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Extending the Graph Theory of Majority Illusions

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Extending the Graph Theory of Majority Illusions

Master's Thesis

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Contents

Abstract

People tend to make decisions and have opinions that are affected by their local neighbourhood. However, this is not always a correct indicator of the global opinion. If someone sees a different local majority winner from the global majority winner, then they are under majority illusion. In this thesis, we extend the existing graph-theoretical approach on majority illusions to directed graphs (digraphs). A graph consists of agents, connections between agents and colours that represent the agents' opinion. In order to determine the local majority winner for digraphs, each agent considers the majority opinion of their out-neighbourhood. We start by considering the proof approaches that were used for simple graphs, namely adapting results on majority colourings and colour swapping. We show that colour swaps can no longer be used in general digraphs, due to asymmetry of edges. Unlike in undirected graphs, we show that there does not always exist a colouring for all digraphs such that a majority of agents sees a weak-majority illusion. Next, some specific types of classes and constraints are determined for which it is possible that a majority of agents are incorrect about what the majority of agents believe. Furthermore, we find some results regarding quota illusions and plurality illusions when agents in a digraph can choose between more than two alternatives. Lastly, we consider the dynamics of majority illusions. For each agent, their colour during the next time step is determined using a majority threshold among (out-)neighbours. We determined constraints for which majority illusions among a majority of agents will remain and when the illusion disappears. When a majority of agents is under majority illusion, the global majority opinion will change over time when a majority threshold update is applied.

1 Introduction

People are often influenced by their surroundings when they make decisions. If everyone around you likes certain music or votes for a certain political party, then you tend to do the same. However, these local opinions do not always give a realistic view of global opinions. It is common to be friends with people who are similar to you, and therefore your local neighbourhood is unlikely to be a representative sample of the general population.

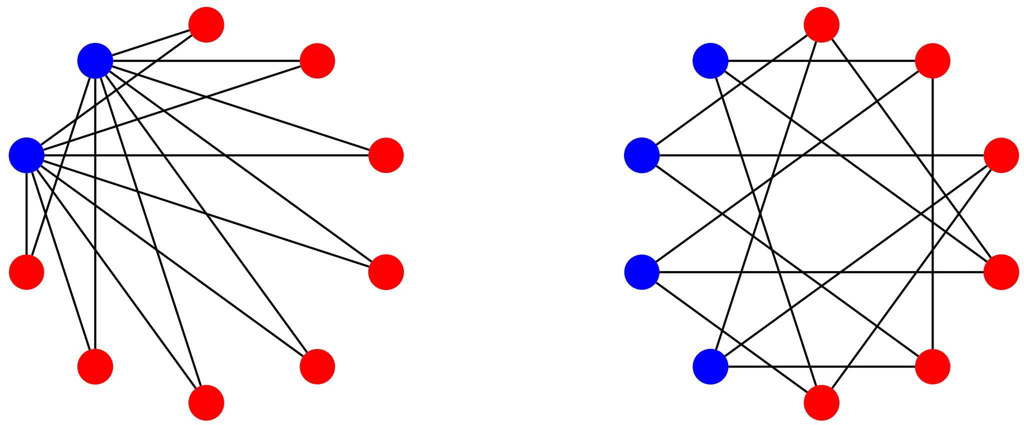
Social networks and their structures are analysed to obtain information on how social phenomena related to opinions and voting can occur. This thesis will focus on majority illusions, which occur when an agent observes a local majority that is different from the global majority (Lerman et al., 2016).

Majority illusions were introduced by Lerman et al. (2016) as a variation of the friendship paradox. Feld's (1991) friendship paradox states that on average people have fewer friends than their friends. Instead of comparing the number of connections that agents have, Lerman et al. (2016) considered how the majority of agents locally can have different opinions or attributes compared to the global reality. Understanding the paradox of friendship has been used, among other practices, to forecast epidemics, predict polling results, and to determine influential nodes to adopt new practices via social contagion (Christakis & Fowler, 2010; Nettasinghe & Krishnamurthy, 2021; Airoldi & Christakis, 2024). Similarly, studying majority illusions will help us understand when this specific network distortion can occur, how to prevent or remove it, and possibly how it can be used to design strategies to combat certain social phenomena.

One of the main focus points of the article by Lerman et al. (2016) is networks that are disassortative in terms of degree. In these networks, high-degree nodes tend to link to nodes with a low degree, which is the case for the friendship paradox. Agents with a high degree impact the observed local opinion of many other agents and thus are over-represented in local opinions. If the high degree nodes have an opinion that is in the minority globally, then this can cause majority illusions. Lerman et al. (2016) used a statistical model to study the occurrence of majority illusions in synthetic and real-life social networks. They found that the influence of high degree nodes and disassortative networks with respect to degree are the main cause of majority illusions and tend to lead to stronger illusions. An example of a degree-disassortative graph where the majority of agents is under majority illusion is shown in Figure 1.

Grandi et al. (2022) focused on computational complexity and determined that checking whether a majority illusion exists and removing it by changing network edges are both NP-complete problems. This research direction has been continued by Grandi et al. (2023), Dippel et al. (2024) and Fioravantes et al. (2024).

Venema-Los et al. (2023) had a different perspective on majority illusions and used a graph-theoretical approach. They considered specific classes of graphs and determined under which circumstance it is possible that there are majority illusions. Whereas Lerman et al. (2016) noticed that nodes with a large degree have a disproportionally large effect on local majorities, Venema-Los et al. (2023) determined that any simple graph can be coloured in such a way that the majority of the agents have a weak-majority illusion. When all nodes have the same degree, the majority of agents can still be under majority illusion. A 3-regular graph with this property can be seen in Figure 1.



(a) A degree-disassortative graph with 10 nodes.

(b) A 3-regular graph with 10 nodes.

Figure 1: Two graphs where the majority of agents is under majority illusion. On the left is shown a degree-disassortative network and on the right is a 3-regular network. In both graphs, the global majority winner is red. All red nodes locally see blue as the winner and thus are under majority illusion.

In a graph, an opinion of an agent is represented by the colour the agent has. Therefore, the field of graph colourings needs to be considered when discussing majority illusions. Especially relevant are weak-majority 2-colourings, where a graph is coloured using two colours in such a way that for each agent at least half their neighbours have the opposite colour from the agent's own colour. In this paper, Section 3 discusses majority illusions in directed graphs.

Kreutzer et al. (2017) determined that every digraph has a weak-majority 4-colouring and then conjectured that every digraph has a weak-majority 3-colouring. In their paper, they worked out that certain classes of graphs under certain assumptions indeed have a weak-majority 3-colouring and discussed some further open problems. This work was continued by Girao et al. (2017); Anholcer et al. (2019); Anastos et al. (2021) and others. The work by Anastos et al. (2021) is especially important for this paper. They determined that any digraph without odd directed cycles is weak-majority 2-colourable. We will use this result in section 3, as it follows that a colouring exists which gives a weak-majority-weak-majority illusion of any digraph without odd directed cycles if at least half the agents do not see a tie as local majority winner or the global majority winner is not a tie.

1.1 Research Questions

This research will work on extending the graph-theoretical framework that is used to discuss majority illusions in social networks. The main focus will be on how directed edges impact the existing theorems found by Venema-Los et al. (2023). Their theorems apply solely to undirected graphs. In a directed graph, edges have a direction that represents a one-sided friendship or information flow. Focusing on directed graphs allows us to apply result to more realistic social networks, because connections are not always bi-directional. This is especially true for social media, where you might follow and be influenced by opinions of one person, but they might not even know of your existence.

Another point of focus in this thesis is the dynamics of majority illusions. Specifically, a majority threshold among neighbours is used to determine the colour of each node in the next time step. We will discuss under which circumstances majority-majority illusions remain and when they disappear.

1.2 Outline

Section 2 introduces the formal definitions and notations relevant within the field of majority illusions. In Section 3 these definitions are adapted for digraphs and then used to determine when majority illusions can occur in digraphs. We start by focusing on general digraphs, then discuss results for specific types of digraphs and eventually discuss other types of illusions when voting between $k > 2$ options. Chapter 4 focuses on dynamics of majority illusions. And lastly, Chapter 5 gives a summary of main contributions and an overview of future work which might be relevant within the field of majority illusions.

2 Theoretical Framework

2.1 Social networks

A social network is represented by a graph $G = \langle V, E \rangle$ with a finite set V of agents (nodes) and a set E of edges between agents. If an agent has an edge to another agent, they consider this agent their neighbour. The set of neighbours of agent i is referred to as N_i and the degree of that agent corresponds to the amount of neighbours the agent has, so $d_i = |N_i|$. Note that in general an agent can be a neighbour of itself. In this thesis, graphs are irreflexive, unweighted and finite. Each agent i has a opinion/colouring c_i on an issue. A coloured graph is represented by $G = \langle V, E, c \rangle$ with nodes V and edges E as described and c the colouring of the entire graph. Unless otherwise specified, we talk about binary opinions and the term coloured graph refers to a binary coloured graph.

2.2 Majority illusion

Now that social networks and different types of graphs are defined, we can look at the definition of majority illusion for both agents and graphs as a whole, as well as their weak and more generalised versions. All of these definitions are based on the ones given in (Venema-Los et al., 2023). These definitions can be applied to graphs which are unweighted, undirected and irreflexive. In Section 3.1.2 these definitions will be adapted so that they can be applied to directed graphs.

In order to understand the formal definition of the majority illusion, we need to discuss the notation and meaning of majority winner. Consider a set S of $n = |S|$ agents and a 2-colouring c. A specific colour $x \in c$ is the majority winner of the set S if more than half of the agents have that colour (so $|\{i \in S | c_i = x\}| > \frac{n}{2}$). If no such colour exists, then the majority winner is said to be a tie. We respectively write $M_S = x$ or $M_S = tie$. To check if an agent is under a majority illusion, you check the local majority winner M_{N_i} and global majority winner M_V. If neither is a tie and they are not the same, then the agent is under a majority illusion. When discussing majority illusions in graphs, we use the term majority-majority illusion and this occurs when more than half (i.e. a majority) of the agents are under majority illusion.

Definition 1 (Majority illusion)**.** [Venema-Los et al. (2023)] Given a coloured graph $C = \langle V, E, c \rangle$, an agent $i \in V$ is under majority illusion if $M_{N_i} \neq tie$ and $M_V \neq tie$ and $M_{N_i} \neq M_V$. A graph is in a majority-majority illusion if more than half of agents are under majority illusion.

This is the strict version of the majority illusion, but for both the agent and the graph a weaker version of the definition is possible. An agent can see a tie in the local majority winner or global majority winner, while those are different from each other. In the graph, exactly half of the agents can be under some form of a majority illusion.

Definition 2 (Weak versions of majority illusions)**.** [Venema-Los et al. (2023)] Given a coloured graph $C = \langle V, E, c \rangle$, agent $i \in V$ is under weak-majority illusion if $M_{N_i} \neq M_V$.
Assume a coloured graph $C = \langle V, E, c \rangle$, then the following four versions of majority illusions can occur:

- Majority-majority illusion: $|\{i \in V | M_{N_i} \neq M_V \text{ and } M_{N_i} \neq tie \text{ and } M_V \neq tie\}| > \frac{|V|}{2}$. Which corresponds to more than half of the agents in the graph being under a strict majority illusion.

- Majority-weak-majority illusion: $|\{i \in V | M_{N_i} \neq M_V\}| > \frac{|V|}{2}$. So, when more than half of the agents are under either a weak or strict majority illusion.
- Weak-majority-majority illusion: $|\{i \in V | M_{N_i} \neq M_V$ and $M_{N_i} \neq tie$ and $M_V \neq tie\}| \geq \frac{|V|}{2}$. At least half of the agents are under a strict majority illusion.
- Weak-majority-weak-majority illusion: $|\{i \in V | M_{N_i} \neq M_V\}| \geq \frac{|V|}{2}$. Thus, at least half of the agents are under either a weak or strict majority illusion.

Definition 2 forms the basis of the theoretical framework for majority-majority illusions which is required to understand this thesis. In Section 3.1, we will see how these definitions need to be adapted to apply to undirected graphs.

2.3 General illusion

Majority voting is a well-known voting rule, but other voting rules can also lead to illusions. Voting rules are defined as a function $\mathcal{R} : S \to 2^C$ which maps a set of agents S and a colouring c to a set of winning colours. A colour is a majority winner of set S when more than half the agents have that colour. However, you might want a colour to only be a winning colour if 75% of the set is that colour. In that case, you want to use the q-quota rule: a colour c is a winner if and only if more than a q-fraction ($0 < q \leq 1$) of nodes have that colour. The q-quota rule can be applied in scenarios with 2 or more colours. Another voting rule that is common in real-life scenarios with multiple voting options is the plurality rule. The plurality rule can informally be explained as the colour(s) with the most votes is/are the plurality winner. In Section 3.5.3 plurality illusions and q-illusions in digraphs will be discussed.

Given a voting rule $\mathcal{R}$ and a k-coloured graph, an agent is under $\mathcal{R}$-illusion when the sets of local winners and global winners according to that voting rule are disjoint and neither set is empty. And the agent is under a weak-$\mathcal{R}$-illusion when the local winner and global winner sets are not the same.

When determining whether a graph is under some illusion, so far majority-majority illusion and its weaker versions have been mentioned. Similarly to how any voting rule $\mathcal{R}$ can be used, it is also possible to use other fractions of agents in the graph. For the majority-majority illusion, more than half of the agents in the graph are under majority illusion. This corresponds to the fraction $p = \frac{1}{2}$, but in general any such fraction $\frac{1}{2} \leq p \leq 1$ can be used to determine whether a graph is under a (weak-)p-(weak-)$\mathcal{R}$-illusion. Given below is its formal definition, which was introduced as definition 5 in the unpublished manuscript of Venema-Los et al. (2025).

Definition 3 ($\mathcal{R}$-illusion)**.** [Venema-Los et al. (2025)] Given a k-colored graph $C = \langle V, E, c\rangle$, agent $i \in V$ is under

- $\mathcal{R}$-illusion for a voting rule $\mathcal{R}$ if $\mathcal{R}(N_i) \cap \mathcal{R}(V) = \emptyset$ and $\mathcal{R}(N_i) \neq \emptyset$ and $\mathcal{R}(V) \neq \emptyset$.
- weak-$\mathcal{R}$-illusion if $\mathcal{R}(N_i) \neq \mathcal{R}(V)$.

A k-coloured graph $C = \langle V, E, c\rangle$ is under

- p-(weak-)$\mathcal{R}$-illusion if more than a p-fraction of the group is under (weak-)$\mathcal{R}$-illusion.
- weak-p-(weak-)$\mathcal{R}$-illusion if at least a p-fraction of the group is under (weak-)$\mathcal{R}$-illusion.

As was the case for the definition of the majority-majority illusion, definition 3 on general illusions can be applied to graphs which are unweighted, undirected, irreflexive and finite. In Section 3.1.3 the definition of $\mathcal{R}$-illusion will be adapted to directed graphs.

2.4 Graph colourings

When determining whether there is a majority illusion, you consider coloured graphs where each agent has a colour based on a vote or opinion they have. Therefore, the field of graph colourings can be used in theorems related to majority illusions, as can be seen in (Venema-Los et al., 2023). Graph colouring most commonly refers to colouring nodes of a graph in a certain way, which is also how it will be used in this paper. In general, a k-colouring is an assignment $c : V \to \{1,...,k\}$. Each node has been given one of the colours $1,...,k$. The notation c_i is used to refer to the colour of node i. Of relevance for majority illusions is the 2-colouring, i.e. when only 2 colours are used to colour the graph. In this thesis, k-colouring refers to any colouring of the nodes of the graph using k colours, this colouring can be proper or improper.

A graph $G = \langle V, E, c\rangle$ is properly 2-coloured when in the assignment $c : V \to \{1,2\}$ for each node $i \in V$ and all its neighbours $j \in N_i$ it holds that $c_i \neq c_j$. Informally, no two adjacent nodes have the same colour, so everyone disagrees with all their neighbours. Any such graph has at least a majority-weak-majority illusion. However, such a graph is not a typical social network. More important in this field is the notion of majority 2-colouring. A majority 2-colouring of a graph $G = \langle V, E, c\rangle$ is a colouring $c : V \to \{1,2\}$ such that for each agent $i \in V$ and all its neighbours $j \in N_i$: $|\{j \in N_i | c_i \neq c_j\}| > \frac{|N_i|}{2}$. Thus, a graph that is coloured in such a way that at least half of the neighbours of each agent have a different colour than the agent's own colour. Note that in the graph colouring literature this definition allows for exactly half of the neighbours having the same colour as the original agent. However, this does not align with the weak and strict majority illusion definitions. Therefore, in this paper, the notion of weak-majority 2-colouring will be used when the graph is coloured in such a way that an agent can have exactly half of its neighbours with the same colour as its own.

An important result is that a weak-majority 2-colouring of a finite, undirected, unweighted, simple graph always exists (Lovász, 1966). The idea is that any 2-colouring of a graph $G = \langle V, E, c\rangle$ that minimises the number of edges between nodes of the same colour is a weak-majority 2-colouring of the graph. Section 3.2 discusses why this result holds for undirected graphs and can no longer be applied to digraphs. Due to the fact that any simple graph has a weak-majority 2-colouring, any simple graph can be in a majority-weak-majority illusion (Venema-Los et al., 2023).

3 Majority Illusions in Digraphs

3.1 Digraphs and defining majority illusions

3.1.1 Digraph definitions

Realistic social networks often do not resemble simple graphs, but rather directed graphs (digraph). A digraph is a graph $G = \langle V, E \rangle$ with a (finite) set V of nodes and a set E of directed edges between nodes. An edge e is an ordered pair (u, v), where u and v are nodes. Read (u, v) as: there is an edge from node u to node v. Informally, agent u considers agent v as a friend.

The definition of degree and neighbours need to be adapted to fit this changed framework. The direction of the edge changes the concept of neighbours to out-neighbours and in-neighbours. If you consider node u, then all nodes v such that $(u, v) \in E$ are out-neighbours N_u^{out} of u. This corresponds to all the nodes that u has an outgoing edge towards. The in-neighbours N_u^{in} of node u are all the nodes v such that $(v, u) \in E$, i.e. all the nodes that u has an incoming edge of.

The degree d_u of a node u in a digraph has also been adapted, it is defined as the addition of the in-degree and the out-degree of that node. The in-degree d_u^{in} refers to the amount of incoming edges. The out-degree d_u^{out} corresponds to the number of outgoing edges.

In this section, multigraphs are not considered, so there cannot be multiple edges going in the same direction between two nodes. Hence, for this section the in-degree of a node u is the number of in-neighbours of u ($d_u^{in} = |N_u^{in}|$) and the out-degree of u is the number of out-neighbours of u ($d_u^{out} = |N_u^{out}|$).

3.1.2 Majority illusions in digraphs

For digraphs, you need to consider how definitions are changed with respect to neighbours. Both colourings and majority illusions refer to neighbours in their definitions. These definitions could be adapted to let neighbours refer to either out-neighbours, in-neighbours, both or even with an entirely new notion. It makes most sense to use the out-neighbours in the updated definitions, as the out-neighbourhood of an agent corresponds to the agents whom the original agent considers as friends and whose opinions are taken into consideration.

This adaptation of using the out-neighbourhood aligns with the changes made to the majority k-colouring definition for digraphs when it was introduced by Kreutzer et al. (2017).

Definition 4 (Majority k-colouring of digraphs)**.** [Kreutzer et al. (2017)] The majority k-colouring of a digraph $G = \langle V, E \rangle$ with k colours is an assignment $c : V \to \{1, ..., k\}$ such that for every node $v \in V$: $|\{w \mid c_v = c_w, w \in N_v^{out}\}| < \frac{|N_v^{out}|}{2}$. For the weak-majority k-colouring the following holds: for every node $v \in V$: $|\{w \mid c_v = c_w, w \in N_v^{out}\}| \leq \frac{|N_v^{out}|}{2}$.

Informally, a weak-majority k-colouring corresponds to a graph which is coloured using k colours such that for each node v the colour of at most half of its out-neighbours has the same colour as v.

For majority illusions, definition 1 (originally definition 1 from (Venema-Los et al., 2023)) is adapted to definition 5 that uses the local majority winners of the out-neighbourhood of node i.

Definition 5 (Majority illusion in digraphs)**.** Given a coloured digraph $C = \langle V, E, c\rangle$, an agent $i \in V$ is under majority illusion if $M_{N_i^{out}} \neq tie$ and $M_V \neq tie$ and $M_{N_i^{out}} \neq M_V$. A graph is in a majority-majority illusion if more than half of agents are under majority illusion.

The change of using out-neighbourhood in the definition also needs to be applied to weak versions of majority illusions. Definition 2 (definition 2 from (Venema-Los et al., 2023)) is adapted to definition 6 given below.

Definition 6 (Weak versions of majority illusions in digraphs)**.** Given a coloured digraph $C = \langle V, E, c\rangle$, agent $i \in V$ is under weak-majority illusion if $M_{N_i}^{out} \neq M_V$.
Digraph C can then have one of the following illusions:

- Majority-majority illusion: $|\{i \in V | M_{N_i}^{out} \neq M_V \text{ and } M_{N_i}^{out} \neq tie \text{ and } M_V \neq tie\}| > \frac{|V|}{2}$. In this case, more than half of the agents in the digraph are under a strict majority illusion.
- Majority-weak-majority illusion: $|\{i \in V | M_{N_i}^{out} \neq M_V\}| > \frac{|V|}{2}$. This corresponds to a digraph in which more than half of the agents are under either a weak or strict majority illusion.
- Weak-majority-majority illusion: $|\{i \in V | M_{N_i}^{out} \neq M_V \text{ and } M_{N_i}^{out} \neq tie \text{ and } M_V \neq tie\}| \geq \frac{|V|}{2}$. At least half of the agents in the digraph are under a strict majority illusion.
- Weak-majority-weak-majority illusion: $|\{i \in V | M_{N_i}^{out} \neq M_V\}| \geq \frac{|V|}{2}$. So, at least half of the agents are under either a weak or strict majority illusion.

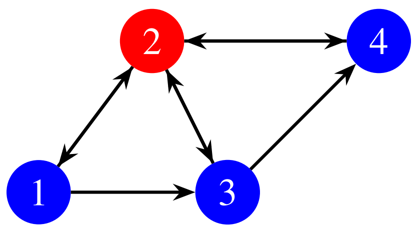


Figure 2: A digraph with 4 nodes for example 1.

Example 1 (Digraphs and majority illusions)**.** *Figure 2 shows a digraph with 4 nodes. This digraph can be formalised as $C = \langle V, E, c\rangle$ with*

- $V = \{1, 2, 3, 4\}$
- $E = \{(1,2), (1,3), (2,1), (2,3), (2,4), (3,2), (3,4), (4,2)\}$
- $c_1 = blue, c_2 = red, c_3 = blue, c_4 = blue.$

The global majority winner M_V is blue, as 3 out of 4 nodes are blue. To determine the local majority winners the out-neighbourhood of each node needs to be considered.

- *Node 1: $N_1^{out} = \{2, 3\}$, which results in a tie for the local majority winner. $M_{N_1^{out}} = tie \neq M_V$. Thus, node 1 is under weak-majority illusion, but not under a strict majority illusion.*

- *Node 2: $N_2^{out} = \{1,3,4\}$. All these nodes are coloured blue, so the local majority winner $M_{N_2^{out}} =$ blue. The local and global majority winner are the same, so node 2 does not have a (weak-)majority illusion.*

- *Node 3: $N_3^{out} = \{4,2\}$, which gives a tie for the local majority winner. $M_{N_3^{out}} = tie \neq M_V$. Thus, node 3 has a weak-majority illusion, but not a strict majority illusion.*

- *Node 4: $N_4^{out} = \{2\}$. This out-neighbour is coloured red, so $M_{N_4^{out}} = red \neq M_V$. Thus, node 4 is under a strict and weak majority illusion.*

In general, more than half the nodes (node 1, 3 and 4) are under at least a weak-majority illusion. Hence, the graph is in a majority-weak-majority illusion.

3.1.3 General illusions in digraphs

Section 2.3 introduced illusions for general voting rules in undirected graphs, which was originally introduced as definition 5 in the unpublished manuscript by Venema-Los et al. (2025). The $\mathcal{R}$-illusion definition for undirected graphs cannot be immediately applied to directed graphs. The definition is adapted by using the out-neighbourhood instead of the neighbourhood of a node, in line with the approach used for updating the definitions of majority illusions and majority colourings.

Definition 7 ($\mathcal{R}$-illusion for digraphs)**.** Given a k-coloured digraph $C = \langle V, E, c \rangle$, agent $i \in V$ is under

- $\mathcal{R}$-illusion for a voting rule $\mathcal{R}$ if $\mathcal{R}(N_i^{out}) \cap \mathcal{R}(V) = \emptyset$ and $\mathcal{R}(N_i^{out}) \neq \emptyset$ and $\mathcal{R}(V) \neq \emptyset$.

- weak-$\mathcal{R}$-illusion if $\mathcal{R}(N_i^{out}) \neq \mathcal{R}(V)$.

A k-coloured digraph $C = \langle V, E, c \rangle$ is under

- p-(weak-)$\mathcal{R}$-illusion if more than a p-fraction of the group is under (weak-)$\mathcal{R}$-illusion.

- weak-p-(weak-)$\mathcal{R}$-illusion if at least a p-fraction of the group is under (weak-)$\mathcal{R}$-illusion.

Definition 7 will be applied in Section 3.5 to define quota illusions and plurality illusions, which are relevant when voting among multiple alternatives.

3.1.4 Nodes without out-neighbours

Venema-Los et al. (2023) worked under the assumption that all nodes would have a neighbour, which is indeed true for most social networks. In digraphs it is more likely that agents do not have out-neighbours, but still not a common occurrence. However, some results related to weak-plurality colourings require us to define the local majority winner $M_{N_i}^{out}$ of an agent $i \in V$ without out-neighbours.

Let us first mention some options that would lead to arbitrary, unwanted and/or unrealistic results:

- Exclude the agent.

- Assume the node is under majority-illusion.

- Assume the node is not under majority-illusion.

- Assume the node sees its own colour as local majority winner.
- Assume the node sees the opposite colour as local majority winner.

We have not mentioned two options so far, which have an underlying reasoning for being a good approach and lead to expected results. Namely, assuming that agents without out-neighbours see a tie as local majority winner or that they see an empty set.

- Assume the node sees a tie: it makes sense to take a non-informative Bayesian prior, i.e. each option has the same prior probability. Hence, agents that have no information will consider the local majority winner a tie. This would result in a weak majority illusion for that agent if the global majority winner $M_V \neq tie$ and no illusions if $M_V = tie$.
- Assume the node sees an empty set: agents do not have any information, because their local neighbourhood is empty. If agents without out-neighbours see an empty set, it makes sense to adapt the $\mathcal{R}-$illusion (definition 7) for majority illusions instead of definition 5. The $\mathcal{R}-$illusion focuses on set comparison and empty sets, whereas definition 5 considers if M_V and $M_{N_i^{out}}$ are equal and whether either is a tie. This requires rewriting the definition of majority-illusion, the new version is given in definition 8. Let us determine what happens to agents without out-neighbours. These agents see an empty set as their local majority winner $M_{N_i^{out}}$, so the agent is not under majority illusion. But, $M_{N_i^{out}} \neq M_V$, so the agent is under weak-majority illusion. This leads to less requirements than when you assume the node sees a tie, because there you need assumptions about the global majority winner M_V not being a tie.

Definition 8 shows a different approach to defining majority illusions. This approach aligns with the general $\mathcal{R}$-illusion and makes use of set-comparison. In appendix A it is shown that definitions 5 and 8 on majority illusions in digraphs are the same (except for notation) if all agents have at least one out-neighbour.

Definition 8 (Majority-illusion in digraphs set-comparison version.)**.** Given a k-coloured digraph $C = \langle V, E, c \rangle$, agent $i \in V$ is under

- majority-illusion if $M_{N_i^{out}} \cap M_V = \emptyset$ and $M_{N_i^{out}} \neq \emptyset$ and $M_V \neq \emptyset$.
- weak-majority-illusion if $M_{N_i^{out}} \neq M_V$.

A k-coloured digraph $C = \langle V, E, c \rangle$ is under

- majority-(weak-)majority-illusion if more than half of the agents is under (weak-)majority-illusion.
- weak-majority-(weak-)majority-illusion if at least a half of the agents is under (weak-)majority-illusion.

Thus, there are two different options for dealing with agents without out-neighbours: either let those agents see a tie as local majority winner or let them see an empty set. Both approaches have a realistic justification and lead to the same results if there are no agents without out-neighbours even though the definition of majority-majority illusion needs to be adapted. In this thesis, we will stick with the Bayesian approach, where both options are equally likely and agents without an out-neighbour see a tie. This approach allows for a more direct comparison to the theorems found by Venema-Los et al. (2023), since definition 1 and 2 on majority-majority illusions for undirected graphs in Venema-Los et al. (2023) align more with definition 5 for digraphs than with definition 8.

3.2 Asymmetry and its effect on colour swaps

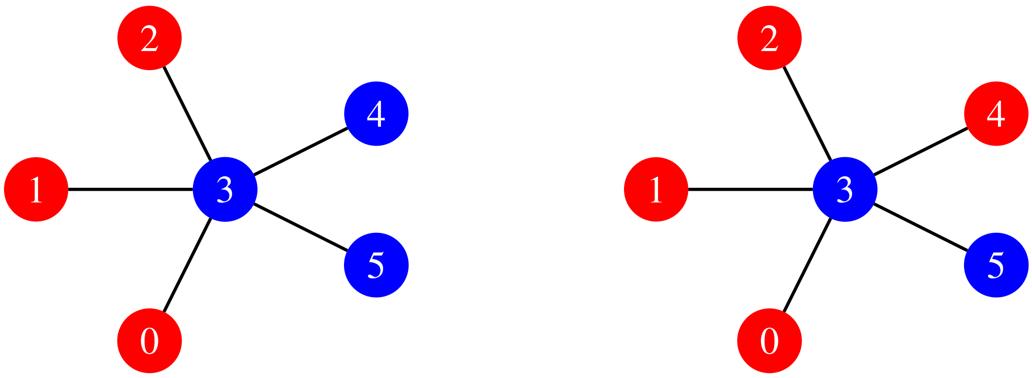
In Theorem 1 of their paper, Venema-Los et al. (2023) showed that for every undirected graph a majority-weak-majority illusion is possible. Their proof made use of colour swaps and existing results on weak-majority 2-colourings. In this section, we will show why the colour swapping approach cannot be applied to digraphs such that results can be proven in the same way as for undirected graphs.

First, recall the definition for monochromatic and dichromatic edges. Monochromatic edges are edges between two nodes with the same colour, while dichromatic edges exist between nodes of different colours. When considering digraphs, it makes sense to consider out-edges, as the definition of majority illusion and majority k-colouring for digraphs makes use of out-neighbours. Node i has a monochromatic out-edge to node j if there exists an edge between i and j and these nodes have the same colour. If there is an edge from i to j, but these nodes are not the same colour, then node i has a dichromatic out-edge to node j.

Theorem 1 of Venema-Los et al. (2023) states that for any undirected graph, a majority-weak-majority illusion is possible. This theorem makes use of the following result attributed to Lovász (1966): every undirected graph is weak-majority 2-colourable. Specifically, any 2-colouring of an undirected graph $G = \langle V, E, c \rangle$ that minimises the amount of monochromatic edges is a weak-majority 2-colouring of the graph. For contradiction, assume that the graph is not weak-majority 2-coloured. Then, there is an agent with strictly more monochromatic edges than dichromatic edges. If you were to swap the colour of that specific agent, then this would reduce the amount of monochromatic edges in the entire graph. This contradicts the assumption that the 2-colouring minimises the total amount of monochromatic edges in the graph. Therefore, any 2-colouring that minimises the number of monochromatic edges in an undirected graph will be a weak-majority 2-colouring of that graph.

Examples 2 and 3 show an undirected graph and a digraph and explain what happens when you apply a colour swap to a node with strictly more monochromatic (out-)edges than dichromatic (out-)edges. First, consider any undirected graph and this explanation on why you can apply colour swaps to eventually reach a weak-majority 2-coloured graph. Let $E(M)$ be the total amount of monochromatic edges, $E_i(M)$ the amount of monochromatic edges for agent $i \in V$ and $E_{V\setminus\{i\}}(M)$ the number of monochromatic edges when excluding all edges to agent i. We have $E(M) = E_{V\setminus\{i\}}(M) + E_i(M)$. Select some agent $i \in V$ that strictly has more monochromatic edges than dichromatic edges. Then, after swapping the colour of agent i, the amount of monochromatic edges connected to agent i will have decreased ($E_i(M) > E'_i(M)$). None of the other nodes changed colour, so the amount of monochromatic edges in the remainder of the graph excluding node i remains the same ($E'_{V\setminus\{i\}}(M) = E_{V\setminus\{i\}}(M)$). Thus, $E'(M) = E'_{V\setminus\{i\}}(M) + E'_i(M) < E_{V\setminus\{i\}}(M) + E_i(M) = E(M)$ and there are strictly less monochromatic edges in the graph than before. During each step, the total amount of monochromatic edges in the graph decreases. Hence, eventually the colour swapping process will terminate when there are no remaining nodes with more monochromatic edges than dichromatic edges. This will give a graph that is weak-majority 2-coloured. However, Example 3 will show that for digraphs the total amount of monochromatic out-edges in the graph is not guaranteed to decrease.

Example 2 (Colour swaps and the total amount of monochromatic edges for undirected graphs)**.** *Consider figure 3, which shows an undirected graph and the same graph after a colour swap is applied to a node with more monochromatic edges than dichromatic edges.*

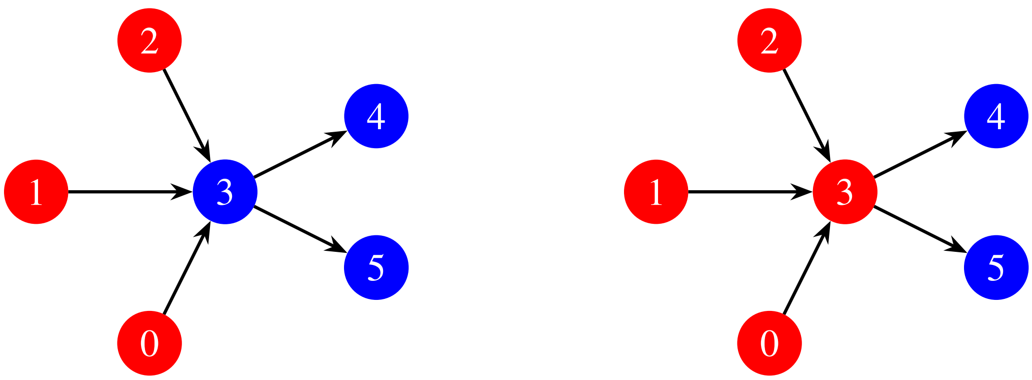


(a) The initial undirected graph. (b) The undirected graph after colour swapping.

Figure 3: An undirected graph before and after colour swapping a node with more monochromatic out-edges than dichromatic out-edges.

The left side of Figure 3 shows an undirected graph where agents 4 and 5 have more monochromatic edges than dichromatic edges. Therefore, the colour of one of these agents is swapped from blue to red (see the right side of Figure 3 with agent 4 colour swapped). The edges in the graph are symmetrical and initially agent 4 had more monochromatic edges than after the colour swap, so the total amount of monochromatic edges in the graph decreases from 2 to 1. In the next step, agent 5 would swap colours to red, and we obtain a (weak-)majority 2-coloured graph.

Example 3 (Colour swaps and the total amount of monochromatic out-edges for digraphs)**.** *Consider figure 3, which shows a digraph and that same digraph after one update step where a node is selected with more monochromatic out-edges than dichromatic out-edges and its colour is changed.*



(a) The initial digraph. (b) The digraph after colour swapping.

Figure 4: A digraph before and after colour swapping a node with more monochromatic out-edges than dichromatic out-edges.

Node 3 has more monochromatic out-edges than dichromatic out-edges. Initially, the total amount of monochromatic out-edges was 2 (edges $(3,5)$ *and* $(3,4)$*). After swapping the colour of node 3, the total number of monochromatic out-edges is 3 (edges* $(0,3)$, $(1,3)$ *and* $(2,3)$*). Due to asymmetry of the edges, the total amount of monochromatic edges in the digraph increases. This indicates that there is no guarantee that eventually a weak-majority 2-colouring can be reached by applying the colour swapping method iteratively.*

Example 3 shows that the total amount of monochromatic out-edges can in fact increase when swapping the colour of a node with strictly more monochromatic out-edges than dichromatic out-edges. Therefore, the colour swapping process might not terminate. There exists an approach that always reduces the total amount of monochromatic edges in the graph, namely:
swap the colour c_i *of agent* $i \in V$ *if (monochromatic in-edges of i + monochromatic out-edges of i)* $>$ *(dichromatic in-edges of i + dichromatic out-edges of i).*

That is, if a node combined has more monochromatic in- and out-edges than dichromatic in- and out-edges, then its colour is swapped. This approach can be applied iteratively and will reach a state where the total amount of monochromatic edges in the graph is locally minimised. However, it is not guaranteed that all agents have less monochromatic out-edges than dichromatic out-edges. Thus, the approach does not lead to the intended result of ensuring that nodes are coloured correctly with respect to the weak-majority 2-colouring. In general, even if an approach can be found to minimise the amount of monochromatic out-edges, the resulting coloured digraph might not be weak-majority 2-coloured as can be seen in Example 4.

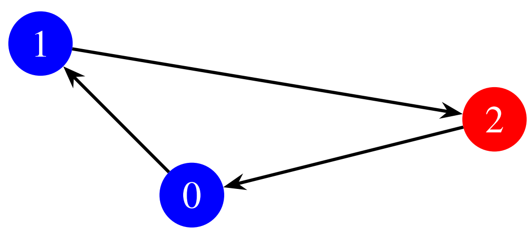


Figure 5: Example of a digraph which does not have a weak-majority 2-colouring.

Example 4 (A digraph with a minimal amount of monochromatic out-edges without a weak-majority 2-colouring.)**.** *Consider a directed 3-cycle as shown in Figure 5. This digraph has a minimal amount of monochromatic out-edges. However, this colouring is not a weak-majority 2-colouring and in fact such a colouring does not exist. There are only 2 possible cases to check: all nodes have the same colour or one node has a different colour. All nodes having the same colour will never result in a weak-majority 2-colouring. Figure 5 shows the case where one node has a different colour. This case has minimal monochromatic out-edges, but is not a weak-majority 2-colouring. Node 1 is not coloured correctly, since it has more monochromatic out-edges than dichromatic out-edges. However, changing its colour to red would cause node 2 to no longer be coloured correctly. Out of the 3 nodes, there will always be one with an out-edge to a node with the same colour.*

In this section, we have seen that proof approaches such as the colour swap can no longer be used for digraphs to guarantee results. The counterexamples indicate problems that arise in digraphs, which will not occur for undirected graphs. It is possible to apply colour swaps to find results on specific digraphs, but this will not work in general. In the remainder of this paper, theorems and lemmas (such as Theorem 3) will often have additional constraints that have to be introduced due to the fact that it is not possible to prove stronger results using a colour swap.

3.3 Majority illusions in general digraphs

Theorem 1 of Venema-Los et al. (2023) states that for any finite, undirected, irreflexive and unweighted graph $G = \langle V, E \rangle$, a majority-weak-majority illusion is possible. However, a majority-weak-majority illusion is no longer always possible when a graph with directed edges is considered.

Theorem 1. *There exists a digraph $G = \langle V, E \rangle$ for which no colouring $c : V \to \{blue, red\}$ is a (weak-)majority-weak-majority illusion.*

Proof. Consider the simple 3-cycle $G = \langle V, E \rangle$ with $V = \{0, 1, 2\}$ and $E = \{(0,1), (1,2), (2,0)\}$. Digraph G can be coloured in $2^3 = 8$ different ways using two colours. However, we only need to check two cases: all nodes have the same colour or one node has a different colour. See Figure 6 for the digraph with colourings corresponding to the two different cases.

- All nodes have the same colour: w.l.o.g. let $c_i = blue$ for all $i \in V$. Then, the global majority winner $M_V = blue$ and local majority winners $M_{N_i^{out}} = blue$ for all agents $i \in V$. $M_V = M_{N_i^{out}}$ for all $i \in V$, so there is no (weak-)majority-weak-majority illusion.

- Two nodes have the same colour and one has a different colour: w.l.o.g. let $c_0 = blue, c_1 = blue, c_2 = red$. Then, the global majority winner M_V is blue. Two out of three nodes have blue as the local majority winner ($M_{N_i^{out}} = blue$ for $i \in \{0, 2\}$). So, only one node is under (weak-)majority illusion. Hence, the graph does not have a (weak-)majority-weak-majority illusion.

In conclusion, no colouring of the 3-cycle gives a (weak-)majority-weak-majority illusion. □

Figure 6: Example of a digraph which does not have a majority-weak-majority illusion irrespective of colouring.

The proof of theorem 1 shows that a 3-cycle does not have a majority-weak-majority illusion independent of colouring. We can now think about whether this also holds for any simple k-cycle. This is not the case, consider for example a simple 2-cycle. If the two nodes have different colours, then $M_V = tie$, but $M_{N_0^{out}} \neq tie$ and $M_{N_1^{out}} \neq tie$. Thus, this gives a majority-weak-majority illusion. However, if you take a simple k-cycle with k odd, then there is no colouring that leads to a majority-weak-majority illusion.

Lemma 1. *For any odd directed cycle $G = \langle V, E \rangle$ there does not exist a 2-colouring $c : V \to \{blue, red\}$ which is a (weak-)majority-weak-majority illusion.*

Proof. Consider a directed k-cycle $G = \langle V, E \rangle$ where $k \in \mathbb{N}$ is an odd number which is coloured using some arbitrary colouring $c : V \to \{blue, red\}$. Assume w.l.o.g. that the global majority winner $M_V = blue$. Then, at least $\frac{k+1}{2}$ of the nodes are blue. Since G is a k-cycle, each node only has exactly one out-neighbour and each node is out-neighbour of some other node. So, at least $\frac{k+1}{2}$ of the nodes $i \in V$ have $M_{N_i^{out}} = blue = M_V$. Thus, less than half of the nodes are under weak-majority illusion. Therefore, there cannot be a colouring that creates a (weak-)majority-weak-majority illusion for any k-cycle with k odd. □

3.4 Majority illusions in specific types of digraphs

3.4.1 Acyclic digraphs

As was shown in lemma 1, odd cycles are the main example of digraphs without a weak-majority 2-colouring and without a weak-majority-weak-majority illusion. It makes sense to consider if there will always be a colouring for any acyclic digraph which gives a weak-majority-weak-majority illusion. Kreutzer et al. (2017) noted the following:

Lemma 2. *[Kreutzer et al. (2017)] Any acyclic digraph $G = \langle V, E \rangle$ has a weak-majority 2-colouring $c : V \to \{blue, red\}$.*

The idea is that any acyclic digraph can be ordered such that each node is preceded by its complete out-neighbourhood. Give the nodes without out-neighbours a colour and then for each node whose entire out-neighbourhood has been coloured, set its colour to the least frequently appearing colour in its out-neighbourhood. This constructs a weak-majority 2-colouring. Example 5 shows this approach for an example graph.

Example 5. *Figure 7 shows an acyclic digraph with a weak-majority 2-colouring coloured according to the approach from Kreutzer et al. (2017). The nodes are ordered, such that out-neighbours are on the right. Start on the right, with all nodes without out-neighbours and colour them blue. The nodes in the middle only blue out-neighbour and thus are coloured red. Finally, the nodes on the left are coloured. The upper node sees mostly blue out-neighbours and thus is coloured red. The other nodes on the left see only red out-neighbours and thus are coloured blue.*

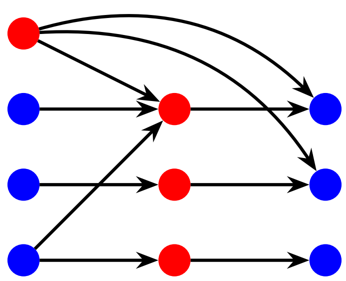


Figure 7: An acyclic digraph (weak-)majority 2-coloured according to the approach from lemma 2.

Lemma 2 cannot be immediately extended to weak-majority-weak-majority illusions, since there are nodes without out-neighbours and there can be ties between the most frequently appearing colour of the out-neighbours. There are two cases for which there is a weak-majority-weak-majority illusion, namely:

- the global majority winner is not a tie;
- the global majority winner is a tie and at most half of the agents see a tie as their local majority winner.

Lemma 3. *Any acyclic digraph $G = \langle V, E \rangle$ has a weak-majority-weak-majority illusion if $M_V \neq tie$ or $|\{i \in V | M_{N_i^{out}} = tie\}| \leq \frac{|V|}{2}$ when $M_V = tie$ for at least one weak-majority 2-colouring $c : V \to \{blue, red\}$.*

Proof. Take any acyclic digraph $G = \langle V, E \rangle$. Then, there exists a weak-majority 2-colouring $c : V \to \{blue, red\}$ according to lemma 2 (Originally from Kreutzer et al. (2017)). For this colouring c there will be two possible cases: $M_V = tie$ or $M_V \neq tie$. For these two scenarios, we will now show that the colouring c gives a weak-majority-weak-majority illusion of the acyclic digraph G if $M_V \neq tie$ or $|\{i \in V | M_{N_i^{out}} = tie\}| \leq \frac{|V|}{2}$ when $M_V = tie$.

- Consider the case where $M_V = tie$. If we assume $|\{i \in V | M_{N_i^{out}} = tie\}| \leq \frac{|V|}{2}$ (i.e. at least half of the nodes do not see a tie as the local majority winner), then for at least half of the agents $i \in V$: $M_V \neq M_{N_i^{out}}$. Under those assumptions, there is a weak-majority-weak-majority illusion.
- Secondly, consider the case where the global majority winner is not a tie, so w.l.o.g. assume $M_V = blue$. For all agents $i \in V$ with $c_i = blue$, $M_{N_i^{out}} \in \{tie, red\}$ by construction. Since $|\{i \in V | c_i = blue\}| > \frac{|V|}{2}$, the majority of agents are under at least weak-majority illusion. Thus, the graph has a majority-weak-majority illusion.

To conclude, if there is at least one weak-majority 2-colouring of G with $M_V = tie$ and $|\{i \in V | M_{N_i^{out}} = tie\}| \leq \frac{|V|}{2}$ or $M_V \neq tie$, then acyclic digraph $G = \langle V, E \rangle$ has a weak-majority-weak-majority illusion. □

We have now shown that acyclic digraphs have a weak-majority-weak-majority illusion if $M_V = tie$ and $|\{i \in V | M_{N_i^{out}} = tie\}| \leq \frac{|V|}{2}$ or $M_V \neq tie$ for some weak-majority 2-colouring. It is likely that such a weak-majority 2-colouring exists.

Conjecture 1. *Any acyclic digraph $G = \langle V, E \rangle$ has a weak-majority 2-colouring such that $M_V = tie$ and $|\{i \in V | M_{N_i^{out}} = tie\}| \leq \frac{|V|}{2}$ or when $M_V \neq tie$. Thus, any acyclic digraph has a weak-majority-weak-majority illusion.*

This conjecture is likely to hold, for the following reasoning: assume that there is a weak-majority 2-colouring such that $M_V = tie$ and $|\{i \in V | M_{N_i^{out}} = tie\}| > \frac{|V|}{2}$. All agents who see a tie as their local majority winner were given an arbitrary colour due to the construction of the weak-majority 2-colouring. The colour of any one of these agents who see a tie as their local majority winner could be swapped.

- The agent whose colour is swapped is not an out-neighbour to any other agent. Then, no other agents need to be changed due to this colour swap. Thus, the global majority winner M_V is no longer a tie. Thus, the digraph will be weak-majority 2-coloured as required.

- The agent whose colour is swapped is an out-neighbour to one or more agents. Let i be the agent whose colour is swapped. This could lead to the following changes among agents for which agent i is among the out-neighbours:
  - An agent keeps their original colour, due to the fact that its local majority winner did not change.
  - An agent now sees a tie as local majority winner. It can either colour swap or not.
  - An agent previously saw a tie. It now has a local majority winner and its colour needs to be the opposite of c_i.

Thus, if among all nodes which see a tie as local majority winner there is at least one node such that it has *no in-neighbours* or *none of the in-neighbours have a tie as local majority winner*, then there exists a weak-majority 2-colouring such that $M_V = tie$ and $|\{i \in V | M_{N_i^{out}} = tie\}| \leq \frac{|V|}{2}$ or when $M_V \neq tie$.

Lemma 3 is in contrast with lemma 1, which showed that odd directed k-cycles can never have a weak-majority-weak-majority illusion. The following section will consider whether the result only holds for acyclic digraphs or also for digraphs without odd directed cycles. We determine under which constraints digraphs without odd directed cycles can have a (weak-)majority-(weak-)majority illusion.

3.4.2 Digraphs without odd directed cycles

Anastos et al. (2021) noted that not only acyclic digraphs are weak-majority 2-colourable, but that this also holds for digraphs without odd directed cycles. In this section, we discuss the lemmas required to prove their theorem and how they can be adapted to results for majority illusions.

First, we need to introduce bipartite digraphs. In a bipartite digraph, all nodes can be divided into two sets U and V, such that nodes in set U only have edges to nodes in set V and vice versa. Anastos et al. (2021) combined the fact that bipartite digraphs are weak-majority 2-colourable and that odd directed cycles are a simple example of digraphs that are not weak-majority 2-colourable to show that *a digraph without odd directed cycles is weak-majority 2-colourable*. In the following section, we will adapt the proofs from Section 2 of (Anastos et al., 2021) to show that *any digraph without odd directed cycle can be coloured in such a way that there is a weak-majority-weak-majority illusion if* $M_V \neq tie$ *or* $|\{i \in V | M_{N_i^{out}} = tie\}| < \frac{|V|}{2}$ when $M_V = tie$.

Bipartite digraphs are weak-majority 2-colourable according to Anastos et al. (2021), so let us consider if there also always exists a colouring such that there is a majority-weak-majority illusion.

Lemma 4. *Any bipartite digraph $G = \langle V, E \rangle$ has a colouring $c : V \to \{blue, red\}$ which gives a majority-weak-majority illusion if $|\{i \in V | d_i^{out} \geq 1\}| > \frac{|V|}{2}$.*

Proof. Take any bipartite digraph $G = \langle V, E \rangle$. Then, by definition, G can be divided into two disjoint sets of nodes B and R such that every edge either goes from a node in set B to one in set R or vice versa. Define the colouring $c : V \to \{blue, red\}$ by setting $c_i = \begin{cases} blue & \text{if } i \in B \\ red & \text{if } i \in R. \end{cases}$

Now consider two cases:

- Consider the case where $M_V = tie$. Each agent $i \in V$ in the set of agents with an out-neighbour $\{i \in V | d_i^{out} \geq 1\}$ has $M_{N_i^{out}} \in \{blue, red\}$. If $|\{i \in V | d_i^{out} \geq 1\}| > \frac{|V|}{2}$, then for more than half of the agents: $M_V \neq M_{N_i^{out}}$. Thus, there is a majority-weak-majority illusion.
- The global majority winner is not a tie, so w.l.o.g. assume $M_V = blue$. Then, $|\{i \in V | c_i = blue\}| > \frac{|V|}{2}$ and for any of those agents $i \in V$ with $c_i = blue$, $M_{N_i^{out}} \in \{red, tie\}$. For a majority of the agents, namely those in the set $\{i \in V | c_i = blue\}$, we have $M_{N_i^{out}} \neq M_V$. So, there is a majority-weak-majority illusion.

In conclusion, any bipartite digraph where more than half of the nodes have an out-neighbour has a colouring $c : V \to \{blue, red\}$ such that there is a majority-weak-majority illusion. □

Now that we know when weak-majority 2-colourings and majority-weak-majority illusions are possible for bipartite digraphs, it makes sense to consider how we can expand on this result. Lemma 5 was observed by Anastos et al. (2021). It discusses the relation between strongly connected components and odd directed cycles. A strongly connected component of a digraph is a subgraph where for any nodes $i, j \in V$ there is a path from i to j and a path from j to i.

Lemma 5. *[Anastos et al. (2021)] Digraph $G = \langle V, E \rangle$ contains no odd directed cycles if and only if all its strong components are bipartite.*

Based on lemma 5, Anastos et al. (2021) noted that any digraph without odd directed cycles is weak-majority 2-colourable.

Theorem 2. *[Anastos et al. (2021)] Let D be a digraph which contains no odd directed cycles. Then D is weak-majority 2-colourable. Moreover, such a colouring can be chosen to extend any given pre-colouring of the sinks of D with colors $\{blue, red\}$.*

The proof-idea behind theorem 2 is to use induction on the number of strongly connected components. The base case is that there is only 1 strongly connected component. The graph in question has no odd directed cycles, so from lemma 5 it follows that the graph is bipartite and thus weak-majority 2-colourable. Then, assume that any graph without odd directed cycles with $s-1$ strongly connected components is weak-majority 2-colourable for any $s \geq 2$. Using this induction hypothesis, Anastos et al. (2021) showed that it is possible to construct a weak-majority 2-colouring for any digraph without odd directed cycles which has s strongly connected components.

Theorem 2 of Anastos et al. (2021) allows us to determine that any digraph without odd directed cycles can have a majority-weak-majority illusion for some 2-colouring if $M_V \neq tie$ or $|\{i \in V | M_{N_i^{out}} = tie\}| < \frac{|V|}{2}$. The exact construction approach does not work for majority-weak-majority illusions. Similar to the acyclic digraph proof of lemma 3, we can use the fact that a weak-majority 2-colouring exists and case-by-case consider what constraints are necessary to determine when a majority-weak-majority illusion is possible.

Theorem 3. *Let $G = \langle V, E \rangle$ be a digraph which contains no odd directed cycles. Then, there exists a colouring $c : V \to \{blue, red\}$ where digraph G has a majority-weak-majority illusion if $M_V \neq tie$ or $|\{i \in V | M_{N_i^{out}} = tie\}| < \frac{|V|}{2}$.*

Proof. Take any digraph $G = \langle V, E \rangle$ which does not contain any odd directed cycles. Then, there exists a colouring $c : V \to \{blue, red\}$ which is a weak-majority 2-colouring of digraph G according to theorem 2 (Anastos et al., 2021). For this colouring, the global majority winner M_V can be a tie or not.

- $M_V = tie$. We will work under the assumption that more than half the agents does not see a tie, which corresponds to $|\{i \in V | M_{N_i^{out}} = tie\}| < \frac{|V|}{2}$. Then, for more than half of the agents $i \in V$ (namely those with $M_{N_i^{out}} \neq tie$): $M_V \neq M_{N_i^{out}}$. Thus, with those assumptions, the graph is coloured as a majority-majority illusion.
- $M_V \neq tie$, w.l.o.g. assume $M_V = blue$. For all agents $i \in V$ with $c_i = blue$, $M_{N_i^{out}} \in \{tie, red\}$ as colouring c is a weak-majority 2-colouring. Since $|\{i \in V | c_i = blue\}| > \frac{|V|}{2}$, the majority of agents are under at least a weak-majority illusion. Thus, the graph has a majority-weak-majority illusion.

Hence, if $M_V \neq tie$ or $|\{i \in V | M_{N_i^{out}} = tie\}| < \frac{|V|}{2}$, then there is a majority-weak-majority illusion for any digraph without odd directed cycles.

□

Theorem 3 requires that at least one of the existing weak-majority 2-colourings of the digraph meets one of the following constraints:

- either the global majority winner is not a tie;
- or more than half of the agents do not see a tie as their local majority winner.

There is no simple approach to determine whether such a weak-majority 2-colouring exists in general, since the construction by Anastos et al. (2021) of the weak-majority 2-colouring is quite involved. Unlike the acyclic digraph lemma 3, where we can look at each specific case and determine how it affects the 2-colouring. However, the constraints are only not met in very specific digraphs and 2-colourings. Thus, for most digraphs without odd directed cycles, there is a possible 2-colouring such that the digraph is under majority-weak-majority illusion.

3.5 Voting between multiple alternatives in digraphs

In these previous sections, we have discussed the majority illusion, an illusion that can occur when you need to vote between two choices. However, you do not always have a binary choice between two alternatives when voting. Even if you do, you might want to include options to vote on both or neither option, which effectively results in choosing between 4 options. In this section, we will discuss two voting rules that can be applied when voting between more than two alternatives and when their respective illusions occur, the plurality and quota illusion. Similar to how weak-majority 2-colourings were relevant to show results regarding majority illusions, it now makes sense to consider weak-majority k-colourings when voting between k alternatives.

3.5.1 Weak-majority k-colouring

Recall how Venema-Los et al. (2023) used the fact that every simple, undirected, unweighted graph has a weak-majority 2-colouring (Lovász, 1966) to show that such a graph always has a colouring for which it is under majority-weak-majority illusion.

In digraphs, it is not possible to make general statements regarding weak-majority 2-colourings as was shown in Theorem 1. However, Kreutzer et al. (2017) determined that every finite, unweighted digraph has a weak-majority 4-colouring. Informally, every digraph has a 4-colouring of the nodes such that for each node $v \in V$, at least half of its out-neighbours receive a different colour to node v.

Theorem 4. *[Kreutzer et al. (2017)] Every digraph $G = \langle V, E \rangle$ has a weak-majority 4-colouring.*

The proof is based on two facts:

- Every digraph has an edge-partitioning into two acyclic subgraphs.
- Every acyclic digraph has a weak-majority 2-colouring.

Appendix B contains an in-depth proof (extending the short proof of Kreutzer et al. (2017)) and an example on how to construct a weak-majority 4-colouring for a digraph. The aim of the appendix is to make this thesis self-contained and give a better understanding of why theorem 4 holds.

Kreutzer et al. (2017) even conjectured that every digraph has a weak-majority 3-colouring and introduced some open problems related to this, which have partly been solved in research papers responding to their paper. In this thesis, we will now consider theorem 4 and determine how this 4-colouring of the digraph relates to quota illusions and plurality illusions.

3.5.2 q-illusion in digraphs

A commonly used voting rule for voting between more than two alternatives is quota voting. During quota voting, a colour is a winner if for some fraction q, more than a q-fraction of the agents have that colour. There can be multiple winners if all those colours appear more than the q-fraction. The quota voting rule is formally defined as follows:

Definition 9 (Quota voting rule)**.** Let quota $0 < q \leq 1$, S be some set of agents and $\mathcal{C}$ a set of colours. $S_c \subseteq S$ with $c \in \mathcal{C}$ is the set of agents with colour c. The quota winner(s) $Q_S = \{c | \frac{|S_c|}{|S|} > q\}$.

The value of the q-fraction can be between 0 and 1, but it is most informative if you take $\frac{1}{k} \leq q \leq 1$ where k is the amount of colours. If there are 100 colours it might be relevant if more than $\frac{1}{100}$-fraction of agents have a certain colour. On the other hand, if there are only 2 colours and more than $\frac{1}{100}$ of the agents have a certain colour, that information is not informative at all.

Agents can be under a q-illusion, which happens when none of the global and local quota winners are the same for that agent. A weak-q-illusion occurs when the global and local quota winners are different. We will now consider how definitions 4 and 5 of Venema-Los et al. (2023) need to be adapted in order to work with digraphs and multiple colours. Definition 10 adapts the definitions from Venema-Los et al. (2023) in a way that they align with definition 7 on general illusions in digraphs.

Definition 10 (q-illusion for agents in digraphs)**.** Given a k-coloured graph $G = \langle V, E, c \rangle$, an agent $i \in V$ is under q-illusion for $q \geq \frac{1}{k}$ if $Q_{N_i^{out}} \cap Q_V = \emptyset$ and $Q_{N_i^{out}} \neq \emptyset$ and $Q_V \neq \emptyset$. Agent $i \in V$ is under weak-q-illusion if $Q_{N_i^{out}} \neq Q_V$.

An agent can be under q-illusion and similarly a graph can be under p-q-illusion. Here, p refers to the fraction of agents in the graph being under q-illusion. The following definition is adapted from definition 6 of Venema-Los et al. (2023) to work for digraphs and is a specific case of the (weak-)p-(weak-)$\mathcal{R}$-illusion (definition 7).

Definition 11 ((weak-)p-(weak-)q illusion)**.** A digraph $G = \langle V, E, c \rangle$ is in a p-(weak-)q illusion if more than a p-proportion of the agents is under (weak-)q-illusion. A graph is in a (weak-)p-(weak-)q illusion if at least a p-proportion of the agents is under (weak-)q-illusion.

We now use the existence of a weak-majority 4-colouring to determine which type of (weak-)p-(weak-)q illusion is possible for any digraph. By definition of the weak-majority 4-colouring, at least half the out-neighbours of any node will have a different colour than the node itself. Therefore, the quota that is most relevant is $q = \frac{1}{2}$. This leads to the following result:

Theorem 5 ($\frac{1}{2}$-weak-$\frac{1}{2}$ illusion)**.** *For every digraph $G = \langle V, E \rangle$ there exist a 4-colouring $c : V \to \{1,2,3,4\}$ such that G has a $\frac{1}{2}$-weak-$\frac{1}{2}$ illusion if $Q_V \neq \emptyset$.*

Proof. Let $G = \langle V, E, c \rangle$ be some arbitrary digraph. Let $c : V \to \{1,2,3,4\}$ be a weak-majority 4-colouring. Such a weak-majority 4-colouring always exists according to theorem 4 which was originally introduced as theorem 1 in Kreutzer et al. (2017). Then, for each agent $i \in V$ we have $|\{j | c_i = c_j, j \in N_i^{out}\}| \leq \frac{|N_i^{out}|}{2}$ (i.e. at most half of the out-neighbours of i have the same colour as i). Consider an arbitrary agent $i \in V$ and now consider its local and global quota winner(s). We have an additional constraint that there exists at least one weak-majority 4-colouring of the digraph such that $Q_V \neq \emptyset$. There will be exactly one global quota winner, since each agent only has one colour, so it is impossible for two colours to appear for more than half of the agents.

$Q_V \neq \emptyset$. For each agent $i \in V$, its own colour c_i either is contained within the set of global quota winners or not, so we will consider both cases.

- Take any agent $i \in V$ with $c_i \in Q_V$. We know that $|\{j | c_i = c_j, j \in N_i^{out}\}| \leq \frac{|N_i^{out}|}{2}$, since the graph is coloured using a weak-majority 4-colouring. Rewriting this formula gives $\frac{|\{j | c_i = c_j, j \in N_i^{out}\}|}{|N_i^{out}|} \leq \frac{1}{2}$. According to definition 9 it follows that $c_i \notin Q_{N_i^{out}}$, because at most half of the out-neighbours of $i \in V$ have the same colour as agent i. We know that $c_i \in Q_V$ and $c_i \notin Q_{N_i^{out}}$, so $Q_{N_i^{out}} \neq Q_V$. Thus, agent $i \in V$ is under weak-$\frac{1}{2}$ illusion.
- Take any agent $i \in V$ with $c_i \notin Q_V$. By construction of the colours of G, $c_i \notin Q_{N_i^{out}}$. The construction of the graph does not exclude any other colours from being possible local quota winners. Hence, it is possible that $Q_{N_i^{out}} = Q_V$ for all agents $i \in V$ with $c_i \notin Q_V$.

By definition, at least half of the agents $i \in V$ have $c_i \in Q_V$ and those agents are under weak-$\frac{1}{2}$ illusion, thus the digraph has a $\frac{1}{2}$-weak-$\frac{1}{2}$ illusion.

□

We need as a requirement that $Q_V \neq \emptyset$ to show any results regarding $\frac{1}{2}$-weak-$\frac{1}{2}$-illusions. In reality, the set of global quota winners can be empty if none of the colours appears more than a $\frac{1}{2}$-proportion of the total number of agents. This case would not lead to a $\frac{1}{2}$weak-$\frac{1}{2}$-illusions, since more than half of the agents could have no local quota winner $Q_{N_i^{out}}$. It is unlikely that a recolouring of digraphs is possible in such a way that $Q_V \neq \emptyset$. Therefore, it is unlikely that all digraphs are under a $\frac{1}{2}$weak-$\frac{1}{2}$-illusion for some 4-colouring.

3.5.3 Plurality illusion in digraphs

Another generalisation of majority illusions that can be applied to cases with multiple alternatives are plurality illusions. This type of illusion uses the plurality voting rule, where the colour that appears most often wins. There can be multiple winners if multiple colours appear the same number of times and more often than all other colours. Formally, this corresponds to the following definition:

Definition 12 (Plurality voting rule)**.** Let S be some set of agents and $\mathcal{C}$ a set of colours. $S_i \subseteq S$ with $i \in \mathcal{C}$ is the set of agents with colour i. The plurality winner(s) $P_S = \{i \mid |S_i| \geq |S_j| \; \forall j \in \mathcal{C} \backslash \{i\}, i \in \mathcal{C}\}$.

This voting rule can also lead to an illusion, an agent can be under (weak)-plurality illusion. This was originally defined in definitions 10 and 11 of the unpublished manuscript by Venema-Los et al. (2025) for undirected, unweighted and irreflexive graphs. These definitions correspond to specific cases of the general definition of $\mathcal{R}$-illusions (see definition 7 for the case for digraphs). The definitions given below correspond to those of Venema-Los et al. (2025), but use the out-neighbourhood as we are working with directed graphs. In these definitions, P_V refers to the set of global plurality winners and $P_{N_i}^{out}$ the set of local plurality winners (of the out-neighbourhood of agent i).

Definition 13 (Plurality illusion for agents in digraphs)**.** Given a coloured digraph $C = \langle V, E, c \rangle$, an agent $i \in V$ is under plurality illusion if $P_{N_i^{out}} \cap P_V = \emptyset$. An agent $i \in V$ is under weak-plurality illusion if $P_{N_i^{out}} \neq P_V$.

If at least a certain fraction $0 < p \leq 1$ of agents is under (weak-)plurality illusion, then the graph is under (weak-)p-(weak)-plurality illusion. Venema-Los et al. (2025) proved in theorem 3 of their unpublished paper that for any undirected graph, a $\frac{1}{k}$-weak-plurality illusion is possible for any $k \geq 2$. Therefore, we will consider weak-$\frac{1}{k}$-weak-plurality illusions as well. They are formally defined as follows:

Definition 14 ($\frac{1}{k}$-plurality illusion)**.** A k-coloured digraph $C = \langle V, E, c \rangle$ with $|V| = n$ is under

- $\frac{1}{k}$-(weak-)plurality illusion if more than $\frac{1}{k}$ of the agents are under (weak-)plurality illusion.
- weak-$\frac{1}{k}$-(weak-)plurality illusion if at least $\frac{1}{k}$ of the agents are under (weak-)plurality illusion.

Now that $\frac{1}{k}$-weak-plurality illusion is defined, we can consider *whether for any directed graph, a $\frac{1}{k}$-weak-plurality illusion is possible for any $k \geq 2$.* We know that this statement holds for undirected graphs, as it was shown in theorem 3 of Venema-Los et al. (2025). However, for digraphs it is not possible to have a $\frac{1}{k}$-weak-plurality illusion for any $k \geq 2$. Theorem 2 shows that there does not always exist a majority-weak-majority illusion of any digraph, which corresponds to the plurality case with $k = 2$.

It is not possible to obtain a $\frac{1}{k}$-weak-plurality illusion for any digraph for all values of $k \geq 2$, but is it possible for specific values of k? Let us consider $k = 4$, since we at least know that every digraph has a weak-majority 4-colouring (see theorem 4 and originally theorem 1 of Kreutzer et al. (2017)). However, this starting point of any digraph having a weak-majority 4-colouring is not very strong. What we are actually interested in are digraphs with a weak-plurality 4-colouring.

Definition 15 shows how a (weak-)plurality k-colouring is defined for digraphs, differing from definition 12 of Venema-Los et al. (2025) about undirected graphs only in the use of out-neighbourhood N_v^{out}.

Definition 15 (Plurality k-colouring)**.** The plurality k-colouring of a digraph $G = \langle V, E \rangle$ with k colours is an assignment $c : V \rightarrow \{1, ..., k\}$ such that for every node $v \in V$: $c_v \notin P_{N_v^{out}}$. The weak-plurality k-colouring of a digraph $G = \langle V, E \rangle$ is an assignment $c : V \rightarrow \{1, ..., k\}$ such that for every node $v \in V$: $P_{N_v^{out}} \neq \{c_v\}$.

Informally, in a digraph with a weak-plurality 4-colouring, the local plurality winner of any node can not be equal to the colour of the node itself. Whereas in a digraph with a weak-majority 4-colouring, each node in the graph can have at most half of its out-neighbours with the same colour as the node itself. Thus, a digraph that is weak-majority 4-coloured does not have to be weak-plurality 4-coloured, as a node's own colour can appear between 25%-50% and be the local plurality winner.

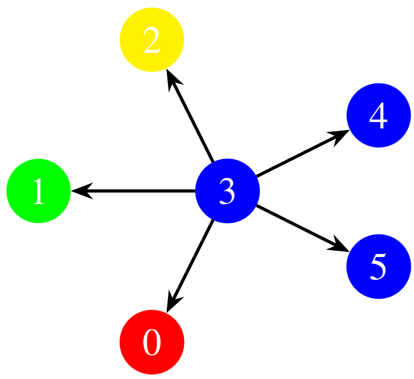


Figure 8: A digraph that is weak-majority 4-coloured, but not weak-plurality 4-coloured.

Example 6 (Weak-majority 4-colouring does not imply weak-plurality 4-colouring.)**.** *Figure 8 shows a digraph which is weak-majority 4-coloured, but not weak-plurality 4-coloured. Consider node 3, since all other nodes do not have out-neighbours and are therefore coloured correctly. Only* $\frac{2}{5}$*-th* ($< 50\%$) *of its out-neighbours are coloured blue. Less than half of the out-neighbours of node 3 have the same colour as node 3. Thus, the digraph is (weak-)majority 4-coloured.*
However, $P_{N_3^{out}} = blue = c_3$*, since blue occurs most among the out-neighbours of agent 3. The colour that appears most among the out-neighbours of node 3 is the same as the colour of node 3. So, the digraph is not weak-plurality 4-coloured.*

Venema-Los et al. (2025) showed in Lemma 3 of their unpublished manuscript that for any $k \geq 2$, an undirected graph has a weak-plurality k-colouring. That is, any k-colouring that minimises the number of monochromatic edges is a weak-plurality k-colouring. Recall that Theorem 3 of Venema-Los et al. (2025) states that for any undirected graph, a $\frac{1}{k}$-weak-plurality illusion is possible for any $k \geq 2$. This theorem relies on Lemma 3 of the same paper. For digraphs, a k-colouring that minimises the number of monochromatic out-edges is no longer guaranteed to be a weak-plurality k-colouring. This was explained for the $k = 2$ case in Section 3.2. Within the literature, weak-plurality k-colourings have not been discussed for digraphs. We want to know if a weak-plurality 4-colouring is always possible for any digraph, since Kreutzer et al. (2017) proved that at least a weak-majority 4-colouring is possible for any digraph. Example 6 showed that digraphs can be coloured in such a way that a weak-majority 4-colouring is not necessarily a weak-plurality 4-colouring. One research direction could be to prove whether a *weak-plurality 4-colouring is always possible for any digraph* and if possible, then aim to extend that result to $\frac{1}{4}$-weak-plurality illusions as was done by Venema-Los et al. (2025).

Since the approach of Venema-Los et al. (2025) cannot immediately be applied to digraphs, another perspective was considered instead. In order to obtain an intuition about the possibilities of there being a $\frac{1}{4}$-weak-plurality illusion for digraphs, code was written according to the algorithms in appendix C. The implementation of the code can also be found on GitHub: `https://github.com/Naomi-Broersma/Majority_illusions`. The code generates arbitrary digraphs and then checks for all possible 4-colourings of those digraphs whether there is a 4-colouring such that the digraph has a $\frac{1}{4}$-weak-plurality illusion. The generated digraphs must be small, since the number of possible 4-colourings that need to be checked is 4^n where n is the number of nodes. The code was unable to find any digraph where a $\frac{1}{4}$-weak-plurality illusion is not possible. However, this may be due to the fact that the digraphs are small and the digraphs are not fully randomly generated.

For small generated digraphs, no counterexample was found. So, we attempted to find an approach that shows that *for any digraph* $G = \langle V, E \rangle$ *with* $C : V \to \{1, 2, 3, 4\}$ *a* $\frac{1}{4}$*-weak-plurality illusion is possible.* However, as discussed in Section 3.2, colour swaps can no longer be used as a general proof approach in digraphs. It also does not work when voting between more than two alternatives. Due to this, we were unable to find a general approach. The easiest way to ensure that agents are under weak-plurality illusion is to have multiple global plurality winners. Then, there is a $\frac{1}{4}$-weak-plurality illusion if less than 75% of agents have those exact global plurality winners as local majority winners. However, a more precise approach and proof are necessary to show that it is always possible to have a $\frac{1}{4}$-weak-plurality illusion in a 4-coloured digraph.

4 Dynamics of Majority Illusions

In previous sections, majority illusions have been discussed from the perspective of static graphs. However, social networks are far from static. Agents gain or lose connections to other agents and can change their opinion on certain topics. A frequently used approach to updating social networks over time are threshold models. Threshold-driven diffusion models are used to show how opinions of agents change, while threshold-driven link change models discuss how due to threshold updates links between agents might appear or disappear. Diffusion and link change can also be combined. However, changing the order of operations (simultaneous, diffusion first or link change first) will generally result in a different network after the update (Baccini et al., 2024). This thesis will only discuss diffusion of the network, as an initial look at how dynamics might impact the existence of certain illusions.

4.1 Threshold models of diffusion

Granovetter (1978) introduced threshold models to discuss how collective behaviour arises. In his paper, people make a binary decisions and a threshold is assigned to each individual. This threshold corresponds to the proportion of people in the population with opinion B, such that when this threshold is reached the individual will change their opinion from opinion A to B. Granovetter (1978) discussed how a small change in the distribution of the population can significantly change the proportion of people who adopt a new opinion/show certain behaviour. Granovetter (1978) raised some issues on using a notion of complete connectedness of the network and how spatial and temporal effects could be taken into consideration.

In current day, these factors are taken into consideration by applying diffusion models to social networks. Agents have their own subset of the population whose opinion is taken into consideration, namely the agents that they have a connection to. And during each discrete time step, the threshold of each agent is checked to determine if the proportion of their neighbours with a certain opinion reach their pre-determined threshold.

There are variations on these threshold models which are based on real-life relevance. First and foremost, there no longer needs to be a binary decision, agents can form any opinion out of a finite number of options (Gao et al., 2016). It is also possible to use weighted edges and use these weights to determine if a threshold is reached, as not every agent has the same impact on the opinion of every other agent (Kempe et al., 2003). Another approach allows agents to switch between opinions in both directions, instead of only reaching a threshold of adopting a new opinion (Adam et al., 2012).

This paper focuses on using majority as a threshold. However, in practice, any value $0 \leq p \leq 1$ can be used as a fraction for an update, and each agent can have its own threshold.

First, let us consider what the majority diffusion update would look like with a binary decision. The model over time can be described by adding a time component, that is, $c_i(t)$ refers to the colour of node i at time t and $M_{N_i}(t)$ corresponds to the majority winner of the local neighbourhood of i at time t. The diffusion update will cause the nodes to change colour to the local majority winner, or stay the original colour in the case of a tie.

Definition 16 (Majority diffusion update)**.** Let $G = \langle V, E \rangle$ be a digraph that is 2-coloured as follows $c : V \to \{blue, red\}$. Then, the colour of agent $i \in V$ at time step $t+1$ will be

$$c_i(t+1) = \begin{cases} red & \text{if } M_{N_i^{out}}(t) = red \\ blue & \text{if } M_{N_i^{out}}(t) = blue \\ c_i(t) & \text{if } M_{N_i^{out}}(t) = tie. \end{cases}$$

Note that Definition 16 also works for the undirected case, since undirected graphs can be viewed as a symmetric directed graph.
The general diffusion update used in this paper aligns with the update of Gao et al. (2016), except for their use of in-neighbours instead of out-neighbours due to a different interpretation of edges between nodes. The colour of each node is changed to be the local plurality winner.

Definition 17 (Plurality diffusion update)**.** Let $G = \langle V, E \rangle$ be a undirected graph that is k-coloured as follows $c : V \to \{1, 2, .., k\}$. Then, the colour of agent $i \in V$ at time step $t+1$ will be

$$c_i(t+1) = \begin{cases} P_{N_i}(t) & \text{if } |P_{N_i}(t)| = 1 \\ c_i(t) & \text{if } |P_{N_i}(t)| > 1 \text{ and } c_i(t) \in P_{N_i}(t) \\ \text{arbitrarily take } c \in P_{N_i}(t) & \text{if } |P_{N_i}(t)| > 1 \text{ and } c_i(t) \notin P_{N_i}(t) \end{cases}$$

Definition 17 also works for undirected graphs. It is most relevant when considering graphs with (weak-)$\frac{1}{k}$-(weak-)plurality illusions, which will be left for further research.

4.2 Majority illusions over time

Now that diffusion models have been introduced, we can consider what happens over time to graphs with a majority-majority illusion. We want to know when a majority-majority illusion is bound to stay and when it would disappear.

Theorem 6. *If at time t, digraph $G = \langle V, E, c \rangle$ is under majority-majority illusion, then $M_V(t+1) \neq M_V(t)$ when applying a majority threshold update.*

Proof. Let $G = \langle V, E \rangle$ be any digraph that is 2-coloured $c : V \to \{blue, red\}$ such that it has a majority-majority illusion at time t, i.e. $|\{i \in V | M_{N_i^{out}}(t) \neq M_V(t) \text{ and } M_V(t) \neq tie \text{ and } M_{N_i^{out}}(t) \neq tie\}| > \frac{|V|}{2}$. W.l.o.g. assume that the global majority winner $M_V(t) = blue$. Then, more than half of the agents $i \in V$ have local majority winner $M_{N_i^{out}}(t) = red$ at time t, since there is a majority-majority illusion. That is, $|\{c_i(t) = blue | i \in V\}| > \frac{|V|}{2}$ and $|\{M_{N_i^{out}}(t) = red | i \in V\}| > \frac{|V|}{2}$.

To obtain the updated colouring of digraph G at time step $t+1$, we will apply a majority diffusion update. As was defined in definition 16: $c_i(t+1) = \begin{cases} red & \text{if } M_{N_i^{out}}(t) = red \\ blue & \text{if } M_{N_i^{out}}(t) = blue \\ c_i(t) & \text{if } M_{N_i^{out}}(t) = tie. \end{cases}$

We can use $|\{M_{N_i^{out}}(t) = red | i \in V\}| > \frac{|V|}{2}$ in combination with the majority diffusion update to obtain $|\{c_i(t+1) = red | i \in V\}| > \frac{|V|}{2}$, because at least half of the agent see a red majority locally and will thus change their colour to red. Thus, $M_V(t+1) \neq M_V(t)$. In conclusion, the global majority winner for consecutive time steps will not be the same after a majority threshold update if the graph has a majority-majority illusion initially. □

The global majority winner of a digraph with a majority-majority illusions will switch winners within one time step. If such a big change occurs in the digraph, one might think that a majority-majority illusion will disappear as well. However, this is not necessarily the case. Specifically, if the digraph is majority 2-coloured, then the digraph will always remain under majority-majority illusion.

Lemma 6. *If at time t, digraph $G = \langle V, E, c \rangle$ is under majority-majority illusion and c is a majority colouring, then G will remain under majority-majority illusion for any time step $T > t$ when applying a majority threshold update. Moreover, $M_V(t+k) = M_V(t)$ when k is even and $M_V(t+k) \neq M_V(t)$ when k is odd.*

Proof. Let graph $G = \langle V, E, c \rangle$ be under majority-majority illusion such that c is a majority colouring at some arbitrary time step t. W.l.o.g. let the global majority winner $M_V(t) = blue$. By the definition of majority-majority illusion, this indicates that at least half of the nodes $i \in V$ have local majority winner $M_{N_i^{out}}(t) = red$.

Now apply a majority threshold update as defined in definition 16: $c_i(t+1) = \begin{cases} red & \text{if } M_{N_i^{out}}(t) = red \\ blue & \text{if } M_{N_i^{out}}(t) = blue \\ c_i(t) & \text{if } M_{N_i^{out}}(t) = tie. \end{cases}$

We are considering a graph with a majority colouring, so the diffusion update can be rewritten as the following: $c_i(t+1) = \begin{cases} red & \text{if } c_i(t) = blue \\ blue & \text{if } c_i(t) = red. \end{cases}$

Since all nodes have been colour swapped, we now know that global majority winner $M_V(t+1) = red$ and at least half of the nodes $i \in V$ have local majority winner $M_{N_i^{out}}(t+1) = blue$. The graph will still have a majority colouring, since each node's own colour and its local majority winner will have swapped.

Time step t was chosen as some arbitrary time step where the assumptions of the graph being under majority-majority illusion and majority-coloured were met. At time step $t+1$ these assumptions were still met. Thus, by the power of induction, for any time step $T > t$: graph G will be under majority-majority illusion.

During each time step, the colours of all the nodes will swap. Therefore, the global majority winner will change colour as well: $M_V(t+k) = M_V(t)$ when k is even and $M_V(t+k) \neq M_V(t)$ when k is odd.

□

Lemma 6 outlines one case in which a majority-majority illusion will always remain. If the digraph is not majority 2-coloured, then there is no guarantee that the majority-majority illusion remains. In theory, it is possible that a majority-majority illusion remains or will reoccur at a later time step.

Since we are talking about finite digraphs, there is only a limited number of possible colourings. This means that at one point in time, a colouring that has previously been encountered will occur again. Since the majority threshold update is deterministic, all colourings after this point will repeat with some given period as long as the graph is solely updated according to the threshold update. However, the colouring at time $t = 0$ might never occur again, since this could have been caused by outside influence, such as initial adopters.

4.3 Majority illusion versus Condorcet's Jury theorem

In this section, we will compare two different ideas that make use of information aggregation: majority illusions and Condorcet's Jury theorem Condorcet (1785). Both make use of a majority decision and give a different perspective on whether it makes sense to use majority voting to make a correct binary decision. This section also aims to substantiate why research into majority illusions is important. Majority-majority illusions are collective distortions of a network and can therefore negatively impact decision-making. First, recall Condorcet's Jury Theorem:

Theorem 7 (Condorcet's Jury theorem)**.** *[(Condorcet, 1785)] Let $n = 2m+1$ individuals make a binary decision, and p = probability that any individual makes the correct decision. We let $h_n(p)$ be the probability that a majority of individuals make the correct decision, where individuals act independently. Then if $p > \frac{1}{2}$ and $n \geq 3$,*

- *$h_n(p) > p$, and*
- *$h_n(p) \to 1$ as $n \to \infty$.*

Informally, if each individual in a group makes a decision independently and is more likely to be correct than not, the probability that the majority of voters are correct is higher than the initial probability of the individual. As the number of individuals tends to infinity, the probability that the majority of agents are correct tends to 100%. Note that this specific case of the theorem only holds if all agents vote independently of each other.

This is unlikely to be the case in real-life scenarios, thus further research focused on variation in voter competence, dependence between voters and indirect voting (Hoeffding, 1956; Miller, 1986; Boland et al., 1989). Boland (1989) reviewed these generalisations and came to the following conclusion: as the dependence between voters increases, the probability of the majority of agents being correct decreases. However, when is there 'too much dependence' between voters such that increasing the population size no longer gives an arbitrarily high probability of making the correct decision?

We will now consider the specific case of graphs with a majority-majority illusion in conjunction with majority threshold updates. There clearly is dependence between agents in a social network, but in order for there to be a majority-majority illusion there does not necessarily have to be a large amount of dependence between agents.

First, let us define in which context we could compare majority-majority illusions with Condorcet's Jury theorem. So far, majority illusions have only been discussed in the context of voting and opinions, in which there is not necessarily a correct answer. However, all theorems and definitions discussed also work when there is one correct answer and each agent has some probability of knowing the correct answer. In order to make a fair comparison, we need to use the same assumptions as were made for Condorcet's Jury theorem. Thus, assume there are at least 3 agents and each agent is more likely to give the correct answer than the incorrect answer given some binary choice. We no longer assume that agents act independently, as was done by Condorcet (1785), since this is not a realistic assumption in real life.

Example 7 (Voting based on own opinions versus local opinions)**.** *Consider any social network $G = \langle V, E, c \rangle$ with a majority-majority illusion. One definite truth that exists in any graph is what the global majority opinion is. W.l.o.g. assume that the global majority winner $M_V = blue$. If you were to ask all agents in the graph to give their opinion about what they think the global majority winner is, they would use their local majority winner as an indication of the global majority opinion. This leads to the following estimations of the global majority opinion:*

- *Condorcet's Jury theorem: we assume that there is no dependence on opinions of other agents, so agents will only use their own opinion. Globally, this corresponds to taking the majority of all agent's opinions to obtain the global majority opinion $M_V = blue$.*
- *Use local majority winners to estimate the global majority winner: agents will no longer act independently and instead make use of the opinions of agents in their local neighbourhood. The vote of agent $i \in V$ will be the local majority winner $M^{out}_{N_i}$ of that agent. Then, taking the majority over the votes of all agents will be their prediction of the global majority opinion. The graph has a majority-majority illusion, so at least half of the agents see a different local majority winner $M^{out}_{N_i}$ than the global majority winner M_V. That is, $M^{out}_{N_i} = red$ for at least half of the agents,, and thus more than half of the agents will vote red as their prediction of the global majority winner. Hence, taking the majority of all agent's votes based on their local prediction results in red being the predicted global majority.*

In conclusion, if a digraph has a majority-majority illusion and people use their local neighbourhood as a predictor of the global majority, then aggregation using majority voting will fail to predict the global majority correctly.

Example 7 suggests that in the existence of a majority-majority illusion, it is better if agents vote independently such that a majority of agents is correct. Example 8 shows how the majority vote can change over time when a majority threshold update is applied to a digraph with a majority-majority illusion.

Example 8 (Majority voting dynamics)**.** *Consider some digraph $G = \langle V, E \rangle$ with colouring $c : V \rightarrow \{blue, red\}$ such that the digraph is under majority-majority illusion at time t. W.l.o.g. assume that the global majority winner $M_V(t) = blue$. From theorem 6 it follows that $M_V(t+1) = red$ if a majority threshold update is applied. The global majority winner has changed over time.*
Consider this result in the context of a binary-decision problem with only one correct solution. Then, at time t and at time $t+1$ there will be different answers using majority aggregation. At only one of the time steps, the majority of agents is correct. Condorcet's Jury theorem states that if the probability of each individual is some value $p > \frac{1}{2}$, then the probability of the majority of agents being correct is greater than p. We no longer have independent voting of agents in realistic scenarios, and applying a majority threshold update changes the resulting global majority winner.

Example 8 shows that updating the opinions of agents in a graph with a majority-majority illusion changes the majority opinion. Under these circumstances, a collective error occurs that leads to inaccuracies in decision making over time. Studying majority illusions and their dynamics is relevant, since we can gain a better understanding of why collective errors occur, when they will disappear, and potentially even how we can use the graph structure to reduce majority illusions in applied settings.

5 Conclusion

5.1 Summary of Main Contributions

In this paper, we extended the graph-theoretical discussion of majority illusions. Previously, the definitions and results were only defined for undirected graphs. The main focus of this thesis was to define majority illusions for digraphs in Section 3.1 and determine for which types of classes a majority of agents could be under majority illusion in Section 3.3 and 3.4. This was done by adapting the existing definition of majority illusions. For digraphs, agents locally consider their out-neighbourhood to determine a local majority winner instead of their full neighbourhood. Initially, results and proof approaches for undirected graphs were taken into account. For undirected graphs, colour swaps can be used to prove (stricter) results, but this is no longer possible for digraphs. In Section 3.2 examples were discussed that illustrate the problem introduced by the asymmetry of edges in digraphs. An approach that turned out to still be relevant was considering how results for majority colourings could be adapted for majority-majority illusions.

In Theorem 1 we showed that not all digraphs have a colouring such that a majority of agents is under weak-majority illusion. In fact, any odd directed cycle is an example of a digraph for which a majority-weak-majority illusion is impossible. This is in stark contrast to undirected graphs, where it was previously shown by Venema-Los et al. (2023) that any undirected network can have a majority-weak-majority illusion. For certain types of digraphs, it was shown in Section 3.4 that a majority-weak-majority illusion is possible. Specifically, digraphs without odd directed cycles have a majority-weak-majority illusion if the global majority winner is not a tie or if less than half the agents see a tie as their local majority winner for some weak-majority 2-colouring.

Not only binary choices were considered, but also decisions between more than two alternatives and voting rules that work for these larger decision spaces. Similar to how each undirected graph has a weak-majority 2-colouring, it was shown by Kreutzer et al. (2017) that each digraph has a weak-majority 4-colouring. In Section 3.5 we considered how this result can be used in the context of $\frac{1}{2}$-weak-$\frac{1}{2}$ illusions and $\frac{1}{4}$-weak-plurality illusions.

Lastly, in Chapter 4 we considered what happens to majority-majority illusions over a period of time assuming a majority threshold update. If the graph is majority 2-coloured, the graph will forever remain under majority-majority illusion. However, the global majority winner will swap each time step. If the graph is not majority 2-coloured, there is no guarantee that the graph will have a majority-majority illusion in later time steps.

5.2 Future Work

This work is a starting point to discuss majority illusions in digraphs. We showed some restrictions on when (weak)-majority-(weak)-majority illusions are possible and when they can no longer exist. A natural direction to take this research would be to consider other restrictions, such as specific classes of graphs and certain out-degree values compared to the number of agents.

We could also extend the definition of (weak)-majority-(weak)-majority illusions to graphs with weighted edges or multiple edges. Mainly, the local majority winner needs to be redefined, as it is no longer informative whether more than half of the (out-)neighbours are of a certain colour. Instead, the summed weight of each colour could be compared for weighted graphs. For multigraphs, we need to count each edge to a node, which could also be represented using weights. The definitions for weighted graphs and multigraphs are more relevant in discussing whether there is some illusion in an applied setting, rather than finding theoretical results.

Plurality illusions also require further investigation, since in Section 3.5.3 no conclusion was reached regarding whether a $\frac{1}{4}$-weak-plurality illusion is possible for any digraph. The proposed algorithm for finding a counterexample can be improved. The code currently checks all possible colourings, but we theoretically do not need to check colourings with analogous reasoning multiple times. If fewer colourings need to be checked, then it becomes possible to check slightly larger digraphs in a reasonable amount of time. Alternatively, it might be possible that a different approach to generating digraphs could lead to a counterexample. However, if a $\frac{1}{4}$-weak-plurality illusion is possible for any digraph this will not lead to any results. Instead, some approach needs to be created that generates such an illusion for any digraph, and it needs to be proven that the approach always works.

The most relevant next step for practical use would be to consider how majority illusions interact with other social phenomena. For instance, the friendship paradox or information gerrymandering. Do majority illusions affect or interact with such phenomena?

Lastly, we could compare local majority voting with individual voting by creating a simulation. Give each agent a colour with some probability $p > 50\%$ of having the correct answer among two alternatives. For the local majority approach, each agent uses their local neighbourhood to determine their answer. For the individual approach, each agent uses their own opinion. The two approaches can be compared in randomly generated graphs. How often do majority-majority illusions occur? And how often does each approach lead to the correct answer? Does the probability of the majority of agents making the correct decision generally improve as the number of agents increases, when people act based on their local neighbourhood instead of independently?

References

Adam, E. M., Dahleh, M. A., & Ozdaglar, A. (2012). On the behavior of threshold models over finite networks. In *Proc. 51st IEEE Conf. Decision and Control* (pp. 2672–2677).

Airoldi, E. M., & Christakis, N. A. (2024). Induction of social contagion for diverse outcomes in structured experiments in isolated villages. *Science*, *384*(6695).

Anastos, M., Lamaison, A., Steiner, R., & Szabó, T. (2021). Majority Colorings of Sparse Digraphs. *The Electronic Journal of Combinatorics*, *28*(2), 2–31. doi: https://doi.org/10.37236/10067

Anholcer, M., Bosek, B., & Grytczuk, J. (2019). Majority coloring of infinite digraphs. *Acta Mathematica Universitatis Comenianae*, *88*(3), 371–376.

Baccini, E., Christoff, Z., & Verbrugge, R. (2024). Dynamic Logics of Diffusion and Link Changes on Social Networks. *Studia Logica*, 1–71. Retrieved from https://doi.org/10.1007/s11225-024-10126-0

Boland, P. J. (1989). Majority systems and the condorcet jury theorem. *Journal of the Royal Statistical Society. Series D (The Statistician)*, *38*(3), 181–189. Retrieved 2025-02-05, from http://www.jstor.org/stable/2348873

Boland, P. J., Proschan, F., & Tong, Y. L. (1989). Modelling dependence in simple and indirect majority systems. *Journal of Applied Probability*, *26*(1), 81–88.

Christakis, N. A., & Fowler, J. H. (2010). Social network sensors for early detection of contagious outbreaks. *PloS one*, *5*(9).

Condorcet, N. (1785). *Essai sur l'application de l'analyse a la probabilite des decisions rendues 'a la pluralite de voix.* Paris.

Dippel, J., la Tour, M. D., Niu, A., Roy, S., & Vetta, A. (2024). Eliminating majority illusion is easy. *arXiv preprint arXiv:2407.20187*.

Feld, S. L. (1991). Why your friends have more friends than you do. *American journal of sociology*, *96*(6), 1464–1477.

Fioravantes, F., Lahiri, A., Lauerbach, A., Sieper, M. D., & Wolf, S. (2024). Eliminating majority illusion.

Gao, Z., Chen, X., Liu, J., & Başar, T. (2016). Periodic behavior of a diffusion model over directed graphs. In *2016 IEEE 55th Conference on Decision and Control (CDC)* (pp. 37–42).

Girao, A., Kittipassorn, T., & Popielarz, K. (2017). Generalized majority colourings of digraphs. *Combinatorics, Probability and Computing*, *26*(6), 850–855.

Grandi, U., Kanesh, L., Lisowski, G., Sridharan, R., & Turrini, P. (2023). Identifying and eliminating majority illusion in social networks. In *Proceedings of the aaai conference on artificial intelligence* (Vol. 37, pp. 5062–5069).

Grandi, U., Lisowski, G., Ramanujan, M., & Turrini, P. (2022). On the Complexity of Majority Illusion in Social Networks. *arXiv preprint arXiv:2205.02056*. Retrieved from https://doi.org/10.48550/ARXIV.2205.02056

Granovetter, M. (1978). Threshold models of collective behavior. *American journal of sociology*, *83*(6), 1420–1443. Retrieved from https://doi.org/10.1086/226707

Hoeffding, W. (1956). On the distribution of the number of successes in independent trials. *The Annals of Mathematical Statistics*, *27*, 713–721.

Kempe, D., Kleinberg, J., & Tardos, E. (2003). Maximizing the spread of influence through a social network. In *Proceedings of the Ninth ACM SIGKDD International Conference on Knowledge Discovery and Data Mining* (p. 137–146). New York, NY, USA: Association for Computing Machinery. Retrieved from https://doi.org/10.1145/956750.956769 doi: 10.1145/956750.956769

Kreutzer, S., Oum, S.-i., Seymour, P., van der Zypen, D., & Wood, D. R. (2017). Majority Colourings of Digraphs. *The Electronic Journal of Combinatorics*, *24*(2), 2–25.

Lerman, K., Yan, X., & Wu, X.-Z. (2016). The 'majority illusion' in social networks. *PloS one*, *11*(2), 1-13. Retrieved from https://doi.org/10.1371/journal.pone.0147617

Lovász, L. M. (1966). On decomposition of graphs. *Studia Sci. Math. Hungar*, *1*, 237–238.

Miller, N. R. (1986). Information, electorates, and democracy: some extensions and interpretations of the condorcet jury theorem. *Information pooling and group decision making*, *2*, 173–192.

Nettasinghe, B., & Krishnamurthy, V. (2021). "what do your friends think?": Efficient polling methods for networks using friendship paradox. *IEEE Transactions on Knowledge and Data Engineering*, *33*(3), 1291-1305.

Venema-Los, M., Christoff, Z., & Grossi, D. (2023). On the Graph Theory of Majority Illusions. In V. Malvone & A. Murano (Eds.), *Multi-Agent Systems* (pp. 17–31). Cham: Springer Nature Switzerland.

Venema-Los, M., Christoff, Z., & Grossi, D. (2025). *On the graph theory of majority illusions: Theoretical results and computational experiments.*

Appendix A Definitions 5 and 8 with at least one out-neighbour for each agent.

In this appendix, we will show that digraph majority illusion definitions 5 and 8 are the same (except for notation) when each agent has at least one out-neighbour. The global majority winner M_V and local majority winner $M_{N_i^{out}}$ are represented as sets in definition 8, while this is not the case in definition 5. The possible winners in definition 8 are $\{blue\}, \{red\}, \{blue, red\}$ and correspond to $blue, red, tie$ respectively.

- We will check the definitions for agent $i \in V$ being under majority illusion. The initial definition 5 (definition 1 of Venema-Los et al. (2023)) is that agent $i \in V$ is under majority illusion if $M_{N_i^{out}} \neq tie$ and $M_V \neq tie$ and $M_{N_i^{out}} \neq M_V$. Definition 8 based on general illusions states that agent $i \in V$ has $M_{N_i^{out}} \cap M_V = \emptyset$ and $M_{N_i^{out}} \neq \emptyset$ and $M_V \neq \emptyset$ when the agent is under majority illusion. We will also check whether the weak-majority illusion definitions 5 and 8 lead to the same results, these both are $M_V \neq M_{N_i^{out}}$. Are these equivalent? We need to check case by case all scenarios for $M_V \in \{blue, red, tie\}$ and $M_{N_i^{out}} \in \{blue, red, tie\}$.
  - $M_V = blue$ for definition 5 or equivalently $M_V = \{blue\}$ for definition 8.
    * If $M_{N_i^{out}} = blue$ or $\{blue\}$, then the agent is not under majority-illusion for both definitions, because $M_{N_i^{out}} = M_V$ (definition 5) and $M_{N_i^{out}} \cap M_V \neq \emptyset$ (definition 8).
    * If $M_{N_i^{out}} = red$ or $\{red\}$, then the agent is under majority-illusion for both definitions, because $M_{N_i^{out}} \neq M_V$ (definition 5) and $M_{N_i^{out}} \cap M_V = \emptyset$ (definition 8).
    * If $M_{N_i^{out}} = tie$ or $\{blue, red\}$, then agent $i \in V$ is not under majority illusion, but is under weak-majority illusion. For definition 5: $M_{N_i^{out}} \neq tie$ is not true, but $M_{N_i^{out}} \neq M_V$ does hold. For definition 8, $M_{N_i^{out}} \cap M_V = \{blue\} \neq \emptyset$, but $M_{N_i^{out}} \neq M_V$ does hold.
  - $M_V = red$ or equivalently $M_V = \{red\}$. The reasoning is analogous to the previous case, since blue and red are arbitrarily chosen colours.
  - $M_V = tie$ or equivalently $M_V = \{blue, red\}$. Agent $i \in V$ is not under majority illusion, but is under weak-majority illusion when the local majority winner is either red or blue (respectively $\{red\}$ or $\{blue\}$) in both definitions.
    * We have $M_V = tie$, which is not allowed for a strict illusion for definition 5. However, if $M_{N_i^{out}}$ is either red or blue, $M_{N_i^{out}} \neq M_V$, which results in a weak-majority illusion using definition 5. If $M_{N_i^{out}} = tie$, then there is no (weak-)majority illusion according to definition 5.
    * Now, let us consider set-definition 8 of majority illusions. We have $M_V = \{blue, red\}$, so if the local majority winner $M_{N_i^{out}}$ is either $\{blue\}$, $\{red\}$ or $\{blue, red\}$, then $M_V \cap M_{N_i^{out}} \neq \emptyset$, so according to definition 8, agent $i \in V$ is not under majority illusion. When $M_{N_i^{out}}$ is $\{blue\}$ or $\{red\}$, $M_V \neq M_{N_i^{out}}$, so there is a weak-majority illusion. If $M_{N_i^{out}} = \{blue, red\}$, then there is no (weak-)majority illusion according to definition 8.

In conclusion, an agent is under (weak-)majority illusion under exactly the same circumstances for both definitions 5 and 8 when agent i has at least one out-neighbour. Extending this to the digraph, if all nodes have at least one out-neighbour, then the definitions of (weak-)majority-(weak-)majority illusion are equivalent. Note that the same reasoning (using M_{N_i} instead of $M_{N_i^{out}}$) will hold for graphs without directed edges.

Appendix B In-depth proof of Kreutzer et al. (2017): every digraph has a weak-majority 4-colouring.

Theorem 1 of Kreutzer et al. (2017) states that every digraph has a weak-majority 4-colouring. We discuss this as theorem 4 in this thesis, and decided to add a more extensive proof of this theorem to this appendix in order to make the paper self-contained.

Theorem (Kreutzer et al. (2017))**.** *Every digraph $G = l\langle V, E\rangle$ has a weak-majority 4-colouring.*

Proof. Let $G = \langle V, E\rangle$ be a digraph. Fix a vertex ordering $x_1, x_2, ..., x_n \in V$. Now 2-colour the nodes from left-to-right such that for each node $u \in V$ at most half of the out-neighbours to the left of u receive the same colour as u. This corresponds to colouring an acyclic subgraph $G' = \langle V', E'\rangle$ which contains all nodes $u \in V$ ($V' = V$) and the set of edges $E' = \{(x_i, x_j) \in E | i > j\}$ As discussed in lemma 2, the required colouring is always possible, because every acyclic digraph has a weak-majority 2-colouring. Any node without out-neighbours in acyclic subgraph G' can be given any colour. We colour from left-to-right and all nodes only have out-neighbours on the left by construction. Thus, any node we encounter can now be given a colour. Simply set its colour to the least frequently occurring colour in their out-neighbourhood.

Using analogous reasoning, it is possible to 2-colour the nodes from right-to-left such that for each node $u \in V$ at most half of the out-neighbours to the right of u receive the same colour as u.

Taking the product of the two 2-colourings gives a 4-colouring. Each colour in this 4-colouring can be represented as $\{\alpha, \beta\}$ where α corresponds to the colour given in the 2-colouring from left-to-right and β the colour given in the 2-colouring from right-to-left. Thus, the 4 possible colours are $\{\alpha_1, \beta_1\}, \{\alpha_1, \beta_2\}, \{\alpha_2, \beta_1\}$ and $\{\alpha_2, \beta_2\}$. At most half of the out-neighbours to the left of any node $u \in V$ will have the same α as u. And at most half of the out-neighbours to the right of node u will have the same β as u. Hence, at most half of the out-neighbours of any node u will have the same colour $\{\alpha, \beta\}$ as u in the 4-colouring. Thus, any digraph has a weak-majority 4-colouring.

□

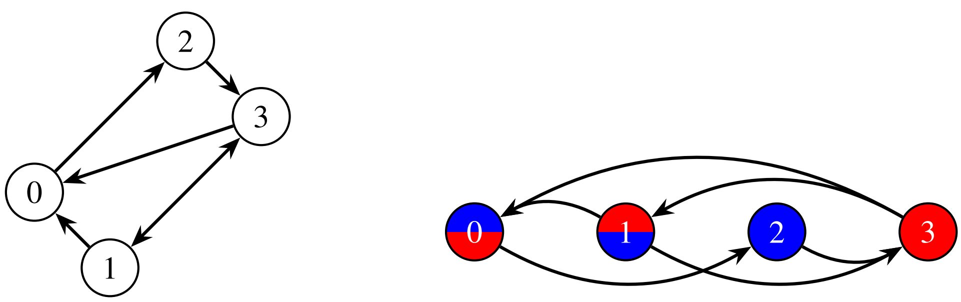


Figure 9: The left of this figure shows a digraph. The right shows the figure weak-majority 4-coloured using the approach described in the appendix above, which was originally proven by Kreutzer et al. (2017).

Example 9 (Weak-majority 4-colouring of a digraph)**.** *Figure 9 shows a digraph (left) and the same digraph with a weak-majority 4-colouring (right). We will now walk through each step discussed in the proof above for this digraph.*

- *Create a vertex ordering: the nodes are ordered as 0, 1, 2, 3.*
- *Colour the digraph from left-to-right (see the upper colour of the nodes). Node 0 does not have any out-neighbours on the left and is arbitrarily given the colour blue. Node 1 has one out-neighbour to the left (blue node 0) and thus needs to be coloured red. Node 2 does not have any out-neighbours and can be given the arbitrary colour blue. Node 3 has two out-neighbours to the left (blue node 0 and red node 1) and due to the fact that the out-neighbours have different colours, node 3 can be given the arbitrary colour red. All nodes have now been coloured from left-to-right such that at most half of their out-neighbours to the left have the same colour as the node itself.*
- *Colour the digraph from right-to-left (see the lower colour of the nodes). Node 3 can be given any colour, since it does not have out-neighbours to the right. Let node 3 be blue. Node 2 has one out-neighbour to the right, the red node 3, so node 2 must be coloured blue. Node 1 also only has node 3 as out-neighbour to the right and needs to be coloured blue as well. Node 0 sees the blue node 2 as an out-neighbour, so node 0 is coloured red. All nodes have now been coloured from left-to-right such that at most half of their out-neighbours to the right have the same colour as the node itself.*
- *There are four different combinations of colours:* $\{blue, blue\}, \{red, red\}, \{blue, red\}$ *and* $\{red, blue\}$*. The product of the two 2-colourings is a weak-majority 4-colouring.*

Appendix C Algorithm counterexample $\frac{1}{4}$-weak-plurality illusion.

This appendix contains an algorithm that checks for each possible 4-colouring whether a given digraph has a $\frac{1}{4}$-weak-plurality illusion. This algorithm was created in order to find a counterexample to the claim: *for any digraph, a $\frac{1}{4}$-weak-plurality illusion is possible.* The implementation of the algorithm can be found on GitHub: `https://github.com/Naomi-Broersma/Majority_illusions`. The code was used on digraphs with a low number of nodes, but did not lead to any counterexamples in this limited search space.

Algorithm 1 Determine all possible colourings of a graph with k different colours

Input: graph G consisting of a list of nodes *Nodes* and list of edges *Edges*, list of k colours C
Output: list of possible ways to colour the nodes *Colourings*

```
procedure DETERMINE_ALL_POSSIBLE_COLOURINGS(G, C)
    Colourings ← empty list
    for node in Nodes do
        Temporary_colourings ← empty list
        if Colourings is an empty list then
            for colour in C do
                Append [colour] to Temporary_colourings
            end for
            Colourings ← Temporary_Colourings
        else
            for sub_colouring in Colourings do
                for colour in C do
                    copy_sub_colouring ← sub_colouring
                    Append colour to copy_sub_colouring
                    Append copy_sub_colouring to Temporary_colourings
                end for
                Colourings ← Temporary_colourings
            end for
        end if
    end for
    return Colourings
end procedure
```

Algorithm 2 Determine if a node is under weak-plurality illusion.

Input: graph G consisting of a list of nodes *Nodes* and list of edges *Edges*, node N, colouring *Colouring*

Output: Boolean *weak_plurality_illusion* that returns *True* if there is a weak-plurality illusion for node N, otherwise *False*

```
procedure CHECK_WEAK_PLURALITY_ILLUSION_NODE(G, N, Colouring)
    plurality_winner_global ← list with colours that occur most frequently among all nodes
    out_neighbours ← list of out-neighbours of N
    colours_neighbours ← empty list
    for neighbour in out_neighbours do
        append colour of neighbour to colours_neighbours
    end for
    if colours_neighbours is an empty list then
        plurality_winner_local ← empty list
    else
        plurality_winner_local ← list of colours that occur most frequently among the out-neighbours of node N
    end if
    weak_plurality_illusion ← True
    if plurality_winner_global == plurality_winner_local then
        weak_plurality_illusion ← False
    end if
    return weak_plurality_illusion
end procedure
```

Algorithm 3 Determine if a graph is under $\frac{1}{k}$-weak-plurality illusion.

Input: graph G consisting of a list of nodes *Nodes* and list of edges *Edges*, colouring *Colouring*, number of colours k

Output: Boolean *k_fraction_weak_plurality_illusion* that returns *True* if there is a $\frac{1}{k}$−weak-plurality illusion for graph G, otherwise *False*

```
procedure CHECK_K_FRACTION_WEAK_PLURALITY_ILLUSION_GRAPH(G, Colouring, k)
    counter_weak_plurality_illusion ← 0
    for node in Nodes do
        weak_plurality_illusion_node ← check_weak_plurality_illusion_node(G, N, Colouring)    ▷ Algorithm 2
        if weak_plurality_illusion_node then
            counter_weak_plurality_illusion ← counter_weak_plurality_illusion + 1
        end if
    end for
    weak_plurality_illusion_fraction ← plurality_illusion_counter / len(Nodes)
    k_fraction_weak_plurality_illusion ← False
    if weak_plurality_illusion_fraction >= 1/k then
        k_fraction_weak_plurality_illusion ← True
    end if
    return k_fraction_weak_plurality_illusion
end procedure
```

Algorithm 4 Determine if there exists a colouring for which the graph is under $\frac{1}{k}$-weak-plurality illusion.

Input: graph G consisting of nodes $Nodes$ and edges $Edges$, list of colours C

Output: Boolean $illusion\_for\_colouring$ that returns $True$ if there exists a colouring for which a $\frac{1}{k}-$weak-plurality illusion for graph G exists, otherwise $False$

```
procedure PLURALITY_ILLUSION_CHECK_PER_COLOURING(G, C)
    illusion_for_colouring ← False
    possible_colourings ← determine_all_possible_colourings(G, C)          ▷ Algorithm 1
    for colouring in possible_colourings do
        k_fraction_weak_plurality_illusion ←
            check_k_fraction_weak_plurality_illusion(G, colouring, |C|)    ▷ Algorithm 3
        if k_fraction_weak_plurality_illusion then
            illusion_for_colouring ← True
            return illusion_for_colouring
        end if
    end for
    return illusion_for_colouring
end procedure
```