University of Groningen

A Post-Newtonian View of the Harmonic Oscillator

Master's Thesis

Supervised by
Prof. Dr. D. Roest and Prof. Dr. M. Seri

Jordi Smit
(S4030729)

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Abstract

Building on the findings of de Neeling [12], we ask whether there exist relativistic systems that preserve the dynamics and symmetries of the classical isotropic harmonic oscillator (IHO). We find that these systems exist at all orders on-shell, and that off-shell an approximate equivalence exists up to at least 2PN order. We show there exists a canonical transformation between such systems back to the IHO. We list the required relation between the coefficients of the SO(N) invariant building blocks of the PN expansion, to preserve a proposed relativistic Fradkin tensor, and thus maintain closed and bounded orbits. We compare our findings to the 1PN expansion of a particle in a de Sitter background, and show by simulating the evolution of 3 Hamiltonians that the right relation among its coefficients indeed preserves a closed and bounded orbit.

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1 Introduction

The Kepler problem and the harmonic oscillator occupy a special place in physics. The former, in which a small body orbits a central, much larger one, is well known for its large symmetry and the simplicity of its solutions. The latter, in which a body moves back and forth in a periodic fashion in some form or another, also has these nice properties. The two systems are cornerstones of many fields within physics; the Kepler problem's inverse square force law, leading to a potential

$$V(r) = -\frac{k}{r}, \quad \text{k} \quad \in \mathbb{R}$$

can also be seen in atomic physics as the Coulomb potential. Most central to the work this thesis is based on is the relation between the potential and the vertices of gravitons in a field theory [12].

The (isotropic) harmonic oscillator, HO or IHO for short, is even more ubiquitous in physics. The reason for this is twofold: firstly, the HO is frequently encountered as a first example for classical mechanics in systems as the simple pendulum; then in solid mechanics as a model for phonons. In a first quantum mechanics course, we learn that the harmonic oscillator can be represented as raising and lowering operators $(a^\dagger, a)$

$$H = a^\dagger a.$$

This formalism is the backbone of quantum field theory. Secondly, almost all potentials can be well approximated by the oscillator potential, since

$$V(x) \approx \frac{1}{2} V''(x_0)(x - x_0)^2$$

is the first non-zero term in the Taylor expansion of a general potential around its minimum.

Both systems have particularly large symmetry groups - $SO(4)$ and $SU(3)$ for 3-dimensional physical space, respectively - and both systems have a hidden symmetry [12][6] . For the Kepler problem, the system is superintegrable due to the presence of the Lagrange-Runge-Lenz vector; the HO's conserved quantities can be nicely organized in a tensorial structure, called the Fradkin tensor [6]. This superintegrability is also expressed by Bertrand's theorem;

Theorem 1. ***(Bertrand's theorem)*** *[16]*
The only scalar central potentials that result in closed orbits for all bounded trajectories are that of the isotropic harmonic oscillator V_{HO} and the Kepler potential V_{Kep}

$$V_{HO} = kr^2 \quad and \quad V_{Kep} = -\frac{k}{r},$$

for some constant $k \in \mathbb{R}$.

The bound and closed orbits we see in the Kepler and harmonic oscillator system are a result of a restricted phase space - a result of a large amount of conserved quantities.

Recently, de Neeling set out a method to find a family of relativistic systems that all obey the symmetries and dynamics of the Kepler problem, by constructing canonical transformations between the relativistic systems and the Kepler system, and demanding the hidden symmetry remain intact. Given the close ties between the Kepler and HO systems, for instance that the solutions to a three-dimensional Kepler problem can be obtained from the solution of a four-dimensional classical (or quantum) oscillator [19], and that they are the only two potentials singled out by Bertrand's theorem, warrants further investigation; is it possible to construct a family of systems that is equivalent to a harmonic oscillator through a canonical transformation and a time reparametrization, that conserves the hidden symmetry seen in the classical IHO? This would show that both systems in Bertrand's theorem have particular relativistic extensions that maintain their classical symmetries, essentially allowing us to write down a relativistic Bertrand's theorem.

To address this question, we first introduce Hamiltonian mechanics and the toolkit given to us by symplectic geometry; key definitions and theorems concerning integrability, canonical transformations, and the classical harmonic oscillator. Building on this introduction we turn to the problem of relativistic systems; we review the post-Newtonian expansion method as formulated in de Neeling's work [12], and compare the treatment of the Kepler system with that of the harmonic oscillator. We discuss the similarities and differences between the systems, and investigate a scenario where we might come accross a relativistic oscillator. Investigating the family of systems that is on-shell equivalent to the HO, we discuss the likeness of the presented arguments with that of the Kepler case. We propose a relativistic Fradkin tensor, and demonstrate that it preserves a local $SU(3)$ symmetry. Finally, we turn to the off-shell equivalence; the real meat and bones of this thesis.

We prove, by demanding that the equivalence between systems holds off-shell up to an n-th order in PN expansion, that there exist systems with the symmetries and dynamics of classical harmonic oscillators, that do not posses the potential $V(r)$ demanded by Bertrand's theorem; we do this up to at least second order. We find that at first- and second-PN order, the Hamiltonian contains two free parameters. The constraints that guarantee the existence of a Fradkin tensor and canonical transformation to the IHO are put forward. Our analysis is completed by performing phase-space simulations at first post-Newtonian order, verifying that the resulting constrained Hamiltonian indeed preserves the classical symmetries of the isotropic harmonic oscillator.

We use (q, p) to denote the canonical coordinates in $T^*\mathbb{R}^3 = \mathbb{R}^3 \times \mathbb{R}^3$, unless otherwise specified. Additionally, we use $r = r(q) = |q|$. Throughout most of this thesis, we use units such that $c^2 = 1$.

2 Symmetry, Hamiltonian mechanics and harmonic oscillators

To discuss systems that are equivalent to the harmonic oscillator, we need to discuss symmetries and the ways available to us to compare one system to another. The study of symmetries in systems is best described from the point of view of Hamiltonian mechanics. One of the key points pertaining to symmetries in physical systems, Noether's theorem, becomes almost trivial to describe in the this formalism; put simply, Noether's theorem states that for every continuous transformation generated by Q that leaves the Hamiltonian H invariant, there exists a conserved quantity associated with that continuous transformation. If we transform the Hamiltonian as

$$H \to H + \delta H, \tag{1}$$

then, with Q generating the transformation, we have that

$$\delta H = \{H, Q\}, \tag{2}$$

which must vanish for the transformation generated by Q to be called a symmetry of the system. Importantly, these symmetries and conserved quantities are not always visible from the point of Newtonian or Lagrangian mechanics - Hamiltonian mechanics enables us to see the effects of these symmetries as a property of the spaces where we define Hamiltonians, as we will see in this section. The most important examples where we can see this happen are central to this work, as well as to the work on which most methods used in this thesis are based; the Kepler problem and the Harmonic oscillator.

In this chapter, we will set out the necessary Hamiltonian formalism, explain canonical transformations and the way in which we will use them in later sections, the link between canonical transformations and symmetry, the effects of symmetry on physical systems (i.e. integrability), and discuss the harmonic oscillator in depth.

In this section, many statements are made as-is, with no proof or example; the point of this thesis is to use these tools and statements and see how they apply to the harmonic oscillator - proofs of these theorems and examples of the cited definitions are readily available in most (under)graduate Mechanics textbooks like [8] or lecture notes, such as [16],[15].

2.1 Hamiltonian formalism

In 1834, Sir William Rowan Hamilton introduced his Principle of stationary action. A little under 200 years later, that same principle and its accompanying variational approach have become ubiquitous in physics. It underlies not only most of theoretical physics, but also economics and some disciplines in mathematics. Developed originally as a theory of optics, the principle can be loosely stated as having the purpose of finding the most ideal path for a system - the path through configuration space that does not change the action. The action principle is based on the action functional S

$$S = \int_{t_i}^{t_f} L(\vec{q}, \dot{\vec{q}}, t) dt, \tag{3}$$

for n generalized coordinates $\vec{q}$ and their velocities $\dot{\vec{q}}$.

The Hamiltonian is defined in terms of the generalized coordinates and momentum q and p, where $p = \frac{\partial L}{\partial \dot{q}_j}$, by employing a Legendre transform of the Lagrangian:

$$H(q, p, t) = \sum_i p_i \dot{q}_i - L(q, \dot{q}, t) \tag{4}$$

where $q = (q_1, q_2, ...q_n)$ and $p = (p_1, p_2, ...p_n)$ are vectors of generalized coordinates and conjugate momenta. Hamilton used this action principle to derive his equations of motion as

$$\dot{q} = \frac{\partial H}{\partial p} \ , \ \dot{p} = -\frac{\partial H}{\partial q}. \tag{5}$$

We can see that the Hamiltonian point of view really treats coordinates and momenta as a new coordinate system together. To formalize this, we use the tools of symplectic geometry. Specifically, we use a $2n$-dimensional manifold (phase space) $\mathcal{P}$, with the coordinates on $\mathcal{P}$ described by the generalized coordinates and momenta $x^\mu = (q^i, p_j)$ with $i, j = 1, ...n$, and $q \in \mathcal{M}$ parameterizing the configuration space. On this manifold we have a symplectic 2-form $\omega = \omega_{\mu\nu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$ and a smooth function $H : \mathcal{P} \mapsto \mathbb{R}$ called the *Hamiltonian.* The coordinates x^μ are the local coordinates on the manifold $\mathcal{P}$. The usual setup which we will follow is that $\mathcal{P} = T^*\mathcal{M}$ - that is, the phase space is the cotangent bundle of an n-dimensional configuration manifold $\mathcal{M}$. The 2-form is then, by Darboux's theorem [23], $\omega = \mathrm{d}p_i \wedge \mathrm{d}q^i$.
The Hamiltonian function then generates dynamics on $\mathcal{P}$ as

$$\dot{x}^\mu = \omega^{\mu\nu} \partial_\nu H. \tag{6}$$

In local (Darboux) coordinates, this can be written as the familiar Hamilton's equations of motion:

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}. \tag{7}$$

In fact, the flow of any function f with evolution parameter λ, can be defined as in (6), by associating a vector field associated with it:

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} = \omega^{\mu\nu}\partial_\nu f(\lambda) =: X_f^\mu, \tag{8}$$

which gives a function when applied to another smooth function g:

$$X_f(g) = \omega^{\mu\nu}\partial_\mu g \partial_\nu f(\lambda) =: \{g, f\}, \tag{9}$$

where the last bracket is called the Poisson bracket. This bracket, developed in 1809 by Poisson (though he did not name them this himself), between two continuous functions $f(q,p)$ and $g(q,p)$ is defined as

$$\{f, g\} \equiv \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}. \tag{10}$$

The Poisson bracket is antisymmetric, linear, and satisfies Leibniz' rules, as well as the Jacobi identity

$$\{X, \{Y, Z\}\} + \{Y, \{Z, X\}\} + \{Z, \{X, Y\}\} = 0.$$

As a consequence of antisymmetry, we have

$$\{F, F\} = 0. \tag{11}$$

The Poisson brackets of the canonical coordinates q, p themselves are important:

$$\{q_i, q_j\} = 0\ ,\ \{p_i, p_j\} = 0\ ,\ \{q_i, p_j\} = \delta_{ij} = -\{p_j, q_i\} \tag{12}$$

The Poisson bracket can be related back to the structure of $\mathcal{P}$ by seeing that

$$\omega^{\mu\nu} = \begin{pmatrix} 0_n & I_n \\ -I_n & 0 \end{pmatrix}, \tag{13}$$

is the 2-form satisfying the required properties. This specific 2-form can always be used when we are working with Darboux coordinates, and when filled into (9) applied to f and g produces the definition of the Poisson bracket in (10).
Taking three smooth functions f, g and h and their vector fields X, we have

$$[X_f, X_g](h) = (X_f X_g - X_g X_f)(h) \tag{14}$$
$$= \{\{h, g\}, f\} - \{\{h, f\}, g\} \tag{15}$$
$$= -\{\{g, f\}, h\} \tag{16}$$
$$= -X_{\{f,g\}}(h), \tag{17}$$

where in going from (16) to (17) antisymmetry and the Jacobi identity were used, and antisymmetry was used to arrive at the concluding equation. This small derivation shows that we can talk about the Lie algebra of vector fields on a phase space, or the Poisson algebra of smooth functions on that same phase space without much distinction.

2.2 Symmetries and Noether's theorem

In classical mechanics, Noether's theorem ties global continuous symmetries to conserved quantities of systems. We call a symmetry a transformation of the phase space manifold that leaves the equations of motion invariant. In the previously introduced Hamiltonian formalism, a continuous symmetry transformation is the flow of a vector field X_f on the phase space $\mathcal{P}$ that leaves H invariant. In terms of the Poisson bracket from before

$$\{H, f\} = \left.\frac{\mathrm{d}f}{\mathrm{d}\lambda}\right|_{\text{flow H}} = 0 \tag{18}$$

This result is known as the infinitesimal Noether's theorem:

Theorem 2. ***Infinitesimal Noether's theorem [16]***
Let ($\mathcal{P}$, ω, H) be a Hamiltonian system and let X_f be the Hamiltonian vector field associated to a smooth function $f : \mathcal{P} \to \mathbb{R}$. Then, f is constant along the flow of H, i.e. $f(\lambda(t))$ is constant if $\lambda(t)$ solves Hamilton's equations.

More precisely, the time evolution of f is given by the Poisson bracket:

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\lambda(t)) = \{H, f\} \tag{19}$$

and f is constant along the flow if and only if $\{H, f\} = 0$. The function f is then called a first integral or conserved quantity of H.
The one-parameter group of diffeomorphisms $\Phi_t : \mathcal{P} \to \mathcal{P}$ with

$$\Phi_t := \{\exp(tX_f) \,|\, t \in \mathbb{R}\}$$

is then called a ***symmetry*** *of the Hamiltonian system. In short, any smooth function in involution with a given Hamiltonian on its phase space is a conserved quantity if $\{H, f\} = 0$.*

A conserved quantity $f(q, p)$ remains constant along the motion, and so the dynamics of the system are confined to the $(2n-1)$-dimensional level set $f^{-1}(c)$, if c is a regular value of f. This makes them extremely useful in simplifying dynamical systems. The presence of multiple conserved quantities is associated with a high degree of integrability.

2.3 Integrability and symmetry

Integrability suggests that one can integrate a system - integrate the equations of motion to find solutions for the paths through the phase space.

Definition 1. Integrable system
A Hamiltonian system ($\mathcal{P}$, ω, H) is called **integrable** if it has $m = \frac{1}{2}\dim\mathcal{P}$ independent first integrals $f_n = (f_1 = H, ...f_m)$ that are in involution (i.e. Poisson commute) with each other. A first integral differs from a constant of motion in that the former does not have explicit time dependence.
An integrable system is a Hamiltonian system for which the equations of motions can be solved by quadratures.

A small class of integrable systems is called maximally superintegrable. A superintegrable system in n-dimensional configuration space has $2n-1$ conserved quantities. Two examples are important; the Kepler system in 2D and 3D, as well as the n-dimensional isotropic harmonic oscillator. These systems have additional structure to their phase space; namely, their phase spaces are foliated by Lagrangian submanifolds, as per the Arnold-Liouville theorem:

Theorem 3. ***Arnold-Liouville[16]***
Let $(\mathcal{M}, \omega, H)$ be an integrable system of dimension $2n$ with integrals of motion $f_n = (f_1 = H, f_2, ...f_n)$. Let $c \in \mathbb{R}^n$ be a regular value of f. Then the corresponding level $f^{-1}(c)$ is a Lagrangian submanifold - that is, for a given regular value, the evolution of the system is limited to a submanifold of the original manifold $\mathcal{M}$.

We must make an important distinction; not all first integrals (another name for conserved quantity) are independent:

Definition 2. Independence of first integrals
A set of first integrals/integrals of motion f_n is called independent if the first integrals are linearly independent, i.e. that one cannot be expressed as a function of others in the same set.

Importantly, since the phase spaces under consideration are compact, having the maximal amount of independent integrals implies periodic motion. Through Bertrand's theorem, we can then conclude that the 2D and 3D isotropic harmonic oscillator must be maximally superintegrable, having $2n-1$ conserved quantities. Let us now investigate those conserved quantities, and discuss hidden symmetries.

2.4 Canonical transformations

In high school, we get told that one of the fundamentals of physics is that the laws of physics should not change when you transform you coordinates "nicely". What is meant by this "nicely" is highlighted very well in symplectic geometry: an advantage of symplectic geometry is that the description of the physical system is independent of the coordinates we have chosen to employ.

In the language of Hamiltonian mechanics, this means that there is a set of Hamiltonian systems $(\mathcal{P}, \omega, H')$ all equivalent to one single $(\mathcal{P}, \omega, H)$, connected by a transformation that satisfies one condition; namely, if we let $\Phi : \mathcal{P} \to \mathcal{P}$ be the transformation, then we have

$$\Phi^* \omega = \omega.$$

Such transformations are called *canonical.* If the new coordinates are denoted (Q, P) then we have for the Hamiltonians

$$H'(Q(q,p), P(q,p)) = H(q,p).$$

Because momenta and coordinates are both coordinates of phase space, there are more canonical transformations than just coordinate transformations; we can independently redefine coordinates in configuration space, momenta, and we can even mix between the two. The only restriction on these transformations is that the symplectic 2-form ω remains the same. We can also write this requirement as

$$\left\{Q^i(q,p), P_j(q,p)\right\} = \delta^i_j \ , \ \left\{Q^i, Q^j\right\} = \{P_i, P_j\} = 0, \tag{20}$$

granted we interpret the above Poisson brackets in the old coordinates q and p.

In this thesis, we will mainly consider infinitesimally small changes to the coordinates; the reason for this will become apparent in later sections. Infinitesimal transformations can be written as

$$Q^i = q^i + \varepsilon A^i(q,p) \ , \ P_i = p_i + \varepsilon B_i(q,p), \tag{21}$$

with small ε as expansion parameter. The Poisson bracket $\{Q^i, P_j\} = \delta^i_j$ truncated at order ε gives

$$\frac{\partial A^i}{\partial q^j} + \frac{\partial B_j}{\partial p_i} = 0, \tag{22}$$

meaning that the functions A^i and B_j are not independent. The ansatz

$$A^i = \frac{\partial G}{\partial p_i} \ , \ B_j = -\frac{\partial G}{\partial q^j} \tag{23}$$

gives just one function $G(q,p)$ to find. This function is called the generating function of the canonical transformation, and is of vital importance later in this thesis. It will be used to prove there exists a canonical transformation from a "relativistic" (i.e. anharmonic) oscillator back to the harmonic oscillator.

The coordinates (Q^i, P_i) label the same point in phase space as (q^i, p_i). This is often called the passive view. For the change of a function $F(q,p;\varepsilon)$ on the phase space transforming under the transformation Φ_G^ε we have, from [12],

$$\frac{\mathrm{d}F(q(-\varepsilon),p(-\varepsilon))}{\mathrm{d}\varepsilon} = \frac{\partial F}{\partial q^i}\frac{\mathrm{d}q^i(-\varepsilon)}{\mathrm{d}\varepsilon} + \frac{\partial F}{\partial p_i}\frac{\mathrm{d}p_i(-\varepsilon)}{\mathrm{d}\varepsilon} \tag{24}$$

$$= -\frac{\partial F}{\partial q^i}\frac{\mathrm{d}G}{\mathrm{d}p_i} + \frac{\partial F}{\partial p_i}\frac{\mathrm{d}G}{\mathrm{d}q^i} \tag{25}$$

$$= \{G,F\} = -\{F,G\}, \tag{26}$$

where there is some explaining required as to how we got from the first to the second equality. We can take ε as flow parameter, changing (q,p) into $(q(\varepsilon),p(\varepsilon)) = \Phi_G^\varepsilon(q,p)$ as a function of ε if we let it range from 0 to some small value. This lets us interpret the coordinates as a function of ε as the *same* coordinate at a *different* point along the flow of G. The coordinates are then

$$q^i(\varepsilon) = q^i(0) + \varepsilon\frac{\partial G}{\partial p_i}\ ,\ p_i(\varepsilon) = p_i(0) + \varepsilon\frac{\partial G}{\partial q^i}. \tag{27}$$

Plugging these into the canonical transformation shows

$$\frac{\mathrm{d}q^i}{\mathrm{d}\varepsilon} = \frac{\partial G}{\partial p_i}\ ,\ \frac{\mathrm{d}p_i}{\mathrm{d}\varepsilon} = -\frac{\partial G}{\partial q^i}, \tag{28}$$

which are Hamiltons equations of motion for a "Hamiltonian" G, with time-evolution parameter ε. We can then put these equations of motion to use in the above derivation of the Poisson bracket $\{G,F\}$.

There is an easy interpretation of this result: the symmetries of a system are the active transformations that leave the Hamiltonian invariant. To see this, see the flow of a function $F(q,p)$ along some vector field X_G:

$$X_G(F) = \{G,F\}, \tag{29}$$

only a sign flip away from the change of the function F under the coordinate change generated by G. Since the generators of continuous symmetries are Hamiltonians in their own right with a different evolution parameter, this completes the interpretation above.

Now for the important bit; higher-order-in-ε transformations - a Taylor series around $\varepsilon = 0$ [22]. To do so, define the adjoint operator

$$\mathrm{ad}_G(\cdot) = \{G, \cdot\} \tag{30}$$

associated with the generating function G. We can differentiate (24) with respect to ε again to obtain the second order derivative

$$\frac{\mathrm{d}^2 F(q(-\varepsilon)), p(-\varepsilon)}{\mathrm{d}\varepsilon^2} = [\mathrm{ad}_G]^2(F(q(-\varepsilon)), p(-\varepsilon)). \tag{31}$$

We can continue this procedure to obtain the Taylor series of $F(q(\varepsilon), p(\varepsilon))$ about $\varepsilon = 0$:

$$F(q(\varepsilon), p(\varepsilon)) = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!}[\mathrm{ad}_G]^n(F), \tag{32}$$

where $[\mathrm{ad}_G]^0(F) = F$ and $F(q(0), p(0)) = F$. This definition of an adjoint operator will become important in our use of generating functions to find canonical transformations relating a family of Hamiltonians to the Hamiltonian representing the isotropic harmonic oscillator, and is identical to that of [12].

2.5 The harmonic oscillator

The main system of this thesis is the isotropic harmonic oscillator. Nearly all potentials can be well approximated by an oscillator, so long as the oscillation around the minimum remains small. The HO shows up in a myriad of places, and needs little to no introduction - here, we will only discuss the relevant information for this thesis.

The harmonic oscillator Hamiltonian $H_{HO} : T^*\mathbb{R}^3 \to \mathbb{R}$ is given by

$$H_{HO} = \frac{p^2}{2m} + m\omega^2 \frac{q^2}{2} \tag{33}$$

with $\omega = \sqrt{\frac{k}{m}}$ and $k, m \in \mathbb{R}$ the mass of the particle described by H and k referred to as the spring constant. The system clearly conserves $H_{HO} = E$, as the Hamiltonian lacks time dependence. Hamiltons equations give

$$\dot{q} = \frac{\partial H_{HO}}{\partial p} = \frac{p}{m} \tag{34}$$

$$\dot{p} = \frac{\partial H_{HO}}{\partial q} = -m\omega^2 q. \tag{35}$$

The Hamiltonian is also composed of only $SO(3)$ invariants p^2 and r^2 and is thus spherically symmetric, so angular momentum $L = p \times q$ is also conserved; explicitly we have

$$\dot{L} = \dot{q} \times p + q \times \dot{p} = -p\frac{\partial H_{HO}}{\partial q} + q\frac{\partial H_{HO}}{\partial p} = -p \cdot 2q + q \cdot 2p = 0 \tag{36}$$

Naturally, the components of the angular momentum vector do not Poisson commute, as

$$\{L_i, L_j\} = \varepsilon_{ijk} L_k, \tag{37}$$

with ε_{ijk} the Levi-Civita tensor. The set (H, L^2, L_z) is in involution. Counting gives 3 angular momentum components as well as the energy providing 4 first integrals, making the harmonic oscillator an integrable system.

But there is another independent integral of the motion. The harmonic oscillator conserves a tensor, the Fradkin tensor (also known as the symmetry tensor)

$$A = \frac{p_i p_j}{2m} + \frac{m\omega^2 q^i q^j}{2}. \tag{38}$$

Explicitly we have, working in the dimensionless situation $m = \omega = 1$ from here on out,

$$\dot{A} = \{A, H_{HO}\} \tag{39}$$

$$= \sum_k \left(\frac{\partial A}{\partial q^k}\frac{\partial H}{\partial p_k} - \frac{\partial A}{\partial p_k}\frac{\partial H}{\partial q^k} \right) + \frac{\partial A}{\partial t} \tag{40}$$

$$= \frac{1}{2}\sum_k \left[\left(q^j \delta_i^k + q^i \delta_j^k\right) p_k - \left(p_j \delta_k^i + p_i \delta_k^j\right) q^k \right] \tag{41}$$

$$= 0. \tag{42}$$

Additionally, A is orthogonal to L:

$$A_{ij} L_j = 0. \tag{43}$$

The n^2 components are not all independent; the trace of A is the Hamiltonian in n dimensions and A is symmetric. In $n = 3$ dimensions, this gives 5 independent components. Combined with the 3 elements of the angular momentum, this gives the 8 generators of $SU(3)$ - the symmetry group of the harmonic oscillator. This can be seen by the commutation relations among its generators [6]:

$$\{L_i, L_j\} = \varepsilon_{ijk} L_k \tag{44}$$

$$\{L_i, A_{jk}\} = \varepsilon_{ijn} A_{nk} + \varepsilon_{ikn} A_{jn} \tag{45}$$

$$\{A_{ij}, A_{kl}\} = (\delta_{ij}\varepsilon_{jln} + \delta_{il}\varepsilon_{jkn} + \delta_{jk}\varepsilon_{iln} + \delta_{jl}\varepsilon_{ikn}) L_n, \tag{46}$$

which defines an algebra isomorphic to $SU(3)$. This Fradkin tensor, then, is the conserved quantity that makes the isotropic harmonic oscillator maximally super-integrable - each of its 5 independent components is conserved individually, thus mandating a closed, bounded orbit.

The development of relativistic versions of the Kepler problem and the close relation to the harmonic oscillator begs the question whether the procedures used to determine these relativistic systems with Kepler dynamics also produce a sort of relativistic harmonic oscillator. That is, given the three different methods to approach a post-Newtonian expansion of the Kepler hamiltonian in [12]; the canonical transformation, the iterative method, and the requirement of an existing hidden symmetry, can we produce a post-newtonian expansion of the harmonic oscillator, who's dynamics are equal to that of the classical harmonic oscillator?

3 The post-Newtonian expansion and relativistic oscillators

In 1859, French astronomer Le Verrier concluded that Mercury's orbit around the sun was incompatible with the Newtonian mechanics, which was until then successfully used to explain the orbit of all other known planets [21]. Many solutions to this problem were put forth - ranging from an unseen planet between the Sun and Mercury (dubbed Vulcan) or another variety of asteroid belt, to concluding that the Sun must be slightly oblate. The former theory had some reason to be believed - similar theories for the perturbation of the orbit of Uranus resulted in the discovery of Neptune [1].

The reason why Mercury's orbit did not line up with the predictions of Newtonian mechanics is perihelion precession; the point furthest away from the Sun in Mercury's elliptical orbit itself rotates with each orbit by 574.10 ± 0.65 arcseconds per century [2].

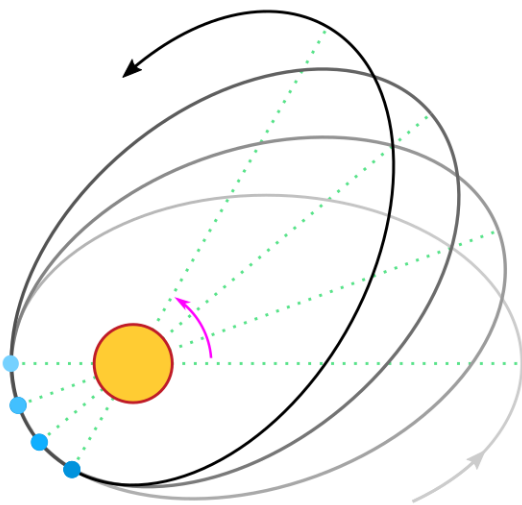

Figure 1: Plot of the precession of the perihelion of Mercury as it orbits the Sun. [4]

In 1915, when Einstein finished his general theory of relativity (GR or sometimes GTR), the peculiar orbit of Mercury was used to test its accuracy; to do so, Einstein first expanded his theory into what is now called a post-Newtonian (PN) expansion. Then came the approximation that our Sun is infinitely heavier than Mercury [1]. Finally ready to calculate the rate of perihelion precession in his new fancy theory of

[1]The actual ratio of M_{Sun} to $M_{Mercury}$ is roughly 1:100.000 [10]

gravity, Einstein calculated a perihelion precession that coincided closely with the observed precession - the theory of general relativity correctly predicted the motion of all planets.

The way Einstein went about his expansion is by slightly distorting the classical Keplerian orbits of the planets. These orbits normally close on themselves, not allowing any precession of the perihelion. A slightly perturbed ellipse does not close in on itself, allowing the precession to occur. This form of expansion is called the Post-Newtonian expansion. In this chapter, the post-Newtonian expansion is discussed, and then applied to spherically symmetric solution to the Einstein's field equations. In the case of Kepler, the Schwarzschild solution is used - in our case, we apply the methods to a de Sitter space, since, as we will see, it gives rise to the terms that show up in a general expansion of the Hamiltonian of a harmonic oscillator.

3.1 Post-Newtonian expansions

In the post-Newtonian expansion, extra terms in the potential come from the effects of GR itself, calculated by using the energy-momentum tensor. In our case, we simply assume such an expansion exists also for the harmonic oscillator.

The post-Newtonian expansion is in orders of $\frac{1}{c^2}$. This introduces the PN expansion of a free relativistic particle as

$$H_{rel} = mc^2 \left(1 + \frac{p^2}{2m^2c^2} - \frac{p^4}{8m^4c^4} + \dots \right) \tag{47}$$

where we expanded the dimensionless parameter p^2/m^2c^2 in powers of $1/c^2$. For the harmonic oscillator the weak field approximation is where r^2 is small, which is close to the origin. In the Kepler version, the potential $V(r) = \frac{k_{kep}}{r}$ is expanded in small k_{kep}; in the HO case, we will expand also in the case of small k_{ho}. By the Virial theorem we have

$$\langle T \rangle = \langle V \rangle \,, \tag{48}$$

where T the kinetic part of the Hamiltonian and V the potential. This is different from the Kepler problem where we have

$$\langle T \rangle = -\frac{1}{2} \langle V \rangle \,, \tag{49}$$

where it relates $\frac{p^2}{m^2c^2} \sim \frac{Gm}{Rc^2}$. In the HO case, it relates $\frac{p^2}{m^2c^2} \sim \frac{kr^2}{mc^2}$. We thus have a parameter we can expand in order $1/c^2$.

Definition 3. Post-Newtonian expansion to N-th order A Hamiltonian $H(q,p;\varepsilon)$ is in a post-Newtonian expansion to the nth order if it is of the form

$$H(q,p;\varepsilon) = \sum_{i=0}^{n} \varepsilon^i H_i(q,p) + \mathcal{O}(\varepsilon^{n+1}) \tag{50}$$

for some regular Hamiltonian functions H_i.

We then have as a relativistic Hamiltonian

$$H_{rel} = \sum_{i=0}^{\infty} \varepsilon^i \Lambda_i(\alpha)\ ,\ \Lambda_i(\alpha) = \sum_{l+m+n=i} \alpha_{lmn}(p^2)^l (pq)^m (r^2)^n\ ,\ (l,m,n) \in \mathbb{N}^3 \tag{51}$$

such that the classical isotropic harmonic oscillator comes out at PN order $i = 0$.

A physical realization of one such a Hamiltonian up to at least 1PN order can be derived from the de Sitter metric:

$$ds^2 = -\left(1 - \frac{\Xi r^2}{3}\right) c^2 \mathrm{d}t + \left(1 - \frac{\Xi r^2}{3}\right)^{-1} \mathrm{d}r^2 + r^2 \mathrm{d}\Omega, \tag{52}$$

where we have used Ξ as the cosmological constant to avoid confusion with the function $\Lambda_i(\alpha)$ above. Assuming motion in the equatorial plane and setting $\theta = \pi/2$ we get the Lagrangian

$$\mathcal{L} = -mc\sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} = -mc\sqrt{-\left[-\left(1 - \frac{\Xi r^2}{3}\right) c^2 - \left(1 - \frac{\Xi r^2}{3}\right)^{-1} \dot{r}^2 - r^2\dot{\phi}^2\right]}. \tag{53}$$

Now, using the weak field limit, expanding

$$\left(1 - \frac{\Xi r^2}{3}\right)^{-1} = \sum_{n=0}^{\infty} \left(\frac{\Xi r^2}{3}\right)^n = 1 + \frac{\Xi r^2}{3} \tag{54}$$

and expanding the square root as

$$\sqrt{1-x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + ... \tag{55}$$

with small x (which is a valid assumption since we have the factor $1/c^2$ in front), we can then approximate the Lagrangian to order $1/c^2$ as

$$\mathcal{L} = -mc^2 + \frac{m}{2}\dot{r}^2 + \frac{m}{2}r^2\dot{\phi}^2 + \frac{m\Xi}{6}r^2 + \frac{m}{8c^2}r^4\dot{\phi}^4 + \frac{m\Xi}{12c^2}r^4\dot{\phi}^2 + \frac{\Xi^2 m}{72c^2}r^4 + \frac{m}{4c^2}r^2\dot{\phi}^2\dot{r}^2 + \frac{m\Xi}{12c^2}r^2\dot{r}^2 + \frac{m}{8c^2}\dot{r}^4 \tag{56}$$

which gives the conjugate momenta as

$$p_r = \frac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{r} + \frac{m}{2c^2}\dot{r}^3 + \frac{m}{2c^2}r^2\dot{\phi}^2\dot{r} + \frac{m\Xi}{6c^2}r^2\dot{r} \tag{57}$$

$$p_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = mr^2\dot{\phi} + \frac{m}{2c^2}r^2\dot{\phi}\dot{r}^2 + \frac{m}{2c^2}r^4\dot{\phi}^3 + \frac{m\Xi}{6c^2}r^4\dot{\phi} \tag{58}$$

Using the definition of the Hamiltonian as (4) we get

$$H = \dot{r}p_r + \dot{\phi}p_\phi - \mathcal{L} \tag{59}$$

With the substitution $p^2 = p_r^2 + p_\theta^2/r^2$ we get then

$$H = mc^2 + \frac{p^2}{2m} - \frac{m\Xi}{6}r^2 + \frac{1}{c^2}\left(-\frac{1}{8m^3}p^4 - \frac{\Xi}{12m}r^2p^2 - \frac{m\Xi^2}{72}r^4\right) + \mathcal{O}(c^{-4}) \tag{60}$$

meaning the 1PN order general relativistic correction to a particle moving in a de-Sitter background is given by

$$H_{1PN} = \left(-\frac{1}{8m^3}p^4 - \frac{\Xi}{12m}r^2p^2 - \frac{m\Xi^2}{72}r^4\right). \tag{61}$$

While this is not immediately useful to us, we will come back to it when discussing the results of the work in the next few sections. A similar derivation can be made for the anti-de Sitter space, which has a negative cosmological constant. The derivation is the same, but there will be sign changes in the resulting 1PN Hamiltonian; a coordinate transformation is also needed to write the line element in the form

$$\mathrm{d}s^2 = -f(r)c^2\mathrm{d}t^2 + (f(r))^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2,$$

where $f(r) = 1 + \frac{r^2}{\alpha^2}$ with $r = \alpha\sinh(\rho)$, and ρ from the cylindrical coordinate system. These coordinates are often called "global coordinates".

3.2 Kepler and the IHO as motion on a 3-sphere

Work has been done to show that the orbits of the Kepler problem can be mapped to geodesic motion on a 3-sphere. The mapping takes us from the phase space $T^*\mathbb{R}^3$ of a regularized Kepler problem with Hamiltonian [11]

$$J = \frac{r}{2}(1 + p^2) - 1 \tag{62}$$

to the phase space $T^*S^3_N$ - the phase space of a particle moving on a 3-sphere without north pole, embedded in $\mathbb{R}^4$. The stereographic projection maps circles to circles, and so does the inverse stereographic projection. In the Kepler case, this means the circles in p-space of Kepler are mapped to circles on the 3-sphere. Additionally, the motion is that of a free Hamiltonian - the SO(4) symmetry seen in Kepler is thus more visible; the orbits of a Kepler problem (great circles on the 3-sphere) are rotated into other orbits (other circles on the 3-sphere). The LRL vector manifests as an isometry.

Such a construction is not so obvious for the harmonic oscillator. Frequently the harmonic oscillator on an n-sphere is encountered in the study of Higgs oscillators. Higgs oscillators appear from the separation of variables of the Klein-Gordon equation in Anti-de Sitter [20]. These are oscillators with the potential term given by

$$V = \frac{1}{2}\omega^2 \lambda^{-1} \tan^2(\chi) \tag{63}$$

with λ the curvature of the sphere, and χ the angular coordinate on the sphere. Unlike the Kepler potential $V = -\lambda^{1/2} \cot(\chi)$, this potential put forth by Higgs in [7] has its singularity on the entire equator of the sphere. The possible orbits of a harmonic oscillator on a 3-sphere are thus constrained to either one of the hemispheres. Since the quantum Higgs oscillator can be written in terms of the Casimir operators of the classical oscillator, the Higgs oscillator has the same hidden symmetry as the classical oscillator [5]. The quantum-statistical properties of the Higgs oscillator on a 2-sphere are set out in [9], but the details of these solutions and its conclusions are outside the scope of this thesis.

Interestingly, the Fradkin tensor has the same form for the quantum harmonic oscillator, just replacing the momentum p and q with their quantum mechanics version:

$$A_{ij} = \frac{1}{2}(\hat{p}_i \hat{p}_j + \hat{p}_j \hat{p}_i) + \omega^2 q_i q_j \tag{64}$$

with $\hat{p}$ the Hermitian momentum operator. For further research, it would be interesting to see if this quantum-Fradkin tensor can also be found for relativistic quantum oscillators, something that can be done for the classical IHO - as we will prove in the next chapters. Given that the potential in (63) has $\lambda^{1/2} r = \tan(\chi)$, it is still proportional to r^2 and we can expect the methods set out in the next section to work.

4 On-shell equivalence

Symmetries and integrability make problems easier to digest, both intuitively and computationally. It is thus natural to try and restore these properties that are present in the non-relativistic problem, to the relativistic variant. Work on this has been done by Perlick, who identified all spacetimes in general relativity that exclusively allow closed and bounded orbits [14]; in brief, Perlick finds that only two forms of a spacetime correlate to Kepler and the harmonic oscillator. The goal of this chapter is to study this family of systems, and prove that (given a certain energy, restricting us to a level set of the Hamiltonian) they are equivalent to the classical harmonic oscillator.

4.1 Keplers Third Law for the Harmonic Oscillator

In the prior work of de Neeling, it is shown that classes of systems that preserve the LRL vector despite containing relativistic corrections to the Kepler potential exist. In the work, a functional relationship between a Hamiltonian H and two smooth functions $f(H)$ and $g(H)$ defines systems described by this H that are orbitally equivalent to the Kepler problem. This structure is reapplied here to the harmonic oscillator.

The full Hamiltonian of these systems is

$$f(H(q,p)) = \frac{p^2}{2m} - g(H(q,p))q^2, \tag{65}$$

with $f, g : \mathbb{R} \to \mathbb{R}$. The form of this functional relation is technically an ansatz; in [13], it is shown that the Kepler variant

$$f(H(q,p)) = \frac{p^2}{2m} - g(H(q,p))\frac{1}{r}$$

comes from a higher-dimensional relativistic structure, which also determined the expansions of f and g. This already looks like the Hamiltonian of a classical harmonic oscillator, just with its force constant k replaced by the function $g(H(q,p))$. As such, we can think of the systems represented here as being one slice of the phase space of harmonic oscillators, with each given value of H giving rise to a harmonic oscillator with a different force constant.

An interesting similarity with [12], where Kepler's third law is violated by taking

$$f_{kep}(H) = \frac{p^2}{2m} - \frac{g_{kep}(H)}{r}, \tag{66}$$

resulting in the orbital period of one particular mass in an orbit around a very heavy Sun as

$$T_{kep} = 2\pi\sqrt{\frac{s^3}{GMg(E)}}, \tag{67}$$

instead of the usual

$$T_{Newton} = 2\pi\sqrt{\frac{s^3}{GM}}. \tag{68}$$

The situation for a harmonic oscillator is similar; the period of oscillation is

$$T = \frac{2\pi}{\sqrt{k/m}}, \tag{69}$$

which is independent of the amplitude (read: semimajor axis s, for the Kepler case). However, if k is a function of E, we will get anharmonic behaviour - its period is then calculated as

$$T = \frac{2\pi}{\sqrt{g(H)/m}} \tag{70}$$

How, then, do we find Hamiltonians satisfying (65), while still being equivalent to a harmonic oscillator? In this section, we will prove that on an energy shell, so for a fixed $H = E$, the flows of the implicitly defined Hamiltonians and that of the harmonic oscillator are proportional to one another. In the next main section, we will prove that we do not need to restrict ourselves to an energy shell by showing that there exists a transformation that maps H back to H_{HO}, showing that this transformation consists of an energy redefinition and a canonical transformation. Important to note is that the construction in this chapter only shows the existence of the $su(3)$ Lie algebra - much like in the case of Kepler, showing this the global symmetry is also $SU(3)$ is a topic possibly explored in further research.

4.2 Proportional flows

Consider the following family of Hamiltonians $H = H_{f,g} : T^*\mathbb{R}_0^d \simeq (\mathbb{R}^d - 0) \times \mathbb{R}^d \to \mathbb{R}$, where d the dimension of physical space, implicitly defined by the functional relation

$$f(H(q,p)) = \frac{p^2}{2m} + \frac{1}{2}g(H(q,p))r^2, \tag{71}$$

where $f, g : \mathbb{R} \to \mathbb{R}$ smooth functions with a power series as

$$f(x) = x + f_1x^2 + f_2x^3 + \dots\ ,\ g(x) = 1 + g_1x + g_2x^2 + \dots\ ,\ f_i, g_i \in \mathbb{R}. \tag{72}$$

The form of these expansions are such that the rest mass term is ignored, and so that at the lowest order the original harmonic oscillator is returned. For now, we assume a solution $H(q,p)$ exists for the functions f,g as given and we will describe its properties as relating to the HO Hamiltonian. For simplicity, take a new Hamiltonian function

$$K(q,p) := f(H(q,p)), \tag{73}$$

which is by definition constant on the flow of H, since it is a function of H. If we take a value $E \in \mathbb{R}$ such that the Hamiltonian is regular, then $H^{-1}(E)$ is called an energy level. Evaluating K on one such energy level gives

$$K\Big|_{H^{-1}(E)(q,p)} = \frac{p^2}{2m} + g(H(q,p))r^2,$$

which is a harmonic oscillator with force constant $k = g(E)$. The flow generated by K on all its energy surfaces can be shown to be proportional to the flow of a HO Hamiltonian:

Theorem 4. *Let $M = T^*\mathbb{R}^d$ with symplectic form $\omega = \sum_k dp_k \wedge dq_k$. Additionally, let f,g be $C^2(\mathbb{R})$ functions, such that the Hamiltonian $H : M \to \mathbb{R}$ is defined by equation* (93)*, and that equation* (73) *holds true. Then, for any regular energy value, that is, an energy $E \in \mathbb{R}$ such that H has no singularities (i.e. "gets stuck" in phase space) we have that the Hamiltonian vector fields $X_K|_{H^{-1}(E)}$ and $X_H|_{H^{-1}(E)}$ of K and H are proportional.*

Proof:

$$X_K = -\frac{\partial K}{\partial q}\frac{\partial}{\partial p} + \frac{\partial K}{\partial p}\frac{\partial}{\partial q} = f'(H)\left(\frac{\partial H}{\partial q}\frac{\partial}{\partial p} - \frac{\partial H}{\partial p}\frac{\partial}{\partial q}\right) = f'(H)X_H, \tag{74}$$

where we used that

$$\frac{\partial K}{\partial q} = g'(H)r^2\frac{\partial H}{\partial q} + 2g(H)q \ , \ \frac{\partial K}{\partial p} = g'(H)r^2\frac{\partial H}{\partial p} + p \text{ and } f'(H) = g'(H)r^2 + 2g(H)q + p. \tag{75}$$

Hence, the vector fields are proportional, showing that the evolution of K is a "sped up" or "slowed down" evolution of H. This will come back in the section on off-shell equivalence, where we will have to allow for a time reparametrization. Additionally, letting

$$\mathcal{E} := \{E \in \mathbb{R} | f'(E) \neq 0 \text{ and } E \text{ is a regular value of } H\}$$

and

$$J : M \times \mathcal{E} \to \mathbb{R}, \quad J(q,p,E) = J_E(q,p) = \frac{p^2}{2m} + g(E)r^2 \tag{76}$$

such that

$$K(H(q,p)) = J(q,p,H(q,p)). \tag{77}$$

Then we have

$$\begin{aligned}
X_K &= -\frac{\partial J(q,p,E)}{\partial q}\frac{\partial}{\partial p} + \frac{\partial J(q,p,E)}{\partial p}\frac{\partial}{\partial q} && (78)\\
&= -\left(\frac{\partial J}{\partial q} + \frac{\partial H}{\partial q}\frac{\partial J}{\partial H}\right)\frac{\partial}{\partial p} + \left(\frac{\partial J}{\partial p} + \frac{\partial H}{\partial p}\frac{\partial J}{\partial H}\right)\frac{\partial}{\partial q} && (79)\\
&= -\left(\frac{\partial J}{\partial q}\frac{\partial}{\partial p} + \frac{\partial J}{\partial p}\frac{\partial}{\partial q}\right) + \frac{\partial J}{\partial H}\left(-\frac{\partial H}{\partial q}\frac{\partial}{\partial p} + \frac{\partial H}{\partial p}\frac{\partial}{\partial q}\right) && (80)\\
&= X_J + \frac{\partial J}{\partial H}X_H && (81)\\
&= X_J + \frac{1}{f'(H)}X_J && (82)
\end{aligned}$$

which, when evaluated on $H(q,p) = E$ gives

$$X_K|_{H^{-1}(E)} = X_{J_E}|_{H^{-1}(E)} + \frac{\frac{\partial J}{\partial E}|_{H^{-1}(E)}}{f'(E)}X_K|_{H^{-1}(E)} \tag{83}$$

rewriting gives

$$X_{J_E} = \left(1 - \frac{\frac{\partial J}{\partial E}}{f'(H)}\right)X_K, \tag{84}$$

meaning that, taken together with (74), it is proven that the evolution of the Hamiltonians satisfying the relation in (93) is equivalent to the evolution of a classical harmonic oscillator. For a given energy E, the trajectories are equivalent to that of a harmonic oscillator with an energy-dependent value of the force constant k, with the possibility of a time-rescaling.

The above statement that the above Hamiltonians are in a way equivalent to a harmonic oscillator, and that all their trajectories are bounded and closed, hints at the existence of a conserved quantity similar to that of the original harmonic oscillator on the energy levels.

4.3 The Relativistic Fradkin Tensor

An obvious choice is for this conserved quantity is the tensor

$$A_{ij}(q,p) = \frac{1}{2}g(H(q,p))(p_ip_j + q^iq^j), \tag{85}$$

since this is the classical Fradkin tensor multiplied by the new potential term. The conserved quantity is in its entirety multiplied by the potential, rather than just the part of the quantity corresponding to the potential term. This is done because the classical Fradkin tensor has on its diagonal the original Hamiltonian - something that is preserved with this proposed relativistic version.

The question is now whether this is actually conserved by Hamiltonians defined by (93);

Theorem 5. *Let* $\mathbb{E}$ *be defined as before, and* $E \in \mathcal{E}$*. On the set*

$$\{(q,p) \in H^{-1}(E) \Big| f'(E) - \frac{\partial J(q,p,E)}{\partial E} \neq 0\}$$

we have that the Hamiltonian defined in (93) *is in involution with every component of the Fradkin tensor* $A(q,p)$ *defined by* (85) *- and so they form integrals of motion of the dynamics following from* H*.*

The proof is by direct computation: Fixing $E \in \varepsilon$ and setting

$$\lambda := \left(1 - \frac{\frac{\partial J(q,p,E)}{\partial E}}{f'(E)}\right)$$

as the proportionality between the flow of $J_E(q,p)$ and $H(q,p)$, which is assumed to be regular and nonvanishing. Using the definition of the Lie derivative, we have that

$$\{A, H\} = \mathcal{L}_{X_H}(A) = \mathcal{L}_{\lambda^{-1}X_{J_E}}(A) = \lambda^{-1}\mathcal{L}_{X_{J_E}}(A) = \lambda^{-1}\{A, J_E\} \tag{86}$$

with all functions evaluated on $H^{-1}(E)$. We are now left to verify that the components of the Fradkin tensor commute with $J_E(q,p)$, from now on dropping the summation indices on q, p:

$$\begin{aligned}
\{A, J\} &= \frac{\partial A}{\partial q}\frac{\partial J_E}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial J_E}{\partial q} && (87)\\
&= \frac{\partial}{\partial q}\left(g(H)p_ip_j + g(H)q^iq^j\right)\frac{\partial J_E}{\partial p} - \frac{\partial}{\partial p}\left(g(H)p_ip_j + g(H)q^iq^j\right)\frac{\partial J_E}{\partial q} && (88)\\
&= (p_ip_j + q_iq_j)\{g(H), J_E\} + \{g(H)(p_ip_j + q_iq_j), J_E\}. && (89)
\end{aligned}$$

Now take into account that on the energy level $H^{-1}(E)$ we have that $A_{ij} = A_{ij}|_E = g(E)(p_ip_j + q_iq_j)$. This means that the second bracket vanishes as the Poisson bracket between a harmonic oscillator and is accompanying Fradkin tensor. The first bracket vanishes because we know $g(H) = g(f^{-1}(K))$, and that by the aforementioned theorem the flows of J_E and K are equal - meaning they Poisson commute. As for the bracket of the angular momentum and the new symmetry tensor, we have

$$\begin{aligned}
\left\{L_i, \bar{A}_{jk}\right\} &= \frac{\partial \bar{A}_{jk}}{\partial p}\frac{\partial L_i}{\partial q} - \frac{\partial \bar{A}_{jk}}{\partial q}\frac{\partial L_i}{\partial p} \\
&= \frac{1}{2}\left(q_j q_k g'(H)\frac{\partial H}{\partial p_s} + p_j\delta_{ks} + p_k\delta_{js}\right)(\varepsilon_{iab}p_b\delta_{as}) \\
&\quad - \frac{1}{2}\left(q_j q_k g'(H)\frac{\partial H}{\partial q_s} + g(H)q_j\delta_{ks} + g(H)q_k\delta_{js}\right)(\varepsilon_{iab}q_a\delta_{bs}) \\
&= (p_k p_n + q_k q_n g(H))\varepsilon_{ijn} + (p_j p_n + q_j q_n g(H))\varepsilon_{ikn} \\
&= \varepsilon_{ijn}\bar{A}_{nk} + \varepsilon_{ikn}\bar{A}_{jn},
\end{aligned}$$

where in going from line 3 to 4 we used that on-shell, all terms proportional to $\frac{\partial H}{\partial q/p}$ vanish; it is seen that the original Poisson bracket of the fradkin tensor A and the angular momentum vector L is the same as the Poisson bracket for the relativistic Fradkin tensor and angular momentum in (45).

A similar computation for $\left\{\bar{A}_{ij}, \bar{A}_{kl}\right\} = \{A_{ij}, A_{kl}\}$ shows that the algebra is maintained:

$$\begin{aligned}
\left\{\bar{A}_{ij}, \bar{A}_{kl}\right\} &= \frac{\partial \bar{A}_{kl}}{\partial p}\frac{\partial \bar{A}_{ij}}{\partial q} - \frac{\partial \bar{A}_{kl}}{\partial q}\frac{\partial \bar{A}_{ij}}{\partial p} \\
&= \frac{1}{2}\left(q_k q_l g'(H)\frac{\partial H}{\partial p_s} + p_k\delta_{ls} + p_l\delta_{ks} + p_k p_l g'(H)\frac{\partial H}{\partial p_s}\right) \\
&\quad \cdot \frac{1}{2}\left(q_i q_j g'(H)\frac{\partial H}{\partial q_s} + q_i g(H)\delta_{js} + q_j g(H)\delta_{is} + p_i p_j g'(H)\frac{\partial H}{\partial q_s}\right) \\
&= f'(H)^2\{H_{ho}, H_{ho}\} + f'(H)\left\{\bar{A}_{ij}, H\right\} + f'(H)\left\{\bar{A}_{kl}, H\right\} + \{A_{ij}, A_{kl}\},
\end{aligned}$$

where we take $f'(H) = \frac{\partial f}{\partial H} = \frac{1}{2}g'(H)r^2$. The first three Poisson brackets vanish trivially, and we are left with

$$\left\{\bar{A}_{ij}, \bar{A}_{kl}\right\} = \{A_{ij}, A_{kl}\} = \frac{1}{4}\left(\delta_{ik}\varepsilon_{jln} + \delta_{il}\varepsilon_{jkn} + \delta_{jk}\varepsilon_{iln} + \delta_{jl}\varepsilon_{ikn}\right)L_n, \tag{90}$$

which is equivalent to the commutation relation in (46). We thus have that all three Poisson brackets are identical to those in the standard harmonic oscillator case, and we can thus conclude that the commutation relations thus define a lie algebra (isomorphic to) $su(3)$.

4.4 Comparison to Kepler

The theorems and their proofs in this section are mostly stated and proven the same way as in [12]. The resulting conclusions are also similar - in the Kepler case, the flows of H and J_E are also proportional to one another, with the same proportionality constant; the difference lies in $\frac{\partial J}{\partial E}/f'(H)$ in (82) being different.

As alluded to before, another difference lies in the form of the "relativistic" hidden symmetry. For the Kepler case,

$$J_E(q,p) = \frac{p^2}{2m} - g(E)\frac{1}{r(q)}, \tag{91}$$

while $K(q,p)$ is defined the same way. An interesting difference is that for the Kepler case, the LRL vector

$$A(q,p) = (p \times L) - mk\frac{q}{r(q)} \tag{92}$$

is replaced with a "relativistic" LRL vector by only applying $g(H)$ to the second term in the original vector; in the harmonic oscillator case, the function $g(H)$ multiplies the *entire* tensor. The reason for this is that the tensor obeys an additional constraint that makes it have (in 3 spatial dimensions) 8 independent components, namely that the trace of the Fradkin tensor is equal to the Hamiltonian. As we will see in the next section, for the trace to be equal to the Hamiltonian of a given system, we will need the ability to have higher powers of p and q as well as mixed terms pq coming from $g(H(q,p))p_i p_j$.

In this section, we proved that for every given value of energy the family of Hamiltonians that satisfies the functional relation in (65) has a flow that is proportional to the flow of a harmonic oscillator and that each in this family possesses its own Fradkin tensor. In the next section, we will explore the solutions satisfying the same functional relation, independent of the energy levels.

5 Off-shell equivalence

The previous section proved that the Hamiltonian implicitly defined by the functional relation (93) has an additional constant of the motion and thus closed orbits when we remain on-shell. However, it does not prove whether or not it is possible to map families of these systems back to one particular harmonic oscillator. Put differently, in the next section we will attempt to generalize the on-shell (i.e. with a given value of the energy) equivalence to an approximate off-shell equivalence.

We begin again with the implicitly defined Hamiltonian:

$$f(H(q,p)) = \frac{p^2}{2m} + g(H(q,p))r^2. \tag{93}$$

The introductory section mentions 3 different methods. Here, we will explain and apply them each. They are, in order: expanding the functional relation and H as a power series; finding generating functions of the canonical transformations relating a solution H of the functional relation back to the HO; and demanding that the system described by H preserves a Fradkin tensor. We begin with an expansion of the functional relation.

5.1 Expansion of functional relation

The goal is to find solutions H to the functional relation in (93) to any post-Newtonian order, by perturbing the existing harmonic oscillator H_{HO}. To find such a solution, we expand the functional relation like

$$H + f_1 H^2 + f_2 H^3 + ... = \varepsilon \frac{p^2}{m} + (1 + g_1 H + g_2 H^2 + ...)\varepsilon r^2 \tag{94}$$

To then find a perturbative solution up to N-th order, we simply plug in the PN expansion

$$H = \sum_{i=1}^{N} \varepsilon^i H_i = H_0 + \varepsilon H_1 + \varepsilon H_2 \tag{95}$$

where ε^i is the ith PN order. We include an additional power of ε in the functional relation to get a dimensionless H. At 0PN, this is trivially solved by

$$H_0 = \varepsilon \frac{p^2}{2m} + \varepsilon r^2 \tag{96}$$

Solving order-by order after plugging in $H = H_0 + \varepsilon^2 H_1$:

$$H_0 + \varepsilon H_1 + f_1(H_0 + \varepsilon H_1)^2 + ... = \varepsilon \frac{p^2}{2m} + (1 + g_1(H_0 + \varepsilon H_1) + ...)\varepsilon r^2 \tag{97}$$

$$H_0 = \varepsilon \frac{p^2}{2m} + \varepsilon r^2 \tag{98}$$

$$\varepsilon H_1 = \varepsilon^2 (f_1 H_0^2 + g_1 r^2 H_0), \tag{99}$$

which gives, after filling in H_0 and dropping ε:

$$H_1 = f_1 p^4 + (2f_1 + g_1) p^2 r^2 + (f_1 + g_1) r^4 \tag{100}$$

The full 1PN perturbative solution to the functional relation is then

$$H = p^2 + r^2 + f_1 p^4 + (2f_1 + g_1) p^2 r^2 + (f_1 + g_1) r^4, \tag{101}$$

leading to a relation among the coefficients in

$$H = \sum_{i=0}^{\infty} \varepsilon^i \Lambda_i(h) \tag{102}$$

in the form of

$$h_{200} + h_{002} = h_{101} \tag{103}$$

Computing the same for 2PN, we find

$$H_2 = (2f_1^2 - f_2) p^6 \tag{104}$$

$$+ (6f_1^2 + g_2 - 3f_1 g_1 - 3f_2) p^4 r^2 \tag{105}$$

$$+ (6f_1^2 + 2g_1 + g_1^2 - 3f_2 - 6f_1 g_1) p^2 r^4 \tag{106}$$

$$+ (2f_1^2 + g_2 + g_1^2 - f_2 - 3f_1 g_1) r^6. \tag{107}$$

This again leads to a relation among the coefficients for the different terms, namely

$$h_{300} - h_{003} = h_{102} - h_{201} \tag{108}$$

Interesting to note is that both the 1PN and 2PN constraint have 2 free parameters; f_1 and g_1 for 1PN, and f_2 and g_2 for 2PN. We will now move on to investigate if it is possible to find a generating function G and time reparametrization τ, such that we can map solutions of 93 to the Hamiltonian of a harmonic oscillator.

5.2 Generating function

There is another perturbative approach; repeated application of canonical transformations Φ and a time reparametrization τ. This method is almost entirely a carbon-copy of theorem 8 in [12], with minor adjustments. Major changes occur only in the resultant generating functions, the real vector space used, and the form of a half-order difference in the order of these generating functions G_i. The concept of a generating function introduced in section 2.4 comes into play here - the goal is to find a generating function that creates a canonical transformation between H_{HO} and H.

Theorem 6. *For given C^∞ functions f, g, the relation (93) can be solved to at least PN order 2 by*

$$H = \Phi^* \tau(\varepsilon H_{HO}),$$

where $\tau : \mathbb{R} \to \mathbb{R}$ and $E \mapsto \tau(E)$ is also C^∞ with $\tau'(0) = 1$ defining a near-identity time reparametrisation and Φ is a near-identity canonical transformation.

We must allow for time reparametrizations due to 4; these will not influence the dynamics as a time reparametrization simply speeds up the motion, and does not change the shape of the equations of motion.

Remark 1. *The Hamiltonian employed here is dimensionless - this means its lowest order term is of order ε. A PN order n then is associated with an order ε^{n+1}. When we put back the units to return to a dimension-full Hamiltonian, this additional ε disappears. This version of the "relative kepler-PN order" is similar to that of [12] and [18], but the construction has been adapted to the harmonic oscillator.*

We begin proving this by introducing the vector spaces in which the i-th order Hamiltonian lives:

$$W_j = \left\{ (p^2)^k (pq)^l (r^2)^m \mid (k, l, m) \in \mathbb{N}^3 \ , \ k + \frac{1}{2}l + m = j + 1 \right\} \tag{109}$$

For example, the classic harmonic oscillator Hamiltonian H_{HO} is in W_0; the Hamiltonian derived in (100) lives in W_1. This vector space comes about as a result of the following requirements:

1. Only p^2, r^2 and $(p \cdot q)$ terms are allowed to show up in $SO(3)$ invariant (i.e. spherically symmetric) Hamiltonians,
2. $q^2 = |q|^2 = r(q)^2$, so the term (pq) must appear with even powers only, hence $k + l/2 + m = j + 1$,

3. the $j = 0$ case should return the lowest order.

We will only focus on those vector spaces W_j with non-negative j and even l; this will result in $F_j \in W_j$ having PN order $j - 1$, although half-order j appear in the eventual generating function. We can conclude that for $F_i \in W_i$ and $F_j \in W_j$ we have

$$\{F_i, F_j\} \in W_{i+j-1/2}, \tag{110}$$

which implies that the direct sum

$$W = \bigoplus_{j \in \mathbb{N}} W_j \tag{111}$$

is closed under the Poisson bracket. Additionally we have

$$F_i F_j \in W_{i+j+1}, \tag{112}$$

showing that W is also closed under multiplication.

The energy rescaling or time reparametrization τ in theorem 6 can be rewritten as a power series

$$\tau(E) = \sum_{n=0}^{\infty} \delta_n E^{n+1} \quad \delta_0 = 1. \tag{113}$$

To ensure that the correct PN orders are counted when applied to a Hamiltonian H, we note that given the $-1/2$ in eq (110) we need the generating function to have a half-order to given even orders in W.

The goal of solving the functional relation order by order will now be reached by applying successive canonical transformations, generated by a generating function G_i, and an energy scaling τ.

$$H := \Phi_n^* \dots \Phi_2^* \Phi_1^* \tau(\varepsilon H_{HO}) \tag{114}$$

We will consider near-identity canonical transformations obtained from the flow of a Hamiltonian vector field generated by the function G, using repeated application of adjoint operators. A function F then transforms as [12, p. 26]

$$F \mapsto \Phi^* F = \sum_{n=0}^{\infty} \frac{1}{n!} [\mathrm{ad}_G]^n F. \tag{115}$$

From 110 we see that there appears a half-order, meaning we will need to "pair up" the right order G with H_{nPN} to get the orders we are looking for. Specifically this implies that, as stated, only even l is permissible in the definition of W_j. This gives us an ansatz for the generating function as

$$G_{i+1/2} = (pq)\Lambda_i(a) \tag{116}$$

where we write $i + 1/2$ as G is composed of $\Lambda_{i-1}(a) \in W_i$ and $(pq) \in W_{1/2}$, and Λ defined as (51).

The full canonical transformation is then defined by

$$H_i = \sum_{n=0}^{i} \sum_{k=0}^{i-n} \sum_{\substack{T\in\mathbb{N}_+^k \\ |T|=i-n}} \frac{\delta_n}{k!} \mathrm{ad}_G^T(H_{HO}^{n+1}) \tag{117}$$

where we used T to indicate the ordered tuple such that $T = (t_k, ...t_2, t_1)$ with $t_1 \leq t_2... \leq t_k$, to give us all possible combinations of Poisson brackets up to an order. We then get the obvious

$$H_0 = H_{HO} \tag{118}$$

as well as

$$H_1 = (\delta_1 H_{HO}^2 + \{G_{1+1/2}, H_{HO}\}). \tag{119}$$

Proceeding with the knowledge that $G_{i+1/2}$ is in $W_{i+1/2}$ and thus that $G_{1+1/2}$ is in $W_{1+1/2}$, we know that

$$G_{1+1/2} = (pq)\Lambda_1(a) = (pq)(a_{100}p^2 + a_{010}pq + a_{001}r^2) \tag{120}$$

Carrying out the Poisson bracket we see

$$H = \delta_1(p^4 + 2p^2r^2 + r^4) + (2a_{100}p^4 - 2a_{001}r^4) \tag{121}$$
$$= (\delta_1 + 2a_{100})p^4 + 2\delta_1 p^2r^2 + (\delta_1 - 2a_{001}) \tag{122}$$

Since we are free to choose a_{100} and a_{001} we let them both be equal to some yet to be determined constant k and so

$$G_{1-1/2} = k(pq)H_{HO}. \tag{123}$$

Interestingly we find the same as before; the coefficients on p^4, r^4 and p^2r^2 are related in the same way as they are in the previous section, that is given H as

$$H = \sum_{i=0}^{\infty} \varepsilon^i \Lambda_i(h) \tag{124}$$

we again have a relation among the coefficients as

$$h_{200} + h_{002} = h_{101} \tag{125}$$

For the second order, we need to investigate the best guess for $G_{2+1/2}$; since we have additional terms in r^2 coming from the implicit Hamiltonian that we want to be eventually equal to, we make the educated guess

$$G_{2+1/2} = pq(aH_{HO}^2 + bH_{HO}r^2). \tag{126}$$

The equation for H_2 is, in accordance with (117), given by

$$H_2 = \delta_2 H_{HO}^3 + \{G_{2+1/2}, H_{HO}\} + \{G_{1+1/2}, \delta_1 H_{HO}^2\} + \frac{1}{2}\{G_{1+1/2}, \{G_{1+1/2}, H_{HO}\}\} \tag{127}$$

Calculating all terms gives

$$\begin{aligned} H_2 = &(\delta_2 + 4k\delta_1 + 8k^2 + a)p^6 &(128)\\ &+ (3\delta_2 + 4k\delta_1 + 24k^2 + a + 3b)p^4 r^2 &(129)\\ &+ (3\delta_2 - 4k\delta_1 - 24k^2 - a + 2b)p^2 r^4 &(130)\\ &+ (\delta_2 - 4k\delta_1 - 8k^2 - a - b)r^2. &(131)\end{aligned}$$

With some basic algebra we note that this too has the relation among the coefficients for the different terms is again

$$h_{300} - h_{003} = h_{102} - h_{201}, \tag{132}$$

just like the 2PN term for the Hamiltonian we calculated through the expansion of the implicit definition of the Hamiltonian in (108). We can also identify the parameters a, b, k, δ_1 and δ_2 with the parameters f_1, f_2, g_1 and g_2 from the series expansion before.

5.3 The Fradkin Tensor

In the last two sections we described a class of Hamiltonians that we have shown to be equivalent to a classical harmonic oscillator - either by being directly proportional to one (on-shell), or by an approximation through canonical transformations and a time reparametrisation. In the following section, we will instead investigate the class of relativistic Hamiltonians, given some assumptions on its form, that have the same symmetry as the harmonic oscillator? Afterwards, these Hamiltonians will be compared to that of the previous sections, to show that all Hamiltonians obeying these symmetries are then equivalent to a harmonic oscillator.

Theorem 7. *Let the family of spherically symmetric two-body Hamiltonians given by*

$$H = \frac{1}{\varepsilon}(\varepsilon H_0 + \varepsilon^2 H_1 + ...), \tag{133}$$

where

$$H_0 = \frac{1}{2m}p^2 + \frac{m\omega^2}{2}r^2 \text{ and for } i \geq 1 \ , \ H_i = \Lambda_i(h) \tag{134}$$

where $\Lambda_i(c)$ *defined as (51). At least up to order 2PN, these Hamiltonians are canonically conjugate to the harmonic oscillator, up to time reparametrisation, if and only if they conserve the tensor quantity*

$$A^k_{ij}(qp) = (\Lambda_k(\alpha))p_i p_j + (\Lambda_k(\beta))q_i q_j + (\Lambda_k(\gamma))(p_i q_j + q_i p_j), \tag{135}$$

which will be referred to as the relativistic Fradkin tensor.

Some coefficients h_{lmn} in the Hamiltonian (133) can be set to zero before we continue; the 1PN correction to the Kepler problem contains a degree of freedom that can be used to remove a term proportional to p_r^2/r - see [3, Appendix A] for details. In short, if we can write

$$H_{1PN} \to H'_{1PN} = H_{1PN} + \{X, H_0\} + ... \tag{136}$$

for some function X that generates a near-identity transformation that satisfies

$$\{X, H_0\} = -A(pr)p^2 - B(pr)r^2 = -\{H_0, X\}, \tag{137}$$

then the terms proportional to $p^3 r$ and pr^3, namely those with coefficients h_{110} and h_{011}, can be removed. Such an X can be found by considering

$$\left\{H_0, p^4\right\} = m\omega^2 r p^3 \ , \ \left\{H_0, r^4\right\} = -\frac{4}{m}pr^3, \tag{138}$$

hence the choice

$$X = -\left(\frac{A}{4m\omega^2}p^4 + \frac{Bm}{4}r^4\right) \tag{139}$$

gives us a valid near-identity (canonical) transformation that can be used to remove the odd-power terms $p^3 r$ and pr^3 at order ε. We make the assumption that this is possible at higher-than-leading order as well.

To prove the approximate equivalence, we will want to show that the Hamiltonian of a given symmetric system can be related back through a time reparametrisation and a canonical transformation to the harmonic oscillator. Given that we have divided every Hamiltonian into separate orders, we are allowed to demand that this transformation exists for every order separately. We take an ansatz for the conserved tensor as (135), and then requiring that it commutes with the Hamiltonian up to an order. This will give a set of relations between the coefficients $\alpha_{lmn}, \beta_{lmn}$ and γ_{lmn} of A_{ij} and h_{lmn} of the Hamiltonian. The ε^k-order term of the relativistic Fradkin tensor is given by

$$A_{ij}^{(k)}(\Lambda_k(\alpha))p_ip_j + (\Lambda_k(\beta))q_iq_j + (\Lambda_k(\gamma))(p_iq_j + q_ip_j), \tag{140}$$

where again we assume the Λ defined as in 51. The Hamiltonian can then be matched to the Hamiltonian constructed in the previous section. For the first non-leading order, the terms of order ε must satisfy

$$\{H, A_{ij}\} = \left\{H_0, \varepsilon A_{ij}^{(1)}\right\} + \left\{\varepsilon H_1, A_{ij}^{(0)}\right\} = 0. \tag{141}$$

We remind ourselves that we are thus working with the following:

$$\begin{aligned}
H^{(0)} &= h_{100}p^2 + h_{001}r^2 \\
H^{(1)} &= h_{200}p^4 + h_{101}p^2r^2 + h_{002}r^4 \\
A_{ij}^{(0)} &= \alpha_{000}p_ip_j + \beta_{000}q_iq_j \\
A_{ij}^{(1)} &= (\alpha_{100}p^2 + \alpha_{010}pq + \alpha_{001}r^2)p_ip_j \\
&\quad + (\beta_{100}p^2 + \beta_{010}pq + \beta_{001}r^2)q_iq_j \\
&\quad + (\gamma_{100}p^2 + \gamma_{010}pq + \gamma_{001}r^2)(p_iq_j + q_ip_j).
\end{aligned}$$

Computing the Poisson brackets, filling in $\alpha_{000} = \beta_{000} = h_{100} = h_{001} = 1$ gives the relations between the coefficients of the Fradkin tensor as

$$\begin{gathered}
\alpha_{010} + 2\gamma_{100} = 0 \ , \ -\beta_{010} - 2\gamma_{001} = 0 \\
2\gamma_{001} - \alpha_{010} = 0 \ , \ \beta_{010} - 2\gamma_{100} = 0 \\
\alpha_{001} + \gamma_{010} - \alpha_{100} = 0 \ , \ \beta_{001} - \beta_{100} - \gamma_{010} = 0 \\
\beta_{010} + 2\gamma_{001} - \alpha_{010} - 2\gamma_{100} = 0
\end{gathered}$$

and the coefficients for the Hamiltonian as

$$\beta_{100} + \gamma_{010} - \alpha_{100} + 2h_{200} - h_{101} = \beta_{001} - \alpha_{001} - \gamma_{010} + 2h_{101} - 2h_{002}.$$

This reduces to

$$\begin{gathered}
\alpha_{010} = -\beta_{010} = -2\gamma_{100} = 2\gamma_{001} \\
\beta_{001} - \beta_{100} = \gamma_{010} = -\alpha_{001} + \alpha_{100} = \gamma_{010}.
\end{gathered}$$

Subtracting the two equations featuring h_{200} and h_{002} from one another and filling in the rest of the constraints, we find that

$$\beta_{100} + \gamma_{010} - \alpha_{100} + 2h_{200} - \beta_{001} + \gamma_{010} + \alpha_{001} + 2h_{002} = 2h_{101}$$

$$\beta_{100} + \underbrace{\beta_{001} - \beta_{100}}_{\gamma_{010}} - \alpha_{100} + 2h_{200} - \beta_{001} + \underbrace{-\alpha_{001} + \alpha_{100}}_{\gamma_{010}} + \alpha_{001} + 2h_{002} = 2h_{101}$$

$$\cancel{\beta_{100}} + \cancel{\beta_{001}} - \cancel{\beta_{100}} - \cancel{\alpha_{100}} + 2h_{200} - \cancel{\beta_{001}} + -\cancel{\alpha_{001}} + \cancel{\alpha_{100}} + \cancel{\alpha_{001}} + 2h_{002} = 2h_{101}$$

This leaves us with one constraint across three variables for the Hamiltonian:

$$h_{200} + h_{002} = h_{101}. \tag{142}$$

The constraints on the Hamiltonian can then be used to create the generating function of the previous section; identifying from the canonical transformation

$$h_{200} = f_1 \ , \ h_{101} = (2f_1 + g_1) \ , \ h_{002} = (f_1 + g_1) \tag{143}$$

we see that it matches the previous constraint. Also, we see by comparing the method of generating functions and the canonical transformation that

$$\delta_1 + 2k = f_1 \ , \ 2\delta_1 = (2f_1 + g_1) \ , \ \delta_1 - 2k = (f_1 + g_1), \tag{144}$$

which also adheres to the constraint in terms of h_{lmn}. We have for the generating function then that

$$\delta_1 = (f_1 + \frac{1}{2}g_1) = \frac{1}{2}(h_{200} + h_{002}) \ , \ k = -\frac{1}{4}g_1 = -\frac{1}{4}(h_{002} - h_{200}) \tag{145}$$

giving the canonical transformation as

$$G_1 = pq\Lambda(a) \ , \ \delta_1 = \frac{1}{2}(h_{200} + h_{002}) \tag{146}$$

with coefficients a as

$$a_{100} = -\frac{1}{4}(h_{002} - h_{200}) \ , \ a_{010} = 0 \ , \ a_{001} = -\frac{1}{4}(h_{002} - h_{200}) \tag{147}$$

and this thus confirms our ansatz

$$G_1 = -\frac{1}{4}g_1 pq(H_{HO}) \ , \ g_1 = (f_1 + g_1) - (f_1) = h_{002} - h_{200}. \tag{148}$$

For the second order, the equation to be fulfilled is

$$\{H, A_ij\} = \{H_{HO}, \varepsilon^2 A^2_{ij}\} + \left\{\varepsilon H_1, \varepsilon A^{(1)}_{ij}\right\} + \left\{\varepsilon^2 H_2, A^{(0)}_{ij}\right\} \tag{149}$$

This gives the relations between all elements as

$$
\begin{aligned}
6h_{003} = {} & -6\alpha_{001}\beta_{100} + 2\alpha_{001}\gamma_{010} + 4\alpha_{001}^2 - 2\beta_{100}\gamma_{010} + 2\beta_{100}^2 - 4\alpha_{001}h_{101} \\
& - 2\alpha_{020}h_{100} - 2\alpha_{101}h_{100} + 4\beta_{100}h_{101} + 4\beta_{200}h_{100} + 3\gamma_{011}h_{100} - \gamma_{110}h_{100} + 2h_{201}
\end{aligned} \tag{150}
$$

$$
\begin{aligned}
6h_{300} = {} & -3\alpha_{001}\beta_{100} - \alpha_{001}\gamma_{010} + \alpha_{001}^2 + \beta_{100}\gamma_{010} + 2\beta_{100}^2 + 2\alpha_{001}h_{101} + \alpha_{020}h_{100} + \alpha_{101}h_{100} \\
& - 2\beta_{100}h_{101} - 2\beta_{200}h_{100} - \gamma_{110}h_{100} + 2h_{201}
\end{aligned} \tag{151}
$$

$$
\begin{aligned}
2h_{102} = {} & -\alpha_{001}\beta_{100} + \alpha_{001}\gamma_{010} + \alpha_{001}^2 - \beta_{100}\gamma_{010} - 2\alpha_{001}h_{101} - \alpha_{020}h_{100} - \alpha_{101}h_{100} + 2\beta_{100}h_{101} \\
& + 2\beta_{200}h_{100} + 3\gamma_{011}h_{100} - 2\gamma_{110}h_{100} + 2h_{201}
\end{aligned} \tag{152}
$$

Taking the combination

$$(150) - (151) - 3(152) \tag{153}$$

and dividing by 6 leaves us with the constraint

$$h_{003} - h_{300} = h_{102} - h_{201} \tag{154}$$

This does follow the same constraint as the previous two methods.

To conclude this section, a short review of our obtained results; the expansion of a Hamiltonian H satisfying the functional relation

$$f(H) = \frac{p^2}{2m} - g(H)r^2$$

has the classical symmetries of a spherically symmetric potential, namely conservation of angular momentum and energy, and additionally conserves the Fradkin tensor up to at least 2PN if the Hamiltonian is of the form

$$H = h_{100}p^2 + h_{001}r^2 + h_{200}p^4 + h_{101}p^2r^2 + h_{002}r^2 + h_{300}p^6 + h_{201}p^4r^2 + h_{102}p^2r^4 + h_{003}r^6$$

and the coefficients are related through

$$h_{101} = h_{200} + h_{002}\ ,\ h_{102} - h_{201} = h_{003} - h_{300}. \tag{155}$$

Additionally, we found the canonical transformations required to map this Hamiltonian back to the isotropic harmonic oscillator. We thus expect Hamiltonians of this form to have closed and bounded orbits, with no perihelion precession.

6 Results and Discussion

In the previous sections we derived the structure of relativistic corrections to the isotropic harmonic oscillator and identified the constraints necessary for the conservation of the hidden symmetry. We now turn to the interpretation of these results. First, we construct a 1PN Hamiltonian, and show the canonical transformation and time reparametrization required to map this Hamiltonian back to the harmonic oscillator, while preserving the relativistic Fradkin tensor by constructing them explicitly.

We then connect these results to the explicit solution to the rate of perihelion precession in the Kepler problem, finding our own measure of deviation from the dynamics of a harmonic oscillator.

Finally, we illustrate our findings by means of numerical simulations of representative relativistic Hamiltonians, confirming that the identified constraints are necessary and sufficient for the preservation of closed orbits and the equivalence of phase space structures under canonical transformations.

Supposing a Hamiltonian

$$H = p^2 + r^2 + \frac{1}{c^2}\left(h_1 p^4 + h_2 p^2 r^2 + h_3 r^4\right), \tag{156}$$

then this Hamiltonian can be canonically transformed into a harmonic oscillator

$$H_{HO} = p^2 + r^2 \tag{157}$$

if its coefficients satisfy

$$h_2 = h_1 + h_3, \tag{158}$$

where the transformation is then generated by

$$G = -\frac{1}{4}(h_3 - h_1)pq(H_{HO}) \ , \ \delta_1 = \frac{1}{2}(h_1 + h_3), \tag{159}$$

which leaves h_1 and h_3 as the two free parameters. Its conserved Fradkin tensor then is

$$A_{ij} = (1 + \Lambda_0(\alpha))p_i p_j + (1 + \Lambda_0(\beta))q_i q_j + (\Lambda_0(\gamma))(p_i q_j + q_i p_j) \tag{160}$$

with the following constraints

$$\alpha_{010} = -2\gamma_{100} = 2\gamma_{001} \ , \ \alpha_{100} - \alpha_{001} = \gamma_{010} \tag{161}$$

$$\beta_{010} = 2\gamma_{100} = -2\gamma_{001} \ , \ \beta_{001} - \beta_{100} = \gamma_{010} \tag{162}$$

Interestingly enough, this lets us set all γ terms to 0, to find

$$\alpha_{100} = \alpha_{001} \ , \ \beta_{100} = \beta_{001} \ , \ \alpha_{010} = \beta_{010} = 0 \tag{163}$$

which leaves us with

$$A^1_{ij} = (\alpha_{100} p^2 + \alpha_{001} r^2) p_i p_j + (\beta_{100} p^2 + \beta_{001} r^2) q_i q_j. \tag{164}$$

If we then take all coefficients equal, we get that, without violating any of the rules we have set out thus far, that the relativistic Fradkin tensor (RFT) is given by

$$A_{ij} = A^{0PN}_{ij} + A^{1PN}_{ij} = (1 + H_{HO}) p_i p_j + (1 + H_{HO}) q_i q_j = g(H)(p_i p_j + q_i q_j), \tag{165}$$

to be exactly equal to the results of section 4.3.

The equation for the precession per revolution for Kepler problems is given by [3]

$$\delta\phi = -n(n+1)\pi \frac{\alpha_n S^{n+1}}{GMm} \tag{166}$$

where M the large mass, m the second mass, G the gravitational constant, n the power of r appearing in the potential $V(r) = \alpha_n r^n$; S refers to the "semilatus rectum of the orbital ellipse", defined as $a(1-e^2)$. The standard result in a Schwarzschild background gives

$$\delta\phi = \frac{6\pi GM}{a(1-e^2)c^2} = \frac{6\pi M^2 \mu^2}{L^2} \ , \ L = \mu\sqrt{GMa(1-e^2)} \tag{167}$$

as the perihelion precession of a Kepler orbit with a 1PN expansion of the potential. Specifically, in the Kepler case in [12] the 1PN Hamiltonian for an Einstein-Maxwell-Dilaton theory is given

$$H_{1PN} = h_1 \frac{p^4}{4\mu^3} + h_2 \frac{\gamma}{\mu} \frac{p^2}{r} + h_3 \frac{\gamma^2}{\mu r^2}, \tag{168}$$

or in the same construction as used in this thesis,

$$H_{1PN} = c_{200} p^4 + c_{110} \frac{p}{r} + c_{020} \frac{1}{r^2}. \tag{169}$$

In this case, the linear combination of $\Delta_{kep} = h_1 + 2h_2 + h_3$ is stated to quantify the deviation away from Keplerian dynamics (i.e. having a conserved LRL vector and closed and bounded orbits). In this EMD theory, the perihelion precession is given by

$$\delta\phi_{1PN,EMD} = -\frac{2\pi\gamma^2}{L^2}\Delta, \tag{170}$$

making it evident that $\Delta = 0$ is indeed no deviation from Kepler dynamics and thus preserves closed and bounded orbits. In this case, we have the constraint on the 1PN Hamiltonian as

$$c_{020} = -2(c_{110} + 2c_{200}) \quad \text{or} \quad c_{020} + 2c_{110} + 4c_{200} = 0 \tag{171}$$

which then translates into $\Delta_{kep} = h_1 + 2h_2 + h_3$ once we take into account the coefficient of $1/4$ on the p^4 term in (168). An equation such as (167) is derived using a Binet coordinate transformation - we will come back to this in a next section.

For our situation, the constraint on a 1PN Hamiltonian is

$$h_{101} = h_{200} + h_{002} \quad \text{then} \quad h_{200} + h_{002} - h_{101} = 0. \tag{172}$$

If we write the constraint in a similar fashion as to the Kepler case we get

$$\Delta = h_{200} + h_{002} - h_{101} \tag{173}$$

as our deviation away from the dynamics of an isotropic harmonic oscillator. Looking at the de Sitter Hamiltonian we derived earlier, we find that this translates to

$$\frac{1}{8m^3} + \frac{m\Xi^2}{72} - \frac{\Xi}{12m} = 0, \tag{174}$$

which can be solved with a standard quadratic formula for any given m. This means that for the de Sitter Hamiltonian derived, in order to display the dynamics of an isotropic harmonic oscillator, we need, assuming a nonzero mass,

$$\Xi = \frac{3}{m^2} \tag{175}$$

giving integer solutions for $m = 1$ as $\Xi = 3$. Of note is that this assumes a unitless cosmological constant - the units of the cosmological constant are normally length^{-2}. While this result is likely of no significance, it is noteworthy that the cosmological constant with units is evaluated as

$$\Xi = 3\left(\frac{H_0}{c}\right)^2 \Omega_\Xi \tag{176}$$

with H_0 the Hubble constant and Ω_Ξ the density parameter for the cosmological constant.

If we want to quantify the perihelion precession like in (167), we need to solve the radial equation of motion for this perturbed potential. According to [20], for any potential $V(r) = \beta/r^n$ the perihelion precession rate is given by

$$\Delta\theta = \frac{\pi}{1-\varepsilon_n} \ , \ \varepsilon_n = \frac{mn(n+2)\beta u_0^{n-2}}{8l^2}. \tag{177}$$

For our situation this would give $n = -4$ and thus

$$\Delta\theta = \frac{\pi}{1-(\frac{m\beta u_0^{-6}}{l^2})} \quad \text{or} \quad \Delta\theta = -\frac{\pi l^2 u_0^6}{m\beta - l^2 u_0^6} \quad \text{or} \quad \Delta\theta = \frac{\pi l^2}{m\beta r_0^6 - l^2} \tag{178}$$

It is not known whether these equations are indeed correct for this system. An alternative approach would see taking an expansion of the orbit equation

$$m\ddot{r} - \frac{l^2}{mr^3} = -\frac{\partial V}{\partial r},$$

since the derivation based on the Binet orbit equation makes the differential equation nonlinear.

6.1 Simulated orbits

Everything so far has been exact - to confirm our findings, we use Mathematica[2] to plot the phase spaces of a few Hamiltonians

$$H = ap^2 + br^2 + xp^4 + yp^2r^2 + zr^4 \tag{179}$$

for varying values of the coefficients a, b, x, y, z.

As can be seen, the first plot is the classical isotropic harmonic oscillator; its phase space is the familiar circle. When we "turn on" the 1PN correction by setting

$$x, y, z \neq 0,$$

we get the second and third plot. The second plot, which *does* fulfill the previously found constraint

$$x + z = y$$

has the same phase space - the phase spaces of an isotropic harmonic oscillator and a 1PN expansion of a relativistic harmonic oscillator obeying the constraints necessary to have a conserved Fradkin tensor are identical. On the other hand, we can see that

[2]See the ancillary files [17]

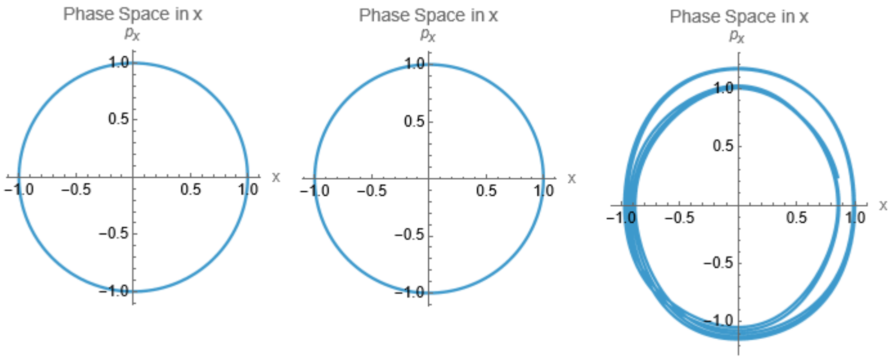

Figure 2: Phase space of several relativistic harmonic oscillators. The first is an isotropic harmonic oscillator. The second obeys the constraints. The third does not.

the phase space of a general non-constraint-obeying RHO is not the same. We can tie this in with Liouiville's theorem; the volume of phase space of the third plot is larger than the other two plots, meaning that by conservation of phase space volume under a canonical transformation, the non-obeying plot is not related through canonical transformations to the other two.

We can also see that a non-obeying oscillator does not have a closed orbit, as expected:

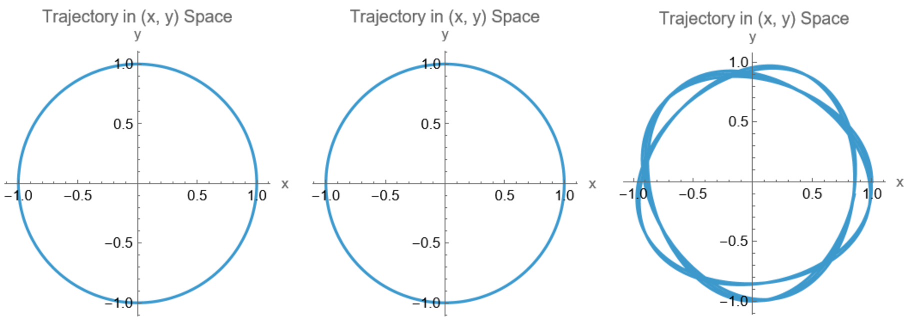

Figure 3: Plotted trajectory of two relativistic oscillators and the isotropic harmonic oscillator. The first is an isotropic harmonic oscillator. The second obeys the constraints. The third does not.

We see that the orbit of a non-conforming oscillator does not close and that the perihelion precesses - this is expected as it cannot be canonically transformed back to the HO, as proven in the previous sections.

7 Conclusions

Having introduced the PN expansion of a harmonic oscillator as an expansion in kr^2/mc^2 in section 3, we spent the past few sections employing various methods to converge onto a family of Hamiltonians

$$H = \sum_{i=0}^{\infty} \varepsilon^i \Lambda_i(h) \ , \ \Lambda_i(\alpha) = \sum_{l+m+n=i} \alpha_{lmn}(p^2)^l (pq)^m (q^2)^n \ , \ (l, m, n) \in \mathbb{N}^3 \tag{180}$$

that, up to at least 2PN, preserve the dynamics of the classical harmonic oscillator - if its coefficients obey the relations

$$h_{200} + h_{002} = h_{101} \quad \text{and} \quad h_{300} - h_{003} = h_{102} - h_{201}. \tag{181}$$

The canonical transformations to map these Hamiltonians back to the harmonic oscillator, are then given by (123) and (126) respectively for the 1PN and 2PN order.

We made no remarks throughout this thesis about anisotropic harmonic oscillators - this could be an area for further study. Additionally, we did not provide an equation for the perihelion precession of non-constraint-obeying Hamiltonians, unlike the one that exists for the Kepler problem. Both of these topics could do with further investigation.

In a broader context, this work shows that the isotropic harmonic oscillator preserves its characteristic symmetries and dynamics under appropriate relativistic corrections. By constructing post-Newtonian Hamiltonians and introducing a relativistic Fradkin tensor, we demonstrated that the hidden symmetries ensuring closed, bounded orbits persist in approximate form. Together with earlier results on the Kepler problem, this confirms that both systems singled out by Bertrand's theorem remain integrable within a relativistic framework. Given the importance of post-Newtonian Kepler dynamics in celestial mechanics and scattering amplitudes, it is natural to explore whether the relativistic harmonic oscillator could find similar applications, for instance in the de Sitter universe used as an example of a realization of a relativistic harmonic oscillator in this thesis.

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