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Model Structures and Infinity Categories

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Abstract

We define the structure of a model category and show that the Quillen structure on sSet is a model structure. We extend this result to define the Joyal model structure and its equivalents and use this to discuss two equivalent constructions of $(\infty, 1)$-categories and go in depth in how to use rigidification for this purpose. Lastly we justify how this is a natural generalisation of ordinary category theory.

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1 Introduction

Model categories arise when attempting to do homotopy outside of the category of topological spaces. It turns out that that they arise in many other categories and form their own rich theory. The most fundamental model structure was defined on the category of simplicial sets by Quillen [16] and forms the basis for a lot of model category theory. A second structure was defined on this category by Joyal [11], and this one turns out to be very useful for describing models of $(\infty, 1)$-categories. Which are generalisations of ordinary categories which allow for a notion of composition up to homotopy.

Model structures can also be defined on very different categories. As was done by Bergner [2], who defined a model structure on the category of simplicially enriched categories. This model turns out to be equivalent to the one defined by Joyal and this equivalence helps show that two different interpretations of $(\infty, 1)$-categories are equivalent [8].

We will start by giving an introduction to Kan-extensions (section 2.4) and simplicial sets (section 2.5); from there we build up these three model structures and show they are equivalent. A lot of this has been done by Lurie [13], however we shall be using various different approaches by Joyal and Tierney [11] (showing the Quillen Model structure is in fact a model structure, section 3.3), and Dugger and Spivak [8] (the Bergner and Joyal models are equivalent, section 5.3). Moreover we shall be using some results of Cisinski [6] regarding the existence of model structures on a presheaf topos (Appendix 3.5) to define the Joyal model structure.

Furthermore we shall show how to derive $(\infty, 1)$-category theory from these notions (chapter 4) and how it is a natural generalisation of ordinary category theory (section 5.1).

2 Prerequisites

2.1 Lifting Properties

A notion we shall need to define model categories is that of a lifting property, this shall be needed to define a weak factorisation system, which is an integral part of the definition of a model category.

Definition 2.1. *Given a category $\mathcal{E}$ and a class of morphisms R in $\mathcal{E}$, we say f has the left lifting property with respect to R if for every $g \in R$ with a commutative square there is a (not necessarily unique) lifting*

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle f}\downarrow & \nearrow & \downarrow{\scriptstyle g} \\ B & \longrightarrow & Y \end{array}$$

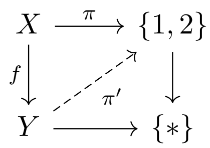
We denote the class of maps with the left lifting property with respect to R by ${}^{\pitchfork}R$ and ${}^{\pitchfork}g :={}^{\pitchfork}\{g\}$ for g a morphism. The dual property is called the right lifting property and denoted by ${}^{\pitchfork}L$. Sometimes we will also use the notation $f \pitchfork g$ meaning $f \in{}^{\pitchfork} g$. It is sometimes written $f \perp G$ if this lifting is unique.

Example 2.2. In the category of sets, a function $f : X \to Y$ is surjective if and only if $f \in (\emptyset \hookrightarrow \{*\})^{\pitchfork}$, indeed suppose f has this lifting property and $y \in Y$, then we define the function $\iota : \{*\} \to Y$ mapping $* \mapsto y$. Hence there is a lifting such that

$$\begin{array}{ccc} \emptyset & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow{\scriptstyle f} \\ \{*\} & \xrightarrow[\iota]{} & Y \end{array}$$

commutes, in other words: there is a function $\iota' : \{*\} \to X$ such that $f \circ \iota' = \iota$. Hence we have $\iota'(*) \in X$ such that $f(\iota'(*)) = y$ and indeed f is surjective. If one believes in the axiom of choice then the converse is easy as we can find a right inverse of f making the lifting problem trivial.

If it moreover true that f is injective precisely if $f \in^{\pitchfork} (\{1,2\} \to \{*\})$. It is clear that every injective function exhibits such a property as every monomorphism in the category of sets is split and therefore the lifting problem is trivial. For the converse, let us pick $a, b \in X$ distinct. Pick any funtion function $\pi : X \to \{1,2\}$ where $\pi(a) = 1$ and $\pi(b) = 2$, then we find a lifting



so that $\pi' \circ f = \pi$. Thus $(\pi' \circ f)(a) = 1$ and $(\pi' \circ f)(b) = 2$ so $f(a) \neq f(b)$.

Combining these two observations we see that

$$\{\text{Surjective functions}\} = (\emptyset \to \{*\})^{\pitchfork} \supseteq \{\text{Injective functions}\}^{\pitchfork}$$

and

$$\{\text{Injective Functions}\} =^{\pitchfork} (\{1,2\} \to \{*\}) \supseteq^{\pitchfork} \{\text{Surjective functions}\}$$

Proposition 2.3. ${}^{\pitchfork}() \dashv ()^{\pitchfork}$ *is a Galois connection on the collection of subclasses of the class of morphisms.*

Proof. Order the subclasses by inclusion, we need to show that ${}^{\pitchfork}I \subseteq J$ if and only if $I \subseteq J^{\pitchfork}$ which is a tautology. ☐

Another notion that shall be useful when deriving the Quillen model structure is that of a saturated class.

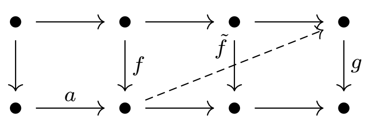
Definition 2.4. *We call a class A of morphisms α-saturated for α an ordinal if it is closed under retracts, pushouts, and if each $f_\gamma \in A$ for $\gamma < \alpha$ then $\operatorname{col}_{\gamma<\alpha} f_\gamma \in A$. When this holds for all ordinals we call A saturated.*

Proposition 2.5. *A saturated class is closed under coproducts and contains all isomorphisms.*

Proposition 2.6. *${}^{\pitchfork}R$ is saturated for any class of morphisms R.*

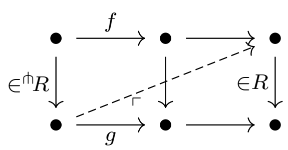
Proof. 1. Given an isomorphism, we can find its inverse, making the lifting problem trivial.

2. Given a retract and $f \in^{\pitchfork} G$ we extend a lifting of f to its retract as $\tilde{f} \circ a$ in



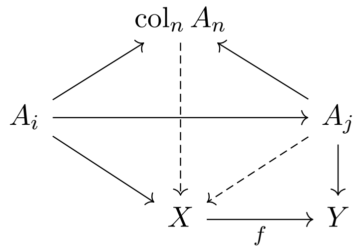
then a diagram chase immediately shows this is indeed a valid lifting.

3. Now suppose we are in the situation



Then the universal property of the pushout gives the desired lifting to make the middle morphism have the left lifting property.

4. Given a chain of morphisms $(A_i \to A_j \in^{\pitchfork} S, i \leq j)$, then for every $f : X \to Y$ in S we have a lifting via



$\square$

Corollary 2.7. *$R^{\pitchfork}$ is closed under retracts and pushouts.*

Proposition 2.8. *Given I a class of morphisms ${}^{\pitchfork}(I^{\pitchfork})$ is the smallest saturated class containing I.*

Proof. I being contained in ${}^{\pitchfork}(I^{\pitchfork})$ is a tautology. Let S be the intersection of all saturated classes containing I. We show S has the left lifting property with respect to $I^{\pitchfork}$. Any arrow in S can be generated from arrows in I via retracts, pushouts, and transfinite composition; each of these preserves the lifting properties. Hence every arrow in S has the desired liftings. $\square$

2.2 Weak Factorisation Systems

A notion that shall be very usefull when defining model categories is that of a *weak factorisation system*, this is a vast generalisation of the epi-mono factorisation in a topos.

Definition 2.9. *Given a category $\mathcal{C}$, a weak factorisation system (wfs) is a pair (L, R) of classes of morphisms in $\mathcal{C}$, such that $L = {}^{\pitchfork} R$ and $R = L^{\pitchfork}$ and every morphism in $\mathcal{C}$ can be factored as rl with $r \in R$ and $l \in L$.*

We call a wfs an orthogonal factorisation system (ofs) if the liftings of L to R are unique.

The condition that $L = {}^{\pitchfork} R$ already tells us a lot about L.

Proposition 2.10. *If (L, R) is a wfs then L is saturated.*

Example 2.11. Example 2.2 together with the well known factorisation of a function f as

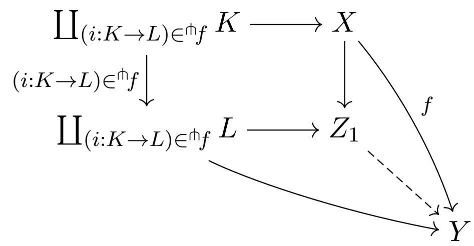
$$X \twoheadrightarrow \mathrm{im} f \hookrightarrow Y$$

that the surjective and injective functions form a weak factorisation system on the category of sets. More generally: in any elementary topos the epimorphisms and monomorphisms form a weak factorisation system.

The main feature of weak factorisation systems we are interested in is that, under some set theoretic considerations, we can take a class of morphisms I and construct a wfs, this is done in full generality in [1, Section 2.4]. We shall only need the case where we really have a set of generating morphisms.

Theorem 2.12 (Small Object Argument)**.** *Let $\mathcal{C}$ be a category with all small colimits and $I \subseteq Mor\, \mathcal{C}$ a set. Then $({}^{\pitchfork}(I^{\pitchfork}), I^{\pitchfork})$ is a wfs.*

The main idea here is that for any morphism $f : X \to Y$ we can consider all morphisms in $i \in^{\pitchfork} f$ between two fixed $i : K \to L$. Then we have a diagram

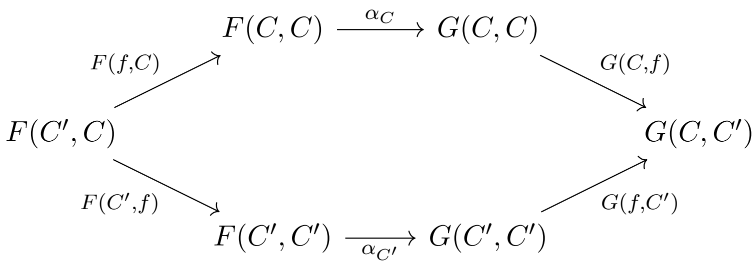


Under some set theoretic consideratiog we can perform transfinite induction by repeating this process for morphisms $Z_1 \to Z_2$, with Z_2 defined in the same way. Where it can be justified [1, Section 2.4] that eventually this factorisation has all the properties we demand of a wfs.

2.3 co/end Notation

We shall also require some of the theory of (co)ends to be able to nicely state the concept of a Kan extension. This will be a useful general framework for describing simplicial categories.

Definition 2.13. *Given two functors $F, G : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$, a dinatural transformation $\alpha : F \ddot{\to} G$ is a collection of morphisms $\alpha_C : F(C, C) \to G(C, C)$ such that for every $f : C \to C'$ α satisfies the coherence property: namely*

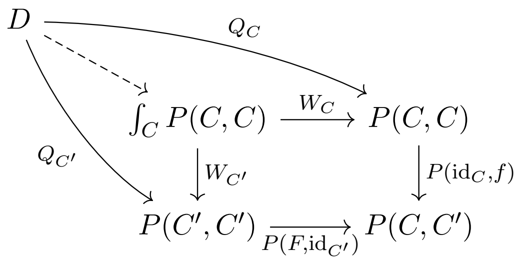


commutes.

Definition 2.14. *Given a funtor $P : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$, a wedge for P is a dinatural transformation $\Delta_D \ddot{\to} P$ where Δ_D is the bifuntor sending $\Delta_D(C', C) = D$ and $\Delta_D(f, f') = \mathrm{id}_D$.*

To avoid confusion with the Δ notation for simplicial sets we shall denote the functor Δ_D by just D.

Definition 2.15. *The end of a functor $P : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ is an object $\int_C P(C, C)$ of $\mathcal{D}$ a terminal wedge $T : \int_C P(C, C) \ddot{\to} P$, that is: for any wedge $W : Q \ddot{\to} P$ and morphism $f : C \to C'$ there is a unique arrow such that*



commutes. The dual notation is denoted $\int^C P(C, C)$.

When the categories are clear we will denote this by $\int_{\mathcal{C}} P$ and $\int^{\mathcal{C}} P$.

Proposition 2.16. *If $\mathcal{C}$ is cocomplete then the end of a funtor $P : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ is isomorphic to the equaliser of the morphisms*

$$\prod_{C \in Ob\,\mathcal{C}} F(C,C) \underset{F_*}{\overset{F^*}{\rightrightarrows}} \prod_{\varphi : C \to C'} F(C,C')$$

where $(F^*)_{(f,C,C')} = F(f,C')$ and $(F_*)_{(f,C,C')} = F(C,f)$.

The proof can be found in [12, remark 1.2.4].

Definition 2.17. *We say a functor F preserves the end of T if FW is the end of FT.*

Proposition 2.18. *If F is continuous then*

$$F\left(\int_{\mathcal{C}} T\right) \cong \int_{\mathcal{C}} FT$$

and

$$\mathcal{D}\left(\int^{C} F(C,C), D\right) \cong \int_{C} \mathcal{D}(F(C,C),D),$$
$$\mathcal{D}\left(D, \int_{C} F(C,C)\right) \cong \int_{C} \mathcal{D}(D,F(C,C))$$

The integral notation is no coincidence: many of the well known formulas from integral calculus have a counterpart in the calculus of coends.

Theorem 2.19 (Fubini)**.** *Given $F : \mathcal{C}^{op} \times \mathcal{C} \times \mathcal{E}^{op} \times \mathcal{E} \to \mathcal{D}$, we have an end $G(E,E') = \int_{C} F(C,C,E,E')$. Then*

$$\int_{\mathcal{C}\times\mathcal{E}} F \cong \int_{E}\int_{C} F(C,C,E,E) \cong \int_{C}\int_{E} F(C,C,E,E).$$

The proof can be found in [12, Section 1.4].

Theorem 2.20 (Dirac Delta for Ends)**.** *The following hold for $K : \mathcal{C}^{op} \to Set$ and $H : \mathcal{C} \to Set$*

$$K \cong \int^{C} KC \times \mathcal{C}(-,C) \cong \int_{C} Set(\mathcal{C}(C,-),KC)\,,$$
$$H \cong \int^{C} HC \times \mathcal{C}(C,-) \cong \int_{C} Set(\mathcal{C}(-,C),HC)\,.$$

2.4 Kan Extensions

There are various things referred to as a "Kan Extension" in the literature. We shall mostly be concerned with the case of the category of presheaves being the free cocompletion of $\mathcal{C}$. Since we have already taken a detour into co/end calculus Kan extensions will be rather easy to define:

Definition 2.21 (Kan Extension)**.** *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{E}$ be functors, $\mathcal{D}$ and $\mathcal{E}$. A left resp. right Kan extension of F is a functor $Cat(\mathcal{C},\mathcal{D}) \to Cat(\mathcal{D},\mathcal{E})$ that is left resp. right adjoint to the precomposition functor $F^* : Cat(\mathcal{D},\mathcal{E}) \to Cat(\mathcal{C},\mathcal{D})$. These are denoted $Lan_F G$ and $Ran_F G$ respectively.*

Theorem 2.22. *Suppose F, G are as above and $\mathcal{D}$ is cocomplete. Then $Lan_F G$ exists.*

Proof. Let

$$(\mathrm{Lan}_F G)(P) := \int^C \coprod_{x\in\mathcal{D}(FC,P)} GC$$

then Dirac Delta for ends gives us

$$\mathrm{Cat}(\mathcal{D},\mathcal{E})\left(\int^C \coprod_{x\in\mathcal{D}(FC,-)} GC, H\right) \cong \int_X \mathcal{D}\left(\int^C \coprod_{\mathcal{D}(FC,X)} GC, HX\right)$$

and Fubini further gives

$$\cong \int_{X\times C} \mathcal{D}\left(\coprod_{\mathcal{D}(FC,X)} GC, HX\right) \cong \int_{X\times C} \mathrm{Set}(\mathcal{D}(FC,X),\mathcal{E}(GC,HX))$$

and using [12, Theorem 1.4.1] and then the covariant Yoneda Lemma

$$\cong \int_C \hom(\mathcal{D}(FC,-),\mathcal{E}(GC,H-) \cong \int_C \mathcal{E}(GC,HFC)$$

which is exactly $\mathrm{Cat}(\mathcal{C},\mathcal{E})(G,HF)$ by the same citation. Then note $HF = F^*H$.

□

Corollary 2.23. *With notation as above we have that $\widehat{\mathcal{C}}$, the category of presheaves over $\mathcal{C}$, is the free cocompletion of $\mathcal{C}$ and $\mathrm{Lan}_y F \dashv \hom(F =,-)$.*

2.5 Simplicial Sets

Definition 2.24. *Let Δ be the category of finite ordinals considered as posetal categories, denoted $[n] = 0 \to 1 \to \cdots \to n$. The category sSet $= \hat{\Delta}$ has objects called simplicial sets, and we often denote $S([n])$ as S_n.*

Definition 2.25. *Let $\iota : \Delta \to Cat$ be the inclusion. We define the nerve to be $N := \iota^*$, that is $NC = \hom(\iota-, C)$ with $N(C)_k$ corresponding to chains of k composable morphisms in C.*

Definition 2.26. *Define a functor*

$$F : \Delta \to Top, \quad [n] \mapsto \left\{(x_0,\dots,x_n) | 0 \le x_i \le 1, \sum_{i=0}^n x_n = 1\right\}$$

we define the geometric realization to be Lan_F and the singular complex functor to be F^*, we denote moreover $|X| := \mathrm{Lan}_F X$ and $SX = F^*X$.

Definition 2.27. *The function $d^i : [n-1] \to [n]$ sending $k \mapsto k + \mathbf{1}_{k\ge i}$, where $\mathbf{1}$ denotes the indicator function. is called the ith coface map, and the function $s^i : [n] \to [n-1]$ sending $k \mapsto k - \mathbf{1}_{k>i}$ is called the ith codegeneracy map. Given a simplicial set X the ith degeneracy map is given by $d_i := Xd^i$ and the ith face map is given by Xs^i.*

Theorem 2.28 (Face and Degeneracy)**.** *The following hold*

$$d_i d_j = d_{j-1} d_i, i < j \tag{1}$$

$$s_i s_j = s_{j+1} s_i, i \le j \tag{2}$$

$$d_i s_j = \begin{cases} 1 & i \in \{j, j+1\}, \\ s_{j-1} d_i & i < j, \\ s_j d_{i-1} & i > j+1. \end{cases} \tag{3}$$

Proof. We prove the dual statements obtained by raising the indices. Note

$$d^j d^i([k]) = d^j([k + \mathbf{1}_{k\geq i}]) = [k + \mathbf{1}_{k\geq i} + \mathbf{1}_{k+\mathbf{1}_{k\geq i}\geq j}]$$

and

$$d^i d^{j-1}([k]) = d^i[k + \mathbf{1}_{k\geq j-1}] = [k + \mathbf{1}_{k\geq j-1} + \mathbf{1}_{k+\mathbf{1}_{k\geq j-1}\geq i}]$$

note that if $i < j$ then the $k \geq j-1$ means $k \geq i$. And we note that if $k \geq j-1$ then $k + \mathbf{1}_{k\geq j-1} \geq j$ So that the second case becomes

$$[k + \mathbf{1}_{k\geq i} + \mathbf{1}_{k+\mathbf{1}_{k\geq i}\geq j}] = d^j d^i[k]$$

The case for the degeneracy maps is similar, lastly consider

$$s^j d^i[k] = s^j[k + \mathbf{1}_{k\geq i}] = [k + \mathbf{1}_{k\geq i} - \mathbf{1}_{k+\mathbf{1}_{k\geq i}>j}]$$

if $i = j$ or $i = j+1$ then the second indicator equals the first so that they cancel. If $i < j$ then the second indicator is nonzero implies the first is as well, so $\mathbf{1}_{k+\mathbf{1}_{k\geq i}>j} = \mathbf{1}_{k\geq j-1}$ and $\mathbf{1}_{k\geq i} = \mathbf{1}_{k+\mathbf{1}_{k\geq j-1}\geq i}$ which gives exactly the terms we need. When $i > j+1$ we do the same trick in reverse. □

Example 2.29. Consider the poset $\Delta_{\leq n}$ of ordinals $\leq n$, there is an obvious inclusion into the category of finite ordinals. Hence we consider the Kan extensions

$$\begin{array}{ccc} \Delta_{\leq n} & \xrightarrow{y} & \mathrm{sSet}_{\leq n} \\ \big\downarrow & & \mathrm{sk}_n \big\downarrow \dashv \big\uparrow \mathrm{tr}_n \\ \Delta & \xrightarrow{y} & \mathrm{sSet} \end{array}$$

where sk_n is called the n skeleton and tr_n is called the n-truncation. The right adjoint is called the n-coskeleton and denoted cosk_n.

Definition 2.30. *We define the simplicial sets $\Delta[n], \Lambda_k[n], \partial\Delta[n]$ where $\Delta[n] = y_n$, $\Lambda_k[n] = \bigcup_{i\neq k} d^i\Delta_n$ the horn, and $\partial\Delta[n] = sk_{n-1}\Delta[n]$. We call horn $\Lambda_k[n]$ inner if $0 < k < n$.*

Definition 2.31. *We call a simplicial set X a Kan complex if every every horn has a filler, that is if we have a morphism $\Lambda_k[n] \to X$ there is a (not necessarily unique) morphism making*

$$\begin{array}{ccc} \Lambda_k[n] & \longrightarrow & X \\ \big\downarrow & \nearrow & \\ \Delta[n] & & \end{array}$$

commute.

A very fundamental object we shall encounter in section 4 is that of an ∞ category, one definition of which is

Definition 2.32. *An ∞-category is a simplicial set where every inner horn has a filler.*

We shall also be considering notions of simplicial theory in categories other than the category of sets. There is a pretty natural generalisation.

Definition 2.33. *Given a category $\mathcal{C}$, a simplicial object in $\mathcal{C}$ is an object of $[\Delta^{op}, \mathcal{C}]$.*

Proposition 2.34. *A simplicial group is a Kan complex, that is: the image of the forgetful functor $[\Delta^{op}, Grp] \to [\Delta^{op}, Set]$ consists of Kan complexes.*

Proof. [16, II.3.8]. □

3 Model Categories

3.1 Motivation

Given a category $\mathcal{C}$ with a subclass of morphisms $\mathcal{W} \subseteq \operatorname{Mor}\mathcal{C}$, we consider the category $\operatorname{Ho}\mathcal{C}$ where we add formal inverses of all elements of $\mathcal{W}$. It turns out that it is very beneficial to add 4 more classes of maps to get the entire picture. This was first considered by Quillen who coined the term "model category" as shorthand for "model for homotopy category" where he expanded on the work of Gabriel and Zisman regarding localizations of categories at such a class $\mathcal{W}$. The problem with the earlier work is that it is hard to find nontrivial example, which is completely avoided by introducing this more intricate structure.

Quillen originally considered this structure to discuss homotopy theory in derived categories in of abelian categories. And to work towards creating theories of homology and cohomology for these settings. This is all discussed in his book [16]. In this section we will define what a model category is together with some basic theorems 3.2; construct the most fundamental model structure 3.3, and extend this result to certain simplicial algebras 3.4.

3.2 Basics of Model Categories

We use the definition used by [17], which is equivalent to the definition used by Quillen [16].

Definition 3.1. *A model category is a bicomplete category $\mathcal{M}$, together with three classes $\mathcal{F}, \mathcal{C}, \mathcal{W} \subseteq$ Mor $\mathcal{M}$ the fibrations, cofibrations, and weak equivalences respectively. Such that $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ are wfs and $\mathcal{W}$ satisfies the 2-3 property: namely if two of f, g, fg are in $\mathcal{W}$ then all are.*

We shall refer to elements of $\mathcal{C}$ as cofibrations, $\mathcal{F}$ as fibrations, and elements of $\mathcal{W}$ as weak equivalences. We shall usually denote maps in $\mathcal{C}$ by $\hookrightarrow$, maps in $\mathcal{F}$ by $\twoheadrightarrow$ and maps in $\mathcal{W}$ by $\xrightarrow{\sim}$, furthermore maps in $\mathcal{C} \cap \mathcal{W}$ will be denoted $\overset{\sim}{\hookrightarrow}$ and maps in $\mathcal{F} \cap \mathcal{W}$ by $\overset{\sim}{\twoheadrightarrow}$.

We shall call an object X fibrant if $X \to 1$ is a fibration, the dual statement lets Y be cofibrant if $0 \to X$ is a cofibration.

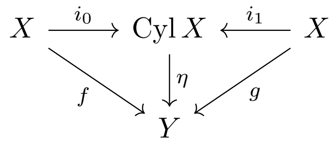
Proposition 3.2. *Every object in a model category is weakly equivalent to a fibrant object.*

Proof. If $X \to 1$ is not a fibration then we can factor it as $X \overset{\sim}{\hookrightarrow} Y \twoheadrightarrow 1$ hence X is weakly equivalent to Y. □

Definition 3.3. *Let $\mathcal{M}$ be a model category, we let $\operatorname{Ho}\mathcal{M}$ be the category where $\operatorname{Ho}\mathcal{M}_0 = \mathcal{M}_0$ and the morphisms are generated by the morphisms of $\mathcal{M}$ and the formal inverses of the weak equivalences in $\mathcal{M}$.*

Definition 3.4. *Given an object X in a model category $\mathcal{M}$ we can factor the morphism $X \coprod X \to X$ in terms of a weak equivalence and a cofibration. We call the object through which we factor the cyliner object $\operatorname{Cyl} X$. The dual notion is called a path object.*

Definition 3.5. *A left homotopy between morphisms f, g is a morphism $\eta : \operatorname{Cyl} X \to Y$ such that*



commutes. The dual notion is called right homotopy. We call two morphisms homotopic if they are both left and right homotopic. We call $f : X \to Y$ a homotopy equivalence if there is $h : Y \to X$ such that fh and hf are homotopic to the identity.

In [16, Chapter 1.1] the following is shown.

Proposition 3.6. *If $\mathcal{M}$ is a model category then* $\operatorname{Ho}\mathcal{M}$ *is equivalent to the category $\mathcal{M}_{cf}$ where the objects of $\mathcal{M}_{cf}$ are the fibrant and cofibrant objects and the morphisms are homotopy equivalences.*

We also have notions of morphisms between model categories

Definition 3.7. *Let $\mathcal{M}, \mathcal{N}$ be model categories. A Quillen adjunction is an adjunction $F \dashv G$ where $F : \mathcal{M} \to \mathcal{N}$ such that F preserves cofibrations and trivial cofibrations.*

We call a Quillen adjunction a Quillen equivalence if HoF is an equivalence of categories.

Example 3.8. The category of topological spaces forms a model category where

1. $f : X \to Y \in \mathcal{W}$ if for all $n \geq 0$ we have $f_* : \pi_n(X, x) \to \pi_n(Y, f(x))$ an isomorphism in the appropriate categories.

2. A fibration is a Serre fibration, that is: f is a fibration if for any CW complex A we have $f \in (A \times 0 \hookrightarrow A \times [0,1])^{\pitchfork}$. So

$$\mathcal{F} = \bigcap_{A \in CW-\text{comp}} (A \times 0 \hookrightarrow A \times [0,1])^{\pitchfork}$$

3. $\mathcal{C} = {}^{\pitchfork}(\mathcal{F} \cap \mathcal{W})$.

This will be denoted $\text{Top}_{\text{Quillen}}$, the proof can be found in [10, Theorem 2.4.19]. In this case $\operatorname{Ho}\text{Top}_{\text{Quillen}}$ corresponds to the homotopy category we know from algebraic topology.

In this category the cylinder objects are easy to determine: $\operatorname{Cyl} X \cong X \times [0,1]$ and from there the notion of homotopy is immediately as we would expect.

It turns out that this model category is equivalent to the model category sSet we will define in section 3.3 [16, Section II.3] via the singular complex adjunction 2.26.

Many concepts from standard category theory also apply to model categories.

Definition 3.9. *We call a model category M monoidal if there is a Quillen bifunctor $\otimes : M \times M \to M$ and a monoidal unit I such that $(M, \otimes)$ is a monoidal category and if X is cofibrant then the cofibrant replacement QI of I has $p_I \otimes X : QI \otimes X \to I \otimes X$ a weak equivalence.*

Definition 3.10. *We call a model category M enriched over a monoidal model category V if M is enriched over V as a category and the copower morphism $\otimes : M \times V \to M$ is a Quillen bifunctor. If moreover $V = \text{sSet}_{Quillen}$ then we call M a simplicial model category.*

3.3 sSet as a Model Category

Here I prove that sSet has a model category structure by following the proof structure presented in [11] in clearer lines. Note this does not rely on any topological properties beyond homotopy equivalences being isomorphisms by expanding upon the method from the Stacks Project [19, Lemma 14.30.2].

Definition 3.11. *We let a morphism f be*

1. *a weak equivalence if $|f|$ is a weak homotopy equivalence, that is, $\pi_n|f|$ is an isomorphism for $n \in \mathbb{Z}_{\geq 0}$.*

2. *a fibration if f is a kan-fibration, that is $f \in \{\Lambda^n_k \hookrightarrow \Delta^n : 0 < k < n\}^{\pitchfork}$;*

3. *a cofibration if it is a monomorphism.*

We denote these classes as $\mathcal{W}, \mathcal{F}, \mathcal{C}$ respectively. We call this the Quillen model category on sSet *(also known as the Kan model or the Kan-Quillen model) and denote it* $\mathrm{sSet}_{Quillen}$.

Lemma 3.12. *$\mathcal{W}$ satisfies the 2-3 property*

Proof. If composable $f, g \in \mathcal{W}$ then clearly $gf \in \mathcal{W}$ as well. When f and fg are in $\mathcal{W}$ then

$$|g| = |f|^{-1}|f||g| = |f|^{-1}|fg|$$

which is a homotopy equivalence, the argument for g and fg follows by symmetry. □

Theorem 3.13. *$(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ is a weak factorisation system*

This follows in its entirety by performing the small object argument on the class of horn inclusions, denote I. What needs to be shown is

Lemma 3.14. $\mathcal{F} = I^{\pitchfork}$

which is immediate by definition of a Kan fibration; and

Lemma 3.15. $\mathcal{C} \cap \mathcal{W} = {}^{\pitchfork}\mathcal{F}$

[11, corollary 1.4.1] proof slightly altered.

Proof. We let f be a trivial cofibration, from the small object argument we know we can factor $f = pj$ where $j \in {}^{\pitchfork}\mathcal{F}$ and p is a fibration. So we have a lifting

$$\begin{array}{ccc} A & \xrightarrow{j} & E \\ {\scriptstyle i}\downarrow & \overset{s}{\nearrow} & \downarrow{\scriptstyle p} \\ B & \xrightarrow{\mathrm{id}_B} & B \end{array} \qquad\qquad \begin{array}{ccccc} A & \xrightarrow{\mathrm{id}} & A & \xrightarrow{\mathrm{id}} & A \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle i} \\ B & \xrightarrow[s]{} & E & \xrightarrow[p]{} & B \end{array}$$

making f a retract of j. But ${}^{\pitchfork}\mathcal{F}$ is closed under retracts, since ${}^{\pitchfork}$ of any class is. Hence f has the left lifting property.

As for the other way around, p is a weak equivalence via the 2-3 property. Hence p is a trivial fibration and hence has the right lifting property with respect to $\mathcal{C}$, therefore we can give the same argument as for the other inclusion. □

Theorem 3.16. *$(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ is a weak factorisation system*

Here we wish to apply the small object argument 2.12 on the set of boundary inclusions, which we will denote I. So we need to show $\mathcal{C} = {}^{\pitchfork}I$ and $\mathcal{F} \cap \mathcal{W} = \mathcal{C}^{\pitchfork}$. The first is obvious. What remains is to show the inclusions $\mathcal{F} \cap \mathcal{W} \subseteq \mathcal{C}^{\pitchfork}$ $\mathcal{F}^{\pitchfork} \subseteq \mathcal{W} \cap \mathcal{C}$. The former shall be shown in Proposition 3.17 and the latter was shown in lemma 3.18.

The following is adapted from [19, Lemma 14.30.2].

Proposition 3.17. $\mathcal{F} \cap \mathcal{W} \subseteq \mathcal{C}^{\pitchfork}$

This will be argued via Zorn's lemma: we shall define some Z' between Z and W and argue a lifting exists from there, then show that this construction can be extended to a chain which is bounded from above by W.

For these purposes let $f : X \to Y \in \mathcal{F} \cap \mathcal{W}$, $g \in \mathcal{C}$, and k be the least integer such that $W_k \setminus Y_k \neq \emptyset$ and $x \in W_k \setminus Y_k$ and Z' the simplicial set with $Z'_k = Z_k \sqcup \{s^i_k x : 0 \leq i \leq k\}$ together with a, b such that

$$\begin{array}{ccc} Z & \xrightarrow{a} & X \\ \big\downarrow & & \big\downarrow{\scriptstyle \sim} \\ W & \xrightarrow[b]{} & Y \end{array}$$

commutes.

Lemma 3.18. $\mathcal{F} \cap \mathcal{W} \subseteq \{\partial\Delta[n] \hookrightarrow \Delta[n]\}^{\pitchfork}$.

Proof. We note that pointwise $\partial\Delta[n]_k = \bigcup_r \Lambda^r_k[n]$ so that $\mathcal{F}$ has the right lifting property with respect to boundary inclusions. □

Lemma 3.19. *There are morphisms φ, β making*

$$\begin{tikzcd} \partial\Delta[n] \arrow[r, "a\circ\varphi|_{\partial\Delta[n]}"] \arrow[d, hook] & X \\ \Delta[n] \arrow[ur, "\beta"'] & \end{tikzcd}$$

commute.

Proof. Let φ be the morphism corresponding to x under the Yoneda lemma and use lemma 3.18 on id_X. □

Lemma 3.20. *For any boundary inclusion $\iota : \partial\Delta[n] \hookrightarrow \Delta[n]$ and element $x \in W_n \setminus Z_n$ there is a pushout*

$$\begin{tikzcd} \partial\Delta[n] \arrow[r] \arrow[d, hook] & Z \arrow[d, "f"] \\ \Delta[n] \arrow[r] & Z' \end{tikzcd}$$

Proof. Given a diagram

$$\begin{tikzcd} \partial\Delta[n] \arrow[r] \arrow[d, "\iota"] & Z \arrow[d, "f"'] \arrow[ddr, bend left, "s"] & \\ \Delta[n] \arrow[r] \arrow[drr, bend right, "t"'] & Z' & \\ & & S \end{tikzcd}$$

we define $f' : Z' \to S$" by mapping $z \in Z_n$ to $s(z)$ and $s^i x \mapsto s^i t\varphi^{-1}(x)$. □

Now we have all the lemmas we need.

Proof of proposition 3.17. By lemma 3.19 we have a commutative diagram

$$\begin{tikzcd} \partial\Delta[n] \arrow[r] \arrow[d] & Z \arrow[d] \arrow[ddr, "a"] & \\ \Delta[n] \arrow[r] \arrow[drr, "\beta"'] & Z' & \\ & & X \end{tikzcd}$$

and by lemma 3.20 the upper left square is a pushout so that there is a unique morphism $Z' \to X$ making this commute.

This shows that we can lift from a strictly larger simplicial set Z'. Setting $Z^1 = Z'$ and letting Z^k be Z^{k-1} together with all the degeneracies of some $x \in W_n \setminus Z^{k-1}_n$, moving on to $W_{n+1} \setminus Z^{k-1}_{n+1}$ when this set is empty. This produces a chain

$$Z \subset Z^1 \subset Z^2 \subset \dots$$

which is bounded from above by W, so via Zorn's lemma this chain stabilises. But this can only happen if for some α and all $N \in \mathbb{N}$ we have $W_N \setminus Z^\alpha_N = \emptyset$, meaning $W = Z^\alpha$. □

We use several properties from [11, section 1.5]

Definition 3.21. *We call A a strong deformation retract of B if there is a monomorphism $i : A \hookrightarrow B$, retraction $r : B \to A$ of i and a homotopy $h : B \times \Delta[1] \to B$ such that $h_0 = \mathrm{id}, h_1 = ir$ and*

$$\begin{array}{ccc} A \times \Delta[1] & \xrightarrow{\pi_A} & A \\ \downarrow & & \downarrow{\scriptstyle i} \\ B \times \Delta[1] & \xrightarrow[h]{} & B \end{array}$$

commutes. We refer to the last condition as h being stationary on A.

The following is adapted from [11, proposition 1.5.4]

Lemma 3.22. *If $i : A \to B$ is a monomorphism such that A is a strong deformation retract of B then i is trivial.*

Proof. Take $r : B \to A$ the retraction and $h : B \times \Delta[1] \to B$ the homotopy making A a strong deformation retract. Then given a lifting problem

$$\begin{array}{ccc} A & \xrightarrow{a} & E \\ {\scriptstyle i}\big\downarrow & \overset{l}{\nearrow} & \big\downarrow{\scriptstyle p} \\ B & \xrightarrow[b]{} & X \end{array}$$

we have

$$\begin{array}{ccccc} A \times \Delta[1] & \xrightarrow{\pi_A} & A & \xrightarrow{a} & E \\ \downarrow & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle p} \\ B \times \Delta[1] & \xrightarrow[h]{} & B & \xrightarrow{b} & P \end{array}$$

then we can lift bh_1 to ar to get a lifting

$$\begin{array}{ccc} A & \xrightarrow{a} & E \\ \downarrow{\scriptstyle i} & \overset{ar}{\nearrow} & \downarrow{\scriptstyle p} \\ B & \xrightarrow[bh_1]{} & P \end{array}$$

as $ari = a$ and $par = bir = bh_1$. The covering homotopy extension property [9, Chapter IV, proposition 2.2] gives us a lifting of bh. This means $i \in^{\pitchfork} \mathcal{F} \subseteq \mathcal{C} \cap \mathcal{W}$. $\square$

Lemma 3.23. $\mathcal{C}^{\pitchfork} \subseteq \mathcal{W}$

Proof. Given $p \in \mathcal{C}^{\pitchfork}$, construct

$$\begin{array}{ccc} 0 & \longrightarrow & E \\ \downarrow & \overset{r}{\nearrow} & \big\downarrow{\scriptstyle p} \\ X & \xrightarrow[\mathrm{id}]{} & X \end{array}$$

making p a homotopy equivalence: we have $pr = \mathrm{id}_Y$ so $|p||r| = \mathrm{id}_{|X|}$, in particular it is homotopic to it.

When we write $1, 2$ for the two inclusions $\Delta[0] \hookrightarrow \Delta[1]$ we have

$$\begin{array}{ccc} E \times \{1, 2\} & \xrightarrow{\langle \mathrm{id}, rp \rangle} & E \\ \downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle p} \\ E \times \Delta[1] & \xrightarrow{p\pi_1} & X \end{array}$$

so that we have a strong deformation retract, making r trivial, hence r and $rp = \mathrm{id}$ are both trivial so Via 2-3 p is too. $\square$

From the result in [15] it then follows:

Corollary 3.24. *The singular complex adjunction (Definition 2.26) is Quillen.*

3.4 Cocomplete Simplicial Algebras

In this section we assume $\mathcal{C}$ is a bicomplete category and $s\mathcal{C}$ its category of simplicial objects. Suppose moreover there is an adjoint pair of functors

$$s\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathrm{sSet} \quad (F \dashv G)$$

and for any indexing category I we have a morphism $\lim_{\longrightarrow I} G(X_\alpha) \to G(\lim_{\longrightarrow I} X_\alpha)$. And this is an isomorphism when I is filtered.

We let

1. $\mathcal{W}$ be the morphisms f such that Gf is a weak equivalence of simplicial sets.
2. $\mathcal{F}$ be the morphisms f such that Gf is a fibration of simplicial sets.
3. $\mathcal{C} =^{\pitchfork} (\mathcal{W} \cap \mathcal{F})$

if moreover ${}^{\pitchfork}\mathcal{F} \subseteq \mathcal{W}$ then $s\mathcal{C}$ forms a closed model category.

Lemma 3.25. $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ *is a weak factorisation system.*

Proof. We already defined $\mathcal{C} =^{\pitchfork} \mathcal{F} \cap \mathcal{W}$ so it suffices to find $I \subseteq \operatorname{Mor} s\mathcal{C}$ that permits the small object agrument and $\mathcal{F} \cap \mathcal{W} = I^{\pitchfork}$.

We claim letting $I = \{F\iota \,|\, \iota : \partial\Delta^n \to \Delta^n\}$ is sufficient. This I permits the small object argument because

$$\hom(\partial\Delta_n, X) = \hom(\bigcup_{k=1}^{n} \Lambda^n_k, X)$$

which embeds into $\bigcup_{k=1}^n \hom(\Lambda^n_k, X)$, which permits the small object argument. Indeed moving the lifting property along the adjunction gives commutative diagrams

$$\begin{array}{ccc} F\partial\Delta^n & \longrightarrow & X \\ \big\downarrow & \nearrow & \big\downarrow f \\ F\Delta^n & \longrightarrow & Y \end{array} \qquad\qquad \begin{array}{ccc} \partial\Delta^n & \longrightarrow & GX \\ \big\downarrow & \nearrow & \big\downarrow Gf \\ \Delta^n & \longrightarrow & GY \end{array}$$

Showing $Gf \in \{\partial\Delta^n \hookrightarrow \Delta^n\}^{\pitchfork}$ and hence a trivial fibration in the model category of simplicial sets, which gives precisely that f is a trivial fibration. $\square$

Lemma 3.26. $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ *is a weak factorisation system.*

Proof. As before, we now take $I = \{F\iota | \iota : \Lambda^n_k \to \Delta^n\}$. From our assumptions it is evident that ${}^{\pitchfork}\mathcal{F} = \mathcal{C} \cap \mathcal{W}$, so again we need only show $I^{\pitchfork} = \mathcal{F}$. Indeed one can again transfer the lifting property along the adjunction as in the previous lemma. $\square$

Example 3.27. Fix R a commutative ring without a unit, we will find a model structure on $s\mathrm{Mod}_R$. There is a clear forgetful functor $G : s\mathrm{Mod}_R \to \mathrm{sSet}$. And this admits a left adjoint as for any simplicial set S the comma category $S \downarrow G$ has initial object given by the trivial simplicial R-module sending $[n] \mapsto 0$.

We need to show that ${}^{\pitchfork}\mathcal{F} \subseteq \mathcal{W}$, that is: given a morphism $f \in^{\pitchfork} \mathcal{F}_{s\mathrm{Mod}_R}$ then by definition $Gf \in^{\pitchfork} \mathcal{F}_{\mathrm{Kan}} \subseteq \mathcal{W}_{\mathrm{Kan}}$ as G is a forgetful functor, hence $f \in \mathcal{W}_{s\mathrm{Mod}_R}$, thus there is a model structure on $s\mathrm{Mod}_R$.

We find the cylinder object of $s\mathrm{Mod}_R$, given a factorisation

$$M \coprod M \longrightarrow \mathrm{Cyl}(M) \xrightarrow{\sim} M$$

the coproduct in $s\mathrm{Mod}_R$ is computed pointwise, and the coproduct in Mod_R is a biproduct, hence we have $G(M \coprod M) \cong GM \times GM$, we find a morphism $GM \coprod GM \to G\mathrm{Cyl}(M)$ via the universal property and applying G to the coinclusion $M \to GM \coprod GM$

$$\begin{array}{ccccc} & & GM & & \\ & \swarrow & & \rightleftarrows & \\ GM \coprod GM & & \dashrightarrow & & GM \times GM \longrightarrow G\mathrm{Cyl}(M) \\ & \nwarrow & & \rightleftarrows & \\ & & GM & & \end{array}$$

so that we must have $G\mathrm{Cyl}(M) \cong \mathrm{Cyl}(GM) \cong GM \times \Delta[1]$. It remains to find a simplicial R-module structure on $M \times \Delta[1]$, for which purpose the action at n being $r(m,t) := (rm, t)$ works.

3.5 Cisinski Toposes

In this section we shall go into a technique to construct a model category out of a presheaf topos, the proofs are outside the scope of this thesis and can be found in [6] and [5].

Theorem 3.28 (Cisinski)**.** *Let $\mathcal{T}$ be a Grothendieck Topos. Then there is a model category structure on $\mathcal{T}$ where the cofibrations are the monomorphisms and the fibrant objects are those objects X where $(X \to 1) \in ({}^{\pitchfork}\mathcal{F}_{\mathcal{T}})^{\pitchfork}$*

the particular subcase we are interested in will take some more definitions to state. Namely we are interested in presheaf toposes. The remainder of this section will be dedicated to describing all the machinery needed to describe the Cisinski Model on a presheaf topos.

In this section we let A be a small category. With a cellular model we will mean a set $M \subseteq \mathrm{Mor}\,\hat{A}$ such that ${}^{\pitchfork}(M^{\pitchfork})$ is the class of monomorphisms in $\hat{A}$.

We shall start by defining a notion of homotopy and build a model category around it.

Definition 3.29. *Given X a presheaf. A cylinder of X is a diagram*

$$\begin{array}{ccc} X \coprod X & \xrightarrow{\langle \mathrm{id}, \mathrm{id} \rangle} & X \\ & {\scriptstyle \langle \partial_0, \partial_1 \rangle} \searrow \quad \nearrow {\scriptstyle \sigma} & \\ & IX & \end{array}$$

More generally, we call an endofunctor $I : A \to A$ with a morphism $\langle \partial_0, \partial_1 \rangle : 1 \coprod 1 \to I$ and a morphism $\sigma : I \to 1$ such that we have a factorisation

$$X \coprod X \xrightarrow{\langle \partial_0 \otimes \mathrm{id}, \partial_1 \otimes \mathrm{id} \rangle} I \otimes X \xrightarrow{\sigma \otimes \mathrm{id}} X$$

and we call I exact if

1. *I commutes with small colimits and preserves monomorphisms.*

2. *When $\varepsilon \in \{0,1\}$ and $j : K \to L$ is a morphism, there is a pullback*

$$\begin{array}{ccc} K & \xrightarrow{\quad j \quad} & L \\ {\scriptstyle \partial_\varepsilon \otimes \mathrm{id}}\downarrow & \lrcorner & \downarrow{\scriptstyle \partial_\varepsilon \otimes \mathrm{id}} \\ I \otimes K & \longrightarrow & I \otimes L \end{array}$$

Under these notions we can define the class of I-Anodyne morphisms similarly to how one may define this in the Quillen structure on sSet.

Definition 3.30. *Given I an exact cylinder and S a class of monomorphisms, we call a class An containing S I-anodyne if for $\varepsilon \in \{0,1\}$*

1. *There is a set Λ such that $An = {}^{\pitchfork}(\Lambda^{\pitchfork})$.*

2. *If $K \to L$ is in An so is $I \otimes K \cup \varepsilon \otimes L \to I \otimes L$.*

3. *If $K \to L$ is in An so is $I \otimes K \cup \partial I \otimes L \to I \otimes L$.*

we call the pair (I, An) a homotopical structure.

furthermore we can now define a notion of homotopy:

Definition 3.31. *Given $f_0, f_1 : K \to X$. An I-homotopy $f_0 \to f_1$ is a morphism $h : I \otimes K \to X$ such that for $\varepsilon \in \{0,1\}$ we have $h(\partial_\varepsilon \otimes \mathrm{id}) = f_\varepsilon$.*

We let $\sim$ be the least equivalence relation where f_0 being I-homotopic to f_1 implies $f_0 \sim f_1$, and let $[K, X] = \hom(K, X)/\sim$.

Definition 3.32. *Given a homotopical structure (I, An) we let a naive fibration be a member of $An^{\pitchfork}$. We call a presheaf X fibrant if $X \to 1$ is naive.*

Then it is proven in [6, Theorem 2.4.19] that the following holds.

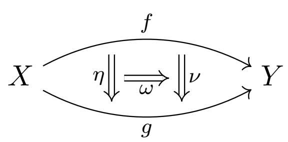
Theorem 3.33 (Cisinski)**.** *Let A be a set and I an exact cylinder on $\hat{A}$ together with a choice of fibrant objects. Then there is a model category structure on $\hat{A}$ where the cofibrations are the monomorphisms, the weak equivalences are the morphisms $f : X \to Y$ such that for every fibrant W we have $f^* : [Y, W] \to [X, W]$ a bijection.*

Usually we will omit the choice of fibrant objects and stick to the largest one allowed by the choice of exact cylinder.

4 Infinity-Categories

4.1 The Idea of Infinity Categories

In this section we will justify the term ∞-category and describe what properties we want this structure to have. Let us start by considering the topological notion of homotopy. If we have two morphisms $f, g : X \to Y$ of topological spaces then a homotopy from f to g is a continuous map $\eta : X \times [0,1] \to Y$ such that $\eta(x, 0) = f(x)$ and $\eta(x, 1) = g(x)$ for all $x \in X$. There is nothing stopping us from defining a homotopy between two homotopies $\eta, \nu : X \times [0,1] \to Y$ to be a continuous map $\omega : X \times [0,1] \times [0,1] \to Y$ satisfying an obvious condition.



This gives us an infinite chain of n homotopies between $n-1$ homotopies. And we will call such a category an n-category. If this process at some point turns trivial, that is all the morphisms above some level k are isomorphisms then we will call this an (n,k)-category.

This notion sounds very similar to the regular definition of a category, and to be able to describe things that are not so easy to describe otherwise we shall require associativity of n morphisms only hold up to an $n+1$ morphism. And this is in line with the example from before: things might be homotopic in many different ways but often we only care whether they are homotopic at all.

Since there is no finite bound on the level of morphisms we can reach with our example of homotopies it might be natural to ask whether we can let $n=\infty$ and what sort of object we then obtain. Let us concern ourselves with the easiest nontrivial case, that being $(\infty,1)$-categories.

Once way to think about this is in terms of liftings against horn inclusions, which we shall discuss in section 4.2. This approach yields very natural generalisations of various categorical notions such as functors and presheaves.

Another way to think about this is it being a category enriched over an $(\infty,0)$-category which we would like to be a groupoid, so we could for instance consider Kan complex enriched categories since it turns out Kan complexes form a model for ∞-groupoids. This is similar to the idea of the Bergner model for $(\infty,1)$-categories (Discussed in 4.3).

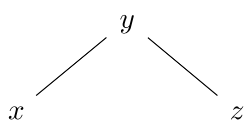
It turns out that a theorem of Ciskinski (discussed in 4.4) gives us a large part of the theory of complete Segal Spaces for free once we have this notion an $(\infty,1)$-category (discussed in 4.4).

4.2 Joyal Structure

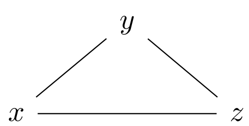
The first notion of an $(\infty,1)$-category will come from definition 2.30, which we will restate here. The idea of this first definition is that a category can be defined as a simplicial set C where every lifting problem

$$\begin{array}{ccc} \Lambda_k[n] & \longrightarrow & C \\ \big\downarrow & \nearrow & \\ \Delta[n] & & \end{array}$$

has a unique lifting f. It is evident this indeed defines a category in the usual sense if we take our objects to be $C[0]$ and morphims $C[1]$. We are in particular interested in the lifting against the inclusion $\Lambda_1^2 \hookrightarrow 2$: this means that if we take three vertices x,y,z with edges as



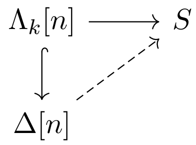
then we can fit them uniquely into a 2-simplex



which is reminiscent of commutativity, and turns into the exact condition if we partially order this diagram by directing the arrows to point only to the right. Then the uniqueness of the lifting ensures associativity.

This can easily be adapted to fit the idea of an ∞-category being only associative up to homotopy by no longer requiring this lifting be unique.

Definition 4.1. *An $(\infty, 1)$-category is a simplicial set S such that every inner horn inclusion has a filler against S, meaning that for $0 < k < n$*



This is more than just an analogy and it turns out a lot of category theoretic notions can be directly generalised to this notion of an $(\infty, 1)$-category.

Let us start with all the surface level notions we would like to have.

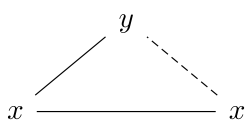
Definition 4.2. *Let S, T be $(\infty, 1)$-categories in the sense of Joyal, a functor $S \to T$ is a natural transformation of simplicial sets $S \Rightarrow T$.*

What this definition is saying is that at every level n there is a function $\eta_n : S_n \to T_n$, so in particular to every vertex ("object") x of S we assign a vertex $\eta_0(x)$, and the same for every n-morphism. So in this sense this really acts like a functor in the way we are used to. As for natural transformations we can apply the same trick. And that in turn gives rise to obvious notions of $(\infty, 1)$-limits and colimits and $(\infty, 1)$-presheaves.

4.3 Bergner Structure

The second definition we shall use uses the fact that the hom-set of an $(\infty, 1)$ category would have to be an $(\infty, 0)$-category, which we can consider an ∞-groupoid in the sense that every morphism is invertible up to a higher morphism.

If in definition 4.2 we instead require that a category has liftings against all horn inclusions we can also fill diagrams as



which exactly gives inverses for every morphism. Hence we have the following definition.

Definition 4.3. *An $(\infty, 1)$-category is a simplicially enriched category S whose hom-set is a Kan complex.*

Which will be the fibrant objects in the Bergner model. Here the analog of functors, natural transformations, composition, and so on are all very natural as these can just be taken to be their enriched counterparts.

4.4 Complete Segal Spaces

A third notion can be obtained from the same starting point as for the Bergner structure, namely for an $(\infty, 1)$-category we would want the hom-sets to be a groupoid, or alternatively the nerve to map into Kan complexes at the level of objects.

Hence we consider bisimplicial sets, that is contravariant functors $\Delta \to \text{sSet}$. Here we try to define an $(\infty, 1)$-category by defining what its nerve is. That is, for any $n \in \mathbb{N}$ we fix a Kan complex $X_{n,-}$ so that $X_{-,-} \in \widehat{\text{sSet}}$. Translating this to homotopic language, we have that the chains of composable n-morphisms are an ∞-groupoid, which is the main idea of complete segal spaces.

A more high-brow approach to this is by considering a Cisinski structure on a presheaf topos $\hat{C}$, by considering the projection $\pi : C \times \Delta \twoheadrightarrow C$ we obtain an adjunction

$$\begin{array}{ccc} C \times \Delta & \xrightarrow{\;y\;} & \widehat{C \times \Delta} \\ \Big\downarrow{\scriptstyle \pi} & & \big(\dashv \big) \\ C & \xrightarrow{\;y\;} & \widehat{C} \end{array}$$

One would expect that together with some niceness conditions this would induce a Quillen equivalence $\hat{C} \simeq_Q \widehat{C \times \Delta}$. And this is in fact true [5, Proposition 2.3.27] if we define cofibrantions and weak equivalences pointwise.

One application of this theorem is that we obtain a model category structure on $\widehat{\text{sSet}}$ that is Quillen equivalent to sSet_{Joy} by letting $\mathcal{C} = \text{Set}$. Let us denote this category by $\text{ssSet}_{\text{CSS}}$. The more conventional approach to defining this structure is through considering the fibrant objects in several other structures.

5 Equivalence of Models for Infinity-Categories

5.1 Motivation

This section will be dedicated to using the theory of model categories to give the right notion of an equivalence between $(\infty, 1)$-categories. It turns out they can be realised as the fibrant objects in certain model categories, and it is well known what the right notion of equivalence of model categories is. Here our aim is to have a structure which encapsulates the situation in the category Toph of topological spaces up to homotopy; where we have homotopy equivalences between morphisms up to 2-homotopies between those homotopies and so on. We will expand upon two different ways of think of $(\infty, 1)$-categories introduced in 4.1 and show that they fit in as the fibrant objects in two model categories (5.2). Furthermore we will explicitly show these two model categories are Quillen-equivalent (5.3), requiring us to develop the theory of necklaces.

5.2 Joyal and Bergner Structures

We adapt the Quillen model on sSet so that the fibrant objects are precisely the ∞-categories. This is Joyal's model on sSet and we shall denote it sSet_{Joy}. Here the three distinguished classes are

1. $\mathcal{W}_{\text{Joy}}$ consists of the morphisms f such that for every ∞-category K the functor $[f, K]$ is an isomorphism.

2. $\mathcal{C}_{\text{Joy}}$ consists of the monomorphisms.

3. $\mathcal{F}_{\text{Joy}} = (\mathcal{W}_{\text{Joy}} \cap \mathcal{C}_{\text{Joy}})^{\pitchfork}$.

Our approach with the small object argument for the Quillen model runs into difficulties here, since there is no known set of generating trivial cofibrations for this structure [1, Remark 6.2.5].

It may be tempting to defer to the class of trivial Joyal-fibrations being precisely the inner fibrations whose induced homotopy functor is an isofibration. This was taken as true because of a paper by Danny Stevenson [20, Theorem B], where it was incorrectly assumed this characterisation describes all trivial Joyal-fibrations, and not just those where the codomain is a quasicategory. The pushout of Δ^2 and Δ^0 over Δ^1 yields a counterexample as presented in [4].

Instead we refer to theorem 3.33, which immediately gives this to be a model for the obvious choice of exact cylinder $I_n = \{0, 1\}$. Note that the fibrant objects in this model are precisely the ∞-categories.

The category $\mathrm{sSet}_{\mathrm{Joy}}$ shall be first example of a model for $(\infty, 1)$-categories. [1, Theorem 5.2.2] offers a large list of Quillen equivalent model categories, any one of which is a model for $(\infty, 1)$-categories. We shall work out a few examples.

Definition 5.1. *We let* sCat *be the category of simplically enriched categories.*

We shall define a model on this category.

Definition 5.2. *We call a simplicial functor $F : \mathcal{C} \to \mathcal{D}$ a Dwyer-Kan equivalence if*

1. *For all objects a, b of $\mathcal{C}$ we have*

$$\hom(a, b) \to \hom(Fa, Fb)$$

   a Quillen equivalence.

2. *$\pi_0 f$ is an equivalence of categories.*

Definition 5.3. *We call a simplicial functor $F : \mathcal{C} \to \mathcal{D}$ a Dwyer-Kan fibration if*

1. *For all objects a, b of $\mathcal{C}$ we have*

$$\hom(a, b) \to \hom(Fa, Fb)$$

   a Quillen fibration.

2. *F is an isofibration, meaning that if $e : Fa \to b$ is a homotopy equivalence then there is a homotopy equivalence $e' : a \to c$ such that $Fe' = e$.*

Then the following was proven in [2]:

Theorem 5.4 (Bergner)**.** *There is a model category structure on* sCat *where*

1. *The weak equivalences are the Dwyer-Kan Equivalences.*

2. *he fibrations are the Dwyer-Kan Fibrations.*

We shall denote this by $\mathrm{sCat}_{Bergner}$.

In this structure the fibrant objects are the $(\infty, 1)$-categories in the sense of 4.3.

We shall show there is a Quillen equivalence between $\mathrm{sCat}_{\mathrm{Bergner}}$ and $\mathrm{sSet}_{\mathrm{Joy}}$. Namely a a pair of functors$\mathfrak{C} : \mathrm{sSet} \to \mathrm{sCat}$ and $\tilde{N} : \mathrm{sCat} \to \mathrm{sSet}$. which are adjoint.

Proposition 5.5. *There is an adjunction $\mathfrak{C} \dashv \tilde{N}$*

We shall call $\mathfrak{C}$ the rigidification and $\tilde{N}$ the homotopy coherent nerve.

Proof. We shall construct these funtors via a Kan-extension so that we get the adjunction for free. Hence we need a functor $\Delta \to \mathrm{sSet}$ and $\Delta \to \mathrm{sCat}$. For the first the Yoneda embedding suffices, the second is a lot harder to construct.

Consider forgetful adjunction between Cat and the category of reflexive directed graphs, call the functors U and F. Consider the comonadic endofunctor $G = FU$ which freely adds all the paths of composable morphisms and let ε and η be the unit and counit of this adjuntion. We define the simplicial object which sends $[n]$ to

$$G[n] \;\; \begin{array}{c} \xleftarrow{\quad \epsilon FU \quad} \\ \xrightarrow{\quad F\eta U \quad} \\ \xleftarrow{\quad FU\epsilon \quad} \end{array} \;\; G^2[n] \;\; \begin{array}{c} \xrightarrow{\quad \epsilon FUFU \quad} \\ \xleftarrow{\quad F\eta UFU \quad} \\ \vdots \\ \xleftarrow{\quad FUF\eta U \quad} \\ \xrightarrow{\quad FUFU\epsilon \quad} \end{array} \;\; \cdots$$

Note that if a simplicial object in Cat leaves the objects invariant then it induces a simplicially enriched category [18]. □

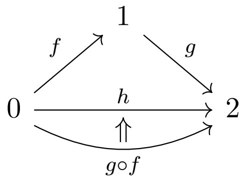
This mimics the behaviour of the truncation-nerve adjunction in ordinary category theory, in fact the behaviour on representables is almost identical.

Our goal now is to prove the following.

Theorem 5.6. *The rigidification adjunction 5.5 is Quillen.*

For this we shall require a more explicit description of $\mathfrak{C}$. From now on we shall define an ordering on simplicial sets based on the distance to the final vertex.

The intuition of our following construction comes from composition being defined only up to homotopy in an ∞-category, namely we have a diagram in Δ^2 considered as an ∞-category.



This is modelled by the composition of the comonadic unit $\eta^n : G^n \Rightarrow \text{id}$. Alternatively, we can cosider all paths in our simplicial sets and let paths be homotopic if they have the same start and endpoint. Our main concern will be describing the simplicial set $\mathfrak{C}(\Delta^n)(i,j)$.

Lemma 5.7. *Let P^n_{ij} be the poset of paths between the vertices i and j in Δ^n, ordered by inclusion: paths a, b have $a \le b$ if b goes through all the same vertices as a. Then $\mathfrak{C}(\Delta^n)(i,j) \cong NP^n_{ij}$. We shall omit the n when this is clear.*

Lemma 5.8. *If $i > j$ then $\mathfrak{C}(\Delta^n)(i,j) \cong (\Delta^1)^{j-i-1}$.*

Proof. Consider NP^n_{ij} as the directed graph with vertices $0, \dots, n$ with an edge $i \to j$ if and only if $i \le j$. Then note $NP^n_{ij} = NP^n_{0(j-i)}$, $NP^n_{01} = \Delta^1$, $NP^n_{0i} = \Delta^1 \times NP^n_{0(i-1)}$. □

Note that $\partial\Delta^n$ and Λ^k_n can be written as a coequaliser and a pushout respectively, since $\mathfrak{C}$ preserves colimits there is an obvious way to compute $\mathfrak{C}(\partial\Delta^n)(i,j)$ and $\mathfrak{C}(\Lambda^k_n)(i,j)$.

These result can be extended to wedge sums of representable simplices, that is identifying the endpoint and starting point of two simplices, see figure 1 for an example.

Definition 5.9. *Let $S, T \in$ sSet where S has final vertex a and T has initial vertex b , then the wedge sum is defined as.*

$$S \vee T = (S \amalg T)/(a \sim b)$$

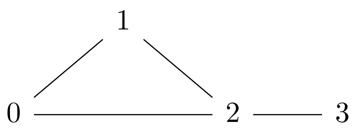
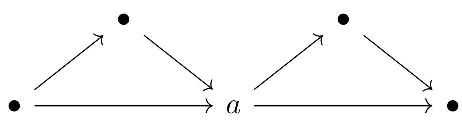


Figure 1: $\Delta^2 \vee \Delta^1$

Lemma 5.10. *Let a be the final vertex of Δ^n, then $\mathfrak{C}(\Delta^n \vee \Delta^m)(i,j) = N(P_{ia} \times P_{aj})$*

Proof. For illustrative purposes let us work with $\Delta^2 \vee \Delta^2$. Note that any path from i to j must pass through a, hence a path from i to j consists of a path from i to a followed by a path from a to j.



□

Our end goal will be to define any simplex as a colimit of chains of wedge sums of representables. Which shall require use to look at a larger class of simplicial sets.

5.3 Necklaces, Flags, and Zig-Zags

Definition 5.11. *We call a simplicial set of the form*

$$\bigvee_{i=1}^{k} \Delta^{n_i}$$

with all $n_i \geq 1$ a necklace. Each Δ^{n_i} is called a bead and the point where Δ^{n_i} and $\Delta^{n_{i+1}}$ are identified is called a joint.

Definition 5.12. *By an ordering on a simplicial set X we mean an antisymmetric relation $\preceq$ on X_0 such that $x \in X_n$ has $x(0) \preceq \cdots \preceq x(n)$ and these vertices determine x uniquely.*

Theorem 5.13. *Let T be a necklace, J_T the set of joints in T and $V_T = T_0$ the set of vertices. We let $a, b \in N_0$ and let*

$$C_T(a,b) := \{S \subseteq V_T : a, b \in S, J_T \subseteq S\}$$

Setting the set of objects be T_0 we obtain a simplicial category NC_T and there is a natural isomorphism with the nerve $\mathfrak{C}(T) \cong NC_T$

Proof. Induction on lemma 5.10. □

So now we are in the situation where we have an easy description of $\mathfrak{C}(T)$ for T a necklace. Our next step is to write every simplicial set as a colimit of necklaces in the spirit of Kan extensions, and then use that $\mathfrak{C}$ is a left adjoint so it must preserve colimits.

Given a simplicial set S if a, b are its initial and final vertex then we write $S_{a,b}$. We furthermore let $\mathcal{N}ec$ be the full subcategory of simplicial sets which are necklaces with the obvious choice of morphisms. Given a necklace we denote α, ω the initial and final vertex, if we let $(\mathcal{N}ec \downarrow S)_{a,b}$ be the category of necklaces over $S_{a,b}$, meaning the objects are pairs $(N, N \to S_{a,b})$. We shall let

$$E_S(a,b) := \operatorname*{colim}_{(T, T\to S) \in (\mathcal{N}ec \downarrow S)_{a,b}} \mathfrak{C}(T)(\alpha, \omega)$$

Theorem 5.14. *For any simplicial set S there is an isomorphism $E_S \cong \mathfrak{C}(S)$.*

Proof. This is proven in [7, Proposition 4.3].

Note that limits in presheaf categories are computed pointwise, so it suffices to show this map is monic and epic. The main idea is that the Yoneda lemma gives us a commutative diagram

$$\begin{array}{ccc} (\mathrm{col}_{\Delta^k \to S} E_{\Delta^k})(a,b) & \xrightarrow{t} & E_S(a,b) \\ \cong\downarrow & & \downarrow \\ (\mathrm{col}_{\Delta^k \to S} \mathfrak{C}(\Delta^k))(a,b) & \xrightarrow[\cong]{} & \mathfrak{C}(S)(a,b) \end{array}$$

where the left vertical map is an isomorphism because $E_{\Delta^k} \cong \mathfrak{C}(\Delta^k)$ is immediate. The commutativity of the diagram forces t to be monic. All that remains to be shown is that t is epic, such that the right vertical map must be an isomorphism.

Given any n-simplex $x \in E_s(a,b)[n]$ the Yoneda lemma gives us a necklace T and a morphism $f :\to S_{a,b}$ which represents x. Yielding a commutative diagram

$$\begin{array}{ccc}
(\mathrm{col}_{\Delta^k \to T}\, \mathfrak{C}(\Delta^k))(\alpha,\omega) & \xrightarrow{\cong} & \mathfrak{C}(T)(\alpha,\omega) \\
\cong\uparrow & & \uparrow \\
(\mathrm{col}_{\Delta^k \to T}\, E_{\Delta^k})(\alpha,\omega) & \longrightarrow & E_T(\alpha,\omega) \\
f\downarrow & & E_f\downarrow \\
(\mathrm{col}_{\Delta^k \to S}\, E_{\Delta^k})(a,b) & \xrightarrow[t]{} & E_s(a,b)
\end{array}$$

by assumption E_f has x in its image, so we need only show the middle horizontal map is surjective. But this is immediate by commutativity of the upper square. □

Since colimits in presheaf categories can be computed as a coproduct modulo an equivalence relation, we now have an explicit way to compute the rigidification of arbitrary presheaves. We shall need a few more definitions to state the specific equivalence relation.

Definition 5.15. *A flagged necklace is a pair* $(T, \overrightarrow{T})$ *where* T *is a necklace and* $\overrightarrow{T}$ *is a chain of inclusions*

$$\overrightarrow{T} = T^0 \subseteq \cdots \subseteq T^n$$

where $J_T \subseteq T^0$ *and* $T^n \subseteq V_T$*. Its length is the number of inclusions. We call a flag flanked if* $T^0 = J_T$ *and* $T^n = V_T$*.*

Now we have all the tools to write out $\mathfrak{C}(S)(a,b)$

Theorem 5.16. *An* n*-simplex in* $\mathfrak{C}(S)(a,b)$ *is an equivalence class* $[T, T \to S, \overrightarrow{T}]_\sim$ *where* T *is a necklace,* $T \to S$ *is a morphism of simplicial sets sending* $\alpha_T \mapsto a$ *and* $\omega_T \mapsto b$*,* $\overrightarrow{T}$ *is a flag of* T*.*

The equivalence relation is the least equivalence relation where $(T \to S, \overrightarrow{T}) \sim (U \to S, \overrightarrow{U})$ *if there is a map of necklaces* $f : T \to U$ *with* $\overrightarrow{U} = f_*(\overrightarrow{T})$*.*

Proof. Note that for a necklace T one has that elements of $\mathfrak{C}(T)$ are identified with a flag of T by theorem 5.13. $\mathfrak{C}(S)$ is hence isomorphic to the colimit of sets of flags and the bottom map in

$$\begin{array}{ccc}
 & E_S(a,b) & \\
\nearrow & & \nwarrow \\
\mathfrak{C}(T)(\alpha_T,\omega_T) & \longrightarrow & \mathfrak{C}(U)(\alpha_U,\omega_U)
\end{array}$$

exists if we can idenfity flags of U with flags of T. □

Lemma 5.17. *Every equivalence class* $[T, f : T \to S, \overrightarrow{T}]_\sim$ *has a representative which is nondegenerate, where the flag is flanked, and* f *is a monomorphism,*

Proof. [7, Lemma 4.12]. □

Now to show how this can be used to compute rigidifications.

Example 5.18. Let X be a finite set and let x be its cardinality. Consider the constant simplicial set $X_n = X$. Then all the necklaces which inject into X necessarily have the form

$$\bigvee_{\sum n_i \le x} \Delta^{n_i}$$

and any such necklace injects into X at level 0 . But for any Δ^k simplicial set we have $\binom{k+1}{n}$ n-simplices at level n. Hence we need to look at simplicial sets where

$$\max\left\{\binom{k+1}{j} : 0 \leq j \leq k+1\right\} \leq x$$

which occurs at $j = \lfloor \frac{k+1}{2} \rfloor$ where we have value

$$\frac{(k+1)!}{(\lfloor \frac{k+1}{2} \rfloor !)^2 \lceil \frac{k+1}{2} \rceil}$$

need to consider k where that value is at most x. For a necklace with ℓ joints we obtain a maximum of

$$\sum_{i=0}^{\ell} \frac{(k_i+1)!}{(\lfloor \frac{k_i+1}{2} \rfloor !)^2 \lceil \frac{k_i+1}{2} \rceil}$$

at any level, so those are all the necklaces we need to consider.

For instance, if $x = 10$ we consider all sums of natural numbers which are less than 10 and check which satisfy the bound. Running this on the computer yields

$$\Delta^0, \Delta^1, \Delta^2, \Delta^3, \Delta^1 \vee \Delta^2, \Delta^4, \Delta^1 \vee \Delta^3,$$
$$\Delta^2 \vee \Delta^3, \Delta^2 \vee \Delta^4, \Delta^1 \vee \Delta^2 \vee \Delta^3$$

hence we can take these necklaces, together with the monomorphisms into X and the obvious flanked flags as representatives for $\mathfrak{C}(X)(a,b)$. Hence one has

$$\begin{aligned}\mathfrak{C}(X)(a,b) \cong \operatorname{colim}\big\{&\emptyset, (\Delta^1)^{\omega_{\Delta^1}-\alpha_{\Delta^1}-1}, (\Delta^1)^{\omega_{\Delta^2}-\alpha_{\Delta^2}-1}, (\Delta^1)^{\omega_{\Delta^3}-\alpha_{\Delta^3}-1}, (\Delta^1)^{\omega-\alpha-1}, (\Delta^1)^{\omega_{\Delta^4}-\alpha_{\Delta^4}-1}\\ &NC_{\Delta^1\vee\Delta^2}(\alpha,\omega), NC_{\Delta^1\vee\Delta^3}(\alpha,\omega), NC_{\Delta^1\vee\Delta^4}(\alpha,\omega),\\ &NC_{\Delta^2\vee\Delta^3}(\alpha,\omega), NC_{\Delta^2\vee\Delta^4}(\alpha,\omega), NC_{\Delta^1\vee\Delta^2,\vee\Delta^3}(\alpha,\omega)\big\}\end{aligned}$$

together with all the morphisms to $\mathfrak{C}(X)$. This does not appear to simplify any further for arbitrary a, b. If we compute $\mathfrak{C}(X)(0,10)$ then we obtain

$$\begin{aligned}\mathfrak{C}(X)(0,10) \cong \operatorname{colim}\big\{&\emptyset, (\Delta^1)^0, (\Delta^1)^1, (\Delta^1)^2, (\Delta^1)^3, (\Delta^1)^0 \times (\Delta^1)^1, (\Delta^1)^0 \times (\Delta^1)^2,\\ &(\Delta^1)^1 \times (\Delta^1)^2, (\Delta^1)^1 \times (\Delta^1)^3, (\Delta^1)^0 \times (\Delta^1)^1 \times (\Delta^1)^2\big\}.\end{aligned}$$

Lemma 5.19. *$\mathfrak{C}$ takes monomorphisms to cofibrations.*

Proof. Because $\mathfrak{C}$ preserves colimits, it suffices to show it takes generating cofibrations to cofibrations. These are the boundary inclusions. Note that if $i > 0$ or $j < n$ then the injections of necklaces $T \to \partial\Delta^n$ with begin point i and endpoint j are exactly the injections into Δ^n. In the remaining case lemma 5.8 yields us the inclusion $f : \partial(\Delta^1)^{n-1} \hookrightarrow (\Delta^1)^{n-1}$. Let U be the functor sSet $\to$ sCat where for $x, y \in U(s)$ we have $\hom(x,x) = \hom(y,y) = \{*\}$, $\hom(x,y) = S, \hom(y,x) = \emptyset$. Then f can be written as the pushout along the obvious maps

$$\begin{array}{ccc} U[\partial(\Delta^1)^{n-1}] & \longrightarrow & \mathfrak{C}(\partial\Delta^n) \\ \big\downarrow U(b) & \ulcorner & f \big\downarrow \\ U[(\Delta^1)^{n-1}] & \longrightarrow & \mathfrak{C}(\Delta^n) \end{array}$$

and is hence a cofibration. $\square$

Lemma 5.20. *If D is a simplicial category then $\tilde{N}D$ is an ∞-category.*

Proof. Transferring along the rigidification adjunction we see that we must only show $D \to 1$ is a fibration, or equivalently that every inner horn inclusion rigidifies to an acyclic cofibration. We already know it is a cofibration so we just need to show it is a weak equivalence.

As before we only need to look at the case $\mathfrak{C}(\Lambda^n_k)(0,n) \hookrightarrow \mathfrak{C}(\Delta^n)(0,n)$. Since we can write horns as a union of faces and $\mathfrak{C}$ preserves colimits, we see that $\mathfrak{C}(\Lambda^n_k)$ is going to be $(\Delta^1)^{n-1}$ with a face missing, which is homotopic to $(\Delta^1)^{n-1}$ and hence the inclusion is a weak equivalence. □

Proposition 5.21. *$\mathfrak{C}$ preserves weak equivalences.*

Proof. We first claim $\mathfrak{C}(X \times \mathrm{cosk}_0[1])$ is a cylinder object for $\mathfrak{C}(X)$, recall the coskeleton functor from example 2.29.

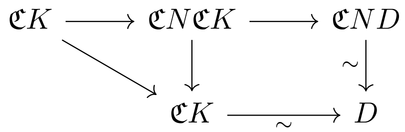
Let D be fibrant, then homotopy classes $[\mathfrak{C}X, D]$ can be computed as the coequaliser of morphisms

$$\mathrm{sCat}(\mathfrak{C}(X \times E^1), D) \rightrightarrows \mathrm{sCat}(\mathfrak{C}X, D)$$

where we can use the adjunction to see that $[\mathfrak{C}(X), D] = [X, ND]_{E^1}$. Since we know ND is an ∞-category we conclude that for every Joyal equivalence $A \to B$ we have $[B, ND]_{E^1} \to [A, ND]_{E^1}$ is a bijection, see theorem 3.33. □

Theorem 5.22. *The rigidification adjunction is a Quillen equivalence.*

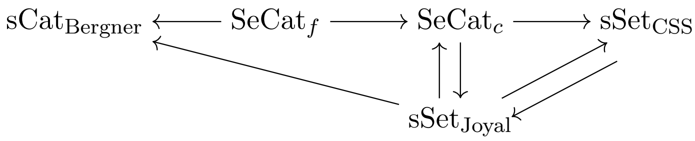
Proof. We see it is a Quillen adjunction by lemma 5.19 and proposition 5.21. To show it is an equivalence we find weak equivalences $\mathfrak{C}N \to \mathrm{id}$ and $N\mathfrak{C} \to \mathrm{id}$ up to homotopy. The first can be found in [8, Proposition 5.8]. For the latter consider the fibrant replacement $\mathfrak{C}(K) \to D$. Since $\mathfrak{C}$ reflects weak equivalences [8, Proposition 8.1] we obtain a commutative diagram



where we get that the required map is a weak equivalence by the 2-3 property. □

6 Conclusion

We described the theory of model categories as it arises in the context of homotopy theory. We examined the Quillen Model category of simplicial sets. And showed how various different interpretations of $(\infty, 1)$-categories are equivalent. There are still more models which we haven't covered and in future work we could for instance have a look a more of them. Bergner has classified 5 models [3] and showed they fit in a diagram as



where SeCat_f and SeCat_c are the fibrantly and cofibrantly generated model category structures on the category of Segal-precategories.

One curiosity is that being Quillen equivalent is not an equivalence relation, and in the diagram we wee that several categories have zig-zags of Quillen equivalences between them but no direct arrow because the adjunctions go the wrong way. For all of them it is not known whether there is a direct Quillen equivalence, and this could be the subject of further work.

Further research could also involve models for (∞, n)-categories. For example there is a similar classification of known models for $(\infty, 2)$-categories presented in [14, Theorem 0.0.3.] where the same composition problem occurs.

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