university of groningen

faculty of science and engineering

Overdamped limits of the Vlasov-Poisson-Fokker-Planck equation in three dimensions

Master Project Mathematics

August 2025

Student: F.Voronine

First supervisor: Prof. J. Peypouquet

Second supervisor: Prof. M. Seri

External supervisor: Prof. O. Tse

Contents

Chapter 1

Introduction

In this thesis, we study the behavior of solutions to the damped Vlasov-Poisson-Fokker-Planck equation, characterizing the evolution of the phase-space distribution of a many-body system where the interactions are given by an inverse square law in the presence of a thermal reservoir. It models the evolution of the distribution of ions interacting in a plasma subject to repulsive Coulomb forces as well that of galaxies interacting in space subject to attractive Newton forces [4]. It is given by

$$\partial_t \mu_t^\gamma + \gamma v \cdot \nabla_x \mu_t^\gamma + \gamma \nabla_v \cdot (\mu_t^\gamma (F(x, \rho_t^\gamma) - \gamma v)) = \gamma^2 \Delta_v \mu_t^\gamma, \tag{VPFP}$$

where $(x, v) \in \mathbb{R}^d \times \mathbb{R}^d$ is a vector with $d = 3$, corresponding to the position and velocity variables respectively, and $\gamma \geq 1$ is a parameter quantifying the amount of damping introduced in the velocity variable. In addition, the solution $\mu^\gamma \in C([0, T], \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d))$ describes the evolution of the state of system $\mu_t^\gamma \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$, corresponding to a probability measure on $\mathbb{R}^d \times \mathbb{R}^d$, subject to the initial condition $\mu_0^\gamma \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$. Moreover, the element $\rho^\gamma \in C([0, T], \mathcal{P}(\mathbb{R}^d))$ represents the evolution of the particle density corresponding to the distribution of the mass of the system, obtained by

$$\rho_t^\gamma(x) := \int_{\mathbb{R}^d} \mu_t^\gamma(x, v)\ dv.$$

Lastly, we consider a force field defined using an interaction potential $\Phi : \mathbb{R}^3 \to \mathbb{R}$ such that

$$F : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^d, \quad F(x, \rho) := -\nabla \Phi * \rho(x), \quad \nabla \Phi(x) := \zeta \frac{x}{|x|^d}, \quad \zeta \in \{-1, +1\},$$

where $\zeta = +1$ gives repulsive Coulomb forces, providing a model for the distribution of ions moving in a plasma, whereas taking $\zeta = -1$ gives attractive Newton forces, giving a model for the distribution of galaxies interacting in space. In both cases, the interaction potential Φ satisfies the Poisson equation

$$\Delta \Phi = -\zeta \delta,$$

up to a positive constant depending on the chosen model, where δ is the Dirac delta function.

In this work, we focus on the behavior of solutions to the damped Vlasov-Poisson-Fokker-Planck equation in the limit as $\gamma \to \infty$, corresponding to the increase in the viscosity of the medium where the particles interact. Under suitable assumptions, the main result of the work consists in showing that the particle density ρ^γ converges in the Wasserstein distance as $\gamma \to \infty$ to the unique solution ρ

to the corresponding Aggregation-Diffusion equation, given by

$$\partial_t \rho_t + \nabla_x \cdot (\rho_t F(x, \rho_t)) = \Delta_x \rho_t. \tag{AD}$$

Here, for $p \geq 1$, the p-Wasserstein distance is a metric on the space of probability measures (see Definition 2.3) with finite p-th moments $\mathcal{P}_p(\mathbb{R}^d)$, corresponding to measures $\rho \in \mathcal{P}(\mathbb{R}^d)$ satisfying

$$\int_{\mathbb{R}^d} |x|^p \, d\rho < \infty.$$

To be specific, the proof of the main result follows the framework outlined in [4] by establishing the existence of a constant $C > 0$, independent of $\gamma \geq 1$ such that we have the inequality

$$W_2^2(\rho_t^\gamma, \rho_t) \leq C \left(W_2^2(\rho_0^\gamma, \rho_0) + \frac{1}{\gamma^2} \right).$$

The proof rests upon the existence of a weak solution to the damped Vlasov-Poisson-Fokker-Planck equation satisfying some restrictive properties. Unfortunately, the well-posedness theory covered in this work does not include all the properties required to prove the main result at a formal level.

However, we do provide conditions so as to guarantee existence and uniqueness of solutions for the damped Vlasov-Poisson-Fokker-Planck equation where the induced force is essentially bounded and the solution has finite second position and velocity moments.

To prove these statements, we fix $\gamma \geq 1$ and apply a change of variables to turn (VPFP) into

$$\partial_t \mu_t + v \cdot \nabla_x \mu_t + \nabla_v \cdot \big(\mu_t (E(x, \rho_t) - \gamma^2 v)\big) = \gamma^4 \Delta_v \mu_t,$$

where $E(x, \rho) := \gamma F(x, \rho)$. This ensures that the results from [1] become applicable.

The thesis is structured as follows. In Chapter 2, we give a derivation of the damped Vlasov-Poisson-Fokker-Planck equation (VPFP) and justify the reason we expect the Aggregation-Diffusion equation (AD) to manifest in the limit as $\gamma \to \infty$. Also, we outline the strategy for proving the main Wasserstein convergence result, serving as a partial blueprint for the remainder of the thesis. In Chapter 3, we obtain uniform estimates in the damping parameter on the second position and velocity moments, which play a critical role in the proof of the main result. These are deduced by considering a natural energy functional and showing that this quantity decays throughout the evolution. In Chapter 4, we outline the framework for taking overdamped limits of the Vlasov-Poisson-Fokker-Planck equation using tools from optimal transport theory. Namely, we exploit the various connections of the field with the theory of partial differential equations to obtain explicit estimates on the quantity of interest in the main result. Combined with results from the previous chapter, the main result is proved. In Chapter 5, we prove existence and uniqueness of smooth classical solutions to the damped Vlasov-Poisson-Fokker-Planck equation satisfying the necessary properties for the proof of the main result.

Note: We use the convention $C > 0$ to denote a constant whose value is irrelevant, depending on the context. In addition, the value of this constant may change from line to line. We use both measure-theoretic integration conventions, integrating against $d\mu(x)$ and $\mu(dx)$, depending on convenience.

Chapter 2

Derivation and result

In this chapter, we provide an informal derivation of the damped Vlasov-Poisson-Fokker-Planck equation, motivated through the study of an interacting particle system [3]. In particular, we write down the equations of motion for a deterministic particle system evolving in a viscous medium and introduce noise into the system. This models the effect of a thermal reservoir acting on the particles, giving a stochastic differential equation whose solution is an Itô process. The structure of the solution can be exploited to write down the evolution equation for the law of the process, corresponding to the damped Vlasov-Poisson-Fokker-Planck equation. In addition, we justify the reason to expect the Aggregation-Diffusion equation to govern the dynamics when taking $\gamma \to \infty$ in the damped Vlasov-Poisson-Fokker-Planck equation [4]. Lastly, we provide a formal statement of the main result described in Chapter 1 and outline the strategy behind its proof.

2.1 Informal derivation of the equation

To motivate the study of the damped Vlasov-Poisson-Fokker-Planck equation, we begin by describing a deterministic interacting particle system evolving in a viscous fluid, subject to interaction forces given by an inverse square law. This serves as a stepping stone to our desired model, so that noise can later be added to the system, modelling the effects of uncertainty in the measurements or randomness in the system. To set the scene, we set the dimension to $d = 3$ and a finite time interval $I = [0, T]$ with $T > 0$. For a fixed viscosity parameter $\gamma \geq 1$, consider a system of N identical particles

$$\mathcal{Z}_I^{\gamma,N} := (Z_t^{\gamma,1}, \dots, Z_t^{\gamma,N})_{t \in I},$$

where $Z_t^{\gamma,i} = (X_t^{\gamma,i}, V_t^{\gamma,i}) \in \mathbb{R}^d \times \mathbb{R}^d$ corresponds to the position and velocity variables of particle i at time $t \in I$. In addition, we make use of an interaction potential $\Phi : \mathbb{R}^d \to \mathbb{R}$ to specify the interactions between the particles. To model ions interacting in a plasma subject to Coulomb forces or galactic systems interacting in space subject to Newton's universal law of gravitation, we take

$$\nabla\Phi(x) = \zeta \frac{x}{|x|^d}, \quad \zeta = \begin{cases} +1 & \text{if } \Phi \text{ is the Coulomb potential,} \\ -1 & \text{if } \Phi \text{ is the Newton potential.} \end{cases}$$

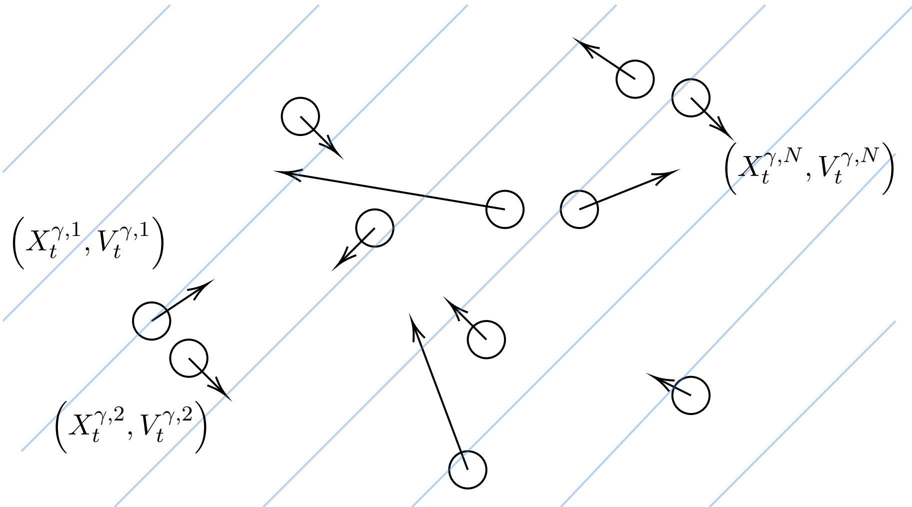

FIGURE 2.1: Particles interacting in a viscous medium, depending on $\gamma \geq 1$.

Writing down Newton's second law to obtain the equations of motion for the damped N-particle system yields the following system of coupled ordinary differential equations

$$\begin{cases} \dfrac{dX_t^{\gamma,i}}{dt} = \gamma V_t^{\gamma,i}, \\ \dfrac{dV_t^{\gamma,i}}{dt} = -\dfrac{\gamma}{N} \displaystyle\sum_{j \neq i} \nabla \Phi(X_t^{\gamma,i} - X_t^{\gamma,j}) - \gamma^2 V_t^{\gamma,i}, \end{cases}$$

for $i = 1, 2, \ldots, N$. For each particle, the first equation defines its velocity whereas the second equation defines its acceleration. The second equation is obtained by normalizing the mass and computing the resultant force of the system. This force is made up of two parts. The first consits of the inverse square law acting between each pair of particles, rescaled by the number of particles. The second part is a damping term that is proportional to the velocity of the particle under consideration.

To model the fact that the particles evolve in the presence of a thermal reservoir, some Brownian noise is added to the velocity variables, introducing randomness into the system. This results in a system of coupled stochastic differential equations

$$\begin{cases} dX_t^{\gamma,i} = \gamma V_t^{\gamma,i} dt, \\ dV_t^{\gamma,i} = -\dfrac{\gamma}{N} \displaystyle\sum_{j \neq i} \nabla \Phi(X_t^{\gamma,i} - X_t^{\gamma,j}) dt - \gamma^2 V_t^{\gamma,i} dt + \sqrt{2\gamma^2} dB_t^i, \end{cases}$$

for $i = 1, 2, \ldots, N$, where the stochastic processes $(B_t^i)_{t \in I}$ correspond to N independent d-dimensional Brownian motions. The solution to the system of stochastic differential equations is a collection of stochastic processes corresponding to the N particle distributions in phase space, collected in $\mathcal{Z}_I^{\gamma,N}$.

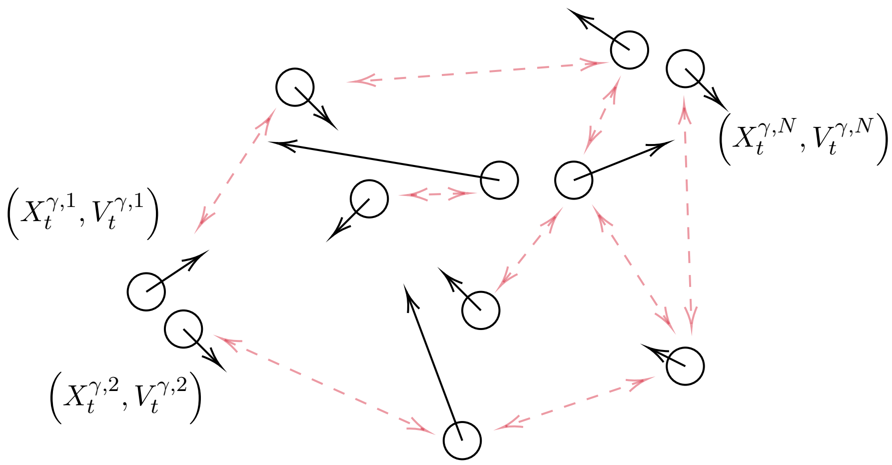

FIGURE 2.2: An interaction potential Φ specifies the interactions between the particles.

It is worth noting that since the particles are allowed to interact, the particle processes $(Z_t^{\gamma,i})_{t\in I}$ are not pairwise independent. However, since we assume that the N particles are identical, we can identify the particle system with the random empirical measure $\mu_t^{\gamma,N} \in \mathcal{P}(\mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d))$, defined by

$$\mu_t^{\gamma,N} := \frac{1}{N}\sum_{i=1}^{N} \delta_{Z_t^{\gamma,i}} = \frac{1}{N}\sum_{i=1}^{N} \delta_{(X_t^{\gamma,i}, V_t^{\gamma,i})},$$

where δ is the Dirac measure. We can apply a similar trick for the position processes only, defining

$$\rho_t^{\gamma,N} := \frac{1}{N}\sum_{i=1}^{N} \delta_{X_t^{\gamma,i}}.$$

This keeps track of the spacial distribution of the particles throughout the evolution. This allows us to rewrite the dynamics governing the stochastic N-particle system using a convolution of the form

$$\begin{cases} dX_t^{\gamma,i} = \gamma V_t^{\gamma,i} dt, \\ dV_t^{\gamma,i} = -\gamma \nabla\Phi * \rho_t^{\gamma,N} dt - \gamma^2 V_t^{\gamma,i} dt + \sqrt{2\gamma^2} dB_t^i. \end{cases} \tag{PS}$$

As the number of particles N grows large, the following two behaviors are to be expected from the system. The first is that the influence that any given particle i has on another particle j becomes increasingly negligible, meaning that the particle distributions become increasingly independent. In other words, for all $t \in I$ and all $i \neq j$, we have

$$\mathrm{Law}(Z_t^{\gamma,i}) \simeq \mathrm{Law}(Z_t^{\gamma,j}),$$

where $\mathrm{Law}(Z)$ denotes the probability density function of a random variable Z. The second expected behavior is that the dynamics become increasingly governed by the macroscopic state of the system, rather than by the microscopic interactions between the particles. This averaging effect is described mathematically by imposing that all particle distributions solve the McKean-Vlasov equation

$$\begin{cases} dX_t^\gamma &= \gamma V_t^\gamma dt \\ dV_t^\gamma &= -\gamma \nabla\Phi * \rho_t^\gamma(X_t^\gamma) - \gamma^2 V_t^\gamma dt + \sqrt{2\gamma^2} dB_t, \end{cases} \tag{MV}$$

where $\rho_t^\gamma = \mathrm{Law}(X_t^\gamma)$. To establish this limiting behavior, three points need to be established.

1. The well-posedness of the particle system (PS).
2. The well-posedness of the limiting equation (MV).
3. Prove convergence of solutions from (PS) to (MV) in some suitable topology as $N \to \infty$.

Proving the above three items establishes a phenomenon known as propagation of chaos, meaning that the limiting system is a good description for the corresponding finite particle system when N is large. This topic has been the subject of intense study ever since Boltzmann's ideas on the matter were formulated in a rigorous framework by Kac and McKean in the first half of the twentieth century.

Unfortunately, establishing propagation of chaos is a very difficult task. While propagation of chaos has been established with other interaction potentials, propagation of chaos for (PS) to (MV) with Newton or Coulomb interactions is an open problem. Despite these challenges, propagation of chaos results have been established by various authors under various regularizing assumptions on the singularity in the interaction potential for the system (PS). For instance, see [2].

From this point on, we assume that the dynamics governed by (MV) is a reasonable approximation for the particle system (PS) and that the problem is well-posed. The remainder of this section is devoted to the derivation of the damped Vlasov-Poisson-Fokker-Planck, describing the evolution of the law of the stochastic process solving (MV). In particular, we begin by writing (MV) as

$$dZ_t^\gamma = b^\gamma(Z_t^\gamma, \mu_t^\gamma)dt + \sigma^\gamma(Z_t^\gamma, \mu_t^\gamma) dB_t,$$

where $Z_t^\gamma = (X_t^\gamma, V_t^\gamma)$ and $\mu_t^\gamma = \mathrm{Law}(Z_t^\gamma)$. In addition, we define

$$\beta^\gamma : \mathbb{R}^{2d} \times \mathcal{P}(\mathbb{R}^{2d}) \to \mathbb{R}^{2d}, \qquad b^\gamma(z,\mu) := \begin{pmatrix} \gamma v \\ -\gamma\nabla\Phi * \rho(x) - \gamma^2 v \end{pmatrix},$$

$$\sigma^\gamma : \mathbb{R}^{2d} \times \mathcal{P}(\mathbb{R}^{2d}) \to \mathbb{R}^{2d\times 2d}, \quad \sigma(z,\mu) := \sqrt{2\gamma^2} \begin{pmatrix} 0_{3\times 3} & 0_{3\times 3} \\ 0_{3\times 3} & I_{3\times 3} \end{pmatrix},$$

where $\rho \in \mathcal{P}(\mathbb{R}^d)$ is the first marginal of μ, defined for all measurable sets $A \subset \mathbb{R}^d$ by

$$\rho(A) := \mu(A \times \mathbb{R}^d).$$

This means that the solution to (MV) is an Itô process with drift vector b^γ and diffusion matrix σ^γ.

A critical observation is that the dynamics in (MV) describing the evolution of the law of Z_t^γ are independent of past states of the system, exhibiting a certain forgetfulness property. In particular, the dynamics only depend of the current law of the process, characterizing the solution to (MV) as a nonlinear Markov process in the sense of McKean.

Thus, for an arbitrary probability measure $\nu \in \mathcal{P}(\mathbb{R}^{2d})$, we can consider the family of transition functions $(P_t^\nu)_{t\in I}$ adapted to the nonlinear process $(Z_t^\gamma)_{t\in I}$, driving the evolution of the nonlinear process with initial condition ν (see [3, Appendix A.4] for its definition and time-inhomogeneous analogue). Using these transition functions, we construct the nonlinear operator

$$S_t : \mathcal{P}(\mathbb{R}^{2d}) \to \mathcal{P}(\mathbb{R}^{2d}), \quad S_t(\nu)(dw) := \iint_{\mathbb{R}^d\times\mathbb{R}^d} P_t^\nu(z, dw)\ \nu(dz),$$

describing the evolution of the law of Z_t^γ with initial condition ν. In particular, note that $(S_t)_{t\in I}$ is a nonlinear semigroup since $S_0 = id$ and using the nonlinear Chapman-Kolmogorov identity, we obtain

$$S_{t+s}(\nu) = \iint_{\mathbb{R}^d\times\mathbb{R}^d} P_s^{S_t(\nu)}(w,\cdot) \left(\iint_{\mathbb{R}^d\times\mathbb{R}^d} P_t^\nu(z, dw)\ \nu(dz) \right) = S_s S_t(\nu).$$

We can use these this nonlinear semigroup to transform the nonlinear Markov process into a time-inhomogeneous Markov process by considering new transition functions of the form

$$\overline{P}_{s,t} : \mathbb{R}^{2d} \to \mathcal{P}(\mathbb{R}^{2d}), \quad \overline{P}_{s,t} := P_{t-s}^{S_s(\mu_0^\gamma)},$$

where $s \le t$ with $s, t \in I$. This was introduced by McKean [14], allowing the use of semigroup methods to track the behavior of the law of the nonlinear Markov process. Namely, we get an evolution system (see [3, Definition A.31]) defined on bounded Borel measurable test functions by

$$T_{s,t} : \mathcal{B}_b(\mathbb{R}^{2d}) \to \mathcal{B}_b(\mathbb{R}^{2d}), \quad T_{s,t}\varphi(z) := \iint_{\mathbb{R}^d\times\mathbb{R}^d} \varphi(w)\ \overline{P}_{s,t}(z, dw) = \iint_{\mathbb{R}^d\times\mathbb{R}^d} \varphi(w)\ P_{t-s}^{\mu_0^\gamma}(z, dw).$$

The test functions, also known as observables, are used to probe information about the current state of the system whereas $T_{s,t}$ describes the behavior of these observables along the evolution. From the evolution system $T_{s,t}$, we obtain the evolution equation governing the law of the process by solving

$$\frac{d}{dt} T_{s,t}\varphi(z) = T_{s,t} L_{\mu_t^\gamma} \varphi(z).$$

Here, we require the (right) infinitesimal generator of the evolution system, defined by

$$L_{\mu_t^\gamma} : B_b(\mathbb{R}^{2d}) \to B_b(\mathbb{R}^{2d}), \quad L_{\mu_t^\gamma}\varphi(z) := \lim_{h\to 0} \frac{T_{t,t+h}\varphi(z) - \varphi(z)}{h} = \lim_{h\to 0} \frac{\int_{\mathbb{R}^{2d}} \varphi(w) P_h^{\mu_t^\gamma}(z, dw) - \varphi(z)}{h}.$$

For the Itô process solving (MV), we may restrict the domain of the evolution system and infinitesimal generator to the set of smooth compactly supported test functions. In this case, provided that the induced force field is bounded, the infinitesimal generator can be computed using Itô's lemma to be

$$L^{\mathrm{MV}}_{\mu_t^\gamma} : C_c^\infty(\mathbb{R}^{2d}) \to C_c^\infty(\mathbb{R}^{2d}), \quad L^{\mathrm{MV}}_{\mu_t^\gamma}\varphi(z) := b^\gamma(z,\mu_t^\gamma)\cdot\nabla\varphi(z) + \frac{1}{2}\sum_{i,j=1}^{2d} a_{ij}^\gamma(z,\mu_t^\gamma)\frac{\partial^2\varphi}{\partial z_i\partial z_j}(z),$$

where $a^\gamma(z,\mu) := \sigma^\gamma(z,\mu)\sigma^\gamma(z,\mu)^T$. Thus, after substituting the above with $w := (x,v)$, we have

$$\begin{aligned}\iint_{\mathbb{R}^d\times\mathbb{R}^d} \varphi(w)\frac{\partial P^{\mu_s^\gamma}_{t-s}}{\partial t}(z,dw) = &-\iint_{\mathbb{R}^d\times\mathbb{R}^d}\left(\left(\gamma\nabla\Phi*\rho_t^\gamma(x)+\gamma^2 v\right)\cdot\nabla_v\varphi(w)\right)P^{\mu_s^\gamma}_{t-s}(z,dw)\\ &+\gamma\iint_{\mathbb{R}^d\times\mathbb{R}^d}\left(v\cdot\nabla_x\varphi(w)\right)P^{\mu_s^\gamma}_{t-s}(z,dw)\\ &+\gamma^2\iint_{\mathbb{R}^d\times\mathbb{R}^d}\left(\Delta_v\varphi(w)\right)P^{\mu_s^\gamma}_{t-s}(z,dw).\end{aligned}$$

Taking $s=0$ and integrating with respect to z against the initial law μ_0^γ with Fubini's theorem,

$$\begin{aligned}\iint_{\mathbb{R}^d\times\mathbb{R}^d} \varphi(w)\frac{\partial S_t(\mu_0^\gamma)}{\partial t}(dw) = &-\iint_{\mathbb{R}^d\times\mathbb{R}^d}\left(\left(\gamma\nabla\Phi*\rho_t^\gamma(x)+\gamma^2 v\right)\cdot\nabla_v\varphi(w)\right)S_t(\mu_0^\gamma)(dw)\\ &+\gamma\iint_{\mathbb{R}^d\times\mathbb{R}^d}\left(v\cdot\nabla_x\varphi(w)\right)S_t(\mu_0^\gamma)(dw)\\ &+\gamma^2\iint_{\mathbb{R}^d\times\mathbb{R}^d}\left(\Delta_v\varphi(w)\right)S_t(\mu_0^\gamma)(dw),\end{aligned}$$

giving a weak solution (see Definition 2.1) to the damped Vlasov-Poisson-Fokker-Planck equation

$$\partial_t\mu_t^\gamma + \gamma v\cdot\nabla_x\mu_t^\gamma + \gamma\nabla_v\cdot\left(\mu_t^\gamma(F(x,\rho_t^\gamma)-\gamma v)\right) = \gamma^2\Delta_v\mu_t^\gamma,$$

with initial condition μ_0^γ and $F(x,\rho) = -\nabla\Phi*\rho$. This concludes the derivation of the equation.

2.2 Informal overdamped limit argument

After providing an informal derivation of the damped Vlasov-Poisson-Fokker-Planck equation (VPFP), we consider the reasons to expect the Aggregation-Diffusion equation (AD) to arise by taking $\gamma\to\infty$. Similarly to the derivation of the main equation, the arguments presented here are informal and should only serve as motivation for the phenomenon under study. For simplicity of presentation, we abuse notation by identifying measures with their Lebesgue density. To start with overdamped limits, note that we can express the dynamics of the damped Vlasov-Poisson-Fokker-Planck equation in the form

$$\partial_t\mu_t^\gamma + \gamma v\cdot\nabla_x\mu_t^\gamma + \gamma\nabla_v\cdot\left(\mu_t^\gamma F(x,\rho_t^\gamma)\right) = \gamma^2\nabla_v\cdot\left(\nabla_v\mu_t^\gamma + v\mu_t^\gamma\right).$$

In addition, the right-hand side in the above equation is of the order of γ^2, which can be expanded to

$$\gamma^2\nabla_v\cdot\left(\nabla_v\mu_t^\gamma(x,v) + v\mu_t^\gamma(x,v)\right) = \gamma^2\nabla_v\cdot\left(\mu_t^\gamma(x,v)\nabla_v\log\left(\frac{\mu_t^\gamma(x,v)}{\mathcal{N}^d(v)}\right)\right),$$

where $\mathcal{N}^d \in \mathcal{P}(\mathbb{R}^d)$ is the standard normal distribution on $\mathbb{R}^d$. This distribution is defined by

$$\mathcal{N}^d : \mathbb{R}^d \to \mathbb{R}, \quad \mathcal{N}^d(v) := \frac{1}{(2\pi)^{d/2}} e^{-|v|^2/2}.$$

Thus, for large values of $\gamma \geq 1$, the quadratic term dominates so that we can expect the factorization

$$\mu_t^\gamma(x, v) \simeq \rho_t^\gamma(x)\mathcal{N}^d(v) := M(\mu_t^\gamma).$$

In addition, we can make use of the momentum field, defined in terms of the phase-space density as

$$m_t^\gamma : \mathbb{R}^d \to \mathbb{R}, \quad m_t^\gamma(x) := \int_{\mathbb{R}^d} v\mu_t^\gamma(x, v)\, dv,$$

to rewrite the dynamics for the damped Vlasov-Poisson-Fokker-Planck in the position variables only. This is obtained by integrating in the velocity variables to obtain the system of equations

$$\begin{cases} \partial_t \rho_t^\gamma + \gamma \nabla_x \cdot m_t^\gamma = 0, \\ \partial_t m_t^\gamma + \gamma \nabla_x \cdot \left(\int_{\mathbb{R}^d} v \otimes v \; \mu_t^\gamma(x, dv) \right) = \gamma \rho_t^\gamma F(x, \rho_t^\gamma) - \gamma^2 m_t^\gamma, \end{cases}$$

where $\otimes$ denotes the outer product. Using the phase-space factorization of the density μ_t^γ, we have

$$\gamma \nabla_x \cdot \left(\int_{\mathbb{R}^d \times \mathbb{R}^d} v \otimes v \; \mu_t^\gamma(x, dv) \right) \simeq \gamma \nabla_x \rho_t^\gamma(x),$$

for large values of $\gamma \geq 1$. Thus, the second equation of the system of equations can be simplified to

$$\partial_t m_t^\gamma + \gamma \nabla \rho_t^\gamma \simeq \gamma \rho_t^\gamma F(x, \rho_t^\gamma) - \gamma^2 m_t^\gamma.$$

From the above equation, we can omit lower-order terms in $\gamma \geq 1$ to get the refined approximation

$$\gamma m_t^\gamma \simeq \rho_t^\gamma F(x, \rho_t^\gamma) - \nabla_x \rho_t^\gamma.$$

Substituting this expression in the momentum dynamics yields the Aggregation-Diffusion equation

$$\partial_t \rho_t^\gamma + \nabla_x \cdot (\rho_t^\gamma F(x, \rho_t^\gamma)) \simeq \Delta_x \rho_t^\gamma,$$

for large values of $\gamma \geq 1$. In other words, the particle densities in the damped Vlasov-Poisson-Fokker-Planck equation become increasingly governed by the Aggregation-Diffusion equation as required.

2.3 Formal statement of the main result

In the remainder of the thesis, we work towards establishing the phenomenon described informally in the previous section in a more rigorous framework. To do so, we first introduce the necessary notions to provide a formal statement of the main result of the work. These notions mainly come from the theory of partial differential equations, statistical mechanics, optimal transport and geometry.

To even start treating partial differential equations in a rigorous context, we first need to specify what it means to solve a partial differential equation. Many different notions of solution exist in the literature, including weak, mild, strong and classical solutions, to name a few. For our purposes, we begin by introducing the notion of weak solution to the Vlasov-Poisson-Fokker-Planck equation and Aggregation-Diffusion equation. These are defined by multiplying the equations by a smooth, compactly supported test function and integrating by parts. This shifts the differential operators away from the solution to the test function. As a result, weak solutions need not be even differentiable.

Definition 2.1. A continuous curve $\mu^\gamma \in C([0,T], \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d))$ is a weak solution to the Cauchy problem for the damped Vlasov-Poisson-Fokker-Planck equation (VPFP) with initial condition $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d)$ if for all $s, t \in (0,T)$ and all $\varphi \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^d)$, we have

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} \varphi \, d\mu_t^\gamma - \iint_{\mathbb{R}^d \times \mathbb{R}^d} \varphi \, d\mu_s^\gamma = \gamma \int_s^t \iint_{\mathbb{R}^d \times \mathbb{R}^d} \left(v \cdot \nabla_x \varphi + (F(x, \rho_r^\gamma) - \gamma v) \cdot \nabla_v \varphi + \gamma \Delta_v \varphi \right) \, d\mu_r^\gamma dr.$$

Similarly, a continuous curve $\rho \in C([0,T], \mathcal{P}_2(\mathbb{R}^d))$ is a weak solution to the Aggregation-Diffusion equation (AD) with initial condition $\rho_0 \in \mathcal{P}_2(\mathbb{R}^d)$ if for all $s, t \in (0,T)$ and all $\varphi \in C_c^\infty(\mathbb{R}^d)$, we have

$$\int_{\mathbb{R}^d} \varphi \, d\rho_t - \int_{\mathbb{R}^d} \varphi \, d\rho_s = \int_s^t \int_{\mathbb{R}^d} \left(F(x, \rho_r) \cdot \nabla_x \varphi + \Delta_x \varphi \right) \, d\rho_r dr.$$

For both equations, we can define stronger notions of solution by imposing that additional properties need to be satisfied. For the damped Vlasov-Poisson-Fokker-Planck equation, we are mainly interested in solutions where the induced force field is bounded. Many other useful features of the solution can be deduced from this property, characterizing it as a strong solution.

Definition 2.2. A continuous curve $\mu \in C([0,T], \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d))$ is a strong solution to the Cauchy problem for the damped Vlasov-Poisson-Fokker-Planck equation (VPFP) with initial condition $\mu_0 \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ if μ is a weak solution to the damped Vlasov-Poisson-Fokker-Planck equation with initial condition μ_0 in the sense of Definition 2.1 and the induced force is bounded, meaning that

$$\sup_{0 \le t \le T} \|F(\cdot, \rho_t))\|_{L^\infty} < \infty.$$

On the other hand, the Aggregation-Diffusion equation has recently been the subject of intense study through the lens of optimal transport theory. This field of mathematics uses measure theory to rigorously define the notion of moving distributions so as to minimize a certain cost function. Whenever the measures are defined on Euclidean spaces equipped with the Euclidean distance, the optimal cost of transportation corresponds to the Wasserstein distance, defined as follows. See [20] and [10] for an introduction to the field.

Definition 2.3. Let $p \geq 1$ and consider probability measures $\mu, \nu \in \mathcal{P}_p(\mathbb{R}^d)$ with finite p-th moment

$$\int_{\mathbb{R}^d} |x|^p \, d\mu < \infty, \qquad \int_{\mathbb{R}^d} |x|^p \, d\nu < \infty.$$

In addition, let $\Pi(\mu, \nu) \subset \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ denote the set of transport plans from μ to ν, consisting of measures with first marginal μ and second marginal ν. This means that for $\pi \in \Pi(\mu, \nu)$, we have

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} (\varphi(x) + \psi(y)) \, d\pi = \int_{\mathbb{R}^d} \varphi(x) \, d\mu + \int_{\mathbb{R}^d} \psi(y) \, d\nu,$$

for all $\varphi, \psi \in C_b(\mathbb{R}^d)$. Then, the Wasserstein distance between μ and ν is defined as

$$W_p^p(\mu, \nu) := \inf_{\pi \in \Pi(\mu,\nu)} \iint_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^p \, d\pi.$$

In particular, a deep result was established by Jordan et al. [16], writing the Aggregation-Diffusion equation as a gradient flow with respect to the 2-Wasserstein distance. Intuitively speaking, this means that solutions evolve so as to minimize a certain energy functional. To define this energy functional, we first need to define a way to quantify the difference in information carried between two probability measures. Typically, we compare measures against another reference measure satisfying some natural property. This can be chosen to be a stationary solution or the usual Lebesgue measure. This comparison of measures with respect to information is captured by the notion of relative entropy.

Definition 2.4. Let E be a Euclidean space and consider probability measures $\mu, \nu \in \mathcal{P}(E)$. Then, the relative entropy between μ and ν is defined as

$$\mathcal{H}(\mu|\nu) := \begin{cases} \iint_{\mathbb{R}^d \times \mathbb{R}^d} h\left(\frac{\mu}{\nu}\right) \, d\nu & \text{if } \mu \ll \nu. \\ +\infty & \text{otherwise,} \end{cases}$$

Here, we define $h(s) := s \log s$, which is conventionally extended to $h(0) := 0$ by continuity.

In particular, the energy functional that is minimized along the evolution of the Aggregation-Diffusion equation can be expressed as a sum of a relative entropy term with a potential energy term, depending on the interaction potential Φ. This is defined as follows.

Definition 2.5. The energy functional for the Aggregation-Diffusion equation (AD) is defined by

$$\mathcal{E} : \mathcal{P}(\mathbb{R}^d) \to \mathbb{R} \cup \{+\infty\}, \quad \mathcal{E}(\rho) := \mathcal{H}(\rho|\mathcal{L}^d) + \frac{1}{2} \int_{\mathbb{R}^d} \Phi * \rho \, d\rho,$$

where $\mathcal{L}^d$ is the d-dimensional Lebesgue measure. Using this energy functional, we can define a stronger notion of solution to the Aggregation-Diffusion equation that is consistent with the gradient flow structure of the equation, known as $\mathcal{E}$-regular solutions. Their main feature is that certain key quantities remain bounded throughout the evolution.

Definition 2.6. Let $\mathcal{E}$ be the energy functional given in Definition 2.5. A continuous curve $\rho \in C([0,T],\mathcal{P}_2(\mathbb{R}^d))$ is an $\mathcal{E}$-regular solution to the Cauchy problem (AD) with initial condition $\rho_0 \in \mathcal{P}_2(\mathbb{R}^d) \cap \operatorname{dom}(\mathcal{E})$ if it is a weak solution to the Aggregation-Diffusion equation in the sense of Definition 2.1, belongs to the Sobolev space throughout the evolution $\rho \in L^1((0,T),W^{1,1}(\mathbb{R}^d))$ and

$$\sup_{0\le t\le T} \mathcal{E}(\rho_t) + \sup_{0\le t\le T} \|F(\cdot,\rho_t)\|_{L^\infty} + \int_0^T \int_{\mathbb{R}^d} \frac{|\nabla \rho_t|^2}{\rho_t}\, dxdt < \infty.$$

Similarly to the Aggregation-Diffusion dynamics, we can also define an energy functional for the damped Vlasov-Poisson-Fokker-Planck equation by considering the phase-space factorization described in Section 2.2. This replaces the reference used in the relative entropy term.

Definition 2.7. The energy for the Vlasov-Poisson-Fokker-Planck equation (VPFP) is defined by

$$\mathcal{E} : \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d) \to \mathbb{R} \cup \{+\infty\}, \quad \mathcal{E}(\mu) := \mathcal{H}(\mu|M(\mu)) + \frac{1}{2}\int_{\mathbb{R}^d} \Phi * \rho \, d\rho,$$

where $M(\mu)$ is the phase-space factorization of μ described in Section 2.2 for large values of $\gamma \ge 1$.

Between the Vlasov-Poisson-Fokker-Planck equation and the Aggregation-Diffusion equation lies an intermediate system, obtained defining modified particle densities which take into account the effect of the velocity variable on the position variable ahead of time. These are known as coarse-grained particle densities, which also play an important role in the existence and uniqueness theory.

Definition 2.8. Let $\lambda \ge 0$ and consider the unique weak solution $\mu^\gamma \in C([0,T],\mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d))$ to the damped Vlasov-Poisson-Fokker-Planck equation (VPFP). Then, the λ-coarse-grained particle density $\rho^{\gamma,\lambda} \in C([0,T],\mathcal{P}(\mathbb{R}^d))$ is defined as

$$\rho_t^{\gamma,\lambda}(x) := \int_{\mathbb{R}^d} \mu_t^\gamma(x-\lambda v, v)\, dv.$$

We refer to the case where $\lambda = 1/\gamma$ as the standard coarse-grained particle density, denoted by $\hat{\rho}^\gamma$.

We are now ready to provide a formal statement of the main result that is proved in the thesis. For now, we assume the well-posedness of both (VPFP) and (AD) and state the overdamped limit result.

Theorem 2.9. Consider the energy functionals given in Definition 2.5 and 2.7 and consider a family of initial conditions $(\mu_0^\gamma)_{\gamma\ge1}$ satisfying the following uniform energy and moment bounds

$$\sup_{\gamma\ge1} \mathcal{E}(\mu_0^\gamma) + \sup_{\gamma\ge1} \iint_{\mathbb{R}^d\times\mathbb{R}^d} |x|^2 + |v|^2 \, d\mu_0^\gamma < \infty.$$

In addition, consider the family of unique strong solutions $(\mu^\gamma)_{\gamma\ge1}$ to the Vlasov-Poisson-Fokker-Planck equation (VPFP) and let $(\rho^\gamma)_{\gamma\ge1}$ and $(\hat{\rho}^\gamma)_{\gamma\ge1}$ denote the corresponding families of particle densities and standard coarse-grained particle densities. Assume that

$$\sup_{\gamma\ge1}\sup_{0\le t\le T} \left\{ \|\rho_t^\gamma\|_{L^\infty} + \|\hat{\rho}_t^\gamma\|_{L^\infty} \right\} < \infty.$$

In addition, assume that the family $(\hat{\rho}^\gamma)_{\gamma\geq 1}$ satisfies the properties of $\mathcal{E}$-regular solutions and induces a family of Lipschitz forces with Lipschitz constants bounded uniformly in $\gamma \geq 1$, meaning that

$$\sup_{\gamma\geq 1} \sup_{0\leq t\leq T} \|F(\cdot, \hat{\rho}_t^\gamma)\|_{\mathrm{Lip}} < \infty.$$

In addition, let $\rho \in C([0,T], \mathcal{P}_2(\mathbb{R}^d))$ be the unique $\mathcal{E}$-regular solution to the Aggregation-Diffusion equation (AD) with initial condition $\rho_0 \in \mathcal{P}_2(\mathbb{R}^d) \cap \mathrm{dom}(\mathcal{E})$. Then, there exists some constant $C > 0$ independent of $\gamma \geq 1$ such that

$$\sup_{0\leq t\leq T} W_2^2(\rho_t^\gamma, \rho_t) \leq C\left(W_2^2(\rho_0^\gamma, \rho_0) + \frac{1}{\gamma^2}\right).$$

To prove the statement, we first need to ensure that the second position and velocity moments remain uniformly bounded in $\gamma \geq 1$ throughout the evolution. The position moments are required to ensure that the 2-Wasserstein distances are well-defined whereas the velocity moments are used to deduce the uniform bound. These results are obtained in Chapter 3. The bound itself is obtained by considering

$$W_2^2(\rho_t^\gamma, \rho) \leq 2W_2^2(\rho_t^\gamma, \hat{\rho}_t^\gamma) + 2W_2^2(\hat{\rho}_t^\gamma, \rho_t).$$

and bounding each term separately. The first term can be bounded by the second velocity moment of μ_t^γ using elementary techniques from optimal transport theory. However, bounding the second term is more involved. A Grönwall argument is applied to a modified evolution-variational inequality

$$\frac{1}{2}\frac{d}{dt}W_2^2(\hat{\rho}_t^\gamma, \rho_t) \leq \lambda W_2^2(\hat{\rho}_t^\gamma, \rho_t) - \mathcal{D}_\Phi(\hat{\rho}_t^\gamma, \rho_t) + \frac{1}{2}\|e_t^\gamma\|_{L^2(\hat{\rho}_t)}.$$

where $\lambda > 0$ is independent of $\gamma \geq 1$. Moreover, we have that $\mathcal{D}_\Phi$ is the modulated interaction energy

$$\mathcal{D}_\Phi(\rho, \nu) := \iint_{\mathbb{R}^d\times\mathbb{R}^d} \Phi(x-y)\ d(\rho-\nu)(y)\ d(\rho-\nu)(x),$$

defined for probability measures $\rho, \nu \in \mathcal{P}(\mathbb{R}^d)$. Lastly, the term e_t^γ in the modified evolution-variational inequality reflects the difference in the forces governing the two systems. We prove that both the modulated interaction energy $\mathcal{D}_\Phi(\hat{\rho}_t^\gamma, \rho_t)$ and the error term e_t^γ can be controlled uniformly in $\gamma \geq 1$ with respect to the Wasserstein distance. These results are treated in Chapter 4.

Chapter 3

Uniform estimates

In this chapter, we obtain estimates on various quantities of interest that can be bounded uniformly in the damping parameter $\gamma \geq 1$. In particular, we establish some useful properties for the energy functional corresponding to the damped Vlasov-Poisson-Fokker-Planck equation (VPFP), defined in Section 2.3. We use these features to prove an energy dissipation inequality, giving natural assumptions to obtain uniform energy bounds throughout the evolution [4]. In addition, we combine these bounds with Grönwall's lemma to obtain uniform bounds on the second moments of the phase-space density. Throughout this section, we assume the well-posedness for the damped Vlasov-Poisson-Fokker-Planck equation (VPFP) with finite second position and velocity moments. This is postponed until Chapter 5. Rather, we focus on bounding the moments uniformly in $\gamma \geq 1$, assuming they exist.

3.1 Uniform energy estimates

To start, we begin by considering the relative entropy term present in the energy functional corresponding to the damped Vlasov-Poisson-Fokker-Planck equation. To help motivate the presence of this term in Definition 2.7, we consider the possible state variables that can contribute to the total energy of the underlying particle system. From a macroscopic perspective, we can consider two kinds of mechanisms giving rise to energy due to the movement of the particles. The first gives rise to energy that is available to do useful work, corresponding to the kinetic energy of the system. The second gives rise to energy that is unavailable to do useful work, corresponding to the entropy of the system. We collect these two mechanisms in a term which we call the kinetic entropy as follows.

Definition 3.1. The kinetic entropy of a probability measure $\mu \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ is defined by

$$H(\mu) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}, \quad H(\mu)(x, v) := \frac{|v|^2}{2}\mu(x, v) + h(\mu(x, v)),$$

where we recall that $h(s) := s \log s$ for $s > 0$, which is conventionally extended to $h(0) := 0$.

The notion of entropy was originally introduced by Clausius [5], which was subsequently investigated further by many physicists of the late nineteenth century. A few years prior, however, Maxwell discovered that the standard normal distribution naturally arises as a good description for the velocity distribution in idealized gases [11, 12]. However, it was Jaynes who first showed that the standard normal distribution minimizes the entropy as defined in Definition 3.1 [8, 9].

Theorem 3.2. Let $\rho \geq 0$ be a non-negative real number. Define the function

$$m_\rho(v) := \rho \mathcal{N}^d(v),$$

where $\mathcal{N}^d \in \mathcal{P}(\mathbb{R}^d)$ is the standard normal distribution on $\mathbb{R}^d$. Then, the function m_ρ solves

$$\min \left\{ H(\sigma) \mid \sigma : \mathbb{R}^d \to [0, \infty) \text{ measurable}, \int_{\mathbb{R}^d} \sigma(v)\, dv = \rho \right\}.$$

Proof. By convexity of the function h, for all non-negative real numbers $s \geq 0$ and $t > 0$, we have

$$h(t) \geq h(s) + h'(s)(t - s).$$

Take $s = m_\rho(v)$ and $t = \sigma(v)$ where σ is a function in the feasible set of the problem. Then, we get

$$\begin{aligned} h(\sigma(v)) &\geq h(m_\rho(v)) + h'(m_\rho(v))(\sigma(v) - m_\rho(v)) \\ &= h(m_\rho(v)) + \left(1 + \log\left(\frac{\rho}{(2\pi)^{d/2}}\right) - \frac{|v|^2}{2}\right)(\sigma(v) - m_\rho(v)). \end{aligned}$$

Using the fact that σ belongs to the feasible set of the minimization problem along with

$$\int_{\mathbb{R}^d} \sigma(v) - m_\rho(v)\, dv = 0,$$

we can integrate both sides of the convexity inequality to deduce that

$$H(\sigma) = \int_{\mathbb{R}^d} \frac{|v|^2}{2}\sigma(v) + h(\sigma(v))\, dv \geq \int_{\mathbb{R}^d} \frac{|v|^2}{2} m_\rho(v) + h(m_\rho(v))\, dv = H(m_\rho).$$

Thus, we have shown that m_ρ minimizes the kinetic entropy as required. □

In other words, for a given particle density, the phase-space distribution with a standard normal velocity profile minimizes the kinetic entropy. For convenience, we give a name to this distribution.

Definition 3.3. The Maxwellian associated to a probability measure $\mu \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ is defined by

$$M(\mu) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}, \quad M(\mu)(x, v) := m_{\rho(x)}(v) = \rho(x)\mathcal{N}^d(v),$$

where $\rho \in \mathcal{P}(\mathbb{R}^d)$ is the first marginal of μ, corresponding to the particle density obtained from μ.

What is interesting is that the Maxwellian associated to a phase-space density serves as a good reference distribution in the relative entropy defined in Definition 2.4. With this choice of reference, the relative entropy coincides with the total energy difference arising from the kinetic entropy.

Remark 3.4. For a probability measure $\mu \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$, a small computation yields

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} H(\mu)\, dz - \iint_{\mathbb{R}^d \times \mathbb{R}^d} H(M(\mu))\, dz = \mathcal{H}(\mu | M(\mu)),$$

This implies that the relative entropy of μ with respect to its Maxwellian is non-negative.

On the other hand, another source of energy comes from the position of the particles with respect to each other. This term is known as the potential energy of the system, which depends on the interaction potential mediating the behavior of the particles between themselves. Note that this type of energy is independent of the velocity profile of the state of the system.

Definition 3.5. The potential energy of a probability measure $\mu \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ is defined as

$$V(\mu) := \frac{1}{2} \int_{\mathbb{R}^d} \Phi * \rho \, d\rho,$$

where $\rho \in \mathcal{P}(\mathbb{R}^d)$ is the first marginal of μ, corresponding to the particle density obtained from μ.

As a result, we can obtain the energy functional defined in Definition 2.7 by summing the potential energy with the energy obtained from the kinetic entropy. Namely, we obtain the equality

$$\mathcal{E}(\mu) := \mathcal{H}(\mu|M(\mu)) + \frac{1}{2} \int_{\mathbb{R}^d} \Phi * \rho \, d\rho = \iint_{\mathbb{R}^d \times \mathbb{R}^d} (H(\mu) - H(M(\mu))) \, dz + V(\mu).$$

Using the fact that the relative entropy with respect to the Maxwellian is non-negative, we prove that that the energy exhibits a dissipative behavior. In fact, we prove two inequalities. The first is later used to prove the uniform bound on the moments, whereas the second is more physically natural.

Theorem 3.6. Let μ^γ be the unique strong solution to the damped Vlasov-Poisson-Fokker-Planck with finite second position and velocity moments and sufficient regularity. Then, we have

$$\begin{aligned} &\mathcal{E}(\mu_t^\gamma) + 4\gamma^2 \int_0^t \iint_{\mathbb{R}^d \times \mathbb{R}^d} \left| \nabla_v \sqrt{\mu_s^\gamma e^{|v|^2/2}} \right|^2 e^{-|v|^2/2} \, dz \leq \mathcal{E}(\mu_0^\gamma) \\ &\mathcal{E}(\mu_t^\gamma) + 2\gamma^2 \int_0^t \mathcal{H}(\mu_s^\gamma | M(\mu_s^\gamma)) \, ds \leq \mathcal{E}(\mu_0^\gamma). \end{aligned}$$

Proof. Using the regularity of the solution μ_t^γ and the fact that Φ is even, we have

$$\frac{d}{dt} \left(\iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|v|^2}{2} \, d\mu_t^\gamma + \frac{1}{2} \int_{\mathbb{R}^d} \Phi * \rho_t^\gamma \, d\rho_t^\gamma \right) = \iint_{\mathbb{R}^d \times \mathbb{R}^d} \left(\frac{|v|^2}{2} + \Phi * \rho_t^\gamma(x) \right) \frac{\partial \mu_t^\gamma}{\partial t} \, dz.$$

Substituting the damped Vlasov-Poisson-Fokker-Planck equation and integrating by parts yields

$$\frac{d}{dt} \left(\iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|v|^2}{2} \, d\mu_t^\gamma + \frac{1}{2} \int_{\mathbb{R}^d} \Phi * \rho_t^\gamma \, d\rho_t^\gamma \right) = -\gamma^2 \iint_{\mathbb{R}^d \times \mathbb{R}^d} v \cdot (\nabla_v \mu_t^\gamma + v \mu_t^\gamma) \, dz.$$

On the other hand, using a similar procedure, we have

$$\begin{aligned} \frac{d}{dt} \left(\iint_{\mathbb{R}^d \times \mathbb{R}^d} h(\mu_t^\gamma) \, dz \right) &= \iint_{\mathbb{R}^d \times \mathbb{R}^d} (1 + \log \mu_t^\gamma) \frac{\partial \mu_t^\gamma}{\partial t} \, dz \\ &= \gamma^2 \iint_{\mathbb{R}^d \times \mathbb{R}^d} (1 + \log \mu_t^\gamma) \left(\nabla_v \cdot (\nabla_v \mu_t^\gamma + v \mu_t^\gamma) \right) \, dz \\ &= -\gamma^2 \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{\nabla_v \mu_t^\gamma}{\mu_t^\gamma} \cdot (\nabla_v \mu_t^\gamma + v \mu_t^\gamma) \, dz. \end{aligned}$$

As a result, summing the two contributions yields

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(\mu_t^\gamma) &= -\gamma^2 \iint_{\mathbb{R}^d\times\mathbb{R}^d} (\nabla_v \mu_t^\gamma + v\mu_t^\gamma)\cdot\left(v + \frac{\nabla_v \mu_t^\gamma}{\mu_t^\gamma}\right)\, dz \\ &= -\gamma^2 \iint_{\mathbb{R}^d\times\mathbb{R}^d} \left| v\sqrt{\mu_t^\gamma} + 2\nabla_v \sqrt{\mu_t^\gamma}\right|^2\, dz \\ &= -4\gamma^2 \iint_{\mathbb{R}^d\times\mathbb{R}^d} \left|\nabla_v \sqrt{\mu_t^\gamma e^{|v|^2/2}}\right|^2 e^{-|v|^2/2}\, dz.\end{aligned}$$

Thus, the first inequality follows by integration

$$\mathcal{E}(\mu_t^\gamma) + 4\gamma^2 \int_0^t \iint_{\mathbb{R}^d\times\mathbb{R}^d} \left|\nabla_v \sqrt{\mu_t^\gamma e^{|v|^2/2}}\right|^2 e^{-|v|^2/2} dz \le \mathcal{E}(\mu_0^\gamma).$$

On the other hand, the logarithmic Sobolev inequality [6] gives

$$2\int_{\mathbb{R}^d} \mu_t^\gamma \log\left(\frac{\mu_t^\gamma}{M(\mu_t^\gamma)}\right)\, dv \le 4\int_{\mathbb{R}^d} \left|\nabla_v \sqrt{\mu_t^\gamma e^{|v|^2/2}}\right|^2 e^{-|v|^2/2}\, dv.$$

Thus, we can use deduce that

$$\frac{d}{dt}\mathcal{E}(\mu_t^\gamma) \le -2\gamma^2 \mathcal{H}(\mu_t^\gamma | M(\mu_t^\gamma)).$$

As a result, we can conclude by using Remark 3.4 to obtain

$$\mathcal{E}(\mu_t^\gamma) + 2\gamma^2 \int_0^t \mathcal{H}(\mu_s^\gamma | M(\mu_s^\gamma)) \le \mathcal{E}(\mu_0^\gamma).$$

This concludes the proof of the required inequalities. □

3.2 Uniform first position moments

The energy dissipation inequalities obtained in Theorem 3.6 provide natural assumptions to obtain uniform bounds on the energy throughout the evolution. In the quest for uniform bounds on the second moments, we start by proving that the first position moments are uniformly bounded as long as the same holds true for the energy. In the process, we also obtain uniform bounds on the second velocity moments. For the remainder of this section, we consider Coulomb interactions between the particles, but the arguments can be adapted to include the Newtonian interaction.

Theorem 3.7. Let $(\mu^\gamma)_{\gamma\ge 1}$ be the family of unique strong solutions to the damped Vlasov-Poisson-Fokker-Planck equation with finite second position and velocity moments and sufficient regularity, where the initial conditions satisfy

$$\sup_{\gamma\ge 1}\mathcal{E}(\mu_0^\gamma) + \sup_{\gamma\ge 1}\iint_{\mathbb{R}^d\times\mathbb{R}^d} |x|\; d\mu_0^\gamma < \infty.$$

Then, it follows that the first position moments are uniformly bounded throughout the evolution

$$\sup_{\gamma\geq 1}\sup_{0\leq t\leq T}\iint_{\mathbb{R}^d\times\mathbb{R}^d}|x|\;d\mu_t^\gamma<\infty.$$

Proof for Coulomb interactions. Consider the following decomposition of the logarithm

$$\log(\mu_t^\gamma)=\log^+(\mu_t^\gamma)-\log^-(\mu_t^\gamma),$$

where we define

$$\log^+(\mu_t^\gamma):=\max\{\log(\mu_t^\gamma),0\},\quad \log^-(\mu_t^\gamma):=\max\{-\log(\mu_t^\gamma),0\}.$$

In particular, on the domains

$$S_1:=\left\{(x,v)\in\mathbb{R}^d\times\mathbb{R}^d:\mu_t^\gamma>e^{-\frac{|x|}{2}-\frac{|v|^2}{4}}\right\},\quad S_2:=\left\{(x,v)\in\mathbb{R}^d\times\mathbb{R}^d:\mu_t^\gamma\leq e^{-\frac{|x|}{2}-\frac{|v|^2}{4}}\right\},$$

we have the following respective bounds

$$\begin{aligned}\iint_{S_1}\mu_t^\gamma\log^-(\mu_t^\gamma)\;dz&\leq\iint_{\mathbb{R}^d\times\mathbb{R}^d}\left(\frac{|x|}{2}+\frac{|v|^2}{4}\right)\;d\mu_t^\gamma,\\ \iint_{S_2}\mu_t^\gamma\log^-(\mu_t^\gamma)\;dz&\leq\iint_{\mathbb{R}^d\times\mathbb{R}^d}e^{-\frac{|x|}{4}-\frac{|v|^2}{8}}\;dz,\end{aligned}$$

The latter follows from the definition of S_2 and the fact that

$$0\leq\sqrt{\mu_t^\gamma}\log^-(\mu_t^\gamma)\leq 1.$$

Thus, it follows that

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d}\frac{|x|}{2}+\frac{|v|^2}{4}+\log^+(\mu_t^\gamma)\;d\mu_t^\gamma-\iint_{\mathbb{R}^d\times\mathbb{R}^d}e^{-\frac{|x|}{4}-\frac{|v|^2}{8}}\;dz\leq\mathcal{E}(\mu_t^\gamma)+\iint_{\mathbb{R}^d\times\mathbb{R}^d}|x|\;d\mu_t^\gamma.$$

Furthermore, using the regularity of the solution, we can apply Young's inequality to deduce that

$$\begin{aligned}\frac{d}{dt}\left(\iint_{\mathbb{R}^d\times\mathbb{R}^d}|x|\;d\mu_t^\gamma\right)&=\gamma\iint_{\mathbb{R}^d\times\mathbb{R}^d}\frac{x}{|x|}\cdot v\;d\mu_t^\gamma\\&=\gamma\iint_{\mathbb{R}^d\times\mathbb{R}^d}\frac{x}{|x|}\cdot\left(v\sqrt{\mu_t^\gamma}+2\nabla_v\sqrt{\mu_t^\gamma}\right)\sqrt{\mu_t^\gamma}\;dxdv\\&\leq\frac{1}{2}+\frac{\gamma^2}{2}\iint_{\mathbb{R}^d\times\mathbb{R}^d}\left|v\sqrt{\mu_t^\gamma}+2\nabla_v\sqrt{\mu_t^\gamma}\right|^2\;dxdv\\&=\frac{1}{2}+2\gamma^2\iint_{\mathbb{R}^d\times\mathbb{R}^d}\left|\nabla_v\sqrt{\mu_t^\gamma e^{|v|^2/2}}\right|^2e^{-|v|^2/2}\;dxdv.\end{aligned}$$

Thus, by integration, we have

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d}|x|\;d\mu_t^\gamma\leq\iint_{\mathbb{R}^d\times\mathbb{R}^d}|x|\;d\mu_0^\gamma+\frac{t}{2}+2\gamma^2\int_0^t\iint_{\mathbb{R}^d\times\mathbb{R}^d}\left|\nabla_v\sqrt{\mu_t^\gamma e^{|v|^2/2}}\right|^2e^{-|v|^2/2}\;dxdvds.$$

Hence, it follows that

$$\begin{aligned}\iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|x|}{2}+\frac{|v|^2}{4}\ d\mu_t^\gamma &\le \iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|x|}{2}+\frac{|v|^2}{4}+\log^+(\mu_t^\gamma)\ d\mu_t^\gamma \\ &\le C+\iint_{\mathbb{R}^d\times\mathbb{R}^d} |x|\ d\mu_t^\gamma+\mathcal{E}(\mu_t^\gamma) \\ &\le C+\iint_{\mathbb{R}^d\times\mathbb{R}^d} |x|\ d\mu_0^\gamma+\frac{T}{2}+\mathcal{E}(\mu_t^\gamma) \\ &\quad +2\gamma^2\int_0^t\iint_{\mathbb{R}^d\times\mathbb{R}^d}\left|\nabla_v\sqrt{\mu_t^\gamma e^{|v|^2/2}}\right|^2 e^{-|v|^2/2}\ dxdvds \\ &< \infty,\end{aligned}$$

uniformly in $\gamma \ge 1$ and $0 \le t \le T$ by Theorem 3.6. Thus, we can conclude that

$$\sup_{\gamma\ge 1}\sup_{0\le t\le T}\iint_{\mathbb{R}^d\times\mathbb{R}^d}|x|+|v|^2\ d\mu_t^\gamma<\infty.$$

This provides uniform bounds on the first position and second velocity moments as required. □

3.3 Uniform second moments

The uniform bounds on the first position and second velocity moments play a key role in the establishment of uniform bounds on the second position moments. In particular, we use a Grönwall-type argument twice to deduce this result. This is formalized in the following theorem.

Theorem 3.8. Let $(\mu^\gamma)_{\gamma\ge 1}$ be the family of unique strong solutions to the damped Vlasov-Poisson-Fokker-Planck equation with finite second position and velocity moments and sufficient regularity, where the initial conditions satisfy

$$\sup_{\gamma\ge 1}\mathcal{E}(\mu_0^\gamma)+\sup_{\gamma\ge 1}\iint_{\mathbb{R}^d\times\mathbb{R}^d}|x|^2\ d\mu_0^\gamma<\infty.$$

Then, it follows that

$$\sup_{\gamma\ge 1}\sup_{0\le t\le T}\left\{\iint_{\mathbb{R}^d\times\mathbb{R}^d}|x|^2\ d\mu_t^\gamma\right\}<\infty.$$

Proof for Coulomb interactions. Again, using the regularity of the solution μ_t^γ, we have that

$$\frac{1}{2}\frac{d}{dt}\left(\iint_{\mathbb{R}^d\times\mathbb{R}^d}|x|^2\ d\mu_t^\gamma\right)=\gamma\iint_{\mathbb{R}^d\times\mathbb{R}^d}x\cdot v\ d\mu_t^\gamma.$$

Thus, we can integrate to obtain

$$\frac{1}{2}\iint_{\mathbb{R}^d\times\mathbb{R}^d}|x|^2\ d\mu_t^\gamma=\frac{1}{2}\iint_{\mathbb{R}^d\times\mathbb{R}^d}|x|^2\ d\mu_0^\gamma+\gamma\iint_0^t\iint_{\mathbb{R}^d\times\mathbb{R}^d}x\cdot v\ d\mu_s^\gamma ds.$$

In addition, we have

$$\frac{d}{dt}\left(\iint_{\mathbb{R}^d\times\mathbb{R}^d} x\cdot v\ d\mu_t^\gamma\right) = \iint_{\mathbb{R}^d\times\mathbb{R}^d} (x\cdot v)\left(-\gamma v\cdot\nabla_x\mu_t^\gamma - \gamma F(x,\rho_t^\gamma)\cdot\nabla_v\mu_t^\gamma + \gamma^2\nabla_v\cdot(\nabla_v\mu + v\mu_t^\gamma)\right) dz$$
$$= \gamma\iint_{\mathbb{R}^d\times\mathbb{R}^d} |v|^2\ d\mu_t^\gamma + \gamma\iint_{\mathbb{R}^d\times\mathbb{R}^d} x\cdot F(x,\rho_t^\gamma)\ d\mu_t^\gamma - \gamma^2\iint_{\mathbb{R}^d\times\mathbb{R}^d} x\cdot v\ d\mu_t^\gamma.$$

To bound the second term, note that

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} x\cdot F(x,\rho_t^\gamma)\ d\mu_t^\gamma = \iint_{\mathbb{R}^d\times\mathbb{R}^d} y\nabla\Phi(y-x)\ d\rho_t^\gamma(x)d\rho_t^\gamma(y)$$
$$-\iint_{\mathbb{R}^d\times\mathbb{R}^d} (x-y)\nabla\Phi(x-y)\ d\rho_t^\gamma(x)d\rho_t^\gamma(y)$$
$$\le -\iint_{\mathbb{R}^d\times\mathbb{R}^d} x\cdot F(x,\rho_t^\gamma)\ d\mu_t^\gamma + \int_{\mathbb{R}^d}\Phi*\rho_t^\gamma\ d\rho_t^\gamma.$$

Thus, we can deduce that

$$\frac{d}{dt}\left(\iint_{\mathbb{R}^d\times\mathbb{R}^d} x\cdot v\ d\mu_t^\gamma\right) \le \gamma\left(\iint_{\mathbb{R}^d\times\mathbb{R}^d} |v|^2\ d\mu_t^\gamma + \frac{1}{2}\int_{\mathbb{R}^d}\Phi*\rho_t^\gamma\ d\rho_t^\gamma\right) - \gamma^2\iint_{\mathbb{R}^d\times\mathbb{R}^d} x\cdot v\ d\mu_t^\gamma.$$

Using the same bounds for the negative part of the logarithm as in the proof of Theorem 3.7, we have

$$\frac{d}{dt}\left(\iint_{\mathbb{R}^d\times\mathbb{R}^d} x\cdot v\ d\mu_t^\gamma\right) \le \gamma\left(\iint_{\mathbb{R}^d\times\mathbb{R}^d} |v|^2\ d\mu_t^\gamma + 4\iint_{\mathbb{R}^d\times\mathbb{R}^d} h(\mu_t^\gamma)\ dz + \frac{1}{2}\int_{\mathbb{R}^d}\Phi*\rho_t^\gamma\ d\rho_t^\gamma\right)$$
$$-4\gamma\iint_{\mathbb{R}^d\times\mathbb{R}^d} h(\mu_t^\gamma)\ dz - \gamma^2\iint_{\mathbb{R}^d\times\mathbb{R}^d} x\cdot v\ d\mu_t^\gamma$$
$$\le 4\gamma\mathcal{E}(\mu_t^\gamma) + 4\gamma\iint_{\mathbb{R}^d\times\mathbb{R}^d} e^{-\frac{|x|}{4}-\frac{|v|^2}{8}}\ dz$$
$$+2\gamma\iint_{\mathbb{R}^d\times\mathbb{R}^d} |x|\ d\mu_t^\gamma - \gamma^2\iint_{\mathbb{R}^d\times\mathbb{R}^d} x\cdot v\ d\mu_t^\gamma$$
$$\le C\gamma - \gamma^2\iint_{\mathbb{R}^d\times\mathbb{R}^d} x\cdot v\ d\mu_t^\gamma.$$

for some constant $C > 0$ independent of $\gamma \ge 1$, which follows by Theorems 3.6 and 3.7. Applying Grönwall's lemma followed by Young's inequality yields

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} x\cdot v\ d\mu_t^\gamma \le \left(\iint_{\mathbb{R}^d\times\mathbb{R}^d}\frac{|x|^2}{2}+\frac{|v|^2}{2}\ d\mu_0^\gamma\right)e^{-\gamma^2 t} + \frac{C}{\gamma}$$
$$\le \frac{C}{\gamma}$$

where the last inequality follows by Theorem 3.7. Thus, we have that

$$\frac{1}{2}\iint_{\mathbb{R}^d\times\mathbb{R}^d} |x|^2\ d\mu_t^\gamma = \frac{1}{2}\iint_{\mathbb{R}^d\times\mathbb{R}^d} |x|^2\ d\mu_0^\gamma + \gamma\int_0^t\iint_{\mathbb{R}^d\times\mathbb{R}^d} x\cdot v\ d\mu_s^\gamma ds$$
$$< \infty,$$

uniformly in $\gamma \geq 1$ and $0 \leq t \leq T$. Thus, we can conclude that

$$\sup_{\gamma \geq 1} \sup_{0 \leq t \leq T} \iint_{\mathbb{R}^d \times \mathbb{R}^d} |x|^2 \, d\mu_t^\gamma < \infty.$$

This gives uniform bounds on the second position moments as required. □

As a closing remark, since the uniform bound on the second position moments implies that the first position moments are also uniformly bounded, we can deduce that both position and velocity moments remain uniformly bounded throughout the evolution. Under the assumptions of Theorem 3.8, we have

$$\sup_{\gamma \geq 1} \sup_{0 \leq t \leq T} \iint_{\mathbb{R}^d \times \mathbb{R}^d} |x|^2 + |v|^2 \, d\mu_t^\gamma < \infty.$$

This fact is key in ensuring that the Wasserstein distances between particle densities are well-defined and serve as a building block for the establishment of the uniform bound given in Theorem 2.9.

Chapter 4

Overdamped limits

In this chapter, we focus on establishing the main result of the thesis, corresponding to a convergence result in the Wasserstein distance of solutions to the damped Vlasov-Poisson-Fokker-Planck equation (VPFP) to the corresponding Aggregation-Diffusion equation (AD). In particular, the proof of Theorem 2.9 passes through an intermediate system, solved by the standard coarse-grained particle density defined in Definition 2.8. Namely, we bound the Wasserstein distances to the intermediate system in terms of the second position and velocity moments of the phase-space density [4]. Combined with the results from Chapter 3, these bounds are made uniform in the damping parameter, under suitable energy assumptions. Throughout this section, we assume the well-posedness of the two equations and postpone the proof for the damped Vlasov-Poisson-Fokker-Planck equation until Chapter 5.

4.1 The coarse-grained system

We begin by investigating the properties of the coarse-grained particle densities introduced in Definition 2.8. These quantities resemble the usual particle density, where the effect of the velocity variable on the position variable is taken into account ahead of time. This ahead-of-time influence is controlled by a parameter $\lambda \geq 0$ through the coarse-graining map, defined by

$$\Gamma^\lambda : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d, \quad \Gamma^\lambda(x, v) := (x + \lambda v, v),$$

Note that this is a volume-preserving diffeomorphism with inverse given by

$$\Gamma^{-\lambda} : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d, \quad \Gamma^{-\lambda}(x, v) := (x - \lambda v, v).$$

This can be verified by checking that $\det D\Gamma^\lambda = 1$. To understand the role of the coarse-graining map, observe that the integrand in Definition 2.8 corresponds to the push-forward measure of the phase-space density by the coarse-graining map. Namely, for a probability measure $\mu \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ that is absolutely continuous with respect to the Lebesgue measure, we have the equality

$$\Gamma^\lambda_\# \mu(x, v) = \mu(x - \lambda v, v).$$

In this measure-theoretic language, we have the following equalities

$$\rho_t^\gamma(x) = \pi^x_\# \mu_t^\gamma(x), \quad \rho_t^{\gamma,\lambda}(x) = (\pi^x \circ \Gamma^\lambda)_\# \mu_t^\gamma(x), \quad \hat{\rho}_t^\gamma(x) = (\pi^x \circ \Gamma^{\frac{1}{\gamma}})_\# \mu_t^\gamma(x),$$

where $\pi^x : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ denotes the projection onto the first component. After shedding light on the coarse-grained particle densities, we consider the dynamics describing their evolution in time.

Theorem 4.1. Let μ^γ be a strong solution to the damped Vlasov-Poisson-Fokker-Planck equation. Then, the standard coarse-grained particle density $\hat{\rho}^\gamma$ is a weak solution to the equation

$$\frac{\partial \hat{\rho}_t^\gamma}{\partial t} + \nabla_x \cdot \hat{J}_t^\gamma = \Delta_x \hat{\rho}_t^\gamma, \quad \hat{J}_t^\gamma(x) := \int_{\mathbb{R}^d} F(x - v/\gamma, \rho_t^\gamma)\, \Gamma_\#^{\frac{1}{\gamma}} \mu_t^\gamma(x, v)\, dv.$$

Proof. Let $\varphi \in C_c^\infty(\mathbb{R}^d)$ be an arbitrary test function. Denote $\varphi^\gamma(x.v) := \varphi(x + v/\gamma)$. Since μ^γ is a weak solution to the Vlasov-Poisson-Fokker-Planck equation, we have that

$$\begin{aligned}
0 &= \iint_{\mathbb{R}^d \times \mathbb{R}^d} \left(\frac{\partial \varphi^\gamma}{\partial t} + \gamma v \cdot \nabla_x \varphi^\gamma + \gamma \nabla_v \varphi^\gamma \cdot (F(x, \rho_t^\gamma) - \gamma v) + \gamma^2 \Delta_v \varphi(x + v/\gamma) \right) d\mu_t^\gamma \\
&= \iint_{\mathbb{R}^d \times \mathbb{R}^d} \left(\frac{\partial \varphi}{\partial t} \circ (\pi^x \circ \Gamma^{\frac{1}{\gamma}}) + \nabla_x \varphi \circ (\pi^x \circ \Gamma^{\frac{1}{\gamma}}) \cdot F(x, \rho_t^\gamma) + \Delta_x \varphi \circ (\pi^x \circ \Gamma^{\frac{1}{\gamma}}) \right) d\mu_t^\gamma \\
&= \int_{\mathbb{R}^d} \left(\frac{\partial \varphi}{\partial t} + \Delta_x \varphi \right) d\hat{\rho}_t^\gamma + \iint_{\mathbb{R}^d \times \mathbb{R}^d} \nabla_x \varphi \cdot F(x - v/\gamma, \rho_t^\gamma)\, d\Gamma_\#^{\frac{1}{\gamma}} \mu_t^\gamma.
\end{aligned}$$

Integrating with respect to t yields the desired the weak formulation of Definition 2.1 as

$$\int_{\mathbb{R}^d} \varphi\, d\hat{\rho}_t^\gamma - \int_{\mathbb{R}^d} \varphi\, d\hat{\rho}_0^\gamma = \int_0^t \int_{\mathbb{R}^d} \Delta_x \varphi\, d\hat{\rho}_s^\gamma ds + \iint_{\mathbb{R}^d \times \mathbb{R}^d} \nabla_x \varphi \cdot F(x - v/\gamma, \rho_s^\gamma)\, d\Gamma_\#^{\frac{1}{\gamma}} \mu_t^\gamma\, ds.$$

As a result, we obtain a weak solution to our desired system with initial condition $\hat{\rho}_0^\gamma$. □

We are now ready to prove the first half of the triangle inequality argument. In particular, we give a bound on the Wasserstein distance between the particle density of the damped Vlasov-Poisson-Fokker-Planck equation and the corresponding standard coarse-grained particle density.

Theorem 4.2. Let μ^γ be the unique strong solution to the Vlasov-Poisson-Fokker-Planck equation and consider the corresponding particle density ρ^γ and the standard coarse-grained particle density $\hat{\rho}_t^\gamma$. Then, we have the bound

$$W_2^2(\hat{\rho}_t^\gamma, \rho_t^\gamma) \leq \frac{1}{\gamma^2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} |v|^2\, d\mu_t^\gamma.$$

Proof. Consider an optimal transport plan $\Pi_t \in \mathcal{P}(\mathbb{R}^{2d} \times \mathbb{R}^{2d})$ from $\Gamma_\#^{\frac{1}{\gamma}} \mu_t^\gamma$ to μ_t^γ with respect to the square Euclidean norm. Such a measure Π_t exists since the initial and final measures are defined on a Polish space and the transportation cost is lower semi-continuous. By projecting onto the position variables, it follows that $(\pi^x \times \pi^x)_\# \Pi_t \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ is a transport plan from $\hat{\rho}_t^\gamma$ to ρ_t^γ. Thus, we obtain the estimate

$$\begin{aligned}
W_2^2(\hat{\rho}_t^\gamma, \rho_t^\gamma) &\leq \iint_{\mathbb{R}^d \times \mathbb{R}^d} |x - \xi|^2\, d(\pi^x \times \pi^x)_\# \Pi_t(x, \xi) \\
&= \iiiint_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d} |\pi^x(x, v) - \pi^x(\xi, \nu)|^2\, d\Pi_t(x, v, \xi, \nu) \\
&\leq \iiiint_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d} |x - \xi|^2 + |v - \nu|^2\, d\Pi_t(x, v, \xi, \nu)
\end{aligned}$$

$$= W_2^2(\Gamma^{\frac{1}{\gamma}}_{\#}\mu_t^\gamma, \mu_t^\gamma).$$

Moreover, consider the transport plan $(\Gamma^{\frac{1}{\gamma}} \times id)_{\#}\mu_t^\gamma \in \mathcal{P}(\mathbb{R}^{2d} \times \mathbb{R}^{2d})$ from $\Gamma^{\frac{1}{\gamma}}_{\#}\mu_t^\gamma$ to μ_t^γ. This yields

$$W_2^2(\Gamma^{\frac{1}{\gamma}}_{\#}\mu_t^\gamma, \mu_t^\gamma) \leq \iint_{\mathbb{R}^d\times\mathbb{R}^d} |\Gamma^{\frac{1}{\gamma}}(x,v) - (x,v)|^2 \, d\mu_t^\gamma = \frac{1}{\gamma^2} \iint_{\mathbb{R}^d\times\mathbb{R}^d} |v|^2 \, d\mu_t^\gamma.$$

Thus, we have shown that

$$W_2^2(\hat{\rho}_t^\gamma, \rho_t^\gamma) \leq \frac{1}{\gamma^2} \iint_{\mathbb{R}^d\times\mathbb{R}^d} |v|^2 \, d\mu_t^\gamma,$$

giving a bound on the Wasserstein distance in terms of the second velocity moment.

□

4.2 Displacement interpolation

The second half of the triangle inequality argument requires significantly more work. As a starting point, we begin by stating a standard result from optimal transport theory, concerning the existence of an optimal transport map whenever the source measure does not give mass to small sets. This is known as Brenier's theorem, which plays an important role in the arguments in this chapter [20].

Theorem 4.3. Let $p \geq 1$ and consider probability measures $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$ such that μ is absolutely continuous with respect to the Lebesgue measure. Then, there exists a convex function $\Psi : \mathbb{R}^d \to \mathbb{R}$ such that $T := \nabla\Psi$ is the unique optimal transport map from μ to ν, meaning that

$$\int_{\mathbb{R}^d} |x - T(x)|^p \, d\mu = \inf_{S_{\#}\mu=\nu} \int_{\mathbb{R}^d} |x - S(x)|^p \, d\mu.$$

Here, the infimum is taken over all transport maps from μ to ν. In addition, the map T is monotone in the sense that for all $x, y \in \mathbb{R}^d$, we have the order inequality

$$(T(x) - T(y)) \cdot (x - y) \geq 0.$$

In addition to Brenier's theorem, we introduce the ideas attributed McCann, concerning the interpolation of probability measures, given a transport map [13]. This entails generating a family of probability measures, consisting of all measures obtained throughout the process of transportation.

Definition 4.4. Let $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$ be probability measures such that μ is absolutely continuous with respect to the Lebesgue measure. In addition, consider the optimal transport map T from μ to ν obtained from Theorem 4.3. For $0 \leq \theta \leq 1$, the displacement function is defined by

$$T^\theta : \mathbb{R}^d \to \mathbb{R}^d, \quad T^\theta(x) := (1-\theta)x + \theta T(x)).$$

We call the measure $T^\theta_{\#}\mu \in \mathcal{P}(\mathbb{R}^d)$ the θ-displacement interpolant between μ and ν.

Moreover, the following theorem relates displacement interpolants with the Wasserstein distance.

Theorem 4.5. Let $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$ be probability measures with finite second moments such that μ is absolutely continuous with respect to the Lebesgue measure. In addition, consider the optimal transport map T from μ to ν and let $0 \leq \theta_1, \theta_2 \leq 1$. Then, we have the equality

$$W_2(T^{\theta_1}_{\#}\mu, T^{\theta_2}_{\#}\mu) = |\theta_1 - \theta_2| W_2(\mu, \nu).$$

Proof. Let $\Pi \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ be an optimal transport plan from μ to ν. Consider the interpolation

$$C^\theta(x,y) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d, \quad C^\theta(x,y) := (1-\theta)x + \theta y.$$

Using the fact that the optimal transport plan Π from μ to ν is supported on the graph of the optimal transport map T from μ to ν, we have

$$C^\theta_{\#}\Pi = C^\theta_{\#}(id \times T)_{\#}\mu = ((1-\theta)id + \theta T)_{\#}\mu = T^\theta_{\#}\mu.$$

Thus, we can deduce that $(C^{\theta_1}, C^{\theta_2})_{\#}\Pi$ is a transport plan from $T^{\theta_1}_{\#}\mu$ to $T^{\theta_2}_{\#}\mu$ and thus

$$W_2(T^{\theta_1}_{\#}\mu, T^{\theta_2}_{\#}\mu) \leq \left(\iint_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|^2 \; d(C^{\theta_1}, C^{\theta_2})_{\#}\Pi \right)^{1/2} = |\theta_2 - \theta_1| \; W_2(\mu, \nu).$$

If the strict inequality holds for some $\theta_1 < \theta_2$, the triangle inequality yields

$$\begin{aligned} W_2(\mu,\nu) &\leq W_2(\mu, T^{\theta_1}_{\#}\mu) + W_2(T^{\theta_1}_{\#}, T^{\theta_2}_{\#}\mu) + W_2(T^{\theta_2}_{\#}, \nu) \\ &< \theta_1 W_2(\mu,\nu) + (\theta_2 - \theta_1) W_2(\mu,\nu) + (1-\theta_2) W_2(\mu,\nu) \\ &= W_2(\mu,\nu), \end{aligned}$$

giving a contradiction. Thus, we can deduce that

$$W_2(T^{\theta_1}_{\#}\mu, T^{\theta_2}_{\#}\mu) = |\theta_1 - \theta_2| W_2(\mu, \nu).$$

This provides our desired result and concludes the proof of the theorem. □

In our quest for the proof of the main theorem, we need to compute the relative entropy of interpolants with respect to the Lebesgue measure. To ensure that this quantity is finite, we must guarantee that the interpolants are absolutely continuous with respect to the Lebesgue measure (see Definition 2.4).

Theorem 4.6. Let $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$ be probability measures that are absolutely continuous with respect to the Lebesgue measure. Consider the optimal transport map T from μ to ν obtained from Theorem 4.3. For $0 \leq \theta \leq 1$, the θ-displacement interpolant $T^\theta_{\#}\mu$ between μ and ν is absolutely continuous with respect to the Lebesgue measure.

Proof. We verify that the θ-displacement interpolant from μ to ν satisfies the definition of absolute continuity with respect to the Lebesgue measure. In particular, let $A \subseteq \mathbb{R}^d$ be a Borel set with zero Lebesgue measure. Let $T = \nabla\Psi$ be the optimal transport map from μ to ν provided by Theorem 4.3.

Define

$$\psi^\theta : \mathbb{R}^d \times \mathbb{R}, \quad S^\theta(x) := (1-\theta)\frac{|x|^2}{2} + \theta\Psi(x).$$

Since the function Ψ is convex, we have that ψ^θ is strictly convex. Thus, we have that $\nabla\psi^\theta$ is strictly monotone and so its inverse is well-defined on its domain. By Cauchy-Schwarz and the monotonicity of the optimal transport map, we obtain

$$\begin{aligned} |\nabla\psi^\theta(x) - \nabla\psi^\theta(y)||x-y| &\geq (\nabla\psi^\theta(x) - \nabla\psi^\theta(y)) \cdot (x-y) \\ &= (1-\theta)|x-y|^2 + \theta(\nabla\Psi(x) - \nabla\Psi(y)) \cdot (x-y) \\ &\geq (1-\theta)|x-y|^2. \end{aligned}$$

From this, we can deduce that the inverse map

$$(\nabla\psi^\theta)^{-1} : \text{ran } \nabla\psi^\theta \to \mathbb{R}^d$$

is Lipschitz with constant less than $(1-\theta)^{-1}$. It follows that the set $(\nabla\psi^\theta)^{-1}(A)$ has zero Lebesgue measure. Hence, the interpolant is absolutely continuous with respect to the Lebesgue measure. □

Given the absolute continuity of interpolants with respect to the Lebesgue measure, we state without proof a deeper result concerning the convexity of the relative entropy along interpolants [13].

Theorem 4.7. Let $p > 1$ and consider probability measures $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$ that are absolutely continuous with respect to the Lebesgue measure. In addition, let T be the optimal transport map from μ to ν obtained from Theorem 4.3. Then, the functions

$$K, L : [0,1] \to \mathbb{R}, \quad I(\theta) := \mathcal{H}(T^\theta_\#\mu|\mathcal{L}^d), \quad L(\theta) := \log(\|T^\theta_\#\mu\|_{L^p})$$

are convex, where $T^\theta_\#\mu$ is the θ-displacement interpolant between μ and ν.

Using the convexity of the relative entropy along interpolants, we show that the displacement interpolants remain essentially bounded, whenever this property holds for the initial and final measures.

Corollary 4.8. Let $\mu, \nu \in \mathcal{P} \cap L^\infty(\mathbb{R}^d)$ be probability measures that are absolutely continuous with respect to the Lebesgue measure with essentially bounded densities. In addition, let T be the optimal transport map from μ to ν obtained from Theorem 4.3. For $0 \leq \theta \leq 1$, the θ-displacement interpolant is essentially bounded with the estimate

$$\|T^\theta_\#\mu\|_{L^\infty} \leq \max\{\|\mu\|_{L^\infty}, \|\nu\|_{L^\infty}\}.$$

Proof. By Theorem 4.7, we have

$$\log(\|T^\theta_\#\mu\|_{L^p}) \leq \max\{\log(\|\mu\|_{L^p}), \log(\|\nu\|_{L^p})\}.$$

Taking the limit superior yields

$$\limsup_{p\to\infty} \log(\|T^\theta_\# \mu\|_{L^p}) \le \max\{\limsup_{p\to\infty} \log(\|\mu\|_{L^p}), \limsup_{p\to\infty} \log(\|\nu\|_{L^p})\}.$$

By continuity of the logarithm and monotonicity of the exponential, we have

$$\|T^\theta_\# \mu\|_{L^\infty} \le \max\{\|\mu\|_{L^\infty}, \|\nu\|_{L^\infty}\}.$$

This shows that the displacement interpolants are essentially bounded as desired. □

The essential boundedness of the displacement interpolants can be leveraged to obtain control in the Wasserstein distance on the modulated interaction energy, described in Section 2.3. The rest of this section is devoted to this aim. To start, we recall the definition of the modulated interaction energy.

Definition 4.9. Let $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$ be probability measures that are absolutely continuous with respect to the Lebesgue measure. Then, the modulated interaction energy between μ and ν is defined by

$$\mathcal{D}_\Phi(\mu, \nu) := \iint_{\mathbb{R}^d\times\mathbb{R}^d} \Phi(x-y)\ d(\mu-\nu)(y)d(\mu-\nu)(x).$$

Before controlling the modulated interaction energy with respect to the Wasserstein distance, we first consider the homogeneous Sobolev norm, defined on the space of signed measures having zero total mass. This norm is suitable for the signed measures appearing in Definition 4.9.

Definition 4.10. Let ω be a signed measure on $\mathbb{R}^d$ with zero total mass, that is $\omega(\mathbb{R}^d) = 0$. Then, the $\dot{H}^{-1}$ homogeneous Sobolev norm is defined by

$$\|\omega\|_{\dot{H}^{-1}} := \sup_{\|\nabla\varphi\|_{L^2}\le 1} \left| \int_{\mathbb{R}^d} \varphi\ d\omega \right|.$$

Remark 4.11. It immediately follows from the definition that if $\nabla\Phi * (\mu-\nu) \in L^2(\mathbb{R}^d)$, we have

$$|\mathcal{D}_\Phi(\mu,\nu)| \le \|\nabla\Phi * (\mu-\nu)\|_{L^2} \|\mu-\nu\|_{\dot{H}^{-1}}.$$

We are ready to prove an estimate on the modulated interaction energy with respect to the Wasserstein distance under an essential boundedness assumption on the density of the measures.

Theorem 4.12. Let $\mu, \nu \in \mathcal{P}_2 \cap L^\infty(\mathbb{R}^d)$ be probability measures that are absolutely continuous with respect to the Lebesgue measure with finite second moments and essentially bounded densities. Then, we have the estimate

$$\|\mu-\nu\|_{\dot{H}^{-1}} \le \sqrt{c_\infty}\ W_2(\mu,\nu), \quad c_\infty := \max\{\|\mu\|_{L^\infty}, \|\nu\|_{L^\infty}\}.$$

Proof. Let $\varphi \in C_c^\infty(\mathbb{R}^d)$ with $\|\nabla\varphi\|_{L^2} \le 1$. By change of variables, we have

$$\int_{\mathbb{R}^d} \varphi(x)\ dT^\theta_\#\mu = \int_{\mathbb{R}^d} \varphi((1-\theta)x + \theta T(x))\ d\mu.$$

Thus, taking derivatives with respect to θ yields

$$\begin{aligned}\frac{d}{d\theta}\int_{\mathbb{R}^d}\varphi\, dT^\theta_\# \mu &= \int_{\mathbb{R}^d\times\mathbb{R}^d} \nabla\varphi((1-\theta)x+\theta T(x))\cdot(T(x)-x)\; d\mu \\ &\leq \left(\int_{\mathbb{R}^d}|T(x)-x|^2\; d\mu\right)^{1/2}\left(\int_{\mathbb{R}^d}|\nabla\varphi|^2 dT^\theta_\#\mu\right)^{1/2} \\ &\leq \sqrt{c_\infty}\; W_2(\mu,\nu),\end{aligned}$$

where the last inequality follows by Corollary 4.8. By the Mean Value Theorem, it follows that

$$\|\mu-\nu\|_{\dot{H}^{-1}} \leq \sqrt{c_\infty}\; W_2(\mu,\nu).$$

This gives the required result, concluding the proof of the theorem. □

4.3 Evolution-variational inequalities

In this section, we make use of a deep connection between the field of optimal transport and the theory of partial differential equations. From an intuitive point of view, the result gives a formula for the time-derivative of the 2-Wasserstein distance along a solution to a continuity equation with a sufficiently regular vector field. For a proof, see [10, Theorem 8.4.7, Remark 8.4.8].

Theorem 4.13. Let $\rho \in C([0,T],\mathcal{P}_2(\mathbb{R}^d))$ be a weak solution to the continuity equation

$$\frac{\partial \rho_t}{\partial t} + \nabla\cdot(\rho_t\xi_t) = 0, \qquad \int_0^T \|\xi_t\|^2_{L^2(\rho_t)}\; dt < \infty,$$

that is absolutely continuous with respect to the Lebesgue measure. For any probability measure $\nu \in \mathcal{P}_2(\mathbb{R}^d)$ with finite second moments, consider the optimal transport map T_t from ρ_t to ν, obtained from Theorem 4.3. Then, for almost every $0 < t < 1$, we have the equality

$$\frac{1}{2}\frac{d}{dt}W_2^2(\rho_t,\nu) = \int_{\mathbb{R}^d}(x-T_t(x))\cdot\xi_t(x)\; d\rho_t.$$

The above theorem serves as a stepping stone to proving an inequality characterizing $\mathcal{E}$-regular solutions to the Aggregation-Diffusion equation, as given in Definition 2.6. In particular, we obtain a bound on the energy differences between measures defined on spacial spaces.

Theorem 4.14. Let $\rho,\nu \in \mathcal{P}_2(\mathbb{R}^d)\cap \mathrm{dom}(\mathcal{E})$ be probability measures with finite second moments, belonging to the domain of the energy functional defined in Definition 2.5. In addition, assume that the measures ρ and ν are absolutely continuous with respect to the Lebesgue measure satisfying

$$\mathcal{D}_\Phi(\rho,\nu) < \infty, \qquad \int_{\mathbb{R}^d}\frac{|\nabla\rho|^2}{\rho}\; dx < \infty, \qquad F(\cdot,\rho)\in L^\infty(\mathbb{R}^d).$$

Consider the optimal transport map T from ρ to ν obtained from Theorem 4.3.

Then, it follows that we have the energy inequality

$$\mathcal{E}(\nu)-\mathcal{E}(\rho)\geq\int_{\mathbb{R}^d}(T(x)-x)\cdot\nabla\rho(x)\;dx+\frac{1}{2}\mathcal{D}_\Phi(\rho,\nu)$$
$$-\int_0^1\int_{\mathbb{R}^d}(T(x)-x)\cdot F(T^\theta(x),\rho)\;d\rho d\theta,$$

for ρ-almost every $x\in\mathbb{R}^d$, where the map T^θ is the displacement function defined in Definition 4.4.

Proof. We begin by bounding the differences in entropy. Using Theorem 4.7, we can use the convexity of the relative entropy along displacement interpolants to deduce that for all $0<\theta<1$, we have

$$\mathcal{H}(\nu|\mathcal{L}^d)-\mathcal{H}(\rho|\mathcal{L}^d)\geq\frac{1}{\theta}\left(\mathcal{H}(T^\theta_\#\rho|\mathcal{L}^d)-\mathcal{H}(\rho|\mathcal{L}^d)\right).$$

In addition, we can use [10, Lemma 10.4.4 (iv)] to deduce that in the limit as $\theta\to0$, we obtain

$$\mathcal{H}(\nu|\mathcal{L}^d)-\mathcal{H}(\rho|\mathcal{L}^d)\geq-\int_{\mathbb{R}^d}\rho(x)\;\mathrm{tr}\;\tilde{\nabla}(T(x)-x)\;dx,$$

where $\mathrm{tr}\;\tilde{\nabla}$ denotes the approximate divergence. By Hölder's inequality, we have that

$$\int_{\mathbb{R}^d}|\nabla\rho|\;dx\leq\left(\int_{\mathbb{R}^d}\frac{|\nabla\rho|^2}{\rho}\;dx\right)^{1/2}<\infty,$$

from which we can deduce that $\rho\in W^{1,1}(\mathbb{R}^d)$. Thus, we can use [10, Lemma 10.4.5] to deduce that

$$\mathcal{H}(\nu|\mathcal{L}^d)-\mathcal{H}(\rho|\mathcal{L}^d)\geq\int_{\mathbb{R}^d}(T(x)-x)\cdot\nabla\rho(x)\;dx$$

holds for ρ-almost every $x\in\mathbb{R}^d$. When it comes to the differences in the potential energy, using the optimal transport map T from ρ to ν with a change of variables, we obtain

$$\frac{1}{2}\int_{\mathbb{R}^d}\Phi*\nu\;d\nu-\frac{1}{2}\int_{\mathbb{R}^d}\Phi*\rho\;d\rho=\frac{1}{2}\int_{\mathbb{R}^d}\Phi*(\nu-\rho)\;d\nu+\frac{1}{2}\int_{\mathbb{R}^d}\Phi*\rho\;d\nu-\frac{1}{2}\int_{\mathbb{R}^d}\Phi*\rho\;d\rho$$
$$=\frac{1}{2}\mathcal{D}_\Phi(\rho,\nu)+\int_{\mathbb{R}^d}\Phi*\rho(T(x))-\Phi*\rho(x)\;d\rho$$
$$=\frac{1}{2}\mathcal{D}_\Phi(\rho,\nu)+\int_0^1\int_{\mathbb{R}^d}\nabla\Phi*\rho(T^\theta(x))\cdot(T(x)-x)\;d\rho d\theta.$$

As a result, we obtain

$$\mathcal{E}(\nu)-\mathcal{E}(\rho)\geq\int_{\mathbb{R}^d}(T(x)-x)\cdot\nabla\rho(x)\;dx+\frac{1}{2}\mathcal{D}_\Phi(\rho,\nu)$$
$$-\int_0^1\int_{\mathbb{R}^d}F(T^\theta(x),\rho)\cdot(T(x)-x)\;d\rho d\theta.$$

This proves our desired result, concluding the proof of the theorem. □

In particular, for systems corresponding to a perturbation of the Aggregation-Diffusion, we can obtain a refined inequality whenever the induced force of the system is Lipschitz in the position variable.

Theorem 4.15. Let $\rho \in C([0,T], \mathcal{P}_2(\mathbb{R}^d) \cap \mathrm{dom}(\mathcal{E}))$ be a weak solution to the continuity equation

$$\frac{\partial \rho_t}{\partial t} + \nabla \cdot (\rho_t(F(\cdot,\rho_t) + e_t)) = \Delta \rho_t, \qquad \int_0^T \|e_t\|^2_{L^2(\rho_t)}\ dt < \infty, \quad \int_0^T \int_{\mathbb{R}^d} \frac{|\nabla \rho_t|^2}{\rho_t}\ dxdt < \infty,$$

that is absolutely continuous with respect to the Lebesgue measure. In addition, assume that for all $0 \le t \le T$, the force field $F(\cdot,\rho_t) \in L^\infty \cap \mathrm{Lip}(\mathbb{R}^d)$ is essentially bounded and Lipschitz, satisfying

$$c_0 := \sup_{0 \le t \le T} \|F(\cdot,\rho_t)\|_{\mathrm{Lip}} < \infty, \qquad \lambda := 1 + c_0.$$

Then, for all $\nu \in \mathcal{P}(\mathbb{R}^d) \cap \mathrm{dom}(\mathcal{E})$ and almost every $0 \le t \le T$, we have the inequality

$$\frac{1}{2}\frac{d}{dt} W_2^2(\rho_t,\nu) \le \mathcal{E}(\nu) - \mathcal{E}(\rho_t) - \frac{1}{2}\mathcal{D}_\Phi(\rho_t,\nu) + \frac{\lambda}{2} W_2^2(\rho_t,\nu) + \frac{1}{2}\|e_t\|^2_{L^2(\rho_t)}. \tag{EVI}$$

We refer to this inequality as the modified evolution-variational inequality.

Proof. Note that the governing equation can be written in the form

$$\frac{\partial \rho_t}{\partial t} + \nabla \cdot (\rho_t \xi_t) = 0, \quad \xi_t := F(\cdot,\rho_t) + e_t - \frac{\nabla \rho_t}{\rho_t}.$$

Using Young's inequality, there exists a constant $C > 0$ such that the vector field ξ_t satisfies

$$\int_0^T \|\xi_t\|^2_{L^2(\rho_t)}\ dt \le C \left(\int_0^T \int_{\mathbb{R}^d} |F(\cdot,\rho_t)|^2\ d\rho_t dt + \int_0^T \int_{\mathbb{R}^d} |e_t|^2\ d\rho_t dt + \int_{\mathbb{R}^d} \left| \frac{\nabla \rho_t}{\rho} \right|^2\ d\rho_t dt \right) < \infty,$$

Hence, combining Theorems 4.13 and 4.14 yields

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt} W_2^2(\rho_t,\nu) &\le \mathcal{E}(\nu) - \mathcal{E}(\rho_t) - \frac{1}{2}\mathcal{D}_\Phi(\rho_t,\nu) + \frac{1}{2} W_2^2(\rho_t,\nu) + \frac{1}{2}\|e_t\|^2_{L^2(\rho)} \\
&\quad + \int_0^1 \int_{\mathbb{R}^d} (T(x) - x) \cdot (F(T_t^\theta(x),\rho_t) - F(x,\rho_t))\ d\rho_t d\theta \\
&\le \mathcal{E}(\nu) - \mathcal{E}(\rho_t) - \frac{1}{2}\mathcal{D}_\Phi(\rho_t,\nu) + \frac{1}{2} W_2^2(\rho_t,\nu) + \frac{1}{2}\|e_t\|^2_{L^2(\rho)} \\
&\quad + c_0 W_2(\rho_t,\nu) \int_0^1 W_2(T_t^\theta{}_\#\mu,\nu)\ d\theta,
\end{aligned}$$

where the last equation follows by Hölder's inequality. Using Theorem 4.5, we can deduce that

$$W_2(T_t^\theta{}_\#\mu,\nu) = \theta W_2(\rho_t,\nu).$$

Thus, we can integrate to conclude that

$$\frac{1}{2}\frac{d}{dt} W_2^2(\rho_t,\nu) \le \mathcal{E}(\nu) - \mathcal{E}(\rho_t) - \frac{1}{2}\mathcal{D}_\Phi(\rho_t,\nu) + \frac{\lambda}{2} W_2^2(\rho_t,\nu) + \frac{1}{2}\|e_t\|^2_{L^2(\rho_t)}.$$

This gives our desired result, concluding the proof of the theorem.

□

4.4 Wasserstein control of EVIs

To apply Grönwall's lemma with the modified evolution-variational inequality (EVI), we must control each term appearing in the inequality with respect to the Wasserstein distance. To relate the framework detailed in Section 4.3 to the Aggregation-Diffusion equation and the coarse-grained system, we begin by choosing a suitable error term. In particular, we take

$$e_t^\gamma(x) := \frac{d\hat{J}_t^\gamma}{d\hat{\rho}_t^\gamma}(x) - F(x, \hat{\rho}_t^\gamma),$$

where the first term corresponds to the Radon-Nikodym derivative of $\hat{J}_t$ with respect to $\hat{\rho}_t^\gamma$ for $\hat{\rho}_t^\gamma$-almost every $x \in \mathbb{R}^d$. In this case, by applying Theorem 4.15 twice with [10, Lemma 4.3.4], we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}W_2^2(\hat{\rho}_t^\gamma, \rho_t) &\leq \frac{1}{2}\frac{d}{dt}W_2^2(\hat{\rho}_t^\gamma, \rho_s)\big|_{s=t} + \frac{1}{2}\frac{d}{ds}W_2^2(\hat{\rho}_t^\gamma, \rho_s)\big|_{t=s} \\ &\leq \lambda^\gamma W_2^2(\hat{\rho}_t^\gamma, \rho_t) - \mathcal{D}_\Phi(\hat{\rho}_t^\gamma, \rho_t) + \frac{1}{2}\left\|e_t^\gamma\right\|_{L^2(\hat{\rho}_t)},\end{aligned}$$

where λ^γ is defined as in Theorem 4.15 for $\hat{\rho}^\gamma$. We now estimate the modulated interaction energy with respect to the Wasserstein distance. This is given in the following theorem.

Theorem 4.16. Let $\mu, \nu \in \mathcal{P} \cap L^\infty(\mathbb{R}^d)$ be probability measures that are absolutely continuous with respect to the Lebesgue measure with essentially bounded densities. Then, we have that

$$|\mathcal{D}_\Phi(\mu, \nu)| \leq c_\infty W_2^2(\mu, \nu).$$

Proof. To begin, let $\varphi \in C_c^\infty(\mathbb{R}^d)$ with $\|\nabla\varphi\|_{L^2} \leq 1$. We use colinearity to maximize the L^2 inner product to deduce that

$$\begin{aligned}\|\mu - \nu\|_{\dot{H}^{-1}} &= \sup_{\|\nabla\varphi\|_{L^2}\leq 1} \left|\int_{\mathbb{R}^d} \varphi(x)(\Delta\Phi * (\mu - \nu))(x)\right| \\ &= \sup_{\|\nabla\varphi\|_{L^2}\leq 1} \left|\int_{\mathbb{R}^d} \nabla\varphi(x) \cdot \nabla\Phi * (\mu - \nu)(x)\, dx\right| \\ &= \|\nabla\Phi * (\mu - \nu)\|_{L^2}.\end{aligned}$$

Using Remark 4.11 with Theorem 4.12, we have that

$$|\mathcal{D}_\Phi(\mu, \nu)| \leq c_\infty W_2^2(\mu, \nu).$$

This gives our desired result, concluding the proof of the theorem. □

Next, we bound the error term due to the discrepancy in the forces of the two systems appearing in the modified evolution-variational inequality. This is bounded in terms of the second velocity moments of the damped Vlasov-Poisson-Fokker-Planck equation, providing a first step towards uniform estimates.

Theorem 4.17. Let μ^γ be the unique strong solution to the damped Vlasov-Poisson-Fokker-Planck equation (VPFP) such that the corresponding particle density $\rho_t^\gamma \in \mathcal{P}\cap L^\infty(\mathbb{R}^d)$ and standard coarse-grained particle density $\hat\rho^\gamma \in \mathcal{P}\cap L^\infty(\mathbb{R}^d)$ have essentially bounded densities with

$$\sup_{0\le t\le T}\left\{\|\rho_t^\gamma\|_{L^\infty}+\|\hat\rho_t^\gamma\|_{L^\infty}\right\}<\infty.$$

Assume the standard coarse-grained particle density satisfies the assumptions of Theorem 4.15. Then,

$$\frac{1}{2}\|e_t^\gamma\|^2_{L^2(\hat\rho_t^\gamma)} \le \frac{c_*^\gamma}{\gamma^2}\iint_{\mathbb{R}^d\times\mathbb{R}^d}|v|^2\,d\mu_t^\gamma,$$

where we define the following constants

$$c_*^\gamma := c_0^\gamma + (c_\infty^\gamma)^2,\quad c_0^\gamma := \sup_{0\le t\le T}\|F(\cdot,\hat\rho_t^\gamma)\|_{\mathrm{Lip}},\quad c_\infty^\gamma = \sup_{0\le t\le T}\max\{\|\rho_t^\gamma\|_{L^\infty},\|\hat\rho_t^\gamma\|_{L^\infty}\}.$$

Proof. Using Hölder's inequality with a change of variables, we obtain

$$\begin{aligned}\|e_t^\gamma\|^2_{L^2(\hat\rho_t^\gamma)} &= \iint_{\mathbb{R}^d\times\mathbb{R}^d} e_t^\gamma(x)\cdot\left(F(x-v/\gamma,\rho_t^\gamma)-F(x,\hat\rho_t^\gamma)\right)\,d\Gamma_{\#}^{\frac{1}{\gamma}}\mu_t^\gamma\\ &\le \|e_t^\gamma\|_{L^2(\hat\rho_t^\gamma)}\left(\iint_{\mathbb{R}^d\times\mathbb{R}^d}|F(x,\rho_t^\gamma)-F(x+v/\gamma,\hat\rho_t^\gamma)|^2\,d\mu_t^\gamma\right)^{1/2}.\end{aligned}$$

Thus, we can deduce that

$$\begin{aligned}\frac{1}{2}\|e_t^\gamma\|^2_{L^2(\hat\rho_t^\gamma)} &\le \frac{1}{2}\iint_{\mathbb{R}^d\times\mathbb{R}^d}|F(x,\rho_t^\gamma)-F(x+v/\gamma,\hat\rho_t^\gamma)|^2\,d\mu_t^\gamma\\ &\le \int_{\mathbb{R}^d}|F(x,\rho_t^\gamma)-F(x,\hat\rho_t^\gamma)|^2\,d\rho_t^\gamma + \iint_{\mathbb{R}^d\times\mathbb{R}^d}|F(x,\hat\rho_t^\gamma)-F(x+v/\gamma,\hat\rho_t^\gamma)|^2\,d\mu_t^\gamma.\end{aligned}$$

By introducing a Dirac delta by integration by parts, the first term can be bounded as

$$\begin{aligned}\int_{\mathbb{R}^d}|F(x,\rho_t^\gamma)-F(x,\hat\rho_t^\gamma)|^2\,d\rho_t^\gamma &\le c_\infty^\gamma\left|\int_{\mathbb{R}^d\times\mathbb{R}^d}\Phi*(\rho_t^\gamma-\hat\rho_t^\gamma)(x)\cdot(-\zeta\Delta\Phi*(\rho_t^\gamma-\hat\rho_t^\gamma)(x))\,dx\right|\\ &= .c_\infty^\gamma\left|\int_{\mathbb{R}^d\times\mathbb{R}^d}\Phi*(\rho_t^\gamma-\hat\rho_t^\gamma)\,d(\rho_t^\gamma-\hat\rho_t^\gamma)\right|\\ &= c_\infty^\gamma|\mathcal{D}_\Phi(\rho_t^\gamma,\hat\rho_t^\gamma)|.\end{aligned}$$

In particular, using Theorems 4.2, 4.12 and 4.16, we obtain

$$\int_{\mathbb{R}^d}|F(x,\rho_t^\gamma)-F(x,\hat\rho_t^\gamma)|^2\,d\rho_t^\gamma \le \frac{(c_\infty^\gamma)^2}{\gamma^2}\iint_{\mathbb{R}^d\times\mathbb{R}^d}|v|^2\,d\mu_t^\gamma.$$

By assumption, the second term can be bounded by

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d}|F(x,\hat\rho_t^\gamma)-F(x+v/\gamma,\hat\rho_t^\gamma)|^2\,d\mu_t^\gamma \le \frac{c_0^\gamma}{\gamma^2}\iint_{\mathbb{R}^d\times\mathbb{R}^d}|v|^2\,d\mu_t^\gamma.$$

Thus, we can conclude that

$$\frac{1}{2}\left\|e_t^\gamma\right\|_{L^2(\hat{\rho}_t^\gamma)}^2 \leq \frac{c_*^\gamma}{\gamma^2}\iint_{\mathbb{R}^d\times\mathbb{R}^d}|v|^2\, d\mu_t^\gamma.$$

This gives our desired result, concluding the proof of the theorem. □

We are ready to prove the main result of the thesis by applying the uniform moments estimates of Chapter 3. Keep in mind that the result assumes the existence of a strong solution to the damped Vlasov-Poisson-Fokker-Planck equation, in addition to the existence of an $\mathcal{E}$-regular solution to the Aggregation-Diffusion equation. In addition, these solution must satisfy some additional properties, which are not completely covered by the well-posedness theory described in Chapter 5. Finding sufficient conditions for these properties to hold forms an open problem, left for future work.

Theorem 2.9. Consider the energy functionals given in Definition 2.5 and 2.7 and consider a family of initial conditions $(\mu_0^\gamma)_{\gamma\geq 1}$ satisfying the following uniform energy and moment bounds

$$\sup_{\gamma\geq 1}\mathcal{E}(\mu_0^\gamma) + \sup_{\gamma\geq 1}\iint_{\mathbb{R}^d\times\mathbb{R}^d}|x|^2+|v|^2\, d\mu_0^\gamma < \infty.$$

In addition, consider the family of unique strong solutions $(\mu^\gamma)_{\gamma\geq 1}$ to the Vlasov-Poisson-Fokker-Planck equation (VPFP) and let $(\rho^\gamma)_{\gamma\geq 1}$ and $(\hat{\rho}^\gamma)_{\gamma\geq 1}$ denote the corresponding families of particle densities and standard coarse-grained particle densities. Assume that

$$c_\infty := \sup_{\gamma\geq 1}\sup_{0\leq t\leq T}\left\{\|\rho_t^\gamma\|_{L^\infty} + \|\hat{\rho}_t^\gamma\|_{L^\infty}\right\} < \infty.$$

In addition, assume that the family $(\hat{\rho}^\gamma)_{\gamma\geq 1}$ satisfies the properties of $\mathcal{E}$-regular solutions and induces a family of Lipschitz forces with Lipschitz constants bounded uniformly in $\gamma \geq 1$, meaning that

$$c_0 := \sup_{\gamma\geq 1}\sup_{0\leq t\leq T}\|F(\cdot,\hat{\rho}_t^\gamma)\|_{\mathrm{Lip}} < \infty, \quad \lambda := 1 + c_0.$$

In addition, let $\rho \in C([0,T],\mathcal{P}_2(\mathbb{R}^d))$ be the unique $\mathcal{E}$-regular solution to the Aggregation-Diffusion equation (AD) with initial condition $\rho_0 \in \mathcal{P}_2(\mathbb{R}^d)\cap\mathrm{dom}(\mathcal{E})$. Then, there exists some constant $C > 0$ independent of $\gamma \geq 1$ such that

$$\sup_{0\leq t\leq T} W_2^2(\rho_t^\gamma,\rho_t) \leq C\left(W_2^2(\rho_0^\gamma,\rho_0) + \frac{1}{\gamma^2}\right).$$

Proof. Using Theorems 4.16 and 4.17, we obtain

$$\frac{1}{2}\frac{d}{dt}W_2^2(\hat{\rho}_t^\gamma,\rho_t) \leq (\lambda + c_\infty)W_2^2(\hat{\rho}_t^\gamma,\rho_t) + \frac{c_*}{\gamma^2}\iint_{\mathbb{R}^d\times\mathbb{R}^d}|v|^2\, d\mu_t^\gamma, \quad c_* := c_0 + c_\infty^2.$$

Since all constants $\lambda, c_\infty, c_* > 0$ are independent of $\gamma \geq 1$, we can apply Grönwall's lemma

$$W_2^2(\hat{\rho}_t^\gamma,\rho_t) \leq \left(W_2^2(\hat{\rho}_0^\gamma,\rho_0) + \frac{2c_*}{\gamma^2}\int_0^t\int_{\mathbb{R}^d\times\mathbb{R}^d}|v|^2\, d\mu_s^\gamma ds\right)e^{2(\lambda+c_\infty)t}.$$

In addition, using the triangle inequality along with two applications of Theorem 4.2, we have

$$\begin{aligned}W_2^2(\rho_t^\gamma, \rho_t) &\le 2\left(W_2^2(\rho_t^\gamma, \hat{\rho}_t^\gamma) + W_2^2(\hat{\rho}_t^\gamma, \rho_t)\right) \\ &\le C\left(W_2^2(\hat{\rho}_0^\gamma, \rho_0) + \frac{1}{\gamma^2}\int_0^t \int_{\mathbb{R}^d\times\mathbb{R}^d} |v|^2 \, d\mu_s^\gamma ds\right) \\ &\le C\left(W_2^2(\rho_0^\gamma, \rho_0) + \frac{1}{\gamma^2}\right),\end{aligned}$$

where the constant $C > 0$ is independent of $\gamma \ge 1$.

□

Chapter 5

Well-posedness for the equation

In this chapter, we study the well-posedness for the Cauchy problem associated to the damped Vlasov-Poisson-Fokker-Planck equation (VPFP). In particular, we look for initial conditions to ensure the existence and uniqueness of a strong solution to the Vlasov-Poisson-Fokker-Planck equation in the sense of Definition 2.2. In particular, we need prove the existence of a weak solution which induces an essentially bounded force field with finite second position and velocity moments. To establish these results, we first fix $\gamma \geq 1$ and apply the change of variables

$$\mu_t(X(x), V(v)) := \mu_t^\gamma(x, v), \quad X(x) := x, \quad V(v) := \gamma v,$$

to obtain the following modified damped Vlasov-Poisson-Fokker-Planck equation

$$\partial_t \mu_t + v \cdot \nabla_x \mu_t + \nabla_v \cdot \big(\mu_t(E(x, \rho_t) - \gamma^2 v)\big) = \gamma^4 \Delta_v \mu_t, \tag{MVPFP}$$

where ρ_t is the first marginal of μ_t and $E(x, \rho) := \gamma F(x, \rho)$, after a relabeling of variables. This corresponds to the setting of [1] by taking $\beta = \gamma^2$ and $\sigma = \gamma^4$. Since the change of variables is invertible, the well-posedness of the modified equation (MVPFP) implies the well-posedness of the original equation (VPFP). Thus, we proceed with the well-posedness for the modified equation.

5.1 The Lions-Perthame representation

To start, we study a related equation, coinciding with the modified equation where the inverse square law is replaced by an arbitrary essentially bounded force field. In particular, we consider the equation

$$\partial_t \mu_t + v \cdot \nabla_x \mu_t + \gamma \nabla_v \cdot \Big(\mu_t(\hat{E}_t - \gamma^2 v)\Big) = \gamma^4 \Delta_v \mu_t, \tag{LVPFP}$$

where the force field $\hat{E} \in L^\infty([0, T], L^\infty(\mathbb{R}^d))$ is essentially bounded at almost all times. Moreover, the initial condition $0 \leq \mu_0 \in L^1 \cap L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ is integrable, essentially bounded and has finite second position and velocity moments. This equation can be viewed as a linear counterpart of (MVPFP) as the force field is independent of the solution μ. Since both (MVPFP) and (LVPFP) are parabolic partial differential equations, the solutions satisfy a maximum principle [1], meaning that we have

$$\|\mu_t\|_{L^\infty} \leq \|\mu_0\| \, e^{d\gamma^2 t}. \tag{MP}$$

To obtain the existence of a weak solution to the Cauchy problem associated to the linearized equation (LVPFP), we use the Green's function corresponding to the fundamental solution to

$$\partial_t \mathcal{G}_t + v \cdot \nabla_x \mathcal{G}_t - \gamma^2 \nabla_v \cdot (v \mathcal{G}_t) - \gamma^4 \Delta_v \mathcal{G}_t = 0, \quad \mathcal{G}_0(x, v, \xi, \nu) = \delta_{(x,v)}(\xi, \nu).$$

This solution can be computed to be

$$\mathcal{G}_t(x, v, \xi, \nu) = g(t, x - \xi - \nu(1 - e^{-\gamma^2 t})/\gamma^2, v - \nu e^{-\gamma^2 t}),$$

where the functions $g : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ and $D : [0, T] \to \mathbb{R}$ are defined by

$$\begin{aligned} g(t, x, v) &:= \frac{1}{(4\pi\gamma^4)^d D(t)^{3/2}} \exp\left(\frac{-1}{4\gamma^4 D(t)} \int_0^t \left| \frac{1 - e^{-\gamma^2 s}}{\gamma^2} v - e^{-\gamma^2 s} x \right|^2 ds \right), \\ D(t) &:= \frac{1}{\gamma^4} \left(\frac{1 - e^{-2\gamma^2 t}}{2\gamma^2} t - \left(\frac{1 - e^{-\gamma^2 t}}{\gamma^2} \right)^2 \right). \end{aligned}$$

Equipped with the above Green's function, we can use the ideas introduced in [15] to write the state of the system as an implicit relation, given by

$$\begin{aligned} \mu_t(x, v) = &\iint_{\mathbb{R}^d \times \mathbb{R}^d} \mathcal{G}_t(x, v, \xi, \nu) \, d\mu_0(\xi, \nu) \\ &+ \int_0^t \iint_{\mathbb{R}^d \times \mathbb{R}^d} \nabla_\nu \mathcal{G}_s(x, v, \xi, \nu) \hat{E}_{t-s}(\xi) \, d\mu_{t-s}(\xi, \nu) ds \\ := &\bar{\mu}_t(x, v) + \tilde{\mu}_t(x, v). \end{aligned}$$

This is known as the Lions-Perthame representation of a solution to (LVPFP). This integral representation is used to establish the existence of a weak solution to the linearized Vlasov-Poisson-Fokker-Planck equation (LVPFP) by applying a fixed-point argument with the operator

$$\begin{aligned} &T : C([0, T], L^1(\mathbb{R}^d \times \mathbb{R}^d)) \to C([0, T], L^1(\mathbb{R}^d \times \mathbb{R}^d)), \\ &\qquad (T\mu)_t(x, v) := \iint_{\mathbb{R}^d \times \mathbb{R}^d} \mathcal{G}_t(x, v, \xi, \nu) \, d\mu_0(\xi, \nu) \\ &\qquad\qquad + \int_0^t \iint_{\mathbb{R}^d \times \mathbb{R}^d} \nabla_\nu \mathcal{G}_s(x, v, \xi, \nu) \hat{E}_{t-s}(\xi) \, d\mu_{t-s}(\xi, \nu) ds. \end{aligned} \tag{FPO}$$

In particular, we aim to apply the Schauder fixed-point theorem,provided below. For a proof, see [17].

Theorem 5.1. Let X be a Banach space and consider a bounded, compact operator $T : X \to X$ such that the set

$$\{x \in X : x = \lambda T x \quad \text{for some } 0 \le \lambda \le 1\} \tag{FP}$$

is bounded. Then T has a fixed point.

Theorem 5.2. There exists a weak solution $\mu \in C([0,T], L^1(\mathbb{R}^d \times \mathbb{R}^d))$ to the linearized Vlasov-Poisson-Fokker-Planck equation (LVPFP), corresponding to a fixed-point of the operator T, defined in (FPO).

Proof. The self-mapping property and boundedness of the operator are straightforward. The same is true for the boundedness of the set in (FP). However, the compactness of the operator can be obtained by applying the Aubin-Lions lemma (see [18, Corollary 4]), concluding the existence proof. □

Unfortunately, the fixed-point argument only provides integrability for the constructed solution μ. However, some regularity on the particle density can be inferred from the boundedness of the second velocity moments. In fact, using our assumptions on the initial condition, the subtle analysis provided in [19] shows that the second position and velocity remain bounded throughout the evolution.

Theorem 5.3. Suppose that the density $\mu_t \in L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ has finite m-th velocity moment, meaning

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} |v|^m \, d\mu_t < \infty,$$

Then, the particle density has the regularity $\rho_t \in L^q(\mathbb{R}^d)$, where $q = (m+d)/d$.

Proof. By splitting the domain of integration using a ball of radius $R > 0$, we obtain

$$\begin{aligned} \rho_t(x) &\le \|\mu_t\|_{L^\infty} \int_{|v|\le R} 1 \, dv + \frac{1}{R^m} \int_{|v|>R} |v|^m \, \mu_t(x,v) \, dv \\ &\le \frac{4\pi R^3}{3} \|\mu_0\|_{L^\infty} e^{d\gamma^2 t} + \frac{1}{R^m} \int_{\mathbb{R}^d} |v|^m \, \mu_t(x,v) \, dv, \end{aligned}$$

where the last inequality follows by (MP). Minimizing the above bound over all $R > 0$ gives

$$\rho_t(x) \le C \, \|\mu_0\|_{L^\infty}^{1/q'} \left(\int_{\mathbb{R}^d} |v|^m \, \mu_t(x,v) \, dv \right)^{1/q},$$

where $q' = (m+d)/m$. Taking the L^q-norm in the position variable, we obtain

$$\|\rho_t\|_{L^q} \le C \, \|\mu_0\|_{L^\infty}^{1/q'} \left(\iint_{\mathbb{R}^d \times \mathbb{R}^d} |v|^m \, d\mu_t \right)^{1/q} < \infty.$$

Thus, we have shown that $\rho_t \in L^q(\mathbb{R}^d)$, concluding the proof of the theorem. □

Remark 5.4. Since the coarse-graining map has unit Jacobian determinant, we can deduce that the coarse-grained density satisfies the same bound as in Theorem 5.3, under the same assumptions,

$$\|\rho_t^\lambda\|_{L^q} \le C \, \|\mu_0\|_{L^\infty}^{1/q} \left(\iint_{\mathbb{R}^d \times \mathbb{R}^d} |v|^m \, d\mu_t \right)^{1/q} < \infty.$$

As a result, we use the Lions-Perthame representation to extend the regularity of the particle density by analyzing the constituent terms separately. To this aim, we first define

$$\bar{\rho}_t(x) := \pi^x_\# \bar{\mu}_t(x), \quad \tilde{\rho}_t(x) := \pi^x_\# \tilde{\mu}_t(x).$$

Similarly, we define the contribution of each term in the integral representation to the induced force

$$\bar{N}_t(x) := \gamma\nabla\Phi * \bar{\rho}_t(x), \quad \tilde{N}_t(x) := \gamma\nabla\Phi * \tilde{\rho}_t(x).$$

By linearity of convolutions, it immediately follows that the induced force can be expressed as

$$N_t(x) := \nabla\Phi * \rho_t(x) = \bar{N}_t(x) + \tilde{N}_t(x), \quad E(x, \rho_t) = -N_t(x).$$

5.2 Coarse-graining via convolutions

To obtain estimates on the particle densities and the induced forces originating from each term in the Lions-Perthame representation, we use the coarse-graining map to define particle densities in a similar spirit as Definition 2.8. Since $\gamma \geq 1$ is fixed, we omit the dependence on this parameter for clarity of presentation. In particular, we define

$$\rho_t^\lambda(x) := (\pi^x \circ \Gamma^\lambda)_\# \mu_t(x), \quad \bar{\rho}_t^\lambda(x) := (\pi^x \circ \Gamma^\lambda)_\# \bar{\mu}_t^\lambda(x), \qquad \tilde{\rho}_t(x) := (\pi^x \circ \Gamma^\lambda)_\# \tilde{\mu}_t(x).$$

In addition, the driving force of the coarse-grained system plays an important role in the arguments to come. As a result, we recall that this force is given by the expression

$$J_t^\lambda(x) := \int_{\mathbb{R}^d} E_t(x - \lambda v)\Gamma^\lambda_\# \mu_t(x, v)\ dv.$$

In addition to the force field J_t^λ, another useful reference field is the one induced by the coarse-grained initial particle density, where the amount of coarse-graining increases with time, defined by

$$R_t(x) := \nabla\Phi * \rho_0^t(x).$$

In addition, a rather long computation shows that [1]

$$\int_{\mathbb{R}^d} \mathcal{G}_t(x - \lambda v, v, \xi, \nu)\ dv = \frac{1}{(2\gamma^4\delta(t,\lambda))^{3/2}}\mathcal{N}\left(\frac{x - \xi - \sigma(t,\lambda)\nu}{\sqrt{2\gamma^4\delta(t,\lambda)}}\right), \tag{5.1}$$

$$\int_{\mathbb{R}^d} \nabla_\nu \mathcal{G}_t(x - \lambda v, v, \xi, \nu)\ dv = -\frac{\sigma(t,\lambda)}{(2\gamma^4\delta(t,\lambda))^2}\nabla\mathcal{N}\left(\frac{x - \xi - \sigma(t,\lambda)\nu}{\sqrt{2\gamma^4\delta(t,\lambda)}}\right), \tag{5.2}$$

$$\int_{\mathbb{R}^d} |\nabla_\nu \mathcal{G}_t(x - \lambda v, v, \xi, \nu)|\ dv \leq \frac{C}{(2\gamma^4\delta(t,\lambda))^{3/2}\sqrt{\gamma^4 t}}\mathcal{M}\left(\frac{x - \xi - \sigma(t,\lambda)\nu}{\sqrt{2\gamma^4\delta(t,\lambda)}}\right), \tag{5.3}$$

where $\mathcal{N}, \mathcal{M} : \mathbb{R}^d \to \mathbb{R}$ are the standard and rescaled normal distribution respectively and the functions $\sigma : [0,T] \times [0,\infty) \to \mathbb{R}$ and $\delta : [0,T] \times [0,\infty) \to \mathbb{R}$ are defined by

$$\mathcal{N}(x) := \frac{1}{(2\pi)^{3/2}}e^{-|x|^2/2}, \quad \mathcal{M}(x) := \sqrt{1 + |x|^2}\mathcal{N}(x),$$

$$\sigma(t,\lambda) := \frac{1 - e^{-\gamma^2 t}}{\gamma^2} + \lambda e^{-\gamma^2 t}, \quad \delta(t,\lambda) := \int_0^t \sigma(\tau,\lambda)^2\ d\tau.$$

As a result, using (5.1), we get the convolution expressions

$$\bar{\rho}_t^\lambda(x) = \frac{1}{(4\gamma^4\pi\delta(t,\lambda))^{3/2}} \exp\left(\frac{-|x|^2}{4\gamma^4\delta(t,\lambda)}\right) * \rho_0^{\sigma(t,\lambda)}(x), \tag{5.4}$$

$$\bar{N}_t(x) = \frac{1}{(4\gamma^4\pi\delta(t,0))^{3/2}} \exp\left(\frac{-|x|^2}{4\gamma^4\delta(t,0)}\right) * R_{\sigma(t,0)}(x). \tag{5.5}$$

In addition, using (5.2), we obtain

$$\tilde{\rho}_t^\lambda(x) = -\int_0^t \frac{\sigma(s,\lambda)}{(2\gamma^4\delta(s,\lambda))^2} \nabla\mathcal{N}\left(\frac{x}{\sqrt{2\gamma^4\delta(s,\lambda)}}\right) * J_{t-s}^{\delta(s,\lambda)}(x)\ ds, \tag{5.6}$$

$$\tilde{N}_t(x) = \int_0^t \frac{\sigma(s,0)}{(2\gamma^4\delta(s,0))^{3/2}} A\left(\frac{x}{\sqrt{2\gamma^4\delta(s,0)}}\right) * J_{t-s}^{\delta(s,0)}(x)\ ds, \tag{5.7}$$

where the matrix A has entries

$$A_{jk}(x) = \frac{\partial^2}{\partial x_j \partial x_k}(-\Delta)^{-1}\mathcal{N}(x), \quad 1 \le j,k \le N.$$

5.3 Convolutional regularity estimates

Given that the quantities of interest in (5.4), (5.5), (5.6) and (5.7) are expressed using convolutions, we recall two important inequalities that can deal with terms of this form. The first is a reformulation of Young's inequality for convolutions, whereas the second theorem is particularly used to bound convolutions where one of the functions contains a singularity at the origin. For a proof, see [7].

Theorem 5.5. Let $1 \le p, q \le \infty$ and define $1/r := 1/p - 1/q \ge 0$. Then, we have the inequality

$$\|f * g\|_{L^r} \le \|f\|_{L^p} \|g\|_{L^{q'}}, \quad \text{where} \quad \frac{1}{q} + \frac{1}{q'} = 1.$$

Theorem 5.6. Let $1 \le p \le \infty$ and $1 \le q < \infty$ such that $1/r := 1/p - 1/q$ satisfies $-1/3 < 1/r < 0$ and assume that there exist constants $C_0, C_1 > 0$ such that

$$|g(x)| \le \frac{C_0}{|x|^{3/q'}}, \qquad |\nabla g(x)| \le \frac{C_1}{|x|^{1+3/q'}}.$$

Then, it follows that for all $f \in L^p(\mathbb{R}^d)$, we have the convolution inequality

$$\left\| \int_{\mathbb{R}^d} (g(x-y) - g(-y)) f(y)\ dy \right\|_{L^r} \le C(p,q)(C_0 + C_1) \|f\|_{L^p}.$$

We consider a uniform-in-time bound on the nonlinear force induced by the second term in the Lions-Perthame representation for the solution to the linearized Vlasov-Poisson-Fokker-Planck equation.

Theorem 5.7. Suppose that the force field $E \in L^\infty((0,T);L^\infty(\mathbb{R}^d))$ is essentially bounded and that the initial density satisfies $0 \le \mu_0 \in L^1 \cap L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ and there exist constants $1 \le p < \infty$ and $3/2 < q \le \infty$ such that

$$S_p(T) := \sup_{0\le t\le T, \lambda \ge 0} \|\rho_t^\lambda\|_{L^p} < \infty, \quad K_q(T) := \operatorname*{esssup}_{0\le t\le T} \|E_t\|_{L^q} < \infty.$$

Then, for all $1 \le r \le \infty$, satisfying

$$\frac{1}{pq'} - \frac{2}{3}\left(\frac{2}{3} - \frac{1}{q}\right) < \frac{1}{r} < \frac{1}{pq'},$$

we have the estimate

$$\sup_{0\le t\le T} \|\tilde{N}_t\|_{L^r} \le C(p,q,r,\gamma,T)\, \|\mu_0\|_{L^\infty}^{1/q}\, K_q(T) S_p(T)^{1/q'}$$

Proof. In light of the convolution obtained in (5.7), we begin by estimating the quantity J_t^λ. In particular, for $\lambda > 0$ and $1 \le q \le \infty$, we can apply Hölder's inequality, followed by a change of variables to obtain

$$|J_t^\lambda(x)| \le \frac{1}{\lambda^{3/q}} \|E_t\|_{L^q}\, \|\mu_t\|_{L^\infty}^{1/q}\, |\rho_t^\lambda(x)|^{1/q'}. \tag{5.8}$$

Using the above bound in combination with the maximum principle given in (MP), we get

$$|\tilde{N}_t(x)| \le K_q(T)\left(\|\mu_0\|_{L^\infty} e^{d\gamma^2 t}\right)^{1/q} \int_0^t \frac{\sigma(s,0)^{1-3/q}}{(2\gamma^4\delta(s,0))^{3/2}} \left| A\left(\frac{x}{\sqrt{2\gamma^4\delta(s,0)}}\right)\right| * |\rho_{t-s}^{\sigma(s,0)}(x)|^{1/q'}\, ds.$$

Applying Theorem 5.5 with $0 < 1/k := 1/pq' - 1/r < 1$, along with a change of variables yields

$$\|\tilde{N}_t\|_{L^r} \le K_q(T) S_p(T)^{1/q'}\, \|A\|_{L^{k'}} \left(\|\mu_0\|_{L^\infty} e^{d\gamma^2 t}\right)^{1/q} \int_0^t \frac{\sigma(s,0)^{1-3/q}}{(2\gamma^4\delta(s,0))^{3/2k}}\, ds.$$

Using the fact that as $s \to 0$, we have

$$\sigma(s,0) \simeq s, \quad \delta(s,0) \simeq s^3/3, \quad 1 - \frac{3}{q} - \frac{9}{2k} > -1,$$

we can deduce that the above integral term converges and thus we have shown that

$$\sup_{0\le t\le T} \|\tilde{N}_t\|_{L^r} \le C(p,q,r,\gamma,T)\, \|\mu_0\|_{L^\infty}^{1/q}\, K_q(T) S_p(T)^{1/q'}.$$

This gives our desired result, concluding the proof of the theorem.

□

We subsequently aim to control the regularity of the particle density along the evolution under certain assumptions on the initial particle density. This is given by the following theorem.

Theorem 5.8. Suppose that the force field $E \in L^\infty((0,T), L^\infty(\mathbb{R}^d))$ is essentially bounded and that the initial condition satisfies $0 \le \mu_0 \in L^1 \cap L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ and there exist constants $1 \le p \le \infty$ and $6 < q \le \infty$ such that

$$Q_p := \sup_{\lambda \ge 0} \|\rho_0^\lambda\|_{L^p} < \infty, \quad K_q(T) := \operatorname*{esssup}_{0 \le t \le T} \|E_t\|_{L^q} < \infty.$$

Then, we have the estimate

$$S_p(T) := \sup_{0 \le t \le T, \lambda \ge 0} \|\rho_t^\lambda\|_{L^p} < \infty.$$

In addition, when the parameters $1 \le k, r \le \infty$ satisfy

$$0 \le \frac{1}{r} < \frac{1}{9} - \frac{2}{3q}, \quad \frac{1}{p} \le \frac{1}{k}, \quad \frac{1}{kq} + \frac{1}{r} > 0,$$

we obtain the inequality

$$S_p(T) \le C(k,p,q,r) Q_p + C(k,p,q,r,\gamma,T) S_k(T)^\theta (K_q(T) \, \|\mu_0\|_{L^\infty}^{1/q})^\kappa,$$

where

$$\theta = \frac{1/pq + 1/r}{1/kq + 1/r}, \quad \kappa = \frac{1/k - 1/p}{1/kp + 1/r}.$$

Proof. We begin by writing

$$\bar{S}_l(T) := \sup_{0 \le t \le T, \lambda \ge 0} \|\bar{\rho}_t^\lambda\|_{L^l}, \quad \tilde{S}_l(T) := \sup_{0 \le t \le T, \lambda \ge 0} \|\tilde{\rho}_t^\lambda\|_{L^l}.$$

These quantities satisfy the following properties

$$S_l(T) \le \bar{S}_l(T) + \tilde{S}_l(T), \quad \bar{S}_l(T) \le S_l(T), \quad \tilde{S}_l(T) \le 2 S_l(T), \tag{5.9}$$

where the first inequality follows by Minkowski's inequality, whereas the second follows from

$$\|\bar{\rho}_t^\lambda\|_{L^l} \le \|\rho_0^{\sigma(t,\lambda)}\|_{L^l}, \tag{5.10}$$

obtained by applying Theorem 5.5 with the convolution expression obtained in (5.4). The last inequality in (5.9) follows from the second along with an application of Minkowski's inequality.

To show that $S_p(T)$ is bounded, observe that it suffices to show that $\tilde{S}_p(T)$ is bounded since we have $\bar{S}_l(T) \le Q_p$ from the second inequality in (5.9). Following a similar approach as in the proof of Theorem 5.7, we can bound the coarse-grained particle density corresponding to the second term in the Lions-Perthame representation obtained in (5.6) using the estimate on J_t^λ obtained in (5.8) and the maximum principle given in (MP) to get

$$|\tilde{\rho}_t^\lambda(x)| \leq K_q(T)\left(\|\mu_0\|\, e^{d\gamma^2 t}\right)^{1/q} \int_0^t \frac{\sigma(s,\lambda)^{1-3/q}}{(2\gamma^4\delta(s,\lambda))^2}\left|\nabla\mathcal{N}\left(\frac{x}{\sqrt{2\gamma^4\delta(s,\lambda)}}\right)\right| * |\rho_{t-s}^{\sigma(s,\lambda)}(x)|^{1/q'}\, ds.$$

At this point, we make the following observation. Assume that

$$0 \leq \frac{1}{r} < \frac{1}{9} - \frac{2}{3q}, \quad q' \leq l \leq \infty, \quad \frac{1}{m} := \frac{1}{l} - \frac{1}{r},$$

along with the fact that $S_{l/q'} < \infty$. If $1/m \geq 0$, we can apply Theorem 5.5 in the above integral, followed by a change of variables to obtain the estimate

$$\|\tilde{\rho}_t^\lambda\|_{L^m} \leq K_q(T)(\|\mu_0\|_{L^\infty}\, e^{d\gamma^2 t})^{1/q}\, \|\nabla\mathcal{N}\|_{L^{r'}}\, S_{l/q'}(T)^{1/q'} \int_0^t \frac{(\sigma(s,\lambda))^{1-3/q}}{(2\gamma^4\delta(s,\lambda))^{1/2+3/2r}}\, ds.$$

Using a similar reasoning as in the end of the proof of Theorem 5.7, we have that as $s \to 0$,

$$\sigma(s,\lambda) \simeq s, \quad \delta(s,\lambda) \simeq s^3/3, \quad 1 - \frac{3}{q} - \frac{3}{2} - \frac{9}{2r} > -1,$$

and thus, the above integral converges. In fact, this integral can be bound independently of $\lambda \geq 0$ since it can be verified that for all $\alpha \geq 0, \beta \leq 2\alpha$ and all $s > 0$, we have that the function $\lambda \mapsto \sigma(s,\lambda)^\beta/\delta(s,\lambda)^\alpha$ is non-increasing on $[0,\infty)$. Thus, we obtain our desired estimate

$$\tilde{S}_m(T) \leq C(q,r,l,\gamma,T)\, \|\mu_0\|_{L^\infty}^{1/q}\, K_q(T) S_{l/q'}(T)^{1/q'}. \tag{5.11}$$

On the other hand, if $-1/9 \leq 1/m < 0$, we can proceed identically except applying Theorem 5.6 in lieu of Theorem 5.5 when bounding the convolution. This covers all the cases that our assumed parameters can satisfy.

Using conservation of mass, we have that $S_1(T) \leq \|\mu_0\|_{L^1} < \infty$. Consider the parameters as in the statement of the theorem $1 \leq k, r \leq \infty$ such that

$$0 \leq \frac{1}{r} < \frac{1}{9} - \frac{2}{3q}, \quad \frac{1}{p} \leq \frac{1}{k}, \quad \frac{1}{kq} + \frac{1}{r} > 0.$$

In particular, we distinguish two cases:

1. If $1/kq' - 1/r \leq 1/p$, define $l = kq'$. In that case, we obtain $1/m = 1/kq' - 1/r \leq 1/p$ and so using the result obtained in (5.11) gives

$$\tilde{S}_m(T) \leq C\, \|\mu_0\|_{L^\infty}^{1/q}\, K_q(T) S_k(T)^{1/q'}.$$

Using interpolation of Lebesgue spaces, we deduce

$$\tilde{S}_p(T) \leq C \tilde{S}_k(T)^{1-\Theta} \tilde{S}_m(T)^\Theta, \quad \text{where} \quad \Theta = \frac{1/k - 1/p}{1/k - 1/q}.$$

Combining the above with the inequalities in (5.9) gives the desired result

$$S_p(T) \leq Q_p + C(\|\mu_0\|_{L^\infty}^{1/q} K_q(T))^{\Theta} S_k(T)^{1-\Theta+\Theta/q'}.$$

2. If $1/kq' - 1/r > 1/p$, define $1/m = 1/r + 1/p < 1/kq'$. Note that this defines the constant $1/l = 2/r + 1/p$. Using the bound in (5.11), we get

$$\tilde{S}_p(T) \leq C \|\mu_0\|_{L^\infty}^{1/q} K_q(T) S_{l/q'}(T)^{1/q'}.$$

Again, by interpolation of Lebesgue spaces, we obtain

$$S_{l/q'}(T) \leq C S_k(T)^{1-\Theta} S_p(T)^{\Theta} \quad \text{where} \quad \Theta = \frac{1/k - q'/l}{1/k - 1/p}.$$

This yields

$$S_p(T) \leq Q_p + C \|\mu_0\|_{L^\infty}^{1/q} K_q(T) S_k(T)^{(1-\Theta)/q'} S_p(T)^{\Theta/q'},$$

where we can apply Young's inequality obtain the desired estimate

$$S_p(T) \leq \frac{1}{1-\Theta/q'} Q_p + C(\|\mu_0\|_{L^\infty}^{1/q} K_q(T) S_k(T)^{(1-\Theta)/q'})^{1/(1-\Theta/q')}.$$

These two cases prove our desired result, concluding the proof of the theorem.

□

5.4 Existence and uniqueness result

Now that we have established some a priori estimates on the solution to the linearized Vlasov-Poisson-Fokker-Planck equation (LVPFP), we prove the existence of a strong solution to the modified Vlasov-Poisson-Fokker-Planck equation (MVPFP) using a mollification argument. In particular, we begin by regularizing the inverse square law using the standard mollifier η^ϵ. Namely, we define

$$E^\epsilon(x, \rho) := \nabla\Phi^\epsilon * \rho, \quad \nabla\Phi^\epsilon(x) := \frac{x}{|x|^d} * \eta^\epsilon \in L^\infty(\mathbb{R}^d).$$

Since the regularized force is essentially bounded, Theorem 5.2 generates a family of weak solutions $(\mu^\epsilon)_{\epsilon>0}$. In what follows, we apply the estimates obtained in Section 5.3 to obtain bounds that are independent of $\epsilon > 0$. As a result, the limiting solution obtained by taking $\epsilon \to 0$ will satisfy the same bounds. This limiting solution exists [1] and corresponds to a weak solution to the modified Vlasov-Poisson-Fokker-Planck equation (MVPFP). However, for clarity of presentation, we omit the dependence on $\epsilon > 0$ to avoid ambiguity with coarse-graining and directly compute uniform estimates for the induced force N_t.

Since the velocity moments remain finite throughout the evolution, Theorem 5.3 implies that

$$\sup_{0\le t\le T} \|\rho_t\|_{L^p} < \infty, \qquad \text{for } 1 \le p \le 5/3. \tag{5.12}$$

Using the above with Theorem 5.6, we can also obtain some regularity on the induced force

$$\sup_{0\le t\le T} \|N_t\|_{L^q} < \infty \qquad \text{for } 3/2 \le q \le 15/4. \tag{5.13}$$

However, we can improve the regularity on the induced force under stricter assumptions on the initial condition. This is formulated in the following theorem.

Theorem 5.9. Suppose that the initial condition $0 \le \mu_0 \in L^1 \cap L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ has finite second velocity moment and there exists a constant $3/2 < p < 6$ such that

$$\sup_{t\ge 0} \|R_t\|_{L^p} < \infty.$$

Then, there exists a weak solution $\mu \in C([0,T], \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d))$ to the modified Vlasov-Poisson-Fokker-Planck equation (MVPFP) where the induced force satisfies

$$\sup_{0\le t\le T} \|N_t\|_{L^p} < \infty.$$

Proof. As a starting observation, note that applying Theorem 5.5 on the induced force from the first term in the Lions-Perthame representation given in (5.5) provides

$$\sup_{t\ge 0}\|\bar{N}_t\|_{L^p} \le \sup_{t\ge 0}\|R_t\|_{L^p} < \infty.$$

For the second term, since $S_{5/3}(T) < \infty$ and we can choose a parameter q such that

$$\frac{3}{5p'} - \frac{2}{3}\left(\frac{2}{3} - \frac{1}{p}\right) < \frac{1}{q} < \frac{3}{5p'},$$

and apply Theorem 5.7 to deduce

$$\sup_{0\le t\le T} \|\tilde{N}_t\|_{L^q} \le C(p,q,\gamma,T)\, \|\mu_0\|_{L^\infty}^{1/p}\, K_p(T) S_{5/3}(T)^{1/p'} < \infty.$$

Indeed, note that the condition $K_p(T) < \infty$ is satisfied by interpolation. In particular, the parameters satisfy $p < q < 6$, from which we can deduce that

$$\sup_{0\le t\le T} \|\tilde{N}_t\|_{L^p} \le C K_p(T)^\alpha < \infty,$$

for some $0 < \alpha < 1$ by interpolation. Combining the two contributions gives the desired result

$$\sup_{0\le t\le T} \|N_t\|_{L^p} < \infty.$$

□

This first result serves as a stepping stone, which combined with Theorem 5.8 gives sufficient conditions for the existence of a weak solution whose particle density is essentially bounded along the evolution.

Theorem 5.10. Suppose that the initial condition $0 \leq \mu_0 \in L^1 \cap L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ with finite second velocity moments, and there exist constants $p \geq 5/3$ and $k > 6$ such that

$$\sup_{t\geq 0} \|\rho_0^t\|_{L^p} < \infty, \quad \sup_{t\geq 0} \|R_t\|_{L^k} < \infty.$$

Then, there exists a weak solution $\mu \in C([0,T], \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d))$ to the modified Vlasov-Poisson-Fokker-Planck equation (MVPFP) such that there exists some $6 < q < k$ with

$$\sup_{0\leq t\leq T} \|N_t\|_{L^q} < \infty.$$

Proof. Without loss of generality, we can assume that $p < \infty$ by interpolation on the initial coarse-grained particle density. For a fixed $6 < q < k$ and $3/2 < r < 6$ such that

$$\frac{1}{pr'} - \frac{2}{3}\left(\frac{2}{3} - \frac{1}{r}\right) < \frac{1}{q} < \frac{1}{pr'}, \tag{5.14}$$

we search for conditions on q and r ensuring the boundedness of induced force throughout the evolution. By Theorem 5.8 with $1/r = 0$ and $k = 5/3$ in the notation of that theorem, we get

$$S_p(T) \leq C\left(1 + K_q(T)^{(5/3)(q)(3/5-1/p)}\right).$$

In addition, using Theorems 5.7 and 5.9, we get for q sufficiently close to 6 so that the first term in the induced force is bounded,

$$\sup_{0\leq t\leq T} \|\tilde{N}_t\|_{L^q} \leq C S_p(T)^{1/r'}, \quad \sup_{0\leq t\leq T} \|N_t\|_{L^q} \leq C(1 + S_p(T)^{1/r'}) \leq C\left(1 + K_q(T)^{(5/3)(q/r')(3/5-1/p)}\right).$$

Thus, note that it is sufficient to ensure that $(5/3)(q/r')(3/5 - 1/p) < 1$ to obtain boundedness of the induced force along the evolution. In particular, take p sufficiently small so that this is achieved. This is possible by interpolation. Subsequently, choose $r < 6$ and $q > 6$ sufficiently close to 6 such that the condition (5.14) is satisfied.

□

We are ready to prove the existence of a solution with an essentially bounded particle density throughout the evolution. However, stronger assumptions on the initial density are imposed.

Theorem 5.11. Suppose that the initial condition $0 \leq \mu_0 \in L^1 \cap L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ with finite m-th velocity moment, for some $m > 6$. Then, there exists a strong solution $\mu \in C([0,T], \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d))$ to the modified Vlasov-Poisson-Fokker-Planck equation (MVPFP).

Proof. Since the initial condition has finite m-th velocity moments, we can apply Remark 5.4 to obtain the first assumption of Theorem 5.10 for some $p > 3$. Using Sobolev embeddings with Remark 5.4, we obtain the second assumption of Theorem 5.10 for some $k > 6$. As a result, we can deduce that

$$\sup_{0\le t\le T} \|N_t\|_{L^q} < \infty,$$

for some $6 < q < k$. Subsequently, we can apply Theorem 5.8 to deduce that $S_p(T) < \infty$. With $p > 3$ and $q > 6$, we can apply Theorem 5.7 with $r = \infty$ to deduce that

$$\sup_{0\le t\le T} \|\tilde{N}_t\|_{L^\infty} < \infty.$$

In addition, the fact that

$$\sup_{0\le t\le T} \|\bar{N}_t\|_{L^\infty} < \infty$$

follows from the fact that this term is obtained by convolution with a Gaussian which is essentially bounded. Thus, we can conclude that

$$\sup_{0\le t\le T} \|N_t\|_{L^\infty} < \infty.$$

This proves the required result, concluding the proof of the theorem.

□

Hence, we have found conditions implying the existence of a strong solution to the modified Vlasov-Poisson-Fokker-Planck equation. Now, we prove uniqueness of such solutions.

Theorem 5.12. Suppose that the initial condition $0 \le \mu_0 \in L^1 \cap L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ has finite second velocity moment. Then, there exists at most one strong solution $\mu \in C([0,T], L^1(\mathbb{R}^d \times \mathbb{R}^d))$ to the Vlasov-Poisson-Fokker-Planck equation (MVPFP).

Proof. Let $\mu, \hat{\mu}$ be two distinct solutions to the modified Vlasov-Poisson-Fokker-Planck equation with essentially bounded induced forces N and $\hat{N}$ respectively. In light of the convolution expression obtained in (5.7), we can express the difference of the induced forces as

$$\hat{N}_t(x) - N_t(x) = \int_0^t \frac{\sigma(s,0)}{(2\gamma^4\delta(s,0))^{3/2}} A\left(\frac{x}{\sqrt{2\gamma^4\delta(s,0)}}\right) * I_{t-s}^{\sigma(s,0)}(x)\ ds, \tag{5.15}$$

where the quantity I_t^λ is the difference of the fluxes, given by

$$I_t^\lambda(x) := \hat{J}_t^\lambda(x) - J_t^\lambda(x).$$

In particular, we estimate this quantity I_t^λ by splitting the integral in the definition of J_t^λ by

$$\begin{aligned} |I_t^\lambda(x)| &\le \int_{\mathbb{R}^d} |\hat{N}_t(x-\lambda v)\hat{\mu}_t(x-\lambda v, v) - N_t(x-\lambda v)\mu_t(x-\lambda v, v)|\ dv \\ &\le \int_{\mathbb{R}^d} |(\hat{N}_t - N_t)(x-\lambda v)|\ \Gamma_\#^\lambda \hat{\mu}_t(x,v)\ dv + C\int_{\mathbb{R}^d} |(\hat{\mu}_t - \mu_t)(x-\lambda v, v)|\ dv \\ &:= \bar{I}_t^\lambda(x) + \tilde{I}_t^\lambda(x), \end{aligned}$$

where we used the fact that N_t is essentially bounded throughout the evolution. In addition, for any $\lambda > 0$ and $1 \leq q \leq \infty$, using Hölder's inequality followed by a change of variables yields

$$\|\bar{I}_t^\lambda\|_{L^{q'}} \leq \frac{1}{\lambda^{3/q}} \|\hat{N}_t - N_t\|_{L^q} \|\mu_t\|_{L^{q'}}. \tag{5.16}$$

Let $q > 2$. Applying Theorem 5.5 with $1/k := 1/q' - 1/q$, followed by a change of variables gives us that, in L^q norm, the first term in (5.15) after substituting the bound on I_t^λ is bounded by

$$\begin{aligned}\int_0^t & \frac{\sigma(s,0)^{1-3/q}}{(2\gamma^4\delta(s,0))^{3/2k}} \|A\|_{L^{k'}} \|\hat{N}_{t-s} - N_{t-s}\|_{L^q} \|\hat{\mu}_{t-s}\|_{L^{q'}} \, ds \\ &\leq C t^{\alpha_1} \sup_{0\leq s\leq t} \|\hat{N}_s - N_s\|_{L^q},\end{aligned}$$

where $\alpha_1 \simeq 1/2$. When it comes to the second term in the expression, using the Lions-Perthame representation, we have

$$\hat{\mu}_t - \mu_t = \int_0^t \iint_{\mathbb{R}^d\times\mathbb{R}^d} \nabla_\nu \mathcal{G}_s(x,v,\xi,\nu)(\hat{N}_{t-s}(\xi)\hat{\mu}_{t-s}(\xi,\nu) - N_{t-s}(\xi)\mu_{t-s}(\xi,\nu)) \, d\xi d\nu ds.$$

For convenience, by defining

$$F_t^\lambda(x) := \int_{\mathbb{R}^d} |\hat{N}_t(x-\lambda v)\hat{\mu}_t(x-\lambda v, v) - N_t(x-\lambda v)\mu_t(x-\lambda v, v)| \, dv,$$

and using the estimate provided in (5.3) and the fact that the forces are essentially bounded, we get

$$\tilde{I}_t^\lambda(x) \leq \int_0^t \frac{C}{(2\gamma^4\delta(s,\lambda))^{3/2}\sqrt{\gamma^4 s}} \mathcal{M}\left(\frac{x}{\sqrt{2\gamma^4\delta(s,\lambda)}}\right) * F_{t-s}^{\sigma(s,\lambda)}(x) \, ds.$$

Thus for any $1 \leq p \leq \infty$, we have the regularity

$$\|\tilde{I}_t^\lambda\|_{L^p} \leq C \int_0^t \frac{1}{\sqrt{\gamma^4 s}} \|F_{t-s}^{\sigma(s,\lambda)}\|_{L^p} \, ds.$$

Moreover, define

$$h_p(t) := \sup_{0\leq s\leq t, \lambda\geq 0} \|\tilde{I}_s^\lambda\|_{L^p}.$$

Using the fact that the initial density has finite second moments which are propagated throughout the evolution, we have the regularity provided by (5.12). Combined with Theorem 5.8 from which we can deduce that $h_p(t)$ is finite for $1 \leq p \leq 5/3$. Hence, we obtain

$$\begin{aligned} h_p(t) &\leq \sup_{0\leq s\leq t, \lambda\geq 0} \int_0^t \frac{C}{\sqrt{\gamma^4 s}} \|I_{t-s}^{\sigma(s,\lambda)}\|_{L^p} \, ds \\ &\leq \sup_{0\leq s\leq t, \lambda\geq 0} \int_0^t \frac{C}{\sqrt{\gamma^4 s}} \left(\|\bar{I}_{t-s}^{\sigma(s,\lambda)}\|_{L^p} + \|\tilde{I}_{t-s}^{\sigma(s,\lambda)}\|_{L^p}\right) \, ds \\ &\leq \sup_{0\leq s\leq t, \lambda\geq 0} \int_0^t \frac{C}{\sqrt{\gamma^4 s}} \|\bar{I}_{t-s}^{\sigma(s,\lambda)}\|_{L^p} \, ds + 2tC h_p(t). \end{aligned}$$

Thus, by taking $\epsilon > 0$ sufficiently small, we get

$$h_p(\epsilon) \leq \sup_{0\leq t\leq \epsilon, \lambda \geq 0} \int_0^t \frac{1}{\sqrt{\gamma^4 s}} \|\bar{I}_{t-s}^{\sigma(s,\lambda)}\|_{L^p}\, ds.$$

However, using a change of variables, we have

$$|\bar{I}_{t-s}^{\sigma(s,\lambda)}(x)| = \frac{1}{\sigma(s,\lambda)^3} \int_{\mathbb{R}^d} |\hat{N}_{t-s}(v) - N_{t-s}(v)| \hat{\mu}_{t-s}\left(v, \frac{x-v}{\sigma(s,\lambda)}\right)\, dv.$$

Thus, from Minkowski's integral and Hölder's inequality, we have

$$\begin{aligned}
\|I_{t-s}^{\sigma(s,\lambda)}\|_{L^p(dx)} &\leq \frac{1}{\sigma(s,\lambda)^3} \int_{\mathbb{R}^d} |\hat{N}_{t-s}(v) - N_{t-s}(v)| \left\| \hat{\mu}_{t-s}\left(v, \frac{x-v}{\sigma(s,\lambda)}\right) \right\|_{L^p(dx)} dv \\
&\leq \frac{1}{\sigma(s,\lambda)^{3/p'}} \left\| \hat{N}_{t-s} - N_{t-s} \right\|_{L^q} \left\| \|\hat{\mu}_{t-s}(x,v)\|_{L^p(dv)} \right\|_{L^{q'}(dx)} \\
&\leq \frac{1}{\sigma(s,\lambda)^{3/p'}} \left\| \hat{N}_{t-s} - N_{t-s} \right\|_{L^q} \left\| \|\hat{\mu}_{t-s}(x,v)\|_{L^1(dv)}^{1/p} \|\hat{\mu}_{t-s}(x,v)^{p-1}\|_{L^\infty(dv)}^{1/p} \right\|_{L^{q'}(dx)} \\
&\leq \frac{C}{\sigma(s,\lambda)^{3/p'}} \left\| \hat{N}_{t-s} - N_{t-s} \right\|_{L^q} \left(\int_{\mathbb{R}^d} \hat{\rho}_t(x)^{q'/p} dx \right)^{1/q'}.
\end{aligned}$$

Whenever $1 \leq q'/p \leq 5/3$, the last integral factor is finite and so we can bound

$$h_p(\epsilon) \leq C \sup_{0\leq t\leq \epsilon} \int_0^t \frac{1}{\sigma(s,\lambda)^{3/p'} \sqrt{\gamma^4 s}} \left\| \hat{N}_{t-s} - N_{t-s} \right\|_{L^p} ds.$$

Choosing $q'/p = 5/3$ such that $1 < p < 6/5$ will guarantee that the above integral is finite and so

$$h_p(\epsilon) \leq C \sup_{0\leq s\leq \epsilon} \left\| \hat{N}_s - N_s \right\|_{L^q}. \tag{5.17}$$

Thus, for all $0 \leq t \leq \epsilon$, we can apply Theorem 5.5 with $1/q = 1/p - 1/r \geq 0$ to deduce

$$\begin{aligned}
&\left\| \int_0^t \frac{\sigma(s,0)}{(2\gamma^4\delta(s,0))^{3/2}} \left| A\left(\frac{x}{\sqrt{2\gamma^4\delta(s,0)}} \right) \right| * \left(\int_{\mathbb{R}^d} |(\hat{\mu}_{t-s} - \mu_{t-s})(x - \sigma(s,0)v, v)|\, dv \right) ds \right\|_{L^p} \\
&\leq \int_0^t \frac{\sigma(s,0)}{(2\gamma^4\delta(s,0))^{3/2}} \left\| A\left(\frac{x}{\sqrt{2\gamma^4\delta(s,0)}} \right) \right\|_{L^r} \left\| \int_{\mathbb{R}^d} |(\hat{\mu}_{t-s} - \mu_{t-s})(x - \sigma(s,0)v, v)| \right\|_{L^p} ds \\
&\leq h_p(\epsilon)\, \|A\|_{L^r} \int_0^t \frac{\sigma(s,0)}{(2\gamma^4\delta(s,0))^{3/2r}}\, ds \\
&\leq C\epsilon^{\alpha_2} \sup_{0\leq s\leq \epsilon} \left\| \hat{N}_s - N_s \right\|_{L^q},
\end{aligned} \tag{5.18}$$

where $\alpha_2 \simeq 1/2$. Thus, combining (5.16) and (5.18) with $\alpha \simeq 1/2$ yields

$$\left\| \hat{N}_\epsilon - N_\epsilon \right\|_{L^q} \leq C\epsilon^\alpha \sup_{0\leq s\leq \epsilon} \left\| \hat{N}_s - N_s \right\|_{L^q}.$$

By taking ϵ sufficiently small, we obtain that $N_t = \hat{N}_t$ for $0 \leq t \leq \epsilon$. In addition, we can use the bound obtained in (5.17) to deduce that $\mu_t = \hat{\mu}_t$ for $0 \leq t \leq \epsilon$. Thus, uniqueness holds for all $0 \leq t \leq T$.

□

Bibliography

[1] F. Bouchut. “Existence and Uniqueness of a Global Smooth Solution for the Vlasov-Poisson-Fokker-Planck System in Three Dimensions”. In: *Journal of Functional Analysis* 111.1 (1993), pp. 239–258.

[2] J. Carrillo, Y.-P. Choi, and Samir Salem. “Propagation of chaos for the Vlasov–Poisson–Fokker–Planck equation with a polynomial cut-off”. In: *Communications in Contemporary Mathematics* 21.04 (2019), p. 1850039.

[3] L.-P. Chaintron and A. Diez. “Propagation of chaos: A review of models, methods and applications. I. Models and methods”. In: *Kinetic and Related Models* 15.6 (2022), p. 895.

[4] Y.P. Choi and O. Tse. “Quantified overdamped limit for kinetic Vlasov–Fokker–Planck equations with singular interaction forces”. In: *Journal of Differential Equations, 330* (2022), pp. 150–207.

[5] R. Clausius. “Ueber verscheidene für die anwendug bequeme formen der hauptgleichungen der mechanischen wärmtheorie”. In: *Annalen der Physik und Cehemie* (1865), pp. 353–400.

[6] L. Gross. “Logarithmic Sobolev inequalities”. In: *Americal Journal of Mathematics, 97 (4)* (1975), pp. 1061–1083.

[7] L. Hörmander. *The Analysis of Linear Partial Differential Operators I.* Springer-Verlag. New York-Berlin, 1983.

[8] E.T. Jaynes. “Information Theory and Statistical Mechanics I”. In: *Physical Review. Series II. 106 (4)* (1957), pp. 620–630.

[9] E.T. Jaynes. “Information Theory and Statistical Mechanics II”. In: *Physical Review. Series II. 108 (2)* (1957), pp. 171–190.

[10] G. Saveré L. Ambrosio N.Gigli. *Gradient Flows in Metric Spaces and in the Space of Probability Measures.* 2. ed. Lectures in Mathematics ETH Zürich. Basel: Birkhäuser, 2008. 334 pp.

[11] J.C. Maxwell. “Illustrations of the dynamical theory of gases. Part I. On the motions and collisions of perfectly elastic spheres.” In: *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science, 4th Series, vol.19* (1860), pp. 19–32.

[12] J.C. Maxwell. “Illustrations of the dynamical theory of gases. Part II. On the process of diffusion of two or more kinds of moving particles among one another.” In: *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science, 4th Series, vol.20* (1860), pp. 21–37.

[13] R. McCann. “A Convexity Principle for Interacting Gases”. In: *Advances in Mathematics* 128.1 (1997), pp. 153–179.

[14] H. McKean. “A class of markov processes associated with nonlinear parabolic equations”. In: *Proceedings of the National Academy of Sciences* 56.6 (1966), pp. 1907–1911.

[15] B. Perthame and P.-L. Lions. "Propagation of moments and regularity for the 3-dimensional Vlasov-Poisson system." In: *Inventiones mathematicae* 105.2 (1991), pp. 415–430.

[16] D. Kinderlehrer R. Jordan and F. Otto. "The Variational Formulation of the Fokker–Planck Equation". In: *SIAM Journal on Mathematical Analysis* 29.1 (1998), pp. 1–17.

[17] J. Schauder. "Der Fixpunktsatz in Funktionalräumen". In: *Studia Math.* (1930), pp. 171–180.

[18] J. Simon. "Compact sets in the space $L^p(O, T; B)$". In: *Annali di Matematica Pura ed Applicata. 146.* (1986), pp. 65–96.

[19] H.D. Victory. "On the existence of global weak solutions for Vlasov-Poisson-Fokker-Planck systems". In: *Journal of Mathematical Analysis and Applications* 160.2 (1991), pp. 525–555.

[20] C. Villani. "Optimal transport – Old and new". In: vol. 338. Jan. 2008, pp. xxii+973.