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faculty of science and engineering

Bounded countermodels for fragments of FLec

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Student: L.I. Sibma

First supervisor: Prof.dr. J. Top

Second supervisor: Dr. D. Frumin and Prof. dr. D.R.S. Ramanayake

Abstract

The bounded finite model property refers to the following problem: given an input formula that is not provable in the logic, is there a finite countermodel that invalidates the formula, and if so, can we give a bound on the size of the model in terms of the size of the input formula? We have investigated this question for the substructural logic FLec, in particular for its fragments of the logical connectives of FLec. FLec algebras are constructed using a closure operator and are shown to form valid FLec algebras that do not validate unprovable formulas. The proof of finiteness of these countermodels gives us a tool to bound these countermodels. We have found bounds for the full fragments for the connectives fusion, conjunction and disjunction and for the non-nested fragment for implication. These bounds are based on the size of the input formula and using the properties of the rules of the logical connectives of FL_{ec}.

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Introduction

Logic is the study of valid reasoning. At its core, logic is concerned with arguments. An argument consists of one or more premises (statements or propositions that provide support) and a conclusion (the statement that follows from the premises). A key concern is ensuring that the connection between premises and conclusion is valid, meaning that if the premises are true, then the conclusion must be True.

One branch of logic is propositional logic. Propositional logic is concerned with how propositions (statements that are either true or false) can be combined and manipulated using logical connectives to form valid arguments or more complex statements. Some examples of propositions are:

1. It is raining.
2. The street is wet.
3. The sky is purple.

In our formal reasoning, we will denote propositions by lowercase letters such as p and q, called propositional variables. These propositions are the building blocks of logical reasoning. Propositional logic focuses on understanding the relationships between such propositions through logical connectives, which allowed us to combine simpler propositions into more complex ones. The primary connectives are:

- Conjunction ($\wedge$): The statement $p \wedge q$ is true if and only if both p and q are true.
- Disjunction ($\vee$): The statement $p \vee q$ is true if at least one of p or q is true.
- Negation ($\neg$): The statement $\neg p$ is true when p is false.
- Implication ($\rightarrow$): The statement $p \rightarrow q$ means that if p is true, then q must also be true.

Using these propositions, we are able to make statements with premises and a conclusion. For example, if we take p to mean 'it is raining', q to be 'the street is wet' and r to be 'the sky is purple', then from the propositions $p, p \rightarrow q$ we may conclude q, 'the street is wet'. These statements do not always make sense in our world. From $q, q \rightarrow p$ we may conclude p, 'the street is wet'. However, there could be several other reasons that the street is wet, maybe some people had a water balloon fight. We even could, from the propositions $p, p \rightarrow r$, infer that when it is raining, the sky is purple.

One way to check the validity of propositions is using a truth table. A truth table shows all possible truth values for the propositions involved, and the resulting truth value of the compound statement. We have fixed truth tables for all the logical connectives based on their meaning and using those, we can check the validity of any proposition. For example, the truth table for the proposition $p \rightarrow (p \wedge q)$ is given by:

p	q	$p \wedge q$	$p \rightarrow p \wedge q$
1	1	1	1
1	0	0	0
0	1	0	1
0	0	0	1

This table shows us that when p is true and q is false, the proposition $p \rightarrow (p \wedge q)$ is false, and in all other cases it is true. We say a formula is valid if it is true under any assignment. An assignment is a map, in this case from the set of propositions to the set $0, 1$. We will see later that these maps can be more complicated in order to reflect the situation we are in. In these cases, instead of talking about propositions, we will talk about formulas.

In this thesis, to find valid formulas, we will be using sequent calculus. Sequent calculus is a formal system developed to represent logical proofs. It provides a way to organise the structure of reasoning in a manner that makes the rules of inference explicit. In sequent calculus, proofs are structured as sequents: a form of statement that expresses the relationship between premises and a conclusion.

A sequent is typically written as $\Gamma \Rightarrow \Delta$, where Γ represents a set of premises and Δ represents a set of conclusions. From now on, we will refer to the premises as the antecedent of the sequent and to the conclusion as the succedent of the sequent. The sequent expresses that given the antecedent Γ, the succedent Δ will follow. In order to manipulate sequents, we will use a set of inference rules. There are four types of rules:

1. Initial sequents.
2. Left and right rules for the logical connectives.
3. Structural rules.
4. Cut rule.

The first type of rules are initial sequents, which are always of the form $\alpha \Rightarrow \alpha$ for any formula α and are used as the starting point of proofs. The second type of rules allow us to introduce new logical connectives into a sequent, either in the antecedent (left rules) or the succedent (right rules). The structural rules allow us to change the structure of a sequent. For example, exchange allows to to change the ordering of two formulas, weakening allows us to duplicate a formula and contraction allows us to delete a duplicate formula. The cut rule allows us to remove a formula if it appears in both the antecedent of one sequent and the succedent of another sequent.

The sequent calculus we will be using is one based on intuitionistic logic. Intuitionistic logic is a system of logic that differs from classical logic in its treatment of truth. It emphasises the idea that the truth of a statement is closely tied to our ability to construct a proof for it. Some key differences between classical and intuitionistic logic are [21, Section 1.4]:

1. Proofs as a construction: The truth of formulas must be established by giving a proof of it. For example, the proof of a formula $p \vee q$ is obtained by giving a proof of either p or q and a proof of the formula $p \wedge q$ is obtained by giving both a proof of p and of q.
2. No excluded middle: In mathematics, we often use *proof by contradiction* to proof a statement. We assume the opposite of what we want to proof and when we reach a contradiction, we have a proof. This technique heavily relies on the fact that for any statement p, either p or $\neg p$ must hold. By the constructive nature of intuitionistic logic, we need to give a proof of either p or $\neg p$, which might not always be possible.
3. Single formula succedent: To reflect the focus of intuitionistic logic on providing an explicit proof of a statement, the succedent Δ of our sequents will always be a single formula φ.

Due to this change in perspective, this logic is more expressive. Although it proves less formulas than the sequent calculus for classical logic, we are able to more accurately describe the world we live in. Another thing we would like to model is resource management. We can do this by removing one or more structural rules from our ruleset. If we remove exchange, the ordering of the formulas in the antecedent becomes relevant. As an example, the sequents $\alpha, \beta \Rightarrow \varphi$ and $\beta, \alpha \Rightarrow \varphi$ could be represented by "when I put on my socks and then my shoes, then I am ready to leave." and "When I put on my shoes and then my socks, I am ready to leave.", respectively, but would have very different meanings. If we remove weakening or contraction, the number of times a formula appears in the antecedent is relevant. The logics resulting from removing one or more structural rules are called substructural logics. The sequent system resulting from removing all structural rules is called the full Lambek calculus (FL), created by Johan Lambek [13]. In this thesis, we will be working with the full commutative (with exchange) Lambek calculus with contraction, which can also be seen as the intuitionistic sequent calculus without weakening. This change necessitates the addition of a new logical connective: Fusion $(\cdot)$. It is also called the *multiplicative conjunction*, and then $\wedge$ is called the *additive conjunction*. To show their difference, consider the following example from *Proof Theory and Algebra in Logic* (Ono, 2019):

Let α, β and γ be the following statements, respectively.

- (α) one has $25.
- (β) one can get this paperback.
- (γ) one can have lunch at that restaurant.

We suppose moreover that either of 'this paperback' and 'lunch at that restaurant' costs \$25. Then, both $\alpha \Rightarrow \beta$ and $\alpha \Rightarrow \gamma$ hold. By applying rules for fusion, $\alpha \cdot \alpha \Rightarrow \beta \cdot \gamma$ can be deduced, but $\alpha \Rightarrow \beta \cdot \gamma$ will not. On the other hand, $\alpha \Rightarrow \beta \wedge \gamma$ will follow by applying rules for conjunction. Then how fusion should be interpreted to see the difference? A possible way is to understand these statements in the context of consumption of resources. In the present case, \$25 is a resource which may be consumed by buying this paperback or by paying for lunch or by something else. Once a resource is consumed, it cannot be used any more. Under this interpretation, the statement $\alpha \cdot \alpha$ will express "one has \$50" and hence "one can get this paperback and at the same time can have lunch". Thus, $\alpha \cdot \alpha \Rightarrow \beta \cdot \gamma$ holds, while $\alpha \Rightarrow \beta \cdot \gamma$ does not since \$25 is not enough to have both of them. On the other hand, $\alpha \Rightarrow \beta \wedge \gamma$ will says that if one has \$25 then one can get this paperback and also can have lunch. Then someone may wonder that conjunction in this interpretation sounds just like disjunction, as in the latter case it says that if one has \$25 then either one can get this paperback or one can have lunch. But this is not the case. For, if the price of lunch at that restaurant has risen to \$30, then $\alpha \Rightarrow \beta \wedge \gamma$ does not hold any more, while $\alpha \Rightarrow \beta \vee \gamma$ still holds. The exact meaning of the former is that either choice is possible but not both [21, Remark 4.2].

Due to their focus on resources, substructural logics have been of interest in both applied and theoretical computer science. In linguistics, the Lambek Calculus [14, 2] is used to analyse the syntax and syntactic types of natural language, as well as context-free grammars and linguistics. In philosophy, Relevant logics [10] offer a nuanced interpretation of the implication, addressing the paradoxes associated with material implication and enhancing logical coherence. In computer science, mathematical fuzzy logics [11, 4] provide a formal framework for modelling degrees of truth; bunched implication logics [19, 8] appear in software program verification and systems modelling. Linear logic [7, 25] is an example of a modal substructural logic, that is also used in quantum mechanics [22, 15].

Due to the expressivity of substructural logics, they have been studied extensively. However, the metamathematical aspects are often intricate to establish and many questions remain open. In this thesis, we will study the *bounded finite model property* for fragments of the full commutative Lambek calculus with contraction. More specifically, we show:

Theorem 0.1. *The substructural logic* FL_{ec} *has the finite model property. Countermodels are constructed using failed proof search. For the fragments of fusion, conjunction, disjunction we can bound the size of the countermodel for the full fragment, and for the implication fragment we can bound the size of the countermodel for formulas without nesting. The size of the countermodel is bounded based on the size of the input formula and using the properties of of the rules for the logical connectives.*

Let us now unpack the statement of this theorem:

Finite model property: The finite model property refers to the following problem: given an input formula that is not provable in the logic, is there a finite countermodel that invalidates the formula? In 1994, Meyer and Ono [18] proved the finite model property for what is essentially the implication fragments of FL_e and FL_{ec}. Buszowski [3] extends this to the logics with conjunction. Meyer [17] added the disjunction, so only fusion was unaccounted for. Okada and Terui [20] obtained countermodels in the setting of phase spaces for the logics FL_X ($X \subseteq \{e, c, w\}$) and their classical counterparts with the exception of $X = \{c\}$. The finite model property in the algebraic setting was established via the finite embeddability property by Van Alten [1] for extensions of intuitionistic linear logic with knotted axioms $x^m \leq x^n$, and by Galatos and Jipsen [6] for extensions of FL with simple sequent rules that do not increase complexity from conclusion to premise (ruling out contraction). None of the above works bound the size of the countermodel.

Fragments for logical connectives of FL_{ec}: Fragments of sequent logic refer to restrictions or specific subsets of formulas within a sequent calculus system. These fragments can involve limitations on the types of logical connectives, the structure of the sequents, or the rules that are allowed. In this thesis, we will focus on fragments that limit the formulas to only use one type of logical connective and with that, only use their corresponding rules in the proofs, together with initial sequents and the structural rules. The fragments allow us to understand the problem on a more specific level.

Countermodels from failed proof search: One example of countermodels from failed proof search is the Kripke countermodel obtained from failed exhaustive proof search in Gentzen's intuitionistic calculus LJ, as demonstrated in *Proof Theory* (Takeuti, 2014) [24, Theorem 8.17]. This method has found widespread application in the proof theory of intermediate and modal logics: exhaustively applying proof rules backwards from the input formula to construct the proof search tree. If no proof is found, a Kripke countermodel is constructed by closing the tree under any necessary graph closure rules, such as transitivity. Kripke models are a natural choice since the proof search tree already exhibits much of the model structure. A bound on the proof search tree in intermediate and modal logic is evident since sequents with the same support set (i.e., ignoring formula multiplicities) are logically equivalent. However, this is not the case for substructural logics. The models are more generalized, so much more structure is needed in their construction. Moreover, bounding the proof search tree in substructural logics is more involved, since sequents with different multiplicities of formulas are inequivalent. In this thesis, we will be constructing countermodels as follows: Given an unprovable formula χ, we construct a proof search tree by exhaustively applying all possible rules of the calculus upwards. We will define an FL_{ec} algebra and its basis elements. We then find all elements of the countermodel by taking intersections.

Overview. In Section 1, we will first give preliminaries on the sequent system of FL_{ec} and then introduce the necessary order and lattice theory. We will then introduce the concept of an FL_{ec} algebra and show that algebras from a closure operator form an FL_{ec} algebra in Section 2. In Section 3, we first create a countermodel that satisfies our desired interpretation and show it invalidates sequents of unprovable formulas. The section finishes by showing that these countermodels are finite. This proof of finiteness gives us a tool to bound the countermodels for fragments of FL_{ec}, which we will do in Section 4. In Section 5, we shortly investigate two ways in which we could improve these bounds.

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1 Prelimaries

In this section, necessarily preliminaries for our results are discussed. We will first introduce the foundation for the logical system FL_{ec}, which serves as the central framework of this thesis. FL_{ec} is a substructural variant of the intuitionistic sequent calculus LJ, obtained by removing the structural rule of weakening. While this modification may seem minor at first glance, this new calculus becomes resource conscious, changing the provability of certain formulas.

We begin by introducing the core components of the sequent calculus FL_{ec}: the syntax of formulas, the structure of sequents, and the inference rules that are used to prove formulas. These include initial sequents, the standard left and right rules for logical connectives, as well as the structural rules of exchange and contraction.

The shift from LJ to FL_{ec} necessitates the introduction of the fusion connective, denoted by $\cdot$, as well as the constants 0 (falsehood) and 1 (truth), with their respective rules. We will see that certain formulas that can be proven in LJ cannot be proven in FL_{ec}.

We then discuss exhaustive proof search in FL_{ec}: a method for systematically constructing all possible derivations of a given formula. This structure will later be used in the construction of a countermodel. This approach highlights several key properties of the system, including the subformula property and the closure behaviour of provable and unprovable sequents.

This section concludes by introducing the foundational notions of partial orders and lattices, beginning with binary relations and their structural properties. We then explore well-quasi orders, which play a key role in finiteness arguments. The latter part of the section introduces semi-lattices and lattices, including their completeness properties and some examples. This knowledge will later be used when we define an FL_{ec} algebra.

1.1 The substructural logic FL_{ec}

1.1.1 Sequents

In this thesis, we will work with a sequent system that has as its basis the sequent system of intuitionistic logic, but without its structural rule *weakening*. This substructural logic is denoted FL_{ec}.

We take a fixed countable set of propositional variables, denoted by lowercase letters such as p, q and r. Formulas are constructed by applying the logical connectives $\wedge, \vee, \rightarrow, \cdot$ repeatedly. We will express formulas using lowercase Greek letters such as $\alpha, \beta, \gamma, \varphi, \psi$. Every formula β that appears in the inductive definition of a formula α is called a subformula of α. For example, if $\alpha = p \vee q$, then p, q and $p \vee q$ are all the subformulas of α.

To prove formulas, we introduce a sequent system FL_{ec}. The basic expressions of sequent systems are sequents. Every sequent in FL_{ec} is an expression of the following form, with $n \geq 0$:

$$\alpha_1, \ldots, \alpha_n \Rightarrow \beta, \tag{1}$$

where each α_i $(i \leq n)$ and β are formulas. Every sequent is a finite sequence of formulas separated by commas which is divided by $\Rightarrow$ into the antecedents $\alpha_1, \ldots, \alpha_n$ and a single or empty succedent β. We denote the empty sequent by ϵ. The set of formulas in the antecedent is called a *Multiset of formulas*, which will be denoted by capital Greek letters such as Γ, Δ. The sequent (1) can be understood as: The conclusion β follows from the assumptions $\alpha_1, \ldots, \alpha_n$.

1.1.2 Inference Rules

The sequent system FL_{ec} consists of three types of rules:

1. Initial sequents
2. Left- and right rules for the logical connectives $\wedge, \vee, \rightarrow, \cdot$
3. Structural rules

Remark 1. The cut-rule is admissible in FL_{ec}. Therefore we will not consider it in any proofs in the rest of this thesis. A proof of this fact can be found in Theorem 4.7 of *Proof Theory and Algebra in Logic* (Ono, 2019) [21], together with the cut-elimination procedure in Chapter 2 of the same work.

Initial sequents are always of the form $\alpha \Rightarrow \alpha$. For each logical connective, there is a rule to introduce this connective to the left or right of the $\Rightarrow$. In intuitionistic logic, the sequents $\alpha_1, \dots, \alpha_n \Rightarrow \beta$ and $\alpha_1 \wedge \cdots \wedge \alpha_n \Rightarrow \beta$ are considered equal expressions [21, Lemma 1.14]. When we remove weakening, this cannot be proven any more. Hence we need to introduce a new connective called *fusion* $(\cdot)$, with the following rules

$$\frac{\Gamma, \alpha, \beta \Rightarrow \varphi}{\Gamma, \alpha \cdot \beta \Rightarrow \varphi} \cdot L \qquad \frac{\Gamma_1 \Rightarrow \alpha \qquad \Gamma_2 \Rightarrow \beta}{\Gamma_1, \Gamma_2 \Rightarrow \alpha \cdot \beta} \cdot R$$

In the sequent calculus FL_{ec}, we consider the sequents $\alpha_1, \dots, \alpha_n \Rightarrow \beta$ and $\alpha_1 \cdot \ldots \cdot \alpha_n \Rightarrow \beta$ to be equal expressions [21, Section 4.2].

In addition to *fusion* we also introduce the two logical constants, falsehood, *0*, and truth, *1*. Their rules are:

$$\frac{}{0 \Rightarrow \epsilon}\ 1\mathrm{R} \qquad \frac{\Gamma \Rightarrow \epsilon}{\Gamma \Rightarrow 0}\ 0\mathrm{R}$$

$$\frac{}{\epsilon \Rightarrow 1}\ 0\mathrm{L} \qquad \frac{\Gamma \Rightarrow \varphi}{\Gamma, 1 \Rightarrow \varphi}\ 1\mathrm{L}$$

The sequents $0 \Rightarrow \epsilon$ and $\epsilon \Rightarrow 1$ are also considered to be initial sequents. The structural rules are:

$$\frac{\Gamma, \alpha, \beta, \Delta \Rightarrow \varphi}{\Gamma, \beta, \alpha, \Delta \Rightarrow \varphi}\ e \qquad \frac{\alpha, \alpha, \Gamma \Rightarrow \varphi}{\alpha, \Gamma \Rightarrow \varphi}\ c$$

Here e denotes *exchange* and c denotes *contraction.* We do not have right rules for exchange and contraction, since that would require at least two formulas in the succedent, which is impossible in FL_{ec}. The complete FL_{ec} calculus can be found in Figure 1.

$$\frac{}{0 \Rightarrow \epsilon}\ 1\mathrm{R} \quad \frac{}{\epsilon \Rightarrow 1}\ 0\mathrm{L} \quad \frac{\Gamma \Rightarrow \varphi}{\Gamma, 1 \Rightarrow \varphi}\ 1\mathrm{L} \quad \frac{\Gamma \Rightarrow \epsilon}{\Gamma \Rightarrow 0}\ 0\mathrm{R} \quad \frac{}{\alpha \Rightarrow \alpha}\ AX \quad \frac{\alpha, \alpha, \Gamma \Rightarrow \varphi}{\alpha, \Gamma \Rightarrow \varphi}\ c$$

$$\frac{\Gamma, \alpha, \beta, \Delta \Rightarrow \varphi}{\Gamma, \beta, \alpha, \Delta \Rightarrow \varphi}\ e \quad \frac{\Gamma, \alpha \Rightarrow \varphi \qquad \Gamma, \beta \Rightarrow \varphi}{\Gamma, \alpha \vee \beta \Rightarrow \varphi} \vee L \quad \frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \alpha \vee \beta} \vee R1 \quad \frac{\Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \vee \beta} \vee R2$$

$$\frac{\Gamma, \alpha \Rightarrow \varphi}{\Gamma, \alpha \wedge \beta \Rightarrow \varphi} \wedge L1 \quad \frac{\Gamma, \beta \Rightarrow \varphi}{\Gamma, \alpha \wedge \beta \Rightarrow \varphi} \wedge L2 \quad \frac{\Gamma \Rightarrow \alpha \qquad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta} \wedge R$$

$$\frac{\Gamma, \alpha, \beta \Rightarrow \varphi}{\Gamma, \alpha \cdot \beta \Rightarrow \varphi} \cdot L \quad \frac{\Gamma_1 \Rightarrow \alpha \qquad \Gamma_2 \Rightarrow \beta}{\Gamma_1, \Gamma_2 \Rightarrow \alpha \cdot \beta} \cdot R \quad \frac{\Gamma_1 \Rightarrow \alpha \qquad \Gamma_2, \beta \Rightarrow \varphi}{\Gamma_1, \Gamma_2, \alpha \to \beta \Rightarrow \varphi} \to L \quad \frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \alpha \to \beta} \to R$$

Figure 1: FL_{ec} sequent calculus

Each rule is structured as follows: Every rule has a *conclusion* meta-sequent, written below the horizontal line, and some number of *premise* meta-sequents, written above the line. Each meta-sequent comprises of multiset structure variables $\Gamma, \Gamma_1, \Gamma_2$ and formulas built from the formula meta-variables α, β, φ. The formula meta-variables α and β are called active formulas, and the formula in the conclusion built from the active formulas is called the principal formula. In instance σr of a rule is a map taking each multiset meta-variable in r to a formula multiset (any formula occurring therein is said to belong to the context of σr), and each formula meta-variable to a formula.

Example 1. As an example, the rule $\to L$ has the multiset meta-variables $\Gamma, \Gamma_1, \Gamma_2$, and formula meta-variables α, β, φ. The active formulas are α and β, and the principal formula is $\alpha \to \beta$. Consider the rule instance $\sigma(\cdot L)$ taking the multiset meta-variable Γ_1 to $(p, q \wedge r)$, Γ_2 to $(q, q \wedge r)$ and taking α, β, φ to the formulas $r, s \vee t, t$ respectively. Its premises are $p, q \wedge r \Rightarrow r$ and $q, q \wedge r, s \vee t \Rightarrow t$ and its conclusion is $p, q, p \wedge q, p \wedge q, r \to (s \vee t) \Rightarrow t$, as can be seen in Figure 2.

$$\dfrac{p, q \wedge r \Rightarrow r \qquad q, q \wedge r, s \vee t \Rightarrow t}{p, q, p \wedge q, p \wedge q, r \to (s \vee t) \Rightarrow t} \to L$$

Figure 2: Application of the rule $\to L$

△

Given a formula χ, our goal is to either find a proof of the sequent $\Rightarrow \chi$, or find a countermodel for it. Let us first consider when a sequent $\Rightarrow \chi$ is provable.

Definition 1.1. *[21, Definition 1.1] A proof P of a sequent $\Delta \Rightarrow \varphi$ is a finite tree-like structure defined inductively in the following way:*

1. *Every uppermost sequent in the proof P is an initial sequent,*
2. *Every sequent in the proof P is obtained by an application of any one of the rules,*
3. *$\Delta \Rightarrow \varphi$ is the single lowest sequent, which is called the base sequent of the proof.*

We write $\Delta \vdash \varphi$ to mean that $\Delta \Rightarrow \varphi$ is provable, i.e. it has a proof in the calculus.

Example 2. Below are the proofs of the sequents $\Rightarrow \alpha \to (\beta \to \alpha)$ and $\Rightarrow (\alpha \to (\alpha \to \gamma)) \to (\alpha \to \gamma)$.

$$\dfrac{\dfrac{\dfrac{\alpha \Rightarrow \alpha}{\alpha, \beta \Rightarrow \alpha}\, w}{\alpha \Rightarrow \beta \to \alpha} \to R}{\Rightarrow \alpha \to (\beta \to \alpha)} \to R \qquad \dfrac{\dfrac{\dfrac{\dfrac{\alpha \Rightarrow \alpha \qquad \dfrac{\alpha \Rightarrow \alpha \qquad \gamma \Rightarrow \gamma}{\alpha, \alpha \to \gamma \Rightarrow \gamma} \to L}{\alpha, \alpha, \alpha \to (\alpha \to \gamma) \Rightarrow \gamma} \to L}{\alpha, \alpha \to (\alpha \to \gamma) \Rightarrow \gamma}\, c}{\alpha \to (\alpha \to \gamma) \Rightarrow \alpha \to \gamma} \to R}{\Rightarrow (\alpha \to (\alpha \to \gamma)) \to (\alpha \to \gamma)} \to R$$

We see that in the proof of the first sequent, we use weakening. As there is no other way to prove this sequent without using weakening, this sequent is not provable in FL_{ec}, we have $\nvdash \alpha \to (\beta \to \alpha)$. The second sequent only uses rules from FL_{ec} and is thus provable in the calculus and we have $\vdash (\alpha \to (\alpha \to \gamma)) \to (\alpha \to \gamma)$. △

Lemma 1.1. *[21, Remark 4.1][Associativity of Fusion, Conjunction and Disjunction] Fusion, Conjunction and Disjunction are associative connectives. In what follows, if we omit parentheses and write $\alpha_1 \cdot \ldots \cdot \alpha_n$, $\alpha_1, \wedge \ldots \wedge \alpha_n$ and $\alpha_1 \vee \ldots \vee \alpha_n$ respectively, we assume left associativity of parentheses.*

Proof. We will only show the proof for associativity of fusion here, the others are proven similarly. To prove associativity, we will prove both $(\alpha \cdot \beta) \cdot \gamma \Rightarrow \alpha \cdot (\beta \cdot \gamma)$ and $\alpha \cdot (\beta \cdot \gamma) \Rightarrow (\alpha \cdot \beta) \cdot \gamma$.

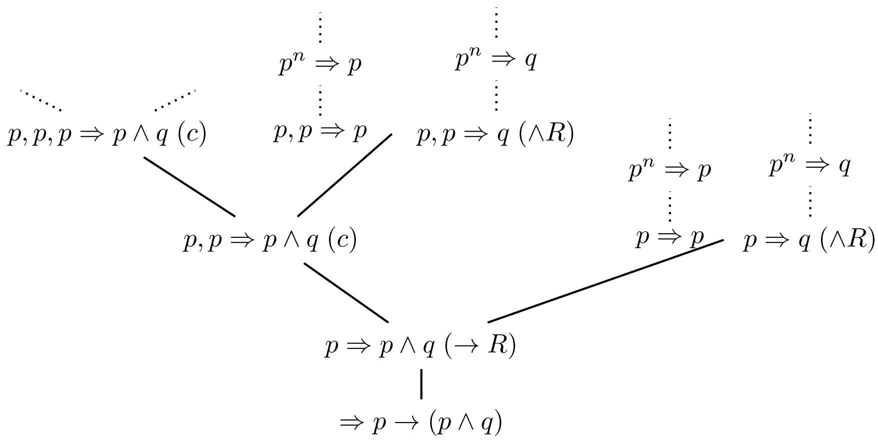
$$\dfrac{\dfrac{\dfrac{\dfrac{\beta \Rightarrow \beta \qquad \gamma \Rightarrow \gamma}{\beta, \gamma \Rightarrow \beta \cdot \gamma} \cdot R \qquad \alpha \Rightarrow \alpha}{\alpha, \beta, \gamma \Rightarrow \alpha \cdot (\beta \cdot \gamma)} \cdot R}{\alpha \cdot \beta, \gamma \Rightarrow \alpha \cdot (\beta \cdot \gamma)} \cdot L}{(\alpha \cdot \beta) \cdot \gamma \Rightarrow \alpha \cdot (\beta \cdot \gamma)} \cdot L \qquad \dfrac{\dfrac{\dfrac{\dfrac{\alpha \Rightarrow \alpha \qquad \beta \Rightarrow \beta}{\alpha, \beta \Rightarrow \alpha \cdot \beta} \cdot R \qquad \gamma \Rightarrow \gamma}{\alpha, \beta, \gamma \Rightarrow (\alpha \cdot \beta) \cdot \gamma} \cdot R}{\alpha, (\beta \cdot \gamma) \Rightarrow (\alpha \cdot \beta) \cdot \gamma} \cdot L}{\alpha \cdot (\beta \cdot \gamma) \Rightarrow (\alpha \cdot \beta) \cdot \gamma} \cdot L$$

□

1.1.3 Exhaustive proof search

Given a formula χ, *exhaustive proof search* in FL_{ec} is the repeated application of all possible left- and right rules upwards, i.e. from conclusion to premises, to construct the proof search tree $\mathcal{T}$ rooted at $\Rightarrow \chi$. Then χ is provable if there is a branch of $\mathcal{T}$ that satisfies the conditions from Definition 1.1. If χ is not provable, its proof search tree will be used to construct its countermodel.

Example 3. Consider the formula $\chi = p \to (p \wedge q)$. Its proof search tree $\mathcal{T}$ looks like:



△

FL_{ec} satisfies the subformula property: every sequent in the proof search tree $\mathcal{T}$ of $\Rightarrow \chi$ will be build from subformulas of χ. Let $\{A_1, \dots, A_d\}$ be the subformulas of χ. Any sequent in the proof search can be written as

$$A_1^{r_1}, \dots, A_d^{r_d} \Rightarrow A_s,$$

where $s \in \{0, \dots, d\}$ and A_0 represents the empty context ϵ. The powers $\{r_1, \dots, r_d\}$ represent the *multiplicities* of the formulas $\{A_1, \dots, A_d\}$, that is, the number of times they appear in the antecedent. A sequent can thus also be represented as the element

$$\big((r_1, \dots, r_d), s\big) \in \mathbb{N}^d \times \{0, \dots, d\}.$$

Another property of the proof search tree $\mathcal{T}$ is that unprovability is upwards closed. In other words, if a sequent $\Gamma \Rightarrow \varphi$ in the proof search tree is unprovable, any sequent above it, by applying any of the rules, is also unprovable. If not, say $\Gamma' \Rightarrow \psi$ appears above $\Gamma \Rightarrow \varphi$ and is provable. Then it has a proof as in Definition 1.1. Since we can go from $\Gamma \Rightarrow \varphi$ to $\Gamma' \Rightarrow \psi$ by application of the rules, we have now also found a proof of $\Gamma \Rightarrow \varphi$.

The opposite holds for provability: it is not upwards closed. For example, the sequent $p \Rightarrow p$ is provable, but after application of the contraction rule, we get the sequent $p, p \Rightarrow p$, which is not provable. It is downwards closed, since if $\Gamma \Rightarrow \varphi$ has a proof, then any sequent reached from it by application of rules then also has a proof.

1.2 Order and Lattice Theory

1.2.1 Partial orders and bad sequences

To begin to study order theory, we must start with binary relations. A *binary relation* over sets X and Y is a set of ordered pairs (x, y) where $x \in X$ and $y \in Y$. Every binary relation R is a subset of the Cartesian product of two sets X and Y. We say X is the domain of R and Y is the codomain of R. If $X = Y$, we call R a *homogeneous relation*. For simplicity, we then call R a relation over X.

A *partial order* is a homogeneous binary relation that is reflexive, transitive, and antisymmetric. We define a partial order on a set P as

Definition 1.2. *[16]* $(P, \preceq) :=$ *partial order if* $\forall x, y, z \in P$

1. $x \preceq x$ *(reflexivity)*
2. $x \preceq y \wedge x \preceq z$ *implies* $x \preceq z$ *(transitivity)*
3. $x \preceq y \wedge y \preceq x$ *implies* $x = y$ *(antisymmetry)*

Remark 2. Here $\preceq$ denotes any binary relation, not necessarily 'less than or equal to'. We call $(P, \preceq)$ a partial ordered set, a poset.

Example 4. Some examples of partial orders are:

- $\mathbb{R}$, ordered by the relation $\leq$.
- The power set $\wp(X)$ of a set X order by the relation $\subseteq$.
- $\mathbb{N}$ ordered by the relation $|$.

△

If the relation is not antisymmetric, we call it a *preorder* or *quasiorder*. If the relation is, moreover, strongly connected ($x \preceq y \vee y \preceq x$), we call it a total order.

Definition 1.3. *[23, Chapter 2] A bad sequence on a quasi-ordered set $(X, \preceq)$ is a sequence $x_0, x_1, \dots$ of elements from X such that $i < j$ implies $x_i \not\preceq x_j$. We say $(X, \preceq)$ is a well-quasi ordered set if every bad sequence on it is finite. In other words, every infinite sequence of elements $x_0, x_1, \dots$ from X contains an increasing pair $x_i \preceq x_j$ with $i < j$.*

Another way of defining a well-quasiorder is to say that they are quasiorderings which do not contain infinite strictly decreasing sequences (of the form $x_0 \succ x_1 \succ x_2 \succ \dots$) nor infinite sequences of pairwise incomparable elements [23, Section 2.1].

Example 5. The simplest example of a well-quasiorder is $(\mathbb{N}, \leq)$ with the standard ordering. Assume that we have a bad sequence starting with n. Since every two elements of $\mathbb{N}$ are comparable, the sequence must be constantly decreasing to ensure $x_i > x_j$ for $i < j$. After at most n elements in the sequence, we reach 0 and the sequence must stop. As n is finite, so is the sequence. △

Example 6. We can extend the example above to $(\mathbb{N}^d, \leq)$ with the componentwise ordering:

$$(x_1, \dots, x_d) \leq (y_1, \dots, y_d) \iff x_i \leq y_i \; \forall i \in \{1, \dots, d\}$$

While for the case $d = 1$ we have no incomparable elements, they exist for larger d. For example, when $d = 2$ consider the elements $(2, 0)$ and $(1, 2)$. Neither of them is larger than the other.

We will prove that $(\mathbb{N}^d, \leq)$ is a well-quasi-order using induction.

- Base case ($d = 1$): This is proven in Example 5.
- Induction hypothesis: We will assume that $(\mathbb{N}^d, \leq)$ is a well-quasiorder.
- Induction step: $(\mathbb{N}^{d+1}, \leq)$ is a well-quasiorder.

We will show the induction step in two steps. First we show that $(\mathbb{N}^{d+1}, \leq)$ has no infinite strictly decreasing sequence. Suppose the sequence starts with the element $x = (x_1, \dots, x_d)$. Every new element in the sequence must be smaller than x, so every coordinate x_i must decrease with at least 1. Let x_j be the smallest coordinate of x. After at most x_j elements in the sequence, there is no smaller element any more, hence our sequence must stop. Hence there cannot exist infinite strictly decreasing sequences. The next thing to show is that $(\mathbb{N}^{d+1}, \leq)$ does not contain an infinite sequence of pairwise incomparable elements. Say it does, then we need, for any two elements $(x_1, \dots, x_{d+1})$ and $(y_1, \dots, y_{d+1})$ in the sequence:

$$\neg((x_1, \dots, x_{d+1}) \leq (y_1, \dots, y_{d+1})) = \begin{cases} x_1 > y_1 \wedge x_2 < y_2 \wedge \dots \wedge x_{d+1} < y_{d+1} \\ \dots \\ x_1 < y_1 \wedge x_2 < y_2 \wedge \dots \wedge x_{d+1} > y_{d+1} \end{cases}$$

where we consider all options for incomparable elements, of which there are a total of $\sum_{i=1}^{d} \binom{d+1}{i}$. If our sequence would be infinite, then at least one of the options would appear an infinite amount of times in our sequence. For each of those options, there are some number of coordinates where $x_i > y_i$, where $i \in \{1, \dots, d+1\}$. Suppose that there are $k < d+1$ coordinates where $x_i > y_i$ holds, then there would be an infinite decreasing sequence in $(\mathbb{N}^k, \leq)$, contradicting the induction hypothesis. Hence every option can only appear a finite number of times in the sequence, implying that our sequence itself is finite. So we conclude that $(\mathbb{N}^d, \leq)$ is a well-quasiorder. △

1.2.2 Lattices

To define a lattice, we must first discuss semi-lattices. A poset $(X, \preceq)$ is a *meet semi-lattice* if for all $x, y \in X$, there exists a greatest lower bound of $\{x, y\}$, denoted $x \wedge y$, called the *meet* of x and y. A poset $(X, \leq)$ is *join semi-lattice* if for all $x, y \in X$, there exists least upper bound of $\{x, y\}$, denoted $x \vee y$, called the *join* of x and y.

Definition 1.4. *[21, Definition 6.2] A lattice is a poset $(X, \preceq)$ that is both a meet- and join semi-lattice, i.e. $\forall x, y \in X$, $\exists x \vee y, x \wedge y$ such that*

1. *$x \preceq x \vee y$ and $y \preceq x \vee y$;*
2. *If $x \preceq z$ and $y \preceq z$ then $x \vee y \preceq z \; \forall z \in X$;*
3. *$x \wedge y \preceq x$ and $x \wedge y \preceq y$;*
4. *If $z \preceq x$ and $z \preceq y$ then $z \preceq x \wedge y \; \forall z \in X$.*

Example 7. Every totally ordered set forms a lattice by defining $x \vee y = \max\{x, y\}$ and $x \wedge y = \min\{x, y\}$. △

Definition 1.5. *[21, Definition 6.4] A lattice $\langle X, \vee, \wedge \rangle$ is a complete lattice if for any possible subset S of X, both the least upper bound of S, denoted $\vee S$ and the greatest lower bound of S, denoted $\wedge S$, exists. We call an element $a \in S$ the least upper bound if*

1. *$x \preceq a$ for all $x \in S$;*
2. *For every $c \in X$, if $x \preceq c$ for all $x \in S$ then $a \preceq c$.*

The greatest lower bound can be defined dually.

2 Construction of the FL_{ec} algebra

The goal of this sections is to define a framework so we can construct countermodels for unprovable sequents. In order to construct a countermodel for a formula, it is important to agree on what even constitutes a model and how to interpret the desired logic in a model. We take models to be FL_{ec} algebras satisfying some additional axioms.

In this section we first introduce the concept of an FL_{ec} algebra and show how it validates formulas. We then explain the construction of an FL_{ec} algebra using a closure operator and explain how our logical connectives should be interpreted in this algebra. We finish the section by showing that the proposed construction indeed satisfies the definition of an FL_{ec} algebra.

2.1 Algebraic models of FL_{ec}

The definition of an FL_{ec} algebra starts with a residuated lattice: a structure $\mathbb{L} = (L, \vee, \wedge, \cdot, 1, \backslash, /)$ such that $(L, \cdot, 1)$ is a monoid, $(L, \vee, \wedge)$ is a lattice and $\backslash$ and $/$ are the residuals. If we add a constant 0, we get an FL-algebra. If the monoid $(L, \cdot, 1)$ is moreover commutative, we get an FL_e-algebra. If then moreover $x \leq x \cdot x$ holds for all $x \in L$, we get an FL_{ec} algebra, where the two residuals $\backslash$ and $/$ have become equal and are now represented by $\rightarrow$ [21, Section 9.1, 9.2].

Definition 2.1. *An FL_{ec} algebra is a tuple $\mathbb{A} = (A, \wedge, \vee, \cdot, \rightarrow, 0, 1)$, where*

- *$(A, \wedge, \vee)$ is a lattice;*
- *$(A, \cdot, 1)$ is a commutative monoid;*
- *$0 \in A$ is an arbitrary element;*
- *$x \cdot y \leq z \Longrightarrow x \leq y \rightarrow z \; \forall x, y, z \in A$;*
- *$x \leq x \cdot x \; \forall x \in A$.*

Let $\mathbb{A}$ be an FL_{ec} algebra. An assignment h on $\mathbb{A}$ is any mapping from the set of all propositional variables to A. The assignment h can naturally be extended to a mapping from the set of all formulas to the set A. Given an FL_{ec} algebra $\mathbb{A}$ and an assignment h, we can interpret each formula φ as an element of $\mathbb{A}$ by setting $[\![\varphi]\!] = h(\varphi)$. We can extend the interpretation to multisets: we interpret the empty multiset as 1, i.e. $[\![\epsilon]\!] = 1$, and the union of multisets as multiplication, i.e. $[\![\varphi, \psi]\!] = [\![\varphi]\!] \cdot [\![\psi]\!]$.

A formula φ is true in an algebra $\mathbb{A}$ if $[\![1]\!] \leq [\![\varphi]\!]$ holds. A formula φ is valid in a class of algebras if it is true in every algebra in that class under any assignment. The class of FL_{ec} algebras is well-known to be sound and complete with regard to the FL_{ec} sequent calculus:

$$\Gamma \vdash \varphi \iff [\![\Gamma]\!] \leq [\![\varphi]\!], \tag{2}$$

is true in every FL_{ec} algebra.

2.2 FL_{ec} algebras from a closure operator

There are several ways to construct an FL_{ec} algebra. One example is the canonical provability model: formulas are interpreted as sets of formula multisets, and the interpretation satisfies the properties:

1. $\varphi \in [\![\varphi]\!]$;
2. $\Delta \in [\![\varphi]\!]$ implies that $\Delta \vdash \varphi$.

Together these conditions give us the canonical provability model: if a formula is valid in the model, then $[\![1]\!] \leq [\![\varphi]\!]$. By the first property, we find $1 \in [\![1]\!] \leq [\![\varphi]\!]$ and then by the second property we have that $\vdash \varphi$ is provable. Due to the canonicity, this model invalidates all the unprovable sequents, but the model is very large and in general infinite. In order to make the model smaller, we want to weaken the second condition to only include sequents that are relevant to us, that is, the ones that appear in the proof search. That is, we require that

$$\Delta \in [\![\varphi]\!] \text{ implies } \big((\Delta \Rightarrow \varphi) \in \mathcal{T} \longrightarrow \Delta \vdash \varphi\big). \tag{3}$$

The implication in Equation (3) cannot be taken as the definition of the interpretation, because it is not compositional in the logical connectives in φ. We can, however, build the model out of the sets that are defined by the condition in Equation (3), and explicitly closing them under certain operations, when needed. In this thesis, we construct a model using the framework of algebras with a closure operator.

Given a set M, let $\wp(M)$ be its power set. We say $\mathcal{C} \subseteq \wp(M)$ is a Moore family if it is closed under arbitrary intersection. Given a collection of sets $(X_i \subseteq M)_{i \in I}$, the smallest Moore family containing it is defined by closing the collection under arbitrary intersection. We refer to the starting sets $(X_i)_{i \in I}$ as the principal closed sets. Given a set $X \in \wp(M)$, we can find the smallest closed set containing X as

$$X \mapsto \cap\{X_i \mid X \subseteq X_i\}.$$

This operator is a closure operator $\mathrm{cl}(-) : \wp(M) \to C$.

Proposition 2.1. *[9] The closure operator satisfies the following properties:*

1. $\mathrm{cl}(-)$ *is monotone;*
2. $X \subseteq \mathrm{cl}(X)$ *for any set* $X \in \wp(M)$*;*
3. $\mathrm{cl}(\mathrm{cl}(X)) = \mathrm{cl}(X)$ *for any set* X*;*
4. $Y = \mathrm{cl}(Y)$ *for any closed set* Y*;*
5. $\mathrm{cl}(X) \subseteq Y \iff X \subseteq Y$ *for any set* X *and any closed set* Y*.*

We now want to define all our operations $(\wedge, \vee, \cdot, \to)$ using this new framework. We define the meet and join as follows: for any $X = (X_i)_{i \in I} \in \wp(M)$, we have

$$\wedge X = \cap_{i \in I} X_i \text{ and } \vee X = \mathrm{cl}(\cup_{i \in I} X_i).$$

We are left to interpret multiplication and implication. If the set M comes with a commutative monoid structure $(M, \cdot, e)$, we can lift the multiplication on M to multiplication on $\mathcal{C}$. We also obtain the residual $\to$ at the same time.

The set $\wp(M)$ has a commutative monoidal structure with the multiplication

$$X \odot Y = \{x \cdot y \mid x \in X, y \in Y\}.$$

with the unit $\{e\}$. We can lift this monoidal structure from $\wp(M)$ to $\mathcal{C}$. The unit becomes $\mathrm{cl}(\{e\})$ and the multiplication becomes

$$X * Y = \mathrm{cl}(X \odot Y).$$

We also get the residual operation $\rightarrow$:

$$X \rightarrow Y = \{m' \mid m \cdot m' \in Y \ \forall m \in X\}.$$

Proposition 2.2. *For closed sets X, Y, Z we have*

$$X \subseteq Y \rightarrow Z \iff X * Y \subseteq Z.$$

Proof. First note that $X*Y \subseteq Z \iff \mathrm{cl}(X \odot Y) \subseteq Z \iff X \odot Y \subseteq Z \iff \{x \cdot y \mid x \in X, y \in Y\} \subseteq Z$ as Z is closed, where we used Proposition 2.1(5).

Now let $x \in X$ be any element of X. Using the definition of $\rightarrow$, we have
$Y \rightarrow Z = \{m' \mid \forall m \in Y, m \cdot m' \in Z\}$. To show $x \in Y \rightarrow Z$, we need to show that $\forall m \in Y, m \cdot x \in Z$. Since $X \odot Y \subseteq Z$, this is indeed the case.

Now assume that $X \subseteq Y \rightarrow Z = \{m' \mid \forall m \in Y, m \cdot m' \in Z\}$. To show $X * Y \subseteq Z$, it suffices to show that $\{x \cdot y \mid x \in X, y \in Y\}$. In other words, taking any element $x \in X$ and $y \in Y$, $x \cdot y \in Z$. This follows immediately from our assumption. □

2.3 C is an FL_{ec} algebra

We will now show that the given interpretation above satisfies Definition 2.1 of an FL_{ec} algebra.

Proposition 2.3. *Given $X, Y \in \mathcal{C}$, $X \wedge Y, X \vee Y, X * Y$ and $X \rightarrow Y$ are elements of $\mathcal{C}$.*

Proof. Let $X, Y \in \mathcal{C}$. We have

- $X \wedge Y = X \cap Y$ is closed by definition of a Moore family, i.e. it is closed under arbitrary intersection.
- $X \vee Y = \mathrm{cl}(X \cup Y)$ is closed since we've defined it to be a closed set.
- $X * Y = \mathrm{cl}(X \odot Y)$ is closed since we've defined it to be a closed set.
- $X \rightarrow Y$: We will show this in the proof of Proposition 3.1.

□

Proposition 2.4. $(\mathcal{C}, *, 1)$ *is a commutative monoid.*

Proof. We need to prove the following:

1. $(X * Y) * Z = X * (Y * Z)$ (associativity).
2. $X * \mathrm{cl}(\{e\}) = X = \mathrm{cl}(\{e\}) * X$ (unitality).
3. $X * Y = Y * X$ (commutativity).

To show associativity, we are to prove $X * (Y * Z) \subseteq (X * Y) * Z$ and $(X * Y) * Z \subseteq X * (Y * Z)$. Here, we will only prove the first, the other follows from symmetry. We have

$$\begin{aligned}
X * (Y * Z) \subseteq (X * Y) * Z &\iff && \text{Proposition 2.1(5)} \\
X \odot (Y * Z) \subseteq (X * Y) * Z &\iff && \text{Proposition 2.2} \\
(Y * Z) \subseteq X \rightarrow ((X * Y) * Z) &\iff && \text{Proposition 2.1(5)} \\
(Y \odot Z) \subseteq X \rightarrow ((X * Y) * Z) &\iff && \text{Proposition 2.2} \\
X \odot (Y \odot Z) \subseteq (X * Y) * Z. &&&
\end{aligned}$$

So we are now left to show that $X \odot (Y \odot Z) \subseteq (X * Y) * Z$. We may first assume this is equal to proving $(X \odot Y) \odot Z \subseteq (X * Y) * Z$. This holds as $\odot$ is an associative operator, where we use that $(M, \cdot, e)$ is a commutative monoid. Then we have that

$$\begin{aligned} (X \odot Y) \odot Z &\subseteq \mathrm{cl}(X \odot Y) \odot Z && \text{Proposition 2.1(2)} \\ &= (X * Y) \odot Z \\ &\subseteq \mathrm{cl}((X * Y) \odot Z) && \text{Proposition 2.1(2)} \\ &= (X * Y) * Z. \end{aligned}$$

To show unitality, we will only show that $X * \mathrm{cl}(\{e\}) = X$, the other case follows by symmetry. We need to show that $X * \mathrm{cl}(\{e\}) \subseteq X$ and that $X \subseteq X * \mathrm{cl}(\{e\})$. Starting with the second inclusion, let $x \in X$. We want to show that $x \in X * \mathrm{cl}(\{e\})$. Note that $x = x \cdot \{e\}$ and since $\{e\} \in \mathrm{cl}(\{e\})$ by Proposition 2.1(2), this is shown. For the first inclusion, it suffices to show that $X \odot \mathrm{cl}\{e\} \subset X$ by Proposition 2.1(5). To prove this, we want to show that $X \odot \mathrm{cl}\{e\} \subseteq \mathrm{cl}(X \odot \{e\})$. We have that

$$\begin{aligned} X \odot \mathrm{cl}\{e\} \subseteq \mathrm{cl}(X \odot \{e\}) && \iff \\ \mathrm{cl}(\{e\}) \subseteq X \to \mathrm{cl}(X \odot \{e\}) && \iff \\ \{e\} \subseteq X \to \mathrm{cl}(X \odot \{e\}) && \iff \\ X \odot \{e\} \subseteq \mathrm{cl}(X \odot \{e\}). \end{aligned}$$

Clearly, our last statement is true by Proposition 2.1(2). This gives us

$$X \odot \mathrm{cl}(\{e\}) \subseteq \mathrm{cl}(X \odot \{e\}) = \mathrm{cl}(X) = X.$$

For commutativity, we have

$$\begin{aligned} X * Y &= \mathrm{cl}(X \odot Y) \\ &= \mathrm{cl}(\{x \cdot y : x \in X, y \in Y\}) \\ &= \mathrm{cl}(\{y \cdot x : x \in X, y \in Y\}) \\ &= \mathrm{cl}(Y \odot X) \\ &= Y * X. \end{aligned}$$

Here we used that $(M, \cdot, e)$ is a commutative monoid. □

Proposition 2.5. *For any closure operator, the collection $\mathcal{C} \subseteq \wp(M)$ of closed sets is a lattice with $\wedge = \cap$ and $\vee X = \mathrm{cl}(\cup X)$, where X is a subset of $\wp(M)$.*

Proof. To prove $(\wp(M), \wedge, \vee)$ is a lattice, let $X, Y \in \mathcal{C}$. We need to show that there exist $X \vee Y$ and $X \wedge Y$ such that

1. $X \subseteq \mathrm{cl}(X \cup Y),\ Y \subseteq \mathrm{cl}(X \cup Y)$;
2. $X \subseteq Z,\ Y \subseteq Z \Rightarrow \mathrm{cl}(X \cup Y) \subseteq Z\ \forall Z \in \mathcal{C}$;
3. $X \cap Y \subseteq X,\ X \cap Y \subseteq Y$;
4. $Z \subseteq X, Z \subseteq Y \Rightarrow Z \subseteq X \cap Y\ \forall Z \in \mathcal{C}$.

For 1, it is easy to see that $X \subseteq X \cup Y \subseteq \mathrm{cl}(X \cup Y)$ and $Y \subseteq X \cup Y \subseteq \mathrm{cl}(X \cup Y)$. For 2, write $X \cup Y = \{m \mid m \in X \text{ or } m \in Y\}$. Let Z be a set such that $X \subseteq Z$, i.e. $\forall m \in X \Rightarrow m \in Z$ and $Y \subseteq Z$, i.e. $\forall m \in Y \Rightarrow m \in Z$. Hence, given any $m \in X \cup Y$, since m is either in X or Y (or both), we find $m \in Z$, so we conclude $X \cup Y \subseteq Z$. As Z is closed, we conclude $\mathrm{cl}(X \cup Y) \subseteq Z$. The statement in 3 follows directly from the definition of intersection. For 4, write $X \cap Y = \{m \mid m \in X \text{ and } m \in Y\}$. Let Z be a set such that $Z \subseteq X$, i.e. $\forall m \in Z \Rightarrow m \in X$ and $Z \subseteq Y$, i.e. $\forall m \in Z \Rightarrow m \in Y$. Hence $\forall m \in Z$, $m \in X$ and $m \in Y$, so $Z \subseteq X \cap Y$. □

Proposition 2.6. *For $X, Y, Z \in \mathcal{C}$, we have*

$$X * Y \subseteq Z \iff X \subseteq Y \to Z.$$

Proof. This has been proven in Proposition 2.2. □

We summarize the results in this section in the following proposition:

Proposition 2.7. *Suppose we have a commutative monoid $(M, \cdot, e)$, and a collection of principal closed sets $(A_i)_{i \in I}$ in $\wp(M)$. Suppose that for any closed sets X, Y, we have that $X \to Y$ is closed and that for any closed set X, $X \subseteq X * X$. Then the collection of closed sets $\mathcal{C} \subseteq \wp(M)$ is an FL_{ec} algebra with the following interpretation:*

$$\llbracket 0 \rrbracket = \mathrm{cl}(\{0\}) \qquad \llbracket 1 \rrbracket = \mathrm{cl}(\{e\}) \qquad \llbracket \varphi \cdot \psi \rrbracket = \mathrm{cl}(\llbracket \varphi \rrbracket \odot \llbracket \psi \rrbracket)$$
$$\llbracket \varphi \wedge \psi \rrbracket = \llbracket \varphi \rrbracket \cap \llbracket \psi \rrbracket \qquad \llbracket \varphi \to \psi \rrbracket = \llbracket \varphi \rrbracket \to \llbracket \psi \rrbracket \qquad \llbracket \varphi \vee \psi \rrbracket = \mathrm{cl}(\llbracket \varphi \rrbracket \cup \llbracket \psi \rrbracket)$$

3 Countermodels for sequents of unprovable formulas

3.1 Construction of the countermodel

From here on, we will consider an unprovable formula χ. We need to construct a specific FL_{ec} algebra that will invalidate the sequent $\Rightarrow \chi$. For the commutative monoid, we take the monoid of contexts. That is, we take the set of multisets of formulas $Mset$. The multiplication is composition of sequences and the unit is the empty sequence. The closure operator is obtained by defining principal closed sets. They are of the form

$$\langle \Delta; \varphi \rangle = \{\Gamma \in Mset \mid (\Gamma, \Delta \Rightarrow \varphi) \in \mathcal{T} \Longrightarrow \Gamma, \Delta \vdash \varphi\},$$

for $\Delta \in Mset$ and a formula φ. By $\mathcal{C}$ we denote the lattice of closed sets. For $X \in \wp(Mset)$, the closure of X is defined as

$$\mathrm{cl}(X) = \cap\{\langle \Delta; \varphi \rangle \mid X \subseteq \langle \Delta; \varphi \rangle\}.$$

Proposition 3.1. *$\mathcal{C}$ as defined above is an FL_{ec} algebra.*

Proof. Most results are already proven by Proposition 2.7. We only have to prove the assumptions that were made in the proposition. We are hence left to prove that $\forall X, Y \in \mathcal{C}, X \to Y$ is closed and that $\Gamma \in Mset$ and that for any closed set X, $X \subseteq X * X$.

To prove the first claim, as $Y \in \mathcal{C}$, we can write $Y = \cap_{i \in I} \langle \Delta_i; \varphi_i \rangle$. Then to show $X \to Y$ is closed, we want to write it as an intersection of principal closed sets. We have

$$\begin{aligned}
\Gamma \in X \to Y &\iff \forall \Gamma' \in X, \Gamma \cdot \Gamma' \in \cap_{i \in I} \langle \Delta_i; \varphi_i \rangle \\
&\iff \forall \Gamma' \in X, \forall i \in I, \Gamma \cdot \Gamma' \in \langle \Delta_i; \varphi_i \rangle \\
&\iff \forall \Gamma' \in X, \forall i \in I, (\Gamma, \Gamma', \Delta_i) \vdash \varphi_i \\
&\iff \forall \Gamma' \in X, \forall i \in I, \Gamma \in \langle \Gamma', \Delta_i; \varphi_i \rangle \\
&\iff \Gamma \in \cap_{i \in I, \Gamma' \in X} \langle \Gamma', \Delta_i; \varphi_i \rangle.
\end{aligned}$$

We can now write

$$X \to Y = \cap_{i \in I, \Gamma' \in X} \langle \Gamma', \Delta_i; \varphi_i \rangle,$$

so we conclude that $X \to Y$ is an intersection of principal closed sets and is hence closed.

To prove the second claim, suppose that $\Gamma \in X$. Showing that $\Gamma \in X * X = \mathrm{cl}(X \odot X)$ means showing that $\Gamma \in \langle \Delta; \varphi \rangle$ for any principal closed set satisfying $\langle \Delta; \varphi \rangle \supseteq X \odot X$, by definition of the closure of a set. Unfolding the definition of $\Gamma \in \langle \Delta; \varphi \rangle$, we must show that $\Gamma, \Delta \Rightarrow \varphi \in \mathcal{T}$ implies $\Gamma, \Delta \vdash \varphi$. Suppose $\Gamma, \Delta \Rightarrow \varphi \in \mathcal{T}$. By exhaustivity of proof search, $(\Delta, \Gamma, \Gamma \Rightarrow \varphi) \in \mathcal{T}$. By the assumption $\Gamma \in X$ we have $\Gamma, \Gamma \in X \odot X$, and hence from $\langle \Delta; \varphi \rangle \supseteq X \odot X$ we get $(\Gamma, \Gamma) \in \langle \Delta; \varphi \rangle$. Hence $\Delta, \Gamma, \Gamma \vdash \varphi$. Using contraction we find that $\Gamma, \Delta \vdash \varphi$. □

We would now like to show that this model indeed invalidates a sequent $\Rightarrow \chi$ for any unprovable formula χ. In order to do this, we first need the following Theorem.

Theorem 3.2 (Okada's Property)**.** *For any formula φ, and empty context ϵ,*

$$\varphi \in [\![\varphi]\!] \subseteq \langle \epsilon; \varphi \rangle.$$

Proof. We will proof this by induction on the lenght of φ.
Base case ($\varphi = 1$). To prove: $1 \in [\![1]\!] = \text{cl}(\{\epsilon\}) \subseteq \langle \epsilon; 1 \rangle$.
For the first inclusion, note that $\epsilon \in [\![1]\!] = \text{cl}(\{\epsilon\})$. We can write $[\![1]\!] = \cap_{i \in I}\{\langle \Delta_i; \psi_i \rangle : \epsilon \subseteq \langle \Delta_i; \psi_i \rangle$. To show $1 \in [\![1]\!]$, let $i \in I$ and we are to show that $(\Delta_i, 1 \Rightarrow \psi_i) \in \mathcal{T}$ implies $\Delta_i, 1 \vdash \psi_i$. Using the left rule of 1, we also have $(\Delta_i, \epsilon \Rightarrow \psi_i) \in \mathcal{T}$ and thus $\Delta_i, \epsilon \vdash \psi_i$ and hence $\Delta_i, 1, \vdash \psi_i$. For the second inclusion, this is equivalent to showing $\{\epsilon\} \subseteq \langle \epsilon; 1 \rangle$. That is, when $(\epsilon \Rightarrow 1) \in \mathcal{T}$, then $\epsilon \vdash 1$ is provable. Since $\vdash 1$ is always provable, the implication holds.

Case $\varphi = \varphi_1 \vee \varphi_2$. To prove: $\varphi_1 \vee \varphi_2 \in [\![\varphi_1 \vee \varphi_2]\!] = \text{cl}([\![\varphi_1]\!] \cup [\![\varphi_2]\!]) \subseteq \langle \epsilon; \varphi_1 \vee \varphi_2 \rangle$.
For the first inclusion, we can write $\text{cl}([\![\varphi_1]\!] \cup [\![\varphi_2]\!]) = \cap_{i \in I}\{\langle \Delta_i; \psi_i \rangle; [\![\varphi_1]\!] \cup [\![\varphi_2]\!] \subseteq \langle \Delta_i; \psi_i \rangle = \cap_{i \in I}\{\langle \Delta_i; \psi_i \rangle; [\![\varphi_1]\!] \subseteq \langle \Delta_i; \psi_i \rangle$ and $[\![\varphi_2]\!] \subseteq \langle \Delta_i; \psi_i \rangle\}$. We want to show that $\forall i \in I, \varphi_1 \vee \varphi_2 \in \langle \Delta_i; \psi_i \rangle$. For this, suppose that $(\Delta_i, \varphi_1 \vee \varphi_2 \Rightarrow \psi) \in \mathcal{T}$. By the left rule of $\vee$, we know that $(\Delta_i, \varphi_1 \Rightarrow \psi) \in \mathcal{T}$ and $(\Delta_i, \varphi_2 \Rightarrow \psi_i) \in \mathcal{T}$. By the induction hypothesis, we know that for $j = 1, 2, \forall i \in I, \varphi_j \in [\![\varphi_j]\!] \subseteq \langle \Delta_i, \psi_i \rangle$ and thus as $(\Delta_i, \varphi_j \Rightarrow \psi_i) \in \mathcal{T}$, that $\Delta_i, \varphi_j \vdash \psi_i$. Hence we know that $\Delta_i, \varphi_1 \vee \varphi_2 \vdash \psi_i$.
For the second inclusion, note that this is equal to $[\![\varphi_1]\!] \cup [\![\varphi_2]\!] \subseteq \langle \epsilon; \varphi_1 \vee \varphi_2 \rangle$ by Proposition 1.5. This is then equivalent to $[\![\varphi_1]\!] \subseteq \langle \epsilon; \varphi_1 \vee \varphi_2 \rangle$ and $[\![\varphi_2]\!] \subseteq \langle \epsilon; \varphi_1 \vee \varphi_2 \rangle$. To show the first one, let $\Gamma \in [\![\varphi_1]\!]$. We want to show $\Gamma \in \langle \epsilon; \varphi_1 \vee \varphi_2 \rangle$. Suppose $(\Gamma \Rightarrow \varphi_1 \vee \varphi_2) \in \mathcal{T}$. Then, by applying the right rule of $\vee$, we know that $(\Gamma \Rightarrow \varphi_1) \in \mathcal{T}$. By the induction hypothesis, $[\![\varphi_1]\!] \subseteq \langle \epsilon; \varphi_1 \rangle$ and thus $\Gamma \vdash \varphi_1$. Then, $\Gamma \vdash \varphi_1 \vee \varphi_2$, hence $\Gamma \in \langle \epsilon; \varphi_1 \vee \varphi_2 \rangle$. The case for φ_2 is proven similarly.

Case $\varphi = \varphi_1 \wedge \varphi_2$. To prove: $\varphi_1 \wedge \varphi_2 \in [\![\varphi_1 \wedge \varphi_2]\!] = [\![\varphi_1]\!] \cap [\![\varphi_1]\!] \subseteq \langle \epsilon; \varphi_1 \wedge \varphi_2 \rangle$.
For the first inclusion, we want to show that both $\varphi_1 \wedge \varphi_2 \in [\![\varphi_1]\!]$ and $\varphi_1 \wedge \varphi_2 \in [\![\varphi_2]\!]$. We will show the first one here. As $[\![\varphi_1]\!]$ is closed, we can write $[\![\varphi_1]\!] = \cap_{i \in I}\langle \Delta_i; \psi_i \rangle$. To show $\varphi_1 \wedge \varphi_2 \in [\![\varphi_1]\!]$, we must show that $\varphi_1 \wedge \varphi_2 \in \langle \Delta_i; \psi_i \rangle \forall i \in I$. So, assume that $(\Delta_i, \varphi_1 \wedge \varphi_2 \Rightarrow \psi_i) \in \mathcal{T}$. We are to show that $\Delta_i, \varphi_1 \wedge \varphi_2 \vdash \psi$. By the left rule of $\wedge$, we must have that $(\Delta_i, \varphi_1 \Rightarrow \psi) \in \mathcal{T}$. By the induction hypothesis, $\varphi_1 \in [\![\varphi_1]\!] \subseteq \langle \Delta_i; \psi_i \rangle$, so $\Delta_i, \varphi_1 \vdash \psi_i$ and hence $\Delta_i, \varphi_1 \wedge \varphi_2 \vdash \psi$. For the second inclusion, suppose that $\Gamma \in [\![\varphi_1]\!] \cap [\![\varphi_2]\!]$. We want to show that $\Gamma \in \langle \epsilon; \varphi_1 \wedge \varphi_2 \rangle$. Assume that $(\Gamma \Rightarrow \varphi_1 \wedge \varphi_2) \in \mathcal{T}$. We want to show that $\Gamma \vdash \varphi_1 \wedge \varphi_2$. By the right rule of $\wedge$, we have that $(\Gamma \Rightarrow \varphi_1) \in \mathcal{T}$ and $(\Gamma \Rightarrow \varphi_2) \in \mathcal{T}$. As, $\Gamma \in [\![\varphi_1]\!] \cap [\![\varphi_2]\!]$, $\Gamma \in [\![\varphi_1]\!]$ and $\Gamma \in [\![\varphi_2]\!]$, so by our induction hypothesis we must have that $\Gamma \in \langle \epsilon; \varphi_1 \rangle$ and $\Gamma \in \langle \epsilon; \varphi_2 \rangle$ and hence $\Gamma \vdash \varphi_1$ and $\Gamma \vdash \varphi_2$. Then $\Gamma \vdash \varphi_1 \wedge \varphi_2$.

Case $\varphi = \varphi_1 \cdot \varphi_2$. To prove: $\varphi_1 \cdot \varphi_2 \in [\![\varphi_1 \cdot \varphi_2]\!] = \text{cl}([\![\varphi_1]\!] \odot [\![\varphi_2]\!]) \subseteq \langle \epsilon; \varphi_1 \cdot \varphi_2 \rangle$.
For the first inclusion, we can write $\text{cl}([\![\varphi_1]\!] \odot [\![\varphi_2]\!]) = \cap_{i \in I}\{\langle \Delta_i; \psi_i \rangle; [\![\varphi_1]\!] \odot [\![\varphi_2]\!] \subseteq \langle \Delta_i; \psi_i \rangle$. By the induction hypothesis, $\varphi_1 \in [\![\varphi_1]\!]$ and $\varphi_2 \in [\![\varphi_2]\!]$ and thus $(\varphi_1, \varphi_2) \in [\![\varphi_1]\!] \odot [\![\varphi_2]\!]$ and $(\varphi_1, \varphi_2) \in \text{cl}([\![\varphi_1]\!] \odot [\![\varphi_2]\!])$ by Proposition 1.2. We want to show that this implies $\varphi_1 \cdot \varphi_2 \in \text{cl}([\![\varphi_1]\!] \odot [\![\varphi_2]\!])$. To show this, we must show $\varphi_1 \cdot \varphi_2 \in \langle \Delta_i; \psi_i \rangle \forall i \in I$. Assume $(\Delta_i, \varphi_1 \cdot \varphi_2 \Rightarrow \psi_i) \in \mathcal{T}$, then by the left rule of $\cdot$, $(\Delta_i, \varphi_1, \varphi_2 \Rightarrow \psi_i) \in \mathcal{T}$. As $(\varphi_1, \varphi_2) \in \text{cl}([\![\varphi_1]\!] \odot [\![\varphi_2]\!])$, we have $(\varphi_1, \varphi_2) \in \langle \Delta_i; \psi_i \rangle \forall i \in I$ and thus $\Delta_i, \varphi_1, \varphi_2 \vdash \psi_i$ and hence $\Delta_i, \varphi_1 \cdot \varphi_2 \vdash \psi_i$. For the second inclusion, note that this is equivalent to showing $[\![\varphi_1]\!] \odot [\![\varphi_2]\!] \subseteq \langle \epsilon; \varphi_1 \cdot \varphi_2 \rangle$ by Proposition 1.5. Let $\Gamma \in [\![\varphi_1]\!] \odot [\![\varphi_2]\!]$. We may assume that $\Gamma = (\Gamma_1, \Gamma_2)$, where $\Gamma_1 \in [\![\varphi_1]\!]$ and $\Gamma_2 \in [\![\varphi_2]\!]$. We want to show that $(\Gamma_1, \Gamma_2) \in \langle \epsilon; \varphi_1 \cdot \varphi_2 \rangle$. Assume that $((\Gamma_1, \Gamma_2) \Rightarrow \varphi_1 \cdot \varphi_2) \in \mathcal{T}$. We want to show that $(\Gamma_1, \Gamma_2) \vdash \varphi_1 \cdot \varphi_2$. Applying the right rule of $\cdot$, we have that $(\Gamma_1 \Rightarrow \varphi_1) \in \mathcal{T}$ and $(\Gamma_2 \Rightarrow \varphi_2) \in \mathcal{T}$. By our induction hypothesis, $\Gamma_i \in [\![\varphi_i]\!] \subseteq \langle \epsilon; \varphi_i \rangle$ for $i = 1, 2$, so $\Gamma_1 \vdash \varphi_1$ and $\Gamma_2 \vdash \varphi_2$ and hence $\Gamma_1, \Gamma_2 \vdash \varphi_1 \cdot \varphi_2$.

Case $\varphi = \varphi_1 \to \varphi_2$. To prove: $\varphi_1 \to \varphi_2 \in [\![\varphi_1 \to \varphi_2]\!] = [\![\varphi_1]\!] \to [\![\varphi_2]\!] \subseteq \langle \epsilon; \varphi_1 \to \varphi_2 \rangle$.
For the first inclusion, let $\Gamma \in [\![\varphi_1]\!]$ and we want to show that $\Gamma, \varphi \to \varphi_2 \in [\![\varphi_2]\!]$. We can write $[\![\varphi_2]\!] = \cap_{i \in I}\langle \Delta_i; \psi_i \rangle$. So we need to show that $\Gamma, \varphi_1 \to \varphi_2 \in \langle \Delta_i; \psi_i \rangle \forall i \in I$. Assume $(\Delta_i, \Gamma, \varphi_1 \to \varphi_2 \Rightarrow \psi_i) \in \mathcal{T}$. Then, using the left rule of $\to$, $(\Gamma \Rightarrow \varphi_1) \in \mathcal{T}$ and $(\Delta_i \varphi_2 \Rightarrow \psi) \in \mathcal{T}$. As $\Gamma \in [\![\varphi_1]\!] \subseteq \langle \epsilon; \varphi_1 \rangle$ by our induction hypothesis, we know $\Gamma \vdash \varphi)_1$. As $\varphi_2 \in [\![\varphi_2]\!] \subseteq \langle \Delta_i; \psi_i \rangle$, $\Delta_i, \varphi_2 \vdash \psi_i$. Hence $\Delta_i, \Gamma, \varphi_1 \to \varphi_2 \vdash \psi_i$. For the second inclusion, let $\Gamma \in [\![\varphi_1]\!] \to [\![\varphi_2]\!]$. We want to show $\Gamma \in \langle \epsilon; \varphi_1 \to \varphi_2 \rangle$. Suppose that $(\Gamma \Rightarrow \varphi_1 \to \varphi_2) \in \mathcal{T}$. We want to show that $\Gamma \vdash \varphi_1 \to \varphi_2$. By applying the right rule of $\to$, we know that $(\Gamma, \varphi_1 \Rightarrow \varphi_2) \in \mathcal{T}$. By our induction hypothesis, $\varphi_1 \in [\![\varphi_1]\!]$ and hence $\Gamma, \varphi_1 \in [\![\varphi_2]\!] \subseteq \langle \epsilon; \varphi_2 \rangle$. Hence $\Gamma, \varphi_1 \vdash \varphi_2$ and thus $\Gamma \vdash \varphi_1 \to \varphi_2$.

Case $\varphi = 0$. To prove: $0 \in [\![0]\!] = \text{cl}(\{0\}) \subseteq \langle \epsilon; 0 \rangle$.
For the first inclusion, we can write $[\![0]\!] = \cap_{i \in I} \{\langle \Delta_i; \psi_i \rangle : \{0\} \subseteq \langle \Delta_i; \psi_i \rangle$. To show $0 \in [\![0]\!]$, let $i \in I$ and we are to show that $(\Delta_i, 0 \Rightarrow \psi_i) \in \mathcal{T}$ implies $\Delta_i, 0 \vdash \psi_i$. Using the left rule of 0, $(\Delta_i \Rightarrow \psi_i) \in T$ and thus $\Delta_i \vdash \psi_i$ and hence $\Delta_i, 0 \vdash \psi_i$.
For the second inclusion, this is equivalent to showing $0 \subseteq \langle \epsilon; 0 \rangle$. That is, when $(0 \Rightarrow 0) \in \mathcal{T}$, then $0 \vdash 0$. $0 \vdash 0$ is always provable by a left and then right rule of 0, so the implication holds ☐

With this result, we are able to show:

Theorem 3.3. *Suppose we have a non-provable formula χ. Then the model $\mathcal{C}$ constructed as above does not validate the sequent $\Rightarrow \chi$.*

Proof. We must show $[\![1]\!] = \text{cl}(\{\epsilon\}) \nsubseteq [\![\chi]\!]$ in $\mathcal{C}$. Suppose on the contrary that $[\![1]\!] \subseteq [\![\chi]\!]$, i.e. $\text{cl}(\{\epsilon\}) \subseteq [\![\chi]\!]$. This implies that $\epsilon \in [\![\chi]\!]$. By Okada's property (Theorem 3.2), we have that $\epsilon \in \langle \epsilon; \chi \rangle$, so either $(\epsilon \Rightarrow \chi) \notin \mathcal{T}$ or $\epsilon \vdash \chi$.

The first one is not true since our prooftree is rooted at $\epsilon \Rightarrow \chi$ and the second is not true since χ is assumed to be non-provable. ☐

Example 8. As an example, consider the simplest sequent $\Rightarrow p$. Obviously, it is not provable, so what does its countermodel look like?

Let us take a look at the prooftree of $\Rightarrow p$. There are not any rules in FL_{ec} that we can apply upwards, so the whole prooftree looks like

$$\Rightarrow p$$

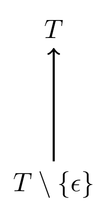
Now, let us determine the principal closed sets. Recall:

$$\langle \Delta; \varphi \rangle = \{\Gamma \in Mset \mid (\Gamma, \Delta \Rightarrow \varphi) \in \mathcal{T} \Rightarrow \Gamma, \Delta \vdash \varphi\}.$$

We have two cases:

1. $\varphi \neq p$. Then $\Gamma, \Delta \Rightarrow \varphi \notin \mathcal{T}$ for any Γ, Δ, so $\langle \Delta, \varphi \rangle = T$ (T is the set of all formula multisets).
2. $\varphi = p$.
   (a) $\Delta = \epsilon$. Then $(\Gamma \Rightarrow p) \in \mathcal{T}$ if and only if $\Gamma = \epsilon$. In that case, we would need $\epsilon \vdash p$, but this is not the case so we get $\langle \epsilon; p \rangle = T \setminus \epsilon$.
   (b) $\Delta \neq \epsilon$. Then $(\Delta, \Gamma \Rightarrow p) \notin \mathcal{T}$ for any Γ, so $\langle \Delta, p \rangle = T$.

So we get two principal closed sets, T and $T \setminus \epsilon$. As all elements are intersections of principal closed sets and $T \cap T \setminus \epsilon = T \setminus \epsilon$, we have precisely two elements in our closed set $\mathcal{C}$. Using the definition of the residual $\rightarrow$, we find that there is an arrow between comparable sets with respect to $\subseteq$, pointing from the smallest to the largest set. Hence our model looks like



Recall that

$$\text{cl}(X) = \cap\{\langle \Delta; \varphi \rangle \mid X \subseteq \langle \Delta; \varphi \rangle\}.$$

We have:

- $[\![p]\!] = \text{cl}(\{p\}) = T \cap T \setminus \epsilon = T \setminus \epsilon$.
- $[\![1]\!] = \text{cl}(\{\epsilon\}) = T$.

As $T \setminus \epsilon \subset T$, we find that $[\![p]\!] < [\![1]\!]$ and hence $[\![1]\!] \nleq [\![p]\!]$, so the model $\mathcal{C}$ does not validate the sequent $\Rightarrow p$. △

Example 9. Consider now the sequent $\Rightarrow p \to q$. The exhaustive proof search looks like:

$$\dfrac{\dfrac{\dfrac{\dfrac{p,\dots,p \Rightarrow q}{\vdots}\,c}{p,p \Rightarrow q}\,c}{p \Rightarrow q}\,c}{\Rightarrow p \to q}\,{\to}R$$

From the sequent $\Rightarrow p \to q$, we can only apply the rule $\to R$ and from then on we can only apply contraction indefinitely. This allows us to determine the principal closed sets. We have 3 cases:

1. $\varphi \neq q, \varphi \neq p \to q$. Then $\Gamma, \Delta \Rightarrow \varphi \notin \mathcal{T}$ for any Γ, Δ, so $\langle \Delta; \varphi \rangle = T$.
2. $\varphi = p \to q$.
   (a) $\Delta = \epsilon$. Then $(\Gamma \Rightarrow \varphi) \in \mathcal{T}$ if and only if $\Gamma = \epsilon$, but $e \not\vdash p \to q$, so $\langle \epsilon; p \to q \rangle = T \setminus \epsilon$.
   (b) $\Delta \neq \epsilon$. Then $(\Delta, \Gamma \Rightarrow p \to q) \notin \mathcal{T}$ for any Γ, so $\langle \Delta; p \to q \rangle = T$.
3. $\varphi = q$.
   (a) $\Delta = \{p^n \ : \ n > 0\}$. Then $(\Delta, \Gamma \Rightarrow q) \in \mathcal{T}$ if and only if $\Gamma = p^k$ or ϵ, but $p^k \not\vdash q$ and $\epsilon \not\vdash q$, so $\langle \Delta; q \rangle = T \setminus \{\epsilon, \{p^k \ : \ k \geq 1\}\}$.
   (b) $\Delta = \epsilon$. Then $(\Delta, \Gamma \Rightarrow q) \in \mathcal{T}$ if and only if $\Gamma = \{p^k : k \geq 1\}$, so $\langle \Delta; q \rangle = T \setminus \{p^k : k \geq 1\}$.
   (c) $\Delta \neq \epsilon, \{p^n \ : \ n > 0\}$. Then $(\Delta, \Gamma \Rightarrow q) \notin \mathcal{T}$ for any Γ, so $\langle \Delta; q \rangle = T$.

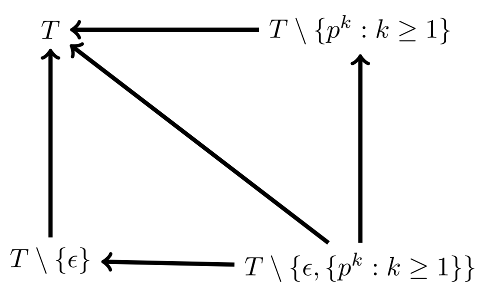
So we get the following principal closed sets:

$$\left\{T, T \setminus \epsilon, T \setminus \{p^k \ : \ k \geq 1\}, T \setminus \{\epsilon, \{p^k \ : \ k \geq 1\}\}\right\}.$$

Taking intersections we do not get any new sets, so we have all closed sets

$$\mathcal{C} = \left\{T, \ T \setminus \epsilon, \ T \setminus \{p^k \ : \ k \geq 1\}, \ T \setminus \{\epsilon, \{p^k \ : \ k \geq 1\}\}\right\}.$$

So our countermodel looks like:



We have

- $[\![p \to q]\!] = [\![p]\!] \to [\![q]\!]$.
- $[\![p]\!] = T \cap T \setminus \epsilon = T \setminus \epsilon$.
- $[\![q]\!] = T \setminus \{\epsilon, \{p^k : k \geq 1\}\}$.
- $[\![1]\!] = \mathrm{cl}(\{\epsilon\}) = T \cap T \setminus \{p^k \ : \ k \geq 1\} = T \setminus \{p^k \ : \ k \geq 1\}$.

Then

$$\begin{aligned}
[\![p]\!] \to [\![q]\!] &= T \setminus \epsilon \to T \setminus \{\epsilon, \{p^k : k \geq 1\}\} \\
&= \{\Gamma' \mid \forall \Gamma \in T \setminus \{\epsilon\}, \Gamma' \cdot \Gamma \in [\![q]\!] = T \setminus \{\epsilon, \{p^k : k \geq 1\}\}\} \\
&= T \setminus \{\epsilon, \{p^k : k \geq 1\}\}
\end{aligned}$$

As $T \setminus \{\epsilon, \{p^k : k \geq 1\}\} \subset T \setminus \{p^k \ : \ k \geq 1\}$, we have that $[\![p \to q]\!] < [\![1]\!]$ and thus $[\![1]\!] \not\leq [\![p \to q]\!]$. $\triangle$

3.2 Finite model property

The goal of this section is to show that the model $\mathcal{C}$ we have constructed throughout Sections 2 and 3 is finite for any unprovable formula χ. In order to do this, we will need another tool: the *contraction extension order*.

3.2.1 Contraction extension order

Definition 3.1 (Contraction extension order). $(\Gamma \Rightarrow \varphi) \lhd (\Gamma' \Rightarrow \varphi')$ *if* $\Gamma \Rightarrow \varphi$ *is obtainable from* $(\Gamma' \Rightarrow \varphi')$ *by one of more applications of the contraction rule. We define* $\unlhd$ *as the reflexive closure of* $\lhd$*. If neither* $(\Gamma \Rightarrow \varphi) \unlhd (\Gamma' \Rightarrow \varphi')$ *nor* $(\Gamma' \Rightarrow \varphi) \unlhd (\Gamma \Rightarrow \varphi')$ *holds, we write* $(\Gamma \Rightarrow \varphi) \perp (\Gamma' \Rightarrow \varphi')$*.*

Example 10. Consider the sequent $p^2, q \Rightarrow p$. The following are true:

- $(p, q \Rightarrow p) \unlhd (p^2, q \Rightarrow p)$
- $(p, q^2 \Rightarrow p) \perp (p^2, q \Rightarrow p)$
- $(p, q \Rightarrow q) \perp (p^2, q \Rightarrow p)$

△

Define $\mathcal{S}$ to be the set of sequents in the proof search tree of $\Rightarrow \chi$ of the form $\Gamma \Rightarrow \varphi$. Recall

Lemma 3.4. $(\mathcal{S}, \unlhd)$ *is a well-quasiorder.*

Proof. $(\mathcal{S}, \unlhd)$ is a well-quasiorder precisely when

- $(\mathcal{S}, \unlhd)$ doesn't contain infinite strictly decreasing sequences.
- $(\mathcal{S}, \unlhd)$ doesn't contain infinite sequences of pairwise incomparable elements.

For the first case, consider a sequence $(\Gamma_0 \rhd \Gamma_1 \rhd \Gamma_2 \rhd \dots)$. By definition of the contraction extension order, we have that for every $i < j$, Γ_j is obtained from Γ_i by application of one of more instances of the contraction rule.

Suppose $\Gamma_0 = (A_1^{n_1}, A_2^{n_2}, \dots, A_d^{n_d})$ where n_i is the multiplicity of A_i in Γ_0. After at most $(n_1 - 1) + (n_2 - 1) + \dots + (n_d - 1)$ elements in the sequence each A_i appears with multiplicity 1 in Γ_j and after that, we cannot apply more contractions. Hence our strictly decreasing sequence can only contain finitely many elements.

For the second case, let us first investigate when two sequents are incomparable. This is the case when:

- $\varphi \neq \varphi'$.
- $\varphi = \varphi'$ but Γ is not obtainable from Γ' by contraction, i.e. $\Gamma \not\subseteq \Gamma'$.

We will first consider the latter case. Recall from Section 1.1.3 that we can write the multisets that appear as antecedents in $\mathcal{T}$ as:

$$(r_1, \dots, r_d) \in \mathbb{N}^d,$$

where the r_i are the corresponding multiplicities of the subformulas A_i or χ. We have already shown that $\mathbb{N}^d$ contains no infinite sequence of incomparable elements in Example 6. This implies that there is no infinite sequence of incomparable multisets in the proof search tree $\mathcal{T}$.

Now, for the former, if χ has d subformulas, by the subformula property there are d possibilities for φ. By multiply the number of pairwise incomparable elements of the latter form by d, we find the total number of pairwise incomparable elements. □

3.2.2 Finite countermodel

We want to show that our countermodel $\mathcal{C}$ is finite. We show that $\mathcal{C}$ is finite by showing that its basis, that is, the collection of principal closed sets, is finite. This is the same as showing that the number of principal opens is finite. Recall that a principal closed set is of the form:

$$\langle \Delta; \varphi \rangle = \{\Gamma \in Mset \mid (\Gamma, \Delta \Rightarrow \varphi) \in \mathcal{T} \Longrightarrow \Gamma, \Delta \vdash \varphi\}.$$

A principal open set is of the form:

$$\langle \Delta; \varphi \rangle^c = \{\Gamma \mid (\Gamma, \Delta \Rightarrow \varphi) \in \mathcal{T} \wedge \Gamma, \Delta \nvdash \varphi\}.$$

If φ is not a subformula of χ, it will not occur in the proof search $\mathcal{T}$. In this case, $\langle \Delta; \varphi \rangle^c$ is empty. So for φ we only consider the empty set or subformulas of χ. Similarly for Δ, we only consider the empty set or combinations of subformulas of χ. To better interpret the principal open sets, we will define a new set.

$$X(\varphi) = \min_{\trianglelefteq}\{\Delta' \mid \Delta' \nvdash \varphi \text{ and } (\Delta' \Rightarrow \varphi) \in \mathcal{T}\}.$$

Let us show that $X(\varphi)$ is finite. Recall from Section 1.1.3 that any multiset that appears as an antecedent in $\mathcal{T}$ can be written as

$$(r_1, \ldots, r_d) \in \mathbb{N}^d,$$

where the r_i are the corresponding multiplicities of the subformulas $\{A_1, \ldots, A_d\}$ of φ. Hence we have that $X(\varphi) \subseteq \mathbb{N}^d$. $\mathbb{N}^d$ is known to be countable and subsets of countable sets are also countable, so we conclude that $X(\varphi)$ is countable. This means we can enumerate the elements as a sequence. If we show this sequence is finite, we are done.

For all $\Delta' \in X(\varphi)$, we have $\Delta' \nvdash \varphi$. Without loss of generality, let Δ' and Δ'' be two distinct elements from $X(\varphi)$. They must be incomparable with respect to $\trianglelefteq$. Otherwise, assume that $\Delta' \triangleleft \Delta''$. This would contradict the minimality of Δ'. If $\Delta'' \triangleleft \Delta'$, it would contradict the minimality of Δ'. Since all elements must be pairwise incomparable, the sequence of elements from $X(\varphi)$ form a bad sequence with respect to the well-quasi-order $\trianglelefteq$. We conclude that $X(\varphi)$ is a finite set.

We can write

$$\begin{aligned}\langle \Delta; \varphi \rangle^c &= \{\Gamma \mid (\Gamma, \Delta \Rightarrow \varphi) \in \mathcal{T} \wedge \Gamma, \Delta \nvdash \varphi\} \\ &= \bigcup_{\Delta' \in X(\varphi)} \{\Gamma \mid \Gamma\Delta \trianglerighteq \Delta'\}.\end{aligned}$$

The support of Δ are the subformulas of χ for which $\Delta_j \neq 0$. For example, if $\chi = p \wedge q$, $\varphi_0 = p, \varphi_1 = q, \varphi_3 = p \wedge q$, then if $\Delta = (2, 0, 1)$, the support of Δ is $p, p \wedge q$. We consider two cases.

In the first case, the support of Δ includes a formula that is not in the support of Δ'. In this case $\{\Gamma \mid \Gamma, \Delta \trianglerighteq \Delta'\} = \emptyset$ for any Γ. This implies $\langle \Delta; \varphi \rangle^c = \emptyset$. Hence we will not consider these Δ. In the second case, the support of Δ includes only formulas that are in the support of Δ' (this may be fewer or equal formulas).

We have three options:

$$\{\Gamma \mid \Gamma, \Delta \trianglerighteq \Delta'\} = \begin{cases} \{\Gamma \mid \Gamma_j = 0 \iff \Delta'_j = 0\}, & \text{if } \Delta \trianglerighteq \Delta' \\ \{\Gamma \mid \Gamma_j = 0 \iff \Delta'_j = 0 \\ \text{ and } \Delta'_j \neq 0 \implies \Gamma_j \geq (\Delta'_j - \Delta_j)\}, & \text{if } \Delta \triangleleft \Delta' \\ \{\Gamma \mid \Gamma_j = 0 \iff \Delta'_j = 0 \\ \text{ and } \Delta'_j \neq 0 \implies \Gamma_j \geq \max\{0, (\Delta'_j - \Delta_j)\}\}, & \text{if } \Delta \perp \Delta' \end{cases}$$

Let us now see how many different $\langle \Delta; \varphi \rangle^c$ there are by varying Δ and φ. By our reasoning above, we only consider the φ and Δ that give us a nonempty $\langle \Delta; \varphi \rangle^c$. Let us first fix φ and range over all possible Δ. We can write

$$\begin{aligned}\langle\Delta;\varphi\rangle^c = &\bigcup_{\Delta'\in X(\varphi),\Delta'\trianglelefteq\Delta}\big(\{\Gamma \mid \Gamma_j = 0 \iff \Delta'_j = 0\}\big)\cup\\ &\bigcup_{\Delta'\in X(\varphi),\Delta'\triangleright\Delta}\big(\{\Gamma \mid \Gamma_j = 0 \iff \Delta'_j = 0 \text{ and } \Delta'_j \neq 0 \implies \Gamma_j \geq (\Delta'_j - \Delta_j)\}\big)\cup\\ &\bigcup_{\Delta'\in X(\varphi),\Delta'\perp\Delta}\big(\{\Gamma \mid \Gamma_j = 0 \iff \Delta'_j = 0 \text{ and } \Delta'_j \neq 0 \implies \Gamma_j \geq \max\{0,(\Delta'_j - \Delta_j)\}\}\big)\end{aligned}$$

We are to show that, ranging over all possible Δ, there are finitely many different sets $\langle\Delta;\varphi\rangle^c$. Since we have written $\langle\Delta;\varphi\rangle^c$ as a finite union, we are left to show that ranging over all possible Δ, there are finitely many different sets in each union. We will treat them separately. For each union, we will fix a $\Delta' \in X(\varphi)$ and range over all possible Δ.

The first union does not depend on Δ, so for each Δ satisfying $\Delta \trianglerighteq \Delta'$, the set $\{\Gamma \mid \Gamma_j = 0 \iff \Delta'_j = 0\}$ contains the same multisets Γ.

For the second union, since $\Delta' \in \mathbb{N}^d$ has finite coordinates, there are only finitely many Δ satisfying the condition $\Delta \triangleleft \Delta'$. Namely, this happens when all of the values of the coordinates of Δ are equal to or below the values of the coordinates of Δ', except the case where all coordinates of Δ are equal to those of Δ'. If $\Delta' = (n_1, \ldots, n_d)$, this gives us a total of

$$\sum_{i=1}^{d}(n_i + 1) - 1$$

different possible Δ such that $\Delta \trianglelefteq \Delta'$. This bounds the number of different values that Γ_j can have when it is nonzero, hence giving us a finite number of different sets for each Δ'.

For the third union, we consider Δ satisfying $\Delta \perp \Delta'$. Since $\Delta, \Delta' \in \mathbb{N}^d$, we can use the fact that $(\mathbb{N}^d, \leq)$ is a well-quasiorder (Example 6) to conclude that given a multiset Δ' there can only be finitely many different Δ that are incomparable to Δ'. We have used this same reasoning in the proof of Lemma 3.4.

Since for each Δ' separately, there are a finite number of different sets $\{\Gamma \mid \Gamma, \Delta \trianglerighteq \Delta'\}$, and we have shown above that $X(\varphi)$ is finite, in all cases above there are finitely many different sets when ranging over all possible Δ. The union above is thus a finite union over finite sets and hence finite.

Now, we have shown that each union will produce finitely many different sets when ranging over all Δ. Since we only consider φ that are subformulas of χ, of which there are a finite amount, there must be finitely many different $\langle\Delta;\varphi\rangle^c$ in total. We conclude that the countermodel must be finite.

4 Bounding the countermodel

In the proof search tree of the sequent $\Rightarrow \chi$, by the subformula property, any sequent $\Gamma \Rightarrow \varphi$ occurring in the proof search tree is built up from subformulas $\{A_1, \ldots, A_d\}$ of χ. Any sequent in the proof can be written as $A_1^{r_1}, \ldots, A_d^{r_d} \Rightarrow A_s$, where $s \in \{0, \ldots, d\}$ and A_0 represents the empty sequent ϵ. If we fix an ordering for these subformulas, we can write it as the element $((r_1, \ldots, r_d), s) \in \mathbb{N}^d \times \{0, \ldots, d\}$. The norm we use on the antecedents is the infinity norm, so if $\Gamma = (r_1, \ldots, r_d)$, then

$$\|\Gamma\| = \|(r_1, \ldots, r_d)\| = \max_{1\leq i\leq d}\{r_i\}.$$

We define $\|\epsilon\| = 0$.

We have seen in Section 3.2.2 that the number of different principal open sets is finite. We did this by showing that for each $\Delta' \in X(\varphi)$, there are finitely many different principal open sets $\langle\Delta;\varphi\rangle^c$. Recall that

$$X(\varphi) = \min_{\trianglelefteq}\{\Gamma \mid \Gamma \not\vdash \varphi \text{ and } (\Gamma \Rightarrow \varphi) \in \mathcal{T}\}.$$

To bound the number of different principal open sets, we thus want to bound the cardinality of $X(\varphi)$. We can do this by calculating a bound on the norm of $X(\varphi)$. The norm of $X(\varphi)$ is defined as

$$\|X(\varphi)\| = \max\{\|\Gamma\| \mid \Gamma \in X(\varphi)\}.$$

If $\|X(\varphi)\| \leq k$, then all coordinates of all $\Gamma \in X(\varphi)$ are bounded by k. Each coordinate has a value between 0 and k and thus $k+1$ option. Since $X(\varphi) \subseteq \mathbb{N}^d$, we find that the cardinality of $X(\varphi)$ is bounded by

$$|X(\varphi)| \leq (\|X(\varphi)\| + 1)^d.$$

The goal of the rest of this section is to find a bound on the norm and with that a bound on the the cardinality of the X-sets of the formulas in the proof search tree of the sequent $\Rightarrow \chi$. In order to make this more manageable, we will not consider the logic FL_{ec} as a whole, but split it into smaller fragments. We do this by considering fragments of FL_{ec}, one for each of the logical connectives. In each fragment, all formulas are built from propositional variables related to each other via only one connective.

Remark 3. We have assumed that we are trying to find a countermodel for an unprovable formula χ and thus our prooftree starts with the sequent $\Rightarrow \chi$. If χ is not of the form

$$(\alpha_1 \to (\alpha_2 \to \ldots (\alpha_{n-1} \to (\alpha_n \to \beta)) \ldots)) \tag{4}$$

for $n > 0$, then there is no rule that will make the antecedent nonempty. If χ is not of the form in Equation (4), then $\langle \Delta; \varphi \rangle = T$ if $\Delta \neq \epsilon$ and $\langle \epsilon; \varphi \rangle = T \setminus \epsilon$.

Any χ not of the form in Equation (4) is unprovable and its countermodel is of size 2. This is not of interest to us, so when we are working in the fragment of the logical connective $*$, we mean that all sequents are of the form $\Rightarrow (\alpha_1 \to (\alpha_2 \to \ldots (\alpha_{n-1} \to (\alpha_n \to \beta)$ and thus $\alpha_1, \ldots, \alpha_n \Rightarrow \beta$, where all the connectives in the α_i and β are $*$. In what follows, we will assume that $\alpha_1 \ldots \alpha_n \Rightarrow \beta$ is the base sequent.

Also note that $\Rightarrow \alpha \to \beta$ is the only sequent with $\alpha \to \beta$ as a succedent and hence $X(\alpha \to \beta) = \emptyset$ and $\|X(\alpha \to \beta)\| = 0$. Recall that we've written

$$\langle \Delta; \varphi \rangle^c = \bigcup_{\Delta' \in X(\varphi)} \{\Gamma \mid \Gamma\Delta \trianglerighteq \Delta'\}.$$

Since $X(\alpha \to \beta) = \emptyset$, we find that $\langle \Delta; \varphi \rangle^c = \emptyset$. This principal open set always exists for any formula, so this will not change the size of the countermodel.

In the rest of this section we will consider the fragments for each of the logical connectives *fusion*, $\cdot$, *implication* $\to$, *conjunction*, $\wedge$ and *disjunction*, $\vee$. They will be structured as follows: first we will look at the case where only propositional variables are used in the sequents and after that, we will incorporate our new logical constants 0 and 1 into them. The rules for 0 and 1 can be found in Figure 3.

$$\frac{}{\epsilon \Rightarrow 1}\,1R \qquad \frac{\Gamma \Rightarrow \varphi}{\Gamma, 1 \Rightarrow \varphi}\,1L \qquad \frac{}{0 \Rightarrow \epsilon}\,0L \qquad \frac{\Gamma \Rightarrow \epsilon}{\Gamma \Rightarrow 0}\,0R$$

Figure 3: Inference rules for the constants 0 and 1.

Let us take a look at the provability of both 0 and 1 considering only these rules. Using the rule $1R$, we can see that $\Delta \Rightarrow 1$ is provable whenever there is a sequence of rules such that Δ can be converted to ϵ. Naturally, $\Delta = \epsilon$ would work, and using the rule $1L$, we see that $\Delta = 1^k$ for $k > 0$ can also be converted to ϵ by applying $1L$ precisely k times. In other words:

$$\Delta \Rightarrow 1 \text{ is provable} \iff \Delta \in \{\epsilon, \{1^k : k > 0\}\}$$

For the sequent $\Delta \Rightarrow 0$, we first apply $0R$ to get the sequent $\Delta \Rightarrow \epsilon$. Then, using the rule $0L$ we find that $\Delta \Rightarrow \epsilon$ is provable precisely when Δ can be converted to 0. In this case that means that Δ can only be equal to 0. In other words:

$$\Delta \Rightarrow 0 \text{ is provable } \iff \Delta = 0$$

We will discuss the provability of 0 and 1 in combination with other logical connectives in their respective subsection.

4.1 Multiplicative fragment of FL_{ec}

In this section, we will be working in the multiplicative fragment of FL_{ec}. That is, the fragment of FL_{ec} where only the connective Fusion, " $\cdot$ ", is used.

The rules that we can apply in this fragment are:

$$\frac{}{\alpha \Rightarrow \alpha}\, AX \qquad \frac{\Gamma, \alpha, \beta \Rightarrow \varphi}{\Gamma, \alpha \cdot \beta \Rightarrow \varphi}\, \cdot L \qquad \frac{\Gamma_1 \Rightarrow \alpha \qquad \Gamma_2 \Rightarrow \beta}{\Gamma_1, \Gamma_2 \Rightarrow \alpha \cdot \beta}\, \cdot R \qquad \frac{\Gamma, \alpha, \alpha \Rightarrow \varphi}{\Gamma, \alpha \Rightarrow \varphi}\, c$$

4.1.1 Fusion with propositional variables

By Remark 3, the base sequent in this fragment is of the form

$$\Delta = \alpha_1, \ldots, \alpha_n \Rightarrow p_i^{\mu_i} \cdot \ldots \cdot p_j^{\mu_j} = \beta,$$

where $\{i, \ldots, j\} \subseteq \{1, \ldots, n\}$. Each α_i is of the form $\alpha_i = p_k^{\lambda_k} \cdot \ldots \cdot p_l^{\lambda_l}$ for some propositional variables $\{p_l, \ldots, p_k\} \subseteq \{p_1, \ldots, p_n\}$. Note here that each propositional variable is grouped together: we can do this because $\cdot$ is an associative connective. Hence changing the ordering of propositional variables in the succedent does not change its provability. In the antecedent we use exchange to group the propositional variables together.

Remark 4. The premise and conclusion of the rule $\cdot L$ are considered equal sequents in the sense that any Γ that makes the premise provable also makes the conclusion provable and vice versa. Hence the sequents $p_k \cdot \ldots \cdot p_l \Rightarrow \varphi$ and $p_k, \ldots, p_l \Rightarrow \varphi$ are equivalent. We will not differentiate between them and to make our writing clear, we will sometimes omit the commas and write $p_k \ldots p_l$.

Theorem 4.1. *Consider the set of n propositional variables $\{p_1, \ldots, p_n\}$. Consider the proof tree of the sequent*

$$\Delta = \alpha_1, \ldots, \alpha_r \Rightarrow p_i^{\mu_i} \cdot \ldots \cdot p_j^{\mu_j} = \beta, \tag{5}$$

where $\{p_i, \ldots, p_j\} \subseteq \{p_1, \ldots, p_n\}$ and $\alpha_i = p_k^{\lambda_k} \cdot \ldots \cdot p_l^{\lambda_l}$ for some propositional variables $\{p_l, \ldots, p_k\} \subseteq \{p_1, \ldots, p_n\}$.

Let φ be any subformula of β. We find that

$$\|X(\varphi)\| \leq \max\{\|\Delta\|, \mu_r + 1\}.$$

Proof. We first examine when the sequent (5) above is provable and when it is unprovable. In the case with one variable, every sequent is of the form

$$p^l \Rightarrow p^m.$$

The first thing to note is that $\Delta \Rightarrow p$ is only provable by $\Delta = p$. So for every p in the succedent of the sequent, we need one p in the antecedent of the sequent, but no more. This sequent is provable precisely when $0 < l \leq m$, since when $l < m$, we can apply contraction κ times until $l + \kappa = m$.

We now look at the case with n variables $\{p_1, \ldots, p_n\}$. We can treat each of them separately, since in this fragment, each propositional variable can only be proven by itself, so they will not interfere with each other. Because of the nature of the rule $\cdot R$, there is a difference between the norm of X-sets of the base sequents and the norm of X-sets of the other sequents in the proof search tree. We will first discuss the X-sets of the base sequents. Recall:

$$X(\varphi) = \min_{\trianglelefteq}\{\Gamma \mid \Gamma \not\vdash \varphi \text{ and } (\Gamma \Rightarrow \varphi) \in \mathcal{T}\}.$$

By assumption, our base sequent (1) is unprovable, and hence $\Delta \in X(\beta)$. Because of Remark 4, we only apply the right fusion rule or contraction in the proof search. If we apply contraction upwards from the base sequent, the antecedent will not be minimal, as Δ was already minimal and can be obtained by applying contraction downwards. If we apply the right fusion rule upwards, β does not appear in the succedent of the sequents and their corresponding antecedents are not elements of $X(\beta)$. Hence $X(\beta) = \{\Delta\}$ for any unprovable base sequent. Since the norm of an X-set is the greatest norm of any element of the set, we find that

$$\|X(\beta)\| = \|\Delta\|.$$

Let us now take a look at all sequents that are not base sequents. In this case, we will be dealing with sequents of the form

$$\Delta \Rightarrow p_i^{\mu_i} \cdot \ldots \cdot p_j^{\mu_j}.$$

In order to investigate all possibilities for Δ, we consider sequents of the form

$$\Delta = p_l^{\lambda_l} \cdot \ldots \cdot p_k^{\lambda_k} \Rightarrow p_i^{\mu_i} \cdot \ldots \cdot p_j^{\mu_j} = \beta, \tag{6}$$

with $\{p_l, \ldots, p_k\}, \{p_i, \ldots, p_j\} \subseteq \{p_1, \ldots, p_n\}$ and $p_r = p_s \iff r = s$. All other possible antecedents are made from Δ by either changing the multiplicities λ_r, removing one of $\{p_l, \ldots, p_k\}$ or adding a different propositional variable $p_r \in \{p_1, \ldots, p_n\}$.

The sequent in Equation 6 is provable when $\{p_l, \ldots, p_k\} = \{p_i, \ldots, p_j\}$ and $\lambda_i \leq \mu_i \ \forall i \in \{l, \ldots k\}$. The sequent becomes unprovable if any combination of the following three holds:

1. $\lambda_i > \mu_i$ for some $i \in \{1, \ldots, k\}$.
2. There exists at least one p_r that does appear in β, but does not appear in Δ.
3. There exists at least one p_s that does appear in Δ, but does not appear in β

We will discuss each item on this list assuming that it happens in isolation and not in combination with others. By our reasoning above, since the propositional variables do not interact with each other, the general case can be proven by an example with any number of variables not equal to 1. The most natural option to show all cases for unprovability is the one with 3 variables p, q and r.

For the first case of unprovability, we assume $\nu_1 > \mu_1$ or $\nu_2 > \mu_2$ or $\nu_3 > \mu_3$. We thus consider the base sequent

$$\Gamma = p^{\nu_1} \cdot q^{\nu_2} \cdot r^{\nu_3} \Rightarrow p^{\mu_1} \cdot q^{\mu_2} \cdot r^{\mu_3}. \tag{7}$$

As we've seen, $\|X((p^{\mu_1} \cdot q^{\mu_2}) \cdot r^{\mu_3})\| = \|p^{\nu_1}, q^{\nu_2}, r^{\nu_3}\| = \max(\nu_1, \nu_2, \nu_3)$. We will now consider $\|X(p^{\mu_1} \cdot q^{\mu_2})\|$ and $\|X(r^{\mu_3})\|$. Since p and q do appear in the antecedent of the base sequent, there exist splittings of $p^{\nu_1}, q^{\nu_2}, r^{\nu_3}$ such that $p^{\mu_1} \cdot q^{\mu_2}$ is provable. For example, we can take the splitting

$$\Gamma_1 = p^{\mu_1}, q^{\mu_2} \quad \Gamma_2 = p^{\nu_1 - \mu_1}, q^{\nu_2 - \mu_2}, r^{\mu_3}.$$

What is then a minimal splitting of Γ only containing p and q, such that $p^{\mu_1} \cdot q^{\mu_2}$ is unprovable? These are the splittings

$$\begin{aligned} \Gamma_1 &= p^{\mu_1+1}, q & \qquad \Gamma_2 &= p^{\nu_1-(\mu_1+1)}, q^{\nu_2-1}, r^{\mu_3} \\ \Gamma_1' &= p, q^{\mu_2+1} & \qquad \Gamma_2' &= p^{\nu_1-1}, q^{\nu_2-(\mu_2+1)}, r^{\mu_3} \end{aligned}$$

Hence $\Gamma_1, \Gamma_1' \in X(p^{\mu_1} \cdot q^{\mu_2})$. Any other Γ_1" is such that either Γ_1" $\trianglerighteq \Gamma_1$ or Γ_1" $\trianglerighteq \Gamma_1'$, or it contains r. The former are clearly not minimal, and for the latter holds that they do not prove $p^{\mu_1} \cdot q^{\mu_2}$ with any multiplicity. So for minimality we can take a splitting of Γ with norm equal to 1. We find that

$$X(p^{\mu_1} \cdot q^{\mu_2}) = \{p^{\mu_1+1}q, pq^{\mu_2+1}, pqr, pr, qr, p, q, r, \epsilon\}.$$

In this case

$$\|X(p^{\mu_1} \cdot q^{\mu_2})\| \leq \max(\mu_1 + 1, \mu_2 + 1).$$

Similarly, $\|X(r^{\mu_3})\| \leq \mu_3 + 1$.

For the second case of unprovability, we assume that one or more propositional variables in the succedent are missing from the antecedent. Without loss of generality, we assume here that p is missing from the antecedent. We consider the base sequent

$$\Gamma = q^{\nu_2} \cdot r^{\nu_3} \Rightarrow p^{\mu_1} \cdot q^{\mu_2} \cdot r^{\mu_3}. \tag{8}$$

We have that $\|X((p^{\mu_1} \cdot q^{\mu_2}) \cdot r^{\mu_3}\| = \|q^{\nu_2}, r^{\nu_3}\| = \max(\nu_2, \nu_3)$. We will now consider $\|X(p^{\mu_1} \cdot q^{\mu_2})\|$ and $\|X(r^{\mu_3})\|$. In this case, there is no splitting of q^{ν_2}, r^{ν_3} such that $p^{\mu_1} \cdot q^{\mu_2}$ is provable. Therefore any splitting of Γ will result in an unprovable sequent with $p^{\mu_1} \cdot q^{\mu_2}$ as its succedent. For minimality, we can take the splittings

$$\begin{aligned}
\Gamma_1 &= q & \Gamma_2 &= q^{\nu_2 - 1}, r^{\nu_3} \\
\Gamma_1 &= r & \Gamma_2 &= q^{\nu_2}, r^{\nu_3 - 1} \\
\Gamma_1 &= qr & \Gamma_2 &= q^{\nu_2 - 1}, r^{\nu_3 - 1} \\
\Gamma_1 &= \epsilon & \Gamma_2 &= q^{\nu_2}, r^{\nu_3}
\end{aligned}$$

and we find that $X(p^{\mu_1} \cdot q^{\mu_2}) = \{qr, q, r, \epsilon\}$. Hence

$$\|X(p^{\mu_1} \cdot q^{\mu_2})\| = 1.$$

For $X(r^{m+1})$, we have that $\|X(r^{\mu_3})\| = \mu_3 + 1$, since r appears in the antecedent of the base sequent, so we are in the same situation as in case 1.

For the third case of unprovability, we assume that one or more propositional variables from the antecedent are missing from the succedent. Without loss of generality, we only add one propositional variable here and since fusion is associative, we add it at the end. We consider the base sequent

$$\Gamma = p^{\nu_1}, q^{\nu_2}, r^{\nu_3}, s^{\nu_4} \Rightarrow (p^{\mu_1} \cdot q^{\mu_2}) \cdot r^{\mu_3}.$$

We have that $\|X((p^{\mu_1} \cdot q^{\mu_2}) \cdot r^{\mu_3})\| = \|p^{\nu_1}, q^{\nu_2}, r^{\nu_3}, s^{\nu_4}\| = \max(\nu_1, \nu_2, \nu_3, \nu_4)$. We now consider $\|X(p^{\mu_1} \cdot q^{\mu_2})\|$ and $\|X(r^{\mu_3})\|$. There exists a splitting of Γ such that $p^{\mu_1} \cdot q^{\mu_2}$ is provable. For unprovability, we could take the splittings

$$\begin{aligned}
\Gamma_1 &= p^{\mu_1 + 1} q & \Gamma_2 &= p^{\nu_1 - (\mu_1 + 1)} q^{\nu_2 - 1} r^{\nu_3} s^{\nu_4} \\
\Gamma_1' &= p q^{\mu_2 + 1} & \Gamma_2' &= p^{\nu_1 - 1} q^{\nu_2 - (\mu_2 + 1)} r^{\nu_3} s^{\nu_4}
\end{aligned}$$

Hence $\Gamma_1, \Gamma_1' \in X(p^{\mu_1} \cdot q^{\mu_2})$. Any other Γ_1'' is such that either $\Gamma_1'' \trianglerighteq \Gamma_1$ or $\Gamma_1'' \trianglerighteq \Gamma_1'$, or it contains at least one of r or s. The former are clearly not minimal, and for the latter holds that they do not prove $p^{\mu_1} \cdot q^{\mu_2}$ with any multiplicity. So for minimality we can take a splitting of Γ with norm equal to 1. We find that

$$X(p^{\mu_1} \cdot q^{\mu_2}) = \{p^{\mu_1 + 1} q, p q^{\mu_2 + 1}, pqrs, pqr, pqs, qrs, prs, pr, ps, qr, qs, p, q, r, s, \epsilon\}.$$

In this case

$$\|X(p^{\mu_1} \cdot q^{\mu_2})\| = \max(\mu_1 + 1, \mu_2 + 1).$$

Similarly, $\|X(r^{\mu_3})\| = \mu_3 + 1$.

To conclude, consider any $\varphi = p_r^{\mu_r} \cdot \ldots \cdot p_s^{\mu_s}$ that appears as the succedent of any sequent that is not the base sequent. We find that

$$\|\varphi\| \leq \max\{\mu_z + 1 \ : \ z \in \{r, \ldots, s\}\}.$$

□

4.1.2 Fusion with truth and falsehood

Before we have only considered propositional variables in the succedent. Since we also have the logical constants, 0 and 1, we will now investigate how adding them to our succedent changes the bound on the norm. The rules for 0 and 1 can be found in Figure 3.

For truth, we first consider the sequent

$$\Delta \Rightarrow 1 \cdot 1 \cdot \ldots \cdot 1 = 1^{\mu}. \tag{9}$$

As before, using Remark 4, we will assume that formulas in Δ are either propositional variables from the set $\{p_1, \ldots, p_n\}$, 0, or 1. Using our knowledge from the rule $\cdot R$, we see that this sequent is provable if for any splitting of the antecedent Δ into Δ_1, Δ_2, we have that

$$\Delta_i \in \{\epsilon, \{1^k \ : \ k > 0\}\} \text{ for } 1 \leq i \leq 2.$$

This can easily be done by taking $\Delta = \epsilon$ or $\Delta = 1^k$ for $k > 0$. This is thus unprovable if Δ also contains any number of propositional variables or 0. Then $\|X(1^{\mu})\|$ is equal to $\|\Delta\|$, where we use again that any rule application either changes minimality of the antecedent (contraction) or the succedent ($\cdot R$). For any 1^{ν} with $\nu < \mu$ that will appear as a succedent in the proof search, we find that $\|X(1^{\nu})\| = 1$. Consider any subset $\Gamma \subset \Delta$ such that $\|\Gamma\| = 1$. If it contains any propositional variables or 0, it is an element of $X(1^{\nu})$ and if it does not, we have $\Gamma = \epsilon$ or $\Gamma = 1$ for which prove the sequent $\Gamma \Rightarrow 1^{\mu}$, so they are not elements of $X(1^{\nu})$.

If we now also consider propositional variables in the succedent, we get sequents of the form

$$\Delta = \alpha_1, \ldots, \alpha_n \Rightarrow p_i^{\mu_i} \cdot \ldots \cdot p_j^{\mu_j} \cdot 1^{\mu} = \beta, \tag{10}$$

where each α_i is of the form $p_l^{\lambda_l} \cdot \ldots \cdot p_k^{\lambda_k}$ as before. We have that Equation 10 is provable when

$$\Delta = p_i^{\lambda_i} \cdot \ldots \cdot p_j^{\lambda_j} \cdot 1^k \text{ or } \Delta = p_i^{\lambda_i} \cdot \ldots \cdot p_j^{\lambda_j} \tag{11}$$

and $\lambda_m \leq \mu_m$ for $i \leq m \leq j$ and $k \geq 0$.

Because the addition of 1 in the antecedent is optional, the cases for unprovability do not change the possibilities for unprovability we have discussed above. So, consider a subformula of $p_i^{\lambda_i} \cdot \ldots \cdot p_j^{\lambda_j} (\cdot 1^k)$ that appears as a succedent in the proof search. Since there is no maximal multiplicity of 1 in Equation (11) for provability, we may take $k = 1$ in any splitting of Δ. Hence the addition of 1 in the succedent does not change the bound on the norm and the results from Theorem 4.1 also hold for sequents like in Equation 10.

For falsehood, we first consider the sequent

$$\Delta \Rightarrow 0 \cdot 0 \cdot \ldots \cdot 0 = 0^{\mu}. \tag{12}$$

The sequent $\Delta \Rightarrow 0^{\mu}$ is provable if and only if 0^{μ} can be derived from Δ using the rules in this fragment. Here this implies that Δ must be of the form

$$\Delta = 0^{\lambda} \text{ or } \Delta' = 0^{\lambda}, 1^k$$

In the second case, where $\Delta' = 0^{\lambda}, 1^k$, we could first apply the rule $1L$ k times to get the sequent $0^{\lambda} \Rightarrow 0^{\mu}$. We may consider the sequents $\Delta \Rightarrow 0^{\mu}$ and $\Delta' \Rightarrow 0^{\mu}$ to be equivalent. These sequents are unprovable if $\mu > \lambda$ or Δ contains any propositional variables.

If only $\mu > \lambda$ holds, then $\|X(0^{\mu})\| = \lambda$ and for any $\nu < \mu$, we find that $\|X(0^{\nu})\| = 1$. If moreover there exist propositional variables in Δ, we find that $\|X(0^{\mu})\| = \|\Delta\|$ and $\|X(0^{\nu})\| = 1$.

If we also add propositional variables to our succedent, we consider sequents of the form

$$\Delta = \alpha_1, \ldots, \alpha_n \Rightarrow p_i^{\mu_i} \cdot \ldots \cdot p_j^{\mu_j} \cdot 0^{\mu} = \beta, \tag{13}$$

where each α_i is of the form $p_l^{\lambda_1} \cdot \ldots \cdot p_k^{\lambda_k}$ as before. We have that Equation 13 is provable when

$$\Delta = p_i^{\lambda_i} \cdot \ldots \cdot p_j^{\lambda_j} \cdot 0^{\lambda}, \tag{14}$$

and $\lambda_m \leq \mu_m$ for $i \leq m \leq j$ and $\lambda \leq \mu$.

In this case, the provability of the succedent does change with the addition of 0. There is a maximum value of λ, namely $\lambda = \mu$. Since the same holds for propositional variables, we can apply our knowledge from before to see that if 0 appears in Δ, there exists a splitting of Δ to get a provable sequent with succedent 0. Hence a splitting of Δ resulting in minimal unprovability much be such that 0 has multiplicit $\mu + 1$. Hence the results from Theorem 4.1 hold for sequents as in Equation 13, but we must take into account the multiplicity of 0 in the base sequent.

4.2 Implicational fragment of FL_{ec}

We continue with the implicational fragment of FL_{ec}: the fragment where only the connective Implication, $\rightarrow$, is used. We are using the rules:

$$\frac{}{\alpha \Rightarrow \alpha}\, AX \qquad \frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \alpha \rightarrow \beta} \rightarrow R \qquad \frac{\Gamma_1 \Rightarrow \alpha \qquad \Gamma_2, \beta \Rightarrow \varphi}{\Gamma_1, \Gamma_2, \alpha \rightarrow \beta \Rightarrow \varphi} \rightarrow L \qquad \frac{\Gamma, \alpha, \alpha \Rightarrow \varphi}{\Gamma, \alpha \Rightarrow \varphi}\, c$$

Remark 5. Similar to the application of the rule $\cdot L$, we have that the sequents $\Gamma \Rightarrow \alpha \rightarrow \beta$ and $\Gamma, \alpha \Rightarrow \beta$ are considered equivalent by application of the rule $\rightarrow R$. We will hence first apply $\rightarrow R$ repeatedly and the resulting sequent has as its succedent a propositional variable p. Suppose this is not the case: Then, since we only use the rules for implication, it would be the case that the succedent is still of the form $\alpha \rightarrow \beta$ to which we can apply the rule $\rightarrow R$. We will assume that all base sequents are of the form

$$\Delta \Rightarrow p.$$

4.2.1 Implication without nesting with propositional variables

Now that we know what the succedent of the base sequent looks like, let us take a look at the antecedent. Unlike the other fragments, we do not have a nice way to represent it. We will first study the case where Δ contains no nesting of propositional variables. Let $\{p_1, \ldots, p_n\}$ be a set of n propositional variables. The set of all possible formula's in Δ can be represented by the set of n vertices and edges in the complete digraph K_n, with loops for every vertex. In other words, Δ can contain the vertices $\{p_1, \ldots, p_n\}$ together with the set of edges $\{p_i \rightarrow p_j \mid 1 \leq i \leq n, 1 \leq j \leq n\}$. We will denote the set containing all possible propositional variables and implications by K^n. The complete digraph with 4 vertices can be seen in Figure 4.

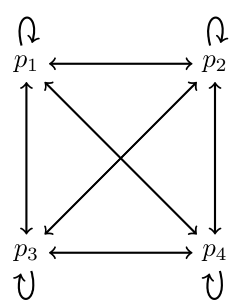


Figure 4: Complete digraph with 4 vertices.

In what follows, we will assume we have the following base sequent:

$$\Delta \Rightarrow p_i, \tag{15}$$

where $p_i \in \{p_1, \ldots, p_n\}$ and $\mathrm{Supp}(\Delta) \subseteq \mathrm{Supp}(K^n)$.

Using this knowledge, we are ready to formulate a theorem about the bound of the X-sets of the propositional variables appearing in the proof search. We will first consider the propositional variables that do not appear as a succedent in the base sequent. After that, we will consider the propositional variable that is the succedent of the base sequent.

Theorem 4.2. *Let Δ be any multiset such that* $\mathrm{Supp}(\Delta) \subseteq \mathrm{Supp}(K^n)$*, and let* $p_i \in \{p_1, \dots, p_n\}$*. Consider the proof tree of the sequent*

$$\Delta \Rightarrow p_i.$$

Let $p_j \in \{p_1, \dots, p_n\}$*. If* $p_j \neq p_i$*, we find that*

$$\|X(p_j)\| \leq 2$$

Proof. Consider a p_j with $j \neq i$. We will first consider a base sequent $\Delta \Rightarrow p_i$ with $\mathrm{Supp}(\Delta) = \mathrm{Supp}(K^n)$, and later show that the same result holds for any Δ such that $\mathrm{Supp}(\Delta) \subset \mathrm{Supp}(K^n)$. By assumption, there is an implication $p_j \to p_m$ to which we can apply the rule $\to L$. Because of how $\to L$ is structured,

$$\Gamma \Rightarrow p_j \in \mathcal{T} \iff \exists \Delta^c \trianglerighteq \Delta \text{ such that } \Gamma \subseteq \Delta^c. \tag{16}$$

Recall:

$$X(p_j) = \min_{\trianglelefteq}\{\Gamma \mid \Gamma \nvdash p_j \text{ and } (\Gamma \Rightarrow p_j) \in \mathcal{T}\}.$$

So elements of $X(p_j)$ are the Γ that satisfy the right hand side of Equation 16 and do not prove p_j. We are left to find out which ones of them are minimal.

Let $\Gamma \subseteq K^n$ (i.e. we assume that all formulas in Γ have multiplicity 1). We want to know whether $\Gamma \vdash p_j$ or not. Suppose it does not: Since $\Gamma \subseteq K^n \subseteq \Delta$, Γ satisfies the right hand side of Equation 16, so it satisfies $\Gamma \Rightarrow p_j \in \mathcal{T}$ and $\Gamma \nvdash p_j$. Since all formulas in Γ have multiplicity 1, it is clearly minimal to do so, hence $\Gamma \in X(p_j)$ and $\|\Gamma\| = 1$. If $\Gamma \vdash p_j$, we must investigate how many contraction we need to apply upwards before the resulting sequent becomes unprovable.

To consider whether sets Γ prove p_j, we first define the following:

Definition 4.1. *[5, Section 1.3] A walk in a graph G is a nonempty sequence of edges* $(e_1, e_2, \dots, e_{n-1})$ *for which there is a sequence of vertices* $(v_1, v_2, \dots, v_n)$ *such that* $e_i = (v_i, v_{i+1})$ *for all* $i < k$*. To denote a walk, we will be writing its edge sequence* $(e_1, \dots, e_{n-1})$*. If* $v_k = v_0$*, the walk is closed.*

The graph of a walk is the corresponding subgraph of K_n*, where all edges of the walk are plotted onto the set of n vertices.*

Example 11. As an example for Definition 4.1 consider the provable sequent $\Gamma \Rightarrow p_1$ with

$$\Gamma = p_4, p_4 \to p_3, p_3 \to p_2, p_2 \to p_4, p_3 \to p_1.$$

The sequent $\Gamma \Rightarrow p_1$ corresponds to the walk $(p_4 \to p_3, p_3 \to p_2, p_2 \to p_4, p_4 \to p_3, p_3 \to p_1)$ and its graph can be seen in Figure 5. △

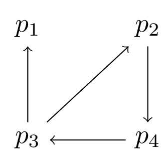


Figure 5: Graph of the walk corresponding to Γ

Definition 4.2. *[5, Section 1.3] A path in a graph G is a nonempty subgraph* $P = (V, E)$ *of the form*

$$V = (v_1, \dots, v_n) \qquad E = (e_1, \dots, e_{n-1}),$$

where all v_i *are distinct propositional variables. If* $v_0 = v_n$*, we call the path a cycle.*

Proposition 4.3. *Let* $\Gamma \subseteq K^n$*. The sequent* $\Gamma \Rightarrow p_j$ *is provable if it satisfies the following properties:*

- Γ *contains exactly one propositional variable outside the implications*
- Γ *corresponds to the graph of a directed walk in K_n that ends in p_j and begins at the propositional variable in Γ*

Proof. Let $\Gamma \subseteq K^n$. Suppose for contradiction that Γ does not contain exactly one propositional variable. Then it must either contain zero propositional variables or more than one. Since the succedent p_j contains no implications, we also need to get rid of them in the antecedent using the application of $\to L$.

If Γ contains no propositional variables, it only contains implications. The sequent with one implication in its succedent is not provable, as its left premise would have an empty succedent.

$$\dfrac{\Rightarrow p_i \qquad p_k \Rightarrow p_j}{p_i \to p_k \Rightarrow p_j} \to L$$

Now consider a sequent with more than one implication. The sequent would be of the form

$$\Gamma = p_1 \to p_2, \ldots, p_{n-1} \to p_n \Rightarrow p_j. \tag{17}$$

Without loss of generality, suppose we apply $\to L$ to the formula $p_m \to p_{m+1}$. The resulting sequents are of the form

$$\Gamma_1 \Rightarrow p_m \qquad\qquad \Gamma_2, p_{m+1} \Rightarrow p_j,$$

where $\Gamma_1, \Gamma_2 = \Gamma \backslash (p_m \to p_{m+1})$. If $\Gamma_1 = \epsilon$ or Γ_1 is a single implication, then this sequent is unprovable and hence the sequent (17) is unprovable. So we assume Γ_1 contains more than one implication. Assuming we do not keep applying contraction to any of the implications, by repeatedly applying $\to L$, there will eventually be a sequent with either an empty succedent or containing a single implication. Hence the sequent (17) is not provable.

Suppose now that Γ contains more than one propositional variable. Without loss of generality, we will assume that there are two, p_i and p_k. The sequent is of the form

$$\Gamma = \Gamma', p_i, p_k, p_m \to p_{m+1} \Rightarrow p_j. \tag{18}$$

If we apply $\to L$ to one of the implications in Γ, say $p_m \to p_{m+1}$, we must divide p_i and p_k between the resulting sequents. If $p_i, p_k \in \Gamma_1$, then Γ_1 still contains two propositional variables. If $p_i \in \Gamma_1, p_k \in \Gamma_2$ or vice versa, then Γ_2, p_{m+1} contains both p_k and p_{m+1} and thus two propositional variables. If $p_i, p_k \in \Gamma_2$, then Γ_2, p_k contains three propositional variables. So after each application of $\to L$ there will be at least one sequent with at least two propositional variables in its succedent. After we have applied $\to L$ to all possible implications, there will be a sequent with at least two propositional variables in its succedent, which is unprovable. So we conclude that the sequent (18) is unprovable.

The second property is proven by induction on the rules in this fragment. We do not consider $\to R$ because of Remark 5. We do not consider contraction, since if we were able to apply contraction to the sequent $\Gamma \Rightarrow p_j$, it would follow that $\Gamma \not\subseteq K^n$ as there would be a formula with multiplicity greater than 1. So we are left with $\to L$.

Consider the provable sequents $\Gamma_1 \Rightarrow p_i$ and $\Gamma_2, p_k \Rightarrow p_j$ and suppose they satisfy the second property in Proposition 4.3. If we apply $\to L$ to these sequents, we get the sequent $\Gamma_1, \Gamma_2, p_i \to p_k \Rightarrow p_j$. The sequents of the premises correspond to the graphs

$$\cdot \xrightarrow{\quad \Gamma_1 \quad} p_i \qquad\qquad p_k \xrightarrow{\quad \Gamma_2 \quad} p_j$$

Then the sequent $\Gamma_1, \Gamma_2, p_i \to p_k \Rightarrow p_j$ corresponds to the graph

$$\cdot \xrightarrow{\quad \Gamma_1 \quad} p_i \cdots\cdots\cdots\cdots\cdots\cdots > p_k \xrightarrow{\quad \Gamma_2 \quad} p_j$$

and hence to the walk $(\text{Walk}(\Gamma_1), p_i \to p_j, \text{Walk}(\Gamma_2))$, where here $\text{Walk}(\Gamma_1), \text{Walk}(\Gamma_2)$ refer to the walks corresponding to Γ_1 and Γ_2 respectively. □

Let us now consider a Γ such that the sequent $\Gamma \Rightarrow p_j$ is provable. We need to investigate how many contractions we can apply before the sequent becomes unprovable. In order to do this, we first need to make a distinction between cycle-formulas and non-cycle-formulas. A formula in Γ is a cycle-formula if, in its corresponding walk, it appears in a cycle as defined in Definition 4.2. All formulas that do not appear in a cycle are called non-cycle-formulas. Propositional variables are always non-cycle-formulas. In Figure 5 the cycle-formulas are

$$\{p_4 \to p_3, p_3 \to p_2, p_2 \to p_4\}$$

and $p_3 \to p_1$ is a non-cycle-formula.

The cycle-formulas have the property that applying contraction to them does not change the provability of the corresponding sequent. As an example, consider the sequent $p, p \to q, q \to p \Rightarrow p$. Its proof is as follows:

$$\dfrac{\dfrac{p \Rightarrow p \qquad q \Rightarrow q}{p, p \to q \Rightarrow q} \to L \qquad p \to p}{p, p \to q, q \to p \Rightarrow p} \to L$$

Suppose we apply contraction to $p \to q$. To prove the resulting sequent $p, (p \to q)^2, q \to p \Rightarrow p$, we first apply contraction to $q \to p$ and then use $\to L$ on both $p \to q$ and $q \to p$, in any order. The resulting branches end in the sequents $p \Rightarrow p$, $q \Rightarrow q$ and $p, p \to q, q \to p \Rightarrow p$ as follows:

$$\dfrac{\dfrac{\dfrac{p, p \to q, q \to p \Rightarrow p \qquad q \to q}{p, (p \to q)^2, (q \to p) \Rightarrow q} \to L \qquad p \Rightarrow p}{p, (p \to q)^2, (q \to p)^2 \Rightarrow p} \to L}{p, (p \to q)^2, q \to p \Rightarrow p} c$$

More generally, if contraction is applied to a cycle-formula, we first apply contraction to all the formulas in the same cycle. Then apply $\to L$ to them in any order to end up with a proof tree with its topmost sequents the original sequent and $p_i \Rightarrow p_i$ for all propositional variables p_i appearing in the cycle. Since we are looking for minimal elements Γ with respect to the contraction extension order, we assume that cycle formulas always appear with multiplicity 1 in Γ.

So, we now focus on non-cycle-formulas. They are either a propositional variable or an implication. For propositional variables we have already seen that sequents that contain more than one propositional variable are not provable, and the same reasoning holds for twice the same propositional variable. Suppose we apply contraction to an implication. The resulting sequent is of the form $\Gamma, (p_i \to p_k)^2 \Rightarrow p_j$. If we apply $\to L$ to this sequent, we get the sequents

$$\Gamma_1 \Rightarrow p_i \qquad\qquad \Gamma_2, p_k \Rightarrow p_j.$$

We will show that it is impossible for both of these sequents to be provable. Since $p_i \to p_k$ is in the antecedent of $\Gamma, (p_i \to p_k)^2 \Rightarrow p_j$, it must be the case that $p_i \to p_k \in \Gamma_1$ or $p_i \to p_k \in \Gamma_2$.

Suppose first that $p_i \to p_k \in \Gamma_1$. If $\Gamma_1 \Rightarrow p_i$ were provable, then it must correspond to the graph of a walk in K_n by Proposition 4.3. In order to do so, there must be a path from p_k to p_i in this walk. However, this in turn implies that $p_i \to p_k$ is part of a cycle, contradicting the fact that we've assumed to contract a non-cycle-formula. So $\Gamma_1 \Rightarrow p_i$ is not provable.

Suppose now that $p_i \to p_k \in \Gamma_2$. If $\Gamma_2, p_k \Rightarrow p_j$ were provable, then it must correspond to the graph of a walk in K_n by Proposition 4.3. Since we start the walk in p_k, by assumption there is an implication $p_k \to p_m \in \Gamma_2$. Since $p_i \to p_k$ is assumed to be an element of the walk, there must be a path from $p_m \to p_i$, again making it part of a cycle, contradicting the fact that we've assumed to contract a non-cycle-formula. So $\Gamma_2, p_k \Rightarrow p_j$ is not provable.

We conclude that after applying contraction once to non-cycle-formulas, the resulting sequent becomes unprovable. So, given the provable sequent $\Gamma \Rightarrow p_j$, the corresponding elements of $X(p_j)$ are Γ plus one contraction applied to a non-cycle-formula. This implies that the norm of all of these elements is 2.

Let us now consider a Δ such that $\text{Supp}(\Delta) \subset \text{Supp}(K^n)$. If there is no formula of the form $p_j \to p_k$ for $1 \leq k \leq n \in \Delta$ then $X(p_k) = \emptyset$, so let us assume there exists $p_j \to p_k \in \Delta$, so that there is a sequent $\Pi \Rightarrow p_j$ in the proof. Let Γ be any subset of Δ such that all multiplicities in Γ are equal to 1. Because we assume $p_j \to p_k \in \Delta$, the right hand side of Equation 16 is satisfied, and we have that $\Gamma \Rightarrow p_j \in \mathcal{T}$. Now, either $\Gamma \nvdash p_j$ and then $\|\Gamma\| = 1$, or $\Gamma \vdash p_j$. In the last case, we can apply the same reasoning as above, and hence in those cases we have that the norm of the resulting multiset is equal to 2.

Hence we conclude that for any Δ such that $\text{Supp}(\Delta) \subseteq \text{Supp}(K^n)$, we have that

$$\|X(p_j)\| \leq 2.$$

□

Theorem 4.4. *Let Δ be any multiset such that $\text{Supp}(\Delta) \subseteq \text{Supp}(K^n)$, and let $p_i \in \{p_1, \ldots, p_n\}$. Consider the proof tree of the sequent*

$$\Delta \Rightarrow p_i.$$

To calculate $\|X(p_i)\|$ we must also consider the base sequent and we find that

$$\|X(p_i)\| \leq \begin{cases} 2, & \textit{if } (p_i \to p_k) \in \Delta \textit{ for } 1 \leq k \leq n \\ 2, & \textit{if } (p_k \to p_l) \in \Delta \textit{ for } 1 \leq k, l \leq n \textit{ and } p_l \in \Delta \\ \max\{2, \|\Delta\|\}, & \textit{if } (p_k \to p_l) \in \Delta \textit{ for } 1 \leq k, l \leq n \textit{ and } p_l \notin \Delta \\ \|\Delta\|, & \textit{otherwise.} \end{cases}$$

Proof. We now consider the norm on the X-set of the propositional variable appearing in the base sequent. Our reasoning is similar to above for all sequents with p_i as succedent that are not the base sequent, so we will only consider the base sequent. We must thus investigate when Δ is minimal.

Let Δ' be the multiset such that $\text{Supp}(\Delta') = \text{Supp}(\Delta)$ and $\|\Delta'\| = 1$. We have the following cases:

1. Δ' contains $p_i \to p_k$ for $1 \leq k \leq n$.
2. Δ' contains $p_k \to p_l$ for $1 \leq k, l \leq n$.
3. None of the above hold.

The first case is the strongest, so when discussing the second case we will assume that the first does not hold. In the first case, we apply contraction to the formula $p_i \to p_k$ and then apply $\to L$ with $\Gamma_1 = \Delta'$. Clearly $\Delta' \trianglelefteq \Delta$ and as $\|\Delta'\| = 1$, we still find $\|X(p_i)\| \leq 2$.

In the second case, we have two options:

1. $p_l \in \Delta'$. In this case, apply contraction to $p_k \to p_l$ and apply $\to L$ with $\Gamma_2 = \Delta' \setminus p_l$ so that $\Gamma_2, p_l \trianglelefteq \Delta$ and $\|\Gamma_2, p_l\| = 1$, so $\|X(p_i)\| \leq 2$.
2. $p_l \notin \Delta'$. In this case, for any choice of Γ_2, we find that $\Gamma_2, p_l \ntrianglelefteq \Delta$. In this case, Δ would be minimal and $\Delta \in X(p_i)$. By our reasoning above, any other element of $X(p_i)$ has norm equal to 2, we find that $\|X(p_i)\| \leq \max\{2, \|\Delta\|\}$.

In the third case, the only sequent appearing with p_i as a succedent is $\Delta \Rightarrow p_i$, so $X(p_i) = \Delta$ and $\|X(p_i)\| = \|\Delta\|$. □

4.2.2 Implication without nesting with truth and falsehood

Let us now consider the situation as above, but with truth (1) and falsehood (0) added. The rules for 0 and 1 can be seen in Figure 3.

As before, we aim to prove a sequent

$$\Delta \Rightarrow \varphi,$$

but now $\varphi \in \{p_1, \ldots, p_n, 0, 1\}$. We will first consider the cases where $\varphi = 0$ and $\varphi = 1$ and then shortly comment on how the addition of 0 and 1 in Δ might change the values of $\|X(p_i)\|$.

As before, given a set of n propositional variables $\{p_1, \ldots, p_n\}$, we can depict all possible formulas in Δ using a complete digraph K_{n+2}, now also adding edges for 0 and 1. For two variables, its graph can be seen in Figure 6.

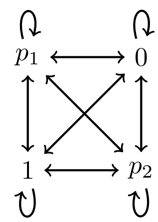


Figure 6: Graph of two variables with 0 and 1.

We have seen in the discussion about the provability of 0 and 1 at the start of this section that 0 behaves a lot like a propositional variable in the sense that a sequent $\Delta \Rightarrow 0$ can only be proven if Δ can be converted to 0, like with a propositional variable. Because of this, we can basically consider 0 as a propositional variable and hence the results we have found in Section 4.2.1 also hold for 0. We summarise this in the following theorem.

Theorem 4.5. *Let Δ be any multiset such that* $\mathrm{Supp}(\Delta) \subseteq \mathrm{Supp}(K^{n+2})$*, and let $p_i \in \{p_1, \ldots, p_n\}$. If we consider the proof tree of the sequent*

$$\Delta \Rightarrow p_i.$$

then if there exists $p_j \to 0 \in \Delta$, we find

$$\|X(0)\| \leq 2.$$

If we consider the sequent

$$\Delta \Rightarrow 0,$$

we find that

$$\|X(0)\| \leq \begin{cases} 2, & \text{if } (0 \to \varphi) \in \Delta \text{ for } \varphi \in \{p_1, \ldots, p_n, 0, 1\} \\ 2, & \text{if } (\psi \to \varphi) \in \Delta \text{ for } \psi, \varphi \in \{p_1, \ldots, p_n, 0, 1\} \text{ and } \varphi \in \Delta \\ \max\{2, \|\Delta\|\}, & \text{if } (\psi \to \varphi) \in \Delta \text{ for } \psi, \varphi \in \{p_1, \ldots, p_n, 0, 1\} \text{ and } \varphi \notin \Delta \\ \|\Delta\|, & \text{otherwise.} \end{cases}$$

The case for when $\varphi = 1$ is a bit more interesting. This is because the sequent

$$\Delta \Rightarrow 1$$

can be proven if Δ can be converted to ϵ or 1^k for $k \geq 1$. Note that Δ can only be converted to ϵ if Δ could have already been converted to 1^k, so we will only consider this case. Consider the set

$$\{\Gamma_i \mid \Gamma_i \text{ can be converted to } 1\}.$$

Then any subset Σ of this set consisting of m elements can be converted to 1^m. Moreover, applying contraction to any element of $\Gamma_i \in \Sigma$ does not change provability, since we can apply contraction to all other elements of Γ_i to get a new subset Σ' that can be converted to 1^{m+1}. First suppose that we have a base sequent

$$\Delta \Rightarrow p_i,$$

where Δ contains an implication $1 \to p_j$ for any $1 \leq j \leq n$, so there are sequents $\Delta' \Rightarrow 1$ in $\mathcal{T}$. Now, if we consider a subset Γ of Δ such that $\|\Gamma\| = 1$. Either $\Gamma \Rightarrow 1$ is unprovable or it is. In the first case, we find an element of $X(1)$ and in the second case we find a provable element that will not become unprovable, so it won't become an element of $X(1)$ by applying any rules. So we find that

$$\|X(1)\| = 1.$$

Consider now the case where the base sequent is

$$\Delta \Rightarrow 1.$$

The reasoning for this case is very similar to the one in the proof for Theorem 4.4. The only difference is that when there is a sequent $\Delta' \Rightarrow 1$ in $\mathcal{T}$ such that $\|\Delta'\|$, the norm of $X(1)$ is now bounded by 1. We can summarise the results in the following theorem.

Theorem 4.6. *Let Δ be any multiset such that $\mathrm{Supp}(\Delta) \subseteq \mathrm{Supp}(K^{n+2})$, and let $p_i \in \{p_1, \ldots, p_n\}$. If we consider the proof tree of the sequent*

$$\Delta \Rightarrow p_i.$$

then if there exists $p_j \to 1 \in \Delta$, we find

$$\|X(1)\| = 1.$$

If we consider the sequent

$$\Delta \Rightarrow 1,$$

we find that

$$\|X(1)\| \leq \begin{cases} 1, & \text{if } (1 \to \varphi) \in \Delta \text{ for } \varphi \in \{p_1, \ldots, p_n, 0, 1\} \\ 1, & \text{if } (\psi \to \varphi) \in \Delta \text{ for } \psi, \varphi \in \{p_1, \ldots, p_n, 0, 1\} \text{ and } \varphi \in \Delta \\ \|\Delta\|, & \text{if } (\psi \to \varphi) \in \Delta \text{ for } \psi, \varphi \in \{p_1, \ldots, p_n, 0, 1\} \text{ and } \varphi \notin \Delta \\ \|\Delta\|, & \text{otherwise.} \end{cases}$$

Lastly, we come back to the case where we consider $X(p_i)$ for a propositional variable $p_i \in \{p_1, \ldots, p_n\}$. In this case, both 0 and 1 will behave the same as the propositional variables. The case for $n + 2$ variables without 0 and 1 is the same as the case for n variables with 0 and 1. Hence we find that our results from Theorem 4.2 and Theorem 4.4 still hold.

4.2.3 Implication with left-nesting

Now we know how non-nested implications work, we can study more intricate formulas. We start by adding nested implications of level 2, with parentheses on the left side. As before, our base sequent is of the form

$$\Delta \Rightarrow p_i,$$

for $p_i \in \{p_1, \ldots, p_n\}$ and Δ contains formulas from the sets

- $\{p_1, \ldots, p_n\}$
- $\{p_k \to p_l \mid 1 \leq k, l \leq n\}$
- $\{(p_u \to p_v) \to p_w \mid 1 \leq u, v, w \leq n\}$

We will denote the set containing all of these elements with multiplicity 1 by L^n.

Theorem 4.7. *Let Δ be any multiset such that $\mathrm{Supp}(\Delta) \subseteq \mathrm{Supp}(L^n)$, and let $p_i \in \{p_1, \ldots, p_n\}$. Consider the proof tree of the sequent*

$$\Delta \Rightarrow p_i.$$

Let $p_j \in \{p_1, \ldots, p_n\}$. If $p_j \neq p_i$, we find that

$$\|X(p_j)\| \leq 2$$

If we apply $\to L$ to the base sequent, our left premise is either a propositional variable or an implication, depending on whether we applied $\to L$ to a non-nested implication or a nested implication respectively. Let us first consider the case where we apply $\to L$ to a non-nested implication $p_k \to p_l$. Then we need to consider the subsets of Δ that will prove p_k. As before, we are able to visualise this in a graph, but we need nodes for all possible non-nested implications, of which there are n^2, giving us a total of $n^2 + n$ nodes. For example, with two variables p and q, we find the graph in Figure 7.

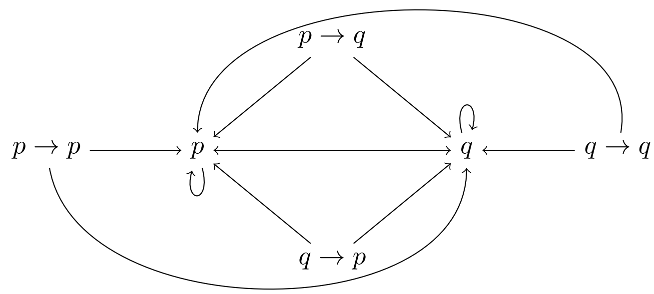


Figure 7: Graph with nested implications for 2 variables: L_n

Proposition 4.8. *Let $\Gamma \subseteq L^n$. The sequent $\Gamma \Rightarrow p_k$ is provable if it satisfies the following properties:*

- Γ *contains at most one propositional variable or nested implication.*
- Γ *corresponds to the graph of a directed walk in L_n that ends in p_k and begins at one of the nodes in L_n.*

We refer to Definition 4.1 for the meaning of 'graph of a walk'.

Proof. Let $\Gamma \subseteq L^n$. We have already discussed the case for none or more than one propositional variables in the proof of Proposition 4.3. First, suppose that Γ contains neither a propositional variable nor a nested implication. Then Γ only contains implications. There must be a sequent in the proof search tree of the form $p_r \to p_s \Rightarrow p_t$, and if we apply $\to L$ to it, we get the sequents $\Rightarrow p_r$ and $p_s \Rightarrow p_t$, of which the former is not provable.

Now suppose that Γ contains both a propositional variable and a nested implication, so we get a sequent of the form

$$\Gamma = \Gamma', p_r, (p_u \to p_v) \to p_w \Rightarrow p_k.$$

Without loss of generality, we apply $\to L$ to the nested implication. We get the sequents

$$\Gamma_1 \Rightarrow p_u \to p_v \qquad\qquad \Gamma_2, p_w \Rightarrow p_k$$

It must be the case that $p_r \in \Gamma_1$ or $p_r \in \Gamma_2$. Clearly it cannot be in Γ_2, since that sequent already contains a propositional variable, making it unprovable. Suppose $p_r \in \Gamma_1$. If we were to prove this sequent without applying $\to R$, we would need a sequence of rules to convert Γ_1 to $p_u \to p_v$. However, we do not have rightly nested implications yet, so this is impossible. Hence we must apply $\to R$, bringing p_u to the antecedent. Now Γ_1, p_u also contains two propositional variables, making the sequent unprovable. So we conclude there cannot be a propositional variable and a nested implication in Γ.

Lastly, suppose there are two nested implications in Γ, so we get a sequent of the form

$$\Gamma = \Gamma', (p_u \to p_v) \to p_w, (p_r \to p_s) \to p_t \Rightarrow p_k.$$

Without loss of generality, we apply $\to L$ to $(p_u \to p_v) \to p_w$ to get the premises

$$\Gamma_1, p_u \Rightarrow p_v \qquad\qquad \Gamma_2, p_w \Rightarrow p_k,$$

where we have already applied $\to R$ to the left premise. Then $(p_r \to p_s) \to p_t$ must be either in Γ_1 or in Γ_2. Since both sequents have the same shape, proving that one of these won't result in a provable sequent proves that neither will. So suppose $(p_r \to p_s) \to p_t \in \Gamma_1$. Then we get the sequent $\Gamma_1', (p_r \to p_s) \to p_t, p_u \Rightarrow p_v$, which is a sequent containing both a propositional variable and a nested implication, which we have shown to be unprovable.

The second condition is proven by induction of the rules in this fragment. We note here that our reasoning in the proof of Proposition 4.3 did not depend on the contents of Γ and hence the same reasoning will work here. □

We are now ready to investigate how many contractions need to be applied to a provable sequent to get unprovability. Let Γ be a multiset such that $\Gamma \Rightarrow p_k$ is unprovable. As before, we have cycle formulas and non-cycle-formulas. The nested implications cannot be part of a cycle: Given a formula $(p_u \to p_v) \to p_w$, to make a cycle using this formula, we would need a formula of the form $p_r \to (p_s \to p_t)$, which we are currently not considering and hence we won't be able to make a cycle. So non-nested implications are the only possible cycle formulas and contracting them will not change provability.

Using our reasoning in the proof of Proposition 4.8, we see that after a contraction of a propositional variable or a nested implication, we get an unprovable sequent. For the reasoning on implications, we refer to the reasoning in Subsection 4.2.1, since the proof did not depend on the contents of Γ.

So we conclude that after a single implication of non-cycle-formulas, we get an unprovable sequent. Hence we find that

$$\|X(p_k)\| \leq 2$$

for any $p_k \neq p_i \in \{p_1, \ldots, p_n\}$.

We move on to the case where we apply $\to L$ to a nested implication $(p_u \to p_v) \to p_w$. This gives us a sequent with $p_u \to p_v$ as a succedent. We have already reasoned in the proof of Proposition 4.8 that the sequent $\Gamma \Rightarrow p_u \to p_v$ is provable without using $\to R$ if there is a sequence of rules that converts Γ to $p_u \to p_v$ to get an initial sequent. This is only possible if Γ is precisely equal to $p_u \to p_v$, in which case we find unprovability after a single contraction. So we may assume that we first apply the rule $\to R$ to get the sequent $\Gamma, p_u \Rightarrow p_v$. Hence we notice a clear relation between $X(p_u \to p_v)$ and $X(p_v)$. Given $\Gamma \in X(p_u \to p_v)$, we must have that $\Gamma, p_u \in X(p_v)$. If we look at it the other way around, given $X(p_v)$, we are able to construct $X(p_u \to p_v)$ by taking the $\Gamma' \in X(p_v)$ such that $\text{mult}(p_u) = 1$ and removing p_u from them. Then $\Gamma' \setminus \{p_u\} \in X(p_u \to p_v)$.

So far we only see implications in the antecedents that prove $p_u \to p_v$. The next step is to see if it possible to prove $p_u \to p_v$ with an antecedent involving any nested implications. Consider a sequent

$$\Gamma \Rightarrow p_u \to p_v.$$

We must apply $\to R$ first, giving us the sequent $\Gamma, p_u \Rightarrow p_v$. If Γ contains any nested implications, we get an unprovable sequent, as we have discussed in the case where Γ contains both a propositional variable and a nested implication. We find that a sequent with an implication without nesting as a succedent cannot be proven if the antecedent contains a nested implication.

So we conclude that there is a clear relation between $X(p_u)$ and $X(p_u \to p_v)$. Because of this relation, we can also relate their cardinalities. We find that for any $\Gamma \in X(p_u \to p_v)$, $\Gamma, p_u \in X(p_v)$. We can write this as a map

$$\begin{aligned} f : X(p_u \to p_v) &\longrightarrow X(p_v) \\ \Gamma &\mapsto \Gamma, p_u \end{aligned}$$

Clearly, for different Γ we get different Γ, p_u, so this map is surjective. Hence we can conclude that

$$|X(p_u \to p_v)| \leq |X(p_v)|.$$

Moreover, since the application of contraction to a non-cycle implication to an element of $X(p_u)$ results in an unprovable sequent, we find that the same holds for elements of $X(p_u \to p_v)$. Hence we also have that

$$\|X(p_r \to p_s)\| \leq 2$$

for any $p_r \to p_s \in \{p_k \to p_l \mid 1 \leq k, l \leq n\}$.

Theorem 4.9. *Let Δ be any multiset such that $\mathrm{Supp}(\Delta) \subseteq \mathrm{Supp}(L^n)$, and let $p_i \in \{p_1, \ldots, p_n\}$. Consider the proof tree of the sequent*

$$\Delta \Rightarrow p_i.$$

We find that

$$\|X(p_i)\| \leq \begin{cases} 2, & \text{if } (p_i \to p_k) \in \Delta \text{ for } 1 \leq k \leq n \\ 2, & \text{if } (p_k \to p_l) \in \Delta \text{ for } 1 \leq k, l \leq n \text{ and } p_l \in \Delta \\ \max\{2, \|\Delta\|\}, & \text{if } (p_k \to p_l) \in \Delta \text{ for } 1 \leq k, l \leq n \text{ and } p_l \notin \Delta \\ 2, & \text{if } ((p_k \to p_i) \to p_j) \in \Delta \text{ for } 1 \leq j, k \leq n \text{ and } p_k \in \Delta \text{ or } p_j \in \Delta \\ \max\{2, \|\Delta\|\}, & \text{if } ((p_k \to p_i) \to p_j) \in \Delta \text{ for } 1 \leq j, k \leq n \text{ and } p_k \notin \Delta, p_j \notin \Delta \\ 2, & \text{if } ((p_j \to p_k) \to p_l \in \Delta \text{ for } 1 \leq j, k, l \leq n \text{ and } p_l \in \Delta \\ \max\{2, \|\Delta\|\}, & \text{if } ((p_j \to p_k) \to p_l) \in \Delta \text{ for } 1 \leq j, k, l \leq n \text{ and } p_l \notin \Delta \\ \|\Delta\|, & \text{otherwise} \end{cases}$$

Proof. We begin by noting that the first 3 cases are the same as in the proof of Theorem 4.4, so we will not discuss them again. Let Δ' be the multiset such that $\mathrm{Supp}(\Delta') = \mathrm{Supp}(\Delta)$ and $\|\Delta'\| = 1$. Let us assume that $(p_k \to p_i) \to p_j \in \Delta$ for some $1 \leq j, k \leq n$. Then we can apply contraction to $(p_k \to p_i) \to p_j$ and apply $\to L$ to get the sequents

$$\begin{array}{cc} \Gamma_1 \Rightarrow p_k \to p_i & \qquad \Gamma_2, p_j \Rightarrow p_i \\ \Gamma_1, p_k \Rightarrow p_i & \qquad \Gamma_2, p_j \Rightarrow p_i \end{array}$$

after an application of $\to R$ to the left premise. If $p_k \in \Delta$, take $\Gamma_1 = \Delta' \setminus p_k$ so that $\Gamma_1, p_k \trianglelefteq \Delta$ and $\|\Gamma_1, p_k\| = 1$, so $\|X(p_i)\| \leq 2$. If $p_l \in \Delta$, take $\Gamma_2 = \Delta' \setminus p_l$ so that $\Gamma_2, p_l \trianglelefteq \Delta$ and $\|\Gamma_2, p_l\| = 1$, so $\|X(p_i)\| \leq 2$. If neither of these two hold, then there exist no Γ_1, Γ_2 such that $\Gamma_1, p_k \trianglelefteq \Delta$ or $\Gamma_2, p_l \trianglelefteq \Delta$ respectively. In this case, Δ would be minimal, so $\Delta \in X(p_i)$. Since all other elements of $X(p_i)$ have norm 2 by our reasoning above, we find that $\|X(p_i)\| \leq \max\{2, \|\Delta\|\}$.

If the above does not hold, but $(p_j \to p_k) \to p_l \in \Delta$ for some $1 \leq j, k, l \leq n$, we have the following: we apply contraction to $(p_j \to p_k) \to p_l$ and then apply $\to L$ to get the premises

$$\Gamma_1 \Rightarrow p_j \to p_k \qquad\qquad \Gamma_2, p_l \Rightarrow p_i.$$

Since the left premise does not have p_i as a succedent, we only consider the right premise. If $p_l \in \Delta$, then we take $\Gamma_2 = \Delta \setminus p_l$ so that $\Gamma_2, p_l \trianglelefteq \Delta$ and $\|\Gamma_2, p_l\| = 1$, so $\|X(p_i)\| \leq 2$. If $p_l \notin \Delta$ then there exists no $\Gamma_2 \subseteq \Delta$ such that $\Gamma_2, p_l \trianglelefteq \Delta$, so Δ would be minimal and $\Delta \in X(p_i)$. Since all other elements of $X(p_i)$ are of norm 2, we find that $\|X(p_i)\| \leq \max\{2, \|\Delta\|\}$.

If none of the above hold, so Δ only contains propositional variables, the only sequent appearing with p_i as a succedent is $\Delta \Rightarrow p_i$, so $X(p_i) = \Delta$ and $\|X(p_i)\| = \|\Delta\|$. □

4.3 Conjunctive fragment of FL_{ec}

In this section, we will be working in the conjunctive fragment of FL_{ec}. That is, the fragment of FL_{ec} where only the connective Conjunction, ”$\wedge$”, is used. We are using the rules

$$\frac{}{\alpha \Rightarrow \alpha}\, AX \qquad \frac{\Gamma, \alpha \Rightarrow \varphi}{\Gamma, \alpha \wedge \beta \Rightarrow \varphi}\, \wedge L1 \qquad \frac{\Gamma, \beta \Rightarrow \varphi}{\Gamma, \alpha \wedge \beta \Rightarrow \varphi}\, \wedge L2$$

$$\frac{\Gamma \Rightarrow \alpha \qquad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta}\, \wedge R \qquad \frac{\Gamma, \alpha, \alpha \Rightarrow \varphi}{\Gamma, \alpha \Rightarrow \varphi}\, c$$

4.3.1 Conjunction with propositional variables

Consider the set of n propositional variables $p_1, \ldots, p_n$. Any conjunction using this set is of the form

$$p_i^{\wedge \mu_i} \wedge \cdots \wedge p_j^{\wedge \mu_j}, \tag{19}$$

where $p_r^{\wedge \mu_r} = p_r \wedge \cdots \wedge p_r$ (μ_r times) and $\{p_i, \ldots, p_j\}$ is a subset of $\{p_1, \ldots, p_n\}$. We will see later that the μ_r do not influence the norm of elements of X-sets, so we will denote such a formula $\Lambda_{\{i,\ldots,j\}}$. Conjunction is commutative, so we can group each propositional variable separately.

Every base sequent in this fragment is of the form

$$\Delta \Rightarrow \Lambda_{\{i,\ldots,j\}},$$

where Δ contains formulas like in Equation 19 and propositional variables $p_i \in \{p_1, \ldots, p_n\}$.

Remark 6. Consider the sequent $\alpha_1, \ldots, \alpha_r \Rightarrow \Lambda_{\{i,\ldots,j\}}$. When we do exhaustive proof search on this sequent, it is it is guaranteed that every subformula of the conjunction in the succedent appears with any combination of subformulas of the α_i in the antecedent.

This then gives us that for any two formulas φ, ψ appearing as succedents in the proof search, they will appear with the same antecedents. The elements of their X-sets will depend on whether φ or ψ can be proven by a subformula of the antecedent.

Definition 4.3. *Let $\Delta \Rightarrow \Lambda_{\{i,\ldots,j\}}$ be the base sequent of our proof search tree, and let $\{p_k, \ldots, p_l\}$ be the propositional variables appearing in the antecedent. Let $p_m \in \{p_k, \ldots, p_l\}$. The* ***Conjunction Sumnorm*** *of p_m, denoted*

$$\left\| \sum p_m \right\|,$$

is equal to the number of times that p_m appears in a formula $\varphi \in \Delta$.

Example 12. Let $\Delta = p, p \wedge q, p \wedge r, q \wedge r$. Then

$$\left\| \sum p \right\| = 3 \qquad \left\| \sum q \right\| = 2 \qquad \left\| \sum r \right\| = 2$$

△

Theorem 4.10. *Let $\Delta \Rightarrow \Lambda_{\{i,\ldots,j\}}$ be the base sequent of our proof search tree, and let $\{p_k, \ldots, p_l\}$ be the propositional variables appearing in the antecedent. Let φ be any formula appearing as a succedent in the proof search. We have*

$$\|X(\varphi)\| \leq \max\left\{2, \max_{m \in \{k,\ldots,l\}} \left\{ \left\| \sum p_m \right\| \right\} \right\}.$$

Proof. We will first take a look at the formula multisets that prove the formula $\Lambda_{\{i,\ldots,j\}}$. By applying $\wedge R$, we find that $\Delta \Rightarrow \Lambda_{\{i,\ldots,j\}}$ is provable when Δ proves each component of the conjunction $\Lambda_{\{i,\ldots,j\}}$ separately. This means that Δ must prove all propositional variables in the set $\{p_i, \ldots, p_j\}$. We conclude that Δ must be a single formula δ. Using the rules $\wedge L1$ and $\wedge L2$, we find that φ must be a conjunction including at least all propositional variables $\{p_i, \ldots, p_j\}$ and possibly other propositional variables from the set $\{p_1, \ldots, p_n\}$. We summarize this knowledge in the following lemma:

Lemma 4.11. *$\Delta \Rightarrow p_i^{\wedge \mu_i} \wedge \cdots \wedge p_j^{\wedge \mu_j}$ is provable if and only if Δ is a single conjunction of the form $\Delta = p_i \wedge \cdots \wedge p_j \wedge \gamma$, where γ is any conjunction of propositional variables in the set $\{p_1, \ldots, p_n\}$.*

From this, we can investigate which formula multisets do not prove $\Lambda_{\{i,\ldots,j\}}$ and hence what the base sequent looks like. We will first give the options and then discuss per option why we have unprovability and how this influences $X(\Lambda_{\{i,\ldots,j\}})$.

1. **Missing propositional variable**: One or more of the propositional variables from $\{p_i, \ldots, p_j\}$ are missing from Δ.
2. **Multiset Δ contains more than one formula**.

Remark 7. For this discussion, we first assume that each case occurs on its own and not together with any other case.

In the first case, there is at least one propositional variable appearing in the succedent that does not appear in the antecedent. Without loss of generality, assume this is p_m. For $\Delta \Rightarrow \Lambda_{\{i,\dots,j\}}$ to be provable, one of the branches of the prooftree should hence end in the initial sequent $p_m \Rightarrow p_m$. Since p_m does not appear in the antecedent, this sequent is unprovable.

First assume that there is at least one propositional variable $p_r \in \{p_i, \dots, p_j\}$ that appears in the antecedent. Since p_r then appears in both the succedent and the antecedent of the base sequent, the sequent $p_r \Rightarrow p_r$ must appear in the proof search tree using exhaustive proof search. Since this sequent is provable, we can apply a single contraction to get the sequent $p_r^2 \Rightarrow p_r$, which is unprovable and is minimal to do so. Hence $p_r^2 \in X(p_r)$. Any other provable sequents of p_r that may appear in the proof search tree are of the form

$$\gamma \wedge p_r \wedge \delta \Rightarrow p_r,$$

where γ and δ are conjunctions of propositional variables in the set $\{p_1, \dots, p_n\}$. For any γ, δ, these antecedents have norm 1 and are unprovable after a single contraction. Any other possible antecedents in the proof search tree will not prove p_r and are thus elements of $X(p_r)$ with norm equal to 1. We conclude that in this case, $\|X(p_r)\| = 2$.

Now suppose that none of the propositional variables in $\{p_i, \dots, p_j\}$ appear in the antecedent. Then there cannot be any provable sequents in the proof search tree. This means that any antecedent appearing in the proof search tree is an element of the X-set of any succedent appearing in the proof search tree with norm equal to 1, i.e. $\|X(\varphi)\| = 1$ if φ is a succedent in $\mathcal{T}$. Hence, for any φ that appears as a succedent in the proof search tree $\mathcal{T}$, we conclude that

$$\|X(\varphi)\| \leq 2.$$

In the second case, we assume Δ is not a single formula, but contains more formulas, $\Delta = \{\varphi_1, \dots, \varphi_m\}$. A sequent of this form is not provable. Assuming we first apply $\wedge R$ before applying $\wedge L$, we have that all topmost sequents in the proof search have a succedent $p_m \in \{p_i, \dots, p_j\}$. We have seen in Section 4.1 that $\Delta \Rightarrow p_m$ can only be proven by $\Delta = p_m$. Translating that to this sequent, $\Delta \Rightarrow p_m$ can only be proven by a formula multiset $\Delta = \varphi$ such that there is a sequence of rules to go from $\varphi \Rightarrow p_m$ to $p_m \Rightarrow p_m$. The only left rules we can apply in this sequent are contraction c and $\wedge L1$ and $\wedge L2$. Clearly contraction will not help us, so we will only apply the left conjunction rules. Hence φ must be of the form displayed in Lemma 4.11. Since Δ contains more than one formula, this sequent is unprovable.

Let us now consider the norm of the X-sets in this case. By Remark 6, the norm of the X-set of the base succedent is equal to the norm of the X-set of all other succedents in the proof search, so we will only consider the application of the rules contraction c, $\wedge L1$ and $\wedge L2$. The antecedent of an unprovable sequent after applying contraction can never be minimal, so we will also not consider that rule. After exhaustively applying the left rules, only one formula of each conjunction will remain in each sequent. Suppose that the propositional variables $\{p_k, \dots, p_l\}$ appear in the antecedent. After the application of the left rules, every antecedent is of the form

$$p_k^{\lambda_k}, \dots, p_l^{\lambda_l}.$$

We want to argue that every one of these appearing as an antecedent is minimal with respect to $\trianglelefteq$. Say it was not minimal, so there is a $p_k^{\nu_k}, \dots, p_l^{\nu_l}$ with $\nu_m \leq \lambda_m$ for $i \leq m \leq j$. This cannot be true, suppose we have that $\nu_k < \lambda_k$. This implies that in one or more conjunctions, we must not have remained with p_k. This means that in these conjunctions, one of $p_{k+1}, \dots, p_l$ is chosen, implying that $\nu_m > \lambda_m$ for $m \in \{i+1, \dots, j\}$. To bound the norm, we must find these $p_k^{\lambda_i}, \dots, p_l^{\lambda_l}$ with the greatest norm. For each propositional variable p_m separately, this is done by taking it from each formula it appears in. For conjunctions it does not appear in, we take one of the others at random. The norm of the resulting formula is equal to the number of times p_m appears in in a formula in the base sequent. We will denote this as $\|\sum p_m\|$ The norm of $X(\Lambda_{\{i,\dots,j\}})$ is then bounded by the maximal over all of the propositional variables appearing in the antecedent. Given any φ appearing as a succedent in the proof search, we have

$$\|X(\varphi)\| \leq \max_{p_m \in \{p_k,\ldots,p_l\}} \left\{ \left\| \sum p_m \right\| \right\}.$$

□

4.3.2 Conjunction with truth and falsehood

Similar to our approach in the Fusion fragment, we will now consider the X-sets for succedents in the proof search tree of a base sequent where truth (1) and falsehood (0) have been added to the succedent. The rules for 0 and 1 can be found in Figure 3.

For truth, we first consider the sequent

$$\Delta \Rightarrow 1^{\wedge k} = 1 \wedge \ldots \wedge 1.$$

Using the rule $\wedge R$, we note that this sequent is provable when the sequent $\Delta \Rightarrow 1$ is provable. We have seen at the beginning of this section that this is the case when Δ can be converted to either ϵ or 1^k for $k > 0$. Using the rules $\wedge L1$ and $\wedge L2$, this means that either $\Delta = \epsilon$, or

$$\forall \varphi \in \Delta \text{ we have that } \varphi = 1 \text{ or } \varphi = 1 \wedge \gamma. \tag{20}$$

where γ is any formula containing only conjunctions as logical connectives. Here we use the fact that the formulas $1 \wedge \gamma$ can 'vanish': we first apply $\wedge L$ as needed to get to 1 to which we can then apply $1L$.

Then $\Delta \Rightarrow 1^{\wedge k}$ is not provable if Δ contains any formula not of the type described in Equation 20. As before, we calculate the Sumnorm of the propositional variables appearing in the antecedent. In this case, we only consider those formulas that do not contain 1 as a component and calculate the Sumnorm of the propositional variables appearing in those formulas.

Let us now consider the sequent

$$\Delta \Rightarrow p_i^{\wedge \mu_i} \wedge \ldots \wedge p_j^{\mu_j} \wedge 1^{\wedge \mu}.$$

This sequent is provable if Δ is a multiset containing:

- $\Delta' = p_i \wedge \ldots \wedge p_j (\wedge 1) \wedge \gamma$.
- $1 \wedge \gamma'$ (which will be 'vanished'),

where γ, γ' are any, possibly empty, conjunctions. The addition of $\wedge 1$ in the first formula is optional. For unprovability, we either have that

1. Δ' does not contain one of $\{p_i, \ldots, p_j\}$
2. Δ contains formulas other than Δ' that do not contain 1.

If only the first holds, we can reapply our reasoning from Section 4.3.1. If Δ' appears without 1, the reasoning is exactly the same and if Δ' appears with 1, we note that while the sequent $1 \Rightarrow 1$ can appear in the proof search tree, its provability will only change when we apply the rule $1L$ to get the sequent $\epsilon \Rightarrow 1$, but since $\|\epsilon\| = 0$, this does not change the result. So, for any φ appearing as a succedent in the proof search, we find

$$\|X(\varphi)\| \leq 2.$$

If moreover the second also holds, we consider all formulas that do not contain 1 as a component and apply Sumnorm to all the propositional variables appearing in those formulas. Then the norm of $X(\varphi)$ is bounded by the maximum over all these Sumnorms.

For falsehood, we first consider the sequent

$$\Delta \Rightarrow 0^{\wedge k} = 0 \wedge \cdots \wedge 0.$$

Using the rule $\wedge R$, we note that this sequent is provable when the sequent $\Delta \Rightarrow 0$ is provable. We have seen at the beginning of this section that this sequent is provable when Δ can be converted to 0 using the rules $\wedge L1$ and $\wedge L2$. In this case, that means that Δ must be a single conjunction that includes 0.

This sequent is unprovable if:

- Δ is a single conjunction not containing 0.
- Δ is a multiset containing at least two formulas that do not contain 1.

If only the first case holds, the norm of Δ is equal to 1 and since there are no provable sequents in $\mathcal{T}$, we have that $\|X(0^{\wedge k})\| = 1$. If second case holds, we consider all formulas that do not contain 1 as a component. We apply Sumnorm to both the propositional variables appearing in these formulas and to 0 itself. The norm is then bounded by the maximum over all these Sumnorms.

Let us now consider the sequent

$$\Delta \Rightarrow p_i^{\wedge \mu_i} \wedge \cdots \wedge p_j^{\wedge \mu_j} \wedge 0^{\wedge \mu}.$$

This sequent is provable if Δ is a multiset containing:

- $\Delta' = p_i \wedge \ldots \wedge p_j \wedge 0 \wedge \gamma$,
- $1 \wedge \gamma'$ (which will be 'erased'),

where γ, γ' are any, possibly empty, conjunctions. The conditions for unprovability are then:

1. Δ' does not contain one of $\{p_i, \ldots, p_j, 0\}$
2. Δ contains formulas other than Δ' that do not contain 1 as a component.

Then our reasoning is the same as above, but when applying the Sumnorm, besides the propositional variables appearing in the antecedent, we also consider the Sumnorm of 0. Again, we only consider those formulas that do not contain 1 as a component.

4.3.3 Conjunction with fusion

Until now, we only considered conjunctions where the multiplicity of each formula in a conjunction is equal to 1. We can easily translate the results of Theorem 4.10 to a case where we allow fusion in the antecedent. For now, we consider the case where only the fusion of propositional variables with themselves is allowed, i.e. we consider propositional variables with a higher exponent. Since we do not change the succedent, the conditions for provability do not change and the results from Lemma 4.11 still hold. The options for unprovability are

1. **Missing propositional variable**: One or more of the propositional variables from $\{p_l, \ldots, p_m\}$ are missing from Δ.
2. **Propositional variable with higher exponent**: One or more of the propositional variables from $\{p_l, \ldots, p_m\}$ only appears with an exponent greater than 1.
3. **Multiset Δ contains more than one formula**.

Remark 8. We will again look at each option assuming it is happening in isolation. In the first and third case, we will not discuss why the resulting sequents are unprovable, since the reasoning is equal to the case without fusion.

For the first case, our antecedent Δ contains one formula. By our assumption in Remark 8, we find that $\|\Delta\| = 1$. In the case where Δ contains at least one propositional variable from $\{p_l, \ldots, p_m\}$, there will be a provable sequent in $\mathcal{T}$ with norm 1, so then $\|X(\varphi)\| = 2$. In the case where Δ contains no propositional variables from the set $\{p_l, \ldots, p_m\}$, there are no provable sequents and hence $\|X(\varphi)\| = 1$. Combining these, we find

$$\|X(\varphi)\| \leq 2$$

for any φ appearing as a succedent in $\mathcal{T}$.

In the second case, we still consider a multiset Δ containing only one single formula, but some of the propositional variables only appear with an exponent higher than 1. For example, consider the sequent $p^2 \wedge q \Rightarrow p \wedge q$. Since a propositional variable can only be proven by itself with multiplicity 1, sequents of this form are unprovable. If the formula in Δ is not a conjunction, it is a propositional variable p_i^k, implying $X(\Lambda_{\{i,\ldots,j\}}) = p_i^k$ and hence $\|X(\Lambda_{\{i,\ldots,j\}})\| = \|p_i^k\| = k$. If the formula is a conjunction, we consider each propositional variable p_m separately and take from all the components with base p_m

the one with the smallest exponent, say it is μ_m. Then $\left\|X(\Lambda_{\{i,\dots,j\}})\right\|$ is bounded by the maximum of these numbers over all propositional variables in the antecedent. For any φ appearing as a succedent in the proof search, we find

$$\|X(\varphi)\| \leq \max_m\{\mu_m\}.$$

In this case, we do not consider whether there are any provable sequents in $\mathcal{T}$, since those must have antecedent with a norm equal to 1 and hence the elements of the X-sets have a norm equal to 2. Since we already assume that there exists a propositional variable in the conjunction with an exponent of at least 2, these provable sequents will not affect the norm of the X-sets.

In the third case, we will combine our knowledge from the second case above and the non-fusion second case. We will consider each propositional variable separately. Take a propositional variable p_m appearing in the antecedent. From each formula in the antecedent, take the p_m^λ where λ is the smallest. For example, from $p_m^2 \wedge p_m$, we would take p_m with multiplicity 1. We can replace Definition 4.3 to take this into account. This definition would also work for the case without fusion.

Definition 4.4. *Let $\Delta \Rightarrow \Lambda_{\{i,\dots,j\}}$ be the base sequent of our proof search tree, and let $\{p_k,\dots,p_l\}$ be the propositional variables appearing in the antecedent. Let $p_m \in \{p_k,\dots,p_l\}$. The* ***Conjunction Sumnorm*** *of p_m, denoted*

$$\left\|\sum p_m\right\|,$$

is an integer defined using the following procedure: In each formula $\varphi \in \Delta$, check if $\varphi = p_m$ or p_m is a component of the conjunction. If so, take from it the p_m^λ where λ is the smallest. To find the Conjunction Sumnorm of p_m, we take the sum over all the values of λ.

Example 13. Let $\Delta = p^2, r^3, p \vee p^2 \vee q, p \vee r, q^3 \vee q^2 \vee r$. Then

$$\left\|\sum p\right\| = 2+1+1 = 4, \qquad \left\|\sum q\right\| = 1+2 = 3, \qquad \left\|\sum r\right\| = 3+1+1 = 5.$$

△

As before, the norm of $X(\Lambda_{\{i,\dots,j\}})$ is bounded by the maximum of these numbers over all propositional variables appearing in the antecedent. Given any φ appearing as a succedent in the proof search, we have

$$\|X(\varphi)\| \leq \max_{p_m \in \{p_k,\dots,p_l\}} \left\{\left\|\sum p_m\right\|\right\}.$$

4.4 Disjunctive fragment of FL_{ec}

In this section, we will be working in the disjunctive fragment of FL_{ec}. That is, the fragment of FL_{ec} where only the connective Disjunction, "$\vee$", is used. We are using the rules

$$\frac{}{\alpha \Rightarrow \alpha}\,AX \quad \frac{\Gamma,\alpha \Rightarrow \varphi \qquad \Gamma,\beta \Rightarrow \varphi}{\Gamma,\alpha\vee\beta \Rightarrow \varphi}\,\vee L \quad \frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \alpha\vee\beta}\,\vee R1$$

$$\frac{\Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha\vee\beta}\,\vee R2 \quad \frac{\Gamma,\alpha,\alpha \Rightarrow \varphi}{\Gamma,\alpha \Rightarrow \varphi}\,c$$

4.4.1 Disjunction with propositional variables

Consider the set of n propositional variables $p_1,\dots,p_n$. Any disjunction using this set is of the form

$$p_i^{\vee\mu_i} \vee \dots \vee p_j^{\vee\mu_j}, \tag{21}$$

where $p_m^{\vee\mu_m} = p_m \vee \dots \vee p_m$ (μ_m times) and $\{p_i,\dots,p_j\}$ is a subset of $\{p_1,\dots,p_n\}$. We will see later that the μ_i do not influence the norm of elements of X-sets, so we will denote such a formula $\Theta_{\{i,\dots,j\}}$. Disjunction is commutative, so we can group each propositional variable separately.

Every base sequent in this fragment is of the form

$$\Delta \Rightarrow \Theta_{\{i,\dots,j\}},$$

where Δ contains formulas like in Equation 21 and propositional variables $p_m \in \{p_1,\dots,p_n\}$.

Remark 9. Consider the sequent $\alpha_1, \ldots, \alpha_r \Rightarrow \Theta_{\{i,\ldots,j\}}$. When we do exhaustive proof search on this sequent, it is it is guaranteed that every subformula of the disjunction in the succedent appears with any combination of subformulas of α_i in the antecedent.

This then gives us that for any two formulas φ, ψ appearing as succedents in the proof search, they will appear with the same antecedents. The elements of their X-sets will depend on whether φ or ψ can be proven by a subformula of the antecedent.

Definition 4.5. *Let $\Delta \Rightarrow \Theta_{\{i,\ldots,j\}}$ be the base sequent of our proof search tree, and let $\{p_k, \ldots, p_l\}$ be the propositional variables appearing in the antecedent. Let $p_m \in \{p_k, \ldots, p_l\}$. The **Disjunction Sumnorm** of p_m, denoted*

$$\left\| \sum p_m \right\|,$$

is equal to the number of times that p_m appears in a formula $\varphi \in \Delta$.

Example 14. Let $\Delta = p, p \vee q, p \vee r, q \vee r$. Then

$$\left\| \sum p \right\| = 3, \qquad \left\| \sum q \right\| = 2, \qquad \left\| \sum r \right\| = 2.$$

△

Theorem 4.12. *Let $\Delta \Rightarrow \Theta_{\{i,\ldots,j\}}$ be the base sequent of our proof search tree, and let $\{p_k, \ldots, p_l\}$ be the propositional variables appearing in the antecedent. Let φ be any formula appearing as a succedent in the proof search. We have*

$$\|X(\varphi)\| \leq \left\{ 2, \max_{m \in \{k,\ldots,l\}} \left\{ \left\| \sum p_m \right\| \right\} \right\}.$$

Proof. We will first take a look at the formula multisets that prove the formula $\Theta_{\{i,\ldots,j\}}$. By applying the rules $\vee R1$ and $\vee L2$, the succedent always becomes a single propositional variable. Because of this, the antecedent cannot be a multiset containing more than one formula. So let $\Delta = \varphi$: it must either a propositional variable or a disjunction of propositional variables. In the first case, this propositional variable must be an element of $\{p_i, \ldots, p_j\}$. In the second case, each component of the disjunction must be an element of the set $\{p_i, \ldots, p_j\}$, but they may be different. This knowledge is summarized in the following lemma:

Lemma 4.13. $\Delta \Rightarrow p_i^{\vee \mu_i} \vee \cdots \vee p_j^{\vee \mu_j}$ *is provable if and only if Δ is a single formula φ, satisfying one of the following:*

- *φ is a propositional variable from the set $\{p_i, \ldots, p_j\}$.*
- *φ is a disjunction containing only propositional variables from the set $\{p_i, \ldots, p_j\}$.*

Example 15. Consider the succedent $p \vee q \vee r$. Some examples for sets that can prove this are

$$\Delta = \{p\}, \{q\}, \{p \vee r\}, \{q \vee q \vee r\}, \{p \vee q \vee r\}, \{p \vee q \vee r \vee r\}.$$

△

From this, we can investigate which formula multisets do not prove $\Theta_{\{i,\ldots,j\}}$ and hence what the base sequent looks like. We will first give the options and then discuss per option why we have unprovability and how this influences $X(\Theta_{\{i,\ldots,j\}})$.

1. **Other propositional variable**: Δ is a propositional variable not equal to $\{p_i, \ldots, p_j\}$.
2. **Extra propositional variable**: At least one propositional variable $p_m \notin \{p_i, \ldots, p_j\}$ is added to the disjunction as in Lemma 4.13.
3. **Multiset Δ contains more than one formula**.

Remark 10. For this discussion, we first assume that each case occurs on its own and not together with any other case.

In the first case, there are no provable sequents in $\mathcal{T}$, so $\|X(\varphi)\| = 1$ for any succedent φ. In the second case, there is at least one propositional variable appearing in the antecedent that does not appear in the succedent. Without loss of generality, assume this is p_m. For $\Delta \Rightarrow \Theta_{\{i,\ldots,j\}}$ to be provable, one of the branches of the prooftree should hence end in the initial sequent $p_m \Rightarrow p_m$. Since p_m does not appear in the succedent, this sequent is unprovable.

Consider the sequent $\Delta \Rightarrow p_i^{\vee\mu_i} \vee \cdots \vee p_j^{\vee\mu_j}$ and suppose that p_r is a propositional variables appearing in both the succedent and the antecedent. By exhaustivity of proof search, the sequent $p_r \Rightarrow p_r$ must appear in $\mathcal{T}$. This is provable, and after a single application of contraction we get the sequent $p_r^2 \Rightarrow p_r$, which is not provable and minimal to do so, giving that $p_r^2 \in X(p_r)$. Any other provable formula is of the form

$$\gamma \vee p_r \vee \delta \Rightarrow p_r,$$

where γ, δ are any disjunctions of propositional variables in the set $\{p_1, \ldots, p_n\}$. These have a norm equal to 1 for any γ, δ and are unprovable after a single contraction. All other antecedents in the proof search do not prove p_r and have a norm equal to 1. Hence $\|X(p_r\| = 2$.

Now suppose that none of the propositional variables $\{p_i, \ldots, p_j\}$ appear in the antecedent Δ. Then there are no provable sequents in the proof search tree $\mathcal{T}$, meaning that any antecedent in the proof search tree is an element of $X(\varphi)$, for any φ appearing as a succedent in the proof search tree. Since these all have a norm equal to 1, we find that the norm of these X-sets is equal to 1. Combining these two options, we find that

$$\|X(\varphi)\| \leq 2,$$

where φ is any formula that appears as a succedent in the proof search tree.

In the third case, we assume Δ is not a single formula, but contains more formulas, $\Delta = \{\varphi_1, \ldots, \varphi_k\}$. A sequent of this form is not provable. After we have exhaustively applied the left- and right rules for disjunction, all topmost sequents in the proof search have a succedent $p_m \in \{p_i, \ldots, p_j\}$. We have seen in the Multiplicative Fragment that $\Delta \Rightarrow p_m$ can only be proven by $\Delta = p_m$. Translating that to this sequent, $\Delta \Rightarrow p_m$ can only be proven by a formula multiset $\Delta = \varphi$ such that there is a sequence of rules to go from $\varphi \Rightarrow p_m$ to $p_m \Rightarrow p_m$. The only left rules we can apply in this sequent are contraction c and $\vee L$. Clearly contraction will not help us, so we will only apply the left disjunction rule. Hence φ must be of the form as discussed in Lemma 4.13. Since Δ contains more than one formula, this sequent is unprovable.

Let us now consider the norm of the X-sets in this case. By Remark 9, the norm of the X-set of the base succedent is equal to the norm of the X-set of all other succedents in the proof search, so we will only consider the application of the rules contraction c and $\vee L$. The antecedent of an unprovable sequent after applying contraction can never be minimal, so we will also not consider that rule. After exhaustively applying the left disjunction rule, only one formula of each disjunction will remain in each sequent. Suppose that the propositional variables $\{p_k, \ldots, p_l\}$ appear in the antecedent. After repeated application of the left rule, every antecedent is of the form

$$p_k^{\lambda_k}, \ldots, p_l^{\lambda_l}. \tag{22}$$

We want to argue that every one of these appearing as an antecedent is minimal with respect to $\trianglelefteq$. Say it was not minimal, so there is a $p_k^{\nu_k}, \ldots, p_l^{\nu_l}$ with $\nu_m \leq \lambda_m$ for $k \leq m \leq l$. Suppose that $\lambda_r < \mu_r$ for $k \leq r \leq l$. This must imply that a conjunction involving p_r has not been resolved. This then implies that a conjunction of the form $p_r \vee p_s$ appears in this antecedent. This antecedent is not of the form as in Equation 22, as it has a different support. This is a contradiction, so we conclude that every antecedent as in Equation 22 must be minimal.

To bound the norm, we must find these $p_k^{\lambda_k}, \ldots, p_l^{\lambda_l}$ with the greatest norm. Because of how $\vee L$ is structured, it is guaranteed that every combination of components from the disjunctions in the antecedent of the base sequent appears as an antecedent in the proof search tree. This is done for each propositional variable p_m separately. We need to count the number of times it appears in any formula $\varphi \in \Delta$. Suppose p_m appears in λ_m formulas in Δ. Since every combination must appear, there is an antecedent where p_m must have multiplicity λ_m. We will denote this number λ_m as

$$\left\| \sum p_m \right\|.$$

The norm of $X(\Theta_{\{l,\ldots,m\}})$ is then bounded by the maximum over all of the propositional variables appearing in the antecedent. Given any φ appearing as a succedent in the proof search, we have

$$\|X(\varphi)\| \leq \max_{p_m \in \{p_k,\ldots,p_l\}} \left\| \sum p_m \right\|.$$

□

Remark 11. In the conjunction fragment, we have considered the case of self-fusion of propositional variables in the antecedent. The same can be done for disjunction with similar techniques, so here we shortly present the results without a detailed explanation. The provability condition in Lemma 4.13 still holds, so we get unprovability if

1. **Other propositional variable**: Δ is a propositional variable not equal to $\{p_i, \ldots, p_j\}$.
2. **Extra propositional variable**: At least one propositional variable $p_m \notin \{p_i, \ldots, p_j\}$ is added to the disjunction as in Lemma 4.13.
3. **Propositional variable with higher exponent**: One or more of the propositional variables appearing in Δ only appears with an exponent greater than 1.
4. **Multiset Δ contains more than one formula**.

The first two cases do not deal with greater exponents, so their reasoning is the same as without fusion. For the third case, if Δ is a single propositional variable, the norm is equal to its multiplicity and if Δ is a disjunction, we consider for each propositional variable the component with the lowest exponent and take the maximum over these values. In the fourth case, we need to reformulate Definition 4.5 as follows:

Definition 4.6. *Let $\Delta \Rightarrow \Theta_{\{i,\ldots,j\}}$ be the base sequent of our proof search tree, and let $\{p_k, \ldots, p_l\}$ be the propositional variables appearing in the antecedent. Let $p_m \in \{p_k, \ldots, p_l\}$. The* ***Disjunction Sumnorm*** *of p_m, denoted*

$$\left\| \sum p_m \right\|,$$

is an integer defined using the following procedure: In each formula $\varphi \in \Delta$, check if $\varphi = p_m$ or p_m is a component of the conjunction. If so, take from it the p_m^λ where λ is the smallest. To find the Disjunction Sumnorm of p_m, we take the sum over all the values of λ.

With this new definition, we find that for any φ appearing as a succedent in $\mathcal{T}$, and $\{p_k, \ldots, p_l\}$ the set of propositional variables appearing in the antecedent,

$$\|X(\varphi)\| \leq \max_{p_m \in \{p_k,\ldots,p_l\}} \left\{ \left\| \sum p_m \right\| \right\}.$$

4.4.2 Disjunction with truth and falsehood

We are now ready to incorporate truth (1) and falsehood (0) into the succedents of the base sequent. The rules for 0 and 1 can be found in Figure 3.

For truth, we first consider the sequent

$$\Delta \Rightarrow 1^{\vee k} = 1 \vee \ldots \vee 1.$$

Using the rule $\vee L$, we find that this sequent is provable when $\Delta \Rightarrow 1$ is provable. This is the case when

1. **Δ is a single disjunction**: All components of the disjunction prove 1.
2. **Δ is a multiset with more than 1 formula**: All combinations from all formulas prove 1.

As an example for the second case, if $\Delta = \alpha_1 \vee \ldots \vee \alpha_n$, then we need that $\alpha_i \in \{\epsilon, 1^k\}$ for $1 \leq i \leq n$, If $\Delta = \alpha_1 \vee \alpha_2, \beta_1 \vee \beta_2$, we would need that $\{\alpha_1, \beta_1\}, \{\alpha_1, \beta_2\}, \{\alpha_2, \beta_1\}, \{\alpha_2, \beta_2\}$ should be a set that proves 1. This implies that $\alpha_i, \beta_i \in \{\epsilon, 1^k\}$ for $1 \leq i \leq 2$.

When we consider unprovability, we find that Δ does not prove 1 if for any $\varphi \in \Delta$, either $\varphi \in \{0, p\}$ for p a propositional variable or φ is a disjunction including 0 or a propositional variable as a component.

To find the norm of $X(1^{\vee k})$, we apply Sumnorm to 0 and all propositional variables appearing in the antecedent. If a propositional variable (or 0) in a formula with a component equal to 1 also appears in a formula with no component equal to 1, the resulting multisets would be more minimal without the propositional variable (or 0) from the formula with 1 as a component. If the propositional variable (or 0) does not appear in any other formula, its maximal multiplicity in any multiset is equal to 1, not changing the norm of the multiset. Hence we may disregard disjunction that have 1 as a component.

Now, let us consider the sequent

$$\Delta \Rightarrow p_i^{\vee \mu_i} \vee \ldots \vee p_j^{\vee \mu_j} \vee 1^{\vee \mu}.$$

Following Lemma 4.13, this is provable when Δ contains either an element from the set $\{p_i, \ldots, p_j, 1\}$ or a disjunction containing only components from the set $\{p_i, \ldots, p_j, 1\}$. Besides this formula, it is possible that Δ contains formulas of the form $\varphi = 1$ or $\psi = 1 \vee \cdots \vee 1 = 1^{\vee l}$. For φ we can apply $1L$ immediately to get rid of it and for ψ, we would first need to apply $\vee L$ precisely l times and then apply $1L$ to all the resulting sequents to get rid of the 1's from them.

Then this sequent is unprovable when one of the conditions above does not hold, i.e.:

1. Δ is a single conjunction containing at least one propositional variable $p_m \notin \{p_i, \ldots, p_j\}$.
2. Δ contains at least two formulas that are not equal to 1 or $1^{\vee l}$.

If the first holds, we may copy our reasoning from Section 4.4.1, and we find that $\|X(\varphi)\| \leq 2$ for any φ appearing as a succedent in $\mathcal{T}$. In the second case, we apply Sumnorm to 0 and all propositional variables appearing in the antecedent and we find that the norm of the X-set for any succedent φ is bounded by the maximum of these Sumnorms.

For falsehood, we first consider the sequent

$$\Delta \Rightarrow 0 \vee \cdots \vee 0 = 0^{\vee l}.$$

Using the rule $\vee L$, we find that this sequent is provable when $\Delta \Rightarrow 0$ is provable. This is the case when Δ is a single conjunction in which each component proves 0, together with possible formulas such as $\varphi = 1$, $\psi = 1^{\vee \lambda}$ for $\lambda \geq 1$. Since $\Delta \Rightarrow 0$ if and only if $\Delta = 0$, we need that

$$\Delta = 0^{\vee k}, (1, \ldots, 1, 1^{\vee \lambda_1}, \ldots, 1^{\vee \lambda_n}),$$

where $k \geq 0$ and $0^{\vee 0} = 0$. We will call the formulas of the form $\{1, 1^{\vee \lambda}\}$ 1-formulas. We have unprovability in the case where the disjunction $0^{\vee k}$ also contains propositional variables or 1, or when there is more than one formula $\varphi \in \Delta$ that is not of the form $1, 1^{\vee \lambda}$ for $\lambda \geq 1$.

First suppose Δ only contains a single disjunction besides the 1-formulas. Since any succedent appears with the same set of antecedents, we find that our base sequent only appears with antecedents with norm 1. There must still be provable sequents in the case where one component of the disjunction is a 0, in which case $\|X(\varphi)\| = 0$ for any succedent φ. If no 0 appears as a component of the disjunction, there are no provable sequents in $\mathcal{T}$, so $\|X(\varphi)\| = 1$. We hence find that

$$\|X(\varphi)\| \leq 2,$$

for any φ that appears as a succedent in $\mathcal{T}$.

If Δ contains at least two formulas φ not of the form $1, 1^{\vee \lambda}$, we apply Sumnorm to all propositional variables in the antecedent of the base sequent and 0, only considering those formulas where there is no component equal to 1, using the same reasoning as for truth. The norm of $X(\varphi)$ is bounded by the maximum of the Sumnorms for any succedent φ.

Now consider the sequent

$$\Delta \Rightarrow p_i^{\vee \lambda_i} \vee \cdots \vee p_j^{\vee \lambda_j} \vee 0^{\lambda}.$$

This sequent is provable if

1. $\Delta \in \{p_i, \dots, p_j, 0\}$, possibly together with 1-formulas.
2. Δ is a single disjunction with only components from the set $\{p_i, \dots, p_j, 0\}$, possibly together with 1-formulas.

For unprovability, we consider the cases:

1. Δ is a propositional variable not equal to $\{p_i, \dots, p_j, 0\}$.
2. Δ is a disjunction with a component not equal to $\{p_i, \dots, p_j, 0\}$.
3. Δ contains at least two formulas that are not 1-formulas.

In the first case, there are no provable sequents in $\mathcal{T}$, so $\|X(\varphi)\| = 1$ for any succedent φ. In the second case, there is at least one propositional variable appearing in the antecedent that does not appear in the succedent. Without loss of generality, assume this is p_m. If one of the components of the disjunction is an element from $\{p_i, \dots, p_j, 0\}$, there will still be a provable sequent in $\mathcal{T}$, so then $\|X(\varphi)\| = 1$. Otherwise, there will be no provable sequents in $\mathcal{T}$ and hence $\|X(\varphi)\| = 1$. So we conclude

$$\|X(\varphi)\| \leq 2,$$

where φ is any formula that appears as a succedent in $\mathcal{T}$.

In the third case, we apply Sumnorm to the propositional variables in the antecedent of the base sequent and 0, only considering those formulas that do not have 1 as a component. The norm of $X(\varphi)$, for any succedent φ, is bounded by the maximum over all Sumnorms.

5 Improving the bound

In this section we will explore some ideas on how to improve the bounds that we have found in Section 4. These ideas have not been worked out in great detail, but may serve as a basis for further research. We will first investigate how much the norm of the base sequent influences the bound of the norm, especially in the fragments of fusion and implication. In the other section we propose a different proof search tree $\mathcal{P}$, that is finite. We will show that the proof search tree $\mathcal{P}$ is related to the proof search tree $\mathcal{T}$.

5.1 Norm of base sequent

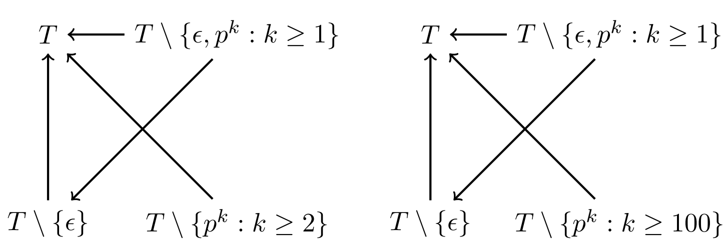
As we have seen in Section 4, the bound on the norm of X-sets in the fragments for fusion and implication possibly includes the norm of the base sequent. However, the norm of the base antecedent does not always accurately describe the size of the countermodel. For example, the sequents $p^2 \Rightarrow p$ and $p^{100} \Rightarrow p$ are both unprovable. By our reasoning above, we have that $\|X(p)\| \leq 2$ for the first, but $\|X(p)\| \leq 100$ for the second, but their actual countermodels may not reflect these numbers.

This brings up the question: For two multisets $\Gamma_1 \trianglelefteq \Gamma_2$ such that $\Gamma_1 \Rightarrow \varphi$ and $\Gamma_2 \Rightarrow \varphi$ are not provable, are the sizes of their countermodel related and if so, how?

Example 16. Coming back to our example, consider the two sequents:

$$\Rightarrow p^2 \to p \text{ and } \Rightarrow p^{100} \to p$$

Their respective countermodels are



Note that while the elements of the countermodel are not the same, they do have the same size. △

Conjecture 5.1. *Let* Γ_1, Γ_2 *be two multisets such that* $\Gamma_1 \Rightarrow \varphi$ *and* $\Gamma_2 \Rightarrow \varphi$ *are unprovable. If* $\Gamma_1 \trianglelefteq \Gamma_2$*, then the size of the size of the countermodel of* $\Gamma_2 \Rightarrow \varphi$ *is related to the size of the countermodel of* $\Gamma_1 \Rightarrow \varphi$*, depending on how many contractions need to be applied to get from* Γ_2 *to* Γ_1*.*

5.2 Finite proof search

Assuming that our formula χ has at least one implication in it, the sequents in our proof search have a nonempty antecedent. Because of this, we can always apply contraction at every step of the exhaustive proof search. This makes our proof tree infinite. Kripke [12] identified that we only need to apply a small number of contractions before applying any other rule. This creates a new calculus, called FL'_{ec}.

5.2.1 FL'_{ec}

To create this new calculus, we define a rule system without contraction as a stand-alone rule. The explicit contraction rule is replaced by a fixed amount of contractions baked-in into each logical rule. To determine how many implicit contractions to add, we need to examine which formulas can get "lost" applying each rule.

Consider the rule $\cdot L$ from FL_{ec}. If $\alpha \cdot \beta \notin \Gamma$, then it might be absent from the premise, but if we first apply a contraction upwards (to get $\Gamma, \alpha \cdot \beta, \alpha \cdot \beta \Rightarrow \varphi$) and only then applying $\cdot L$ upwards to get $\Gamma, \alpha \cdot \beta, \alpha, \beta \Rightarrow \varphi$, this problem is solved. In addition to $\cdot L$, we add the rule $\cdot L_1$ to FL'_{ec}:

$$\frac{\Gamma, \alpha \cdot \beta, \alpha, \beta \Rightarrow \varphi}{\Gamma, \alpha \cdot \beta \Rightarrow \varphi}\ \cdot L_1$$

Now consider the rule $\to L$ from FL_{ec}. In addition to $\alpha \to \beta$, the formulas not in $\Gamma_1 \cap \Gamma_2$ could be missing from the premises. Again, we can fix this by first applying a number of contractions upwards and only then applying $\to L$. We get the following variations of $\to L$ in FL'_{ec}:

$$\frac{\Gamma_1, \Delta \Rightarrow \alpha \qquad \Gamma_2, \Delta, \beta \Rightarrow \varphi}{\Gamma_1, \Delta, \Gamma_2, \alpha \to \beta \Rightarrow \varphi}\ \to L \qquad \frac{\Gamma_1, \Delta, \alpha \to \beta \Rightarrow \alpha \qquad \Gamma_2, \Delta, \beta \Rightarrow \varphi}{\Gamma_1, \Delta, \Gamma_2, \alpha \to \beta \Rightarrow \varphi}\ \to L_1$$

$$\frac{\Gamma_1, \Delta \Rightarrow \alpha \qquad \Gamma_2, \Delta, \alpha \to \beta, \beta \Rightarrow \varphi}{\Gamma_1, \Delta, \Gamma_2, \alpha \to \beta \Rightarrow \varphi}\ \to L_2 \qquad \frac{\Gamma_1, \Delta, \alpha \to \beta \Rightarrow \alpha \qquad \Gamma_2, \Delta, \alpha \to \beta, \beta \Rightarrow \varphi}{\Gamma_1, \Delta, \Gamma_2, \alpha \to \beta \Rightarrow \varphi}\ \to L_3$$

The whole sequent calculus can be seen in Figure 8. The notation $A^?$ means that this occurrence of A may be omitted or retained in each premise independently of the choice made in other premises; the notation $A^{??}$ means that the occurrence of A may be omitted or retained and the same choice must be made in every premise of the rule instance.

$$\frac{}{0 \Rightarrow \epsilon}\ 1\mathrm{R} \qquad \frac{}{\epsilon \Rightarrow 1}\ 0\mathrm{L} \qquad \frac{\Gamma \Rightarrow \varphi}{\Gamma, 1 \Rightarrow \varphi}\ 1\mathrm{L} \qquad \frac{\Gamma \Rightarrow \epsilon}{\Gamma \Rightarrow 0}\ 0\mathrm{R} \qquad \frac{}{\alpha \Rightarrow \alpha}\ AX \qquad \frac{\alpha, \alpha, \Gamma \Rightarrow \varphi}{\alpha, \Gamma \Rightarrow \varphi}\ c$$

$$\frac{\Gamma, \alpha, (\alpha \vee \beta)^{??} \Rightarrow \varphi \qquad \Gamma, \beta, (\alpha \vee \beta)^{??} \Rightarrow \varphi}{\Gamma, \alpha \vee \beta \Rightarrow \varphi}\ \vee L \qquad \frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \alpha \vee \beta}\ \vee R1 \qquad \frac{\Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \vee \beta}\ \vee R2$$

$$\frac{\Gamma, \alpha, (\alpha \wedge \beta)^{?} \Rightarrow \varphi}{\Gamma, \alpha \wedge \beta \Rightarrow \varphi}\ \wedge L1 \qquad \frac{\Gamma, \beta, (\alpha \wedge \beta)^{?} \Rightarrow \varphi}{\Gamma, \alpha \wedge \beta \Rightarrow \varphi}\ \wedge L2 \qquad \frac{\Gamma \Rightarrow \alpha \qquad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta}\ \wedge R$$

$$\frac{\Gamma, \alpha, \beta, (\alpha \cdot \beta)^{?} \Rightarrow \varphi}{\Gamma, \alpha \cdot \beta \Rightarrow \varphi}\ \cdot L \qquad \frac{\Gamma_1 \Rightarrow \alpha \qquad \Gamma_2 \Rightarrow \beta}{\Gamma_1, \Gamma_2 \Rightarrow \alpha \cdot \beta}\ \cdot R$$

$$\frac{\Gamma_1, \Delta, (\alpha \to \beta)^{?} \Rightarrow \alpha \qquad \Gamma_2, \Delta, (\alpha \to \beta)^{?}\beta \Rightarrow \varphi}{\Gamma_1, \Gamma_2, \alpha \to \beta \Rightarrow \varphi}\ \to L \qquad \frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \alpha \to \beta}\ \to R$$

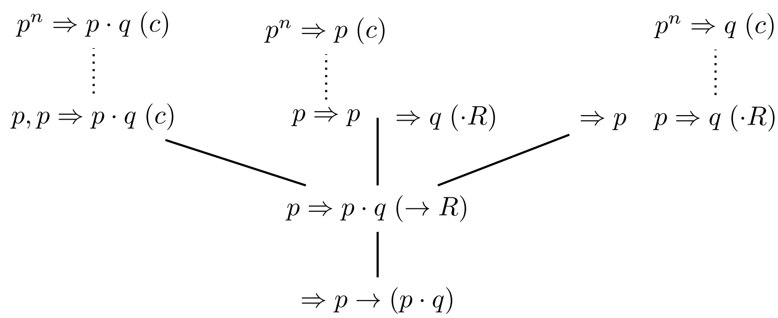
Figure 8: FL'_{ec} sequent calculus

In the system FL'_{ec}, contraction is height-preserving admissible. This means that no sequent in the proof is contractible to a sequent closer to the root. This creates a new proof tree $\mathcal{P}$: we write down a premise of a rule instance in FL'_{ec} only if that premise is not contractible to any sequent closer to the root. In case of rule instances with more than one premise, we only write those that are not contractable to another sequent in the proof. The contractibility ordering on the sequents of $\mathcal{P}$ is a well-quasi ordering, as defined in Definition 1.3. Since all sequents must be incomparable by construction, every branch of the proof tree is a bad sequence and is hence finite. Since only finitely many rules can be applied upwards to any sequent, the proof tree is finitely branching, and hence by König's Lemma, the prooftree $\mathcal{P}$ of any sequent $\Rightarrow \chi$ is finite.

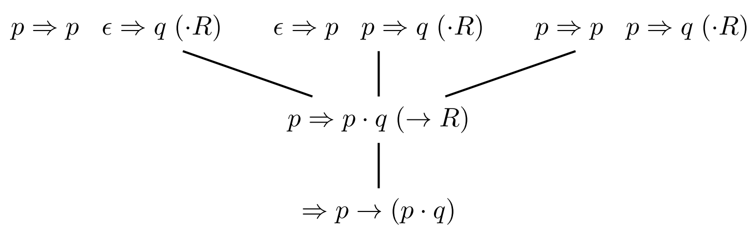
Example 17. Now that we have defined a new proof tree $\mathcal{P}$, let us compare the proof trees $\mathcal{T}$ and $\mathcal{P}$ for some sequents. We have seen the (infinite) proof tree $\mathcal{T}$ for the sequent $\Rightarrow p \to (p \wedge q)$ in Example 3. Its finite proof tree $\mathcal{P}$ looks like

$$\dfrac{\dfrac{p \Rightarrow p \qquad p \Rightarrow q}{p \Rightarrow p \wedge q}\,\wedge R}{p \to (p \wedge q)}\,\to R$$

Let us now consider a more difficult example, looking at the proof search trees for the sequent $\Rightarrow p \to (p \cdot q)$. The proof search tree $\mathcal{T}$ is a lot less compact. Since we can not depict an infinite tree, we do not show the application of $\cdot R$ after the first contraction of p.

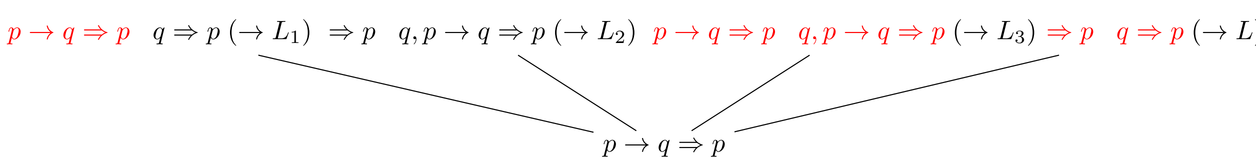


The proof search tree $\mathcal{P}$ looks like:



△

Example 18. Let us now consider a sequent where a left rule needs to be applied: $p \to q \Rightarrow p$. Since we now have 4 different left rules for $\to L$, one might expect a larger proof tree $\mathcal{P}$. However, the constriction on the contractibility of the sequents plays a big role in the size of the proof tree. Below you can find a depiction of the applications of all the four $\to L$ rules, with sequents that would not be written down in red.



Clearly, we can not apply any rules to the sequents $q \Rightarrow p$ and $\Rightarrow p$. As it turns out, the application of any left implication rule, with any choice of $\Gamma_1, \Gamma_2, \Delta$, results in two sequents that are contractible to another sequent in the proof. Hence this is the finished proof search tree. △

5.2.2 Relation between $\mathcal{P}$ and $\mathcal{T}$

Now that we have defined both the infinite proof search tree $\mathcal{T}$ and the finite proof search tree $\mathcal{P}$, we would like to relate them.

Theorem 5.2. *$\Gamma \Rightarrow \psi \in \mathcal{T}$ if and only if $\exists \Delta \Rightarrow \psi \in \mathcal{P}$ such that $\Delta \trianglelefteq \Gamma$.*

Proof. From right to left is trivial as $\mathcal{P} \subset \mathcal{T}$. From left to right is done by induction on the depth of the node in $\mathcal{T}$. The base case is the root of the proof and since $\mathcal{T}$ and $\mathcal{P}$ have the same root, the claim holds. For the induction hypothesis, suppose that the claim holds for a sequent $\Psi \Rightarrow \psi$ at depth N in $\mathcal{T}$ that is the conclusion of an instance σr of a rule $r \in \mathrm{FL}_{ec}$ with premises $\{\Psi_1 \Rightarrow \psi_1\}, \{\Psi_2 \Rightarrow \psi_2\}$.

We will show that for every rule instance σr of a rule $r \in \mathrm{FL}_{ec}$, the claim also holds for the premises of that rule. We will assume here that every premise is written down. Suppose it was not: If the claim holds for a sequent $\Delta \Rightarrow \psi$, but it was not written down because there is a sequent $\Delta' \Rightarrow \psi$ such that $\Delta' \trianglelefteq \Delta$, then it follows that $\Delta' \trianglelefteq \Delta \trianglelefteq \Gamma$, so the claim still holds.

For contraction, suppose the claim holds for $\Gamma, \alpha \Rightarrow \beta$. We want to show the claim also holds for $\Gamma, \alpha, \alpha \Rightarrow \beta$. By the induction hypothesis, $\exists \Delta \Rightarrow \beta \in \mathcal{P}$ such that $\Delta \trianglelefteq \Gamma, \alpha$. Then $\Delta \trianglelefteq \Gamma, \alpha \trianglelefteq \Gamma, \alpha, \alpha$, so the claim holds.

The rules $\vee R1$ and $\vee R2$ are very similar. Suppose the claim holds for $\Gamma \Rightarrow \alpha \vee \beta$. We want to show the claim holds for $\Gamma \Rightarrow \alpha$ respectively $\Gamma \Rightarrow \beta$. By the induction hypothesis $\exists \Delta \Rightarrow \alpha \vee \beta \in \mathcal{P}$ such that $\Delta \trianglelefteq \Gamma$. If we now apply the FL'_{ec} rules $\vee R1$ and $\vee R2$ respectively, we get the sequents $\Delta \Rightarrow \alpha$ and $\Delta \Rightarrow \beta$ respectively in $\mathcal{P}$, for which we have $\Delta \trianglelefteq \Gamma$, proving the claim.

For the rule $\rightarrow R$, suppose that the claim holds for $\Gamma \Rightarrow \alpha \rightarrow \beta$. We want to show the claim holds for $\Gamma, \alpha \Rightarrow \beta$. By the induction hypothesis $\exists \Delta \Rightarrow \alpha \rightarrow \beta \in \mathcal{P}$ such that $\Delta \trianglelefteq \Gamma$. If we now apply the rule $\rightarrow R$ from FL'_{ec}, we get the sequent $\Delta, \alpha \Rightarrow \beta$, and since $\Delta \trianglelefteq \Gamma$, we have that $\Delta, \alpha \trianglelefteq \Gamma, \alpha$, so $\Delta, \alpha \Rightarrow \beta$ proves the claim.

For the rule $\wedge R$, suppose the claim holds for $\Gamma \Rightarrow \alpha \wedge \beta$. We want to show the claim holds for both $\Gamma \Rightarrow \alpha$ and $\Gamma \Rightarrow \beta$. By the induction hypothesis $\exists \Delta \Rightarrow \alpha \wedge \beta \in \mathcal{P}$ such that $\Delta \trianglelefteq \Gamma$. If we now apply the FL'_{ec} rule $\wedge R$, we get the sequents $\Delta \Rightarrow \alpha$ and $\Delta \Rightarrow \beta$ in $\mathcal{P}$, for which we have $\Delta \trianglelefteq \Gamma$, proving the claim.

For the rule $\cdot R$, we will need to use the following Lemma:

Lemma 5.3. *Suppose $\Delta \trianglelefteq \Gamma_1, \Gamma_2$ Then $\exists \Delta_1, \Delta_2, \Delta'$ such that*

1. $\Delta = \Delta_1, \Delta_2, \Delta'$.
2. $\Delta_1, \Delta' \trianglelefteq \Gamma_1$.
3. $\Delta_2, \Delta' \trianglelefteq \Gamma_2$.

Suppose the theorem holds for $\Gamma_1, \Gamma_2 \Rightarrow \alpha \cdot \beta$. We want to show it also holds for both premises $\Gamma_1 \Rightarrow \alpha$ and $\Gamma_2 \Rightarrow \beta$. By the induction hypothesis, $\exists \Delta \Rightarrow \alpha \cdot \beta$ such that $\Delta \trianglelefteq \Gamma_1, \Gamma_2$. By Lemma 5.3, $\exists \Delta_1, \Delta_2, \Delta'$ such that $\Delta = \Delta_1, \Delta_2, \Delta'$ and $\Delta_1, \Delta' \trianglelefteq \Gamma_1$ and $\Delta_2, \Delta' \trianglelefteq \Gamma_2$. So we get the sequent $\Delta_1, \Delta_2, \Delta' \Rightarrow \alpha \cdot \beta \in \mathcal{P}$. Applying the rule $\cdot R$ from FL'_{ec}, we get the sequents $\Delta_1, \Delta' \Rightarrow \alpha$ and $\Delta_2, \Delta' \Rightarrow \beta$. Since $\Delta_1, \Delta' \trianglelefteq \Gamma_1$ and $\Delta_2, \Delta' \trianglelefteq \Gamma_2$, these sequents prove the claim.

For the rules $\wedge L1$ and $\wedge L2$, suppose the claim holds for $\Gamma, \alpha \wedge \beta \Rightarrow \varphi$. We want to show the claim also holds for $\Gamma, \alpha \Rightarrow \varphi$ and $\Gamma, \beta \Rightarrow \varphi$ respectively. By the induction hypothesis, $\exists \Delta \Rightarrow \varphi \in \mathcal{P}$ such that $\Delta \trianglelefteq \Gamma, \alpha \wedge \beta$. Since Δ thus must contain a copy of $\alpha \wedge \beta$, we can write $\Delta = \Delta', \alpha \wedge \beta \trianglelefteq \Gamma, \alpha \wedge \beta$. To proceed, we need the following Lemma.

Lemma 5.4. *Suppose $\Delta, \varphi \trianglelefteq \Gamma, \varphi$. Let $\Delta(\varphi)$ be the multiplicity of φ in Δ. Then*

1. $\Delta(\varphi) = \Gamma(\varphi) \Rightarrow \Delta \trianglelefteq \Gamma$.
2. $\Delta(\varphi) < \Gamma(\varphi) \Rightarrow \Delta, \varphi \trianglelefteq \Gamma$

We have two cases. In the first case, $\Delta'(\alpha \wedge \beta) = \Gamma(\alpha \wedge \beta)$. From Lemma 5.4.1, we get that $\Delta' \trianglelefteq \Gamma$. We get the sequent $\Delta', \alpha \wedge \beta \Rightarrow \varphi \in \mathcal{P}$, so we can apply the version of $\wedge L1$ respectively $\wedge L2$ with no extra $\alpha \wedge \beta$ in the premise to get the sequent $\Delta', \alpha \Rightarrow \varphi$ and $\Delta', \beta \Rightarrow \varphi$ respectively. These sequents prove the claim.

In the second case, $\Delta'(\alpha \wedge \beta) < \Gamma(\alpha \wedge \beta)$. From Lemma 5.4.2, we get that $\Delta', \alpha \wedge \beta \trianglelefteq \Gamma$. We get the sequent $\Delta', \alpha \wedge \beta \Rightarrow \varphi \in \mathcal{P}$, so we can apply the version of $\wedge L1$ respectively $\wedge L2$ with an extra $\alpha \wedge \beta$ in the premise to get the sequents $\Delta', \alpha \wedge \beta, \alpha \Rightarrow \varphi$ and $\Delta', \alpha \wedge \beta, \beta \Rightarrow \varphi$ respectively. These sequents prove the claim.

For the rule $\cdot L$, suppose the claim holds for $\Gamma, \alpha \cdot \beta \Rightarrow \varphi$. We want to show the claim also holds for $\Gamma, \alpha, \beta \Rightarrow \varphi$. By the induction hypothesis, $\exists \Delta \Rightarrow \varphi \in \mathcal{P}$ such that $\Delta \trianglelefteq \Gamma, \alpha \cdot \beta$. Since Δ thus must contain a copy of $\alpha \cdot \beta$, we can write $\Delta = \Delta', \alpha \cdot \beta \trianglelefteq \Gamma, \alpha \cdot \beta$.

We have two cases. In the first case, $\Delta'(\alpha \cdot \beta) = \Gamma(\alpha \cdot \beta)$. From Lemma 5.4.1, we get that $\Delta' \trianglelefteq \Gamma$. We get the sequent $\Delta', \alpha \cdot \beta \Rightarrow \varphi \in \mathcal{P}$, so we can apply $\cdot L$ with no extra $\alpha \cdot \beta$ in the premise to get the sequent $\Delta', \alpha, \beta \Rightarrow \varphi$. This sequent proves the claim.

In the second case, $\Delta'(\alpha \cdot \beta) < \Gamma(\alpha \cdot \beta)$. From Lemma 5.4.2, we get that $\Delta', \alpha \cdot \beta \trianglelefteq \Gamma$. We get the sequent $\Delta', \alpha \cdot \beta \Rightarrow \varphi \in \mathcal{P}$, so we can apply $\cdot L$ with an extra $\alpha \cdot \beta$ in the premise to get the sequent $\Delta', \alpha \cdot \beta, \alpha, \beta \Rightarrow \varphi$. This sequent proves the claim.

For the rule $\vee L$, suppose the claim holds for $\Gamma, \alpha \vee \beta \Rightarrow \varphi$. We want to show the claim also holds for $\Gamma, \alpha \Rightarrow \varphi$ and $\Gamma, \beta \Rightarrow \varphi$. By the induction hypothesis, $\exists \Delta \Rightarrow \varphi \in \mathcal{P}$ such that $\Delta \trianglelefteq \Gamma, \alpha \vee \beta$. Since Δ thus must contain a copy of $\alpha \vee \beta$, we can write $\Delta = \Delta', \alpha \vee \beta \trianglelefteq \Gamma, \alpha \vee \beta$.

We have two cases. In the first case, $\Delta'(\alpha \vee \beta) = \Gamma(\alpha \vee \beta)$. From Lemma 5.4.1, we get that $\Delta' \trianglelefteq \Gamma$. We get the sequent $\Delta', \alpha \vee \beta \Rightarrow \varphi \in \mathcal{P}$, so we can apply the version of $\vee L1$ respectively $\vee L2$ with no extra $\alpha \vee \beta$ in the premise to get the sequent $\Delta', \alpha \Rightarrow \varphi$ and $\Delta', \beta \Rightarrow \varphi$ respectively. These sequents prove the claim.

In the second case, $\Delta'(\alpha \vee \beta) < \Gamma(\alpha \vee \beta)$. From Lemma 5.4.2, we get that $\Delta', \alpha \vee \beta \trianglelefteq \Gamma$. We get the sequent $\Delta', \alpha \vee \beta \Rightarrow \varphi \in \mathcal{P}$, so we can apply the version of $\vee L1$ respectively $\vee L2$ with an extra $\alpha \vee \beta$ in the premise to get the sequent $\Delta', \alpha \vee \beta, \alpha \Rightarrow \varphi$ and $\Delta', \alpha \vee \beta, \beta \Rightarrow \varphi$ respectively. These sequents prove the claim.

For the rule $\to L$, assume that the claim holds for $\Gamma_1, \Gamma_2, \alpha \to \beta \Rightarrow \varphi$. We want to show that the claim holds for both $\Gamma_1 \Rightarrow \alpha$ and $\Gamma_2, \beta \Rightarrow \varphi$. By the induction hypothesis, $\exists \Delta \Rightarrow \varphi \in \mathcal{P}$ such that $\Delta \trianglelefteq \Gamma_1, \Gamma_2, \alpha \to \beta$. By Lemma 5.3, we can write $\Delta = \Delta_1, \Delta_2, \Delta'$ with $\Delta_1, \Delta' \trianglelefteq \Gamma_1, \alpha \to \beta$ and $\Delta_2, \Delta' \trianglelefteq \Gamma_2$. We consider 4 cases:

1. $\Gamma_1(\alpha \to \beta) = \Gamma_2(\alpha \to \beta) = 0$.
2. $\Gamma_1(\alpha \to \beta) > 0, \Gamma_2(\alpha \to \beta) = 0$.
3. $\Gamma_1(\alpha \to \beta) = 0, \Gamma_2(\alpha \to \beta) > 0$.
4. $\Gamma_1(\alpha \to \beta) > 0, \Gamma_2(\alpha \to \beta) > 0$.

Case 1: Δ_2, Δ' contain no $\alpha \to \beta$, so $\Delta_1(\alpha \to \beta) = 1$. We can thus write $\Delta_1 = \Delta_1', \alpha \to \beta$. So we have $\Delta_1', \alpha \to \beta, \Delta' \trianglelefteq \Gamma_1, \alpha \to \beta$. Since $\Delta_1'(\alpha \to \beta) = \Gamma_1(\alpha \to \beta) = 0$, Lemma 5.4.1 implies that $\Delta_1', \Delta \trianglelefteq \Gamma_1$.

We now have the sequent $\Delta_1', \Delta_2, \Delta', \alpha \to \beta \Rightarrow \varphi \in \mathcal{P}$. If we now apply the version of $\to L$ without extra $\alpha \to \beta$ in the premises, we get the sequents $\Delta_1', \Delta' \Rightarrow \alpha$ and $\Delta_2, \Delta', \beta \Rightarrow \varphi$. Since $\Delta_1', \Delta \trianglelefteq \Gamma_1$ and $\Delta_2, \Delta' \trianglelefteq \Gamma_2$, these sequents prove the claim.

Case 2: Δ_2, Δ' contain no $\alpha \to \beta$, so $\Delta_1(\alpha \to \beta) \geq 1$. We can write $\Delta_1 = \Delta_1', \alpha \to \beta$, but in this case $\Delta_1'(\alpha \to \beta) \geq 0$. If $\Delta_1'(\alpha \to \beta) = 0$, then $\Delta_1', \Delta'(\alpha \to \beta) < \Gamma_1(\alpha \to \beta)$, so $\Delta_1', \Delta, \alpha \to \beta \trianglelefteq \Gamma_1$ by Lemma 5.4.1. If $\Delta_1'(\alpha \to \beta) > 0$, we either have that $\Delta_1'(\alpha \to \beta) < \Gamma_1(\alpha \to \beta)$ or $\Delta_1'(\alpha \to \beta) = \Gamma_1(\alpha \to \beta)$. For the former, we have that $\Delta_1', \Delta', \alpha \to \beta \trianglelefteq \Gamma_1$ and for the latter we have that $\Delta_1', \Delta' \trianglelefteq \Gamma_1$, using Lemma 5.4. Using $\Delta = \Delta_1, \Delta_2, \Delta' = \Delta_1', \Delta_2, \Delta', \alpha \to \beta$, we get the sequent $\Delta_1', \Delta_2\Delta', \alpha \to \beta \Rightarrow \varphi \in \mathcal{P}$.

For the cases where $\Delta_1', \Delta, \alpha \to \beta \trianglelefteq \Gamma_1$, we apply the version of $\to L$ with an extra $\alpha \to \beta$ in the left premise to get the sequents

$$\Delta_1', \Delta', \alpha \to \beta \Rightarrow \alpha, \qquad \Delta_2, \Delta', \beta \Rightarrow \varphi.$$

Since $\Delta_1', \Delta', \alpha \to \beta \trianglelefteq \Gamma_1$ and $\Delta_2, \Delta' \trianglelefteq \Gamma_2$, so $\Delta_2, \Delta', \beta \trianglelefteq \Gamma_2, \beta$, these sequents prove the claim.

For the case where $\Delta_1', \Delta' \trianglelefteq \Gamma_1$, we apply the version of $\to L$ without extra $\alpha \to \beta$ in the premises, to get the sequents

$$\Delta_1', \Delta' \Rightarrow \alpha, \qquad \Delta_2, \Delta', \beta \Rightarrow \varphi.$$

Since $\Delta_1', \Delta' \trianglelefteq \Gamma_1$ and $\Delta_2, \Delta' \trianglelefteq \Gamma_2$, so $\Delta_2, \Delta', \beta \trianglelefteq \Gamma_2, \beta$, these sequents prove the claim.

Case 3: As $\Gamma_2(\alpha \to \beta) > 0$, at least one of Δ_2, Δ' contains a copy of $\alpha \to \beta$ and as $\Gamma_1(\alpha \to \beta) = 0$, one of Δ_1, Δ' contains precisely one copy of $\alpha \to \beta$. So we can look at the cases:

1. $\Delta'(\alpha \to \beta) = 1,\ \Delta_1(\alpha \to \beta) = 0,\ \Delta_2(\alpha \to \beta) \geq 0$
2. $\Delta'(\alpha \to \beta) = 0,\ \Delta_1(\alpha \to \beta) = 1,\ \Delta_2(\alpha \to \beta) > 0$

In the first case, we can write $\Delta' = \Delta'', \alpha \to \beta$, with $\Delta''(\alpha \to \beta) = 0$. This implies $\Delta_1, \Delta''(\alpha \to \beta) = \Gamma_1(\alpha \to \beta) = 0$, so we get that $\Delta_1, \Delta'' \trianglelefteq \Gamma_1$ by Lemma 5.4.2 and that $\Delta_2, \Delta'', \alpha \to \beta \trianglelefteq \Gamma_2$. We find the sequent $\Delta_1, \Delta_2, \Delta'', \alpha \to \beta \Rightarrow \varphi \in \mathcal{P}$. If we apply the version of $\to L$ with an extra $\alpha \to \beta$ added to the right premise, we get the sequents

$$\Delta_1, \Delta'' \Rightarrow \alpha, \qquad \Delta_2, \Delta'', \alpha \to \beta, \beta \Rightarrow \varphi,$$

which prove the claim.

In the second case, we can write $\Delta_1 = \Delta_1', \alpha \to \beta$ with $\Delta_1'(\alpha \to \beta) = 0$. This implies $\Delta_1', \Delta' \trianglelefteq \Gamma_1$ by Lemma 5.4.2. If we apply the version of $\to L$ without extra $\alpha \to \beta$ in the premises, we get the sequents

$$\Delta_1', \Delta' \Rightarrow \alpha, \qquad \Delta_2, \Delta', \beta \Rightarrow \varphi,$$

which prove the claim.

Case 4: we consider the subcases

1. $\Delta'(\alpha \to \beta) = 0,\ \Delta_1(\alpha \to \beta) \geq 1,\ \Delta_2(\alpha \to \beta) \geq 1$
2. $\Delta'(\alpha \to \beta) > 0,\ \Delta_1(\alpha \to \beta) \geq 0,\ \Delta_2(\alpha \to \beta) \geq 0$.

In the first case, we can write $\Delta_1 = \Delta', \alpha \to \beta$ to get $\Delta_1', \Delta', \alpha \to \beta \trianglelefteq \Gamma_1, \alpha \to \beta$. We get the sequent $\Delta_1', \Delta_2, \Delta', \alpha \to \beta \Rightarrow \varphi \in \mathcal{P}$. If $\Delta_1'(\alpha \to \beta) = \Gamma_1(\alpha \to \beta)$, then $\Delta_1', \Delta' \trianglelefteq \Gamma_1$ by Lemma 5.4.1. Applying the version of $\to L$ with no extra $\alpha \to \beta$ to get the sequents

$$\Delta_1', \Delta' \Rightarrow \alpha \qquad \Delta_2, \Delta', \beta \Rightarrow \varphi,$$

which prove the claim.

If $\Delta_1'(\alpha \to \beta) < \Gamma_1(\alpha \to \beta)$, then $\Delta_1', \Delta', \alpha \to \beta \trianglelefteq \Gamma_1$ by Lemma 5.4.2. Applying the version of $\to L$ with an extra $\alpha \to \beta$ in the left premise to get the sequents

$$\Delta_1', \Delta', \alpha \to \beta \Rightarrow \alpha, \qquad \Delta_2, \Delta', \beta \Rightarrow \Gamma_2,$$

which prove the claim.

In the second case, we can write $\Delta' = \Delta'', \alpha \to \beta$ to get $\Delta_1, \Delta'', \alpha \to \beta \trianglelefteq \Gamma_1, \alpha \to \beta$ and $\Delta_2, \Delta'', \alpha \to \beta \trianglelefteq \Gamma_2$. If $\Delta_1, \Delta''(\alpha \to \beta) = \Gamma_1(\alpha \to \beta)$, then $\Delta_1, \Delta'' \trianglelefteq \Gamma_1$ by Lemma 5.4.1. If $\Delta_1, \Delta''(\alpha \to \beta) < \Gamma_1(\alpha \to \beta)$, then $\Delta_1, \Delta'', \alpha \to \beta \trianglelefteq \Gamma_1$ by Lemma 5.4.2. We have the sequent $\Delta_1, \Delta_2, \Delta'', \alpha \to \beta \Rightarrow \varphi \in \mathcal{P}$.

In the case where $\Delta_1, \Delta''(\alpha \to \beta) = \Gamma_1(\alpha \to \beta)$, we apply $\to L$ with an extra $\alpha \to \beta$ in the right premise. This gives us the sequents

$$\Delta_1, \Delta'' \Rightarrow \alpha, \qquad \Delta_2, \Delta'', \alpha \to \beta, \beta \Rightarrow \varphi,$$

which prove the claim.

In the case where $\Delta_1, \Delta''(\alpha \to \beta) < \Gamma_1(\alpha \to \beta)$, we apply $\to L$ with an extra $\alpha \to \beta$ in both premises. This gives us the sequents

$$\Delta_1, \Delta'', \alpha \to \beta \Rightarrow \alpha, \qquad \Delta_2, \Delta'', \alpha \to \beta, \beta \Rightarrow \varphi,$$

which prove the claim. □

Using Theorem 5.2, we are able to prove a slightly different version of the finiteness of the countermodel. Recall that the principal open sets are of the form

$$\langle \Delta; \varphi \rangle^c = \{\Gamma \mid (\Gamma, \Delta \Rightarrow \varphi) \in \mathcal{T} \text{ and } \Gamma, \Delta \not\vdash \varphi\}.$$

Using Theorem 5.2, we can rewrite this as

$$\langle \Delta; \varphi \rangle^c = \{\exists \Delta' : \Delta' \trianglelefteq \Gamma, \Delta \text{ and } (\Delta' \Rightarrow \varphi) \in \mathcal{P} \wedge \Gamma, \Delta \not\vdash \varphi\}.$$

We define the set

$$Z(\varphi) = \{\Delta_m \mid \Delta_m \not\vdash \varphi \text{ and } \nexists \Delta \text{ s.t. } \Delta \not\vdash \varphi \wedge \Delta \lhd \Delta_m\}.$$

Let us show that $Z(\varphi)$ is finite. We will start by showing that elements of $Z(\varphi)$ only consist of multisets containing subformulas of χ. Let us assume the contrary, say χ has as subformulas $\varphi_0, \ldots, \varphi_{d-2}, \varphi_{d-1} = \chi$ and Δ_m contains the formula $\psi \neq \varphi_i$ for any $i \in \{0, \ldots, d-1\}$. Then all Γ such that $\Delta, \Gamma \trianglerighteq \Delta_m$ must also contain at least as many copies of ψ as Δ_m does, since we assume Δ only contains subformulas of χ (using $\langle \Delta; \varphi \rangle^c = \{\Gamma \mid (\Gamma, \Delta \Rightarrow \varphi) \in \mathcal{T} \text{ and } \Gamma, \Delta \not\vdash \varphi\}$).

Now, coming back to our definition of $\langle \Delta; \varphi \rangle^c$, we note that in the first set of this intersection, the Γ cannot contain any formulas that are not subformulas of χ, since Δ' must be such that $\Delta' \Rightarrow \varphi \in \mathcal{P}$ and thus contains only subformulas of χ. Hence $\Gamma, \Delta \trianglerighteq \Delta'$ can only be true if Γ contains only subformulas of χ.

In conclusion, Δ_m that contain formulas that are not subformulas of χ will change $\{\Gamma \mid \Gamma, \Delta \not\vdash \varphi\}$, but these Γ will be filtered out in the intersection as they cannot appear in the first set of the intersection. Thus these Δ_m do not change the 'value' of $\langle \Delta; \varphi \rangle^c$, and hence we may assume that all $\Delta_m \in Z(\varphi)$ only contain subformulas of χ.

Hence we have that $Z(\varphi) \subseteq \mathbb{N}^d$. $\mathbb{N}^d$ is known to be countable and subsets of countable sets are also countable, so we conclude that $Z(\varphi)$ is countable. This means we can enumerate the elements as a sequence. If we show this sequence is finite, we are done.

For all $\Delta_m \in Z(\varphi)$, we have $\Delta_m \not\vdash \varphi$. Without loss of generality, let Δ_m and Δ_n be two distinct elements from $Z(\varphi)$. They must be incomparable with respect to $\trianglelefteq$. Otherwise, assume that $\Delta_m \lhd \Delta_n$. This would contradict the minimality of Δ_n. If $\Delta_n \lhd \Delta_m$, it would contradict the minimality of Δ_m. Since all elements must be pairwise incomparable, the sequence of elements from $Z(\varphi)$ a bad sequence w.r.t. the wqo $\trianglelefteq$ and is thus finite.

We can write

$$\begin{aligned}\{\Gamma \mid \Gamma, \Delta \not\vdash \varphi\} &= \{\Gamma \mid \exists \Delta_m \in Z(\varphi) : \Gamma, \Delta \trianglerighteq \Delta_m\} \\ &= \bigcup_{\Delta_m \in Z(\varphi)} \{\Gamma \mid \Gamma, \Delta \trianglerighteq \Delta_m\}.\end{aligned}$$

We now see a similar union as in Section 3.2.2. By the same reasoning as in that section, it can be shown that we have finitely many different principal open sets and our countermodel is of finite size.

Conclusion

In this thesis, we have studied the bounded finite model property for the substructural logic FL_{ec}. More specifically, we studied its fragments for each of the four logical connectives fusion, implication without nesting, disjunction and conjunction. The research showed similar results for fusion and implication, and for conjunction and disjunction. This follows from the fact that the rules from both of these groups share a similar property.

For fusion and implication, applying the the rules $\cdot R$ and $\to L$ upwards causes the antecedent of the sequents to split up into two separate sets. Except for the base sequent, this allowed us to get multisets with norm 1 as antecedents in our proof search tree, which were elements of our X-sets in the case of unprovability. If sequents were provable, we need to consider how many contractions need to be applied before we reach unprovability. In the fusion fragment, we have seen that the maximal norm of antecedents of provable sequents depends on the exponents of the propositional variables in the succedents. The unprovable sequents resulting from this have a maximal norm that is one above these exponents. In the implication fragment without nesting, we have seen that the antecedents of provable sequents always have a minimal norm equal to 1, and are unprovable after a single contraction.

For conjunction and implication, it is the case that for any antecedent appearing in the proof search tree and any succedent appearing in the proof search tree, there will be a sequent in $\mathcal{T}$ with this antecedent and succedent. In the case of a single formula antecedent, the norm of the X-sets depends on whether there are provable sequents in $\mathcal{T}$, which is the case if a propositional variable appears in both the antecedent and succedent of the base sequent. In this case, the norm is bounded by 2 and in the case of no provable sequent, the norm is bounded by 1. In the case of a multi formula antecedent, we had to consider the number of appearances of propositional variables in formulas in the antecedent. Since there are no provable sequents in $\mathcal{T}$ in this case, we only needed to consider a single succedent to get a result for all succedents in $\mathcal{T}$. The addition of 0 and 1 tweaked the result in the sense that formulas with 1 as a component do not contribute to the minimal multisets in the proof search tree.

We have also shorty investigated the cases of implication with level 2 nested formulas with parentheses on the left, and conjunction and disjunction with self-fusion in the antecedent. For implication, our results showed a similar behaviour to the case without nesting. A next step would be to add level 2 nested formulas with parentheses on the right, making the single headed arrows in Figure 7 into double headed arrows. This will allow for more possible cycles, and the expected norm of X-sets is the same as the one we have observed. For conjunction and disjunction, we needed a new definition of Sumnorm, providing us with a similar result as before. A next step could be to also consider the fusion of different propositional variables in the antecedent or self-fusion in the succedent.

In Section 5, we have investigated two concepts that could help us to improve the bounds we have found in Section 4. These have not been worked out in great detail, so they could be a good starting place for future work done on the topic. In the case of norm of the base sequent, we have seen an interesting result in the most basic case with a single propositional variable and one implication. It would be interesting to see if this result also holds more generally. For the finite proof search, studying the norm of $Z(\varphi)$ directly will probably give us similar results to those we have already observed. One could reformulate the construction of the countermodel with the finite proof search in mind to make better use of the ideas in Section 5.2.

The goal of this research would be to find a bound on the size of countermodels for the full logic, and I hope that the techniques and results from this thesis can be a tool in order to achieve this goal.

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