university of groningen / faculty of science and engineering

Tableau Calculus for Fuzzy Logic

Bachelor's Thesis Mathematics

July 18, 2025

Student: J.L.Q. Lin
First supervisor: Prof. J.Top
Second assessor: Prof. L.C.Verbrugge

This thesis explores a tableau calculus to prove the validity of statements for Basic Logic, a Fuzzy Logic framework characterized by the use of continuous t-norms. To introduce tableau methods, we first examine a tableau calculus for Classical Propositional Logic, establishing its soundness and completeness. Building on this foundation, we construct a tableau calculus for Basic Logic that is also sound and complete. This construction relies on the Decomposition Theorem, which states that any continuous t-norm can be expressed as an ordinal sum of the Łukasiewicz and product t-norms.

Contents

1 Introduction

Fuzzy Logic extends Classical Propositional Logic by allowing reasoning with degrees of truth, rather than limiting statements to being simply true or false. Within this framework, Basic Logic, which is a formal system based on continuous t-norms, plays a central role. Unlike Classical Propositional Logic, where propositions take truth values in the discrete set $\{0, 1\}$, Basic Logic allows truth values to range continuously over the entire unit interval $[0, 1]$.

The goal of this thesis is to develop proof system a systematic approach to determine the validity of logical formulas for Basic Logic called a tableau calculus. We show that this tableau system is both sound (it never proves something false) and complete (it can prove every semantically valid formula).

To establish the soundness and completeness of the tableau calculus for Basic Logic, we first formalize the language of Basic Logic by translating natural language statements into the logical formulas, starting with atomic sentences and building up through logical connectives illustrated by examples. Following that, we introduce logical validity and a tableau calculus for Classical Propositional Logic, and prove that this system is both sound and complete.

This foundation for Classical Propositional Logic provides a preliminary understanding of tableau calculus, which we aim to generalize and extend to Basic Logic. Building on this, we introduce key fuzzy logic concepts essential to Basic Logic, such as Łukasiewicz Fuzzy Logic, continuous t-norms, and the Decomposition Theorem. Using these tools, we develop a tableau calculus specifically designed for Basic Logic, and conclude with formal proofs of its soundness and completeness. The following chapters correspond to these steps and provide a detailed treatment of each topic.

2 Natural language to language of logic

The aim of this thesis is to design a systematic logic-based method confirm the validity of statements. To show the validity in a systematic approach, we need a consistent process to translate statements in our natural language into the language of logic. This formalization allows us to make use of proven theorems and inference methods to analyze and solve the problem at hand. This formalization consists of breaking down an inference into atomic sentences (to which we assign *truth values*) and logical connectives. Decomposing these statements into simpler terms allows us to analyze the inference in a formal framework. In the following subsections, we describe each component in detail.

2.1 Atomic sentences

This process starts with the most basic building blocks of logic: atomic sentences. These are the simplest expressions that are formed from symbols that represent objects and the properties or relations they have.
To define them formally, let us introduce the following definitions. The following definitions are adapted from [1].

Definition 2.1. An **individual constant** is a symbol used to refer to a specific fixed object. Individual constants are recognized by their initial lowercase letter.

Definition 2.2. A **predicate symbol** is a symbol used to express a property of individual constants, or a relation between individual constants. Predicate symbols are recognized by their initial uppercase letter.

Definition 2.3. A sentence formed by a predicate symbol followed by the correct number of individual constants is called an **atomic sentence**.

Example 2.4. Suppose we want to translate the natural language sentence:

"The weather is sunny outside."

Let us introduce the individual constant `weather` to refer to the weather outside, and the predicate symbol `Sunny` to express the property of being sunny. Then, the corresponding atomic sentence in formal logic is:

$$\texttt{Sunny}(\texttt{weather}).$$

Definition 2.5. To such an atomic sentence, we can assign a **truth value** to indicate its degree of truth. In classical logic, the set of truth values is typically $\{0, 1\}$, where 1 denotes truth and 0 denotes false. In many-valued logic, the set of truth values may include additional values to represent varying degrees of truth.

Definition 2.6. A subset of the set of truth values is called the set of **designated values**. An atomic sentence is considered true if its truth value lies within this designated subset.

2.2 Connectives

While atomic sentences allow us to express basic facts, real-world examples often involve a combination of these facts. For example, we may want to state that two things are true at once, that one thing implies another. To do this, we use logical connectives.

Logical connectives are symbols used to form complex sentences by combining atomic sentences (or complex sentences). In this section, we will discuss the following logical connectives: conjunction, disjunction, negation, and implication.

Definition 2.7. The connective **conjunction** is placed between two atomic sentences and is used to declare that both sentences are true. We will denote the conjunction by the symbol $\wedge$.

Definition 2.8. The connective **disjunction** is often placed between two atomic sentences and is used to declare that at least one of the sentences is true. We will denote the disjunction by the symbol $\vee$.

Definition 2.9. The connective **negation** is often placed before an atomic sentence and is used to declare that the antonym of the sentence is true. We will denote the negation by the symbol $\neg$.

Definition 2.10. The connective **implication** is often placed between two atomic sentences and is used to declare that if the first sentence is true, then the second sentence must also be true. We will denote the implication by the symbol $\rightarrow$.

Remark 2.11. Other sources, such as [2], also include the biconditional connective $\leftrightarrow$ and the falsity constant $\perp$. However, these can be expressed using the connectives we have already introduced:

$$A \leftrightarrow B = (A \rightarrow B) \wedge (B \rightarrow A), \quad \perp = A \rightarrow 0,$$

where A and B are arbitrary formulas, and 0 denotes falsity. Therefore, we do not consider these connectives.

2.3 Example

We continue Example 2.4 to illustrate the application of the logical tools developed in this thesis. This example will serve as a running case study throughout the remainder of the paper.

Example 2.12. Consider the inference: "Sunny weather implies no rain, or no rain implies sunny weather." In this inference, the atomic sentences are defined as follows:

- $p =$ "sunny weather",
- $q =$ "no rain".

These are connected using the logical connectives of implication and disjunction, resulting in the following expression:

$$(p \to q) \vee (q \to p).$$

3 Tableau Methods in Propositional Logic

Now that we have a consistent method for translating natural language statements into formal logic, we want to determine whether certain conclusions follow logically from given premises.

To accomplish this, we begin by defining the foundations of Classical Propositional Logic, which provides a way to evaluate logical consequences. We then introduce tableau calculus, which is a systematic method for evaluating logical consequences. Finally, we demonstrate their practical application through examples.

3.1 Logical Validity in Classical Propositional Logic

We begin by introducing basic definitions that define classical propositional logic, which is a logic where each formula is either true or false. These notions, inspired by [3], will are required for understanding the tableau calculus.

Definition 3.1. A **logic** is defined by a triple structure $\langle \mathcal{V}, \mathcal{D}, \{f_\sharp : \sharp \in \mathcal{C}\}\rangle$ where

- $\mathcal{V}$ is the set of truth values,
- $\mathcal{D}$ is a set of designated values (with $\mathcal{D} \subseteq \mathcal{V}$),
- $\mathcal{C}$ is the set of connectives,
- for each n-ary connective $\sharp \in \mathcal{C}$, there is a truth function $f_\sharp : \mathcal{V}^n \to \mathcal{V}$.

Definition 3.2. Consider the language $\mathcal{L}$ in Classical Propositional Logic based on only operators conjunction ($\wedge$), disjunction ($\vee$), negation ($\neg$), and implication ($\to$). Let $\mathcal{P} = \{p_i : i \in \mathbb{N}\}$ be the countable set of propositional atoms. A **well-formed formula** (wff) in $\mathcal{L}$ is inductively defined as follows:

(1) All propositional atoms p in $\mathcal{P}$ are wffs.

(2) If A and B are wffs, then so are $A \wedge B$, $A \vee B$, $\neg A$, and $A \to B$.

(3) Nothing is a wff unless it is generated by finitely many repeated applications of **(1)** and **(2)**.

Definition 3.3. Given a set of propositional atoms $\mathcal{P}$ and a logic $\langle \mathcal{V}, \mathcal{D}, \{f_\sharp : \sharp \in \mathcal{C}\}\rangle$, the **interpretation** is a function $v : \mathcal{P} \to \mathcal{V}$ which is extended to all formulas by an inductive definition as follows, using truth functions f_c for each n-ary connective c,

$$v(\sharp(A_1, \ldots, A_n)) = f_\sharp(v(A_1), \ldots, v(A_n)).$$

By the notion of Definition 3.1, Classical Propositional Logic is defined by the triple structure $\langle \mathcal{V}, \mathcal{D}, \{f_\sharp : \sharp \in \mathcal{C}\}\rangle$ where $\mathcal{V} = \{0,1\}$, $\mathcal{D} = \{1\}$, $\mathcal{C} = \{\wedge, \vee, \neg, \rightarrow\}$, equipped with truth functions $f_\wedge, f_\vee, f_\neg, f_\rightarrow$ which can be described by the truth tables

$f_\neg$	
0	1
1	0

$f_\wedge$	0	1
0	0	0
1	0	1

$f_\vee$	0	1
0	0	1
1	1	1

$f_\rightarrow$	0	1
0	1	1
1	0	1

Moreover, the extension of all interpretations $v : \mathcal{P} \rightarrow \mathcal{V}$ to all wffs is given by

$$\begin{aligned} v(A \wedge B) &= \min(v(A), v(B)), & v(A \vee B) &= \max(v(A), v(B)), \\ v(\neg A) &= 1 - v(A), & v(A \rightarrow B) &= \max(1 - v(A), v(B)). \end{aligned} \tag{1}$$

Definition 3.4. Let Σ be a finite set of wffs. We say A is a **logical consequence** of Σ ($\Sigma \models A$) iff every interpretation v has property that if $v(B) \in \mathcal{D}$ for all $B \in \Sigma$, then $v(A) \in \mathcal{D}$.
If A is a logical consequence of $\emptyset$, then A is called **logically valid**.

Proposition 3.5. Let Σ be a finite set of well-formed formulas and A be any wff. The following are equivalent:

1. A is a logical consequence of Σ.
2. $\left(\bigwedge_{B\in\Sigma} B\right) \rightarrow A$ is logically valid.

Proof. By De Morgan's laws, we know that $\Sigma \rightarrow A = \neg\Sigma \vee A$.
Moreover, from equation (1), we know that

$$v\left(\bigwedge_{B\in\Sigma} B\right) = \min_{B\in\Sigma}\{v(B)\}.$$

Combining both equalities, we know the following

$$\begin{aligned} v\left(\left(\bigwedge_{B\in\Sigma} B\right) \rightarrow A\right) = v\left(\neg\left(\bigwedge_{B\in\Sigma} B\right) \vee A\right) &= \max\left\{1 - v\left(\bigwedge_{B\in\Sigma} B\right), v(A)\right\} \\ &= \max\left\{1 - \min_{B\in\Sigma}\{v(B)\}, v(A)\right\}. \end{aligned} \tag{2}$$

$(\Rightarrow)$: Let A be a logical consequence of Σ. Notice the following:
if $v(B) = 1$ for all $B \in \Sigma$, then $v(A) = 1$. Else $v(B) = 0$ for some $B \in \Sigma$.
Inputting both cases in to (2), we obtain that $v\left(\left(\bigwedge_{B\in\Sigma} B\right) \rightarrow A\right) = 1$.

$(\Leftarrow)$: Assume that $v\left(\left(\bigwedge_{B\in\Sigma} B\right) \to A\right) = 1$. From equation 2, this is equivalent to $\max\left\{1 - \min_{B\in\Sigma}\{v(B)\}, v(A)\right\} = 1$. For this to be true, we need $v(A) = 1$ when $v(B) = 1$ for all $B \in \Sigma$. $\square$

Example 3.6. Recall Example 2.12, where we investigate whether the formula $(p \to q) \vee (q \to p)$ is logically valid. To determine this, let us observe the following computation.

$$\begin{aligned} v((p \to q) \vee (q \to p)) &= \max\{v(p \to q), v(q \to p)\} \\ &= \max\{\max\{1 - v(p), v(q)\}, \max\{1 - v(q), v(p)\}\}. \end{aligned}$$

Observe that if $v(p) = 1$, then the second term

$$\max\{1 - v(q), v(p)\} = 1.$$

On the other hand, if $v(p) = 0$, then the first term

$$\max\{1 - v(p), v(q)\} = 1.$$

Thus, the entire expression

$$\max\left\{\max\{1 - v(p), v(q)\}, \max\{1 - v(q), v(p)\}\right\} = 1.$$

regardless of the values of $v(p)$ and $v(q)$. Hence, the formula $(p \to q) \vee (q \to p)$ is logically valid.

3.2 Tableau Calculus for Classical Propositional Logic

As we saw in Example 3.6, we proved the logical validity of the inference through a case distinction. While this approach works in simple examples, increasing the complexity of Σ or A can quickly make the proof too cumbersome. To address this, we are interested in a more systematic method.

Instead of proving by case distinctions, we can proceed by contradiction on Definition 3.4. That is, we assume that there exists an interpretation v such that $v(B) \in \mathcal{D}$ for all $B \in \Sigma$, but $v(A) \notin \mathcal{D}$. We then attempt to expand this assumption in a tree-like structure, inspired by [3], in order to reach a contradiction. To formalize this process, we first define what we mean by a tree.

Definition 3.7. A **tree** has the structure that looks generally looks like this

$$
\begin{tikzcd}
 & & \cdot \arrow[d] & \\
 & & \cdot \arrow[dl] \arrow[dr] & \\
 & \cdot \arrow[dl] \arrow[dr] & & \cdot \\
 \cdot & & \cdot &
\end{tikzcd}
$$

The components of a tree are defined as follows:

- A **node** n is a single element in the tree.[1]
- The node with no predecessor is called the **root**.
- A node with no successor is called a **leaf**.
- A **branch**, denoted by $\mathcal{B}$, is a path from the root to leaf $l_{\mathcal{B}}$.
- The **height** of a node n on a branch $\mathcal{B}$ is the number of steps from n to the leaf $l_{\mathcal{B}}$, and is denoted by $h(n, \mathcal{B})$.

Definition 3.8. Given a finite set of formulas Σ and a formula A, we construct a **tableau** $\mathcal{T}$ whose root consists of all wffs in Σ together with $\neg A$. The tableau is then expanded from the root using tableau rules, which are defined below.

Definition 3.9. Let $\mathcal{E}$ be any set of wffs. The **tableau rules** for Classical Propositional Logic on the language $\mathcal{L}$ are given by:

$$
\begin{tikzcd}
\begin{array}{c}\mathcal{E}\\ A \wedge B\end{array} \arrow[d] \\
\begin{array}{c}\mathcal{E}\\ A\\ B\end{array}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \begin{array}{c}\mathcal{E}\\ \neg(A \wedge B)\end{array} \arrow[dl] \arrow[dr] & \\
\begin{array}{c}\mathcal{E}\\ \neg A\end{array} & & \begin{array}{c}\mathcal{E}\\ \neg B\end{array}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \begin{array}{c}\mathcal{E}\\ A \vee B\end{array} \arrow[dl] \arrow[dr] & \\
\begin{array}{c}\mathcal{E}\\ A\end{array} & & \begin{array}{c}\mathcal{E}\\ B\end{array}
\end{tikzcd}
\qquad
\begin{tikzcd}
\begin{array}{c}\mathcal{E}\\ \neg(A \vee B)\end{array} \arrow[d] \\
\begin{array}{c}\mathcal{E}\\ \neg A\\ \neg B\end{array}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\begin{array}{c}\mathcal{E}\\ \neg\neg A\end{array} \arrow[d] \\
\begin{array}{c}\mathcal{E}\\ A\end{array}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \begin{array}{c}\mathcal{E}\\ A \to B\end{array} \arrow[dl] \arrow[dr] & \\
\begin{array}{c}\mathcal{E}\\ \neg A\end{array} & & \begin{array}{c}\mathcal{E}\\ B\end{array}
\end{tikzcd}
\qquad
\begin{tikzcd}
\begin{array}{c}\mathcal{E}\\ \neg(A \to B)\end{array} \arrow[d] \\
\begin{array}{c}\mathcal{E}\\ A\\ \neg B\end{array}
\end{tikzcd}
$$

[1] In this thesis, each node contains a set of wffs and each node is generated by a tableau rule. This contrasts with [3], where each wff represents a node. This choice is necessary for our proof by induction on the height of a node in Theorem 4.51.

With the tableau rules in place, we need to determine whether a given tableau leads to a contradiction. To do this, we introduce the definitions that allow us to conclude when a tableau leads to a contradiction.

Definition 3.10. A branch of a tableau is **closed** iff there are formulas of the form A and $\neg A$ on two of its nodes. We denote a closed branch by the symbol $\otimes$. If all branches are closed, then the tableau itself is **closed**.

Definition 3.11. A branch of a tableau is **complete** iff it is closed or all tableau rules which can be applied on the branch have been applied. If all branches are complete, then the tableau itself is **complete**.

Definition 3.12. If there exists a closed and complete tree whose root consists of the formulas in Σ together with the negation of A, then A is called a **proof-theoretic consequence** of Σ, which we denote by $\Sigma \vdash A$.
If A is a proof-theoretic consequence of $\emptyset$, then A is **provable**.

Proposition 3.13. Let Σ be a finite set of wffs and A be any wff. The following are equivalent:

- A is a proof-theoretic consequence of Σ.
- $\left(\bigwedge_{B\in\Sigma} B\right) \to A$ is provable.

Proof. This can be deduced by observing the tableau rules for negated implication and conjunction in Definition 3.9, which enable us to transform one root into the other. $\square$

Example 3.14. The tableau calculus offers a systematic way of testing whether an statement is provable. To illustrate how this method works in practice, let us revisit the statement from Example 3.6

$$(p \to q) \vee (q \to p),$$

We will construct a tableau with the negation of this formula at the root and systematically apply the tableau rules to determine whether the tableau closes.

$$
\begin{array}{c}
\neg((p \to q) \vee (q \to p)) \\
\Big\downarrow \\
\neg(p \to q) \\
\neg(q \to p) \\
\Big\downarrow \\
p \\
\neg q \\
\neg(q \to p) \\
\downarrow \\
p \\
\neg q \\
q \\
p \\
\otimes
\end{array}
$$

As this tableau only develops in a single closed branch, this implies that the formula is indeed provable.

3.3 Soundness & Completeness for Classical Propositional Logic

Notice that Examples 3.6 and 3.14 show that the inference introduced in Example 2.12 is both logically valid and provable. A natural question that arises is whether logical validity and provability always coincide.

Intuitively, the connection makes sense. If we construct a tableau whose initial list consists of the premises together with the negation of the conclusion, and the tableau closes completely, then we have shown that this combination leads to a contradiction. Since the premises are assumed to be true, it must be the negation of the conclusion that is false. Hence, the conclusion must be true.

Conversely, if an inference is logically valid, we expect that any tableau constructed from the premises and the negation of the conclusion will eventually close.

To prove both relations, we follow a similar approach to that of [3], where mathematical statements about 'faithfulness' and 'interpretations induced by branches' are introduced first, as these play a vital role in proving the soundness and completeness of our tableau calculus.

Definition 3.15. Let v be any interpretation and let $\mathcal{B}$ be a branch of a tableau. We say that v is **faithful** to $\mathcal{B}$ if and only if, for every wff A on $\mathcal{B}$, it holds that $v(A) = 1$.

The following lemma plays a central role in the proof of the Soundness Theorem.

Lemma 3.16. Let v be an interpretation faithful to a branch $\mathcal{B}$. If a tableau rule from Definition 3.9 is applied to $\mathcal{B}$, then v is faithful to at least one of the branches generated by this application.

Proof. The proof is an induction of all the tableau rules mentioned in Definition 3.9 using the extension of the interpretation v in (1).

$(\wedge)$: suppose that v is faithful to $\mathcal{B}$ containing $A \wedge B$. In other words, $v(A \wedge B) = 1$. On the same branch, we obtain A and B after applying the tableau rules. Moreover, for $1 = v(A \wedge B) = \min(v(A), v(B))$ we need $v(A) = 1$ and $v(B) = 1$. Hence, v makes all formulas on $\mathcal{B}$ true.

$\neg(\wedge)$: suppose that v is faithful to $\mathcal{B}$ containing $\neg(A \wedge B)$. In other words, $v(\neg(A \wedge B)) = 1$. Applying the tableau rules, we obtain two branches. One containing $\neg A$ and the other containing $\neg B$. Moreover, for $1 = v(\neg(A \wedge B)) = 1 - \min(v(A), v(B))$, we need $v(A) = 0$ or $v(B) = 0$. Hence, v makes all formulas on $\mathcal{B}$ true.

$(\vee)$: suppose that v is faithful to $\mathcal{B}$ containing $A \vee B$. In other words, $v(A \vee B) = 1$. Applying the tableau rules, we obtain two branches. One containing A and the other containing B. For $v(A \vee B) = 1$, we need $v(A) = 1$, or $v(B) = 1$. In the first case, v is faithful to the left branch. In the second, it is faithful to the right.

$\neg(\vee)$: suppose that v is faithful to $\mathcal{B}$ containing $\neg(A \vee B)$. In other words, $v(\neg(A \vee B)) = 1$. On the same branch, we obtain $\neg A$ and $\neg B$ after applying the tableau rules. Moreover, for $1 = v(\neg(A \vee B)) = 1 - \max(v(A), v(B))$ we need $v(\neg A) = 1$ and $v(\neg B) = 1$. Hence, v makes all formulas on $\mathcal{B}$ true.

$\neg(\neg)$: suppose that v is faithful to $\mathcal{B}$ containing $\neg\neg A$. In other words, $v(\neg\neg A) = 1$. On the same branch, we obtain A after applying the tableau rules. Moreover, for $1 = v(\neg\neg A) = 1 - (1 - v(A))$ we need $v(A) = 1$. Hence, v makes all formulas on $\mathcal{B}$ true.

$(\rightarrow)$: suppose that v is faithful to $\mathcal{B}$ containing $A \rightarrow B$. In other words, $v(A \rightarrow B) = 1$. Applying the tableau rules, we obtain two branches. One containing $\neg A$ and the other containing B. For $v(A \rightarrow B) = 1$, we need $v(\neg A) = 1$, or $v(B) = 1$. In the first case, v is faithful to the left branch. In the second, it is faithful to the right.

$\neg(\rightarrow)$: suppose that v is faithful to $\mathcal{B}$ containing $\neg(A \rightarrow B)$. In other words, $v(\neg(A \rightarrow B)) = 1$. On the same branch, we obtain A and $\neg B$ after applying the tableau rules. Moreover, for $1 = v(\neg(A \rightarrow B)) = 1 - \max(1 - v(A), v(B))$ we need $v(A) = 1, v(\neg B) = 1$. Hence, v makes all formulas on $\mathcal{B}$ true.

Notice that expanding a branch using the tableau rules in Definition 3.9 will always lead to at least one faithful branch. This concludes the proof. $\square$

Theorem 3.17 (Soundness Theorem)**.** For finite Σ, if A is a proof-theoretic consequence of Σ, then A is a logical consequence of Σ.

Proof. We prove the contrapositive. Suppose that $\Sigma \not\models A$. Then there is an interpretation, v, which makes all premises in Σ true, and A false. Now consider a completed tableau for the inference. As v is faithful to the initial list, we can use a repeated application of the Lemma 3.16 so that we can obtain at least one branch $\mathcal{B}$ to which v is faithful. For $\mathcal{B}$ to be closed, it would have to contain some formulas of the form A and $\neg A$. By the faithfulness of v, we must have $v(A) = v(\neg A) = 1$, which cannot be the case. Hence, we arrive at a contradiction, and thus, is open. In other words, $\Sigma \not\vdash A$. □

Now that we have proven that a proof-theoretic consequence leads to a logical equivalence, we also want to show the other way. For that, we require the following preliminary results.

Definition 3.18. Let $\mathcal{B}$ be an open branch of a tableau. The **interpretation induced by** $\mathcal{B}$ is any interpretation, v, such that for every propositional atom, p, if p appears on $\mathcal{B}$, then $v(p) = 1$, and if $\neg p$ appears on $\mathcal{B}$, then $v(p) = 0$. If p does not appear at all, then $v(p)$ can be anything.

Lemma 3.19. Let $\mathcal{B}$ be an open complete branch of a tableau and let v be the interpretation induced by $\mathcal{B}$. Then the following implications hold.

- if A is on $\mathcal{B}$, then $v(A) = 1$.
- If $\neg A$ is on $\mathcal{B}$, then $v(A) = 0$.

Proof. Let v be the interpretation induced by $\mathcal{B}$. The proof is based on the induction of the composition of A.

Base case: if A is a propositional atom, then by Definition 3.18, $v(A) = 1$ when A appears on $\mathcal{B}$ and $v(A) = 0$ when $\neg A$ appears on $\mathcal{B}$

Induction hypothesis: let B, C be any wff. If B, C appear on $\mathcal{B}$, then $v(B) = v(C) = 1$. If $\neg B, \neg C$ appear on $\mathcal{B}$, then $v(B) = v(C) = 0$.

Induction step: let A be one of the following wffs:

$$B \wedge C, \neg(B \wedge C), B \vee C, \neg(B \vee C), \neg B, \neg(\neg B), B \to C, \neg(B \to C).$$

$A = B \wedge C$: since $\mathcal{B}$ is complete, the conjunction rule has been applied. Hence, both B and C are on the branch. By induction hypothesis, $v(B) = v(C) = 1$. Hence, $v(A) = v(B \wedge C) = 1$.

$A = \neg(B \wedge C)$: since $\mathcal{B}$ is complete, the negated conjunction rule has been applied. Hence, $\neg B$ or $\neg C$ is on the branch. By induction hypothesis, $v(B) = 0$ or $v(C) = 0$. In either case, $v(A) = v(\neg(B \wedge C)) = 0$.

$A = B \vee C$: since $\mathcal{B}$ is complete, the disjunction rule has been applied. Hence, B or C are on the branch. By induction hypothesis, $v(B) = 1$ or $v(C) = 1$. Either case, $v(A) = v(B \vee C) = 1$.

$A = \neg(B \vee C)$: since $\mathcal{B}$ is complete, the negated disjunction rule has been applied. Hence, $\neg B$ and $\neg C$ appear on the branch. By induction hypothesis, $v(B) = 0$ and $v(C) = 0$. Therefore, $v(A) = v(\neg(B \vee C)) = 0$.

$A = \neg B$: since $\neg B$ is on $\mathcal{B}$. Then, our induction hypothesis and construction of v, $v(B) = 0$. Hence, $v(A) = v(\neg B) = 1$.

$A = \neg(\neg B)$: $v(A) = V(\neg(\neg B)) = v(B) = 1$.

$A = \neg(B \rightarrow C)$: since $\mathcal{B}$ is complete, the negated implication rule has been applied. Hence, both B and $\neg C$ are on the branch. By induction hypothesis, $v(B) = 1$ and $v(C) = 0$. Therefore, $v(A) = v(\neg(B \rightarrow C)) = 0$.

$A = B \rightarrow C$: since $\mathcal{B}$ is complete, the implication rule has been applied. Hence, $\neg B$ or C is on the branch. By induction hypothesis, $v(B) = 1$ or $v(C) = 0$. Hence, $v(A) = v(B \rightarrow C) = 1$.

As the induction step hold, we can assume that if A is on b, then $v(A) = 1$, and if $\neg A$ is on b, then $v(A) = 0$, where v is induced by $\mathcal{B}$. □

This Lemma lets us proof the completeness of our tableau calculus in a straightforward way.

Theorem 3.20 (Completeness Theorem)**.** For finite Σ, if A is a logical consequence of Σ then A is a proof-theoretic consequence of Σ.

Proof. We prove the contrapositive. Suppose that $\Sigma \nvdash A$. Consider an open and complete tableau for the inference and choose an open branch. Lemma 3.19 tells us that there exists an induced interpretation that makes all the members of Σ true and A false. Hence, $\Sigma \nvDash A$. □

Theorems 3.17 and 3.20 show that our tableau calculus for Classical Propositional Logic is both sound and complete. That is, $\Sigma \vDash A$ if and only if $\Sigma \vdash A$. As a result, our tableau calculus provides a systematic and reliable procedure to determine the validity of propositional arguments.

4 Fuzzy Logic

In Classical Propositional Logic, atomic sentences are either true or false. For instance, in Example 2.12, we considered the inference:

> "Sunny weather implies no rain or no rain implies sunny weather."

Here, atomic sentences are assigned a truth value from the set $\{0, 1\}$, where 0 represents falsehood and 1 represents truth.

However, not all propositions in our natural language are either absolutely true or absolutely false. Consider instead the sentence "The weather is warm." Unlike "sunny weather," the notion of warmth is vague. For example, we might agree that

- if it is 0°C outside, then "It is warm" is clearly false, so we assign it a truth value of 0,
- if it is 30°C outside, then "It is warm" is clearly true, so we assign it a truth value of 1,
- but if it is x°C for some x with $0 < x < 30$, then it would make sense for the truth value of "It is warm" to lie somewhere between 0 and 1.

To account for such degrees of truth, we turn to many-valued logics, for which we allow more than just the two truth values 0 and 1. A particularly well-known and useful form of many-valued logic is Fuzzy Logic, in which the set of truth values is the interval $[0, 1]$. This allows for a smooth representation of vagueness.

Unlike Classical Propositional Logic, where only the designated value is 1, in Fuzzy Logic our designated values is the interval $[\epsilon, 1] : \epsilon > 0$. This corresponds to the idea that a proposition may be true enough even if its truth value is not exactly 1.

The only component still missing from our Definition 3.1 is the set of connectives along with their corresponding truth functions. However, in Fuzzy Logic the choice of these truth functions is not fixed in advance. Instead, it depends on a function known as a t-norm.

To introduce the notion of a t-norm and how the logical connectives can be constructed from it, we now turn our attention to a well-studied example of Fuzzy Logic: Łukasiewicz many-valued logic.

4.1 Łukasiewicz Fuzzy Logic

In Łukasiewicz Fuzzy Logic, the truth functions for the basic connectives are defined as follows:

$$\begin{aligned} f_\wedge(x, y) &= \min(x, y), \\ f_\vee(x, y) &= \max(x, y), \\ f_\neg(x) &= 1 - x, \\ f_\to(x, y) &= \min(1, 1 - x + y). \end{aligned} \tag{3}$$

Intuitively, these truth functions make sense: a conjunction is as true as its least true conjunct, and a disjunction is as true as its most true disjunct, the negation of x becomes more true as the x becomes less true.
The implication is defined by the fact that if $x \leq y$, then the implication is absolutely true; otherwise, the truth value of the implication decreases proportionally to how much x exceeds y.

The truth functions introduced above for conjunction, disjunction, negation, and implication define Łukasiewicz Logic, one of the most well-known and extensively studied systems in the field of many-valued logic. Originally proposed by Jan Łukasiewicz in the early 20^{th} century in [4], this logic extends Propositional Logic by allowing truth values from the entire interval $[0, 1]$ rather than just $\{0, 1\}$. Its significance lies not only in its historical impact, but also in its elegant treatment of vagueness, its applicability in fuzzy control systems described in [5].

A nice property of Łukasiewicz Logic is that the truth functions can be derived from a single operation called the t-norm. We will now see how this works by formally introducing t-norms in the next section.

4.2 *t*-norm

We start this section by defining a t-norm and examining its properties. Afterwards, we discuss the concept of continuity for a t-norm and show that it induces the unique operation called the residuum. The following definitions and results are adapted from [6].

Definition 4.1. A ***t*-norm** is a binary operation $\star : [0, 1]^2 \to [0, 1]$ satisfying the following axioms for all $x, y, z \in [0, 1]$:

A1 Commutativity: $x \star y = y \star x$,

A2 Associativity: $x \star (y \star z) = (x \star y) \star z$,

A3 Monotonicity: $x \star y \leq x \star z$ if $y \leq z$,

A4 Boundary Condition: $x \star 1 = x$.

Corollary 4.2. Let $\star$ be a t-norm. From the axioms we deduce that $0 \star x = 0$ and $x_1 \star y_1 \leq x_2 \star y_2$ if $x_1 \leq x_2$ and $y_1 \leq y_2$.

Definition 4.3. A t-norm $\star$ is a **continuous t-norm** if it is a continuous mapping in the usual sense.

Proposition 4.4. Let $\star$ be a (left) continuous t-norm. Then there is a unique operation $\Rightarrow: [0,1]^2 \to [0,1]$ satisfying for all $x, y, z \in [0,1]$

$$(x \star z) \leq y \text{ iff } z \leq (x \Rightarrow y).$$

This unique operation is given by $x \Rightarrow y = \sup\{z : x \star z \leq y\}$ and is called the **residuum** of the $\star$.

Proof. This proposition states two things. The first is the "iff" connection the second is the uniqueness.
If $(x \star z) \leq y$, then $z \leq \sup\{z : x \star z \leq y\} = (x \Rightarrow y)$.
If $z \leq (x \Rightarrow y)$, then using the fact that $\star$ is monotonic, commutative and continuous, we have

$$x \star z \leq x \star (x \Rightarrow y) = x \star \sup\{z : x \star z \leq y\} = \sup\{x \star z : x \star z \leq y\} \leq y.$$

As for the uniqueness of the residuum, assume that there exists two residuum functions, $\Rightarrow_1$ and $\Rightarrow_2$, such that for all $x, y, z \in [0,1]$, we have

$$(x \star z) \leq y \text{ iff } z \leq (x \Rightarrow_1 y) \text{ and } (x \star z) \leq y \text{ iff } z \leq (x \Rightarrow_2 y).$$

Combining this, we obtain

$$z \leq (x \Rightarrow_1 y) \text{ iff } z \leq (x \Rightarrow_2 y).$$

Then take $z_1; = x \Rightarrow_1 y$ and $z_2 := x \Rightarrow_2 y$. With these variables, we have the following

$$\begin{aligned} &\text{As } (x \Rightarrow_1 y) \leq (x \Rightarrow_1 y), \text{ we have } (x \Rightarrow_1 y) \leq (x \Rightarrow_2 y). \\ &\text{As } (x \Rightarrow_2 y) \leq (x \Rightarrow_2 y), \text{ we have } (x \Rightarrow_2 y) \leq (x \Rightarrow_1 y). \end{aligned}$$

Combining the two inequalities, we obtain that

$$(x \Rightarrow_1 y) = (x \Rightarrow_2 y).$$

□

The next Lemma will state some properties about the residuum.

Lemma 4.5. Let $\star$ be a continuous t-norm with residuum $\Rightarrow$. The following hold for all $x, y \in [0,1]$,

1. $x \leq y$ iff $x \Rightarrow y = 1$.
2. $1 \Rightarrow x = x$ and $x \Rightarrow 1 = 1$.
3. If $x \leq y$, then $x = y \star (y \Rightarrow x)$.

Proof.

1. Suppose $x \leq y$. Then $x \Rightarrow y = \sup\{z : x \star z \leq y\}$. As $z \in [0,1]$, we have $x \star z \leq x \star 1 = x \leq y$. Therefore, the supremum is equal to 1.

   Suppose $(x \Rightarrow y) = 1$, we know that $x = x \star 1 \leq y$.

2. $1 \Rightarrow x = \sup\{z : 1 \star z \leq x\} = x$ and $x \Rightarrow 1 = \sup\{z : x \star z \leq 1\} = 1$.

3. Take arbitrary $x, y \in [0,1] : x \leq y$. Define the continuous function $f : [0,1] \Rightarrow [0,y]; f(z) = y \star z$. Notice that $f(1) = y$ and $f(0) = 0$. By the intermediate value theorem, there exists z_0 such that $0 \leq f(z_0) = x \leq y$. As a result, we have that

   $$y \Rightarrow x = \sup\{z : z \star y \leq x\} = z_0.$$

   Moreover, $x = f(z_0) = y \star z_0 = y \star (y \Rightarrow x)$.

$\square$

Next, we will show that continuous t-norms and their residua are all we need to derive the truth functions for connectives $\{\wedge, \vee, \neg, \rightarrow\}$. Before that, we first introduce the language in which these operations will be interpreted. This leads us to define Basic Logic.

Definition 4.6. Following Definition 3.1, Basic Logic is defined by the triple structure $\{\mathcal{V}, \mathcal{D}, f_\sharp : \sharp \in \mathcal{C}\}$ where $\mathcal{V} = [0,1], \mathcal{D} = [1], \mathcal{C} = \{\&, \rightarrow\}$ equipped with truth functions $f_\&(x,y) = x \star y$ and $f_\rightarrow(x,y) = x \Rightarrow y$, where $\star$ is any continuous t-norm and $\Rightarrow$ its residuum.

In Classical Propositional Logic, we define well-formed formulas to structure the language $\mathcal{L}$, where the set of connectives is $\{\wedge, \vee, \neg, \rightarrow\}$. We now introduce the corresponding formation rules for BL, where the language $\mathcal{F}$ is structured using the connectives $\{\&, \rightarrow\}$.

Definition 4.7. Let $\mathcal{P}$ be the set of propositional atoms. We define the set of well-formed formulas $\mathcal{F}$ of Basic Logic as follows. $\mathcal{F}$ is the least set such that

- $\mathcal{P} \cup \{\bar{0}\}$, where $\bar{0}$ represents completely false,
- if $\psi, \varphi \in \mathcal{F}$, then $\psi \& \varphi, \psi \rightarrow \varphi \in \mathcal{F}$.

We relate the languages $\mathcal{L}$ and $\mathcal{F}$ through the following definition.

Definition 4.8. Let $\psi, \varphi \in \mathcal{F}$. Then the connectives $\{\wedge, \vee, \neg\}$ are defined as follows

$$\begin{aligned}
\psi \wedge \varphi &:= \psi \& (\psi \to \varphi), \\
\psi \vee \varphi &:= ((\psi \to \varphi) \to \varphi) \wedge ((\varphi \to \psi) \to \psi)), \\
\neg\psi &:= \psi \to \bar{0}.
\end{aligned}$$

Proposition 4.9 (Łukasiewicz t-norm and residuum)**.** The continuous t-norm corresponding to the Logic of Łukasiewicz is defined by the binary operation $\star_{\text{Ł}} : [0,1]^2 \to [0,1]$

$$x \star_{\text{Ł}} y := \max(0, x + y - 1).$$

Its residuum $\Rightarrow_L$ is given by

$$x \Rightarrow_{\text{Ł}} y := \min(1, 1 - x + y).$$

Furthermore, the truth functions for the connectives $\wedge, \vee, \neg$ defined in 3 are derivable from Definitions 4.6 and 4.8. In other words

$$\begin{aligned}
\min\{x, y\} &= f_{\wedge}(x, y) = f_{\&}(x, f_{\to}(x, y)), \\
\max\{x, y\} &= f_{\vee}(x, y) = \min\{(f_{\to}(x, f_{\to}(x, y))), f_{\to}(y, f_{\to}(y, x))\}, \\
1 - x &= \quad f_{\neg}(x) = f_{\to}(x, 0).
\end{aligned}$$

Proof. First, we verify that $\star_{\text{Ł}}$ is indeed a t-norm. Notice that this function is well-defined as $x \star_{\text{Ł}} y = \max(0, x + y - 1) \in [0,1]$ for all $x, y \in [0,1]$.
Next, we want to proof that $\star_{\text{Ł}}$ abides every axiom in Definition 4.1. To do so, take arbitrary elements $x, y, z \in [0,1]$ and observe the following

A1 $x \star_{\text{Ł}} y = \max(0, x + y - 1) = \max(0, y + x - 1) = y \star_{\text{Ł}} x$.

A2

$$\begin{aligned}
x \star_{\text{Ł}} (y \star_{\text{Ł}} z) &= x \star_{\text{Ł}} \max(0, y + z - 1) = \max(0, x + \max(0, y + z - 1) - 1) \\
&= \max(0, \max(x - 1, x + y + z - 2)) = \max(0, x + y + z - 2), \\
&\text{and} \\
(x \star_{\text{Ł}} y) \star_{\text{Ł}} z &= \max(0, x + y - 1) \star_{\text{Ł}} z = \max(0, \max(0, y + z - 1) + z - 1) \\
&= \max(0, \max(z - 1, x + y + z - 2)) = \max(0, x + y + z - 2).
\end{aligned}$$

A3 if $y \leq z$, then $x \star_{\text{Ł}} y = \max(0, x + y - 1) \leq \max(0, x + z - 1) = x \star_{\text{Ł}} z$.

A4 $x \star_{\text{Ł}} 1 = \max(0, x + 1 - 1) = x$.

Therefore, $\star_{\text{L}}$ is indeed a t-norm

As for the the continuity of the Łukasiewicz t-norm, let $(x_n, y_n) \to (x, y)$ be a sequence in $[0,1]^2$. We want to show

$$\lim_{n\to\infty} \max(0, a_n + b_n - 1) = \lim_{n\to\infty} x_n \star_{\text{L}} y_n = x \star_{\text{L}} y = \max(0, x + y - 1).$$

Next, we define the new sequence $z_n := x_n + y_n$ and constant $z = x + y$. With these, we can define the function

$$f : [0,2] \to [0,1] \text{ where } f(t) = \max(0, t - 1).$$

Notice that this function is piecewise linear and has a kink at $t = 1$. Therefore, let us proof that this function is also continuous at $t = 1$.

$$f(1) = \max(0,0) = 0, \quad \lim_{t\to 1^-} f(t) = 0, \quad \lim_{t\to 1^+} f(t) = 0.$$

As a result, we can conclude that f is a continuous function on $[0,2]$. Combining this with the fact this function has the property $x \star_{\text{L}} y = f(z)$, we have

$$\lim_{n\to\infty} x_n \star_{\text{L}} y_n = \lim_{n\to\infty} f(x_n + y_n) = f(x + y) = x \star_{\text{L}} y.$$

As (x_n, y_n) is an arbitrary sequence in $[0,1]^2$, we can conclude that $\star_{\text{L}}$ is a continuous t-norm.

Next, we verify that $\Rightarrow_L$ is indeed the residue of $\star_{\text{L}}$. When $x \leq y$, Lemma 4.5 tells us that $x \Rightarrow_{\text{L}} y = 1$. When $x > y$, will prove that $x \Rightarrow_{\text{L}} y = 1 - x + y$ by contradiction. So suppose that $x \Rightarrow_{\text{L}} y \neq 1 - x + y$. In other words,

$$\sup A \neq 1 - x + y.$$

where $A = \{z : \max(0, x + z - 1) = x \star_{\text{L}} z \leq y\}$.
Notice that $1 - x + y \in A$ as $x + (1 - x + y) - 1 = y \leq y$. Next, assume that there exists an element $a \in A$ such that $a > 1 - x + y$. As $a \in A$, we have that a must satisfy $\max(0, x + a - 1) \leq y$. If $x + a - 1 \leq 0$, then $a \leq 1 - x \leq 1 - x + y$, which we assumed is not true. If $x + a - 1 > 0$, then $x + a - 1 \leq y$, which implies that $a \leq 1 - x + y$, which we assumed is not true. Hence, we arrive at a contradiction. Therefore, $x \Rightarrow_{\text{L}} y = 1 - x + y$.

To show that $\min\{x, y\} = f_{\&}(x, f_{\to}(x, y))$, assume that $x \leq y$, then $f_{\to}(x, y) = 1$ which results in

$$f_{\&}(x, f_{\to}(x, y)) = f_{\&}(x, 1) = x = \min\{x, y\}.$$

If $x > y$, then $f_{\to}(x, y) = 1 - x + y$ which results in

$$f_{\&}(x, f_{\to}(x, y)) = f_{\&}(x, 1 - x + y) = \max\{0, x + 1 - x + y - 1\} = y = \min\{x, y\}.$$

To show that $\max\{x,y\} = f_\vee(x,y) = \min\{(f_\to(f_\to(x,y),y)), f_\to(f_\to(y,x),x)\}$, assume that $x \leq y$, then $f_\to(x,y) = 1$ and $f_\to(y,x) = \min\{1, 1-y+x\} = 1-y+x$. If we substitute these equalities and use Lemma 4.5, we obtain

$$\min\{f_\to(1,y), f_\to(1-y+x,x)\} = \min\{y, \min\{1, 1-1+y-x+x\}\} = y = \max\{x,y\}.$$

If $x > y$, then $f_\to(y,x) = 1$ and $f(x,y) = \min\{1, 1-x+y\} = 1-x+y$. If we substitute these equalities and use Lemma 4.5, we obtain

$$\min\{f_\to(1-x+y,y), f_\to(1,x)\} = \min\{\min\{1, 1-1+x-y+y\}, x\} = x = \max\{x,y\}.$$

To show that $1-x = f_\to(x,0)$, notice that

$$f_\to(x,0) = x \Rightarrow 0 = \min\{1, 1-x+0\} = 1-x.$$

This concludes the proof. $\square$

Proposition 4.9 shows that the truth functions of Łukasiewicz Logic are derivable from its t-norm. Similarly how we defined the logical validity for the language $\mathcal{L}$ in Classical Propositional Logic, we now aim to do the same for the language $\mathcal{F}$ in BL. In other words, we seek to define logical validity for all continuous t-norms and their residua.

Definition 4.10. Let $\star$ be a continuous t-norm and $\Rightarrow$ its residuum. An **interpretation** $V_* : \mathcal{P} \to [0,1]$ is a function assigning propositional atoms to $[0,1]$ and can be extended to $\mathcal{F}$ in the following way:

- $V_\star(\bar{0}) = 0$,
- for any $\psi, \varphi \in \mathcal{F}$,
  - $V_\star(\psi \& \varphi) = V_\star(\psi) \star V_\star(\varphi)$,
  - $V_\star(\psi \to \varphi) = V_\star(\psi) \Rightarrow V_\star(\varphi)$.

Definition 4.11. A wff ψ in $\mathcal{F}$ is **logically valid** iff $V_\star(\psi) = 1$ for all continuous t-norms $\star$ and all interpretations $V : \mathcal{P} \to [0,1]$.

Example 4.12. Again, we want to check if the wff $(p \to q) \vee (q \to p)$ is logically valid. In other words, we want to verify $V((p \to q) \vee (q \to p)) = 1$. Using Definitions

4.6 and 4.8, by substituting $\psi = p \to q, \varphi = q \to p$ we find the following

$$\begin{aligned}
&V\big(\psi \vee \varphi\big)\\
&= V\big([(\psi \to \varphi) \to \varphi] \wedge [(\varphi \to \psi) \to \psi]\big)\\
&= V\Big(\big[(\psi \to \varphi) \to \varphi\big] \& \big[((\psi \to \varphi) \to \varphi) \to ((\varphi \to \psi) \to \psi)\big]\Big)\\
&= V\Big((\psi \to \varphi) \to \varphi\Big) \star V\Big(\big[(\psi \to \varphi) \to \varphi\big] \to \big[(\varphi \to \psi) \to \psi\big]\Big)\\
&= \Big(V(\psi \to \varphi) \Rightarrow V(\varphi)\Big) \star \Big(V((\psi \to \varphi) \to \varphi) \Rightarrow V((\varphi \to \psi) \to \psi)\Big)\\
&= \Big([V(\psi) \Rightarrow V(\varphi)] \Rightarrow V(\varphi)\Big) \star \Big[\Big([V(\psi) \Rightarrow V(\varphi)] \Rightarrow V(\varphi)\Big) \Rightarrow \Big([V(\varphi) \Rightarrow V(\psi)] \Rightarrow V(\psi)\Big)\Big].
\end{aligned}$$

Now there are two cases. Either $V(p) \leq V(q)$ or $V(p) > V(q)$.

- Suppose $V(p) \leq V(q)$. Then, by Lemma 4.5, we know that $V(\psi) = V(p) \Rightarrow V(q) = 1$. Moreover, from Lemma 4.5, we know that $1 \Rightarrow x = x$, $x \Rightarrow x = 1$. Combining these equalities, we obtain that $[1 \Rightarrow x] \Rightarrow x = 1$ and $[x \Rightarrow 1] \Rightarrow 1 = 1$. Using this result, $V(\psi \vee \varphi)$ trivializes to $1 \star 1 = 1$.
- The opposite case can be proven similarly

As a result, we conclude that that our $(p \to q) \vee (q \to p)$ is logically valid.

4.3 Decomposition Theorem

In this section, we will show that every continuous t-norm can be decomposed into a t-norm called the **ordinal sum**, whose components are isomorphic to the three fundamental t-norms: the Łukasiewicz, product, and Gödel t-norms, which are given by

$$\begin{aligned}
\textbf{Łukasiewicz}: &\quad x \star_{\text{Ł}} y = \max(0, x + y - 1),\\
\textbf{Product}: &\quad x \star_P y = x \cdot y,\\
\textbf{Gödel}: &\quad x \star_G y = \min(x, y).
\end{aligned} \tag{4}$$

The order we prove this is similar to the approach in [6]. In Proposition 4.9, we saw that $\star_{\text{Ł}}$ is a continuous t-norm. Similarly, we can prove that $\star_P$ and $\star_G$ are also continuous t-norms. We will now examine some properties that these three t-norms have that will help to define the ordinal sum.

Proposition 4.13. 1. For all $x, y \in [0,1]$ and any t-norm $\star$, $x \star y \leq x \star_G y$.

2. The only t-norm satisfying $x \star x = x$ for all $x \in [0,1]$ is the Gödel t-norm.

Proof. 1. By the boundary axiom of $\star$, we know that

$$x \star y \leq x \star 1 = x, \qquad x \star y \leq 1 \star y = y.$$

Combining these two, we get $x \star y \leq \min(x, y) = x \star_G y$.

2. Let $\star$ be a t-norm satisfying $x \star x = x$. Without loss of generality, assume that $x \le y$. As $\star$ is monotonic, we have $x \star_G y = x = x \star x \le x \star y$. Moreover, from 1. we know that $x \star y \le x \star_G y$. Combining these two, we conclude that $x \star y = x \star_G y$.

□

As these fundamental t-norms are continuous, each admits a unique residuum, which can be derived in the following proposition.

Proposition 4.14. The following operations $\Rightarrow\colon [0,1]^2 \to [0,1]$ are the residua of the Łukasiewicz t-norm, the product t-norm, and the Gödel t-norm respectively

1. Łukasiewicz implication: $x \Rightarrow_{\text{Ł}} y = 1 - x + y$ if $y < x$ and $x \Rightarrow_{\text{Ł}} y = 1$ otherwise. Alternatively, $x \Rightarrow_{\text{Ł}} y = \min\{1, 1 - x + y\}$.
2. Product implication: $x \Rightarrow_P y = \frac{y}{x}$ if $y < x$ and $x \Rightarrow_P y = 1$ otherwise.[2]
3. Gödel implication: $x \Rightarrow_{\text{G}} y = y$ if $y < x$ and $x \Rightarrow_G y = 1$ otherwise.

Proof. In Proposition 4.9, we saw how the residuum of the Łukasiewicz t-norm is derived. The derivations for the Product and Gödel implications proceed similarly and are therefore omitted. □

Now that we have a better understanding of the three fundamental t-norms, we want to discuss how we decompose an arbitrarily continuous t-norm into them. To do so, we need the following concepts.

Notation 4.15. Let $\star$ be a t-norm and $x \in [0,1]$. We define $x^n_\star$ inductively as $x^0_\star = 1$ and $x^{n+1}_\star = x^n_\star \star x$.

Definition 4.16. Let $\star$ be a t-norm. An element $x \in [0,1]$ is called an **idempotent element** of $\star$ iff $x \star x = x$. An element $x \in (0,1)$ is called a **nilpotent element** of $\star$ iff there exists some $n \in \mathbb{N}$ such that $x^n_\star = 0$.

These idempotent elements allow us to extend Lemma 4.5 as follows:

Lemma 4.17. For each continuous t-norm $\star$ and its residuum $\Rightarrow$, for all $x, y \in [0,1]$,

4. If $x \le u \le y$ with $u \in [0,1]$ being an idempotent element, then $(x \star y) = x$.
5. If $x \le y^n_\star$ for every $n \in \mathbb{N}$, then there is an idempotent $e \in [0,1]$ such that $x \le e \le y$.

[2]This implication is often referred as the Goguen implication.

Proof. 4. Let $x \leq u \leq y$ with u idempotent. As $x \leq u$, we can use previous statement and the commutativity of $\star$ to show that.

$$x \star u = u \star (u \Rightarrow x) \star u = u \star (u \Rightarrow x) = x$$

Next, notice that we have both $x \star y \leq x \star 1 = x$ and $x = x \star u \leq x \star y$, and thus $x = x \star y$.

5. Take $e = \inf\{y_\star^n : n \in \mathbb{N}\}$ such that $x \leq e \leq y_\star^n$. By the continuity of $\star$,

$$e \star e = \inf\{y_\star^n : n \in \mathbb{N}\} \star \inf\{y_\star^n : n \in \mathbb{N}\} = \inf\{y_\star^{n+m} : n + m \in \mathbb{N}\} = e.$$

$\square$

Moreover, Definition 4.24 provides a classification of various types of t-norms based on their properties.

Definition 4.18. Let $\star$ be a t-norm.

1. $\star$ is called **strict** iff it is continuous and satisfies $x \star y < x \star z$ whenever $x > 0$ and $y < z$.
2. $\star$ is called **nilpotent** iff it is continuous and each $u \in (0, 1)$ is a nilpotent element of $\star$.
3. $\star$ is called **Archimedean** iff for each $(x, y) \in (0, 1)^2$ there exists an $n \in \mathbb{N}$ such that $x_\star^n < y$.

The purpose of classifying t-norms into these types is to establish that certain classes are equivalent, up to isomorphism, to the fundamental t-norms. This correspondence will be demonstrated in the Decomposition Theorem 4.36. To that end, we require the following mathematical statements.

Proposition 4.19. Let $\star$ be a t-norm. Then $\star$ is Archimedean iff for all $x \in (0, 1)$,

$$\lim_{n\to\infty} x_\star^n = 0.$$

Proof. Suppose $\star$ is Archimedean. Assume, for contradiction, that $\lim_{n\to\infty} x_\star^n \neq 0$. Then $\lim_{n\to\infty} x_\star^n = y$ by the continuity of $\star$. As x is Archimedean, we know that there exists some $m \in \mathbb{N}$ such that $x_\star^m < y = \lim_{n\to\infty} x_\star^n$. If we apply the Archimedean property again, then we know that there is some $s \in \mathbb{N}$ such that $y_\star^s < x_\star^m$. However, notice that $y_\star^s = \lim_{n\to\infty} x_\star^n$. As a result, we arrive at a contradiction.

Suppose $\star$ has the limit property. Then for all $x \in (0, 1)$, we have $\lim_{n\to\infty} x_\star^n = 0$. From Corollary 4.2, we can deduce that $x_\star^{n+1} \leq x_\star^n$, so $(x_\star^n)$ is strictly decreasing and bounded below by 0. By the limit property, $x_\star^n \to 0$. Now let $x, y \in (0, 1)$. Since $x_\star^n \to 0$, there exists $n \in \mathbb{N}$ such that $x_\star^n < y$. $\square$

Proposition 4.20. Every Archimedean t-norm has no idempotent elements except 0 and 1.

Proof. Let $\star$ be an Archimedean t-norm. Assume, for contradiction, $x \in (0,1)$ is an idempotent element. In other words, $x \star x = x$. Now take an element $y \in (0,x)$. Notice that for all $n \in \mathbb{N}$, $x_\star^n = x > y$. This contradicts the fact that $\star$ is Archimedean. Therefore, the only idempotent elements of $\star$ are 0 and 1. $\square$

Definition 4.21. Let $0 \leq a < b \leq 1$. We call $[a,b]$ a **contact interval** of the t-norm $\star$ if u, v are idempotent elements of $\star$ and no element of (a,b) is an idempotent. Let $\mathcal{I}_\star$ be called the family of contact intervals of $\star$.

This allows us to partition the domain of a t-norm $\star$ as follows.

$$[0,1] = \mathcal{I}_\star \cup \mathcal{J}_\star,$$

where $\mathcal{I}_\star$ is defined as in Definition 4.21 and $\mathcal{J}_\star = [0,1] \backslash \mathcal{I}_\star$ is the complement of the contact intervals in $[0,1]$.
If we now restrict ourselves to a single contact interval $[a,b] \in \mathcal{I}_\star$, we cannot call $\star_{[a,b]}$ a t-norm anymore as its domain is not the interval $[0,1]$. This observation motivates the following definition.

Definition 4.22. Let $a, b \in \mathbb{R}$ with $a < b$. The function $\star : [a,b]^2 \to [a,b]$ is a **proto-*t*-norm** iff it satisfies the following axioms

A1 Commutativity: $x \star y = y \star x$.

A2 Associativity: $x \star (y \star z) = (x \star y) \star z$.

A3 Monotonicity: $x \star y \leq x \star z$ if $y \leq z$.

A4 Boundary condition: $x \star b = x$.

Corollary 4.23. From these axioms, we can conclude $x \star a = a$.

Proof. Suppose not, then $x \star a > a$. Using **A1, A2, A4**, we can derive the following inequality

$$a = b \star a > x \star a > a.$$

$\square$

Similarly to Definition 4.18, we can define idempotent and nilpotent elements, as well as corresponding classes, for proto-t-norms.

Definition 4.24. Let $\star$ be a proto-t-norm. An element $x \in [a,b]$ is called an **idempotent element** of $\star$ iff $x \star x = x$. An element $x \in (a,b)$ is called a **nilpotent element** of $\star$ iff there exists some $n \in \mathbb{N}$ such that $x_\star^n = a$.

Definition 4.25. Let $\star : [a,b]^2 \to [a,b]$ be a proto-t-norm.

1. $\star$ is called **strict** iff it is continuous and satisfies $x \star y < x \star z$ whenever $x > a$ and $y < z$.
2. $\star$ is called **nilpotent** iff it is continuous and each $u \in (0,1)$ is a nilpotent element of $\star$.
3. $\star$ is called **Archimedean** iff for each $(x,y) \in (a,b)^2$ there exists an $n \in \mathbb{N}$ such that $x_\star^n < y$.

Since most of the axioms in Definitions 4.1 and 4.22 are similar, it is natural to assume two t-norms are equivalent if they behave similarly on their respective domains. This motivates the notion of isomorphisms between proto-t-norms. However, because proto-t-norms are also monotonic, such isomorphisms must preserve the order on its domain. This leads to the following definition.

Definition 4.26. Let $a,b,c,d \in \mathbb{R}$ with $a < b$ and $c < d$, and let $\star_1 : [a,b]^2 \to [a,b]$ and $\star_2 : [c,d]^2 \to [c,d]$ be functions. An **isomorphism** from $\star_1$ to $\star_2$ is a strictly increasing bijection $f : [a,b] \to [c,d]$ such that for every $x, y \in [a,b]$ we have

$$f(x \star_1 y) = f(x) \star_2 f(y). \tag{5}$$

Moreover, $\star_1$ and $\star_2$ are isomorphic iff there is some isomorphism from $\star_1$ to $\star_2$.

As we have defined isomorphisms as strictly increasing bijections with property (5), the following proposition can be proven.

Proposition 4.27. An isomorphism between two t-norms is also a homeomorphism.

Proof. Consider an isomorphism $f : [a,b] \to [c,d]$. Take an arbitrary point $x_0 \in [a,b]$ and define the two the two one-sided limits

$$L := \lim_{x \to x_0^-} f(x) = \sup\{f(x) : x \leq x_0\} \quad \text{and} \quad R := \lim_{x \to x_0^+} f(x) = \inf\{f(x) : x > x_0\},$$

with property $L \leq f(x_0) \leq R$. If $L = f(x_0) = R$, then f would be continuous at f.

Suppose, for contradiction, that $L < f(x_0)$, then there exists some value $y \in [c,d]$: $L < y < f(x_0)$. As f is a strictly increasing bijection, its inverse f^{-1} is also strictly increasing. This means that $L < y < f(x_0)$ implies that $f^{-1}(L) < f^{-1}(y) < f^{-1}(f(x_0)) = x_0$. As f is a bijection, it means that there exists some $x \in [a, x_0)$ such that $f(x) = y$. However, this contradicts the fact that L is the supremum of $\{f(x) : x \leq x_0\}$. As a result, we have $L = f(x_0)$. Similarly, we can show that $R = f(x_0)$. As $x_0 \in [a,b]$ was chosen arbitrary, we have that f is continuous on $[a,b]$. $\square$

In addition, isomorphisms preserve several other properties, which we summarize in the following lemma.

Lemma 4.28. Let $a, b, c, d, k, l \in \mathbb{R}$ with $a < b$ and $c < d$, and let $\star_1 : [a, b]^2 \to [a, b]$, $\star_2 : [c, d]^2 \to [c, d]$, and $\star_3 : [k, l]^2 \to [k, l]$ be binary operations. Let f be an isomorphism from $\star_1$ to $\star_2$ and g be an isomorphism from $\star_2$ to $\star_3$ Then:

1. f^{-1} is an isomorphism from $\star_2$ to $\star_1$,
2. the composition $g \circ f$ is an isomorphism from $\star_1$ to $\star_3$.
3. if $\star_1$ is continuous, then so is $\star_2$.
4. if $\star_1$ is a proto-t-norm, then so is $\star_2$.
5. if $\star_1$ is an Archimedean proto-t-norm, then so is $\star_2$.
6. if $u \in [a, b]$ is a nilpotent element of $\star_1$, then $f(u)$ is a nilpotent element of $\star_2$.
7. if $u \in [a, b]$ is an idempotent element of $\star_1$, then $f(u)$ is an idempotent element of $\star_2$.

Proof.
1. We want to show that f^{-1} is an isomorphism from $\star_2$ to $\star_1$. In other words, we want to show that f^{-1} is a strictly increasing bijection with the property $f^{-1}(x \star_2 y) = f^{-1}(x) \star_1 f^{-1}(y)$.

   f^{-1} is a bijection by definition. Moreover, take $x, y \in [c, d]$ with $x < y$. Suppose, for contradiction, $f^{-1}(x) \geq f^{-1}(y)$. If $f^{-1}(x) = f^{-1}(y)$, then we arrive at a contradiction as f^{-1} is an injection. Therefore, assume that $f^{-1}(x) > f^{-1}(y)$. As f is strictly increasing and $f^{-1}(x), f^{-1}(y) \in [a, b]$, we have $x = f(f^{-1}(x)) > f(f^{-1}(y)) = y$. This contradicts our assumption. Hence, f^{-1} is also strictly increasing. Lastly, we need to show that $f^{-1}(x \star_2 y) = f^{-1}(x) \star_1 f^{-1}(y)$. As f is a bijection, there exists $u = f^{-1}(x), v = f^{-1}(y) \in [a, b]$. Using this, we get

   $$f^{-1}(x \star_2 y) = f^{-1}(f(u) \star_2 f(v)) = f^{-1}(f(u \star_1 v)) = u \star_1 v = f^{-1}(x) \star_1 f^{-1}(y).$$

2. As f and g are both bijections, it follows that $g \circ f$ is also a bijection. Moreover, to show it that it is also strictly increasing, take elements $x, y \in [a, b]$ such that $x < y$. As f, g are strictly increasing, we have $f(x) < f(y)$ and $(g \circ f)(x) g(f(x)) < g(f(y)) = (g \circ f)(y)$. Hence, we can conclude that $g \circ f$ is also strictly increasing. Lastly, notice that

   $$\begin{aligned}(g \circ f)(x \star_1 y) = g(f(x \star_1 y)) &= g(f(x) \star_2 f(y)) = g(f(x)) \star_3 g(f(y)) \\ &= (g \circ f)(x) \star_3 (g \circ f)(y).\end{aligned}$$

   As a result, we can conclude that $g \circ f$ is indeed an isomorphism from $\star_1$ to $\star_3$.

3. For convenience of notation, we write $\star(x, y) := x \star y$. Notice that for all $x, y \in [c, d]$, we have $f^{-1}(x), f^{-1}(y) \in [a, b]$. Using this, we can define

$$\begin{aligned}\star_2 : [c,d]^2 \to [c,d] \text{ where } \star_2(x,y) :&= x \star_2 y = f(f^{-1}(x) \star_1 f^{-1}(y)) \\ &= f(\star_1(f^{-1}(x), f^{-1}(y))) \\ &= (f \circ \star_1)(f^{-1}(x), f^{-1}(y)).\end{aligned}$$

Using Proposition 4.27 and the assumption, we know that f, f^{-1} and $\star_1$ are all continuous functions. Therefore, $\star_2$ is a composition of continuous functions. Hence, $\star_2$ is also a continuous function.

4. Suppose $\star_1$ is a proto-t-norm. To show that $\star_2$ is a proto-t-norm, we will check show that $\star_2$ satisfies the axioms in Definition 4.22. Therefore, take $x, y, z \in [c, d]$ with inverses $u = f^{-1}(x), v = f^{-1}(y), w = f^{-1}(z) \in [a, b]$.

   A1 $x \star_2 y = f(u) \star_2 f(v) = f(u \star_1 v) = f(v \star_1 u) = f(v) \star_2 f(u) = y \star_2 v$.

   A2 $x \star_2 (y \star_2 z) = f(u) \star_2 (f(v) \star_2 f(w)) = f(u) \star_2 f(v \star_1 w) = f(u \star_1 (v \star_1 w)) = f((u \star_1 v) \star_1 w) = f(u \star_1 v) \star_2 f(w) = (f(u) \star_2 f(v)) \star_2 f(w) = (x \star_2 y) \star_2 z$.

   A3 If $y \leq z$, then $f(v) \leq f(w)$. As f is a strictly increasing bijection, $f(v) \leq f(w)$ implies $v = f^{-1}(f(v)) \leq f^{-1}(f(w)) = w$. Using the monotonicity axiom of $\star_1$, we know $u \star_1 v \leq u \star_1 w$. As f is strictly increasing, we have $x \star_2 y = f(u) \star_2 f(v) = f(u \star_1 v) \leq f(u \star_1 w) = f(u) \star_2 f(w) = x \star_2 z$.

   A4 Notice that $f(b) = d$ as f is a strictly increasing bijection. Therefore, $x \star_2 d = f(u) \star_2 f(b) = f(u \star_1 b) = f(u) = x$.

5. Suppose $\star_1$ is Archimedean. Take two arbitrary elements $x, y \in (c, d)$ and denote their inverses by $u = f^{-1}(x), v = f^{-1}(y) \in (a, b)$. As $\star_1$ is Archimedean, there exists some $n \in \mathbb{N}$ such that $u_{\star_1}^{(n)} < v$. As f is strictly monotone, we know that this implies $f(u_{\star_1}^{(n)}) < f(v)$ Next, notice that $f(u_{\star_1}^{(n)}) = f(u)_{\star_2}^{(n)}$, which can easily be proven using induction. Combining these facts, we get $x_{\star_2}^{(n)} = f(u)_{\star_2}^{(n)} < f(v) = y$. As $x, y \in (c, d)$ were chosen arbitrarily, we conclude that $\star_2$ is also Archimedean.

6. Suppose there exists a $n \in \mathbb{N}$ such that $a = u_{\star_1}^n$. Applying f to both sides, and using 5, we obtain

$$c = f(a) = f(u_{\star_1}^n) = f(u_{\star_1}^{n-1}) \star_2 f(u_{\star_1}) = \cdots = f(u)_{\star_2}^n.$$

7. Suppose $u \star_1 u = u$. By 5, we have

$$f(u) = f(u \star_1 u) = f(u) \star_2 f(u).$$

□

Now that the main properties of isomorphisms between proto-t-norms are established, we introduce a lemma for constructing isomorphisms.

Lemma 4.29. Let $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$, let $\star_2 : [c, d]^2 \to [c, d]$ be a function and let $f : [a, b] \to [c, d]$ be a strictly increasing bijection. Define $\star_1 : [a, b]^2 \to [a, b]$ by

$$x \star_1 y := f^{-1}(f(x) \star_2 f(y)),$$

for $x, y \in [a, b]$. Then f is an isomorphism from $\star_1$ to $\star_2$.

Proof. Applying f on both sides of the equation, we get

$$f(x \star_1 y) = f(f^{-1}(f(x) \star_2 f(y))) = f(x) \star_2 f(y).$$

From this and the assumption, we can conclude that f is an isomorphism from $\star_1$ to $\star_2$. □

Example 4.30. As an application of Lemma 4.29, consider the linear transformation

$$f(x) = c + (d - c) \cdot \frac{x - a}{b - a}.$$

This function represents a rescaling on the length of interval $[a, b]$ to match the length of $[c, d]$, followed by a translation to align the intervals. In particular, this function is a strictly increasing bijection. Hence, given a proto-t-norm $\star_2 : [c, d]^2 \to [c, d]$, we can define a proto-$t$-norm $\star_1 : [a, b]^2 \to [a, b]$ as

$$x \star_1 y = a + (b - a) \cdot \frac{\left(c + (d - c) \cdot \frac{x-a}{b-a}\right) \star_2 \left(c + (d - c) \cdot \frac{y-a}{b-a}\right) - c}{d - c}.$$

Recall that we can partition any domain of a t-norm $\star$ into the union of $\mathcal{I}_\star$ and $\mathcal{J}_\star$, where $\mathcal{I}_\star$ represents the set of contact intervals of $\star$ and $\mathcal{J}_\star$ represents the set of sub-intervals in $[0, 1]$ which are not contact intervals. Lemma 4.28 tells us that $\star$ restricted to $[a, b] \in \mathcal{I}_\star$ is an Archimedean proto-t-norm.
We will show that $\star_{[a,b]}$, dependent on its class, is isomorphic to either the product t-norm or the Łukasiewicz t-norm. Before we propose this proposition, we will state some properties about Archimedean continuous t-norms.

Lemma 4.31. Let $\star$ be an Archimedean continuous t-norm. For $x \in (0, 1)$ and $n \in \mathbb{N}$, there is a unique y such that $y_\star^n = x$.

Proof. Assume that $n > 1$. Consider the following continuous function $f : [0, 1] \to [0, 1]; f(y) = y_\star^n$. As $x \in [0, 1]$, by the intermediate value theorem, there exists some $y \in [0, 1]$ such that $f(y) = x$. To show that this y is unique, let us assume that $f(y) = f(z) = x$ for some $z \in [0, 1]$. Now, consider another continuous function $g : [0, 1] \to [0, 1]; g(t) = y \star t$. By the intermediate value theorem, there exists some

$t \in [0,1]$ such that $y \star t = z$. If $t = 1$, then $y = z$, which contradicts our assumption. If $t \in [0,1)$, we can say

$$y_\star^n = z_\star^n = y_\star^n \star t_\star^n.$$

Continuing this argument, we find $y_\star^n = y_\star^n \star t_\star^{m\cdot n}$ to hold. By Proposition 4.19, we know that $\lim_{m\to\infty} t_\star^{m\cdot n} = 0$. As a result

$$x = z_\star^n = y_\star^n \star 0 = 0.$$

As we considered $x \in (0,1)$, we arrive at a contradiction. $\square$

Definition 4.32. Let $\star : [0,1]^2 \to [0,1]$ be a continuous t-norm. For each $x \in [0,1]$, $x_\star^{(\frac{1}{n})}$ is the unique $y \in [0,1]$ with $y_\star^n = x$. Moreover, for a rational number $r = \frac{m}{n}$, we use the following notation.

$$x_\star^r = x_\star^{\frac{m}{n}} = (x_\star^{\frac{1}{n}})_\star^m.$$

Lemma 4.33. Let $\star : [0,1]^2 \to [0,1]$ be an Archimedean continuous t-norm.

1. If $\frac{m}{n} = \frac{m'}{n'}$, then $x_\star^{\frac{m}{n}} = x_\star^{\frac{m'}{n'}}$.
2. $x_\star^r \star x_\star^s = x_\star^{r+s}$ for all $x \in [0,1], r, s$ positive rational.
3. If $x > 0$, then $\lim_{n\to\infty} x_\star^{\frac{1}{n}} = 1$.

Proof.

1. As $\frac{m}{n} = \frac{m'}{n'}$, there exists some $k \in \mathbb{Z}$ such that $m' = km$ and $n' = kn$. Using this,
$$x_\star^{\frac{m'}{n'}} = (x_\star^{\frac{1}{n'}})_\star^{m'} = (x_\star^{\frac{1}{kn}})^{km} = ((x_\star^{\frac{1}{kn}})_\star^k)_\star^m = (x_\star^{\frac{1}{n}})_\star^m = x_\star^{\frac{m}{n}}.$$
2. Let $r = \frac{m}{n}$ and $s = \frac{k}{n}$. then
$$x_\star^r \star x_\star^s = (x_\star^{\frac{1}{n}})_\star^m \star (x_\star^{\frac{1}{n}})_\star^k = (x_\star^{\frac{1}{n}})_\star^{m+k} = x_\star^{r+s}.$$

Consider the function $f : [0,1] \to [0,1]; f(y) = y_\star^n$. Notice that this function is continuous and strictly increasing on $(0,1)$ with inverse $f^{-1}(x) = x_\star^{\frac{1}{n}}$. As a result, the inverse is also strictly increasing. Moreover, the limit $\lim_{n\to\infty} x_\star^{\frac{1}{n}}$ is idempotent. From $x > 0$ and $\star$ being an Archimedean t-norm, we can conclude that the limit must be the trivial idempotent 1. $\square$

The practical applications of these lemmas lie in the proof of the following proposition.

Proposition 4.34. Let $\star$ be an Archimedean continuous t-norm.

1. If $\star$ is strict, then $\star$ is isomorphic to the product t-norm $\star_P$.
2. If $\star$ is nilpotent, then $\star$ is isomorphic to the Łukasiewicz t-norm $\star_{\text{Ł}}$.

Remark 4.35. It suffices to consider only t-norms instead of proto-t-norms, since an isomorphism can always be constructed between them, as in Example 4.30.

Proof. This proof is an adaptation of the proof mentioned in [7]. We will prove both statements as follows. We will construct a function f from $\star$ to both t-norms restricted on their respective dense subsets. Then prove that f_1 is (i) continuous, (ii) a strictly increasing bijection, (iii) has property $f_1(x \star_1 y) = f_1(x) \star_2 f_1(y)$. After that, we use [8, Lemma 3.1] to uniquely extend our isomorphisms to their whole domain.

1. Consider the function $f_1 : \mathcal{D}_1 \to \mathcal{C}_1$, where $f_1, \mathcal{D}_1, \mathcal{C}_1$ are defined as follows:

$$\mathcal{D}_1 := \left\{ \frac{1}{2}_\star^r : r \in \mathbb{Q}^+ \right\}, \quad \mathcal{C}_1 := \left\{ \frac{1}{2^r} : r \in \mathbb{Q}^+ \right\}, \quad f_1\left(\frac{1}{2}_\star^r\right) := \frac{1}{2^r}.$$

   (i) Take an arbitrary convergent sequence $(\frac{1}{2}_\star^{r_n}) \to \frac{1}{2}_\star^{r_0}$. We want to show that
$$f_1\left(\frac{1}{2}_\star^{r_n}\right) \to f_1\left(\frac{1}{2}_\star^{r_0}\right).$$
   To do so, consider the mapping $g : \mathbb{Q}^+ \to g(\mathbb{Q}^+); g(r) = \frac{1}{2}_\star^r$. This mapping is continuous and strictly decreasing, which implies that it is a bijection onto its range. From this, we can conclude that if $\frac{1}{2}_\star^{r_n} \to \frac{1}{2}_\star^{r_0}$, then $r_n = g^{-1}(g(r_n)) \to g^{-1}(g(r_0)) = r_0$. Hence,
$$\lim_{n\to\infty} f_1\left(\frac{1}{2}_\star^{r_n}\right) = \lim_{n\to\infty} \frac{1}{2^{r_n}} = \frac{1}{2^{r_0}} = f_1\left(\frac{1}{2}_\star^{r_0}\right).$$

   (ii) It is obvious to see that f_1 is a bijection, as an inverse can easily be constructed. To show that it is a strictly increasing function, consider $\frac{1}{2}_\star^r, \frac{1}{2}_\star^s \in \mathcal{D}_1 : \frac{1}{2}_\star^r < \frac{1}{2}_\star^s$. This can only be true when $r > s$. If that is true, then $\frac{1}{2^r} < \frac{1}{2^s}$. Hence, it is strictly increasing.

   (iii) Again, consider $\frac{1}{2}_\star^r, \frac{1}{2}_\star^s \in \mathcal{D}_1$ and observe the following
$$f_1\left(\frac{1}{2}_\star^r \star \frac{1}{2}_\star^s\right) = f_1\left(\frac{1}{2}_\star^{r+s}\right) = \frac{1}{2^{r+s}} = \frac{1}{2^r} \cdot \frac{1}{2^s} = \frac{1}{2^r} \star_P \frac{1}{2^s} = f_1\left(\frac{1}{2}_\star^r\right) \star_P f_1\left(\frac{1}{2}_\star^s\right).$$

2. If $\star$ is nilpotent, then we will first show that $\star$ is isomorphic to $\star_{CP} : [\frac{1}{4}, 1]^2 \to [\frac{1}{4}, 1]$ and afterwards prove that $\star_{CP}$ is isomorphic to $\star_{\text{Ł}}$. As $\star$ is nilpotent,

there exists a maximal $d \in (0,1)$ with property $d \star d = 0$. With this, we define the following:

$$\mathcal{D}_2 := \{d_\star^r : r \in \mathbb{Q} \cap [0,2)\}, \mathcal{C}_2 := \left\{\frac{1}{2^r} : r \in \mathbb{Q} \cap [0,2)\right\}, f_2(d_\star^r) := \begin{cases} \frac{1}{2^r} & \text{if } r < 2, \\ \frac{1}{4} & \text{otherwise.} \end{cases}$$

We will show that f_2 (i) is continuous, (ii) is a strictly increasing bijection, (iii) has property $f_2(x \star y) = f_2(x) \star_{\mathrm{L}} f_2(y)$.

(i) Take an arbitrary convergent sequence $(d_\star^{r_n}) \to d_\star^{r_0} \in \mathcal{D}_2$, where $r_0 \neq 2$. By the continuity of $\frac{1}{2^r}$, we know that

$$f_2(d_\star^{r_n}) = 1 - \frac{1}{2^{r_n}} \to 1 - \frac{1}{2^{r_0}} = f_2(d_\star^{r_0}).$$

If $r_0 = 2$, then approaching the limit from below, we obtain

$$f_2(d_\star^{r_n}) = \frac{1}{2^{r_n}} \to \frac{1}{2^2} = \frac{1}{4} = f_2(d_\star^2).$$

(ii) It is obvious to see that f_2 is a bijection, as an inverse an easily be constructed. To show that it is strictly increasing, consider $d_\star^r, d_\star^s \in \mathcal{D}_2$: $d_\star^r < d_\star^s$. This can only be true when $r > s$. If that is true, then $\frac{1}{2^r} < \frac{1}{2^s}$. Hence, it is strictly increasing.

(iii) Again, consider $d_\star^r, d_\star^s \in \mathcal{D}_2$ and observe the following

$$f_2(d_\star^r \star d_\star^s) = f_2(d_\star^{r+s}) = \begin{cases} \frac{1}{2^{r+s}} & \text{if } r+s < 2, \\ \frac{1}{4} & \text{otherwise.} \end{cases}$$

$$f_2(d_\star^r) \star_{CP} f_2(d_\star^s) = \max\left\{\frac{1}{4}, \frac{1}{2^{r+s}}\right\} = \begin{cases} \frac{1}{2^{r+s}} & \text{if } r+s < 2, \\ \frac{1}{4} & \text{otherwise.} \end{cases}$$

It can easily be shown that $\mathcal{D}_{1,2}$ and $\mathcal{C}_{1,2}$ are dense using the density of $\mathbb{Q}$ in $\mathbb{R}$. As we have constructed isomorphisms between dense subsets, we know that there exists a unique extension from their whole domains. Hence, they are isomorphic.

To show that $\star_{CP} : [\frac{1}{4}, 1]^2 \to [\frac{1}{4}, 1]$ is isomorphic to $\star_{\mathrm{L}}$, consider the function $h : [0,1] \to [\frac{1}{4}, 1]; h(x) = 2^{2(x-1)}$. As this mapping is an exponential mapping, we know that this is a continuous strictly increasing bijection. Moreover, h satisfies the isomorphism property (5) which can be observed below.

$$h(x \star_{\mathrm{L}} y) = h(\max\{0, x+y-1\}) = \max\{h(0), h(x+y-1)\} = \max\left\{\frac{1}{4}, 2^{2(x+y-2)}\right\},$$

$$h(x) \star_{CP} f(y) = \max\left\{\frac{1}{4}, 2^{2(x-1)} \cdot 2^{2(y-1)}\right\} = \max\left\{\frac{1}{4}, 2^{2(x+y-2)}\right\}.$$

Therefore, this h is an isomorphism from $\star_{\mathrm{L}}$ to $\star_{CP}$.

$\square$

This proposition describes the behavior of any continuous t-norm $\star$ in the family of contact intervals $\mathcal{I}\star$. To fully understand $\star$, we also need to analyze its behavior in the complement set $\mathcal{J}\star$. The following theorem provides a complete characterization of $\star$ over the entire interval $[0, 1]$.

Theorem 4.36 (Decomposition Theorem)**.** Let $\star$ be a continuous t-norm. Then the following statements are valid.

1. For each contact interval I, the restriction of $\star$ to I is isomorphic either to $\star_{Ł}$ or to $\star_P$.
2. For all elements $x, y \in [0, 1]$, if there is no contact interval I such that $x, y \in I$, then $x \star y = x \star_G y$, where $\star_G$ is the Gödel t-norm.
3. There are countably many contact intervals of $\star$.

Proof.

1. This follows from Proposition 4.34
2. Without loss of generality, assume that $x < y$. If there is no contact interval $I \in \mathcal{I}_\star$ such that $x, y \in I$, then it means that $[x, y]$ is no contact interval. In other words, there exists some idempotent element $e \in (x, y)$. Using Lemma 4.17, we can conclude $x \star y = x = \min(x, y) = x \star_G y$.
3. Take an arbitrary contact interval $[a_i, b_i] \in \mathcal{I}_\star$. As $\mathbb{Q}$ is countable, we know that the $\mathbb{Q} \cap [0, 1]$ is also countable. Moreover, using the density of $\mathbb{Q}$ in $\mathbb{R}$, we know that there exists a rational $q_i \in (a_i, b_i)$. Using this rational, we construct function $f : \mathcal{I}_\star \to \mathbb{Q} \cap [0, 1]; f([a_i, b_i]) = q_i$. As contact intervals are pairwise disjoint, we know that this function is an injection. This function tells us that there exists an injection from the set of contact intervals into a countable set. Hence, the set of contact intervals is also countable.

$\square$

Since $\star$ is locally isomorphic to one of the three fundamental t-norms on $[a, b] \in \mathcal{I}_\star \cup \mathcal{J}_\star$, we now seek a global function to describe $\star$. This function is called the ordinal sum.

Definition 4.37. Let $([a_n, b_n])$ with $0 \leq a_n < b_n \leq 1$ for all $n \in \mathfrak{C}$ be a family of pairwise distinct intervals such that every two distinct interval has disjoint interiors.[3] Then the **ordinal sum $\boldsymbol{\sum([a_n, b_n], *_n)}$ of a family of $\boldsymbol{t}$-norms $\boldsymbol{(*_n)}$** is the function $\star : [0, 1]^2 \to [0, 1]$ defined by

$$x \star y = \begin{cases} a_n + (b_n - a_n) \cdot (\frac{x-a_n}{b_n-a_n} *_n \frac{y-a_n}{b_n-a_n}) & \text{if } x, y \in [a_n, b_n], \\ \min(x, y) & \text{otherwise.} \end{cases}$$

[3]Where $\mathfrak{C}$ is any countable indexing set.

The following two theorems show the equivalence between continuous t-norms and ordinal sums of t-norms, each isomorphic to either the Łukasiewicz or the product t-norm.

Theorem 4.38. Every continuous t-norm is the ordinal sum of a family of t-norms each of which is isomorphic to either the Łukasiewicz t-norm of the product t-norm.

Proof. The following proof is an adaptation from [6]. Take an arbitrary continuous t-norm $\star$. For each contact interval $[a_n, b_n] \in \mathcal{I}_\star$, we know that $\star_{[a_n,b_n]}$ is isomorphic to either $\star_{\text{Ł}}$ or $\star_P$. In other words, there exists an isomorphism f_n from $\star_n$ to $\star_{\text{Ł}\vee P}$. Using this isomorphism, we can write $\star_n$ in the following way

$$x \star_n y = f_n^{-1}(f_n(x) \star_{\text{Ł}\vee P} f_n(y)).$$

Next, let us define the two new strictly increasing bijections

$$h_n : [a_n, b_n] \to [0,1]; h_n(x) = \frac{x - a_n}{b_n - a_n} \quad \text{and} \quad g_n : [0,1] \to [0,1]; g_n = f_n \circ h_n^{-1}.$$

With these functions, we can apply Lemma 4.29 to construct the t-norm $*_n$, which is isomorphic to either the Łukasiewicz t-norm or the product t-norm, as follows

$$x *_n y = g_n^{-1}(g_n(x) \star_{\text{Ł}\vee P} g_n(y)).$$

By construction, $f_n^{-1} = h_n^{-1} \circ g_n^{-1}$ and $f_n = g_n \circ h_n$. Thus

$$\begin{aligned}
x \star_n y &= f^{-1}(f_n(x) \star_{\text{Ł}\vee P} f_n(y)) \\
&= (h_n^{-1} \circ g_n^{-1})((g_n \circ h_n)(x) \star_{\text{Ł}\vee P} (g_n \circ h_n)(y)) \\
&= h_n^{-1}\left(g_n^{-1}\left(g_n\left(\frac{x - a_n}{b_n - a_n}\right) \star_{\text{Ł}\vee P} g_n\left(\frac{y - a_n}{b_n - a_n}\right)\right)\right) \\
&= h_n^{-1}\left(\frac{x - a_n}{b_n - a_n} *_n \frac{y - a_n}{b_n - a_n}\right) \\
&= a_n + (b_n - a_n) \cdot \left(\frac{x - a_n}{b_n - a_n} *_n \frac{y - a_n}{b_n - a_n}\right).
\end{aligned}$$

When $\star$ acts on intervals in $\mathcal{J}_\star$, then by the Decomposition Theorem, it behaves like the Gödel t-norm. This concludes the proof. □

Theorem 4.39. Every ordinal sum of family of t-norms, each of which is isomorphic to either the Łukasiewicz t-norm or the product t-norm is a continuous t-norm.

Proof. Let $\star$ be the ordinal sum $\sum_{n \in \mathfrak{C}}([a_n, b_n], *_n)$ where $*_n$ is isomorphic to either the Łukasiewicz t-norm or the product t-norm. We need to show that $\star$ satisfies the following

A. $\star$ is a well-defined function from $[0,1]^2$ to $[0,1]$,

B. $\star$ is a continuous function,

C. $\star$ is a t-norm.

We will denote the isomorphisms for each $*_n$, by g_n, and we will denote by $h_n : [a_n, b_n] \to [0,1]$ a function $h_n(x) = \frac{x-a_n}{b_n-a_n}$ for $x \in [a_n, b_n]$. Note that $g_n, h_n, g_n^{-1}, h_n^{-1}$ are all strictly increasing bijections. Let us do so step by step

A. For $\star$ to be well defined, we need $x \star y \in [0,1]$ for all $x, y \in [0,1]$. Take $x, y \in [a_n, b_n]$ and notice that $\frac{x-a_n}{b_n-a_n}, \frac{y-a_n}{b_n-a_n} \in [0,1]$. Hence, $(\frac{x-a_n}{b_n-a_n} *_n \frac{y-a_n}{b_n-a_n}) \in [0,1]$ form which we conclude that $a_n + (b_n - a_n) \cdot (\frac{x-a_n}{b_n-a_n} *_n \frac{y-a_n}{b_n-a_n}) \in [a_n, b_n] \subseteq [0,1]$. If x, y not both in any contact interval of $\star$, then $\min\{x, y\} \in [0,1]$.

B. We will show that for every sequence $(x_k)_{k\in\mathbb{N}}$ and every sequence $(y_k)_{k\in\mathbb{N}}$, if $\lim_{k\to\infty} x_k = x$ and $\lim_{k\to\infty} y_k = y$, then $\lim_{k\to\infty} x_k \star y_k = x \star y$.
Suppose that there is at least one interval $[a_n, b_n]$ for some $n \in \mathfrak{C}$ such that $x, y \in [a_n, b_n]$, otherwise we deal with the Gödel t-norm which is continuous

(1) Suppose that $x, y \in \{a_n, b_n\}$ for some $n \in \mathfrak{C}$. We show the case $x = y = a_n$ as the other cases are analogous. Take arbitrary sequences $(x_k)_{k\in\mathbb{N}}, (y_k)_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} x_k = x$ and $\lim_{k\to\infty} y_k = y$. We take subsequences (a) $(x'_k)_{k\in\mathbb{N}} \uparrow x$ and (b) $(x''_k)_{k\in\mathbb{N}} \downarrow x$ and for all $k \in \mathbb{N}$, $x''_k < b_n$. At least one of them exists.

(a) Take $(y'_k)_{k\in\mathbb{N}}$ such that for every $k \in \mathbb{N}$, $x'_k \leq y'_k$. Then by Proposition 2.4, $\lim_{k\to\infty} x'_k \star y'_k \leq \lim_{k\to\infty} x'_k \star_G y'_k \lim_{k\to\infty} x'_k = x = x \star y$.

(b) Select $(y''_k)_{k\in\mathbb{N}}$ such that for every $k \in \mathbb{N}$, either (B1) $y''_k < a_n < x''_k$, or (B2) $a_n < y''_k < x''_k$. At least (B1) or (B2) is true.

(b1) $\lim_{k\to\infty} x''_k \star y''_k = \lim_{k\to\infty} y''_k = y = x \star y$.

(b2) $\lim_{k\to\infty} x''_k \star y''_k = x \star y$ by Lemma 2.25 since $g_n, h_n, g_n^{-1}, h_n^{-1}$ are all strictly increasing bijections, and by Theorem 2.5.

(2) Suppose that $x_0, y_0 \in (a_n, b_n)$ for some $n \in \mathfrak{C}$. Take arbitrary sequences $(x_k)_{k\in\mathbb{N}}, (y_k)_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} x_k = x_0$ and $\lim_{k\to\infty} y_k = y_0$. We take subsequences $(x'_k)_{k\in\mathbb{N}}$ and $(y'_k)_{k\in\mathbb{N}}$ such that for every $k \in \mathbb{N}, a_n < x'_k, y'_k < b_n$. Then $\lim_{k\to\infty} x'_k \star y'_k = x_0 \star y_0$ by Lemma 2.25 and Theorem 2.5.

(3) Suppose there is $[a_n, b_n]$ such that one of its endpoint, say c, is such that $\min\{x_0, y_0\} < c < \max\{x_0, y_0\}$. Suppose that $\min\{x_0, y_0\} = x_0$ and $\max\{x_0, y_0\} = y_0$. The other case is analogous.
Take arbitrary sequences $(x_k)_{k\in\mathbb{N}}, (y_k)_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} x_k = x_0$ and $\lim_{k\to\infty} y_k = y_0$. We take subsequences $(x'_k)_{k\in\mathbb{N}}$ and $(y'_k)_{k\in\mathbb{N}}$ such that

for every $k \in \mathbb{N}, x'_k < c < y'_k$. Then $\lim_{k\to\infty} x'_k \star y'_k = \lim_{k\to\infty} x'_k = x_0 = x_0 \star y_0$.

(4) Suppose that $b_n < x_0, y_0 < a_{n+1}$, where b_n is either 0 or the right endpoint of an interval and a_{n+1} is either 1 or the left endpoint of an interval. Suppose that $x_0 \leq y_0$. The other case $y_0 \leq x_0$ is similar. Take arbitrary sequences $(x_k)_{k\in\mathbb{N}}, (y_k)_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} x_k = x_0$ and $\lim_{k\to\infty} y_k = y_0$. We take subsequences $(x'_k)_{k\in\mathbb{N}}$ and $(y'_k)_{k\in\mathbb{N}}$ such that for every $k \in \mathbb{N}, x'_k < y'_k$. Then $\lim_{k\to\infty} x'_k \star y'_k = \lim_{k\to\infty} x'_k = x_0 = x_0 \star y_0$. This completes the proof that $\star$ is continuous.

C. Before we show that $\star$ is a t-norm, we will show that $\star$ satisfies the following

$$x \star y \leq x : x, y \in [0,1]. \tag{6}$$

Suppose that $x, y \in [a_n, b_n]$. Then

$$x \star y = a_n + (b_n - a_n) \cdot g_n^{-1}\left(g_n\left(\frac{x-a_n}{b_n-a_n}\right) \star_{\text{Ł}\vee P} g_n\left(\frac{y-a_n}{b_n-a_n}\right)\right).$$

As $g_n\left(\frac{y-a_n}{b_n-a_n}\right) \leq 1$, we know that

$$g_n\left(\frac{x-a_n}{b_n-a_n}\right) \star_{\text{Ł}\vee P} g_n\left(\frac{y-a_n}{b_n-a_n}\right) \leq g_n\left(\frac{x-a_n}{b_n-a_n}\right) \star_{\text{Ł}\vee P} 1 = g_n\left(\frac{x-a_n}{b_n-a_n}\right)$$

Combined, we conclude the following

$$x \star y \leq a_n + (b_n - a_n) \cdot g_n^{-1}\left(g_n\left(\frac{x-a_n}{b_n-a_n}\right)\right) = x.$$

The case that there is no $[a_n, b_n]$ that x, y belong to, $x \star y = \min(x, y) \leq x$. This completes the proof of the claim. Now, we will show that $\star$ satisfies the axioms in Definition 4.1.

A1 It is clear from Definition 4.37 to see that $\star$ is indeed commutative.

A2 To show that $\star$ is associative, we need to show that $(x\star y)\star z = x\star(y\star z)$ for all $x, y, z \in [0,1]$. We will divide this into five cases. Consider arbitrary contact interval $[a_n, b_n]$.

(a) Suppose $x, y, z \in [a_n, b_n]$. Then

$$\frac{x \star y - a_n}{b_n - a_n} = g_n^{-1}\left(g_n\left(\frac{x-a_n}{b_n-a_n}\right) \star_{\text{Ł}\vee P} g_n\left(\frac{y-a_n}{b_n-a_n}\right)\right).$$

Using this, we know that $(x \star y) \star z$ is equal to

$$a_n + (b_n - a_n) g_n^{-1}\left(g_n\left(\frac{x-a_n}{b_n-a_n}\right) \star_{\text{Ł}\vee P} g_n\left(\frac{y-a_n}{b_n-a_n}\right) \star_{\text{Ł}\vee P} g_n\left(\frac{z-a_n}{b_n-a_n}\right)\right).$$

Similarly, $x \star (y \star z)$ is equal to

$$a_n+(b_n-a_n)g_n^{-1}\left(g_n\left(\frac{x-a_n}{b_n-a_n}\right) \star_{Ł\vee P} g_n\left(\frac{y-a_n}{b_n-a_n}\right) \star_{Ł\vee P} g_n\left(\frac{z-a_n}{b_n-a_n}\right)\right).$$

(b) Suppose $x, y \in [a_n, b_n]$ and $z \notin [a_n, b_n]$. There are two subcases: either $z < a_n$ or $z > b_n$. The former yields $(x \star y) \star z = z$ and $x \star (y \star z) = z$. The latter yields $(x \star y) \star z = x \star y$ and $x \star (y \star z) = x \star y$.

(c) Suppose $y, z \in [a_n, b_n]$ and $x \notin [a_n, b_n]$. The proof is similar to (b).

(d) Suppose $x, z \in [a_n, b_n]$ and $y \notin [a_n, b_n]$. The proof is similar to (b).

(e) Suppose there is no $[a_n, b_n]$ such that at least two x, y, z belong to it. Then $(x \star y) \star z = \min(x, y, z) = x \star (y \star z)$.

A3 For $\star$ to be monotonic, it needs to satisfy $x \star y \leq x \star z$ for arbitrarily elements $x, y, z \in [0, 1]$ with $y \leq z$. Again, we will divide this into four cases. Take an arbitrary contact interval $[a_n, b_n]$.

(a) Suppose that $x, y, z \in [a_n, b_n]$. Then $x \star y \leq x \star z$ by the fact that $g_n, h_n, g_n^{-1}, h_n^{-1}$ are all strictly increasing bijections.

(b) Suppose that $x, y \in [a_n, b_n]$ and $z \notin [a_n, b_n]$. Since $x \star y \leq x$ and $x \star y \leq y \leq z$, by definition of the ordinal sum, $x \star y \leq \min(x, z) = x \star z$.

(c) Suppose $x, z \in [a_n, b_n]$ and $y \notin [a_n, b_n]$. Then $y < a_n$ and $x \star z \geq a_n$. Thus, $x \star z \geq y = \min(x, y) = x \star y$.

(d) Suppose there is no interval $[a_n, b_n]$ such that x and at least one of y, z belong to it. Then $x \star y = \min(x, y) \leq \min(x, z) = x \star z$.

A4 If there exists a contact interval $[a_n, 1]$ such that $x \in [a_n, 1]$, then $x \star 1 = x$ by properties of $g_n, h_n, g_n^{-1}, h_n^{-1}$ and t-norms. Otherwise, $x \star 1 = \min(x, 1) = x$.

This completes the proof of the theorem. □

Together, the two theorems yield the following corollary.

Corollary 4.40. The following two statements are equivalent:

1. The function $\star : [0, 1]^2 \to [0, 1]$ is a continuous t-norm.
2. The function $\star$ is the ordinal sum of a family of t-norms each of which is isomorphic to either the Łukasiewicz t-norm or the product t-norm.

Similarly, as every continuous t-norm admits a unique residuum, we now show how the residuum can be defined for an ordinal sum."

Theorem 4.41. Let $\star$ be the ordinal sum $\sum_{n\in\mathfrak{C}}([a_n,b_n],*_n)$ of t-norms $(*_n)_{n\in\mathfrak{C}}$ such that $*_n$ is either isomorphic to the $\star_{\text{Ł}}$ or $\star_P$. Let g_n denote the isomorphisms for $*_n$. Then the residuum of $\star$ is given by

$$x \Rightarrow y = \begin{cases} 1 & \text{if } x \le y \\ a_n + (b_n - a_n)\, g_n^{-1}\left(g_n\left(\frac{x-a_n}{b_n-a_n}\right) \Rightarrow_{\text{Ł}\vee P} g_n\left(\frac{y-a_n}{b_n-a_n}\right)\right) & \text{if } x > y \text{ and } x, y \in [a_n, b_n] \\ y & \text{otherwise} \end{cases} \tag{3}$$

Proof. The case for $x \le y$ is obvious. Let $x > y$.

Suppose that $x, y \in [a_n, b_n]$. By Proposition 4.4 and the fact that we can write $[0,1] = [0,a_n) \cup [a_n,b_n] \cup (b_n,1]$, we know that

$$x \Rightarrow y = \max\left\{\sup_{z\in[0,a_n)}\{x\star z \le\ y\},\ \sup_{z\in[a_n,b_n]}\{x\star z\le y\},\ \sup_{z\in(b_n,1]}\{x\star z\le y\}\right\}.$$

It is easy to check that

$$\sup_{z\in[0,a_n)}\{x\star z\le y\} = a_n, \text{ and } \{z\in(b_n,1] : x\star z\le y\} = \emptyset.$$

Therefore, all that is left to compute is $\sup_{z\in[a_n,b_n]}\{x\star z\le y\}$. Suppose that on $[a_n,b_n], *_n$ is isomorphic to the $\star_P$. Then

$$x\star z = a_n + (b_n-a_n)\cdot g_n^{-1}\left(g_n\left(\frac{x-a_n}{b_n-a_n}\right)\cdot g_n\left(\frac{z-a_n}{b_n-a_n}\right)\right) \le y.$$

Consider the right inequality. By fixing x and writing z in terms of y, we are able to obtain the following inequality

$$z \le a_n + (b_n-a_n)\cdot g_n^{-1}\left(\frac{g_n\left(\frac{y-a_n}{b_n-a_n}\right)}{g_n\left(\frac{x-a_n}{b_n-a_n}\right)}\right).$$

Since $a_n \le a_n + (b_n-a_n)\, g_n^{-1}\left(\frac{g_n\left(\frac{y-a_n}{b_n-a_n}\right)}{g_n\left(\frac{x-a_n}{b_n-a_n}\right)}\right)$, we conclude that

$$x\Rightarrow y = a_n + (b_n-a_n)\cdot g_n^{-1}\left(\frac{g_n\left(\frac{y-a_n}{g_n-a_n}\right)}{g_n\left(\frac{x-a_n}{b_n-a_n}\right)}\right).$$

Suppose that on $[a_n,b_n], *_n$ is isomorphic to the Łukasiewicz t-norm. Then,

$$x \star z = a_n + (b_n - a_n) \cdot g_n^{-1} \left(\max \left\{ 0, g_n \left(\frac{x - a_n}{b_n - a_n} \right) + g_n \left(\frac{z - a_n}{b_n - a_n} \right) - 1 \right\} \right) \leq y.$$

Again, we can write z in terms of y by fixing x to obtain.

$$z \leq a_n + (b_n - a_n) \cdot g_n^{-1} \left(1 - g_n \left(\frac{x - a_n}{b_n - a_n} \right) + g_n \left(\frac{y - a_n}{b_n - a_n} \right) \right)$$

As $a_n \leq a_n + (b_n - a_n) \cdot g_n^{-1} \left(1 - g_n \left(\frac{x-a_n}{b_n-a_n} \right) + \ g_n \left(\frac{y-a_n}{b_n-a_n} \right) \right)$, we can conclude that

$$x \Rightarrow y = a_n + (b_n - a_n) \cdot g_n^{-1} \left(1 - g_n \left(\frac{x - a_n}{b_n - a_n} \right) + g_n \left(\frac{y - a_n}{b_n - a_n} \right) \right).$$

Thus,

$$x \Rightarrow y = a_n + (b_n - a_n) \cdot g_n^{-1} \left(g_n \left(\frac{x - a_n}{b_n - a_n} \right) \Rightarrow_{\text{Ł}\vee P} g_n \left(\frac{y - a_n}{b_n - a_n} \right) \right).$$

Suppose now that there is no $[a_n, b_n]$ such that $x, y \in [a_n, b_n]$. Then $x \Rightarrow y = \sup\{z : x \star z \leq y\}$. If $z \leq y$, then $x \star z \leq x \star y = y$. If $z > y$, then by 4.13, we know $x \star z \leq \min\{x, z\}$, $x \star z \leq \min\{x, z\}$ and thus we have $x \star z \leq \min\{y, z\}$ by assumption of $x > y$.

Suppose now that there is no $[a_n, b_n]$ such that $x, y \in [a_n, b_n]$ and $x > y$. By Proposition 4.4, we write

$$x \Rightarrow y = \sup\{z : x \star z \leq y\}.$$

Take an arbitrary $z \in [0, 1] : x \star z \leq y$. Notice that $x, z \notin [a_n, b_n]$ for $n \in \mathfrak{C}$ as $x \star z \leq \min\{x, z\} \in [a_n, b_n]$ and $y < a_n$. By Definition 4.37, $\min\{x, z\} = x \star z \leq y$. As a $x > y$, the residuum transforms to

$$x \Rightarrow y = \sup\{z : x \star z \leq y\} = \sup\{z : \min\{x, z\} \leq y\} = \sup\{z : z \leq y\} = y.$$

This completes the proof of the theorem. □

4.4 Tableau Calculus

Having shown that any continuous t-norm can be expressed as an ordinal sum of t-norms, each isomorphic to either the Łukasiewicz or the product t-norm, we now turn to the construction of the tableau calculus. Similarly to [6], the tableau rules will be designed with the framework given by the Decomposition Theorem 4.36 in mind. Before presenting the tableau rules themselves, we first introduce the formal framework required to prove the soundness and completeness of the tableau calculus.

This formal framework begins with a method to construct a tableau from a statement we wish to prove. Unlike in Classical Propositional Logic, where tableau formulas are wffs in language $\mathcal{L}$, tableau formulas in Basic Logic take the form of binary relations between terms.

The reason for this difference is because in Classical Propositional Logic, the truth values of wffs are either 0 or 1, so their validity can be handled through syntactic manipulation of logical formulas. In Basic Logic however, truth values of wffs range over $[0, 1]$, where logical operations are defined via continuous t-norms and their associated residua. Therefore, reasoning about wffs in Basic Logic necessarily involves reasoning about equality relations between truth degrees. This motivates the following definitions.

Definition 4.42. Let $L_0 = Par \cup \{+, -, \min, \max, \leq, <, 0, 1\}$ and $L_1 = L_0 \cup \{\star, \Rightarrow\}$ be signatures, where Par is the set of constants (parameters), $0, 1$ are constants, $+, -, \min, \max \star, \Rightarrow$ are binary function symbols and $\leq, <$ are binary relation symbols. Let Var be an infinite set of variables.

1. If x, y are L_1 terms, then $x \leq y, x < y, x = y$ are **tableau formulas**.
2. Let E be the set of tableau formulas $s \leq t, s < t, s = t$, where s, t are L_0-terms. Let $\sigma : Var \to [0, 1]$ be a mapping, and $\mathcal{M} = (\mathbb{R}, +, -, \min, \max, \leq, <, 0, 1, \rho)$ be an L_0-structure, where $+, -, \min, \max, 0, 1, \leq, <$ are interpreted as usual and $\rho : Par \to [0, 1]$ is a function. The pair $(\mathcal{M}, \sigma)$ is a **solution** of E iff $M, \sigma \vDash \bigwedge E$. By convention $\bigwedge \emptyset = \top$ (verum). We denote by $[\![z]\!]^{\mathcal{M},\sigma}$ the value of L_0-term z in $\mathcal{M}$ under the assignment σ.

Definition 4.43. Let $\mathcal{F}$ be the set of formulas of Basic Logic and $\mathbb{T}$ be the set of L_1-terms. Let $\mu : \mathcal{P} \to Var$ (we will write $\mu(p)$ as μ_p) be a one-to-one mapping assigning variables to propositional atoms. Then, we define a **translation function** $\tau : \mathcal{F} \to \mathbb{T}$, inductively:

1. $\tau(\bar{0}) = 0, \quad \tau(\bar{1}) = 1,$	4. $\tau(\psi \to \varphi) = \tau(\psi) \Rightarrow \tau(\varphi),$
2. $\tau(p) = \mu_p$ for every $p \in \mathcal{P}$,	5. $\tau(\psi \vee \varphi) = \max\{\tau(\psi), \tau(\varphi)\},$
3. $\tau(\psi \,\&\, \varphi) = \tau(\psi) \star \tau(\varphi),$	6. $\tau(\psi \wedge \varphi) = \min\{\tau(\psi), \tau(\varphi)\}.$

Now that we have a method to translate the wffs in $\mathcal{F}$ into tableau formulas, we are ready to define the tableau itself. From Definition 4.11, a formula ψ is logically valid in Basic Logic if and only if $V_\star(\psi) = 1$ for every valuation $V_\star$. Following the general approach of tableau calculus in Classical Propositional Logic, we start by assuming $\tau(\psi) < 1$ and attempt to derive a contradiction. We now present the formal definitions required to construct tableaux for BL.

Definition 4.44. A **tableau** $\mathcal{T}$ for a formula ψ in Basic Logic is a tree whose root is given by $\{\tau(\psi) < 1\}$. To expand the tableau, we proceed as follows. Select a leaf from any open branch $l_{\mathcal{B}}$ that contains a formula containing either $x \star y$ or $x \Rightarrow y$ with set of parameters Par consisting of elements $a_0 < b_0 \leq a_1 < b_1 \leq \cdots \leq a_{n-1} < b_{n-1}$, representing the contact intervals that have been introduced so far (this set is initially empty). Then there are two cases, either x, y belong to some contact interval $[a, b] \in \mathcal{I}_\star$ or they do not. We call these cases **inclusion case** and **exclusion case** respectively.

Inclusion case: if we assume that $x, y \in [a_n, b_n]$, then there are 5 possible cases.

Case 1. If no parameters have been selected in previous step, then $\mathcal{I} = \{0 \leq a_n < b_n \leq 1\}$,

Case 2. $\mathcal{I} = \{0 \leq a_n < b_n \leq a_0\}$,

Case 3. $\mathcal{I} = \{a_n = a_i < b_n = b_i\}$ for each $0 \leq i \leq n-1$,

Case 4. $\mathcal{I} = \{b_i \leq a_n < b_n \leq a_{i+1}\}$ for each $0 \leq i \leq n-2$,

Case 5. $\mathcal{I} = \{b_{n-1} \leq a_n < b_n \leq 1\}$.

Exclusion: If we assume that x, y are not in the same contact interval, then there are 6 possible cases.

Case 6. If no parameters have been selected in previous step, then $\mathcal{J} = \emptyset$,

Case 7. $\mathcal{J} = \{0 \leq x \leq a_0, x \neq y\}$,

Case 8. $\mathcal{J} = \{y \leq a_i \leq x \leq b_i, x \neq y\}$ for each $0 \leq i \leq n-1$,

Case 9. $\mathcal{J} = \{a_i \leq x \leq b_i \leq y, x \neq y\}$ for each $0 \leq i \leq n-1$,

Case 10. $\mathcal{J} = \{b_i \leq x \leq a_{i+1}\}$ for each $0 \leq i \leq n-2$,

Case 11. $\mathcal{J} = \{b_{n-1} \leq x \leq 1\}$.

With these inclusion and exclusion cases, the **tableau rules** are defined as follows:

($\star$) : node $l_{\mathcal{B}}$ expands to its new successor $l'_{\mathcal{B}}$ in the following way:

ŁP rule:	If case 1 - 5 are assumed, then the successor $l'_{\mathcal{B}}$ is defined as follows: $l'_{\mathcal{B}} := l_{\mathcal{B}}[\max\{a_n, x+y-b_n\}/x \star y] \cup \mathcal{I} \cup \{a_n \leq x, y \leq b_n\}$.
min rule:	If case 6 - 11 are assumed, then the successor $l'_{\mathcal{B}}$ is defined as follows: $l'_{\mathcal{B}} := l_{\mathcal{B}}[\min\{x, y\}/x \star y] \cup \mathcal{J}$.

(⇒) : node $l_{\mathcal{B}}$ expands to its new successor $l'_{\mathcal{B}}$ in the following way:

All rule:	This rule can always be applied to define the successor $l'_{\mathcal{B}}$ as follows: $l'_{\mathcal{B}} := l_{\mathcal{B}}[1/x \Rightarrow y] \cup \{x \leq y\}$.
ŁP rule:	If cases 1 - 5 are assumed, then the successor $l'_{\mathcal{B}}$ is defined as follows: $l'_{\mathcal{B}} = l_{\mathcal{B}}[b_n - x + y/x \Rightarrow y] \cup \mathcal{I} \cup \{a_n \leq y < x \leq b_n\}$.
min rule:	If cases 6 - 11 are assumed, then the successor $l'_{\mathcal{B}}$ is defined as follows: $l'_{\mathcal{B}} = l_{\mathcal{B}}[y/x \star y] \cup \mathcal{J} \cup \{y < x\}$.

Where $l_{\mathcal{B}}[\delta/\epsilon]$ represents every occurrence of $\epsilon \in l_{\mathcal{B}}$, gets replaced by δ.[4]

Similarly to Classical Propositional Logic, we wish to define provability for any formula $\psi \in \mathcal{F}$. Therefore, we now state the following definitions.

Definition 4.45. For each branch $\mathcal{B}$ of a tableau and each node $N \in \mathcal{B}$, we consider the set $N|_{L_0}$ of L_0-formulas in n.

- A solution $(\mathcal{M}, \sigma)$ of N_{L_0} is **nice** iff for every marked x^Γ, y^Γ of L_0-terms, where $\Gamma \in \mathcal{B}$, there are no parameters a, b with $a < b$ occurring at any node of $\mathcal{B}$ such that we have $[\![a]\!]^{\mathcal{M},\sigma} \leq [\![x^\Gamma]\!]^{\mathcal{M},\sigma}, [\![y^\Gamma]\!]^{\mathcal{M},\sigma} \leq [\![b]\!]^{\mathcal{M},\sigma}$.
- We say that $\mathcal{B}$ is **closed** if for some node $N \in \mathcal{B}$, $N|_{L_0}$ has no nice solution, otherwise it is **open**. Tableau $\mathcal{T}$ is **closed** if it only contains closed branches. A tableau $\mathcal{T}$ is **open** if it contains an open branch.

Definition 4.46. A branch of a tableau is **complete** iff it is closed or all tableau rules which can be applied on the branch have been applied. If all branches are complete, then the tableau itself is **complete**.

Definition 4.47. A wff $\psi \in \mathcal{F}$ is **provable** iff its tableau $\mathcal{T}$ with root $\{\tau(\psi) < 1\}$ closed and complete.

Example 4.48. Again, we wish to verify the provability wff $(p \to q) \vee (q \to p)$. Consider the root

$$\{\tau(\psi) < 1\} = \{\max\{\mu_p \Rightarrow \mu_q, \mu_q \Rightarrow \mu_p\} < 1\}.$$

This tableau formula contains two L_1-terms, which would need application of 3 rules ($\Rightarrow$: All, $\Rightarrow$: ŁP-1, $\Rightarrow$: min-6), followed by an application of 0, 7, and 7 rules ($\Rightarrow$: All, $\Rightarrow$: ŁP-1, $\Rightarrow$: min-6), ($\Rightarrow$: ŁP-2, $\Rightarrow$: ŁP-3, $\Rightarrow$: ŁP-5, $\Rightarrow$: min-7, $\Rightarrow$: min-8, $\Rightarrow$: min-9, $\Rightarrow$: min-11), ($\Rightarrow$: ŁP-2, $\Rightarrow$: ŁP-3, $\Rightarrow$: ŁP-5, $\Rightarrow$: min-7, $\Rightarrow$: min-8, $\Rightarrow$: min-9, $\Rightarrow$: min-11). In total, this would result in 15 different branches. Due to the size of the paper, we illustrate only a single branch. This branch will describe the rules ($\Rightarrow$: ŁP-1$-$ $\Rightarrow$: min-7) in order.

[4] In the proofs of soundness and completeness mentioned in Section 4.5, we will replace $l_{\mathcal{B}}$ by γ as we talk about arbitrary nodes.

$$\{ \quad \max\{\mu_p \Rightarrow \mu_q, \mu_q \Rightarrow \mu_p\} < 1 \quad \}$$

$$\downarrow$$

$$\begin{aligned}\{ \quad &\max\{b_0 - \mu_p + \mu_q, \mu_q \Rightarrow \mu_p\} < 1,\\ &a_0 \le \mu_q < \mu_p \le b_0,\\ &0 \le a_0 < b_0 < 1 \quad \}\end{aligned}$$

$$\downarrow$$

$$\begin{aligned}\{ \quad &\max\{b_0 - \mu_p + \mu_q, \mu_p\} < 1,\\ &a_0 \le \mu_q < \mu_p \le b_0,\\ &0 \le a_0 < b_0 < 1,\\ &0 \le \mu_q \le a_0, \mu_q \ne \mu_p,\\ &\mu_p < \mu_q \quad \}\end{aligned}$$

This tableau closes as both $\mu_q < \mu_p$ (also $a_0 < \mu_q$ and $\mu_q < a_0$). Similarly, we can prove that the other 14 cases also close. Hence, this formula is provable in Basic Logic.

4.5 Soundness & Completeness

Both soundness and completeness theorems have been adapted from [6] in a way to match the syntax for this thesis. Before we prove the soundness and completeness of our tableau calculus, we will first define the following t-norm and its residue.

Definition 4.49. Let $\mathcal{B}$ be a branch of a tableau $\mathcal{T}$. Let $0 \le a_0 < b_0 \le \cdots \le a_{n-1} < b_{n-1} \le 1$ be parameters introduced by the tableau rules in $\mathcal{B}$. Let $l_{\mathcal{B}}$ be the leaf of the branch. Suppose that $(\mathcal{M}, \sigma)$ is a nice solution of $l_{\mathcal{B}}$.

1. We expand the model $\mathcal{M}$ to a L_1-stucture $\mathcal{M}_{\mathcal{B}} = (\mathbb{R}, +, -, \min, \max, \le, <, 0, 1, \rho, \star_{\mathcal{B}}, \Rightarrow_{\mathcal{B}})$ where the functions $\star_{\mathcal{B}}$ and $\Rightarrow_{\mathcal{B}}$ are defined on $\mathbb{R}^2$ as follows:

$$v \star_{\mathcal{B}} w = \begin{cases} \max\{\rho(a_k), v + w - \rho(b_k)\} & \text{if } v, w \in [\rho(a_k), \rho(b_k)], 0 \le k \le n-1,\\ \min\{v, w\} & \text{otherwise.}\end{cases} \tag{7}$$

$$v \Rightarrow_{\mathcal{B}} w = \begin{cases} 1 & \text{if } v \le w,\\ \rho(b_k) - v + w & \text{if } \rho(a_k) \le w < v \le \rho(b_k), 0 \le k \le n-1,\\ w & \text{otherwise.}\end{cases} \tag{8}$$

These functions are well-defined, but k in $\star_{\mathcal{B}}$ is not always unique.

2. A node S of a branch $\mathcal{B}$ is **$\mathcal{B}$-satisfiable via $\mathcal{M}, \sigma$** iff $\mathcal{M}_{\mathcal{B}}, \sigma \vDash \bigwedge S$ where the L_1-structure $\mathcal{M}_{\mathcal{B}}$ is constructed from $\mathcal{M}$ as in point 1.

Now we say that branch $\mathcal{B}$ is **satisfiable** iff there exists $\mathcal{M}, \sigma$ such that all nodes of $\mathcal{B}$ are $\mathcal{B}$-satisfiable via $\mathcal{M}, \sigma$. We denote by $[\![z]\!]^{\mathcal{M}_{\mathcal{B}},\sigma}$ the value of L_1-term z in $\mathcal{M}_{\mathcal{B}}$ under the assignment σ.

Remark 4.50. We constructed these functions such that $\star_{\mathcal{B}}$ is the ordinal sum $\Sigma_{0\leq k\leq n-1}([\rho(a_k), \rho(b_k)], *_k)$, where $*_k$ isomorphic to the Łukasiewicz t-norm, and its residuum is derived from Theorem 4.41. As a result, we can conclude that both functions restricted to $[0, 1]$ are continuous.

Theorem 4.51. Let $\mathcal{T}$ a tableau with a root $\{\tau(\psi) < 1\}$. If $\mathcal{T}$ has an open branch, then there is a model $\mathcal{N} = ([0, 1], \star, \Rightarrow, \min, \max, 0, 1, V)$, where $\star$ is a continuous t-norm, $\Rightarrow$ is the residuum of $\star$, and $V : \mathcal{P} \to [0, 1]$ such that $V_\star(\psi) < 1$.

Proof. Suppose that branch $\mathcal{B}$ of $\mathcal{T}$ is open. In other words, there exists a nice solution $(\mathcal{M}, \sigma)$ of leaf $\mathcal{B}$. Define the following functions

$$V(p) := \sigma(\mu_p) : p \in \mathcal{P}, \quad \star := \star_{\mathcal{B}}|_{[0,1]}, \quad \Rightarrow := \Rightarrow_{\mathcal{B}} |_{[0,1]}.$$

By Theorems 4.39 and 4.41, $\star_{\mathcal{B}}$ is a continuous t-norm and $\Rightarrow_{\mathcal{B}}$ is its residuum when both functions are restricted to $[0, 1]$. All that remains to show is that $V_\star(\psi) < 1$.

Claim 1. Let θ be a subformula of ψ. Then $V_\star(\theta) = \sigma_{\mathcal{M}_{\mathcal{B}}}(\tau(\theta))$.

Proof. We prove by induction on θ. Base case: let $\theta = \bar{0}$ or $\theta = \bar{1}$. By Definitions 4.10, 4.43, 4.49

$$V_\star(\bar{0}) = 0 = \sigma_{\mathcal{M}_{\mathcal{B}}}(0) = \sigma_{\mathcal{M}_{\mathcal{B}}}(\tau(0)),$$
$$V_\star(\bar{1}) = 1 = \sigma_{\mathcal{M}_{\mathcal{B}}}(1) = \sigma_{\mathcal{M}_{\mathcal{B}}}(\tau(1)).$$

Assume the induction hypothesis for θ and φ. We show the claim for $\theta \to \varphi$. The other cases are familiar.

$$V_\star(\theta \to \varphi) = V_\star(\theta) \Rightarrow V_\star(\varphi) = V_\star(\theta) \Rightarrow_{\mathcal{B}} V_\star(\varphi) = \sigma_{\mathcal{M}_{\mathcal{B}}}(\tau(\theta)) \Rightarrow_{\mathcal{B}} \sigma_{\mathcal{M}_{\mathcal{B}}}(\tau(\varphi))$$
$$= \sigma_{\mathcal{M}_{\mathcal{B}}}(\tau(\theta) \Rightarrow \tau(\varphi)) = [\![\tau(\theta \to \varphi)]\!]^{\mathcal{M}_{\mathcal{B}},\sigma}.$$

□

Claim 2. Every node $m_{\mathcal{B}}$ of $\mathcal{B}$ is $\mathcal{B}$-satisfiable via $\mathcal{M}, \sigma$.

Proof. As $\mathcal{B}$ is open, we know that there exists some node $m_{\mathcal{B}}$ with a solution $(\mathcal{M}, \sigma)$ We will prove this claim by use of induction on $h(m_{\mathcal{B}}, \mathcal{B})$.
Base case: If $h(m_{\mathcal{B}}, \mathcal{B}) = 0$, $m_{\mathcal{B}}$ is the leaf of the branch. As we assume $\mathcal{B}$ to be open, we know that there exists a solution $(\mathcal{M}, \sigma)$ such that $\mathcal{M}, \sigma \vDash \bigwedge m_{\mathcal{B}}$.
Induction hypothesis: Suppose that $m_{\mathcal{B}} \in \mathcal{B}$ is $\mathcal{B}$-satisfiable via $\mathcal{M}, \sigma$.
Induction step: Let $m'_{\mathcal{B}} \in \mathcal{B}$ be the predecessor of $m_{\mathcal{B}}$. In other words, $h(m'_{\mathcal{B}}, \mathcal{B}) =$

$h(m_{\mathcal{B}}, \mathcal{B}) + 1$. As $(m_{\mathcal{B}} \cup m'_{\mathcal{B}}, m_{\mathcal{B}} - m'_{\mathcal{B}} \subseteq m_{\mathcal{B}}$, we have $\mathcal{M}_{\mathcal{B}}, \sigma \vDash \bigwedge(m_{\mathcal{B}} \cup m'_{\mathcal{B}})$ and $\mathcal{M}_{\mathcal{B}}, \sigma \vDash \bigwedge(m_{\mathcal{B}} - m'_{\mathcal{B}})$.[5]

We will prove this by induction on the height of node $m_{\mathcal{B}}$ of $\mathcal{B}$. As $\mathcal{B}$ is open, $l_{\mathcal{B}}$ is $\mathcal{B}$-satisfiable via $\mathcal{M}, \sigma$. Let $m'_{\mathcal{B}}$ be the node on $\mathcal{B}$ such that $h(m'_{\mathcal{B}}, \mathcal{B}) = h(m_{\mathcal{B}}, \mathcal{B}) + 1$, where $m_{\mathcal{B}}$ is assumed inductively to be $\mathcal{B}$-satisfiable via $\mathcal{M}, \sigma$. Since $\mathcal{M}_{\mathcal{B}}, \sigma \vDash m_{\mathcal{B}}, \mathcal{M}_{\mathcal{B}}, \sigma \vDash \bigwedge(m_{\mathcal{B}} \cup m'_{\mathcal{B}})$ and $\mathcal{M}_{\mathcal{B}}, \sigma \vDash \bigwedge(m_{\mathcal{B}} - m'_{\mathcal{B}})$. By inspecting $m_{\mathcal{B}}$ and $m'_{\mathcal{B}}$, we know which tableau formulas γ are in $m'_{\mathcal{B}} - m_{\mathcal{B}}$ and thus by Definition 3.8, which branch expansion rule has been applied to generate $m_{\mathcal{B}}$ from its predecessor $m'_{\mathcal{B}}$. We know which parameters of $\{a_i, b_i : i \in \mathbb{N}\}$ occur in $m'_{\mathcal{B}}$. We show the claim for one rule and one case. The other rules and cases can be shown analogously. Suppose that $0 \leq a_0 < b_0 \leq a_1 < b_1 \leq 1 \in m'_{\mathcal{B}}$. Suppose that the active term in $m'_{\mathcal{B}}$ is $x \Rightarrow y$, where x, y are L_0-terms, and the rule and case used to generate $m_{\mathcal{B}}$ is $(\Rightarrow)\min$. Case 9, where $i = 0$. We have x, y marked and let us denote $x^{m'_{\mathcal{B}}}$ and $y^{m'_{\mathcal{B}}}$ as u, v respectively. Thus, in $m_{\mathcal{B}} - m'_{\mathcal{B}}$ we have $b_0 \leq u \leq a_1, v < u, \gamma[v/u \Rightarrow v]$ for all $\gamma \in m'_{\mathcal{B}} - m_{\mathcal{B}}$. Since $(\mathcal{M}, \sigma)$ is nice, there are no parameters a, b with $a < b$ occurring in $l_{\mathcal{B}}$ such that $[\![u]\!]^{\mathcal{M}_{\mathcal{B}},\sigma}, [\![v]\!]^{\mathcal{M}_{\mathcal{B}},\sigma} \in [\rho(a), \rho(b)]$. By Definitions 4.42, 4.45, and 4.49,

$$[\![u \Rightarrow v]\!]^{\mathcal{M}_{\mathcal{B}},\sigma} = [\![u]\!]^{\mathcal{M}_{\mathcal{B}},\sigma} \Rightarrow_{\mathcal{B}} [\![v]\!]^{\mathcal{M}_{\mathcal{B}},\sigma} = [\![v]\!]^{\mathcal{M}_{\mathcal{B}},\sigma}.$$

So for every formula $\gamma \in m'_{\mathcal{B}} - m_{\mathcal{B}}$, we have $\mathcal{M}_{\mathcal{B}}, \sigma \vDash \gamma$ iff $\mathcal{M}_{\mathcal{B}}, \sigma \vDash \gamma[v/u \Rightarrow v]$. Since $\gamma[v/u \Rightarrow v] \in m_{\mathcal{B}} - m'_{\mathcal{B}}$, the induction hypothesis gives $\mathcal{M}_{\mathcal{B}}, \sigma \vDash \gamma[v/u \Rightarrow v]$. So $\mathcal{M}_{\mathcal{B}}, \sigma \vDash \gamma$. This holds for all $\gamma \in m'_{\mathcal{B}} - m_{\mathcal{B}}$. Therefore, $\mathcal{M}_{\mathcal{B}}, \sigma \vDash \bigwedge(m'_{\mathcal{B}} - m_{\mathcal{B}})$. This completes the proof of the claim. □

From these two claim, we can conclude that the root of $\mathcal{T}$ is $\mathcal{B}$-satisfiable via $\mathcal{M}, \sigma$, and thus $[\![\tau(\psi)]\!]^{\mathcal{M}_{\mathcal{B}},\sigma} = V_\star(\psi) < 1$.

□

Next, we will prove the converse. Before we do so, we need the following Lemmas alongside their proofs which have been adapted from [6].

Lemma 4.52. Let $\star_1, \star_2$ be continuous t-norms and let f be an isomorphism from $\star_1$ to $\star_2$. Let $V_1, V_2 : [0,1] \to [0,1]$ be interpretations such that $f(V_1(p)) = V_2(p)$ for every $p \in \mathcal{P}$. Then $f\left((V_1)_{\star_1}(\psi)\right) = (V_2)_{\star_2}(\psi)$ for every $\psi \in \mathcal{F}$.

Proof. The cases for atomic formulas are obvious. Assume the induction hypothesis for θ, φ. Then by properties of a t-norm, the induction hypothesis, the semantics of Basic Logic and the fact that f is an isomorphism from $\star_1$ to $\star_2$,

$$(V_2)_{\star_2}(\theta \& \varphi) = f\left((V_1)_{\star_1}(\theta \& \varphi)\right).$$

[5] In this context $m_{\mathcal{B}} - m'_{\mathcal{B}}$ are the L_1-terms appearing in $m_{\mathcal{B}}$ but not in $m'_{\mathcal{B}}$.

Let $\psi = \theta \to \varphi$. Take an arbitrary $z \in [0,1]$. Then, by the semantics of Basic Logic and Proposition 4.4, $z \leq (V_2)_{\star_2}(\theta \to \varphi)$ iff $z \star_2 (V_2)_{\star_2}(\theta) \leq (V_2)_{\star_2}(\varphi)$. By the induction hypothesis, this is equivalent to $z \star_2 f\left((V_1)_{\star_1}(\theta)\right) \leq f\left((V_1)_{\star_1}(\varphi)\right)$. Since f is a bijection, there exists $t \in [0,1]$ such that $t = f^{-1}(z)$. Therefore,

$$z \star_2 f\left((V_1)_{\star_1}(\theta)\right) \leq f\left((V_1)_{\star_1}(\varphi)\right) \text{ iff } f(t) \star_2 f\left((V_1)_{\star_1}(\theta)\right) \leq f\left((V_1)_{\star_1}(\varphi)\right).$$

By properties of f, this is equivalent to $t \star_1 (V_1)_{\star_1}(\theta) \leq (V_1)_{\star_1}(\varphi)$. By Proposition 4.4, the semantics of BL, we have $t \leq (V_1)_{\star_1}(\theta \to \varphi)$, and by properties of f, we have further equivalence $z \leq f\left((V_1)_{\star_1}(\theta \to \varphi)\right)$. Since z is arbitrary,

$$(V_2)_{\star_2}(\theta \to \varphi) = f\left((V_1)_{\star_1}(\theta \to \varphi)\right).$$

□

Lemma 4.53. Let $\star_1$ and $\star_2$ be the ordinal sums $\sum_{n \in \mathfrak{C}}([a_n, b_n], *_{1,n})$ and $\sum_{n \in \mathfrak{C}}([a_n, b_n], *_{2,n})$, respectively, of t-norms $(*_{1,n})_{n \in \mathfrak{C}}$ and $(*_{2,n})_{n \in \mathfrak{C}}$, respectively, such that for each $n \in \mathfrak{C}$, $*_{1,n}$ and $*_{2,n}$ are either both isomorphic to the Łukasiewicz t-norm or both isomorphic to the product t-norm. Let $V_1 : \mathcal{P} \to [0,1]$. Then there exists $V_2 : \mathcal{P} \to [0,1]$ and a strictly increasing bijection $f : [0,1] \to [0,1]$ such that for every $\psi \in \mathcal{F}$,

$$(V_2)_{\star_2}(\psi) = f\left((V_1)_{\star_1}(\psi)\right).$$

Proof. Let $\star_0$ be the ordinal sum $\sum_{n \in \mathfrak{C}}([a_n, b_n], *_{0,n})$ where $*_{0,n}$ is isomorphic to Łukasiewicz t-norm iff $*_{1,n}$ is isomorphic to the Łukasiewicz t-norm, and $*_{0,n}$ is the product t-norm iff $*_{1,n}$ is isomorphic to the product t-norm. First, we will find an isomorphism f_1 from $\star_1$ to $\star_0$ and $V_0 : \mathcal{P} \to [0,1]$.

Let $h_n : [a_n, b_n] \to [0,1]$ be the function such that $h_n(x) = \frac{x - a_n}{b_n - a_n}$ defined for each $n \in \mathfrak{C}$. Let $g_n : [0,1] \to [0,1]$ be a strictly increasing bijection such that $x *_{1,n} y = g_n^{-1}(g_n(x) \star_{Ł \vee P} g_n(y))$. We define function $f_1 : [0,1] \to [0,1]$ in the following way: $f_1(x) = \left(h_n^{-1} \circ g_n \circ h_n\right)(x)$ if $x \in (a_n, b_n)$ for some (necessarily unique) $n \in \mathfrak{C}$ and $f_1(x) = x$, otherwise. By Lemma 4.28, the two t-norms $\star_1$ and $\star_0$ are isomorphic via f_1.

We define $V_0 : \mathcal{P} \to [0,1]$ as $V_0(p) = f_1(V_1(p))$ for all $p \in \mathcal{P}$. By the previous lemma, for all $\psi \in \mathcal{F}$

$$(V_0)_{\star_0}(\psi) = f_1\left((V_1)_{\star_1}(\psi)\right)$$

Similarly, we can find a strictly increasing bijection $f_2 : [0,1] \to [0,1]$ and $V_2 : \mathcal{P} \to [0,1]$ such that $V_2(p) = f_2^{-1}(V_0(p))$ for all $\psi \in \mathcal{F}$. Then for all $\psi \in \mathcal{F}$,

$$f_2^{-1}\left((V_0)_{\star_0}(\psi)\right) = (V_2)_{\star_2}(\psi).$$

Therefore, f is $f_2^{-1} \circ f_1$ and V_2 is $\left(f_2^{-1} \circ f_1\right)(V_1)$.

This completes the proof of the lemma. □

The following theorem has been adapted from [9, Theorem 3]. A detailed proof can be found in the original source.

Theorem 4.54. Let $\star_{kL}$ be a t-norm whose idempotents are $0, \frac{1}{2}, \frac{2}{k}, \ldots, \frac{k-1}{k}, 1$ for some $0 < k < \omega$ and is isomorphic to the Łukasiewicz t-norm on each of its contact intervals $[i/k, (i+1)/k]$ for $0 \leq i \leq k-1$. If $\varphi \in \mathcal{F}$ is not valid, then for some $0 < k < \omega$, φ is not valid.

Remark 4.55. By Corollary 4.40, each $\star_{k\text{Ł}}$ is a continuous t-norm

These mathematical statements give the foundations to proving the converse of Theorem 4.51 which has been adapted from [6].

Theorem 4.56. Let $\mathcal{T}$ be a tableau with a root $\{\tau(\psi) < 1\}$. If there is a model $\mathcal{N} = ([0,1], \star, \Rightarrow, \min, \max, 0, 1, V)$, where $\star$ is a continuous t-norm and $\Rightarrow$ is its residuum, $V : PROP \to [0,1]$ such that $V_\star(\psi) < 1$, then $\mathcal{T}$ has an open branch.

Proof. Suppose that there is a model $\mathcal{N} = ([0,1], \star, \Rightarrow, \min, \max, 0, 1, V)$, where $\star$ is a continuous t-norm and $\Rightarrow$ is its residuum, and $V : \mathcal{P} \to [0,1]$, such that $V_\star(\psi) < 1$. Therefore, by Theorem 4.54, there exists $k < \omega$ and $V' : \mathcal{P} \to [0,1]$ such that $V'_{\star_{kL}}(\psi) < 1$ and by Lemma 4.53, there is an ordinal sum $\star_0$ with isomorphic components to $\star_{\text{Ł} \vee P}$ and $V_0 : \mathcal{P} \to [0,1]$ such that $V_{\star_0}(\psi) < 1$, where $\star_0$ is the ordinal sum $\sum_{i=0}^{k-1} ([i/k, (i+1)/k], *_i)$ of Łukasiewicz t-norms. We know that for every node $n_\mathcal{B}$ of a branch $\mathcal{B}$, we have $n_\mathcal{B}|_{L_0} \subseteq l_\mathcal{B}$, where $l_\mathcal{B}$ is the leaf of the branch. Therefore, a branch is open if its leaf $l_\mathcal{B}$ has a nice solution. That is, we need to find a branch $\mathcal{B}$ and a nice solution $(\mathcal{M}, \sigma)$ of $l_\mathcal{B}$. First, we take $\sigma(\mu_p) = V(p)$ for all $p \in \mathcal{P}$ and $\sigma(x) = 0$ for all $x \in \mathrm{Var} - \mu(\mathcal{P})$. Now in **(1)** below, we construct the structure $\mathcal{M}$; the process reduces to constructing the mapping $\rho : Par \to [0,1]$. At the same time we will be selecting nodes on a branch, we will call it $\mathcal{B}$. Then, in **(2)** we show that branch $\mathcal{B}$ is open.

We write Λ for the root of the tableau. We let $\circ$ range over $\{\star, \Rightarrow\}$, and $(\#, \sharp)$ range over $\{(\star, \&), (\Rightarrow, \to), (\min, \wedge), (\max, \vee)\}$– that is, $\#$ is the L_1-function symbol corresponding to the fuzzy connective $\sharp$, and vice versa. Note that when $\# \in \{\min, \max\}$, it is an L_0-function symbol.

Let φ be a subformula of ψ. We define an L_1-term φ_Γ for each node Γ of the tableau by induction on $d(\Gamma)$. We let $\varphi_\Lambda = \tau(\varphi)$. If Γ is a successor of Γ', the active term at Γ' was $x \circ y$, and the branch expansion rule that constructed Γ from Γ' replaced $x \circ y$ in formulas in Γ' by the L_0-term z, then we define $\varphi_\Gamma = \varphi_{\Gamma'}[z/x \circ y]$.

Remark 4.57. Note that for an L_1-term t ,

$$t[z/x \circ y] = \begin{cases} t, & \text{if } t \in \mathrm{Var} \cup \{0,1\}, \\ z, & \text{if } t = x \circ y, \\ f\left(t_1[z/x \circ y], \ldots, t_n[z/x \circ y]\right) & \text{if } t = f\left(t_1, \ldots, t_n\right) \neq x \circ y. \end{cases}$$

Claim 1. For each node Γ of the tableau, each term $t\#u$ occurring in Γ, where $\# \notin L_0$, is equal to $(\theta\sharp\varphi)_\Gamma$ for some (not necessarily unique) subformula $\theta\sharp\varphi$ of ψ.

Proof. By induction on $d(\Gamma)$. The only non-atomic terms occurring in the root $\{\tau(\psi) < 1\}$ are of the form $\tau(\theta\sharp\varphi)$ for some subformula $\theta\sharp\varphi$ of ψ, and $\tau(\theta\sharp\varphi) = (\theta\sharp\varphi)_\Lambda$ by definition above. If Γ is a successor of Γ', suppose that the active term at Γ' was $x \circ y$, and that the branch expansion rule that constructed Γ from Γ' replaced $x \circ y$ in formulas in Γ' by L_0-term z. By inspection of the tableau rules, we see that $t\#u = (t'\#u')\,[z/x \circ y]$ for some $t'\#u'$ occurring in Γ'. By induction hypothesis, $t'\#u' = (\theta\sharp\varphi)_{\Gamma'}$ for some subformula $\theta\sharp\varphi$ of ψ. So $t\#u = (\theta\sharp\varphi)_{\Gamma'}[z/x \circ y] = (\theta\sharp\varphi)_\Gamma$. This completes the proof of the claim. $\square$

(1) We will assign values of parameters occurring on $\mathcal{B}$ under ρ to elements of $\{0, 1/k, \ldots, (k-1)/k, 1\}$. Suppose we have already selected the sequence of nodes $n_0, \ldots, n_l$, where $l \geq 0, n_0$ is the root of $\mathcal{T}$ and n_{i+1} is a successor of n_i for all $0 \leq i \leq l$. That is, branch $\mathcal{B}$ is partially defined and ρ is defined for all parameters in these nodes. By comparing n_l and its successors, we deduce what the active term is and thus which of the branch expansion rules has been applied. Suppose it is $(\star)$; the other case is similar. Let Γ_1 be the set of L_1-terms in n_l that are not in any of the successors of n_l. Let the active term at n_l be $x \star y$, say. This term occurs in n_l. So by the claim above, we can choose a subformula $\varphi\&\theta$ of ψ with $x \star y = (\varphi\&\theta)_{n_l}$. We know whether or not there is $0 \leq i \leq k-1$ such that $V_{\star_0}(\varphi), V_{\star_0}(\theta) \in [i/k, (i+1)/k]$. If there is such i, we choose the least one and select the subrule **Ł**. of $(\star)$, and depending on the relation of the values of parameters occurring on n_l to $i/k, (i+1)/k$, we select the node, say n_l', resulting from Cases 1-5. We then assign to the parameters at n_l' that do not occur in n_l, say a, b with $a < b$, values $\rho(a) = i/k, \rho(b) = (i+1)/k$. Suppose now that there is no such i. Then there are $0 \leq i, j \leq k-1$ such that $i \neq j$ and $V_{\star_0}(\varphi) \in [i/k, (i+1)/k], V_{\star_0}(\theta) \in [j/k, (j+1)/k]$. We know the relations among the values of parameters occurring in n_l, thus we know which of Cases 6-11 match these values under $V_{\star_0}$. Therefore, we can select the subsequent node. In this case there are no new parameters to assign. We have now selected node n_{l+1} in the path from the root. The procedure terminates at a leaf, where there are no L_1-terms occurring, at which point we selected all nodes in branch $\mathcal{B}$. We also partially defined the function ρ. To the parameters that have not received values under ρ in this procedure, we assign 0. So $\rho : Par \to [0,1]$ is now fully defined. We have now constructed an L_0-structure $\mathcal{M} = (\mathbb{R}, +, -, \min, \max, \leq, <, 0, 1, \rho)$.

(2) To show that $\mathcal{M}, \sigma \models \bigwedge l_\mathcal{B}$, where $l_\mathcal{B}$ is the leaf of branch $\mathcal{B}$, it is sufficient to prove that for all nodes $n_\mathcal{B}$ in $\mathcal{B}$, we have $\mathcal{M}_\mathcal{B}, \sigma \models \bigwedge n_\mathcal{B}$ and $(\mathcal{M}, \sigma)$ is a nice solution. First, we show Claim 4, which we will need to prove Claim 5, completing this direction of the lemma.

Claim 2. Let the node $\Gamma \in \mathcal{B}$ be arbitrary. Then

1. For each subformula $\theta\sharp\varphi$ of ψ, if $(\theta\sharp\varphi)_\Gamma$ is not an L_0-term, then $(\theta\sharp\varphi)_\Gamma = \theta_\Gamma \# \varphi_\Gamma$.
2. For each subformula φ of ψ, if φ_Γ is an L_0-term, then $[\![\varphi_\Gamma]\!]^{\mathcal{M}_{\mathcal{B}},\sigma} = V_{\star_0}(\varphi)$.
3. For each subformula φ of ψ, we have $[\![\varphi_\Gamma]\!]^{\mathcal{M}_{\mathcal{B}},\sigma} = [\![\varphi_\Lambda]\!]^{\mathcal{M}_{\mathcal{B}},\sigma}$.
4. For each subformula φ of ψ, φ_Γ occurs in Γ.
5. For each subformula φ of ψ, we have $[\![\tau(\varphi)]\!]^{\mathcal{M}_{\mathcal{B}},\sigma} = V_{\star_0}(\varphi)$.

Proof. We prove 1 - 4 by induction on $d(\Gamma)$. First, let $\Gamma = \Lambda$.

1. By definition of τ.

2. By induction on φ.
The base case. For $\varphi = p \in \mathcal{P}$, by definitions of p_Λ, the translation function τ and σ, we have

$$[\![p_\Lambda]\!]^{\mathcal{M}_{\mathcal{B}},\sigma} = [\![\tau(p)]\!]^{\mathcal{M}_{\mathcal{B}},\sigma} = [\![\mu_p]\!]^{\mathcal{M}_{\mathcal{B}},\sigma} = \sigma\left(\mu_p\right) = V_{\star_0}(p).$$

The proof for $\varphi = \overline{0}, \overline{1}$ is easy.
The inductive case. Suppose $\theta\sharp\varphi$ is a subformula of ψ such that $(\theta\sharp\varphi)_\Lambda = \theta_\Lambda \# \varphi_\Lambda$ is an L_0-term, and assume the result for θ, φ. Then $\#$ is an L_0 function symbol, that is, $\sharp \in \{\wedge, \vee\}$. If $\sharp = \wedge$, we have $[\![(\theta \wedge \varphi)_\Lambda]\!]^{\mathcal{M}_{\mathcal{B}},\sigma} = [\![\min\{\theta_\Lambda, \varphi_\Lambda\}]\!]^{\mathcal{M}_{\mathcal{B}},\sigma} = \min\left\{[\![\theta_\Lambda]\!]^{\mathcal{M}_{\mathcal{B}},\sigma}, [\![\varphi_\Lambda]\!]^{\mathcal{M}_{\mathcal{B}},\sigma}\right\}$. By the induction hypothesis,

$$\min\left\{[\![\theta_\Lambda]\!]^{\mathcal{M}_{\mathcal{B}},\sigma}, [\![\varphi_\Lambda]\!]^{\mathcal{M}_{\mathcal{B}},\sigma}\right\} = \min\left\{V_{\star_0}(\theta), V_{\star_0}(\varphi)\right\} = V_{\star_0}(\theta \wedge \varphi).$$

The other case can be proven analogously.
3. Obvious.
4. By definition, $\varphi_\Lambda = \tau(\varphi)$, which is a subterm of $\tau(\psi)$, and $\Lambda = \{\tau(\psi) < 1\}$.

Now suppose that Γ is the successor of $\Gamma' \in \mathcal{B}$. Suppose that the active term at Γ' was $x \circ y$ for some L_0-terms x, y, and the branch expansion rule that constructed Γ from Γ' replaced $x \circ y$ in formulas in Γ' by some L_0-term z. Assume inductively that 1 - 4 hold for Γ'. We prove them for Γ.

1. Suppose that $(\theta\sharp\varphi)_\Gamma$ is not an L_0-term. Then by the induction hypothesis,

$$(\theta\sharp\varphi)_\Gamma = (\theta\sharp\varphi)_{\Gamma'}[z/x \circ y] = \left(\theta_{\Gamma'} \# \varphi_{\Gamma'}\right)[z/x \circ y] = \theta_{\Gamma'}[z/x \circ y] \# \varphi_{\Gamma'}[z/x \circ y] = \theta_\Gamma \# \varphi_\Gamma.$$

Otherwise $(\theta\sharp\varphi)_\Gamma$ would be an L_0-term.

2. In general $\star_{\mathcal{B}}$ interpreted in $\mathcal{M}_{\mathcal{B}}$ may be different from $\star_0$. Let ζ be a subformula of ψ, and suppose that $\zeta_\Gamma = \zeta_{\Gamma'}[z/x \circ y]$ is an L_0-term. We need to prove that $[\![\zeta_\Gamma]\!]^{\mathcal{M}_{\mathcal{B}},\sigma} = V_{\star_0}(\zeta)$. We will do this by induction on ζ. There are three cases following Remark preceding Claim 3.

Case 1: ζ is atomic. Then $\zeta_{\Gamma'}$ is obviously an L_0-term. Then, $\zeta_\Gamma = \zeta_{\Gamma'}[z/x \circ y] = \zeta_{\Gamma'}$, so inductively, $[\![\zeta_\Gamma]\!]^{\mathcal{M}_\mathcal{B},\sigma} = [\![\zeta_{\Gamma'}]\!]^{\mathcal{M}_\mathcal{B},\sigma} = V_{\star_0}(\zeta)$.

Case 2: $\zeta = \theta \sharp \varphi$ and $\zeta_{\Gamma'} = x \circ y$. Since $(\theta\sharp\varphi)_{\Gamma'} = \zeta_{\Gamma'} = x \circ y$, which is not an L_0–term, by the induction hypothesis for Part 1 , we have $(\theta\sharp\varphi)_{\Gamma'} = \theta_{\Gamma'} \# \varphi_{\Gamma'}$. As this is $x \circ y$, we have $\theta_{\Gamma'} = x$ and $\varphi_{\Gamma'} = y$, and $\# = \circ$. As x, y are L_0-terms, so by the induction hypothesis we have $V_{\star_0}(\theta) = [\![x]\!]^{\mathcal{M}_\mathcal{B},\sigma}$ and $V_{\star_0}(\varphi) = [\![y]\!]^{\mathcal{M}_\mathcal{B},\sigma}$. In **(1)** we chose some subformula $\theta'\sharp\varphi'$ of ψ with $(\theta'\sharp\varphi')_{\Gamma'} = x \circ y$, and arranged that if $V_{\star_0}(\theta') = [\![x]\!]^{\mathcal{M}_\mathcal{B},\sigma}$ and $V_{\star_0}(\varphi') = [\![y]\!]^{\mathcal{M}_\mathcal{B},\sigma}$, then

$$[\![z]\!]^{\mathcal{M}_\mathcal{B},\sigma} = V_{\star_0}\left(\theta'\right) \#_0 V_{\star_0}\left(\varphi'\right), \tag{$\dagger$}$$

where $\#_0$ is $\star_0$ if $\# = \star$, and the residuom of $\star_0$ if $\#$ is $\Rightarrow$. Again, by the induction hypothesis for Part $1, \theta'_{\Gamma'} = x$ and $\varphi'_{\Gamma'} = y$ are L_0-terms. So by the induction hypothesis $V_{\star 0}(\theta') = [\![x]\!]^{\mathcal{M}_\mathcal{B},\sigma}$ and $V_{\star 0}(\varphi') = [\![y]\!]^{\mathcal{M}_\mathcal{B},\sigma}$. So ($\dagger$) holds. Therefore,

$$\begin{aligned}[\![\zeta_\Gamma]\!]^{\mathcal{M}_\mathcal{B},\sigma} &= [\![z]\!]^{\mathcal{M}_\mathcal{B},\sigma} = V_{\star_0}\left(\theta'\right) \#_0 V_{\star_0}\left(\varphi'\right) = [\![x]\!]^{\mathcal{M}_\mathcal{B},\sigma} \#_0 [\![y]\!]^{\mathcal{M}_\mathcal{B},\sigma} \\ &= V_{\star_0}(\theta) \#_0 V_{\star_0}(\varphi) = V_{\star_0}(\theta\sharp\varphi) = V_{\star_0}(\zeta).\end{aligned}$$

Case 3: $\zeta = \theta\sharp\varphi$ and $\zeta_{\Gamma'} \neq x \circ y$. If $\zeta_{\Gamma'}$ is an L_0–term, we use the proof of Case 1. So we assume that it is not an L_0-term. By the induction hypothesis to Part $1, \zeta_{\Gamma'} = \theta_{\Gamma'} \# \varphi_{\Gamma'}$. This is not $x \circ y$, so

$$\zeta_\Gamma = \zeta_{\Gamma'}[z/x \circ y] = (\theta_{\Gamma'} \# \varphi_{\Gamma'})\,[z/x \circ y] = \theta_{\Gamma'}[z/x \circ y] \# \varphi_{\Gamma'}[z/x \circ y] = \theta_\Gamma \# \varphi_\Gamma.$$

This is an L_0-term, so $\# \in \{\min, \max\}$ and hence $\# \in \{\wedge, \vee\}$. Assume $\sharp = \wedge$; the other case can be proven analogously. Then $\# = \min$ and $\theta_\Gamma, \varphi_\Gamma$ are L_0-terms, so by induction hypothesis,

$$\begin{aligned}[\![\zeta_\Gamma]\!]^{\mathcal{M}_\mathcal{B},\sigma} &= [\![\min\{\theta_\Gamma, \varphi_\Gamma\}]\!]^{\mathcal{M}_\mathcal{B},\sigma} = \min\left\{[\![\theta_\Gamma]\!]^{\mathcal{M}_\mathcal{B},\sigma}, [\![\varphi_\Gamma]\!]^{\mathcal{M}_\mathcal{B},\sigma}\right\} \\ &= \min\{V_{\star_0}(\theta), V_{\star_0}(\varphi)\} = V_{\star_0}(\theta \wedge \varphi) = V_{\star_0}(\zeta).\end{aligned}$$

3. In **(1)** we chose some subformula $\theta'\sharp\varphi'$ of ψ with $(\theta'\sharp\varphi')_{\Gamma'} = x \circ y$, and arranged that if $V_{\star_0}(\theta') = [\![x]\!]^{\mathcal{M}_\mathcal{B},\sigma}$ and $V_{\star_0}(\varphi') = [\![y]\!]^{\mathcal{M}_\mathcal{B},\sigma}$, then

$$[\![z]\!]^{\mathcal{M}_\mathcal{B},\sigma} = [\![x \circ y]\!]^{\mathcal{M}_\mathcal{B},\sigma} \tag{$\dagger$}$$

By the induction hypothesis for Part $1, \theta'_{\Gamma'} = x$ and $\varphi'_{\Gamma'} = y$. As x, y are L_0– terms, by the inductive hypothesis for Part 2 we have $V_{\star_0}(\theta') = [\![x]\!]^{\mathcal{M}_\mathcal{B},\sigma}$ and $V_{\star_0}(\varphi') = [\![y]\!]^{\mathcal{M}_\mathcal{B},\sigma}$. So ($\dagger$) holds. Thus by ($\dagger$) and the induction hypothesis,

$$[\![\varphi_\Gamma]\!]^{\mathcal{M}_\mathcal{B},\sigma} = [\![\varphi_{\Gamma'}[z/x \circ y]]\!]^{\mathcal{M}_\mathcal{B},\sigma} = [\![\varphi_{\Gamma'}]\!]^{\mathcal{M}_\mathcal{B},\sigma} = [\![\varphi_\Lambda]\!]^{\mathcal{M}_\mathcal{B},\sigma}.$$

4. By the induction hypothesis and inspection of the tableau rules.

This completes the induction. Now we prove Part 5. Let $l_\mathcal{B}$ be the leaf of $\mathcal{B}$. By Part 4, $\varphi_{l_\mathcal{B}}$ occurs in $l_\mathcal{B}$. Since only L_0-terms occur in $l_\mathcal{B}, \varphi_{l_\mathcal{B}}$ is an L_0-term. By definition of φ_Λ and Parts 3 and 2, $[\![\tau(\varphi)]\!]^{\mathcal{M}_\mathcal{B},\sigma} = [\![\varphi_\Lambda]\!]^{\mathcal{M}_\mathcal{B},\sigma} = [\![\varphi_{l_\mathcal{B}}]\!]^{\mathcal{M}_\mathcal{B},\sigma} = V_{\star_0}(\varphi)$. This proves the claim.

□

Claim 5. For every node $n_\mathcal{B}$ of $\mathcal{B}, \mathcal{M}_\mathcal{B}, \sigma \models \bigwedge n_\mathcal{B}$ and $(\mathcal{M}, \sigma)$ is a nice solution.

Proof. By induction on $d\,(n_\mathcal{B})$. For the base case: $d\,(n_\mathcal{B}) = 0$, i.e. $n_\mathcal{B} = \Lambda$. By Claim 4 Part 5 and the initial assumption, $[\![\tau(\psi)]\!]^{\mathcal{M}_\mathcal{B},\sigma} = V_{\star_0}(\psi) < 1$. Therefore $\mathcal{M}_\mathcal{B}, \sigma \models \tau(\psi) < 1$, i.e. $\mathcal{M}_\mathcal{B}, \sigma \models \bigwedge n_\mathcal{B}$.

Assume the induction hypothesis for a non-leaf node $n_\mathcal{B}$ of branch $\mathcal{B}$. We need to show that if $\mathcal{M}_\mathcal{B}, \sigma \models \bigwedge n_\mathcal{B}$, then $\mathcal{M}_\mathcal{B}, \sigma \models \bigwedge n'_\mathcal{B}$, where $n'_\mathcal{B} \in \mathcal{B}$ and $d\,(n'_\mathcal{B}) = d\,(n_\mathcal{B}) + 1$.

By inspecting $n_\mathcal{B}, n'_\mathcal{B}$, we know which tableau formulas γ are in $n_\mathcal{B} - n'_\mathcal{B}$. Suppose that $(\varphi\&\theta)_{n_\mathcal{B}} = x \star y$ is the active term of $\gamma \in n_\mathcal{B} - n'_\mathcal{B}$; the other cases are similar. By **(1)** above, we know which case and which subrule of which branch expansion rule are used to generate $n'_\mathcal{B}$. Thus, suppose that it was Rule $(\star)$ Ł. Case 1. We will show one more case below and other cases can be proven analogously. Let a, b be the new parameters occurring at $n'_\mathcal{B}$. Thus, $\{0 \le a < b \le a_0\} \subseteq n'_\mathcal{B} - n_\mathcal{B}$, where a_0 is a parameter occurring at $n_\mathcal{B}$. By **(1)** above, we know that $0 \le \rho(a) < \rho(b) \le \rho\,(a_0)$. The other elements of $n'_\mathcal{B} - n_\mathcal{B}$ are (a) $a \le x \le b$, (b) $a \le y \le b$, (c) $\gamma[\max\{a, x + y - b\}/x \star y]$ for all $\gamma \in n_\mathcal{B} - n'_\mathcal{B}$. By **(1)** above and Claim 4 Part 2, the L_0-formulas in (a) and (b) are $\mathcal{B}$-satisfiable via $\mathcal{M}, \sigma$ and x, y are not marked since $V_{\star_0}(\theta), V_{\star_0}(\varphi) \in [\rho(a), \rho(b)]$. Now we show that L_1-formulas in (c) are $\mathcal{B}$-satisfiable via $\mathcal{M}, \sigma$. Let $z = \max\{a, x + y - b\}$. By **(1)** above, z was substituted for the active term $x \star y$ in the step from $n_\mathcal{B}$ to $n'_\mathcal{B}$. So $(\varphi\&\theta)_{n'_\mathcal{B}} = (\varphi\&\theta)_{n_\mathcal{B}}[z/x \star y] = (x \star y)[z/x \star y] = z$. By Claim 4 Part 3, $[\![z]\!]^{\mathcal{M}_\mathcal{B},\sigma} = [\![(\varphi\&\theta)_{n'_\mathcal{B}}]\!]^{\mathcal{M}_\mathcal{B},\sigma} = [\![(\varphi\&\theta)_{n_\mathcal{B}}]\!]^{\mathcal{M}_\mathcal{B},\sigma} = [\![x \star y]\!]^{\mathcal{M}_\mathcal{B},\sigma}$. Now by the induction hypothesis, all $\gamma \in n_\mathcal{B} - n'_\mathcal{B}$ are $\mathcal{B}$-satisfiable via $\mathcal{M}, \sigma$. The formulas in (c) are of the form $\gamma[z/x \star y]$ for $\gamma \in n_\mathcal{B} - n'_\mathcal{B}$. Since $[\![z]\!]^{\mathcal{M}_\mathcal{B},\sigma} = [\![x \star y]\!]^{\mathcal{M}_\mathcal{B},\sigma}$, all formulas in (c) are $\mathcal{B}$-satisfiable via $\mathcal{M}, \sigma$.

Now, suppose that the rule that was selected is Rule $(\star)$ min. Case 6. The marked terms are $x^{n'_\mathcal{B}}$ and $y^{n'_\mathcal{B}}$. We denote $x^{n'_\mathcal{B}}$ and $y^{n'_\mathcal{B}}$ as u, v, respectively. Thus $n'_\mathcal{B} - n_\mathcal{B} = \{0 \le u \le a_0\} \cup \{\gamma[\min\{u, v\}/u \star v] : \gamma \in n_\mathcal{B}\}$. By **(1)** above, we know that there are $0 \le i, j \le k - 1$ such that $i \ne j, (i+1)/k \le \rho\,(a_0)$, and $V_{\star_0}(\theta) \in [i/k, (i +$

$1)/k], V_{\star_0}(\varphi) \in [j/k, (j+1)/k]$. By **(1)** above and Claim 4 Part 2, the L_0-formulas in $n'_{\mathcal{B}} - n_{\mathcal{B}}$ are $\mathcal{B}$-satisfiable via $\mathcal{M}, \sigma$. By **(1)** above and Claim 4 Part 3, the L_1-formulas in $n'_{\mathcal{B}} - n_{\mathcal{B}}$ are $\mathcal{B}$-satisfiable via $\mathcal{M}, \sigma$ using the same argument as above. Thus, $\mathcal{M}_{\mathcal{B}}, \sigma \models \bigwedge n'_{\mathcal{B}}$.

We show that $(\mathcal{M}, \sigma)$ is a nice solution with respect to all parameters in $n'_{\mathcal{B}}$. By **(1)** and Claim 4 Part 2, if terms x, y are marked by $\Gamma \in \mathcal{B}$, then there is no $i < k$ such that $[\![x]\!]^{\mathcal{M}_{\mathcal{B}},\sigma}, [\![y]\!]^{\mathcal{M}_{\mathcal{B}},\sigma} \in [i/k, (i+1)/k]$. Since for all parameters $a_j < b_j \in \mathcal{B}$, we have $\rho\,(a_j) = i/k$ and $\rho\,(b_j) = (i+1)/k$ for some $i < k$, it follows that there is no pair of parameters $a_j < b_j$ such that $\mathcal{M}_{\mathcal{B}}, \sigma \models a_j \leq x, y \leq b_j$. So $(\mathcal{M}_{\mathcal{B}}, \sigma)$ is nice.

$\square$

By Claim 5, in particular $\mathcal{M}, \sigma \models \bigwedge n_{\mathcal{B}}|_{L_0}$ for all $n_{\mathcal{B}} \in \mathcal{B}$. We have now proved that in particular $\mathcal{M}, \sigma \models \bigwedge l_{\mathcal{B}}$. Thus, there is a nice solution $(\mathcal{M}, \sigma)$. Therefore $\mathcal{B}$ is open.

$\square$

5 Conclusion and Discussion

In this thesis, we began by defining a method to translate natural language statements into the formal language of logic. Following this, we introduced a tableau calculus for Classical Propositional Logic to prove the validity of formulas.

Continuing from these foundations, we presented a sound and complete tableau calculus for Basic Logic, based on the framework of continuous t-norms and their ordinal decomposition. Our system generalizes the Classical Propositional Tableau by adapting it to many-valued logic, replacing manipulations of formulas in Classical Propositional Logic with inequalities between truth values in Basic Logic.

Through this development, we have demonstrated how Fuzzy Logic, via continuous t-norms, can be integrated into tableau methods, providing a method for reasoning in many-valued logical systems.

It is worth noting that this thesis works under several strong restrictions. Exploring our tableau is both sound and complete while relaxing these assumptions would be an interesting direction for future research. For example, what happens when we extend our logic to first-order or even higher-order BL, which would involve introducing quantifiers such as $\forall$ and $\exists$, as well as introducing additional connectives such as strong conjunction and disjunction. These features would complicate the studies further, as new definitions would be required to define logical validity and provability.

Another restriction we imposed on ourselves is that the designated values of Basic Logic is the singleton set 1. However, one may wonder what happens if we instead consider the designated values to be the interval $[\epsilon, 1]$ with $\epsilon < 1$. This would be natural in contexts where a more flexible interpretation of truth is desired.

Finally, the design of our tableau system makes it suitable for creating an algorithm that checks the validity of formulas in Basic Logic. This is possible because we restrict ourselves to finite sets of formulas and only have a finite amount of rules which we can apply.

These prospects highlight both the richness of Basic Logic as a formal system and the versatility of tableau methods for advancing theoretical and practical work in Fuzzy Logic.

References

[1] David Barker-Plummer, Jon Barwise, and John Etchemendy. *Language, Proof, and Logic: Second Edition.* Center for the Study of Language and Information/SRI, 2nd edition, 2011.

[2] Nicola Olivetti. Tableaux for łukasiewicz infinite-valued logic. *Studia Logica*, 73: 81–111, 2003.

[3] Graham Priest. *An Introduction to Non-Classical Logic: From If to Is.* Cambridge University Press, New York, 2008.

[4] Jan Łukasiewicz. O logice trójwartościowej. *Studia Filozoficzne*, 270(5), 1988.

[5] Angel Barriga and Nashaat Mohamed Hussein Hassan. *Application of Fuzzy Logic and Lukasiewicz Operators for Image Contrast Control*, pages 133–154. Springer Berlin Heidelberg, Berlin, Heidelberg, 2011.

[6] Agnieszka Kulacka. *Propositional Fuzzy Logics: Tableaux and Strong Completeness.* Ph.d. thesis, Imperial College London, 2017.

[7] Petr Hájek. *Metamathematics of Fuzzy Logic.* Kluwer Academic Publishers, Dordrecht, Boston and London, 1998.

[8] Kyriakos Keremedis and Eliza Wajch. On densely complete metric spaces and extensions of uniformly continuous functions in **ZF**, 2019.

[9] Matthias Baaz, Petr Hájek, Franco Montagna, and Helmut Veith. Complexity of t-tautologies. *Annals of Pure and Applied Logic*, 113(1):3–11, 2001. First St. Petersburg Conference on Days of Logic and Computability.