

These rotations satisfy the properties listed in Lemma 3.3, so they generate a subgroup of SO(3) isomorphic to F_2. For the sake of a successful visualization, the length of words in the group is restricted to four. The set of all such words is denoted by G^4. In Figure 13, the orbit of p under the elements from the set G^4 is presented. Different colours represent the fact that given point was obtained by acting on p with a word from G^4 that ends with:

1. Blue: $W_p(\sigma)$, points obtained by acting on p with words in G^4 that end with σ,

2. Green: $W_p(\tau)$, points obtained by acting on p with words in G^4 that end with τ,
3. Red: $W_p(\sigma^{-1})$, points obtained by acting on p with words in G^4 that end with σ^{-1},
4. Purple: $W_p(\tau^{-1})$, points obtained by acting on p with words in G^4 that end with τ^{-1},

as defined in the proof of Theorem7.

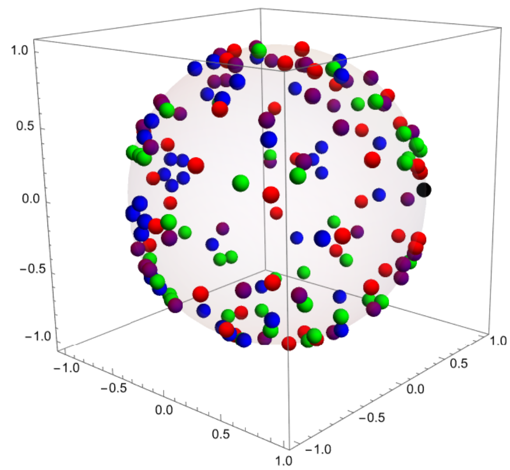

Figure 13: Orbit of the point p under the action of the elements from the set G^4.

Now, in Figure 14 all sets are translated to visually separate all of the subsets of the orbit that were obtained from p by acting on it with words that end with different generator. This step is not necessary in the theoretical construction of the decomposition.

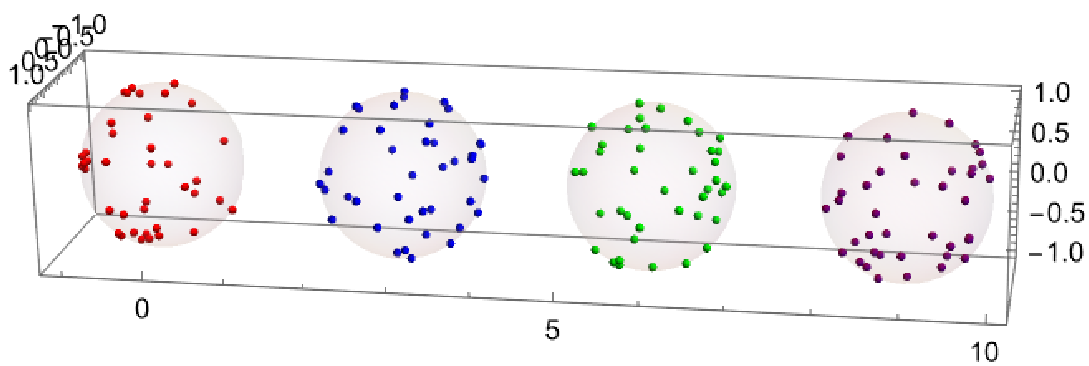

Figure 14: Subsets of the orbit translated.

Consider sets $W_p(\sigma^{-1})$ and $W_p(\tau^{-1})$ and rotate all of the points contained in them by σ and τ, respectively. This rotation can be seen in Figure 15.

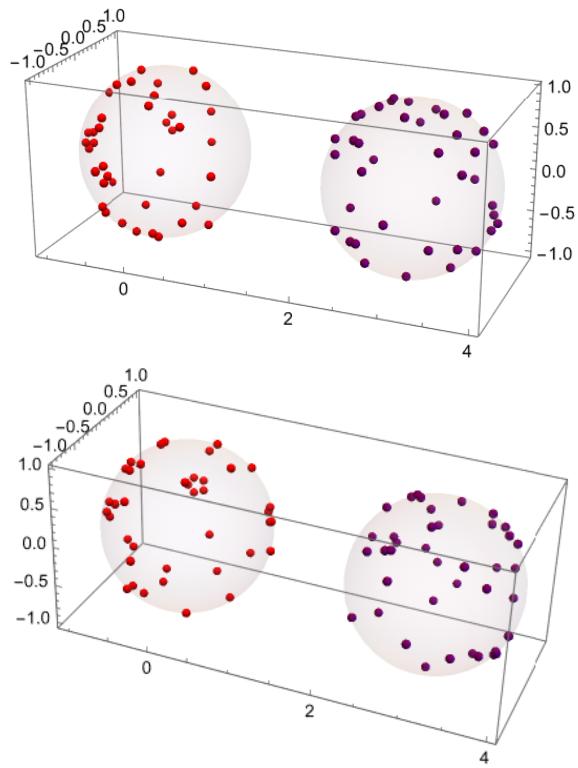

Figure 15: Upper figure: sets $W_{\sigma^{-1}}(p)$ and $W_{\tau^{-1}}(p)$. Lower figure: sets $\sigma W_{\sigma^{-1}}(p)$ and $\tau W_{\tau^{-1}}(p)$.

The paradoxical nature of this process follows from the fact that, in the infinite version of the Banach-Tarski paradox, words that are considered are of infinite length. Thus, while the set $W_p(\sigma^{-1})$ contains all of the points obtained by rotating point p by words that end with σ^{-1}, the set $\sigma W_p(\sigma^{-1})$ contains all of the points obtained by rotating p by words that do not end with σ. The same procedure holds for the set $W_p(\tau^{-1})$. Unfortunately, this process cannot be accurately presented in the finite approximation. Points in sets $\sigma W_p(\sigma^{-1})$ and $\tau W_p(\tau^{-1})$ can be now reassigned to their new classes based on what element of the generating set ends the rotation that is acting on p after the groups ware rotated by σ or τ. New reassignment can be seen in Figure 16.

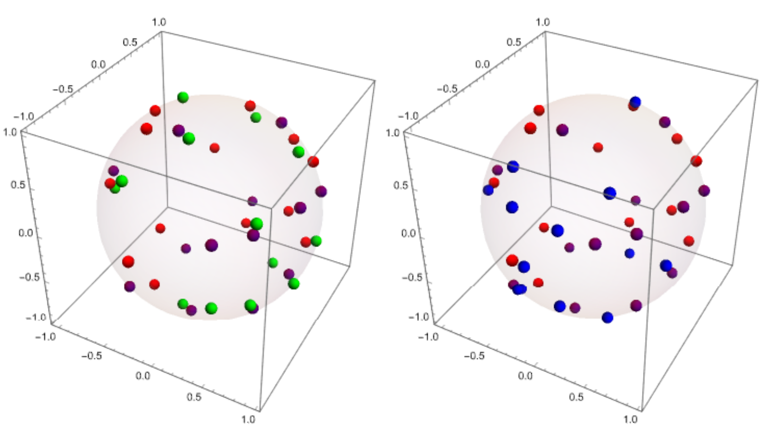

Figure 16: On the left: set $\sigma W_p(\sigma^{-1})$. On the right: set $\tau W_p(\tau^{-1})$.

Now, if the set $W_p(\sigma)$ is combined with the set $\sigma W_p(\sigma^{-1})$, then in fact we obtain the whole initial orbit of p. The point p is being acted on by all rotations that end with σ, due to the set $W_p(\sigma)$, and all of the rotations that do not end withσ, from the set $\sigma W_p(\sigma^{-1})$. In this way, we obtain the fist copy of the orbit. Sets $W_\tau(p)$ and $\tau W_{\tau^{-1}}(p)$ can also be combined, which gives the second copy of the initial orbit concluding the Banach-Tarski paradox. Two copies of the initial orbit can be seen in Figure 17.

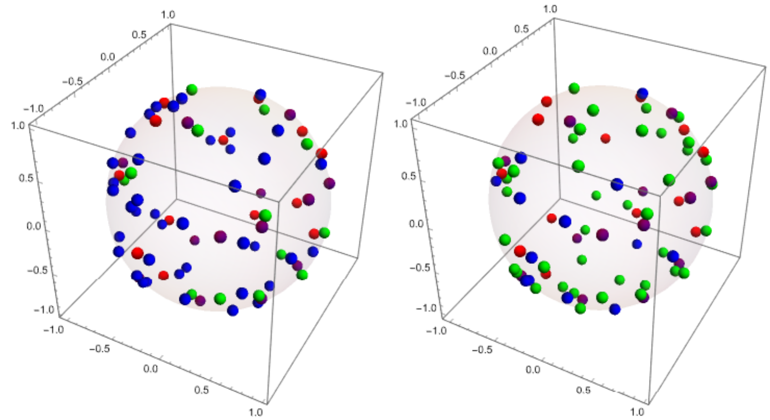

Figure 17: Two copies of the initial orbit.

The visualisation of the Banach-Tarski paradox in Figure 18 can be understood as beginning stages of the construction of the paradoxical decomposition of the sphere. This process would have to be repeated for all orbits, and under all elements of a group isomorphic to F_2.

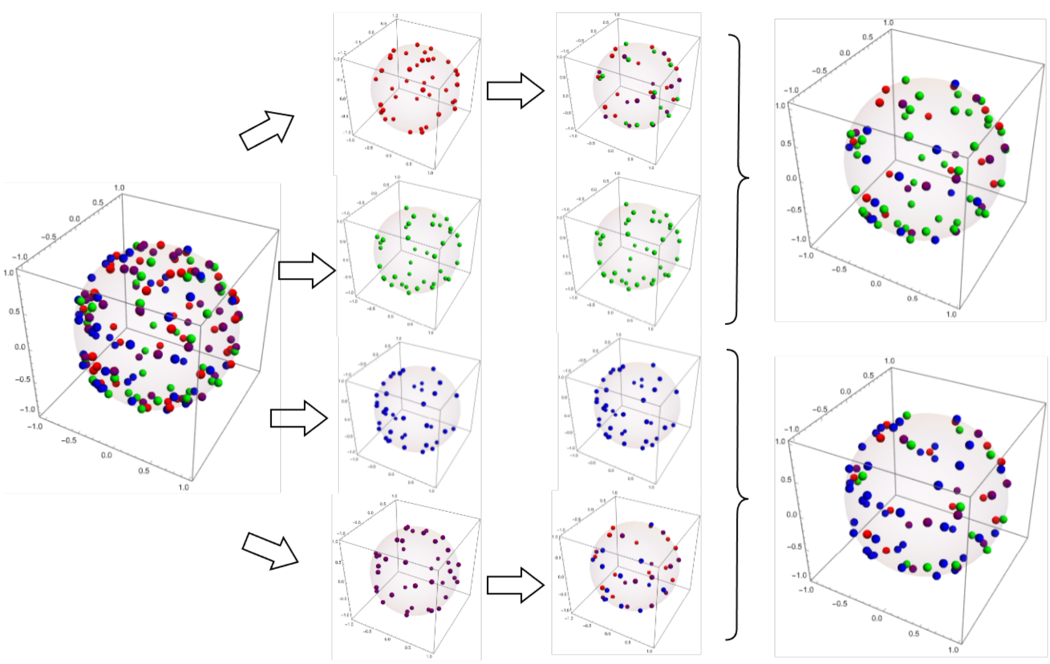

Figure 18: Visual description of the initial stage of the decomposition of the sphere in the Banach-Tarski paradox.

6.2 Visualisation for the unit square

Visualization of the paradoxical decomposition, that is created in the proof of the Banach-Tarski paradox, can also be preformed for the unit square. Similarly to the sphere, the whole construction cannot be presented, as sets into which the square is decomposed are dense and uncountable. To provide a reasonable visual representation of the paradox, the decomposition is performed for the orbit of one point $q = (\frac{1}{\pi}, \frac{1}{e})$. The group that acts on q is $\hat{F}$ generated by two shears σ_1, σ_2, defined as in 14. The length of the words in $\hat{F}$ that act on q is restricted to maximum six.

Before the decomposition is considered, on Figure 19 there is presented a reason why it is necessary to consider the action of F on a square under the equivalence relation. It can be clearly noticed that the point q, in most of the instances, is not mapped back to J^2 by elements from the group F.

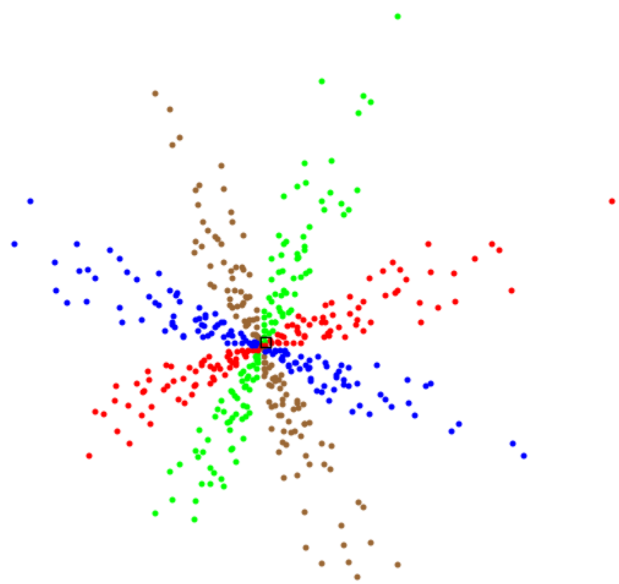

Figure 19: The orbit of q under words in F of maximum length 6.

However, in Figure 20 the orbit of q under $\hat{F}$ is presented. Indeed, now the point q gets mapped back to the square by all elements of the group. Orbit is decomposed into subsets that can be distinguished by different colours:

1. Blue: $W_q(\sigma_1^{-1})$, orbit of q under words in $\hat{F}$ that end with σ_1^{-1},
2. Brown: $W_q(\sigma_2^{-1})$, orbit of q under words in $\hat{F}$ that end with σ_2^{-1},
3. Green: $W_q(\sigma_2)$, orbit of q under words in $\hat{F}$ that end with σ_2,
4. Red: $W_q(\sigma_1)$, orbit of q under words in $\hat{F}$ that end with σ_1.

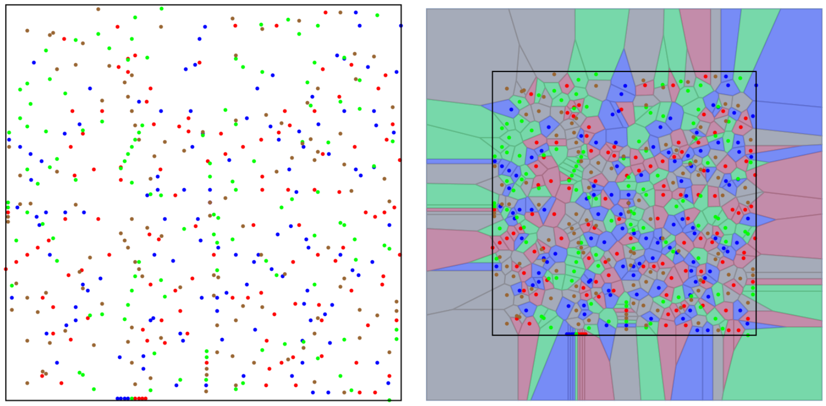

Figure 20: On the left: Orbit of q under words in $\hat{F}$ of maximum length 6. On the right: the same orbit as on the left figure, but with the Voronoi mesh.

If all words in $\hat{F}$ were considered, then the orbit of q would be dense in a square. To encapsulate this idea in the version using a finite number of points, we generate a Voronoi mesh on the square based on the position of points in the orbit of q. Voronoi mesh is one of the ways in which the plane can be partitioned based on some finite set of points. Each point is contained in exactly one region called a Voronoi cell. All cells are pairwise disjoint and consist of all of the points in the plane that are the closest to a given point from the set.

For the visualization purposes, subsets of the orbit can be separated into separate squares. In Figure 21 subsets are again presented with the Voronoi mesh.

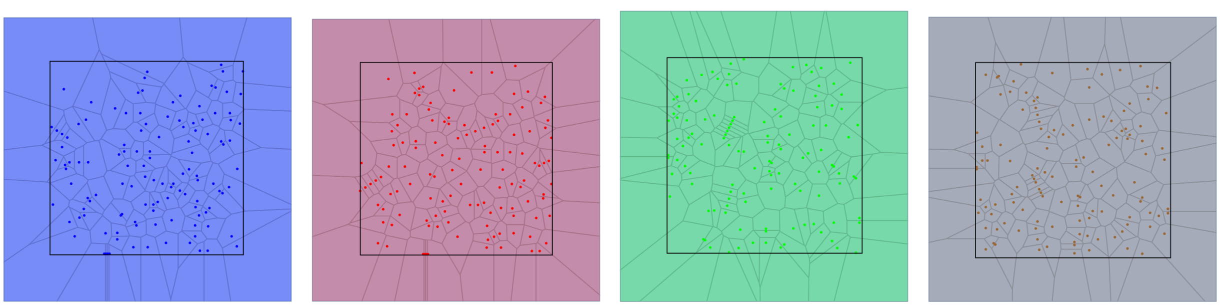

Figure 21: Orbit of q under $\hat{F}$ separated into four subsets.

Now, the subsets of the orbit $W_q(\sigma_1^{-1})$ and $W_q(\sigma_2^{-1})$ are acted on by the elements σ_1 and σ_2, respectively. This means that now the set $\sigma_1 W_q(\sigma_1^{-1})$ consists of all of the points that can be obtained by acting on q with words that do not end with σ_1 and the set $\sigma_2 W_q(\sigma_2^{-1})$ consists of all of the points that can be obtained by acting on q with words that do not end with σ_2. Points in these sets can be rearranged into new classes based on the element of the generating set that ends the word that acts on q after the shear operations are performed.

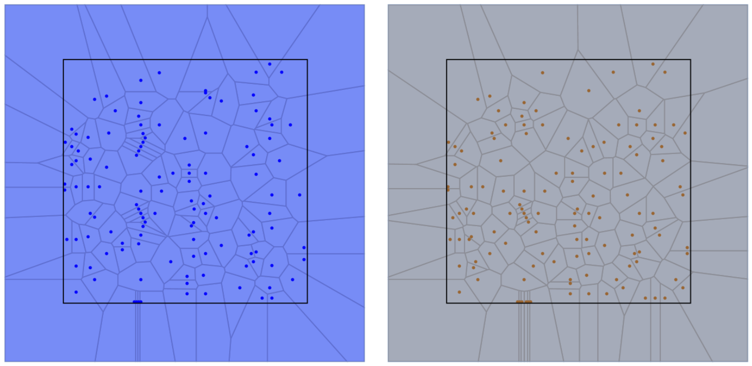

Figure 22: On the left: set $\sigma_1 W_q(\sigma_1^{-1})$. On the right: set $\sigma_2^{-1} W_q(\sigma_2^{-1})$.

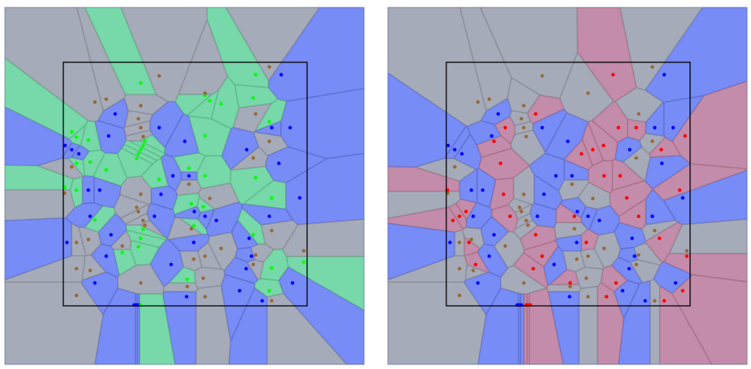

Figure 23: Sets from Figure 22 with points reassigned to their new classes.

Finally, the sets $W_q(\sigma_1)$ and $\sigma_1 W_q(\sigma_1^{-1})$ are combined to obtain the first copy of the unit square. Second copy is obtained by taking the union of sets $W_q(\sigma_2)$ and $\sigma_2 W_q(\sigma_2^{-1})$. These copies can be seen in Figure 24.

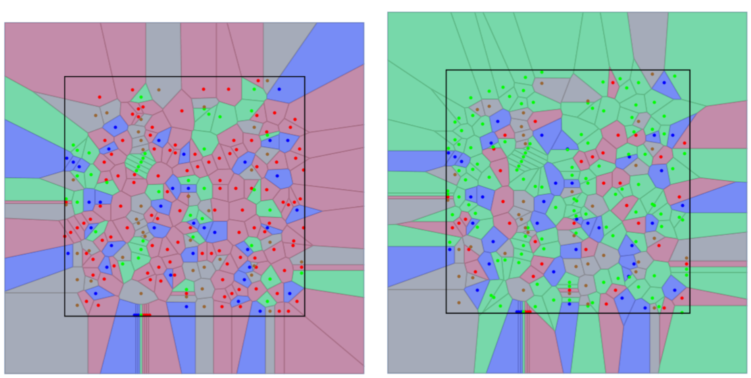

Figure 24: Two copies of the unit square obtained by the Banach-Tarski paradox.

7 Conclusion

7.1 Comment on the Axiom of Choice

The Axiom of Choice is a pivotal part of the proof of not only the Banach Tarski Paradox, but also of many other results involving uncountably infinite sets. Axiom of Choice was first stated by Ernst Zermelo in 1904 [19] in order to formalise the proof of the Well-ordering theorem. Results obtained by assuming it, for example the Banach-Tarski paradox, started the debate about the integrity of this axiom with the rest of the set theory axioms, until Kurt Godel proved its consistency [7]. The main issue that arises while assuming the Axiom of Choice is the fact that it allows for the choice of elements from arbitrary collection of sets, but it does not describe how the choices are made, thus proofs taking advantage of the Axiom of Choice are not constructive. Choice function acts like a black box that results in a set of elements, but the process of choosing is unknown. Though the Axiom of Choice was considered to be controversial at first, it turns out that assuming it allow for proving many fascinating and important results such as the Banach-Tarski paradox. In the context of the Banach-Tarski paradox, the Axiom of Choice is used to select representatives of uncountably many orbits.

7.2 Further generalizations

The Banach-Tarski paradox has been challenging the intuition of mathematicians for over a century as it is a fascinating result bringing together notions from the measure theory, geometry, topology and algebra. This thesis investigated various generalizations of this paradox including the classic realization for the unit ball and sphere in $\mathbb{R}^3$, spheres and balls in dimensions $n \geq 3$, rational sphere in $\mathbb{R}^3$, and a square in $\mathbb{R}^2$. Even though the Banach-Tarski paradox was proven more than hundred years ago, results inspired by it are still an active area of research of many mathematicians. They are a particularly rich source for counterexamples in the areas of mathematics involving measure theory, but at the same time they spark the interest of geometers who are fascinated by constructions created while considering proofs. What is more, there are still many open problems involving the existence of finitely additive, invariant measures on certain spaces or uniqueness of measures that are being worked on using methods inspired by the ones used to provide proofs for Banach-Tarski results [17].

Generalizations presented in this thesis are by no means an exhaustive list. Worth investigating are Banach-Tarski-like results in the hyperbolic space [17], as well as in non-Archimedean vectors spaces [11]. What is more, the set up of the Banach-Tarski paradox can be modified. In this thesis only a free subgroup of SO(3) with two generators was considered to prove the result for the unit ball. However, considering free groups with more generators, or some other infinite group that is not free, could lead to the decomposition with some desirable properties. Even though many sets can be concluded to be paradoxical by considering a free action of some paradoxical group on it, from the geometric point of view finding an explicit decomposition of set allows for deeper understanding of the the structure of sets and relations between them. Modern mathematical methods allow for the fact that in recent years many open problems posed by mathematicians, who were first investigating paradoxical sets were solved. It can be expected that the development in this field will progress and there are many fascinating results inspired by the Banach-Tarski paradox to be discovered.

8 References

[1] Sebastian Agata. "Free groups of rotations acting without fixed points on the rational unit sphere". In: *Mathematical Sciences* (2008). ISSN: 0716-8446.

[2] Stefan Banach. "Sur le probleme de la mesure". In: *Fundamenta Mathematicae* (1923).

[3] Stefan Banach and Alfred Tarski. "Sur la decomposition des ensembles de points en parties respectivement congruentes". In: *Fundamenta Mathematicae* (1924).

[4] Joel Lee Brenner. "Quelques groupes libres de matrices". In: *C.R. Acad. Sci. Paris* (1955).

[5] Anthony Fletcher. *Growth Functions and Automatic Groups*. 1997.

[6] James Foley et al. *Computer Graphics: Principles and Practice*. Addison-Wesley, 1990.

[7] Kurt Godel. *The Consistency of The Axiom Of Choice and Of The Generalized Continuum Hypothesis With The Axioms Of Set Theory*. Princeton, New Jersey University Press, 1940.

[8] Leong Yee Hang. *Geometric Dissection*. 2020.

[9] Andrew J. Hanson. *Visualizing Quaternions*. Morgan Kaufmann Publishers, 2006.

[10] John Von Neumann. "Zur allgemeinen Theorie des Masses". In: *Fundamenta Mathematicae* (1929).

[11] Kamil Orzechowski. "The Banach-Tarski paradox for some subsets of finite-dimensional normed spaces over non-Archimedean valued fields". In: *arXiv* (2024).

[12] Raphael M. Robinson. "On the decomposition of spheres". In: *ICM* (1947).

[13] Ivan N. Sanov. "A property of a representation of a free group". In: *Doklady Akademii Nauk SSSR* (1947).

[14] Kenzi Sato. "A free group acting without fixed points on the rational unit sphere". In: *Fundamenta Mathematicae* (1995).

[15] Stanisław Świerczkowski. "On a free group of rotations of thh Euclidean Space". In: *The Mathematical Institute of the Polish Academy of Sciences* (1958).

[16] Alfred Tarski. "Algebraische Fassung des Maßproblems". In: *Fundamenta Mathematicae* (1938).

[17] Stan Wagon and Grzegorz Tomkowicz. *Banach-Tarski Paradox*. Cambridge University Press, 2016. ISBN: ISBN 978-1-107-04259-9.

[18] Stanley Wagon. "The use of Shears to Construct Paradoxes in R^2". In: *American Mathematical Society* (1982).

[19] Ernst Zermelo. "Beweis, dass jede Menge wohlgeordent werden kann". In: *Mathematische Annalen* (1904).

A Code

This code can be adjusted to create animations and figures for the Banach-Tarski decomposition of a square.

```
(*generators of the free group F*)
a = {
  {1, 2},
  {0, 1}
 }; b = {
  {1, 0},
  {2, 1}
 };
A = Inverse[a];
B = Inverse[b];
vector1 = {1/Pi, 1/E};

(*all reduced words in the group F*)
AllWords[s_] := Flatten[Table[Tuples[{a, b, A, B}, k], {k, 0, s}], 1];
BadWords[s_] := Module[{AW = AllWords[s]},
  Join[
   Cases[AW, {y___, a, A, x___}],
   Cases[AW, {y___, A, a, x___}],
   Cases[AW, {y___, b, B, x___}],
   Cases[AW, {y___, B, b, x___}]
   ]
  ]
GoodWords[s_] := Complement[AllWords[s], BadWords[s]]

(*all reduced words in the group F that end with a certain generator x\
 of max length s*)
Subscript[StartWith, X_][s_] := Cases[GoodWords[s], {X, q___}]

(*equivalence relation*)
Hat[{x_, y_}] := {x - Floor[x], y - Floor[y]}
(*words in F under equivalence relation*)
Subscript[Orbit, X_][s_] :=
  Hat /@ DeleteDuplicates[
    Apply[Dot, Append[#, vector1]] & /@ Subscript[StartWith, X][s]];
```

This code can be adjusted to create animations and figures for the Banach-Tarski decomposition of a sphere.

```
(*genrators of a free group*)
a = RotationMatrix[ArcCos[1/3], {0, 0, 1}];
b = RotationMatrix[ArcCos[1/3], {1, 0, 0}];
A = Inverse[RotationMatrix[ArcCos[1/3], {0, 0, 1}]];
B = Inverse[RotationMatrix[ArcCos[1/3], {1, 0, 0}]];
vector1 = {1/Sqrt[2], 1/Sqrt[2], 0};

(*all words in the group*)
AllWords[s_] := Flatten[Table[Tuples[{a, b, A, B}, k], {k, 0, s}], 1];
BadWords[s_] := Module[{AW = AllWords[s]},
```

```
  Join[
   Cases[AW, {y___, a, A, x___}],
   Cases[AW, {y___, A, a, x___}],
   Cases[AW, {y___, b, B, x___}],
   Cases[AW, {y___, B, b, x___}]
   ]
  ]

GoodWords[s_] := Complement[AllWords[s], BadWords[s]]
Subscript[StartWith, X_][s_] := Cases[GoodWords[s], {X, q___}]
Subscript[rotation, X_][s_] :=
  DeleteDuplicates[ Apply[Dot, #] & /@ Subscript[StartWith, X][s]];

(*all words that end with certain generator of maximum length s*)
Subscript[finalpoints, X_][s_, v_] :=
  Dot[#, v] & /@ Subscript[rotation, X][s];
```