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# Uncertainty Quantification and Stability Analysis of the Lotka-Volterra Model Using the Stochastic Galerkin Approach

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## Abstract

The classical Lotka-Volterra model describes the population evolution of two interacting species, one a predator, and the other its prey. The paper investigates the model under parametric uncertainty using the Stochastic Galerkin method, where the deterministic system is transformed into a set of coupled ordinary differential equations via generalized Polynomial Chaos Expansion with Legendre polynomials. Analytical and numerical computations are performed for low-dimensional truncation orders to determine the steady states of the system and their stability. Results show that steady states under uncertainty can be stable, unstable, or neither at the same time and additional fixed points emerge with increasing stochastic complexity. A sensitivity analysis further explores the effects of varying model parameters on equilibrium behavior and population evolutions.

# Contents

# 1 Introduction

Mathematical modeling plays a crucial role in understanding complex natural phenomena, particularly in the study of ecological systems [5]. One of the most classical and widely analyzed models is the Lotka-Volterra system, which captures the nonlinear interaction between a prey and a predator species [14]. Introduced by Alfred J. Lotka in 1920 [12] and by Vito Volterra in 1926 [23] independently, the model provides a simple yet powerful tool in analyzing the oscillatory dynamics and long-term behavior of the prey-predator interaction [3].

Despite its theoretical simplicity and elegance, the standard Lotka-Volterra model makes strong assumptions about the environment and the interaction of the two species, often considering all parameters to be known and fixed [13]. In real-world ecosystems, however, biological interactions are subject to various forms of uncertainty, ranging from fluctuating environmental conditions to incomplete knowledge or incorrect measurements of key parameters such as reproduction rate, hunting rate, or mortality [5]. Therefore, incorporating uncertainty in the model is essential for making realistic predictions and gaining deeper insights into system dynamics [5].

In recent years, uncertainty quantification has gained attraction across applied sciences, especially in fields where models depend on parameters subject to variability, such as fluid dynamics and biology [27]. Some of the most common techniques used to quantify uncertainty include the generalized Polynomial Chaos Expansion (gPCE) [26], Monte Carlo simulations [27], and the Stochastic Galerkin projection method [6]. While stochastic extensions of the Lotka-Volterra equations have been investigated, often using the Monte Carlo approach, there is limited literature applying Stochastic Galerkin methods directly to this problem [5]. The method converts a stochastic differential system of equations into a deterministic higher-dimension system that enables an efficient computation and analytical analysis of how uncertainty affects the system and its stationary values [6].

The thesis explores the Lotka-Volterra equations under parametric uncertainty using the Stochastic Galerkin projection method. In this work, we introduce uncertainty in the prey's natural growth rate, rather than in other parameters such as the predator mortality or the interaction coefficients. This choice is motivated by both ecological and modeling considerations. The prey's growth rate is strongly affected by environmental conditions such as food availability and disease, both of which include natural variability [4, 9]. Moreover, since the size of the prey population directly affects the predator population, uncertainty in this parameter can have a significant effect on the system's long-term behavior, making it an appropriate focus for uncertainty and sensitivity analysis [16]. By introducing uncertainty into the prey's natural growth rate and applying generalized Polynomial Chaos Expansion (gPCE), the model is reformulated into a system of ordinary differential equations (ODEs) of higher dimensionality. Using gPCE, a variable is rewritten as an infinite sum of a deterministic part multiplied by a known polynomial [8]. Since infinite sums are not used in practice, the expansion is approximated by truncating it at a finite order $k$, known as the truncation order [8]. This allows for both qualitative and quantitative analyses of the model's dynamics under uncertainty [5, 6].

While the steady states and their stability are well understood for the standard Lotka-Volterra model [3], it remains unclear how these properties are affected when uncertainty is introduced into the system [5]. This work aims to address this gap by studying the behavior of the Stochastic Galerkin system for truncation orders $k = 0, 1, 2$. A numerical implementation is used to solve the truncated system and identify unique stationary points, followed by analytical stability assessments based on the eigenvalues of the Jacobian matrix. Finally, a sensitivity analysis is performed by varying the fixed parameters of the model. First, the predator's hunting rate $\beta$, the predator's mortality rate $\gamma$ and the effect $\delta$ that the prey population has on the predator's are varied individually to assess their effect on the system dynamics at equilibrium. Furthermore, a similar analysis is performed for the stochastic components of the prey's growth rate, investigating the impact of uncertainty on equilibrium behavior.

## 2 The Lotka-Volterra model

The Lotka-Voterra model is a mathematical model used to describe the dynamics of two interacting populations, one a predator and the other its prey. The equations that model the interaction of the two species bear the names of two scientists: Lotka and Volterra. Alfred James Lotka (1880-1949) was a Polish-American chemist and statistician and Vito Volterra (1860-1940) an Italian mathematician [3].

### 2.1 Short bibliographies of Lotka and Volterra

Alfred J. Lotka is considered as one of the founders of mathematical demography. He was born in Lwów, Austria-Hungary, then part of Poland, and died in New York, USA. Despite not pursuing a career in academia, his work was profoundly influential to the academic world, with all authors investigating his field of work regularly referencing him and his discoveries. Lotka's advanced studies in physics and chemistry began at the University of Birmingham, where he also obtained his B. Sc. in 1901, and later a D. Sc. in 1912. Between the two, he worked at Leipzig University and Cornell University, where in 1909 he also received his M. Sc. in Physics [3, 22].

Figure 1: Alfred J. Lotka

Lotka wrote six books and published almost one hundred papers on various topics in chemistry, physics, epidemiology and biology. Between 1907 and 1939, he worked in mathematical demography, time in which he also derived a model proving that undamped, permanent oscillations arise in biological systems in 1920 [12]. He further had publications in the field of bibliometrics and contributions in scientometrics – the scientific study of scientific publications. Around half of his publications were on population issues, specifically on the concepts of stable age-distribution of a population and of natural rate of increase of a population. Furthermore, towards the end of his professional career, he was a statistician for the Metropolitan Life Insurance Company, New York [3, 22].

Figure 2: Vito Volterra

Vito Volterra was born in Ancona, then part of the Papal States, and died in Rome, Italy. His work was focused around integral equations and calculus. In fact, in 1896 he published papers on what are known today as integral equations of *Volterra* type, and had later contributions in integral and integro-differential equations [3]. Moreover, several papers published by Volterra initiated the modern theory of functional analysis. After being a professor of rational mechanics at the University of Pisa, where he also performed his studies, in 1892, he became professor of mechanics at the University of Turin and then, in 1900, professor of mathematical physics at the University of Rome La Sapienza [3]. After World War I, Volterra started studying the ecological problem of a predator population interacting with the prey one [3].

His work progressed in the following years, publishing more papers, with the aim of obtaining a mathematical theory of the struggle for existence of interacting species. This led to him being a plenary speaker at the First Congress of Romanian Mathematicians in Cluj, Romania in 1929, presenting *On the mathematical theory of the struggle for existence.* In fact, Volterra had a lot of Romanian mathematician friends [3]. This is where his work overlapped with Lotka's work. Lotka published his paper in 1920 [12] and Volterra in 1926 [23], both reaching the same conclusion, that the interaction of two species would give rise to an intriguing oscillation of their populations [3]. Volterra published his first paper in 1881, just before turning twenty-one, and his last in 1939-1940, when he was almost eighty years old.

Although the two were in different fields of studies and lived in different countries, their mutual interest in mathematical modeling resulted in their professional partnership. The authors even exchanged a few letters during their work on the model [3]. While Lotka firmly established his goal of deriving equations modeling two animal species, Volterra generalized the equations for $n$ species, including their past interactions [3, 24]. A comprehensive derivation of the generalized Lotka-Volterra equations can be found in Chapter 3 of [17] or Volterra's paper [24].

# 3 Simple population models

The equations of the standard Lotka-Volterra model derived by the two authors are

$$\begin{aligned} \frac{dy}{dt} &= \alpha y - \beta yz, \\ \frac{dz}{dt} &= \delta yz - \gamma z, \end{aligned} \tag{1}$$

where $y(t)$ is the population density of the prey at time $t$, and $z(t)$ the population density of the predator at time $t$. The prey's parameters, $\alpha$ and $\beta$, describe, respectively, the maximum prey per capita growth rate, and the effect of the presence of predators on the prey death rate. The predator's parameters, $\gamma$ and $\delta$, respectively describe the predator's per capita death rate, and the effect of the presence of prey on the predator's growth rate. Physically, $\beta$ is the hunting rate of the predator on the prey and $\gamma$ is the predator's mortality rate. Note that all four parameters are positive.

Building up to model (1), we begin by analyzing a simplified case involving only one of the two species, namely the prey. Starting with the basic ODE describing exponential population growth, we then incorporate a quadratic term of the population $y$ to represent limiting factors such as food scarcity or habitat limitations. With this foundation, we extend the analysis to the two-species Lotka–Volterra model (1).

## 3.1 Single-Species Exponential Growth

The simplest mathematical model that describes the population growth of an animal species assumes that the rate of increase of population is proportional to the size of the population at any time. In mathematical terms, this describes a basic Ordinary Differential Equation (ODE) of the form

$$\frac{dy}{dt} = \alpha y(t), \tag{2}$$

where $\alpha$ is the population growth rate, defined as the difference between the birth and death rates. Here, the birth rate is defined as the number of individuals born in a population per unit time, while the death rate represents the average number of individuals that die per unit time [10]. Both rates are measured per unit time, so $\alpha$ has units of inverse time $t$. The function $y(t)$ denotes the population density at time $t$, represented by a scalar since only one species is considered.

Since the end goal involves analyzing the Lotka-Volterra model (1), the parameter $\alpha$ is often more chaotic rather than constant. Furthermore, the biological measurements are subject to error, as there is no way of accurately determining the birth rate of a species, or similarly, the spontaneous death rate. Thus, we want to add uncertainty to the model to overtake the possible miscalculations.

Assume that $\alpha$ is a uniformly distributed variable. We assume uniform distribution due to simplicity and the fact that it is bounded, meaning that we can assume $\alpha$ is positive and can be written as

$$\alpha := a + b\omega, \tag{3}$$

where $a, b \in \mathbb{R}$, and $\omega$ is uniformly distributed $\omega \sim \mathcal{U}([-1, 1])$. To ensure that the exponential growth rate $\alpha$ remains strictly positive for all values of $\omega \sim \mathcal{U}([-1, 1])$, we impose the condition $a - |b| > 0$. Note that $\omega$ is a scalar, thus it is dimensionless. Substitution with (3) in (2), results in the following uncertain ordinary differential equation with parametric uncertainty:

$$\frac{dy}{dt}(t, \omega) = (a + b\omega)y(t, \omega), \quad a - |b| > 0, \ \omega \sim \mathcal{U}([-1, 1]). \tag{4}$$

Note that since $\omega$ can take different values, function $y(t)$ in (2) is now dependent on both time and $\omega$, so it becomes $y(t, \omega)$, as given in (4).

To perform uncertainty quantification, we make use of the stochastic Galerkin projection method. This is an intrusive method, meaning that new equations are derived, ones that take into account the uncertainty. Stochastic Galerkin projection relies on generalized Polynomial Chaos Expansion (gPCE) of the involved variables, which in this case is $y(t)$. As stated above, this affects $y$ in the sense that it is now not only depending on time $t$, but also on the random variable $\omega$. Using the gPCE a variable can be written as an infinite sum of a deterministic part multiplied by a known polynomial, which depends on the random parameter $\omega$. Since in practice we cannot work with infinite sums, we truncate it in the following way:

$$y(t, \omega) \approx \sum_{i=0}^{k} \hat{y}_i(t)\phi_i(\omega), \tag{5}$$

where $k \in \mathbb{N}$ is the truncation order, taken to be finite. Using (5) and the known definitions for mean and variance given in [18], we derive the following expressions for the mean $\mu_y$ and standard deviation $\sigma_y$ for the population $y$:

$$\mu_y = \mathbb{E}[y] = \hat{y}_0, \quad \sigma_y = \sqrt{Var[y]} = \sum_{i=1}^{k} \hat{y}_i^2, \tag{6}$$

which are used in Section 5.

The goal now is to derive an ODE-system for the unknown stochastic Fourier coefficients $\hat{y}_0, \ldots, \hat{y}_k$. Since the random parameter $\omega$ is uniformly distributed on $[-1, 1]$, the corresponding family of orthogonal polynomials with respect to the uniform probability density function $\mathbf{1}_{[-1,1]}$ is the *Legendre* polynomials. For other types of distributions, different polynomial families are used. For example, *Hermite* polynomials are used when the random variable follows a standard normal distribution, due to their orthogonality with respect to the Gaussian probability density function. Similarly, *Jacobi* polynomials are used for $\beta$-distributions and *Laguerre* for exponential or $\gamma$-distributions [15].

*Legendre* polynomials are a known family of orthogonal polynomials first introduced in 1782 by Adrien-Marie Legendre and are obtained through well-known linear algebra methods based on Sturm-Liouville theory [2]. They are most commonly obtained as a result of the solution of Legendre differential equation by power series [2]. This paper denotes the polynomials as $\{\phi_i\}_{i=0}^{\infty}$, and as an example, we have that the first three standard (unnormalized) Legendre polynomials are $\phi_0(\omega) = 1, \phi_1(\omega) = \omega, \phi_2(\omega) = \frac{1}{2}(3\omega^2 - 1)$. Substituting equation (5) into the uncertain ODE (4) yields

$$\frac{d}{dt}\sum_{i=0}^{k}\hat{y}_i(t)\phi_i(\omega) = (a + b\omega)\sum_{i=0}^{k}\hat{y}_i(t)\phi_i(\omega)$$

$$\iff \sum_{i=0}^{k}\frac{d\hat{y}_i}{dt}\phi_i(\omega) = \sum_{i=0}^{k}a\hat{y}_i(t)\phi_i(\omega) + \sum_{i=0}^{k}b\hat{y}_i(t)\omega\phi_i(\omega). \tag{7}$$

Next, project (7) onto a test function $\phi_s(\omega)$, for $s = 0, \ldots, k$, by multiplying (7) by $\phi_s(\omega)$ and integrating over $\omega \in [-1, 1]$. Note that the test function is also taken to be a Legendre polynomial. The projection of (7) onto the test function $\phi_s$ yields the following expression:

$$\underbrace{\int_{-1}^{1}\sum_{i=0}^{k}\frac{d\hat{y}_i}{dt}\phi_i(\omega)\phi_s(\omega)d\omega}_{P\left(\frac{d\hat{y}}{dt}\right)} = \underbrace{a\int_{-1}^{1}\sum_{i=0}^{k}\hat{y}_i(t)\phi_i(\omega)\phi_s(\omega)d\omega}_{P(a\hat{y})} + \underbrace{b\int_{-1}^{1}\sum_{i=0}^{k}\hat{y}_i(t)\omega\phi_i(\omega)\phi_s(\omega)d\omega}_{P(b\omega\hat{y})}, \tag{8}$$

where $P(\cdot)$ denotes the projection operator onto the Legendre polynomial basis.

It is a known fact that Legendre polynomials are orthogonal, giving

$$\int_{-1}^{1}\phi_i(\omega)\phi_s(\omega)d\omega = h_s\delta_{is}, \tag{9}$$

where $h_s = \frac{2}{2s+1}$ is the normalization factor (we assume $\phi_i(1) = 1$), and $\delta_{is}$ is the Kronecker-delta function,

$$\delta_{is} = \begin{cases} 1, & \text{if} \quad i = s, \\ 0, & \text{if} \quad i \neq s. \end{cases} \tag{10}$$

A straightforward computation for $P\left(\frac{d\hat{y}}{dt}\right)$ and $P(a\hat{y})$ in (8) gives

$$P\left(\frac{d\hat{y}}{dt}\right): \quad \sum_{i=0}^{k}\frac{d\hat{y}_i}{dt}\int_{-1}^{1}\phi_i(\omega)\phi_s(\omega)d\omega = h_s\frac{d\hat{y}_s}{dt}, \tag{11}$$

$$P(a\hat{y}): \quad \sum_{i=0}^{k}a\hat{y}_i\int_{-1}^{1}\phi_i(\omega)\phi_s(\omega)d\omega = ah_s\hat{y}_s. \tag{12}$$

Next, for $P(b\omega\hat{y}) = \sum_{i=0}^{k}\hat{y}_i\int_{-1}^{1}\omega\phi_i(\omega)\phi_s(\omega)d\omega$, we make use of the three-term recurrence relation of orthogonal polynomials. This is an appropriate tool since the Legendre polynomials are orthogonal by definition. The formal recurrence relation states as follows.

$$\phi_{i+1}(\omega) = (A_i\omega + B_i)\phi_i(\omega) - C_i\phi_{i-1}(\omega)$$

$$\iff \omega\phi_i(\omega) = \frac{1}{A_i}(\phi_{i+1}(\omega) + C_i\phi_{i-1}(\omega) - B_i\phi_i(\omega)). \tag{13}$$

The coefficients $A, B$ and $C$ dependent on index $i$ that correspond to the choice of Legendre polynomials are

$$A_i := \frac{2i+1}{i+1}, \quad B_i := 0, \quad C_i := \frac{i}{i+1}. \tag{14}$$

Thus, using (14), the expression (13) translates to our problem as

$$\omega\phi_i(\omega) = \frac{i+1}{2i+1}\phi_{i+1}(\omega) + \frac{i}{2i+1}\phi_{i-1}(\omega). \tag{15}$$

Using (15), we obtain

$$\begin{aligned}\int_{-1}^{1} \omega\phi_i(\omega)\phi_s(\omega)d\omega &= \int_{-1}^{1} \frac{i+1}{2i+1}\phi_{i+1}(\omega)\phi_s d\omega + \int_{-1}^{1} \frac{i}{2i+1}\phi_{i-1}(\omega)\phi_s(\omega)d\omega \\ &= \left(\frac{s}{2s-1}\delta_{i,s-1} + \frac{s+1}{2s+3}\delta_{i,s+1}\right)h_s, \end{aligned} \tag{16}$$

where $\delta$ is the Kronecker-delta function (10).

With the result (16) and the orthogonality property of Legendre polynomials (9), one obtains

$$P(b\omega\hat{y}): \quad b\sum_{i=0}^{k}\hat{y}_i \int_{-1}^{1} \omega\phi_i(\omega)\phi_s(\omega)d\omega = bh_s\frac{s}{2s-1}\hat{y}_{s-1} + bh_s\frac{s+1}{2s+3}\hat{y}_{s+1}. \tag{17}$$

Note that since all three expressions (11), (12) and (17) involve the factor $h_s$, we divide all three equations by $h_s$ and summing them we obtain the following ODE-system describing the evolution of $\hat{y}(t)$ over time $t$, as required:

$$\frac{d\hat{y}_s}{dt} = ay_s + \frac{bs}{2s-1}\hat{y}_{s-1} + b\frac{s+1}{2s+3}\hat{y}_{i+1}, \quad s = 0, \ldots, k. \tag{18}$$

In what follows, a quadratic term of the animal population $y$ is added to the basic ODE, representing the limiting factors of the model. This term is incorporated to approach the Lotka-Volterra model (1) which involves a nonlinear relationship between the two species. The added term increases the complexity of the system and hence applying the same method as in this section requires further computations to obtain a similar system to (18).

### 3.2 Single-Species Logistic Growth

As an intermediate step towards the Lotka-Volterra model with uncertainty, we are interested in the role of adding a quadratic component of the animal population to the ODE. Consider then

$$\frac{dy}{dt} = \alpha y(t) + y(t)^2, \tag{19}$$

where $\alpha$ is the exponential growth of the population $y(t)$ and $\alpha$ and $y$ are defined as in Section 3.1. Again, suppose $\alpha$ is uniformly distributed, so it is positive and bounded and let $\alpha = a + b\omega$, where $\omega \in \mathcal{U}[-1,1]$ uniformly distributed. As given in Section 3.1, we impose the condition $a - |b| > 0$ and give that $\omega$ is dimensionless. The uncertain ODE is then

$$\frac{dy}{dt}(t,\omega) = (a+b\omega)y(t,\omega) + y^2(t,\omega), \quad a - |b| > 0, \ \omega \in \mathcal{U}[-1,1]. \tag{20}$$

Note that as in (4), function $y(t)$ is now dependent on both time $t$ and random variable $\omega$, so it becomes $y(t,\omega)$, as given in (20).

Using the truncated gPCE (5), we have

$$\underbrace{\frac{d}{dt}\left(\sum_{i=0}^{k}\hat{y}_i(t)\phi_i(\omega)\right)}_{\mathcal{T}_k\left(\frac{d\hat{y}}{dt}\right)} = \underbrace{(a+b\omega)\sum_{i=0}^{k}\hat{y}_i(t)\phi_i(\omega)}_{\mathcal{T}_k(\alpha y)} + \underbrace{\left(\sum_{i=0}^{k}\hat{y}_i(t)\phi_i(\omega)\right)^2}_{\mathcal{T}_k(y^2)}, \tag{21}$$

where $\mathcal{T}_k(\cdot)$ denotes the truncation operator of the generalized Polynomial Chaos Expansion to finite order $k$.

The term $\mathcal{T}_k(\alpha y)$ in (21) is discussed in Section 3.1, hence the focus is on the term $\mathcal{T}_k(y^2)$. Note that this term involves the product of two sums, leading to a double sum, for which we have two indices. Denote the second index as $j$. Then, we have

$$\mathcal{T}_k(y^2): \quad \left(\sum_{i=0}^{k}\hat{y}_i(t)\phi_i(\omega)\right)^2 = \sum_{i=0}^{k}\sum_{j=0}^{k}\hat{y}_i(t)\hat{y}_j(t)\phi_i(\omega)\phi_j(\omega). \tag{22}$$

As in Section 3.1, the sum is projected using test function $\phi_s(\omega)$ for $s = 0, \dots, k$. The projection is performed by multiplying (22) by $\phi_s(\omega)$ and then integrating over the random variable $\omega \in [-1, 1]$, obtaining

$$\int_{-1}^{1} \sum_{i=0}^{k} \sum_{j=0}^{k} \hat{y}_i(t)\hat{y}_j(t)\phi_i(\omega)\phi_j(\omega)\phi_s(\omega)d\omega = \sum_{i=0}^{k} \sum_{j=0}^{k} \hat{y}_i(t)\hat{y}_j(t) \underbrace{\int_{-1}^{1} \phi_i(\omega)\phi_j(\omega)\phi_s(\omega)d\omega}_{=:C_{sij}}, \tag{23}$$

where $C_{sij}$ is known as the triple product integral of the Legendre polynomials. As given in Section 3.1, the first three standard (unnormalized) Legendre polynomials are $\phi_0(\omega) = 1, \phi_1(\omega) = \omega, \phi_2(\omega) = \frac{1}{2}(3\omega^2 - 1)$, orthogonal by definition. The computation of $C_{sij}$ for $s, i, j = 0, 1, 2$ is then as follows. Note that the triple product integral is symmetric, i.e., $C_{100} = C_{010} = C_{001}$ and similar.

$$C_{000} = \int_{-1}^{1} \phi_0(\omega)\phi_0(\omega)\phi_0(\omega)d\omega = \int_{-1}^{1} 1 \cdot 1 \cdot 1 d\omega = \omega\Big|_{-1}^{1} = 2,$$

$$C_{001} = C_{010} = C_{100} = \int_{-1}^{1} \phi_0(\omega)\phi_0(\omega)\phi_1(\omega)d\omega = \int_{-1}^{1} 1 \cdot 1 \cdot \omega \quad d\omega = \frac{\omega^2}{2}\Big|_{-1}^{1} = 0,$$

$$C_{011} = C_{101} = C_{110} = \int_{-1}^{1} \phi_0(\omega)\phi_1(\omega)\phi_1(\omega)d\omega = \int_{-1}^{1} 1 \cdot \omega \cdot \omega \quad d\omega = \frac{\omega^3}{3}\Big|_{-1}^{1} = \frac{2}{3},$$

$$C_{111} = \int_{-1}^{1} \phi_1(\omega)\phi_1(\omega)\phi_1(\omega)d\omega = \int_{-1}^{1} \omega^3 d\omega = \frac{\omega^4}{4}\Big|_{-1}^{1} = 0,$$

$$C_{002} = C_{020} = C_{200} = \int_{-1}^{1} \phi_0(\omega)\phi_0(\omega)\phi_2(\omega)d\omega = \int_{-1}^{1} \frac{3}{2}\omega^2 - \frac{1}{2} d\omega = \frac{\omega^3}{2} - \frac{\omega}{2}\Big|_{-1}^{1} = 0,$$

$$\begin{aligned} C_{012} = C_{021} = C_{102} = C_{120} = C_{201} = C_{210} &= \int_{-1}^{1} \phi_0(\omega)\phi_1(\omega)\phi_2(\omega)d\omega = \\ &= \int_{-1}^{1} \frac{3}{2}\omega^3 - \frac{1}{2}\omega d\omega = \frac{3}{8}\omega^4 - \frac{1}{4}\omega^2\Big|_{-1}^{1} = 0, \end{aligned}$$

$$C_{022} = C_{202} = C_{220} = \int_{-1}^{1} \frac{9}{4}\omega^4 - \frac{3}{2}\omega^2 + \frac{1}{4} d\omega = \frac{9}{20}\omega^5 - \frac{1}{2}\omega^3 + \frac{1}{4}\omega\Big|_{-1}^{1} = \frac{2}{5},$$

$$C_{112} = C_{121} = C_{211} = \int_{-1}^{1} \frac{3}{2}\omega^4 - \frac{1}{2}\omega^2 d\omega = \frac{3}{10}\omega^5 - \frac{1}{6}\omega^3\Big|_{-1}^{1} = \frac{4}{15},$$

$$C_{122} = C_{212} = C_{221} = \int_{-1}^{1} \frac{9}{4}\omega^5 - \frac{3}{2}\omega^3 + \frac{1}{4}\omega d\omega = \frac{3}{8}\omega^6 - \frac{3}{8}\omega^4 + \frac{1}{8}\omega^2\Big|_{-1}^{1} = 0,$$

$$C_{222} = \int_{-1}^{1} \frac{27}{8}\omega^6 - \frac{27}{8}\omega^4 + \frac{9}{8}\omega^2 - \frac{1}{8} d\omega = \frac{27}{56}\omega^6 - \frac{27}{40}\omega^4 + \frac{9}{24}\omega^2 - \frac{1}{8}\omega\Big|_{-1}^{1} = \frac{4}{35}.$$

Observe that the triple product integral vanishes whenever $s+i+j$ is odd. This is because, in such cases, the integrand becomes an odd function, and the integral of an odd function over a symmetric interval around zero is zero. Since we integrate over $[-1, 1]$, the interval is indeed symmetric around zero. The formal proof is found in [19].

The triple product integral $C_{sij}$ of the form

$$\int_{-1}^{1} \phi_i(\omega)\phi_j(\omega)\phi_s(\omega)d\omega, \quad 0 \le s, i, j \le k, \tag{24}$$

where $k$ is the truncation order of the gPCE can be computed by hand for low orders $s, i, j = 0, 1, 2$ as presented above, since computations are quite fast. For higher orders, however, it becomes cost ineffective to compute analytically, thus it is usually computed numerically.

However, if one wishes to evaluate (24) analytically for all $0 \le s, i, j \le k$, for a finite $k$, another approach is constructing a suitable Gaussian quadrature rule tailored to the Legendre polynomial basis. Since each basis function $\phi_n(\omega)$ is a polynomial of degree $n = 0, \ldots, k$, the product $\phi_s(\omega)\phi_i(\omega)\phi_j(\omega)$ is a polynomial of degree $s + i + j \leq 3k$. Therefore, to integrate all such triple products, one requires a quadrature rule that integrates the polynomials of degree up to $3k$. Note however, that this method provides an approximation and not the exact integral [11].

Gaussian quadrature with $N$ nodes integrates polynomials of degree up to $2N - 1$. Choose $N = \frac{3k+1}{2}$, ensuring integration of all triple products. Next, let $\{\omega_n\}_{n=1}^N$ be the quadrature nodes with corresponding weights $\{w_n\}_{n=1}^N$, associated with the Legendre polynomials. The method using Legendre polynomials is known as the Gauss-Legendre quadrature [11]. With this rule, each triple product integral $C_{sij}$ can be approximated as

$$\int_{-1}^{1} \phi_s(\omega)\phi_i(\omega)\phi_j(\omega)d\omega \approx \sum_{n=1}^{N} w_n \phi_s(\omega_n)\phi_i(\omega_n)\phi_j(\omega_n).$$

This quadrature construction enables an efficient computation of all necessary integrals for the stochastic Galerkin system.

Turning back to obtaining an ODE-system for the unknown $\hat{y}_0, \ldots, \hat{y}_k$, combining (23) with the result (18) obtained in Section 3.1, gives the following system of equations for the truncated $\hat{y}(t, \omega)$:

$$\frac{d\hat{y}_s}{dt} = ah_s y_s + \sum_{i=0}^{k}\sum_{j=0}^{k} C_{sij}\hat{y}_i\hat{y}_j + bh_s\frac{s}{2s-1}\hat{y}_{s-1} + bh_s\frac{s+1}{2s+3}\hat{y}_{i+1}, \quad s = 0, \ldots, k. \tag{25}$$

Dividing (25) by $h_s$ for simplification gives

$$\frac{d\hat{y}_s}{dt} = ay_s + \frac{1}{h_s}\sum_{i=0}^{k}\sum_{j=0}^{k} C_{sij}\hat{y}_i\hat{y}_j + b\frac{s}{2s-1}\hat{y}_{s-1} + b\frac{s+1}{2s+3}\hat{y}_{i+1}, \quad s = 0, \ldots, k. \tag{26}$$

Hence, the required ODE-system (26) is obtained, and we observe that the quadratic component of $y(t, \omega)$ only adds the term involving the triple product integral $C_{sij}$ to the system (18).

Note that the system (26) is the first equation of the Lotka-Volterra model (1) with uncertainty, for only one species $y$. Thus, we are one step closer to obtaining the full two-species Lotka-Volterra model with uncertainty. Before expanding the second equation of (1), we first make some assumptions on the model.

# 4 Lotka-Volterra equations

The simplest Lotka-Volterra model (1) derived by Alfred Lotka and Vito Volterra involves two animal species, namely a predator and its prey. A standard example of such an interaction is foxes and rabbits in a flatland. Note the following assumptions about the environment and interaction that are made for simplicity [3]:

- The predator only feeds on the prey and nothing else.
- The prey has an unlimited third food supply at all times.
- In the absence of predators, the prey population $y$ would follow the basic ODE $\frac{dy}{dt} = \alpha y(t)$ explored in the previous sections. The coefficient $\alpha$ was named by Volterra the "coefficient of auto-increase" [3].
- In the absence of prey, the predator population $z$ would decrease proportionally in a form given by $\frac{dz}{dt} = -\gamma z(t)$.
- When both the prey and predator are present, a decrease in prey population $y$ and increase in predator population $z$ occur at respectively proportional rates with coefficients $\beta$ for prey and $\delta$ for predators.

Thus, with the above assumptions, the Lotka-Volterra model for two species is represented by the following pair of prey-predator equations [12, 23]:

$$\frac{dy}{dt} = \alpha y - \beta yz,$$
$$\frac{dz}{dt} = \delta yz - \gamma z,$$

as given by (1). Remember that $y(t)$ is the population density of the prey at time $t$, and $z(t)$ the population density of the predator at time $t$. Parameters $\alpha$ and $\gamma$ denote the natural evolution of the prey and predator populations, respectively and $\beta$ and $\gamma$ denote the effect of the predator on the prey population, and respectively the effect of the prey on the predator population. All four parameters are assumed to be strictly positive.

As given in (3), we include uncertainty in parameter $\alpha := a + b\omega$, where $a, b \in \mathbb{R}$, and $\omega$ uniformly distributed $\omega \sim \mathcal{U}(-1, 1)$. Again, we impose the condition $a - |b| > 0$, so that $\alpha$ is strictly positive, and take all other parameters $a, b, \beta, \delta$ and $\gamma$ to be fixed. Thus, the uncertain Lotka-Volterra model is of the form

$$\begin{aligned} \frac{dy}{dt} &= (a + b\omega)y - \beta yz, \\ \frac{dz}{dt} &= \delta yz - \gamma z, \end{aligned} \tag{27}$$

for $a - |b| > 0,\ \omega \in \mathcal{U}[-1, 1]$. Using the gPCE as in (5), obtain the following approximations of $y(t, \omega)$ and $z(t, \omega)$:

$$\begin{aligned} y(t, \omega) &\approx \sum_{i=0}^{k} \hat{y}_i(t)\phi_i(\omega), \\ z(t, \omega) &\approx \sum_{j=0}^{k} \hat{z}_j(t)\phi_j(\omega). \end{aligned} \tag{28}$$

Similar to 6, we use the definition for the truncated gPCE (28) and the standard definitions for mean and variance given in [18] to obtain the following expressions for the mean $\mu$ and standard deviation $\sigma$ for the prey and predator populations $y$ and $z$, respectively, that are used in Section 5 for the numerical imlementation of the steady states:

$$\begin{aligned} \mu_y &= \mathbb{E}[y] = \hat{y}_0, \quad \sigma_y = \sqrt{Var[y]} = \sum_{i=1}^{k} \hat{y}_i^2, \\ \mu_z &= \mathbb{E}[z] = \hat{z}_0, \quad \sigma_z = \sqrt{Var[z]} = \sum_{i=1}^{k} \hat{z}_i^2. \end{aligned} \tag{29}$$

The goal is to obtain an ODE system of equations similar to (18) and (26) for both $\frac{d\hat{y}_i}{dt}$ and $\frac{d\hat{z}_i}{dt}$. Let us first look at the first equation of the uncertain Lotka-Volterra system (27). Using the above substitution (28), the first equation of (27) becomes

$$\sum_{i=0}^{k} \frac{d}{dt}\hat{y}_i(t)\phi_i(\omega) = (a + b\omega)\sum_{i=0}^{k} \hat{y}_i(t)\phi_i(\omega) - \beta \sum_{i=0}^{k}\sum_{j=0}^{k} \hat{y}_i(t)\hat{z}_j(t)\phi_i(\omega)\phi_j(\omega). \tag{30}$$

Using the three-term recurrence (13), the expression (30) becomes

$$
\begin{aligned}
\sum_{i=0}^{k} \frac{d}{dt}\hat{y}_i(t)\phi_i(\omega) &= a\sum_{i=0}^{k}\hat{y}_i(t)\phi_i(\omega) + b\left[\sum_{i=0}^{k}\frac{1}{A_i}\hat{y}_i(t)(\phi_{i+1}(\omega)+C_i\phi_{i-1}(\omega)-B_i\phi_i(\omega))\right] \\
&\quad - \beta\sum_{i=0}^{k}\sum_{j=0}^{k}\hat{y}_i(t)\hat{z}_j(t)\phi_i(\omega)\phi_j(\omega) \\
\iff \sum_{i=0}^{k} \frac{d}{dt}\hat{y}_i(t)\phi_i(\omega) &= \sum_{i=0}^{k}\left[\left(a - b\frac{B_i}{A_i}\right)\hat{y}_i(t)\phi_i(\omega) - \beta\sum_{j=0}^{k}\hat{y}_i(t)\hat{z}_j(t)\phi_i(\omega)\phi_j(\omega)\right] \\
&\quad + b\sum_{i=0}^{k}\hat{y}_i(t)\left[\frac{C_i}{A_i}\phi_{i-1}(\omega) + \frac{1}{A_i}\phi_{i+1}(\omega)\right].
\end{aligned} \tag{31}
$$

Project (31) using test function $\phi_s(\omega)$. Using the orthogonality property (9), we obtain

$$
\begin{aligned}
h_s\frac{d}{dt}\hat{y}_s(t) &= h_s(a - b\frac{B_i}{A_i})\hat{y}_s(t) - \beta\int_{-1}^{1}\sum_{i=0}^{k}\sum_{j=0}^{k}\hat{y}_i(t)\hat{z}_j(t)\phi_i(\omega)\phi_j(\omega)\phi_s(\omega)d\omega \\
&\quad + bh_s\frac{C_{s+1}}{A_{s+1}}\hat{y}_{s+1}(t) + bh_s\frac{1}{A_{s-1}}\hat{y}_{s-1}(t).
\end{aligned} \tag{32}
$$

Substitute in (32) the coefficients $A, B, C$ for Legendre polynomials given in (14) and divide (32) by $h_s$:

$$
\frac{d\hat{y}_s}{dt} = a\hat{y}_s(t) - \frac{\beta}{h_s}\sum_{i=0}^{k}\sum_{j=0}^{k}C_{sij}\hat{y}_i(t)\hat{z}_j(t) + b\frac{s}{2s-1}\hat{y}_{s-1}(t) + b\frac{s+1}{2s+3}\hat{y}_{s+1}(t), \tag{33}
$$

where $C_{sij} := \int_{-1}^{1}\phi_i(\omega)\phi_j(\omega)\phi_s(\omega)d\omega$ is the triple product integral for Legendre polynomials given in (24).

Next, let us look at the second equation of the uncertain Lotka-Volterra model (27). Substitution with the truncated gPCE (28) gives

$$
\sum_{j=0}^{k}\frac{d}{dt}\hat{z}_j(t)\phi_j(\omega) = \delta\sum_{i=0}^{k}\sum_{j=0}^{k}\hat{y}_i(t)\hat{z}_j(t)\phi_i(\omega)\phi_j(\omega) - \gamma\sum_{j=0}^{k}\hat{z}_j(t)\phi_j(\omega).
$$

Using the same method as above and in the previous section, project using test function $\phi_s(\omega)$:

$$
h_s\frac{d}{dt}\hat{z}_s(t) = \delta\int_{-1}^{1}\sum_{i=0}^{k}\sum_{j=0}^{k}\hat{y}_i(t)\hat{z}_j(t)\phi_i(\omega)\phi_j(\omega)\phi_s(\omega)d\omega - h_s\gamma\hat{z}_s(t)
$$

$$
\iff \frac{d\hat{z}_s}{dt} = \frac{\delta}{h_s}\sum_{i=0}^{k}\sum_{j=0}^{k}C_{sij}\hat{y}_i(t)\hat{z}_j(t) - \gamma\hat{y}_i(t), \quad \text{s=0}, \ldots, k.. \tag{34}
$$

Combining (33) and (34), the ODE-system used to find a solution for the uncertain Lotka-Volterra model is

$$
\begin{aligned}
\frac{d\hat{y}_s}{dt} &= a\hat{y}_s(t) - \frac{\beta}{h_s}\sum_{i=0}^{k}\sum_{j=0}^{k}C_{sij}\hat{y}_i(t)\hat{z}_j(t) + b\frac{s}{2s-1}\hat{y}_{s-1}(t) + b\frac{s+1}{2s+3}\hat{y}_{s+1}(t) \\
\frac{d\hat{z}_s}{dt} &= \frac{\delta}{h_s}\sum_{i=0}^{k}\sum_{j=0}^{k}C_{sij}\hat{y}_i(t)\hat{z}_j(t) - \gamma\hat{y}_i(t), \quad \text{for s=0}, \ldots, k.
\end{aligned} \tag{35}
$$

Since the uncertain Lotka-Volterra equations cannot be solved analytically partly due to the infinite sums, the truncated gPCE is used with $k < \infty$, which helps tremendously. However, for each value of $k$, the system involves $2(k+1)$ equations, since we are working with a prey-predator system involving two species. This means that for $k \geq 2$, the system becomes too technical to solve analytically, so we use numerical tools such as Python and Mathematica to derive the steady states for the cases $k = 0, 1, 2$.

For example, the equations for $k = 2$ are

$$\begin{aligned}
\frac{d\hat{y}_0}{dt} &= a\hat{y}_0 + \frac{b}{3}\hat{y}_1 - \beta(\hat{y}_0\hat{z}_0 + \frac{1}{3}\hat{y}_1\hat{z}_1 + \frac{1}{5}\hat{y}_2\hat{z}_2),\\
\frac{d\hat{y}_1}{dt} &= a\hat{y}_1 + b\hat{y}_0 + \frac{2}{5}b\hat{y}_2 - \beta(\hat{y}_0\hat{z}_1 + \hat{y}_1\hat{z}_0 + \frac{2}{5}\hat{y}_1\hat{z}_2 + \frac{2}{5}\hat{y}_2\hat{z}_1),\\
\frac{d\hat{y}_2}{dt} &= a\hat{y}_2 + \frac{2}{3}b\hat{y}_1 - \beta(\hat{y}_0\hat{z}_2 + \frac{2}{3}\hat{y}_1\hat{z}_1 + \hat{y}_2\hat{z}_0 + \frac{2}{7}\hat{y}_2\hat{z}_2),\\
\frac{d\hat{z}_0}{dt} &= \delta(\hat{y}_0\hat{z}_0 + \frac{1}{3}\hat{y}_1\hat{z}_1 + \frac{1}{5}\hat{y}_2\hat{z}_2) - \gamma\hat{z}_0,\\
\frac{d\hat{z}_1}{dt} &= \delta(\hat{y}_0\hat{z}_1 + \hat{y}_1\hat{z}_0 + \frac{2}{3}\hat{y}_1\hat{z}_2 + \frac{2}{5}\hat{y}_2\hat{z}_1) - \gamma\hat{z}_1,\\
\frac{d\hat{z}_2}{dt} &= \delta(\hat{y}_0\hat{z}_2 + \frac{2}{3}\hat{y}_1\hat{z}_1 + \hat{y}_2\hat{z}_0 + \frac{2}{7}\hat{y}_2\hat{z}_2) - \gamma\hat{z}_2.
\end{aligned} \tag{36}$$

For the cases $k = 0, 1$, terms of higher order than $k$ in system (36) are set to zero. In what follows, we take exact values for the parameters in order to solve the equations numerically and find explicit stationary points for the cases $k = 0, 1, 2$, for the chosen values of parameters. Moreover, an analytical stability analysis is performed on some of the stationary points, using linearization techniques.

# 5 Numerical and Analytical Methods for the Uncertain Lotka–Volterra Model

The previously derived stochastic Galerkin-projected Lotka-Volterra system (35) incorporating uncertainty in the prey's growth parameter (3) is

$$\begin{aligned}
\frac{d\hat{y}_s}{dt} &= a\hat{y}_s(t) - \frac{\beta}{h_s}\sum_{i=0}^{k}\sum_{j=0}^{k} C_{sij}\hat{y}_i(t)\hat{z}_j(t) + b\frac{s}{2s-1}\hat{y}_{s-1}(t) + b\frac{s+1}{2s+3}\hat{y}_{s+1}(t)\\
\frac{d\hat{z}_s}{dt} &= \frac{\delta}{h_s}\sum_{i=0}^{k}\sum_{j=0}^{k} C_{sij}\hat{y}_i(t)\hat{z}_j(t) - \gamma\hat{y}_i(t), \quad \text{for s=0}, \dots, k.
\end{aligned}$$

## 5.1 Stability Analysis of Stationary Points for $k = 0, 1, 2$

In [8], a similar approach for uncertainty quantification using polynomial chaos expansion is used, and it even provides an example using the Lotka-Volterra equations. However, the article's goal is slightly different than of this report's, where the focus is on stationary points. Nevertheless, we use the same parameters as in [8]:

$$\begin{aligned}
&- a, b = 0.95, 0.05,\\
&- \beta, \delta, \gamma = 0.1, 0.75, 1.5,\\
&- y_0(t) = 10 \text{ (initial value of prey population)},\\
&- z_0(t) = 5 \text{ (initial values of predator population)},\\
&- t = 25 \text{ (time of simulation), discretized in 512 points .}
\end{aligned} \tag{37}$$

Note that in [8], uncertainty is assumed in both $\alpha$ and $\beta$ parameters, while this report focuses on only $\alpha-$uncertainty. Including uncertainty in the $\gamma-$parameter would provide similar results for the other animal species, namely the prey. Remember that $\gamma$ is the predator's mortality rate. Including uncertainty in either $\beta$ or $\delta$ would affect the long-term behavior of both animal species more than uncertainty in $\alpha$ and $\gamma$ since $\beta, \delta$ are the effects that the animal species have on each other.

In the following sections, the cases with $k = 0, 1, 2$ are investigated. While for $k = 0$ the computations for determining the exact stationary points depending on the parameters might be straightforward and do not require extensive work since the system (35) for $k = 0$ is the standard model (1), that is not the case for $k = 1$ and above. For example, for $k = 1$, the equations involve 4 coupled variables and the solution is more intricate. Similarly, complexity increases for higher truncation orders $k > 1$ that capture more detailed effects of uncertainty.

Thus, we use numerical tools to solve the system and find its stationary points. The programming languages used in this paper are Python and Mathematica. Python is used to find the values of the steady states and produce figures showing the evolution of the animal populations. It provides more than enough pre-existing functions and module extensions to solve the uncertain Lotka-Volterra equations and approximate the values of the stationary points for a given truncation order $k$. For example, Python's SciPy module includes the extension *special.legendre*, which directly provides the Legendre polynomials. This is helpful to compute the exact triple integral values $C_{sij}$ (24) using *scipy.integrate* with the *quad* extension. Mathematica is used to verify that the steady states indeed result in the zero right hand side of (35), and to obtain the eigenvalues of the Jacobian matrices explored in this section, using the specific set of parameters (37).

Before emerging into the analysis, note that for a given truncation order $k$ of the gPCE (28), the Stochastic Galerkin formulation (35) of the Lotka-Volterra model (1) consists of $2(k+1)$ coupled ODEs. This is because $\hat{y}_0, \ldots, \hat{y}_k$ and $\hat{z}_0, \ldots, \hat{z}_k$ in (28) give rise to $k+1$ and $k+1$ equations, respectively, for a finite $k$. The system (35) is nonlinear due to the bilinear interactions of the form $\hat{y}_i \hat{z}_j$ arising from the projections (28) onto the polynomial chaos basis [26].

In general, a nonlinear system of degree $d$ with $n$ variables can have up to $d^n$ unique stationary points [20]. When evaluating the stationary points, all derivatives are set to zero, i.e.,

$$\frac{d\hat{y}_s}{dt} = 0, \quad \frac{d\hat{z}_s}{dt} = 0, \quad \forall s = 0, \ldots, k.$$

Thus, for the quadratic system (35), we expect to have at most $2^{2(k+1)}$ unique stationary points. However, due to system symmetries, mode coupling structure and physical constraints such as non-negative populations, many of these equilibria are either non-physical or degenerate [20]. By degenerate we mean that the steady states might lack uniqueness or do not correspond to a meaningful or independent solution. By non-physical, we refer to the fact that we are considering animal populations, which cannot have negative values. While the system (35) has $2(k+1)$ variables and is quadratic, suggesting a theoretical upper bound of $2^{2(k+1)}$ equilibria, it is empirically observed that each Galerkin mode pair $\hat{y}_i \hat{z}_j$ tends to contribute a binary-like state at equilibrium, leading to a total of $2^{k+1}$ meaningful solutions [25].

In what follows, an analytical and numerical analysis is performed to determine the steady states and their stability of the uncertain Lotka-Volterra model (35) for truncation orders $k = 0, 1, 2$. Thus, in this paper's analysis we work with 2, 4, and 8 unique stationary points for the cases with $k = 0, 1$, and 2, respectively.

## Case study: k=0

For $k = 0$, the uncertain Lotka-Volterra system (35) is

$$\begin{aligned} \frac{d\hat{y}_0}{dt} &= a\hat{y}_0 - \beta\hat{y}_0\hat{z}_0, \\ \frac{d\hat{z}_0}{dt} &= \delta\hat{y}_0\hat{z}_0 - \gamma\hat{z}_0. \end{aligned} \tag{38}$$

A stationary point or steady state of (38), or similarly, for any value of $k$ is a point $(\hat{y}^*, \hat{z}^*)$

such that $\frac{d\hat{y}}{dt}(\hat{y}^*, \hat{z}^*) = \frac{d\hat{z}}{dt}(\hat{y}^*, \hat{z}^*) = 0$. Thus, set $\frac{d\hat{y}_0}{dt} = \frac{d\hat{z}_0}{dt} = 0$ and solve for $\hat{y}_0$ and $\hat{z}_0$:

$$\hat{y}_0(a - \beta\hat{z}_0) = 0 \iff \hat{y}_0 = 0 \text{ or } \hat{z}_0 = \frac{a}{\beta},$$
$$\hat{z}_0(\delta\hat{y}_0 - \gamma) = 0 \iff \hat{z}_0 = 0 \text{ or } \hat{y}_0 = \frac{\gamma}{\delta}.$$

Hence, the case with $k = 0$ has two steady states $(\hat{y}^*, \hat{z}^*)$:

$$\text{trivial point: } (0, 0), \qquad \text{non-trivial point: } \left(\frac{\gamma}{\delta}, \frac{a}{\beta}\right). \tag{39}$$

To perform the stability analysis of (39), we look at the Taylor expansion of (38) around each stationary point and ignore higher order terms as explained in [21]. First define $f$ as

$$\begin{cases} \frac{d\hat{y}_0}{dt} = a\hat{y}_0 - \beta\hat{y}_0\hat{z}_0 & = f_1(\hat{y}, \hat{z}) \\ \frac{d\hat{z}_0}{dt} = \delta\hat{y}_0\hat{z}_0 - \gamma\hat{z}_0 & = f_2(\hat{y}, \hat{z}) \end{cases} \Rightarrow f(\hat{y}, \hat{z}) = \begin{bmatrix} a\hat{y}_0 - \beta\hat{y}_0\hat{z}_0 \\ \delta\hat{y}_0\hat{z}_0 - \gamma\hat{z}_0 \end{bmatrix}. \tag{40}$$

Next, compute the Jacobian of $f$ (40) with respect to $\hat{y}$ and $\hat{z}$ and evaluate it at the stationary points,

$$J(\hat{y}, \hat{z}) = \begin{bmatrix} a - \beta\hat{z}_0 & -\beta\hat{y}_0 \\ \delta\hat{z}_0 & \delta\hat{y}_0 - \gamma \end{bmatrix},$$
$$J(0,0) = \begin{bmatrix} a & 0 \\ 0 & -\gamma \end{bmatrix}, \quad J\left(\frac{\gamma}{\delta}, \frac{a}{\beta}\right) = \begin{bmatrix} 0 & \frac{-\beta\gamma}{\delta} \\ \frac{a\delta}{\beta} & 0 \end{bmatrix}.$$

The linearization of the system is:

$$\begin{bmatrix} \dot{\hat{y}} \\ \dot{\hat{z}} \end{bmatrix} = \underbrace{J(\hat{y}^*, \hat{z}^*)}_{=:A} \begin{bmatrix} \hat{y} \\ \hat{z} \end{bmatrix}.$$

To determine the stability of a stationary point $(\hat{y}^*, \hat{z}^*)$, we examine the eigenvalues $\lambda \in \sigma(A)$, where $A$ is the Jacobian evaluated at a stationary point and $\sigma(A)$ is the spectrum of $A$, i.e., the set of all eigenvalues of matrix $A$. The change of notation is only for simplicity.

A stationary point, also called a steady state is said to be stable if a small perturbation of the solution from the fixed point decays in time; it is said to be unstable if a small perturbation grows in time [7, 21]. The steady state is stable if all the eigenvalues of the Jacobian matrix have a negative real part, and unstable if at least one eigenvalue has a positive real part. If $\mathrm{Re}(\lambda) = 0$, then the stationary point might be either stable or unstable, or neither simultaneously, in which case it is called semi-stable. A comprehensive explanation of the method and derivation is found in [21].

For the first case in (39), we investigate the eigenvalues $\lambda$ of $J(0, 0)$ and obtain $\lambda_1 = a > 0$ and $\lambda_2 = -\gamma < 0$, for $a, \gamma > 0$. As per the definition of stability given above, we conclude that the trivial stationary point $(0, 0)$ is unstable due to the positivity of one of the eigenvalues.

Similarly, we investigate the eigenvalues of $J(\frac{\gamma}{\delta}, \frac{a}{\beta})$ and obtain $\lambda^2 = -a\gamma$, so $\lambda_{1,2} = \pm\iota\sqrt{a\gamma}$, where $\iota = \sqrt{-1}$. This results in a semi-stable or neutral stable stationary point at $(\frac{\gamma}{\delta}, \frac{a}{\beta})$.

Note that these results correspond to the stationary points and their stability for the standard Lotka-Volterra model (1) without uncertainty, given in [3].

With the parameter values (37), we plot the evolution of the prey and predator populations for $k = 0$. Figure 3 depicts this evolution.

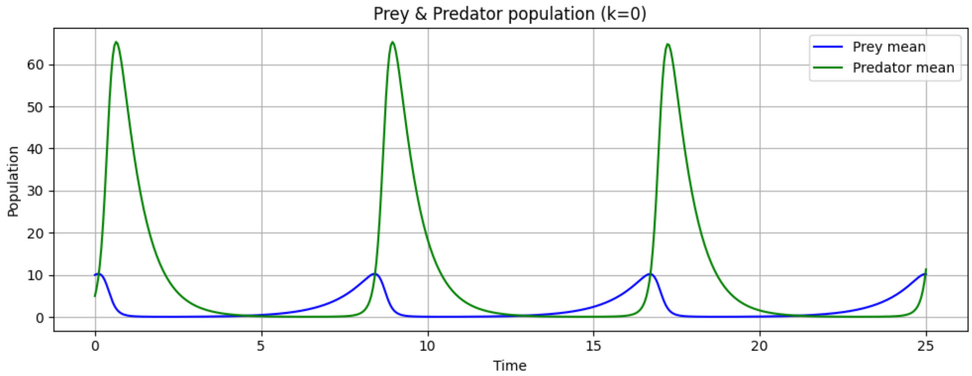


Figure 3: Prey & predator populations for a period of t=25, k=0.

Note that using the expressions (29) for $k=0$, the standard deviation is zero for both the prey and the predator populations, therefore Figure 3 only depicts the mean of the two species' populations.

Moreover, the stationary points are indeed $(0,0)$ and $(\frac{\gamma}{\delta}, \frac{a}{\beta}) = (2, 9.5)$ with the chosen values for parameters (37). As per the expressions (29), we have that $\mu_y = 2$ and $\mu_z = 9.5$ for the non-trivial case, and since there are no $y_1$ and $z_1$, standard deviation is indeed zero for both the trivial and non-trivial cases (39).

To properly analyze the behavior of the two species' populations and detect possible patterns, we further study the cases of the uncertain system (35) for $k=1$ and $k=2$ respectively.

## Case study: k=1

The uncertain Lotka-Volterra system (35) evaluated for $k=1$ is

$$\begin{aligned}
\frac{d\hat{y}_0}{dt} &= a\hat{y}_0 + \frac{b}{3}\hat{y}_1 - \beta(\hat{y}_0\hat{z}_0 + \frac{1}{3}\hat{y}_1\hat{z}_1), \\
\frac{d\hat{y}_1}{dt} &= a\hat{y}_1 + b\hat{y}_0 - \beta(\hat{y}_0\hat{z}_1 + \hat{y}_1\hat{z}_0), \\
\frac{d\hat{z}_0}{dt} &= \delta(\hat{y}_0\hat{z}_0 + \frac{1}{3}\hat{y}_1\hat{z}_1) - \gamma\hat{z}_0, \\
\frac{d\hat{z}_1}{dt} &= \delta(\hat{y}_0\hat{z}_1 + \hat{y}_1\hat{z}_0) - \gamma\hat{z}_1.
\end{aligned} \tag{41}$$

The equations yield $2^{k+1} = 4$ unique stationary points. To find all steady states $(\hat{y}^*, \hat{z}^*)$ of (41), we implement a numerical method that solves for the zeros of equations (41). That is, we set all equations (41) to zero and solve for $\hat{y}_0, \hat{y}_1, \hat{z}_0$ and $\hat{z}_1$. Using Mathematica, we obtain that the four steady states $(\hat{y}^*, \hat{z}^*)$ of the uncertain model (41) are

$$(\hat{y}_1^*, \hat{z}_1^*) = ([0,0],[0,0]), \tag{42}$$

$$(\hat{y}_2^*, \hat{z}_2^*) = \left(\left[\frac{\gamma}{\delta}, 0\right], \left[\frac{a}{\beta}, \frac{b}{\beta}\right]\right), \tag{43}$$

$$(\hat{y}_3^*, \hat{z}_3^*) = \left(\left[\frac{\gamma}{2\delta}, -\frac{\sqrt{3}\gamma}{2\delta}\right], \left[\frac{3a-\sqrt{3}b}{6\beta}, \frac{-\sqrt{3}a+b}{2\beta}\right]\right), \tag{44}$$

$$(\hat{y}_4^*, \hat{z}_4^*) = \left(\left[\frac{\gamma}{2\delta}, \frac{\sqrt{3}\gamma}{2\delta}\right], \left[\frac{3a+\sqrt{3}b}{6\beta}, \frac{\sqrt{3}a+b}{2\beta}\right]\right). \tag{45}$$

The numerical implementation showed that, as expected, one of the steady states of (41) is the trivial point (42), and what should be expected is that setting $\hat{y}_1 = \hat{z}_1 = 0$ combined with the non-trivial point given in (39) should also result in a stationary point. However, it was found that (43) is actually a steady state for (41). Therefore, to analyze the stability of (42) and (43) analytically, we define function $f$ in the following way:

$$\begin{cases} \frac{d\hat{y}_0}{dt} = a\hat{y}_0 + \frac{b}{3}\hat{y}_1 - \beta(\hat{y}_0\hat{z}_0 + \frac{1}{3}\hat{y}_1\hat{z}_1) & =: f_1(\hat{y},\hat{z}), \\ \frac{d\hat{y}_1}{dt} = a\hat{y}_1 + b\hat{y}_0 - \beta(\hat{y}_0\hat{z}_1 + \hat{y}_1\hat{z}_0) & =: f_2(\hat{y},\hat{z}), \\ \frac{d\hat{z}_0}{dt} = \delta(\hat{y}_0\hat{z}_0 + \frac{1}{3}\hat{y}_1\hat{z}_1) - \gamma\hat{z}_0 & =: f_3(\hat{y},\hat{z}), \\ \frac{d\hat{z}_1}{dt} = \delta(\hat{y}_0\hat{z}_1 + \hat{y}_1\hat{z}_0) - \gamma\hat{z}_1 & =: f_4(\hat{y},\hat{z}) \end{cases}$$

$$\Rightarrow f(\hat{y},\hat{z}) := \begin{bmatrix} a\hat{y}_0 + \frac{b}{3}\hat{y}_1 - \beta(\hat{y}_0\hat{z}_0 + \frac{1}{3}\hat{y}_1\hat{z}_1) \\ a\hat{y}_1 + b\hat{y}_0 - \beta(\hat{y}_0\hat{z}_1 + \hat{y}_1\hat{z}_0) \\ \delta(\hat{y}_0\hat{z}_0 + \frac{1}{3}\hat{y}_1\hat{z}_1) - \gamma\hat{z}_0 \\ \delta(\hat{y}_0\hat{z}_1 + \hat{y}_1\hat{z}_0) - \gamma\hat{z}_1 \end{bmatrix}. \tag{46}$$

The Jacobian of $f$ (46) evaluated at $(\hat{y}^*,\hat{z}^*)$ is then

$$J(\hat{y}^*,\hat{z}^*) = \begin{bmatrix} a-\beta z_0 & \frac{b}{3}-\frac{\beta}{3}z_1 & -\beta y_0 & -\frac{\beta}{3}y_1 \\ b-\beta z_1 & a-\beta z_0 & -\beta y_1 & -\beta y_0 \\ \delta z_0 & \frac{\delta}{3}z_1 & \delta y_0 - \gamma & \frac{\delta}{3}y_1 \\ \delta z_1 & \delta z_0 & \delta y_1 & \delta y_0 - \gamma \end{bmatrix}. \tag{47}$$

Next we evaluate the Jacobian matrix (47) at the trivial steady state (42) and at the non-trivial point (43):

$$J([0,0],[0,0]) = \begin{bmatrix} a & \frac{b}{3} & 0 & 0 \\ b & a & 0 & 0 \\ 0 & 0 & -\gamma & 0 \\ 0 & 0 & 0 & -\gamma \end{bmatrix}, \quad J\left(\left[\frac{\gamma}{\delta},0\right],\left[\frac{a}{\beta},\frac{b}{\beta}\right]\right) = \begin{bmatrix} 0 & 0 & \frac{-\beta\gamma}{\delta} & 0 \\ 0 & 0 & 0 & \frac{-\beta\gamma}{\delta} \\ \frac{\delta a}{\beta} & \frac{\delta b}{3\beta} & 0 & 0 \\ \frac{\delta b}{\beta} & \frac{\delta a}{\beta} & 0 & 0 \end{bmatrix}. \tag{48}$$

The linearization of the system is again

$$\begin{bmatrix} \dot{\hat{y}} \\ \dot{\hat{z}} \end{bmatrix} = J(\hat{y}^*,\hat{z}^*)\begin{bmatrix} \hat{y} \\ \hat{z} \end{bmatrix},$$

where now both $\hat{y}$ and $\hat{z}$ are 2-dimensional, hence the system is 4-dimensional. To determine the stability of each steady state, we investigate the eigenvalues of the Jacobian matrices given in (48).

The eigenvalues $\lambda$ of $J([0,0],[0,0])$ given in (48) are

$$\lambda_{1,2} = -\gamma < 0, \quad \lambda_{3,4} = \frac{1}{3}\left(3a \pm \sqrt{3}b\right). \tag{49}$$

Note that $\frac{1}{3}\left(3a \pm \sqrt{3}b\right) > 0$ since $3a > |\sqrt{3}b|$. Thus, we conclude that the trivial stationary point (42) is unstable due to the positivity of $\lambda_{3,4}$, agreeing with the case $k = 0$ that the trivial point is unstable.

For the non-trivial point (43), however, the analysis is more technical. Using Mathematica, we find that the eigenvalues $\lambda$ of $J\left(\left[\frac{\gamma}{\delta},0\right],\left[\frac{a}{\beta},\frac{b}{\beta}\right]\right)$ are

$$\begin{aligned} \lambda_1 &= -\frac{\sqrt{-3a\beta^2\gamma\delta^2 - \sqrt{3}b\beta^2\gamma\delta^2}}{\sqrt{3}\beta\delta}, \\ \lambda_2 &= \frac{\sqrt{-3a\beta^2\gamma\delta^2 - \sqrt{3}b\beta^2\gamma\delta^2}}{\sqrt{3}\beta\delta}, \\ \lambda_3 &= -\frac{\sqrt{-3a\beta^2\gamma\delta^2 + \sqrt{3}b\beta^2\gamma\delta^2}}{\sqrt{3}\beta\delta}, \\ \lambda_4 &= \frac{\sqrt{-3a\beta^2\gamma\delta^2 + \sqrt{3}b\beta^2\gamma\delta^2}}{\sqrt{3}\beta\delta}. \end{aligned} \tag{50}$$

Substitution with parameters (37) in (50) yields

$$\lambda \approx \begin{bmatrix} 1.212\iota \\ -1.212\iota \\ 1.175\iota \\ -1.175\iota \end{bmatrix}, \quad \iota = \sqrt{-1}. \tag{51}$$

Thus, since the Jacobian at the non-trivial point (43) given in (48) has purely imaginary eigenvalues with both positive and negative signs, the non-trivial steady state (43) is a center which is neither stable nor unstable. We conclude that the point (43) is semi-stable, agreeing with the semi-stable non-trivial stationary point for $k = 0$ given in (39).

Next, we are interested in the population evolution of the two species for $k = 1$ with the chosen parameter values (37). Figure 4 depicts this evolution.

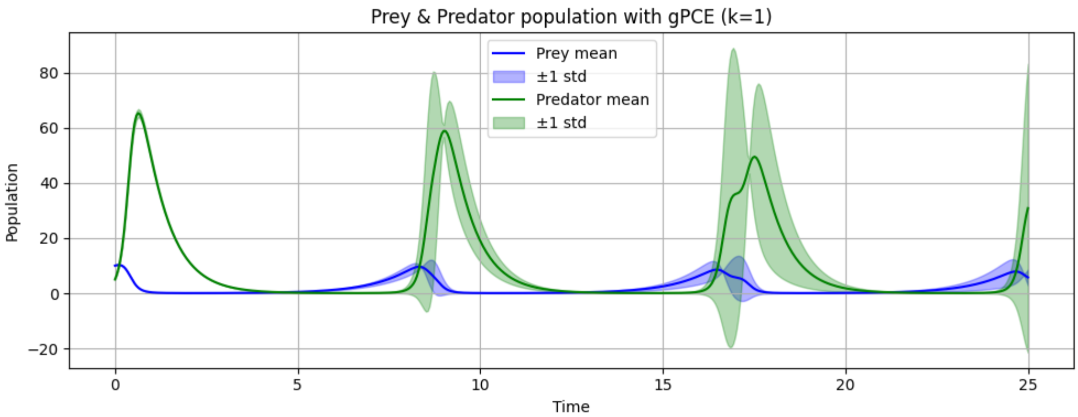


Figure 4: Prey & predator populations for a period of t=25, k=1.

Notice the shaded regions in Figure 4, representing the uncertainty, scaled by the standard deviation of each population. As seen in Figure 4, the predator population is affected by the uncertainty more than the prey. This is due to the fact that uncertainty is included in the prey's growth rate $\alpha$, as given in (3). If we were to include uncertainty in the predator's mortality rate $\gamma$ instead, the uncertainty would affect the prey population more than the predator's. This is left as further study, since the aim of this paper is to investigate uncertainty added in the Lotka-Volterra model (1) in the $\alpha-$parameter.

Table 1 expresses the stable states of (41) along with the mean $\mu$ and standard deviation $\sigma$ at each stationary point for the prey population $y$ and predator population $z$. Notice the two stationary points that were not explored above. All stationary points are found numerically using Python using five fixed points between 0 and 15 for both $y$ and $z$ and performing all possible combinations of the initial values and then running the program. This is the case for all simulations performed for different values of $k$.

| k=1 | Steady state $(\hat{y}^*, \hat{z}^*)$ | mean $\mu_y$ | s.d. $\sigma_y$ | mean $\mu_z$ | s.d. $\sigma_z$ |
|---|---|---|---|---|---|
| 1 | ([0.000, 0.000],[0.000, 0.000]) | 0.000 | 0.000 | 0.000 | 0.000 |
| 2 | ([1.000, 1.7321],[4.8943, 8.4772]) | 1.000 | 1.732 | 4.894 | 8.477 |
| 3 | ([2.000, 0.000],[9.500, 0.500]) | 2.000 | 0.000 | 9.500 | 0.500 |
| 4 | ([1.000, -1.7321],[4.6057, -7.9772]) | 1.000 | 1.732 | 4.606 | 7.977 |

Table 1: Steady states for $k = 1$.

Using the values of the steady states in Table 1, we can perform the stability analysis of points 2 (44) and 4 (45), which were not discussed previously. We examine the eigenvalues of the Jacobian matrix (47) at the steady states.

For the second stationary point in Table 1, corresponding to (44), using Mathematics, we obtain that the eigenvalues are approximately

$$\lambda \approx \begin{bmatrix} 0.921 \\ -1.5 \\ 1.212\iota \\ -1.212\iota \end{bmatrix}, \quad \iota = \sqrt{-1}. \tag{52}$$

Note that the real eigenvalues 0.921 and −1.5 are $\frac{1}{3}\left(3a - \sqrt{3}b\right)$ and $-\gamma$, respectively, obtaining a similarity to (49). Similarly, the last two eigenvalues in (52) correspond to the first two eigenvalues in (51). Due to the positivity of the first eigenvalue, the second steady point (44) is unstable for the set of parameters (37).

For the fourth and last point in Table 1, corresponding to (45), the eigenvalues of the Jacobian evaluated at that point are approximately

$$\lambda \approx \begin{bmatrix} 0.979 \\ -1.5 \\ 1.175\iota \\ -1.175\iota \end{bmatrix}, \quad \iota = \sqrt{-1}. \tag{53}$$

Note again that the real eigenvalues in (53) are $\frac{1}{3}\left(3a + \sqrt{3}b\right)$ and $-\gamma$ respectively as found for two of the eigenvalues for for the trivial point (42) in (49), and the last two eigenvalues in (53) correspond to the last two eigenvalues in (51). Thus, with a similar reasoning as for the second steady state in Table 1, given by (44), we conclude that the steady state (45) is unstable as well due to the positivity of $\frac{1}{3}\left(3a + \sqrt{3}b\right)$ for the parameters (37). This means that for $k = 1$ for the specific set of parameters (37), the system has three unstable and one semi-stable equilibria.

## Case study: k=2

To conclude the fact that there is indeed a pattern in stationary points and their stability, we compute the steady states of the uncertain Lotka-Volterra equations (35) for $k = 2$ and analyze their stability. The pattern we aim to show is that the trivial point is a steady state of the uncertain Lotka-Volterra model (35) for any truncation order $k$ and moreover, that it is unstable for all $k$. Similarly, we aim to show that the Jacobian matrix evaluated at each steady state has recurring eigenvalues for different truncation orders $k$. Thus, the uncertain Lotka-Volterra system (35) for $k = 2$ as given in (36) is:

$$\begin{aligned}
\frac{d\hat{y}_0}{dt} &= a\hat{y}_0 + \frac{1}{3}b\hat{y}_1 - \beta(\hat{y}_0\hat{z}_0 + \frac{1}{3}\hat{y}_1\hat{z}_1 + \frac{1}{5}\hat{y}_2\hat{z}_2), \\
\frac{d\hat{y}_1}{dt} &= a\hat{y}_1 + b\hat{y}_0 + \frac{2}{5}b\hat{y}_2 - \beta(\hat{y}_0\hat{z}_1 + \hat{y}_1\hat{z}_0 + \frac{2}{5}\hat{y}_1\hat{z}_2 + \frac{2}{5}\hat{y}_2\hat{z}_1), \\
\frac{d\hat{y}_2}{dt} &= a\hat{y}_2 + \frac{2}{3}b\hat{y}_1 - \beta(\hat{y}_0\hat{z}_2 + \frac{2}{3}\hat{y}_1\hat{z}_1 + \hat{y}_2\hat{z}_0 + \frac{2}{7}\hat{y}_2\hat{z}_2), \\
\frac{d\hat{z}_0}{dt} &= \delta(\hat{y}_0\hat{z}_0 + \frac{1}{3}\hat{y}_1\hat{z}_1 + \frac{1}{5}\hat{y}_2\hat{z}_2) - \gamma\hat{z}_0, \\
\frac{d\hat{z}_1}{dt} &= \delta(\hat{y}_0\hat{z}_1 + \hat{y}_1\hat{z}_0 + \frac{2}{3}\hat{y}_1\hat{z}_2 + \frac{2}{5}\hat{y}_2\hat{z}_1) - \gamma\hat{z}_1, \\
\frac{d\hat{z}_2}{dt} &= \delta(\hat{y}_0\hat{z}_2 + \frac{2}{3}\hat{y}_1\hat{z}_1 + \hat{y}_2\hat{z}_0 + \frac{2}{7}\hat{y}_2\hat{z}_2) - \gamma\hat{z}_2.
\end{aligned}$$

There are $2^{(k+1)} = 8$ unique stationary points corresponding to the equations above. The numerical computation of finding the zeros of system (36) has shown that two of these stationary points of the form $(\hat{y}^*, \hat{z}^*)$ are

$$(\hat{y}^*, \hat{z}^*) = ([y_0^*, y_1^*, y_2^*], [z_0^*, z_1^*, z_2^*]) = ([0,0,0],[0,0,0]) \quad \text{(trivial)}, \tag{54}$$

$$= \left(\left[\frac{\gamma}{\delta}, 0, 0\right], \left[\frac{a}{\beta}, \frac{b}{\beta}, 0\right]\right) \qquad \text{(non-trivial)}. \tag{55}$$

Thus, to analyze the stability of (54) and (55), we use the same method presented for cases $k = 0$ 5.1 and $k = 1$ 5.1. That is, using (36), define $f$ in the following way:

$$\begin{cases} \frac{d\hat{y}_0}{dt} = a\hat{y}_0 + \frac{1}{3}b\hat{y}_1 - \beta(\hat{y}_0\hat{z}_0 + \frac{1}{3}\hat{y}_1\hat{z}_1 + \frac{1}{5}\hat{y}_2\hat{z}_2) & =: f_1(\hat{y},\hat{z}), \\ \frac{d\hat{y}_1}{dt} = a\hat{y}_1 + b\hat{y}_0 + \frac{2}{5}b\hat{y}_2 - \beta(\hat{y}_0\hat{z}_1 + \hat{y}_1\hat{z}_0 + \frac{2}{5}\hat{y}_1\hat{z}_2 + \frac{2}{5}\hat{y}_2\hat{z}_1) & =: f_2(\hat{y},\hat{z}), \\ \frac{d\hat{y}_2}{dt} = a\hat{y}_2 + \frac{2}{3}b\hat{y}_1 - \beta(\hat{y}_0\hat{z}_2 + \frac{2}{3}\hat{y}_1\hat{z}_1 + \hat{y}_2\hat{z}_0 + \frac{2}{7}\hat{y}_2\hat{z}_2) & =: f_3(\hat{y},\hat{z}), \\ \frac{d\hat{z}_0}{dt} = \delta(\hat{y}_0\hat{z}_0 + \frac{1}{3}\hat{y}_1\hat{z}_1 + \frac{1}{5}\hat{y}_2\hat{z}_2) - \gamma\hat{z}_0 & =: f_4(\hat{y},\hat{z}), \\ \frac{d\hat{z}_1}{dt} = \delta(\hat{y}_0\hat{z}_1 + \hat{y}_1\hat{z}_0 + \frac{2}{3}\hat{y}_1\hat{z}_2 + \frac{2}{5}\hat{y}_2\hat{z}_1) - \gamma\hat{z}_1 & =: f_5(\hat{y},\hat{z}), \\ \frac{d\hat{z}_2}{dt} = \delta(\hat{y}_0\hat{z}_2 + \frac{2}{3}\hat{y}_1\hat{z}_1 + \hat{y}_2\hat{z}_0 + \frac{2}{7}\hat{y}_2\hat{z}_2) - \gamma\hat{z}_2 & =: f_6(\hat{y},\hat{z}) \end{cases}$$

$$\Rightarrow f(\hat{y},\hat{z}) := \begin{bmatrix} a\hat{y}_0 + \frac{1}{3}b\hat{y}_1 - \beta(\hat{y}_0\hat{z}_0 + \frac{1}{3}\hat{y}_1\hat{z}_1 + \frac{1}{5}\hat{y}_2\hat{z}_2) \\ a\hat{y}_1 + b\hat{y}_0 + \frac{2}{5}b\hat{y}_2 - \beta(\hat{y}_0\hat{z}_1 + \hat{y}_1\hat{z}_0 + \frac{2}{5}\hat{y}_1\hat{z}_2 + \frac{2}{5}\hat{y}_2\hat{z}_1) \\ a\hat{y}_2 + \frac{2}{3}b\hat{y}_1 - \beta(\hat{y}_0\hat{z}_2 + \frac{2}{3}\hat{y}_1\hat{z}_1 + \hat{y}_2\hat{z}_0 + \frac{2}{7}\hat{y}_2\hat{z}_2) \\ \delta(\hat{y}_0\hat{z}_0 + \frac{1}{3}\hat{y}_1\hat{z}_1 + \frac{1}{5}\hat{y}_2\hat{z}_2) - \gamma\hat{z}_0 \\ \delta(\hat{y}_0\hat{z}_1 + \hat{y}_1\hat{z}_0 + \frac{2}{3}\hat{y}_1\hat{z}_2 + \frac{2}{5}\hat{y}_2\hat{z}_1) - \gamma\hat{z}_1 \\ \delta(\hat{y}_0\hat{z}_2 + \frac{2}{3}\hat{y}_1\hat{z}_1 + \hat{y}_2\hat{z}_0 + \frac{2}{7}\hat{y}_2\hat{z}_2) - \gamma\hat{z}_2 \end{bmatrix}. \tag{56}$$

The Jacobian matrix of $f$ (56) evaluated at $(\hat{y},\hat{z})$ is

$$J(\hat{y},\hat{z}) = \begin{bmatrix} a-\beta\hat{z}_0 & \frac{b}{3}-\frac{\beta}{3}\hat{z}_1 & -\frac{\beta}{5}\hat{z}_2 & -\beta\hat{y}_0 & -\frac{\beta}{3}\hat{y}_1 & -\frac{\beta}{5}\hat{y}_2 \\ b-\beta\hat{z}_1 & a-\beta\hat{z}_0-\frac{2}{5}\beta\hat{z}_2 & \frac{2}{5}b-\frac{2}{5}\beta\hat{z}_1 & -\beta\hat{y}_1 & -\beta\hat{y}_0-\frac{2}{5}\beta\hat{y}_2 & -\frac{2}{5}\beta\hat{y}_1 \\ -\beta\hat{z}_2 & \frac{2}{3}b-\frac{2}{3}\beta\hat{z}_1 & a-\beta\hat{z}_0-\frac{2}{7}\beta\hat{z}_2 & -\beta\hat{y}_2 & -\frac{2}{3}\beta\hat{y}_1 & -\beta\hat{y}_0-\frac{2}{7}\beta\hat{y}_2 \\ \delta\hat{z}_0 & \frac{\delta}{3}\hat{z}_1 & \frac{\delta}{5}\hat{z}_2 & \delta\hat{y}_0-\gamma & \frac{\delta}{3}\hat{y}_1 & \frac{\delta}{5}\hat{y}_2 \\ \delta\hat{z}_1 & \delta\hat{z}_0+\frac{2}{3}\delta\hat{z}_2 & \frac{2}{5}\delta\hat{z}_1 & \delta\hat{y}_1 & \delta\hat{y}_0+\frac{2}{5}\delta\hat{y}_2-\gamma & \frac{2}{3}\delta\hat{y}_1 \\ \delta\hat{z}_2 & \frac{2}{3}\delta\hat{z}_1 & \delta\hat{z}_0+\frac{2}{7}\delta\hat{z}_2 & \delta\hat{y}_2 & \frac{2}{3}\delta\hat{y}_1 & \delta\hat{y}_0+\frac{2}{7}\delta\hat{y}_2-\gamma \end{bmatrix}. \tag{57}$$

Evaluation the Jacobian (57) at the steady states (54), (55) results in

$$J([0,0,0],[0,0,0]) = \begin{bmatrix} a & \frac{b}{3} & 0 & 0 & 0 & 0 \\ b & a & \frac{2}{3}b & 0 & 0 & 0 \\ 0 & \frac{2}{3}b & a & 0 & 0 & 0 \\ 0 & 0 & 0 & -\gamma & 0 & 0 \\ 0 & 0 & 0 & 0 & -\gamma & 0 \\ 0 & 0 & 0 & 0 & 0 & -\gamma \end{bmatrix}, \tag{58}$$

$$J\left(\left[\frac{\gamma}{\delta},0,0\right],\left[\frac{a}{\beta},\frac{b}{\beta},0\right]\right) = \begin{bmatrix} 0 & 0 & 0 & \frac{-\beta\gamma}{\delta} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-\beta\gamma}{\delta} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{-\beta\gamma}{\delta} \\ \frac{\delta a}{\beta} & \frac{\delta b}{\beta} & 0 & 0 & 0 & 0 \\ \frac{\delta b}{\beta} & \frac{\delta a}{\beta} & \frac{2}{5}\frac{\delta b}{\beta} & 0 & 0 & 0 \\ 0 & \frac{2}{3}\frac{\delta b}{\beta} & \frac{\delta a}{\beta} & 0 & 0 & 0 \end{bmatrix}. \tag{59}$$

The linearization of the system is

$$\begin{bmatrix} \dot{\hat{y}} \\ \dot{\hat{z}} \end{bmatrix} = J(\hat{y}^*,\hat{z}^*)\begin{bmatrix} \hat{y} \\ \hat{z} \end{bmatrix},$$

where $\hat{y}$ and $\hat{z}$ are 3-dimensional, hence the system is 6-dimensional.

The eigenvalues $\lambda$ of $J([0,0,0],[0,0,0])$ given in (58) are

$$\lambda_{1,2,3} = -\gamma > 0, \quad \lambda_4 = a > 0, \quad \lambda_{5,6} = \frac{1}{5}(5a \pm \sqrt{15}b). \tag{60}$$

Note that $\frac{1}{5}(5a \pm \sqrt{15}b) > 0$ since $5a > |\sqrt{15}b|$. Thus, we conclude that the trivial steady state (54) is unstable for $k = 2$ for the chosen set of parameters (37), agreeing with the cases $k = 0, 1$.

For the non-trivial steady state (55), as for case $k = 1$, the computation of the eigenvalues is more technical. Using Mathematica, we find that the eigenvalues $\lambda$ of $J\left(\left[\frac{\gamma}{\delta}, 0, 0\right], \left[\frac{a}{\beta}, \frac{b}{\beta}, 0\right]\right)$ are

$$\begin{aligned}
\lambda_1 &= \iota\sqrt{a\gamma}, \\
\lambda_2 &= -\iota\sqrt{a\gamma}, \\
\lambda_3 &= -\frac{\sqrt{-5a\beta^2\gamma\delta^2 - \sqrt{15}b\beta^2\gamma\delta^2}}{\sqrt{5}\beta\delta}, \\
\lambda_4 &= \frac{\sqrt{-5a\beta^2\gamma\delta^2 - \sqrt{15}b\beta^2\gamma\delta^2}}{\sqrt{5}\beta\delta}, \\
\lambda_5 &= -\frac{\sqrt{-5a\beta^2\gamma\delta^2 + \sqrt{15}b\beta^2\gamma\delta^2}}{\sqrt{5}\beta\delta}, \\
\lambda_6 &= \frac{\sqrt{-5a\beta^2\gamma\delta^2 + \sqrt{15}b\beta^2\gamma\delta^2}}{\sqrt{5}\beta\delta}.
\end{aligned} \tag{61}$$

Substitution with parameters (37) in (61) yields:

$$\lambda \approx \begin{bmatrix} 1.212\iota \\ -1.212\iota \\ 1.194\iota \\ -1.194\iota \\ 1.169\iota \\ -1.169\iota \end{bmatrix}, \quad \iota = \sqrt{-1}. \tag{62}$$

Thus, since the Jacobian at the non-trivial point (55) given in (59) has purely imaginary values with both positive and negative signs (62), the stable state (55) is a center, which is semi-stable for the set of parameters (37).

Figure 5 expresses the population evolution of the two species modeled by the uncertain Lotka-Volterra model (35) for $k = 2$.

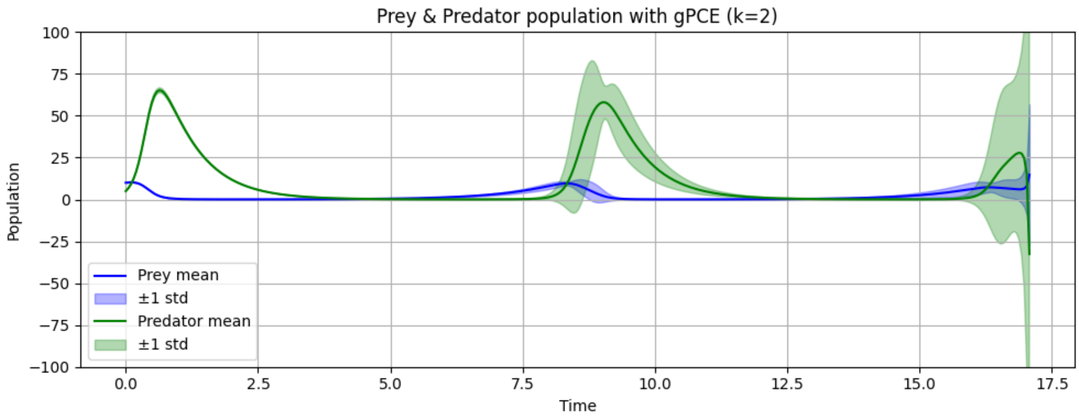


Figure 5: Prey & predator populations for a period of t=25, k=2.

In Figure 5 the shaded uncertainty region corresponds to the standard deviation of each animal population. Notice that as time increases, the predator population in green becomes increasingly sensitive to perturbations. Here, by perturbation we refer to the uncertainty included in the prey's evolution rate $\alpha$, as given in (3). Moreover, the analytical stability analysis was performed for the two steady states (54), (55), but the remaining six stationary points for $k = 2$, together with the respective mean and standard deviation at each equilibrium point are presented in Table 2

| k=2 | Steady state $(\hat{y}^*, \hat{z}^*)$ | mean $\mu_y$ | s.d. $\sigma_y$ | mean $\mu_z$ | s.d. $\sigma_z$ |
|---|---|---|---|---|---|
| 1 | ([0.000, 0.000, 0.000], [0.000, 0.000, 0.000]) | 0.000 | 0.000 | 0.000 | 0.000 |
| 2 | ([0.695, 0.034, 2.129], [3.305, 0.547, 10.116]) | 0.695 | 2.129 | 3.305 | 10.131 |
| 3 | ([0.491, 1.107, 1.288], [2.425, 5.512, 6.304]) | 0.491 | 1.699 | 2.424 | 8.373 |
| 4 | ([2.000, 0.000, 0.000], [9.500, 0.500, 0.000]) | 2.000 | 0.000 | 9.500 | 0.500 |
| 5 | ([1.509, 1.128, -1.259], [7.262, 5.607, -5.791]) | 1.509 | 1.690 | 7.262 | 8.061 |
| 6 | ([0.491, -1.128, 1.259], [2.238, -5.107, 5.791]) | 0.491 | 1.690 | 2.238 | 7.721 |
| 7 | ([1.509, -1.107, -1.288], [7.076, -5.012, -6.304]) | 1.509 | 1.699 | 7.076 | 8.053 |
| 8 | ([1.305, -0.034, -2.129], [6.195, -0.047, -10.116]) | 1.305 | 2.129 | 6.195 | 10.116 |

Table 2: Steady states for $k = 2$.

As for the case $k = 1$ in Section 5.1, a numerical analysis was performed to determine the stability of the remaining stationary points $2, 3, 5, 6, 7$ and 8 in Table 2. Using Mathematica, it was found that all the mentioned steady states are unstable due to the positivity of at least one eigenvalue of the Jacobian matrix (57) evaluated at each steady state.

We conclude that for the specific set of parameters (37), the uncertain Lotka-Volterra model (35) for $k = 2$ has one semi-stable and seven unstable equilibria.

## 5.2 Observed results

The uncertain Lotka-Volterra model (35) was analyzed for truncation orders $k = 0, 1, 2$. For these cases, steady states of the system have been both analytically and numerically derived, which presented some patterns in their values and stability for increasing orders $k$. Moreover, for a given truncation order $k$, the system (35) involves $2(k + 1)$ equations having $2^{k+1}$ steady states.

For $k = 0$, the uncertain Lotka-Volterra system (35) resulted in the two-dimensional ODE-system (38), which corresponds to the standard model (1) derived by the two authors, given in [3]. For the standard model, steady states together with their stability are known, and the model has one semi-stable and one unstable equilibrium derived in (39). Note that these results correspond to the specific set of parameters (37). Following the definitions (29) for mean and standard deviation with gPCE, we have that for $k = 0$, standard deviation is zero for both the prey and the predator populations. Therefore, Figure 3 depicts the evolution of the two species' populations by mean only. This is also consistent with the fact the for $k = 0$, the model with uncertainty is the standard Lotka-Volterra model (1), thus uncertainty does not affect the system for $k = 0$. Recall that the standard deviation is the effect of the uncertainty.

For $k = 1$, it is derived in Section 5.1 that the system (35) consists of the four-dimensional ODE-system (41). The system (41) has four steady states presented in Table 1. An analytical computation of the Jacobian matrix (47) of the right-hand side of the system (41), followed by a numerical computation evaluating the Jacobian at each steady state, showed that the uncertain system for $k = 1$ has one semi-stable and three unstable equilibria for the chosen set of parameters (37). The steady states for $k = 1$ in Table 1 include the trivial point, which was proven to be unstable, remaining consistent with the case $k = 0$. Furthermore, it has been found that some of the eigenvalues of the Jacobian matrix have the same values at different steady states, as explained in Section 5.1 for $k = 1$. The eigenvalues are (52), (51) and (53) for points $2, 3$ and 4, respectively, in Table 1. The steady state (43) has values comparable to the non-trivial steady state (39) in the case $k = 0$, and both points were found to be semi-stable due to the purely imaginary eigenvalues of the Jacobian evaluated at those points.

For the case $k = 0$ in Section 5.1, the system (35) consists of the six-dimensional ODE-system (36). The six equations result in 8 steady states presented in Table 2. Similar to the case $k = 1$, analytical and numerical computations were performed to determine the steady states and their stability. Consistent to the cases $k = 0, 1$, the trivial point (54) proved to be unstable for the parameters (37). Moreover, the non-trivial point (55) with similar values to the nontrivial point (39) and point (43) is semi-stable for $k = 2$ as well, for the parameters (37). We obtained numerically that the other six steady states that have not been explicitly explored for $k = 2$ in Table 2 are unstable due to the positivity of at least one eigenvalue of the

Jacobian (57) evaluated at each steady state. Thus, the uncertain Lotka-Volterra system (35) for $k = 2$ has one semi-stable and seven unstable equilibria for the specific set of parameters (37). Moreover, the eigenvalues (51) and (62) have similar values, showing another similarity between the steady states for varying truncation orders $k$.

Although the results are not presented in this paper, a numerical study was performed for the uncertain Lotka-Volterra system (35) for $k = 3$, which resulted in a consistent pattern with the conclusions made above. That is, the trivial point remains unstable, the non-trivial steady state (55) corresponding to $k = 3$ remains semi-stable, and all other steady states remain unstable for the specific set of parameters (37). Therefore, one can assume that the pattern persists for all truncation orders $k \geq 0$ for parameters (37).

Since the results are obtained when using specific values of prey, and respectively predator parameters (37), it is expected that the choice of these values also affects the stationary points. Moreover, there may exist direct relations between the parameters and stationary points. This hypothesis is tested in what follows.

# 6 Analysis on parameters

The analysis of the steady states of the uncertain Lotka-Volterra model (19) and their stability is performed for the specific set of parameters (37) given in [8]. We are interested in observing the effect, if any, that each parameter has on the values or stability of the steady states. Thus, we vary $\beta, \delta, \gamma, a$ and $b$ in (37) individually and analyze the values and stability type of the steady states.

## 6.1 Varying $\beta$

First, let us look at parameter $\beta$, originally chosen to be 0.1 in (37). Recall that $\beta$ is the effect that the predator population has on the prey's death rate, thus it is the hunting rate of the predator on the prey. We vary $\beta$ in such a way as to be able to detect a pattern in the possible changes in the results, by increasing or decreasing $\beta$ by a certain factor. Physically, this means that the predator is hunting more or less, respectively.

Take $\beta$ *doubled, tripled* and *halved*. This allows for an accurately drawn conclusion of whether there is a pattern in the change in values of the steady states.

### k=0

| $\beta$ | Steady state $(\hat{y}^*, \hat{z}^*)$ | mean $\mu_y$ | s.d. $\sigma_y$ | mean $\mu_z$ | s.d. $\sigma_z$ |
|---|---|---|---|---|---|
| 0.2 | (0.000, 0.000) | 0.000 | 0.000 | 0.000 | 0.000 |
| | (2.000, 4.750) | 2.000 | 0.000 | 4.750 | 0.000 |
| 0.3 | (0.000, 0.000) | 0.000 | 0.000 | 0.000 | 0.000 |
| | (2.000, 3.167) | 2.000 | 0.000 | 3.167 | 0.000 |
| 0.05 | (0.000, 0.000) | 0.000 | 0.000 | 0.000 | 0.000 |
| | (2.000, 19.000) | 2.000 | 0.000 | 19.000 | 0.000 |

Table 3: Steady states for k=0, varying $\beta$.

Table 3 presents the steady states for $k = 0$, varying parameter $\beta$. The observed pattern is that there is no change in the prey population at equilibrium, however the change in the predator population at equilibrium is inversely proportional to the change in $\beta$. That is, for doubling $\beta$ we observe that the predator population is halved, while when halving $\beta$, the population is doubled. Similarly, when increasing $\beta$ by a factor of 3, the stationary points decrease by the same factor. The pattern follows for any change in $\beta$. Therefore, we conclude that the $\beta-$parameter only affects the predator population.

Recall that $\beta$ is the effect of the predator on the prey. Thus at first, it might seem that changing the value of $\beta$ would affect the prey population and not the predator one. However, it is previously derived that the non-trivial equilibrium point for $k = 0$ is $(\frac{\gamma}{\delta}, \frac{a}{\beta})$, derived in (39). Therefore, at equilibrium, $\beta$ only affects $\hat{z}_0^*$, which explains the pattern seen in Table 3.

Moreover, we wish to determine whether the changes made affect the stability of the fixed points. Thus, we perform a stability analysis similar to Section 5.1 and observe that the stability type of the corresponding fixed points that were tested in Section 5.1 for $k = 0$ does not change. That is, the trivial point remains unstable, and the non-trivial point (39) explored is semi-stable, due to it being a center. In what follows, cases for $k = 1, 2$ are analyzed to strengthen the conclusions made.

## k=1

| $\beta$ | Steady state $(\hat{y}^*, \hat{z}^*)$ | mean $\mu_y$ | s.d. $\sigma_y$ | mean $\mu_z$ | s.d. $\sigma_z$ |
|---|---|---|---|---|---|
| 0.2 | ([0.000, 0.000],[0.000, 0.000]) | 0.000 | 0.000 | 0.000 | 0.000 |
| | ([1.000, 1.7321],[2.4472, 4.2386]) | 1.000 | 1.732 | 2.447 | 4.239 |
| | ([2.000, 0.000],[4.750, 0.250]) | 2.000 | 0.000 | 4.750 | 0.250 |
| | ([1.000, -1.7321],[2.3028 -3.9886]) | 1.000 | 1.732 | 2.303 | 3.989 |
| 0.3 | ([0.000, 0.000],[0.000, 0.000]) | 0.000 | 0.000 | 0.000 | 0.000 |
| | ([1.000, 1.7321],[1.6314, 2.8257]) | 1.000 | 1.732 | 1.631 | 2.826 |
| | ([2.000, 0.000],[3.1667, 0.1667]) | 2.000 | 0.000 | 3.167 | 0.167 |
| | ([1.000, -1.7321],[1.5352, -2.6591]) | 1.000 | 1.732 | 1.535 | 2.659 |
| 0.05 | ([0.000, 0.000],[0.000, 0.000]) | 0.000 | 0.000 | 0.000 | 0.000 |
| | ([1.000, 1.7321],[9.7887 16.9545]) | 1.000 | 1.732 | 9.789 | 16.954 |
| | ([2.000, 0.000],[19.000, 1.000]) | 2.000 | 0.000 | 19.000 | 1.000 |
| | ([1.000, -1.7321],[9.2113 -15.9545]) | 1.000 | 1.732 | 9.211 | 15.954 |

Table 4: Steady states for k=1, varying $\beta$.

As seen in Table 4, the observed pattern persists for $k = 1$, and the values of the prey population at equilibrium remain constant when $\beta$ is varied, while the ones of the predator population change inversely proportional to the parameter change. Moreover, the stability analysis resulted, as for $k = 0$, that the change in $\beta$ does not affect the stability type of the steady states.

Since for $k = 0$ the standard deviation is zero, there is no change observed in the standard deviation when varying $\beta$. The uncertain Lotka-Volterra system for $k = 1$ (41) allows us to comment on the changes observed in the standard deviation when varying $\beta$. As seen in Table 4, the standard deviation for the prey remains constant, similar to the prey's steady states. For the predator population however, we observe the same pattern of inverse proportion, when applying the expression (29). The changes in the mean for both populations are expected due to the straightforward expression (29).

## k=2

Table 5 depicts the evolution of the stationary points of the uncertain Lotka-Volterra model for $k = 2$ (36) when varying parameter $\beta$. As expected, the predator population at equilibrium varies inversely proportional to the changes in $\beta$, while the prey population is not affected. Similarly, the mean and standard deviation of both the prey and predator populations behave as for the case $k = 1$ when $\beta$ is varied computed with the expressions (29). Furthermore, there is no change in the stability of the steady states.

| $\beta$ | Steady state $(\hat{y}^*, \hat{z}^*)$ | mean $\mu_y$ | s.d. $\sigma_y$ | mean $\mu_z$ | s.d. $\sigma_z$ |
|---|---|---|---|---|---|
| 0.2 | ([0.000, 0.000, 0.000],[0.000, 0.000, 0.000]) | 0.000 | 0.000 | 0.000 | 0.000 |
| | ([0.6952, 0.0337, 2.1285],[1.6526, 0.2734, 5.0579]) | 0.695 | 2.129 | 1.653 | 5.065 |
| | ([0.491, 1.1074, 1.2882],[1.2122, 2.7558, 3.1518]) | 0.491 | 1.699 | 1.2122 | 4.187 |
| | ([2.000, 0.000, 0.000],[4.750, 0.250, 0.000 ]) | 2.000 | 0.000 | 4.750 | 0.250 |
| | ([1.509, 1.1275, -1.2587],[3.6309, 2.8036, -2.8956]) | 1.509 | 1.690 | 3.6309 | 4.030 |
| | ([0.491, -1.1275, 1.2587],[1.1191, -2.5536, 2.8956]) | 0.491 | 1.690 | 1.1191 | 3.861 |
| | ([1.3048 -0.0337 -2.1285],[3.0974 -0.0234 -5.057]) | 1.305 | 2.129 | 3.097 | 5.058 |
| | ([1.509 -1.1074 -1.2882],[3.5378 -2.5058 -3.1518]) | 1.509 | 1.699 | 3.538 | 4.027 |
| 0.3 | ([0.0000, 0.0000, 0.0000],[0.0000, 0.0000, 0.0000]) | 0.000 | 0.000 | 0.000 | 0.000 |
| | ([0.6952, 0.0337, 2.1285],[1.1017, 0.1823, 3.372 ]) | 0.695 | 2.129 | 1.102 | 3.377 |
| | ([0.491, 1.1074, 1.2882],[0.8082, 1.8372, 2.1012]) | 0.491 | 1.699 | 0.808 | 2.791 |
| | ([2.000, 0.000, 0.000],[3.1667, 0.1667, 0.0000]) | 2.000 | 0.000 | 3.167 | 0.167 |
| | ([1.509, 1.1275, -1.2587],[2.4206, 1.8691, -1.9304]) | 1.509 | 1.690 | 2.421 | 2.687 |
| | ([1.3048 -0.0337 -2.1285],[2.065, -0.0156, -3.372]) | 1.305 | 2.129 | 2.065 | 3.372 |
| | ([0.491, -1.1275, 1.2587],[0.746, -1.7024, 1.9304]) | 0.491 | 1.690 | 0.746 | 2.574 |
| | ([1.509 -1.1074 -1.2882],[2.3585 -1.6705, -2.1012]) | 1.509 | 1.699 | 2.359 | 2.684 |
| 0.05 | ([0.0000, 0.0000, 0.0000],[0.0000, 0.0000, 0.0000]) | 0.000 | 0.000 | 0.000 | 0.000 |
| | ([0.491, -1.1275, 1.2587],[4.4763, -10.2144 , 11.5822]) | 0.491 | 1.690 | 4.476 | 15.443 |
| | ([0.6952, 0.0337, 2.1285],[6.6102 , 1.0937, 20.2318]) | 0.695 | 2.129 | 6.610 | 20.261 |
| | ([0.491, 1.1074, 1.2882],[4.8489, 11.0231, 12.6073]) | 0.491 | 1.699 | 4.849 | 16.747 |
| | ([2.000, 0.000, 0.000],[19.000, 1.000, 0.000]) | 2.000 | 0.000 | 19.000 | 1.000 |
| | ([1.509, 1.1275, -1.2587],[14.5237, 11.2144, -11.5822]) | 1.509 | 1.690 | 2.421 | 16.122 |
| | ([1.509 -1.1074 -1.2882],[14.1511, -10.0231, -12.6073]) | 1.509 | 1.699 | 14.151 | 16.106 |
| | ([1.3048 -0.0337 -2.1285],[12.3898, -0.0937, -20.2318]) | 1.305 | 2.129 | 12.390 | 20.232 |

Table 5: Steady states for k=2, varying $\beta$.

## 6.2 Varying $\delta$

Next, we are interested in observing the behavior of the stationary points and their stability when varying parameter $\delta$. Recall that $\delta$ is the effect of the presence of prey on the predator's growth rate. Since $\beta$ and $\delta$ play symmetric roles in the coupling between the two species in the system (35) and the behavior of the system when varying $\beta$ is consistent for all orders $k = 0, 1, 2$ and all factors of $\beta$, we can assume that the pattern when increasing $\delta$ by a factor of 3 remains consistent to the factors of 2 and $\frac{1}{2}$. Thus, with a similar method as before, we choose to double and halve $\delta$. Recall that originally, $\delta$ is taken to be 0.75, as given in (37).

| $\delta$ | Steady state $(\hat{y}^*, \hat{z}^*)$ | mean $\mu_y$ | s.d. $\sigma_y$ | mean $\mu_z$ | s.d. $\sigma_z$ |
|---|---|---|---|---|---|
| 1.5 | ([0.000, 0.000, 0.000], [0.000, 0.000, 0.000]) | 0.000 | 0.000 | 0.000 | 0.000 |
| | ([0.3476, 0.0169, 1.0642], [3.3051, 0.5469, 10.1159]) | 0.348 | 1.064 | 3.305 | 10.131 |
| | ([0.2455, -0.5638, 0.6294], [2.2381, -5.1072, 5.7911]) | 0.245 | 0.845 | 2.238 | 7.721 |
| | ([0.2455, 0.5537, 0.6441], [2.4245, 5.5116, 6.3037]) | 0.245 | 0.849 | 2.424 | 8.373 |
| | ([1.000, 0.000, 0.000], [9.500, 0.500, 0.000]) | 1.000 | 0.000 | 9.500 | 0.500 |
| | ([0.7545, 0.5638, -0.6294], [7.2619, 5.6072, -5.7911]) | 0.755 | 0.845 | 7.262 | 8.061 |
| | ([0.6524, -0.0169, -1.0642], [6.1949, -0.0469, -10.1159]) | 1.305 | 2.129 | 6.195 | 10.116 |
| | ([0.7545, -0.5537, -0.6441], [7.0755, -5.0116, -6.3037]) | 0.755 | 0.849 | 7.076 | 8.053 |
| 0.375 | ([0.000, 0.000, 0.000], [0.000, 0.000, 0.000]) | 0.000 | 0.000 | 0.000 | 0.000 |
| | ([1.3904, 0.0675, 4.257], [3.3051, 0.5469, 10.1159]) | 1.390 | 4.257 | 3.305 | 10.131 |
| | ([0.9819, -2.2551, 2.5175], [2.2381, -5.1072, 5.7911]) | 0.982 | 3.380 | 2.238 | 7.721 |
| | ([0.9820, 2.2147, 2.5765], [2.4245, 5.5116, 6.3037]) | 0.982 | 3.398 | 2.424 | 8.373 |
| | ([4.000, 0.000, 0.000], [9.500, 0.500, 0.000]) | 4.000 | 0.000 | 9.500 | 0.500 |
| | ([3.0181, 2.2551, -2.5175], [7.2619, 5.6072, -5.7911]) | 3.018 | 3.380 | 7.262 | 8.061 |
| | ([2.6096, -0.0675, -4.257], [6.1949, -0.0469, -10.1159]) | 2.610 | 4.257 | 6.195 | 10.116 |
| | ([3.018, -2.2147, -2.5765], [7.0755, -5.0116, -6.3037]) | 3.018 | 3.398 | 7.076 | 8.053 |

Table 6: Steady states for k=2, varying $\delta$.

Looking at the first column in Table 6, we observe that in this case, the predator population at equilibrium remains constant for all values of $\delta$. Contrary to this, the prey population at equilibrium behaves inversely proportional to the change in $\delta$. That is, for $\delta$ doubled, the values are halved, and for $\delta$ halved, the values are doubled. The pattern is consistent for any change performed to the parameter. Therefore, we conclude that the $\delta-$parameter only affects the stationary points of the prey population.

Recall that $\delta$ is the effect of the prey on the predator. Therefore, intuitively one would expect that changing the value of $\delta$ would affect the values of the predator population and not the prey. However, as previously mentioned, the non-trivial point involves $(\frac{\gamma}{\delta}, \frac{a}{\beta})$ for $(\hat{y}_0^*, \hat{z}_0^*)$, explaining the behavior of both the fact that it only affects the population of the prey species, and that the relationship is inversely proportional due to $\delta$ appearing in the denominator.

Moreover, the mean and standard deviation obtained with the formulae (29) result in the expected values. The stability analysis performed on the stationary points proved that the stability type of the points remains consistent to the results obtained in Section 5.1 for any change in $\delta$.

## 6.3 Varying $\gamma$

Although the two cases above for varying $\beta$ and $\delta$ provide enough insight to make an educated guess of the effect that varying parameter $\gamma$ has on the stationary points and their stability, let us look at the case $k = 2$ of system (35) derived in (36), when $\gamma$ is doubled and halved. Recall that $\gamma$ is the predator's mortality rate. Therefore, physically this means that the predator population dies at a faster or respectively slower rate. Initially, it is taken to be 1.5, as given in (37).

| $\gamma$ | Steady state $(\hat{y}^*, \hat{z}^*)$ | mean $\mu_y$ | s.d. $\sigma_y$ | mean $\mu_z$ | s.d. $\sigma_z$ |
|---|---|---|---|---|---|
| 3.0 | ([0.000, 0.000, 0.000], [0.000, 0.000, 0.000]) | 0.000 | 0.000 | 0.000 | 0.000 |
| | ([1.3904, 0.0675, 4.2570], [3.3051, 0.5469, 10.1159]) | 1.390 | 4.257 | 3.305 | 10.131 |
| | ([0.9819, -2.2551, 2.5175], [2.2381, -5.1072, 5.7911]) | 0.982 | 3.380 | 2.238 | 7.721 |
| | ([0.9820, 2.2147, 2.5765], [2.4245, 5.5116, 6.3037]) | 0.982 | 3.398 | 2.424 | 8.373 |
| | ([4.000, 0.000, 0.000], [9.500, 0.500, 0.000]) | 4.000 | 0.000 | 9.500 | 0.500 |
| | ([3.0181, 2.2551, -2.5175], [7.2619, 5.6072, -5.7911]) | 3.018 | 3.380 | 7.262 | 8.061 |
| | ([2.6096, -0.0675, -4.257], [6.1949, -0.0469, -10.1159]) | 2.610 | 4.257 | 6.195 | 10.116 |
| | ([3.018, -2.2147, -2.5765], [7.0755, -5.0116, -6.3037]) | 3.018 | 3.398 | 7.076 | 8.053 |
| 0.75 | ([0.000, 0.000, 0.000], [0.000, 0.000, 0.000]) | 0.000 | 0.000 | 0.000 | 0.000 |
| | ([0.3476, 0.0169, 1.0642], [3.3051, 0.5469, 10.1159]) | 0.348 | 1.064 | 3.305 | 10.131 |
| | ([0.2455, -0.5638, 0.6294], [2.2381, -5.1072, 5.7911]) | 0.245 | 0.845 | 2.238 | 7.721 |
| | ([0.2455, 0.5537, 0.6441], [2.4245, 5.5116, 6.3037]) | 0.245 | 0.849 | 2.424 | 8.373 |
| | ([1.000, 0.000, 0.000], [9.500, 0.500, 0.000]) | 1.000 | 0.000 | 9.500 | 0.500 |
| | ([0.7545, 0.5638, -0.6294], [7.2619, 5.6072, -5.7911]) | 0.755 | 0.845 | 7.262 | 8.061 |
| | ([0.6524, -0.0169, -1.0642], [6.1949, -0.0469, -10.1159]) | 1.305 | 2.129 | 6.195 | 10.116 |
| | ([0.7545, -0.5537, -0.6441], [7.0755, -5.0116, -6.3037]) | 0.755 | 0.849 | 7.076 | 8.053 |

Table 7: Steady states for k=2, varying $\gamma$.

Notice that the predator population at equilibrium remains constant for all values of $\gamma$, while the prey population has a proportional relationship to the changes in $\gamma$. This is explained by the stationary value $\frac{\gamma}{\delta}$ of $\hat{y}_0^*$. With the same reasoning as for the cases when $\beta$ and $\delta$ are varied, using expressions (29), one obtains the values of the mean and the standard deviation at equilibrium as seen in Table 7. Moreover, the stability type of all steady states is consistent for any change in the $\gamma-$parameter.

Three out of the four parameters involved in the Lotka-Volterra model have been analyzed. This leaves parameter $\alpha$, through which uncertainty has been added to the system. In what follows, an analysis of how the remaining parameter affects the values of the stationary points and their stability is explored.

## 6.4 Varying $a$ and $b$

As given in (3), uncertainty is added to the Lotka-Volterra system (1) in the $\alpha-$parameter in the form $\alpha := a + b\omega$, with $\omega \in \mathcal{U}([-1, 1])$. Since $a$ and $b$ are assumed to be constants with the condition $a - |b| > 0$, they can be varied to observe the system's behavior at equilibrium, given that they satisfy the condition.

First, let us examine parameter $a$. Take $a$ to be doubled and halved, with the original value of 0.95, as given in (37).

| $a$ | Steady state $(\hat{y}^*, \hat{z}^*)$ | mean $\mu_y$ | s.d. $\sigma_y$ | mean $\mu_z$ | s.d. $\sigma_z$ |
|---|---|---|---|---|---|
| 1.9 | ([0.000, 0.000, 0.000], [0.000, 0.000, 0.000]) | 0.000 | 0.000 | 0.000 | 0.000 |
| | ([0.4909, -1.1229, 1.2657], [4.5696, -10.4184, 11.8371]) | 0.491 | 1.692 | 4.570 | 15.769 |
| | ([0.6956, 0.0168, 2.1296], [6.6095, 0.5467, 20.2344]) | 0.696 | 2.130 | 6.609 | 20.242 |
| | ([0.4909, 1.1129, 1.2804], [4.7559, 10.8229, 12.3496]) | 0.491 | 1.696 | 4.756 | 16.421 |
| | ([2.000, 0.000, 0.000], [19.000, 0.500, 0.000]) | 2.000 | 0.000 | 19.000 | 0.500 |
| | ([1.5091, 1.1229, -1.2657], [14.4304, 10.9184, -11.8371]) | 1.509 | 1.692 | 14.430 | 16.104 |
| | ([1.5091, -1.1129, -1.2804], [14.2441, -10.3229, -12.3496]) | 1.509 | 1.696 | 14.244 | 16.096 |
| | ([1.3044, -0.0168, -2.1296], [12.3905, -0.0467, -20.2344]) | 1.304 | 2.130 | 12.391 | 20.234 |
| 0.475 | ([0.000, 0.000, 0.000], [0.000, 0.000, 0.000]) | 0.000 | 0.000 | 0.000 | 0.000 |
| | ([0.4914, -1.1361, 1.2455], [1.0723, -2.4508, 2.7687]) | 0.491 | 1.686 | 1.072 | 3.698 |
| | ([0.6937, 0.0680, 2.1238], [1.6533, 0.5474, 5.0553]) | 0.694 | 2.125 | 1.653 | 5.085 |
| | ([0.4915, 1.0953, 1.3048], [1.2586, 2.8547, 3.2816]) | 0.492 | 1.704 | 1.259 | 4.349 |
| | ([2.000, 0.000, 0.000], [4.750, 0.500, 0.000]) | 2.000 | 0.000 | 4.750 | 0.500 |
| | ([1.5086, 1.1361, -1.2455], [3.6777, 2.9508, -2.7687]) | 1.509 | 1.686 | 3.678 | 4.046 |
| | ([1.5085, -1.0953, -1.3048], [3.4914, -2.3547, -3.2816]) | 1.508 | 1.704 | 3.491 | 4.039 |
| | ([1.3063, -0.068, -2.1238], [3.0967, -0.0474, -5.0553]) | 1.306 | 2.125 | 3.097 | 5.055 |

Table 8: Steady states for k=2, varying $a$.

As seen in Table 8, contrary to other parameters, there is no overall pattern in the changes in equilibrium values when varying $a$. Some of the predator values change proportionally to $a$, however this is not the case for all steady states. The stability type of the points however, is not affected by variations in parameter $a$. That is, most points, including the trivial equilibrium (54), are unstable, and one of the non-trivial steady states, namely (55) is semi-stable. Looking at the last column of the table, we observe that the standard deviation of the predator population is highly affected by the changes in $a$. Since $a$ is one of the parameters of the added uncertainty, this means that uncertainty highly affects the predator population at equilibrium. As further study, one could examine the effect of varying $a^2$. That is double or halve $a^2$ and analyze the results. However, this goes beyond of the scope of this paper.

We further study the effect that parameter $b$ has on the steady states of the system (36) and their stability. Similar to variations of the previous parameters, we choose to double and halve $b$. Recall that $b$ is originally 0.05, as given in (37).

| $b$ | Steady state $(\hat{y}^*, \hat{z}^*)$ | mean $\mu_y$ | s.d. $\sigma_y$ | mean $\mu_z$ | s.d. $\sigma_z$ |
|---|---|---|---|---|---|
| 0.1 | ([0.000, 0.000, 0.000], [0.000, 0.000, 0.000]) | 0.000 | 0.000 | 0.000 | 0.000 |
| | ([0.4914, -1.1361, 1.2455], [2.1446, -4.9015, 5.5373]) | 0.491 | 1.686 | 2.145 | 7.395 |
| | ([0.6937, 0.0680, 2.1238], [3.3066, 1.0947, 10.1105]) | 0.694 | 2.125 | 3.307 | 10.170 |
| | ([0.4915, 1.0953, 1.3048], [2.5173, 5.7093, 6.5631]) | 0.492 | 1.704 | 2.517 | 8.699 |
| | ([2.000, 0.000, 0.000], [9.500, 1.000, 0.000]) | 2.000 | 0.000 | 9.500 | 1.000 |
| | ([1.5086, 1.1361, -1.2455], [7.3554, 5.9015, -5.5373]) | 1.509 | 1.686 | 7.355 | 8.093 |
| | ([1.5085, -1.0953, -1.3048], [6.9827, -4.7093, -6.5631]) | 1.508 | 1.704 | 6.983 | 8.078 |
| | ([1.3063, -0.0680, -2.1238], [6.1934, -0.0947, -10.1105]) | 1.306 | 2.125 | 6.193 | 10.111 |
| 0.025 | ([0.000, 0.000, 0.000], [0.000, 0.000, 0.000]) | 0.000 | 0.000 | 0.000 | 0.000 |
| | ([0.4909, -1.1229, 1.2657], [2.2848, -5.2092, 5.9186]) | 0.491 | 1.692 | 2.285 | 7.885 |
| | ([0.6956, 0.0168, 2.1296], [3.3047, 0.2734, 10.1172]) | 0.696 | 2.130 | 3.305 | 10.121 |
| | ([0.4909, 1.1129, 1.2804], [2.378, 5.4115, 6.1748]) | 0.491 | 1.696 | 2.378 | 8.210 |
| | ([2.000, 0.000, 0.000], [9.500, 0.250, 0.000]) | 2.000 | 0.000 | 9.500 | 0.250 |
| | ([1.5091, 1.1229, -1.2657], [7.2152, 5.4592, -5.9186]) | 1.509 | 1.692 | 7.215 | 8.052 |
| | ([1.5091, -1.1129, -1.2804], [7.1220, -5.1615, -6.1748]) | 1.509 | 1.696 | 7.122 | 8.048 |
| | ([1.3044, -0.0168, -2.1296], [6.1953, -0.0234, -10.1172]) | 1.304 | 2.130 | 6.195 | 10.117 |

Table 9: Steady states for k=2, varying $b$.

We observe in Table 9 that similar to varying $a$, varying parameter $b$ affects the values of the steady states, but there is no clear pattern between the changes applied to $b$ and the changes seen in the values of the steady states. This applies to both the prey and predator populations. Furthermore, as observed for all parameters, the stability of the steady states is not affected by any change in $b$.

Further work could involve varying a linear combination of parameters $a$ and $b$. For example, one could investigate how changing $a + b^2$ by a given factor affects the steady states of the uncertain Lotka-Volterra system (35). However, as mentioned previously, this goes beyond the analysis in this paper.

# 7 Conclusion

The thesis investigated the classical Lotka-Volterra prey-predator model under parameter uncertainty, using the Stochastic Galerkin projection method based on generalized Polynomial Chaos Expansion (gPCE). Focusing on uncertainty in the prey's natural growth rate, the model was reformulated into a deterministic system of coupled ODEs, where solutions were derived for low truncation orders $k = 0, 1, 2$. This approach allows for an accurate prediction of how a biological model behaves in a real-world scenario and how equilibrium points behave in the overall system dynamics.

The results confirmed that introducing uncertainty increases the complexity of the system, not only in dimensionality but also in behavior. For all cases studied, the trivial point remains unstable, along with additional non-trivial steady states, aligning with existing deterministic analyses of the Lotka-Volterra model involving uncertainty [5, 13]. More interestingly, one recurring non-trivial point $(\frac{\gamma}{\delta}, \frac{a}{\beta})$ was found to remain consistently semi-stable, due to purely imaginary eigenvalues with flipped signs. This result is consistent with the oscillatory behavior of the classic Lotka-Volterra model and supports the hypothesis that uncertainty does not affect the stability of the stationary points.

The analysis for the higher truncation order cases showed that increased stochastic complexity leads to additional equilibrium points, most of which were found to be unstable. Furthermore, it was observed that for a given truncation order $k$, the uncertain ODE-system has dimension $2(k + 1)$, resulting in $2^{k+1}$ unique steady states [25]. These points arise from the nonlinear coupling introduced by the gPCE truncation and are a direct result of projecting nonlinear dynamics onto polynomial chaos bases, as also observed in other applications of the gPCE method [6, 26].

Moreover, a parameter sensitivity analysis proved that varying fixed population parameters $\beta, \delta, \gamma$ intuitively affects the population dynamics at equilibrium. However, the effects are only on one animal species at a time. That is, due to the known point $(\frac{\gamma}{\delta}, \frac{a}{\beta})$, the $\beta-$parameter only affects the predator population, while the $\delta, \gamma-$parameters only affect the prey population at equilibrium. Furthermore, it was seen that adding uncertainty to the system affects the deviation of the predator population significantly more that of the prey's. If, instead, uncertainty were added to the predator equation in (1) rather than the prey's, the observed variability would be expected to shift toward the prey population. These findings highlight the utility of the Stochastic Galerkin projection method for both quantifying parametric uncertainty and providing qualitative insight into nonlinear biological systems [5, 6].

In contrast to the fixed parameters study, the sensitivity analysis performed on the stochastic components $a$ and $b$ concluded in no clear patterns between changing the parameters and the population sizes at equilibrium. However, the increase in standard deviation, especially in the predator population, at equilibrium, proves that adding uncertainty to the model affects the predictability of animal populations at a given time. That is, introducing uncertainty into the prey equation of the Lotka–Volterra model significantly influences the long-term dynamics of the predator population.

While the analysis was limited to low truncation orders $k = 0, 1, 2$ due to high complexity, the observed patterns suggest a consistent behavior of the stochastic Lotka-Volterra model under uncertainty. This hypothesis is strengthened by the case study performed for $k = 3$.

Further work on the subject could explore the effect of having more than one uncertain parameter. This could be included either in parameter $\beta$, which would leave both parameters of the predator equation with fixed parameters $\delta$ and $\gamma$, or one of the two could include uncertainty, which would make both equations of the coupled system uncertain. Moreover, since there were no visible patterns in the steady states when varying the stochastic components $a$ and $b$, one could investigate the effect of varying a linear combination of the two. For example, a factor of $ab$ or $a + b^2$ could result in a clear pattern in the variations of the steady states. Additionally, Volterra generalized the standard model to $n$ species [24]. One could further investigate the generalized model with or without uncertainty.

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# A Python Code

```python
import numpy as np
from math import sqrt
from math import factorial as f
import matplotlib.pyplot as plt
from scipy.integrate import quad, solve_ivp
from scipy.optimize import root
from scipy.special import legendre
from itertools import product
from tqdm import tqdm

# SETTINGS
k = 2                                       # gPCE truncation order
a, b = 0.95, 0.05                            # alpha(omega) = a + b*omega
beta, delta, gamma = 0.1, 0.75, 1.5         #constant parameters

n = k + 1  # number of coefficients

# Compute C using integration quad from scipy
def compute_C_tensor(k):
    C = np.zeros((k+1, k+1, k+1))
    for s in range(k+1):
        for i in range(k+1):
            for j in range(k+1):
                C[s,i,j] = quad(legendre(s)*legendre(i)*legendre(j), -1, 1)[0]
    return C

C = compute_C_tensor(k)

def h(s):
    return 2 / (2*s+1)

# ODE system
def ode_system(t, U):
    Y = U[:n]
    Z = U[n:]
    dYdt = np.zeros_like(Y)
    dZdt = np.zeros_like(Z)

    for s in range(n):
        sum_C = sum(Y[i] * Z[j] * C[s, i, j] for i in range(n) for j in range(n))
        Y_splus1 = Y[s+1] if s + 1 <= k else 0
        Y_sminus1 = Y[s-1] if s - 1 >= 0 else 0

        dYdt[s] = a * Y[s] + b * (((s+1)/(2*s+3)) * Y_splus1 + (s / (2*s-1))
        * Y_sminus1) - beta / h(s) * sum_C
        dZdt[s] = delta / h(s) * sum_C - gamma * Z[s]

    return np.concatenate([dYdt, dZdt])

# Initial conditions
Y0 = np.zeros(k+1)
Z0 = np.zeros(k+1)
Y0[0] = 10
Z0[0] = 5
```

```python
U0 = np.concatenate([Y0, Z0])

# Time integration
t_span = (0, 25)
t_eval = np.linspace(*t_span, 512)

#Integrate and solve equations
sol = solve_ivp(ode_system, t_span, U0, t_eval=t_eval)
t = sol.t
Y_sol = sol.y[:n]
Z_sol = sol.y[n:]

# Compute mean and standard deviation
mean_y = Y_sol[0]
std_y = np.sqrt(np.sum(Y_sol[1:]**2, axis=0))
mean_z = Z_sol[0]
std_z = np.sqrt(np.sum(Z_sol[1:]**2, axis=0))

# Plot
plt.figure(figsize=(10, 4))
plt.plot(t, mean_y, label='Prey mean', color='blue')
plt.fill_between(t, mean_y - std_y, mean_y + std_y, color='blue', alpha=0.3,
label='±1 std')
plt.plot(t, mean_z, label='Predator mean', color='green')
plt.fill_between(t, mean_z - std_z, mean_z + std_z, color='green', alpha=0.3,
label='±1 std')
plt.xlabel("Time")
plt.ylabel("Population")
plt.title(f"Prey & Predator population with gPCE (k={k})")
plt.grid(True)
plt.legend()
plt.tight_layout()
plt.show()

# ----------- FIND ALL UNIQUE STATIONARY POINTS -----------

def stationary_system(U):
    return ode_system(0, U)

print(f"Values of a, b, beta, delta, gamma: {a,b,beta,delta,gamma}.")
print(f"k={k}.")

print("\n Searching for stationary points...")

num_guesses = 5  # control grid density (higher = more expensive)
Y_ranges = [np.linspace(0, 15, num_guesses) for _ in range(n)]
Z_ranges = [np.linspace(0, 15, num_guesses) for _ in range(n)]

found_points = []
tol = 1e-5  # tolerance for uniqueness

for y0 in tqdm(product(*Y_ranges)): #product:all combinations of guesses
#(ex: for k=2, 5^3 combinations); tqdm: shows progress (time & no of iterations)
    for z0 in product(*Z_ranges):
        U0 = np.concatenate([y0, z0])
        sol = root(stationary_system, U0, method='hybr')
```

```python
        if sol.success:
            U_stat = sol.x
            # Check uniqueness
            if all(np.linalg.norm(U_stat - p) > tol for p in found_points):
                found_points.append(U_stat)

print(f"\n Found {len(found_points)} unique stationary point(s):")
for i, U_stat in enumerate(found_points):
    Y_stat = U_stat[:n]
    Z_stat = U_stat[n:]
    mean_y = Y_stat[0]
    std_y = np.sqrt(np.sum(Y_stat[1:]**2))
    mean_z = Z_stat[0]
    std_z = np.sqrt(np.sum(Z_stat[1:]**2))

    print(f"\n Stationary Point {i+1}")
    print("Y:", np.round(Y_stat, 4))
    print("Z:", np.round(Z_stat, 4))
    print(f"Prey: mean = {mean_y:.3f}, std = {std_y:.3f}")
    print(f"Predator: mean = {mean_z:.3f}, std = {std_z:.3f}")
```