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Model-Theoretic Consistency of Transfinite Induction

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Abstract

In this thesis we explore model-theoretic methods of determining the validity of the proof technique and logical inference tool known as transfinite induction. We explore basic model theory and specific theories therein, such as Peano Arithmetic and its arithmetic Hierarchy of formulas as well as Zermelo-Frankel-Choice set theory. In those theories we show the validity of transfinite induction in fragments of PA, as well as the validity of ϵ-induction and transfinite induction in ZFC. We also provide the context and necessary conditions for the use of transfinite induction in general mathematics.

Contents

1. Introduction

1.1 Motivation

Induction is one of the most common proof techniques across the entirety of mathematics; allowing mathematicians to conclude many properties of objects indexed by the natural numbers. Yet its limits are obvious: the natural numbers. There are many objects across mathematics which don't necessarily have a countable structure; topology and set theory constantly deal with such spaces. We wish to show in this thesis that with careful construction one can extend our intuition of induction beyond the limits of the natural numbers; leading to the technique of Transfinite Induction.

Given a new proof technique that operates as an extension of an intuitive one, it might be natural to ask: does this produce consistent results and is it a genuinely valid method of inference? One could simply add it as a logical axiom and determine it to be true, but the validity of any axiom precisely lies in the truth of its consequences. The schema of induction in theories of arithmetic does not determine the truth of natural induction, rather, we know induction to be true in the natural numbers and recognise that such an axiom schema is necessary to complete our theory of numbers. Moreover, if it is not necessary, then best not add it. In that spirit, the following thesis seeks to show that transfinite induction is provable in both basic arithmetic and set theory, meaning so long as you accept common systems of inference and construction, then you must accept this one as well. We explain the scope and limitations of this technique for Bachelor's students not familiar with it.

1.2 Outline

In order to demonstrate the validity of a proof technique beyond just intuition, it serves to construct a rigorous meta-mathematical system with which you can have a proper view of your inference processes. We assume the reader has comfortable knowledge of basic propositional logic, First-Order Logic (FOL) and simple Set

Theory, so that we can introduce the methods and structures of basic Model Theory. This includes the concepts of Languages, Theories and Models seen in Section 2.2; as well as some differences and relations between these structures. We also introduce the concept of ordinals in Section 2.1, their intuitional meaning and the rigorous set construction of these objects. Once the underlying machinery of Model theory is expanded on, we follow with the specific construction of Theories of Arithmetic (PA) in Section 2.3 and Set Theory (ZFC) in Section 2.4 and specific necessary properties within. Then we show the provability of the inference scheme of transfinite induction for both these theories in Section 3.1 and 3.2. Finally we present a proof using transfinite induction in the familiar setting of the real numbers in 3.3, to show an example of how it can be used.

2. Background

2.1 Ordinals

Ordinals and their properties are an integral part of the functioning of Transfinite Induction and the proofs that make use of it. Ordinals are a lesser known relative to Cardinals and are, in fact, logically prior to many rigorous definition of Cardinals. Cardinals classically denote the different infinite "sizes" of possible sets, whereas ordinals denote the form of ordering that a given set has. In finite sets there is a one-to-one correspondence between cardinality and ordering in sets, but from $\aleph_0$ (the cardinality of $\mathbb{N}$) onward, sets of the same cardinality can have differing orders.

It's important to note that in referring to orders, there is a strict relational definition underlying this:

Definition 2.1.1. For a given set X, it is said to have a Total Ordering if it has a binary reflexive and transitive relation $\prec$ with the following properties, with respect to elements $x, y, z \in X$:

1. $x \prec x$ (reflexivity)
2. if $x \prec y$ and $y \prec z$ then $x \prec z$ (Transitive)
3. if $x \prec y$ and $y \prec x$ then $x = y$ (anti-symmetric)
4. for all pairs, $x \prec y$ or $y \prec x$

This relation has all the essential properties of the lesser/greater than or equal to sign ($\leq$) and for simplicity we will use this sign from this point forward to represent an ordering relation. The use of this relation henceforth need not represent the canonical ordering of the natural numbers (0,1,2,3,4...). Ordinal numbers themselves are associated to sets with a somewhat stronger condition of ordering called a Well-Ordering:

Definition 2.1.2. A Well-Ordered set X is a Totally Ordered set such that $\forall S \subset X, \exists a \in S$ such that $\forall b \in S,\ a \leq b$.

Suppes (1960) [13] has a more complete treatment of different forms of ordered sets from an axiomatic perspective.

2.1.1 Transfinite Orderings

It is illustrative to first understand finite order types in order to understand the distinguishing feature of transfinite sets. For a well-ordered set with finite cardinality, it is simple to create a new ordering just by permuting the given order of elements. Then one can create a bijection from the original ordering to the other matching placement to placement (1st to 1st, 2nd to 2nd etc.). This is known as an order-preserving bijection. Given that any such reordering of the finite set still has such a function, we say that all finite sets of the same cardinality have the same order type.

Consider $\mathbb{N}$ the natural numbers and its canonical ordering:

$$0, 1, 2, 3, 4, 5, 6 \ldots$$

Let us then then give the label ω for the order-type that all well-ordered sets have an order-isomorphism (order-bijection) with $\mathbb{N}$. Next consider the following re-ordering of $\mathbb{N}$:

$$1, 2, 3, 4, 5, 6, \ldots, 0$$

That is the normal ordering of $\mathbb{N}$ except that zero is now "greater than" all other natural numbers, call this $\tilde{\mathbb{N}}$. We will now show that this is in fact a distinct order-type to ω.

Proof. Suppose there existed an order-isomorphism g from $\mathbb{N}$ to $\tilde{\mathbb{N}}$. This would imply there exists $\alpha \in \mathbb{N}$ such that $\alpha = g^{-1}(0)$. For g to be order preserving this means that $\forall n \in \mathbb{N}\ \alpha \geq n$. But it must hold that $\alpha \leq \alpha+1$, implying that $\alpha = \alpha+1$. Hence we arrive at a contradiction. □

Similar proofs can show that the following are separate order types as well:

$$\begin{aligned} &2, 3, 4, 5, 6, 7, 8, \ldots 0, 1 \\ &1, 3, 5, 7, \ldots 0, 2, 4, 6, 8, \ldots \end{aligned} \tag{2.1}$$

The previous ordering types are denoted by $\omega + 1, \omega + 2, \omega \cdot 2$ respectively. This notation of ordinals seems to imply an ordering and arithmetic to these "numbers", which will be explained in the following section.

2.1.2 Von Neumann's Set Construction and Arithmetic

The previous section provided a somewhat intuitive explanation for the origin of ordinals, but the intuitive construction falls short once you reach an ordering on the scale of ω^2. To solve this problem, we can follow the Von Neumann construction of ordinals, one that the reader may have seen prior as a set-theoretic construction of the natural numbers (as opposed to an elementary arithmetic construction).

Definition 2.1.3. A set is called *transitive* if each of its elements are also a subset

This implies that for any set within a transitive one, its elements are also in the larger transitive set. One can see that the empty set is vacuously transitive, you can then define a successor operation on these sets:

Definition 2.1.4. The successor operation $Succ()$ on a set X is defined as follows:

$$Succ(X) = X \ \cup \ \{X\}$$

Then from starting with the empty set we can begin a construction of finite ordinals with the following assignment:

$$\begin{aligned}
&\varnothing = 0 \\
&Succ(0) = \{\varnothing\} = 1 \\
&Succ(1) = \{\varnothing, \{\varnothing\}\} = 2 \\
&Succ(2) = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\} = 3 = \{0, 1, 2\} \\
&\vdots
\end{aligned} \tag{2.2}$$

This current construction can only give finite cardinality sets; to introduce transfinite ordinals, we must define the supremum operation for this transitive set context. If X is a set of ordinals, then $\sup X = \bigcup_{A \in X} A$. From the example above of the ordinal representing 3, we have $\sup 3 = \sup\{0, 1, 2\} = 2$, which gives the normal answer for the supremum of a finite set of numbers. The real use of this definition comes when we consider $\mathbb{N}$:

$$\sup \mathbb{N} = \sup\{0, 1, 2, 3, 4...\} = \bigcup_{n \in \mathbb{N}} n := \omega$$

Remark. A construction quite similar to this one is called Von-Neumann's Cumulative Hierarchy, that uses the power set operation instead of the successor. These sets are then indexed by ordinals and are used to create a transitive domain for all mathematics [10].

In this case, ω is known as a *limit ordinal* with a straightforward definition:

Definition 2.1.5. An ordinal is known as a limit ordinal if it is not a successor of any previous ordinal.

Through this definition, we can see the next biggest ordinal to always be the set that contains all previous ordinals within them. $\omega + 1 = \mathbb{N} \cup \{\omega\}$.
This construction allows us to reach past even ω^2 to $\omega^2 + \omega$ then to $\omega^2 \cdot 2$ and further to ω^3; it is convention to write ordinal multiplication on the right hand side. At this point, it may be obvious that the form of these transfinite ordinals seems to be polynomials in ω and this is not coincidental. This form provides both a natural ordering of the ordinals themselves but also a method to perform arithmetic on them.

Cantor Normal Form

Viewing ordinals in this natural identification with $Pol_n(\omega)$ we can then borrow the natural lexicographic ordering that is found among them. Recall that a function *dominates* another if there exists a point after which the function is greater for all numbers after that point. One can show that polynomials dominate one another in a pattern that follows the following inductive definition of the ordering [9]:

Let Pol_0 be the non-negative natural numbers with regular ordering. Then let every $\alpha \in Pol_{n+1}$ be of the form:

$$\omega^{\alpha_0} \cdot a_0 + \cdots + \omega^{\alpha_{k-1}} \cdot a_{k-1} + \beta$$

Where $a_i \neq 0 \in Pol_0$ and $\alpha_i \in Pol_n \backslash Pol_{n-1}$. Given that $\alpha_0 \geq_n \alpha_1 \geq_n \cdots \geq_n \alpha_{k-1}$ and that $\beta \in Pol_n$. The ordering $\geq_n$ is then defined in such a way that $\alpha \geq_{n+1} \mu$ if for a given:

$$\mu = \omega^{\mu_0} \cdot b_0 + \cdots + \omega^{\mu_{s-1}} \cdot b_{s-1} + \delta$$

then it must be that either there exists an $i \leq k$ such that:

$$[\forall\ t < k, \alpha_k = \mu_k\ \wedge\ a_k = b_k]\ \wedge\ [\alpha_i \geq_n \mu_i\ \vee a_i > b_i] \tag{2.3}$$

or

$$[k = s \text{ and }\ \forall\ t \leq k,\ \alpha_k = \mu_k, a_k = b_k]\ \wedge\ [\beta \geq_n \delta] \tag{2.4}$$

Put simply, the ordering is found by comparison of exponents followed by coefficients in that priority. If you expand β similarly and continue that process for all lower exponents, we would then say that α is in its Cantor normal form.

The reader may have thought that the transitive definition allows us to then take $\bigcup Pol_n$ and ask about its supremum; this would then provide us the next rung of ordinals classically denoted ϵ_0. This is the first ordinal which satisfies the equation $\alpha^\omega = \alpha$. Further ordinals cannot be as easily represented by polynomials, which has implications on their representability in certain systems (e.g. arithmetic).

Remark. Note that all of these ordinals, even ϵ_0, still describe a countable set or orderings on a countable set. Most such ordinals are in fact not describable in this way, such as the Church-Kleene ordinal [6]. The set of all countable ordinals is called ω_1 and is itself the definition of $\aleph_1$.

Arithmetic on these ordinals follows its own set of rules. Where $\alpha + 1$ simply means the successor of alpha. While if γ is a limit ordinal, then $\alpha + \gamma$ is the smallest ordinal ν such that $\nu > \alpha + \beta,\ \forall \beta < \gamma$. This causes some strange noncommutative properties to the ordinal-addition, namely $\omega + \omega^2 = \omega^2 \neq \omega^2 + \omega$. In general, right addition by a larger limit ordinal comes out to be just that ordinal, while left addition gives the expected outcome. Multiplication has a similar noncommutative but associative definition: $\alpha \cdot 0 = 0$ while $\alpha \cdot (\beta + 1) = \alpha \cdot \beta + \beta$. If γ is a limit ordinal then $\alpha \cdot \gamma$ is the smallest ordinal ν such that $\nu > \alpha \cdot \delta,\ \forall \delta < \gamma$. This is why the finite coefficients of ordinals are always written on the right, as under this definition $\forall n \in \mathbb{N},\ n \cdot \omega = \omega$.

2.2 Metamathematical Structures

In the process of formalising mathematics during the 20th century it became necessary to make explicit certain logical and syntactical structures that mathematicians took to be obvious in previous centuries. Whitehead and Russell's *Principia Mathematica* [15] was the first serious, and failed, attempt to derive all mathematical truths from a concrete axiomatic foundation. David Hilbert's attempts to construct a consistent system for the truths of analysis through a finite construction [5] also petered out when Gödel published his famous incompleteness theorems. Yet their work fuelled the propagation of model theory and proof theory, as well as helping to create the framework through which to best understand the relative truth of undecidable statements, like the continuum hypothesis. It also provides a metamathematical framework to view and investigate non-standard proof techniques such as Transfinite Induction. Much of the following sections on this were adapted from both Takeuti ([14]) and Chang & Keisler ([2]).

2.2.1 Languages

To make the initial steps in formalising mathematics one must first formalise the language of First-Order Logic (FOL) with which most statements are made.

Definition 2.2.1. A First-Order language **L** consists of sequences of the following symbols:

i. Constants:

   (a) Individual constants $c_0, c_1, c_2, \ldots$

   (b) Function constants (n-ary) $f_0^n, f_1^n, f_2^n, \ldots$

   (c) Predicate/Relation constants (k-ary), $P_k^0, P_k^1, \ldots$

ii. Variables:

   (a) Free variables: $a_0, a_1, a_2, \ldots$

   (b) Bound variables: $x_0, x_1, x_2, \ldots$

iii. Logical symbols:

   (a) Connectives: $\wedge$ and $\neg$

   (b) Quantifiers: $\forall$ and $\exists$

iv. Grammatical symbols: such as commas or parentheses

Constants themselves are placeholders for whatever context the language may be operating in; they could be numbers or symbols representing some other structure entirely. Functions are as expected and map tuples of constants to constants, while predicates are functions with technical semantic meanings that are defined separately e.g. a|b is a 2-ary predicate indicating relative divisibility. Languages in this sense do not indicate "true" things but are prerequisite for expressing truth.

The reader might also wonder why only the inclusion of "and" and "not" as logical symbols; this is because they generate all other connectives through their combination:

$$\begin{aligned} &\phi \vee \psi : \neg(\neg\phi \wedge \neg\psi) \\ &\phi \to \psi : \neg\phi \vee \psi \\ &\phi \leftrightarrow \psi : (\phi \to \psi) \wedge (\psi \to \phi) \end{aligned} \tag{2.5}$$

A simple sequence of these symbols can give nonsensical sentences that have no meaning in any traditional sense; therefore that we will restrict our considerations to *formulas* and *terms*:

Definition 2.2.2. terms are defined as follows:

1. A variable is a term
2. A constant is a term
3. If f_k^n is a n-ary function symbol, then $f_k^n(t_1, \dots t_n)$ is a term so long as t_i is a term for all $i \leq n$
4. Any sequence of symbols are a term so long as it is shown to be some finite application of these rules.

Definition 2.2.3. A predicate constant $P^k(t_1, ..., t_k)$, where t_i are terms, is called an atomic formula. Then formulas are inductively defined as follows:

i. If φ is an atomic formula then it is a formula

ii. if φ and ψ are formulas, then so is $\neg\varphi$ and $\varphi \wedge \psi$

iii. If φ is a formula which contains the free variable a then $\forall x\varphi$ and $\exists x\varphi$ are formulas, and x replaces the instances of a found in φ.

iv. Formulas are only produced through finite applications of rules i. - iii. .

Definition 2.2.4. A *sentence* is a formula with no free variables

Remark. Sentential Models and Logic are their own domain of study within Model Theory. These are Languages defined prior to quantifiers, consisting solely of predicate logic. Though they have many of the same qualities of FOL models, they have essential differences in effective completeness [2].
For us to develop a deductive process from these languages, it is necessary to define a syntactical notion of tautology of sentences. For a finite sequence of sentences $(\psi_0, \psi_1, \dots, , \psi_n)$ a mapping to some $\tilde{a} \in \{0,1\}^{n+1}$ is called an *assignment*.

Definition 2.2.5. Given a sentence φ and assignment $\tilde{a} \in \{0,1\}^{n+1}$, its value is defined:

i) If φ is the sentence ψ_k, then its value is $\tilde{a}_k$

ii) If φ is $\neg\psi_k$, then its value is $1 - \tilde{a}_k$

iii) if φ is the sentence $(\psi_k \wedge \psi_m)$ then its value is $\min(\tilde{a}_k, \tilde{a}_m)$

Note that as quantifiers necessitate the existence of free variables and therefore are not necessary to define in relation with sentences.

Definition 2.2.6. The sentence ψ is a *tautology* if it has the value 1, for all assignments $\tilde{a} \in \{0,1\}^{n+1}$ on its constituent sentences $(\psi_0, \psi_1, \ldots, , \psi_n)$

To make use of the meta-framework that we are residing in, a formal and explicit system of Logical axioms and rules of inferences is essential. Therefore, the following is a series of logical axioms that one uses when deducing in a language **L**:

1. Every tautology of **L** is a Logical Axiom
2. Of Quantifiers:
    i. For ν, ρ formulas, and v not a free variable in ν;
    $$(\forall v(\nu \rightarrow \rho)) \rightarrow (\nu \rightarrow \forall v \rho)$$
    ii. For formula ν, if one can obtain the formula ρ by substitutive all occurrences of the free variable v in ν;
    $$(\forall v)\nu \rightarrow \rho$$
3. Of Identity:
    (a) $x \equiv x$
    (b) $x \equiv y \ \rightarrow g(a_1, \ldots, x, \ldots a_n) \equiv g(a_1, \ldots, y, \ldots a_n)$ (where g is some predicate, function or atomic formula)

In combination with the inference rules of *modus ponens* and Generalisation ($\phi \rightarrow \forall x \phi$), they create a formal system of proof on sentences in a language, a proof being some sequential application of axioms and inferences to produce another formula or sentence. This provides a key piece of notation core to the proof theory. A set of sentences Σ proves a formula σ:

$$\Sigma \vdash \sigma$$

if there exists some proof from the sentences of Σ to σ. This a syntactical notion of the "correctness" of a formula, a relation dependent on a formal system of deduction. If $\Sigma = \emptyset$, then any deducible sentence is called a Theorem of **L**. If a set of sentences allows you to deduce all formulas in a language we say that it is inconsistent; referring to the principle of *ex contradictione sequitur quodlibet*. One can then call Σ consistent if it is not inconsistent.

2.2.2 Models and Theories

Models of a language consist of a universe that interprets the truthfulness of possible statements made in a language, providing a context in which these are true. Formally a *Model* is a pair $\mathcal{U} = < A, \mathfrak{g} >$, where A is called a universe and g the interpretation. The elements of A correspond to the constants, whereas each n-ary predicate constant relates to a $P \subset A^n$ and function constants relate to a function $f : A^m \to A$. $\mathfrak{g}$ maps the symbols of L to appropriate subsets, functions and elements of A.

Truthfulness can be defined within Models as a relation between a statement in **L** and some model $\mathcal{U}$, this is known as satisfaction ($\mathcal{U}$ satisfies σ). The exact definition and method of interaction between the interpretation $\mathfrak{g}$ is quite involved (see [2]), but a brief explanation suits our purposes. A satsifaction relation uses the interpretation of a model and language to see if a given formula is realised in the model with a given assignment of constants within said model. Some satisfaction relation can be seen as a set of tuples with a formula and the constants that satisfy it. It has additional conditions called *Tarski's truth conditions* [1], though this simply formalises satisfaction as closed under logic.

The main notion to understand is that while there is significant overlap between the concepts of deducibility and satisfaction, they are fundamentally different and their difference is incredibly important. Certain non-first-order properties of Models form the leverage which causes such separation, and form the reason Model Theory exists as a separate field of study. This separation is the difference between "a true statement" and "a provable statement", the lack of implication from the former to the latter (in some cases) being the essence of Gödel's famous incompleteness theorems. If the reader wishes to see proofs of this, Takeuti [14], Hájek & Pudlák [4] and Kotlarski [9] all have different versions of this.

Theories are the formal structure that contrasts to models, being closed under the ($\vdash$) syntactical deduction. Theories are a set of sentences of **L**. A theory is complete if it is maximally consistent and it is equivalent to any other theory if they have the same set of consequences (sentences of **L** that are deducible). Axioms therefore are just some set of sentences which generate a Theory; the theory being finitely axiomizable if there exists a finite set of axioms that generate it. Given the $\wedge$ operation, if a theory does have a finite axiomatization it also necessarily has a single axiom that generates it.

Some theory $\mathcal{Y}$ is the theory of a model $\mathcal{U}$ if it has the same consequences as the sentences that are satisfied in $\mathcal{U}$. As an example if one takes the axioms of group actions ($\mathcal{Y}$ = identity, inverse, associativity), then the model $\mathcal{U} =< G, +, 0 >$ is a model of group theory. An interesting distinction to make as compared to a regular view of such a triple is that this model contains all the true formulas one can make about said group, not just an implicit structure that one can work on.

2.3 The Theory of Peano Arithmetic

Arithmetic is historically the most basic form of mathematics that all further areas of mathematics assume some degree of commandment of; a necessary prerequisite that any unifying foundation of mathematics must contain. It is this primacy of arithmetic that makes statements about its metamathematical properties far-reaching; any system must contain it to be considered math and therefore be an extension of it. This section introduces a particular pair of model and theory that serves as the foundation for formalised arithmetic in FOL, Peano Arithmetic. Its properties and construction was largely adapted from Hájek & Pudlák (1993) [4]

In the previous section, it was necessary to talk about a non-specified language with general properties, but henceforth we must specify the terms within the language of arithmetic and how it relates to $\mathbb{N}$.

Definition 2.3.1. Let $\mathcal{L}_0$ be the first-order language containing the following symbols:

i. the constant 0

ii. the functions: $+$, $*$, $'$ (where $'$ is the successor)

iii. the predicate sybmvols: $=$, $\leq$

iv. free variables: x, y, a, m etc.

Successor function here works similarly to the von Neumann transitive set operation, where 0 stands in for the empty set. By our construction, $\mathbb{N}$ with addition and multiplication and its ω ordering is the so-called standard model of this language.

Remark. $\omega + 1$ orderings of $\mathbb{N}$ are non-standard and non-isomorphic models of this language [4]; the reasoning for this being essentially the same as shown in Section 2.1.1, yet replaced with use of the successor function on $\mathbb{N}$.

2.3.1 Arithmetical Hierarchy of formulas

One can fragment or split types of formulas within $\mathcal{L}_0$ (and any FOL language), via the number of changes of quantifiers from existential to universal and vice versa. This practice is necessary to explore different aspects of arithmetic and formalise the scope of different functions and how much of number theory can be found in limited fragments, e.g., what type of formula in this language would exponentiation satisfy? This splitting of the language is also essential to the main proofs regarding transfinite induction in further sections.

Definition 2.3.2. A formula is called a bounded formula if each bound variables in the formula is of the form: $(\forall x < y)\psi$. Such formulas are said to be in the set Δ_0. You can then inductively define an *arithmetical hierarchy* of formulas with the following rules: $\Sigma_0 = \Delta_0 = \Pi_0$; Π_{n+1} is then formulas $(\forall x_1 \ldots \forall x_k \ \varphi)$, where $\varphi \in \Sigma_n$. Σ_{n+1} similarly is all formulas $(\exists y_1 \ldots \exists y_m \ \sigma)$, where $\sigma \in \Pi_n$. Formulas in either set can be seen as Δ_0 formulas preceded by an alternating series of universal and existential quantifiers. As one can always create superfluous quantifiers, any Σ_n or Π_n formula is also in all higher indexed sets; therefore when referring to the class of a formula, it is the least number of quantifier changes that express it.

Remark. Sets of natural numbers defined by said formulas are also given such denominations and if they can be defined by both Σ_n and Π_n formulas, then we say that it is Δ_n

Example. The set of even numbers can be defined by the universal formula $\forall k(n \neq (2 * k)')$ or the existential formula $\exists m(n = 2 * m)$. This means that even numbers are at most a Δ_1 set.

Proposition 2.3.3. For any n:

(1) Π_n, Σ_n are closed under $\wedge$ and $\vee$ operations

(2) Δ_n sets are closed under complementation

Proof. (1) Let us only consider Π_n as the proof for Σ_n follows the same. Take the formulas $\forall y \ \varphi(y, w)$ and $\forall k \ \psi(m, k)$ where $\varphi, \psi \in \Sigma_{n-1}$. The the formula $\forall y(\varphi(y, w)) \vee \forall k(\psi(m, k))$ has a prenex normal form of $\forall y \ \forall k(\varphi(y, w) \vee \psi(m, k)) \in \Pi_n$. The $\wedge$ conjunction operates similarly, therefore (1) holds.

(2) Consider a set X defined by the formula $\phi \in \Sigma_n$. Then X^c can be described by $\neg\phi$. As $\phi = \exists \mathbf{x} \forall \mathbf{y} \ldots \varphi$ for some bounded formula (note that the first quantifier depends on the parity of n), then $\neg\phi = \forall \mathbf{x} \exists \mathbf{y} \ldots \neg\varphi$ which is an Π_n formula. As X is a Δ_n set then there is also an equivalent Π_n formula that describes it, and it's negation must also be Σ_n. Therefore X^c is Δ_n set. ☐

2.3.2 Axioms of Peano Arithmetic

Given now $\mathcal{L}_0$ and appropriate fragments of the language, we can now introduce the central theory PA for which $\mathbb{N}$ is a model. There are many ways to present the axioms of PA and different authors make different choices, some being slightly weaker or stronger or completely equivalent. For example Takeuti (1975)[14] includes the first 6, while Hájek & Pudlák [4] uses all 7. The basic axioms are:

1. $\forall x(x' \neq 0)$
2. $\forall x(x + 0 = x)$
3. $\forall x(x * 0 = 0)$
4. $\forall x \forall y(x' = y' \to x = y)$
5. $\forall x \forall y(x + y' = (x + y)')$
6. $\forall x \forall y(x * y' = x * y + x)$

If one includes the following axiom they also have what's known as Robinson arithmetic:

7. $\forall x \exists y(x = 0 \ \vee x = y')$

Lastly to complete PA one has the following induction schema:

$$(\varphi(0) \wedge \forall x(\varphi(x) \to \varphi(x'))) \to \forall x \varphi(x)$$

This is called a *schema* as it must be applied separately to all formulas φ and each one is an axiom of PA. This means PA is not finitely axiomatized by this theory and it is in fact not finitely axiomizable in general [8], but if one considers second order logic there exists finite axioms for the same model $\mathbb{N}$ called the Peano Axioms. This is because second order logic allows one to vary over all formulas as well as constants. This schema may be applied individually to different parts of the hierarchy to create fragments of PA, in general such a theory is written as $I\Sigma_n$ or $I\Delta_0$, called induction on bounded formulas.

Theorem 2.3.4. $I\Sigma_n$ proves the *least number schema* for Σ_n formulas:

$$I\Sigma_n \ \vdash \ \exists x \psi(x) \to \exists a(\psi(a) \wedge (\forall y < a)\neg\psi(y))$$

The following proof is adapted from the one given in Hájek & Pudlák [4].

Proof. Given that $\exists x\psi(x)$ where $\psi(x) \in \Sigma_n$ and further assume that $\forall x(\psi(x) \rightarrow (\exists y < x)\psi(y))$ to aim for contradiction. Then define the formula:

$$\delta(x) = \forall y < x\neg\psi(y)$$

Let us then apply induction on x in δ. The $\delta(0)$ case is clearly true. The assume $\delta(m)$ is true and consider $\delta(m')$, if $\psi(m)$ were true, then by our initial assumption $\exists y < m\psi(y)$, which by $\delta(m)$ cannot be true, therefore $\neg\psi(m)$ and we can infer $\delta(m')$. Therefore by the induction schema we have $\forall x\delta(x)$ which implies $\forall x\neg\psi(x)$ and we reach a contradiction. □

Furthermore we also require what is known as the *collection principle* schema:

Theorem 2.3.5. $I\Sigma_n$ proves the collection principle for Σ_n formulas, given $\varphi \in \Pi_{n-1}$:

$$I\Sigma_n \vdash \forall u([\forall x \leq u\ \exists y\varphi(x,y)] \rightarrow \exists v[\forall x \leq u, \forall y \leq v\varphi(x,y)])$$

Proof. Briefly sketched as in Hájek & Pudlák, we use bounded induction on v in $\exists t\forall x \leq v, \exists y \leq t\varphi(x,y)$ to get $\forall v \leq u, \exists t, \forall x \leq v, \exists y \leq t\varphi(x,y)$ then infer the consequent. □

Coding formulas and a Truth Predicate

For future purposes it helps to introduce a brief definition of Tarski's Truth predicate (more specifically the partial truth). The detail with which one can go into when defining it is immense and could be the source of it's own project, if the reader wishes a thorough explanation Hájek & Pudlák [4] has a form of this.

To create such a predicate with the intuitive meaning of "ϕ is true", it is useful to create a unique coding (bijection) of formulas to numbers as to completely arithmetise the process.

Definition 2.3.6. A Gödel numbering ([12]) is a bijection between each symbol in $\mathcal{L}_{\prime}$ and a number. For a formula σ given by the sequence of symbols $(a_1, \ldots, a_m)$, it's coding is given by:

$$\ulcorner\sigma\urcorner = 2^{a_1} \cdot 3^{a_2}\ \cdots \cdot p_m^{a_m}$$

There's a strict non-singularity to the different numberings one could possibly make, but for a fixed coding this provides a unique and reversible number to each formula. The creation of a complete coding for all basic formulas and symbols can be proven in $I\Sigma_1$ (see [4] for proof, but note that in Hájek & Pudlák a dot is used to represent this coding).

From Section 2.2.2, one can see then that a satisfaction relation is contained within a model, in our case $\mathbb{N}$, but the coding of formulas allows us to relate logical formulas within the theory of PA to constants within the theory of PA. And as predicates act on terms, this means one can define a predicate relating a formula (given by it's code) and the terms for which it's true (constants, functions etc.). Moreover note that the bijection of $\mathbb{N}$ with $\mathbb{N}^k$ means that one can define a unique number assigning any tuple of naturals to a singular natural.

Remark. It is vital to state that this can only be constructed in "stronger systems"; one of Tarski's main theorems on Undefinability [3] makes it such that PA cannot define its own truth predicate. Nor can any system containing Robinson's arithmetic (see p. 16)

I claim without proof that it is possible to construct a satisfaction predicate Tr_0 on formulas in Δ_0 that is itself in Δ_1 (see Hájek & Pudlák Chapter I, 1. d) [4]). This predicate operates on the coding of a formula and a list of assigned constants for which it is true (the empty set if said formula is a sentence). We can then meta inductively create such predicates for all higher fragments. Then given $Tr_n = Tr_{\Pi_n}$ and $\varphi \in \Pi_n$:

$$Tr_{n+1}(\ulcorner \exists y \varphi(x,y) \urcorner, s) := \exists y Tr_n(\ulcorner \varphi(x,y) \urcorner, s[y])$$

Where s is some assignment of constants and $s[y]$ is an assignment containing y. In the hierarchy for $n \geq 1, Tr_n \in \Sigma_n$ or Π_n

Ordinals in PA

It is also possible to define transfinite ordinals in PA, the method of doing so borrows exactly from our use of polynomials in Section 2.1.2. It is relatively simple to define exponents as the composition of the (*) function, then use an inductive application of the composition of polynomials in exponents to define the higher class of polynomials Pol_n, done identically as shown in Section 2.1.2. These functions then have the same properties as discussed in Section 2.1.2, giving us a rigorous formulation of ordinals up to ϵ_0 in PA, more specifically as Σ_1 formulas. The identification of the polynomials can be done via formula as done in Kotlarski [9], $\exists n \ Tr_1(\ulcorner Pol_n(\alpha) \urcorner)$ up to ϵ_0.

Furthermore we also adopt the ordering relation $\bigcup_{n\in\mathbb{N}} \leq_n$ defined previously as a minor addition to the language of PA.

2.4 The Theory of Zermelo-Frankel Set Theory

Set theory was developed in the early 20th century as a system for unifying all disparate forms of mathematics under a single far reaching language. Many mathematicians contributed to its current construction such as Ernest Zermelo and Abraham Frankel(the namesake's of ZFC) and other's like Skolem. The construction was meant to fulfill all the necessary properties one now uses in sets along while avoiding paradoxical objects like *Bertrand's Paradox*.

It is necessary as well with PA to define the specific FOL language of Set Theory. This language contains only a single fundamental relation and along with all the usual logical symbols of FOL. Let us define $\mathcal{L}_\in$ to be the FOL language such that:

Definition 2.4.1. Let $\mathcal{L}_\in$ to be the FOL language such that:

i. Every constant a is a set

ii. The only fundamental predicate is the $(\in)$ inclusion relation

iii. Variables: x, y, z

Remark. Given that the definition of models and languages relied on some form of sets and relations, it is useful to ask how circular defining set theory by using set theory is. Chang [2] deals with this; the solution is to use an intuitive set theory that is prior to the following constructed version. One could assume ZF in making these definitions but then one loses the power to speak on the effective consistency of said system as you're constricted by it.

2.4.1 Axioms of ZF(C)

The following axioms are all canonical though some of them have many names for the same formula, they are adapted from Kunen [10] ; we present them in order with intuitive explanations of their consequences following. ZF defines the following 8 axioms:

1. **Extensionality**: $\forall x \forall y[\forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y]$

2. **Regularity/ Foundation**: $\forall x[\exists y(y \in x) \rightarrow \exists w(w \in x \wedge \neg \exists z(z \in x \wedge z \in w))]$

3. **Scheme of Comprehension**: for formula ψ with variables in w, x and $\vec{a}$:

$$\forall w \forall \vec{a} \exists y \forall x[x \in y \leftrightarrow x \in w \wedge \psi(x, w, \vec{a})]$$

4. **Pairing**: $\forall w \forall z \exists x[w \in x \land z \in x]$

5. **Union**: $\forall X \exists A \forall Y \forall x[x \in Y \land Y \in X \to x \in A]$

6. **Scheme of Replacement**: For a formula ψ with variables $x, y, B, \vec{a}$:

$$\forall B \forall \vec{a}[\forall x(x \in B \to \exists! y \psi) \to \exists A \forall x(x \in B \to \exists z(z \in A \land \psi))]$$

7. **Infinity**: $\exists N[\emptyset \in N \land \forall x(x \in N \to Succ(x) \in N)]$

8. **Power set**: $\forall x \exists y[(z \subset x \to z \in y]$

Extensionality extends set equality to be an if and only if statement, as the converse of Extensionality: $(x = y \to \forall z(z \in x \leftrightarrow z \in y))$m is necessarily true. *Regularity* ensures that there is a disjoint element in any set, while the *scheme of comprehension* allows the creation of sets via logical formulas. *Pairing* ensures that for any pair there is a set that contains said pair of sets. The axiom of *union* states that any set has a corresponding set that contains the elements of its own constituents, constructive the union of the elements of the original set; therefore a combination of pairing and union can provide the set containing the union of any two sets. The *scheme of replacement* says that the image of a set under a definite function given by a formula will be a subset of a set. *Infinity* defines the existence of an infinite set, more specifically the least such set being ω or $\mathbb{N}$. While the *power set* axiom defines the power set of any given set.

Finally the last axiom that has been controversial to some but accepted by most, is the *Axiom of Choice*. We will present an equivalent statement under ZF, the proof of equivalence can be found in Jech [7]:

9. **Well-Ordering**: $\forall X \exists R(R \text{ well orders } X)$

Where R is a binary relations satisfying Definition 2.1.2. R can also be a seen as a set of ordered pairs of elements in X. As with PA, ZFC is clearly not finitely axiomizable due to the schemes of Comprehension and Replacement.

There are a number of extensions of ZFC that try to deal with things such as proper classes (the class of all sets) and conglomerates (containing all classes) in order to contain very abstract fields such as Category Theory fully within an axiomatised system. Examples being Von Neumann-Bernays-Gödel set theory and Tarski-Grothendieck set theory respectively.

3. Transfinite Induction

We can now introduce the main interest of this thesis, namely *Transfinite Induction* or abbreviated as TI. Transfinite Induction is an extension of natural inductive inferences to all ordering-types, even beyond countable sets. As a technique of proof its most common use case is in the abstract settings of Model Theory and Set Theory, but in section (name section) we will show its use in more familiar settings. In general the schema for TI is given as follows:

$$(\psi(0) \wedge \forall\beta(\forall\gamma < \beta\ \psi(\gamma) \rightarrow \psi(\beta))) \rightarrow \forall\alpha\ \psi(\alpha) \tag{3.1}$$

In some setting it is useful to restrict the size of ordinals with which you consider the applicability of TI. Therefore it is useful to define the restricted schema $TI(\delta)$:

$$(\psi(0) \wedge \forall\beta < \delta(\forall\gamma < \beta\ \psi(\gamma) \rightarrow \psi(\beta))) \rightarrow \forall\alpha < \delta\ \psi(\alpha) \tag{3.2}$$

It is immediate to see that the unrestricted case certainly implies the restricted one. In this format one can consider Natural Induction to be equivalent to the restricted schema TI_ω, though this is technically a form of strong induction. When proofs using TI is given it often follows a number of steps similar to natural induction:

Step 1: Prove $\psi(0)$
Step 2: Prove the successor case: $\psi(\alpha) \rightarrow \psi(\alpha + 1)$ (3.3)
Step 3: Prove the limit case: $(\forall\gamma < \beta\ \psi(\gamma)) \rightarrow \psi(\beta)$

While the successor step is often used, it is technically not necessary to implement the schema, though it may be useful to do so dependent on the structure you wish to induct on. For example the explicit construction of ordinal arithmetic operations is done via transfinite induction (see Section 2.1.2), and was intentionally left unmentioned. This is not a circular definition as the arithmetic of ordinals is not necessary to use transfinite induction, though it may make some applications easier once defined. Considering that ordinals extend to all definable cardinalities, this means that

TI extends induction not only to non-trivial countable sets but also uncountable sets and the continuum as well (Though which order-type it has is undecidable without further assumptions).

3.1 TI in Arithmetic

The following set of proofs about PA and transfinite induction were adapted from Kotlarski [9] but originally given by Mints [11]. In order to show that ordinal induction works for PA we will first show that PA proves the *Least ordinal scheme* $L(\rho, \Sigma_n)$:

$$(\exists\gamma < \rho A(\gamma)) \rightarrow \exists\alpha(A(\alpha) \wedge [\forall\beta < \alpha\neg A(\beta)] \quad A \in \Sigma_n$$

Structurally this is the exact same as the least number scheme for finite ordinals, but extended to all ordinals up to a certain size. Furthermore this scheme is equivalent to TI, but it is only necessary to show one direction; for which the following proof is mine.

Lemma 3.1.1. $L(\rho, \Sigma_n) \vdash TI(\rho, \Pi_n)$

Proof. Given for some $\psi \in \Pi_n$ we know that $\psi(0) \wedge \forall\beta < \rho(\forall\gamma < \beta\ \psi(\gamma) \rightarrow \psi(\beta))$. Assume $\exists\omega < \rho\neg\psi(\omega)$, then we can apply the scheme of least ordinal onto this so that $\exists\alpha(\neg\psi(\alpha) \wedge [\forall\beta < \alpha\neg(\neg\psi(\beta))]$. As such an ordinal must be less than ρ by first assumption we can then apply the limit inference assumed in the beginning. Therefore we know $\psi(\alpha)$ and arrive at a contradiction. □

The reverse implication follows the exact same lines as the finite ordinal case, though care must be taken in ensuring the scheme is applied to formulas according to its quantifier.

Lemma 3.1.2. $I\Sigma_m \vdash L(\omega^k, \Sigma_m)$

Proof. Given a formula $\delta(\cdot) \in \Sigma_m$ and assume $\exists\gamma < \omega^k\ \delta(\gamma)$. The aim is to construct the smallest such ordinal α as the scheme demands. From Section 2.3.2 we know our ordinal in the assumption must be of the form $\gamma = \omega^{k-1}\cdot m_{k-1}\cdots+\omega^0\cdot m_0$. Therefore we can construct from the original assumption the following formula $M_0(m_{k-1})$:

$$\exists\langle m_{k-2}, \ldots, m_0\rangle\ \delta(\omega^{k-1} \cdot m_{k-1} \cdots + \omega^0 \cdot m_0)$$

then by the assumption that $\exists a_{k-1} M_0(a_{k-1})$ we can use Theorem 2.3.4 and find the minimal such a_{k-1}. Then we implement a sequence of such formulas where the previously found minimal coefficient is implemented, therefore $M_1(m_{k-2})$:

$$\exists\langle m_{k-3},\dots,m_0\rangle\ \delta(\omega^{k-1}\cdot a_{k-1}\cdots+\omega^0\cdot m_0)$$

k iterations of the minimising schema leaves with:

$$\exists\alpha(\alpha=\omega^{k-1}\cdot a_{k-1}\cdots+\omega^0\cdot a_0\ \wedge\ \delta(\alpha)\ \wedge\forall\beta<\alpha\neg\delta(\alpha)).$$

□

This statement clearly does not show TI for all ordinals, as it is limited by the finite application of minimum schema. We use the next lemma to extend the statement to ordinal powers.

Lemma 3.1.3. for $\nu\geq\omega$ an ordinal $I\Delta_0+L(\nu,\Sigma_{m+1})\vdash L(\omega^\nu,\Sigma_m)$

Proof. As before let $\delta(\cdot)$ be a formula in Σ_m, we must translate this problem into one involving ordinals less than ν in a higher class. We construct the formula for the smallest possible exponents of the expansion of the ordinal such that $\exists\alpha<\omega^\nu\ \delta(\alpha)$. Let the first such formula $O_0(\alpha_0)$ be:

$$\begin{gathered}[\exists a_0\neq 0\ \exists t\ \exists\langle a_1,\dots,a_{t-1}\rangle\exists\langle\alpha_1,\dots,\alpha_{t-1}\rangle\exists\alpha\ (\nu>\alpha_0>\cdots>\alpha_{t-1})\ \wedge\\ (\alpha=\omega^{\alpha_0}\cdot a_0+\cdots+\omega^{\alpha_{t-1}}\cdot a_{t-1})\ \wedge\ \delta(\alpha)]\&\\ [\forall\sigma<\alpha_0\forall t\ \forall\langle a_0,\dots,a_{t-1}\rangle\forall\langle\alpha_1,\dots,\alpha_{t-1}\rangle\forall\alpha\ (\nu>\sigma>\cdots>\alpha_{t-1})\\ \alpha=\omega^{\sigma}\cdot a_0+\cdots+\omega^{\alpha_{t-1}}\cdot a_{t-1}\ \rightarrow\ \neg\delta(\alpha)]\end{gathered}$$

As a conjunction of Σ_m and Π_m formulas then $O_0(\alpha_0)\in\Delta_m\subset\Sigma_{m+1}$. We can then also create a formula on the minimal coefficient as done in the previous proof, $M_0(a_0)$:

$$\begin{gathered}\exists\alpha_0[O_0(\alpha_0)]\&\ [\exists t\ \exists\langle a_1,\dots,a_{t-1}\rangle\exists\langle\alpha_1,\dots,\alpha_{t-1}\rangle\exists\alpha\ (\nu>\alpha_0>\cdots>\alpha_{t-1})\ \wedge\\ (\alpha=\omega^{\alpha_0}\cdot a_0+\cdots+\omega^{\alpha_{t-1}}\cdot a_{t-1})\ \wedge\ \delta(\alpha)]\&\\ [\forall b<a_0\forall t\ \forall\langle a_0,\dots,a_{t-1}\rangle\forall\langle\alpha_1,\dots,\alpha_{t-1}\rangle\ \forall\alpha\ (\nu>\sigma>\cdots>\alpha_{t-1})\\ \alpha=\omega^{\alpha_0}\cdot b+\cdots+\omega^{\alpha_{t-1}}\cdot a_{t-1}\ \rightarrow\ \neg\delta(\alpha)]\end{gathered}$$

Both of these formulas are consequences of the least ordinal and least number schemas under assumption. Therefore we can state:

$$\exists\alpha < \omega^\nu\ \delta(\alpha) \rightarrow \exists\alpha_0, a_0[O_0(\alpha_0) \wedge M_0(a_0)]$$

We then recursively construct O_r, M_r. Given that $O_1, \ldots, O_{r-1}$ and $M_1, \ldots, M_{r-1}$ are already known, then O_r is given by:

$$\begin{gathered}\exists\langle a_0, \ldots, a_{r-1}\rangle\exists\langle\alpha_0, \ldots, \alpha_{r-1}\rangle[\forall i < r\ Tr_{m+1}(\ulcorner O_i(\alpha_i) \wedge M_i(a_i)\urcorner)]\& \\ [\exists a_r \exists t\ \exists\langle a_{r+1}, \ldots, a_{t-1}\rangle\exists\langle\alpha_{r+1}, \ldots, \alpha_{t-1}\rangle\exists\alpha(\alpha_r > \cdots > \alpha_{t-1}) \wedge \\ (\alpha = \omega^{\alpha_0} \cdot a_0 + \cdots + \omega^{\alpha_{t-1}} \cdot a_{t-1})\ \wedge\ \delta(\alpha)]\& \\ [\forall\sigma < \alpha_r \forall t\ \forall\langle a_r, \ldots, a_{t-1}\rangle\forall\langle\alpha_{r+1}, \ldots, \alpha_{t-1}\rangle\forall\alpha\ (\sigma > \cdots > \alpha_{t-1}) \\ \alpha = \omega^{\alpha_o} \cdot a_0 + \cdots + \omega^{\sigma} \cdot a_r + \cdots + \omega^{\alpha_{t-1}} \cdot a_{t-1}\ \rightarrow\ \neg\delta(\alpha)]\end{gathered}$$

M_r being defined completely analogously. This formula is still a Σ_{m+1} formula as $\forall i < r\ Tr_{m+1}(\ulcorner O_i(\alpha_i) \wedge M_i(a_i)\urcorner)$ does not affect the class by the collection principle. It follows then that the formula $T(\iota) = \exists i\ Tr(\ulcorner O_i(\iota)\urcorner)$ is also in said class as it is closed under existential quantifiers. As our initial assumption implies that some ι fulfills $T(\iota)$ and by the assumption of $L(\nu, \Sigma_{m+1})$ we have there is a least ordinal that satisfies this and that there is an i for which ι satisfies. Given that Cantor normal form arranges in order of largest to smallest ordinal, this means i is the least index and ι the least exponent. Then we choose the sequences $\langle\alpha_0, \ldots, \alpha_i\rangle$, $\langle a_0, \ldots, a_i\rangle$ such that $\forall s \leq i\ Tr_{m+1}(\ulcorner O_s(\alpha_s) \wedge M_s(a_s)\urcorner)$. Then it is obvious that $\alpha = \omega^{\alpha_0} \cdot a_0 + \cdots + \omega^{\alpha_i} \cdot a_i$ is the least such ordinal that has $\delta(\alpha)$. □

This lemma then gives us the tools to quickly show that the least ordinal schema, and therefore TI, holds in all of PA. First we briefly introduce an inductive power tower for ordinals. Where $\omega_0(\nu) = \nu$ and $\omega_{n+1}(\nu) = \omega^{\omega_n(\nu)}$. It follows that for any input the limit of this sequence is ϵ_0, safely restricting the size of the ordinals.

Theorem 3.1.4. $I\Sigma_{k+l} \vdash L(\omega_{k+1}(n), \Sigma_l)$

Proof. By Lemma 3.1.2 we know that $L(\omega^n, \Sigma_{k+l})$ is true. Then by k iterations of Lemma 3.1.3 we get our desired result. □

Then we can see that for any size tower and any desired class of formula, there is a fragment of PA which shows this.

Corollary 3.1.5. $\forall n \in \mathbb{N}$ and $\forall \varrho < \epsilon_0$:

$$PA \vdash TI(\varrho, \Sigma_n)$$

3.2 TI in Set Theory

Transfinite induction follows a lot more simply from ZF then it does from PA, this is due to the transitive set definition of ordinals in the cumulative hierarchy created by Von Neumann. In fact a much stronger form of induction, called well-founded induction or Noetherian induction holds in ZF. For our purposes we will show a slightly weaker version (whose proof is identical to Kunen [10]) called bounded $\epsilon-$induction:

$$\forall x \in A[(\forall y \in x \ \psi(y) \to \psi(x)] \to \forall z \in A \ \psi(z)$$

Well-founded relations are ones with some minimal element in relation and from the axiom of Regularity it follows that $\in$ is a well-founded relation.

Theorem 3.2.1. $ZF \vdash \epsilon-$induction

Proof. Given a transitive set A and a formula ψ, assuming that $\forall x \in A[(\forall y \in x \ \psi(y) \to \psi(x)]$. We then construct using the axiom of comprehension the set $W = \{w \in A : \neg\psi(w)\}$, first assume $W \neq \emptyset$ then using the axiom of regularity to find the $\in$-minimal element such that $w \in W \wedge \neg\exists z(z \in W \wedge z \in w)$. This then means that all $z \in w \ \psi(z)$ leading to a contradiction by assumption. □

By recognising that ordinals as defined in Section 2.1.2 are in fact transitive sets, we can easily extend ϵ-induction to the transfinite form.

Theorem 3.2.2. $ZF \vdash TI(\nu)$ for some ordinal ν

Proof. Here we again assume the antecedent of the schema of transfinite induction and take some arbitrary formula ψ; $\psi(0 = \emptyset) \wedge \forall\beta < \nu(\forall\gamma < \beta \ \psi(\gamma) \to \psi(\beta))$. Via Von-Neumann's construction of ordinals we recognise that for any ordinal $\alpha < \beta \leftrightarrow \alpha \in \beta$. Therefore:

$$[\forall\beta < \nu(\forall\gamma < \beta \ \psi(\gamma) \to \psi(\beta)] \leftrightarrow [\forall\beta \in \nu(\forall\gamma \in \beta \ \psi(\gamma) \to \psi(\beta))]$$

Then applying Theorem 3.2.2 we can conclude $\forall\beta < \nu \ \psi(\beta)$. □

This then shows transfinite induction as a scheme for any ordinal. To conclude that the scheme is true in an unbounded sense we would have to use $\epsilon-$induction in the unbounded sense. This requires a construction of proper classes which can be done in NBG set theory, as the concept of "all ordinals" cannot be contained within a set. In practice almost nothing requires the application of all ordinals at once, since the size of such an object would exceed any definable cardinal.

3.3 An Example of TI on the Reals

So far we have only shown Transfinite Induction in very constructed environments, which raises the question of the applicability for the "mathematician on the street". With this in mind, we wish to show an example proof of a statement on the real numbers that can only be proven using Transfinite Induction or a similar ordinal process. Necessarily when using transfinite induction on a structure it must have some form of well-ordering, so as to have an order-type with which to apply your ordinal analysis. It's quite easy to see that the regular canonical ordering of the real numbers is not in fact a well-ordering. The existence of a well-ordering on the reals only follows from use of the axiom of choice or equivalently the well-ordering principle. What specific ordering type it has is then itself undecidable in ZFC without further assumption such as the Continuum or General Continuum Hypothesis. An explicit construction of the ordering itself is not even possible in ZFC + GCH.

Luckily transfinite induction only requires the property of well-ordering to be used. Therefore even without specific knowledge of the reals order-type or the orderings construction one can still assert it's existence and use the TI schema. This of course limits the specificity of the statements one can prove, but not necessarily how interesting they are.

Proposition 3.3.1. There a exists a set $\mathfrak{U} \subset \mathbb{R}$ such that for each $r \in \mathbb{R}/\{0\}$:

$$\exists!\{x, y\} \subset \mathfrak{U} \text{ such that } |x - y| = r$$

In plain English, there is a set of points such that for each real number, it appears exactly once as a distance between the points in said set (except 0). A similar statement but with the natural numbers can be made and the proof follows very simply with natural induction. For this proof it requires the reader to understand the connection between cardinality and ordinals, specifically the cardinality of an infinite set is the least such ordinal that is in bijection (not necessarily order preserving) with said set. Therefore $\#\mathbb{N} = \omega$, and we let such an ordinal for $\mathbb{R}$ to be named $\mathfrak{c}$.

Proof. We label the well-ordered real numbers $\mathbb{R}_\alpha$, with each $r_\alpha < r_\gamma$ if $\alpha < \gamma$. Next we will build a series of nested sets $\mathfrak{U}_\alpha \subset \mathbb{R}_\alpha$ that satisfies the properties of the proposed set, up to the the number r_α. We then seek to perform transfinite induction with this:

Base case: For $\mathfrak{U}_0$ take the set $\{0, r_0\}$.
Limit induction step: Assume that $\forall\gamma < \beta$ where $\beta < \mathfrak{c}$, we have minimal $\mathfrak{U}_\gamma$. Then take the union of all such sets $A = \bigcup_{\gamma<\beta} \mathfrak{U}_\gamma$, then we ask whether r_β appears as a distance in said set. If it does then we let $A = \mathfrak{U}_\beta$, else we wish to assert there is some $x \in \mathbb{R}$ such that it adds the distance r_β and does not repeat any previous distance. We then define the following sets given some $a \in A$:

$$
\begin{aligned}
D_a &= \{r \in \mathbb{R} : |r - a| = r_\gamma,\ \gamma < \beta\} \\
D_A &= \bigcup_{a \in A} D_a \\
R_\beta &= \{(r_1, r_2) \in \mathbb{R}^2 : |r_1 - r_2| = r_\beta\}
\end{aligned}
\tag{3.4}
$$

Clearly D_A is the set of all such points that would cause a repetition of distances already in A. While it is obvious that R_β is bijective with $\mathbb{R}$ and therefore has the cardinality of the continuum $\mathfrak{c} = \#\mathbb{R}$; furthermore $(D_A)^2 \cap R_\beta \neq \emptyset$. On the other-hand if D_A is not finite then it is in bijection with $(D_A)^2$ and if it is finite then the claim easily follows.

We know for each r_γ and a that there are only two such points: a itself can be indexed to at most β (there could be less but there is at least a surjection from β to A). Moreover each D_a has a surjection with $\beta \cdot 2$, considering that simple ordinal arithmetic of similar transfinite ordinals does not change the cardinality of the underlying set this means that $\#(D_A)^2 \leq \#\beta < \mathfrak{c}$. This must be true as $\mathfrak{c}$ is the minimal ordinal with said cardinality and $\beta < \mathfrak{c}$. Therefore $R_\beta/(D_A)^2 \neq \emptyset$, so there must exist some pair (r_1, r_2) such that $A \cup \{r_1, r_2\} = \mathfrak{U}_\beta$.

Therefore using the TI schema we know that $\exists\ \mathfrak{U}_\alpha, \forall\alpha < \mathfrak{c}$. Then we take $\mathfrak{U} = \bigcup_{\alpha<\mathfrak{c}} \mathfrak{U}_\alpha$ and this set clearly fulfills the properties of the proposition. □

4. Conclusion

To conclude, we have introduced many essential topics of meta-mathematics and algebraic logic, such as Theories, Models and Languages, though there is much more depth to be found in these topics not broached in this thesis. The reader is encouraged to further research into areas such as Gödel's incompleteness ([4], [9]) theorems and Tarski's undefinability theorems [3]. These topics lay large theoretical restrictions on the ability of mathematics to produce any self-contained "truth" and invite further thought on the philosophy of mathematics and what a sufficient system of inference may be.

Transfinite induction as we have seen, is a natural extension of the induction principles commonly seen throughout mathematics. Further, in some of the most commonly accepted mathematical systems, PA and ZFC, we have shown that the technique's conclusions are consistent and valid. Moreover the technique plays a special role in investigating these larger meta-mathematical structures; the limits of infinite constructions and the consistency of said systems. Beyond that it has potential application on any structure not able to be indexed by the natural numbers, so long as you take careful precautions on the construction of its ordering. Most other systems of math that have wide consideration have Zermelo-Frankel set theory as a sub-theory of the system and so the consistency of transfinite induction follows simply; yet investigation into it's applicability in systems of type-theory would be warranted as an extension of this thesis.

Bibliography

[1] Dave Barker-Plummer, Jon Barwise, and John Etchemendy. *Language, Proof, and Logic*. en. CSLI Publications, 2011. ISBN: 978-1-57586-632-1.

[2] C. C. Chang and H. J. Keisler. *Model Theory*. en. Elsevier, June 1990. ISBN: 978-0-08-088007-5.

[3] Monika Gruber. *Alfred Tarski and the "Concept of Truth in Formalized Languages": A Running Commentary with Consideration of the Polish Original and the German Translation*. en. Springer, Sept. 2016. ISBN: 978-3-319-32616-0.

[4] Petr Hájek and Pavel Pudlák. *Metamathematics of First-Order Arithmetic*. en. Cambridge University Press, Mar. 2017. ISBN: 978-1-316-73945-7.

[5] D. Hilbert. "Mathematische Probleme". deu. In: *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse* 1900 (1900), pp. 253–297. URL: `https://eudml.org/doc/58479` (visited on 06/01/2025).

[6] Peter G. Hinman. *Recursion-Theoretic Hierarchies*. en. Cambridge University Press, Mar. 2017. ISBN: 978-1-107-16824-4.

[7] Thomas Jech. *The Axiom of Choice*. en. Dover Publications, 2008. ISBN: 978-0-486-46624-8.

[8] Richard Kaye. *Models of Peano Arithmetic*. en. Clarendon Press, 1991. ISBN: 978-0-19-853213-2.

[9] Henryk Kotlarski. "Transfinite Induction". en. In: *A Model–Theoretic Approach to Proof Theory*. Ed. by Henryk Kotlarski et al. Cham: Springer International Publishing, 2019, pp. 73–87. ISBN: 978-3-030-28921-8. DOI: `10.1007/978-3-030-28921-8_4`. URL: `https://doi.org/10.1007/978-3-030-28921-8_4` (visited on 04/30/2025).

[10] Kenneth Kunen. *Set Theory*. en. College Publications, 2011. ISBN: 978-1-84890-050-9.

[11] G. E. Mints. “Exact estimates of the provability of transfinite induction in the initial segments of arithmetic”. en. In: *Journal of Soviet Mathematics* 1.1 (Jan. 1973), pp. 85–91. ISSN: 1573-8795. DOI: 10.1007/BF01117473. URL: https://doi.org/10.1007/BF01117473 (visited on 06/01/2025).

[12] Ernest Nagel and James Roy Newman. *Gödel's Proof*. en. Routledge & Kegan Paul, 1959.

[13] Patrick Suppes. *Axiomatic Set Theory*. en. Courier Corporation, Jan. 1972. ISBN: 978-0-486-61630-8.

[14] Gaisi Takeuti. *Proof Theory: Second Edition*. en. Courier Corporation, Oct. 2013. ISBN: 978-0-486-32067-0.

[15] Alfred North Whitehead and Bertrand Russell. *Principia Mathematica*. en. Merchant Books, 2009. ISBN: 978-1-60386-184-7.