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Multiplicative Calculus and Other Alternatives to the Derivative

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Abstract

This paper serves as an introduction to the many alternatives to calculus, constructed by changing the definition of the derivative. For each new derivative we define, we will assign new terminology and notation, discuss its intuitive meaning, find its differentiation rules, relate it to the ordinary derivative and derive a function approximation reminiscent of the tangent line. With a primary focus on the 'multiplicative derivative', we will reconstruct various other concepts from calculus like the mean value theorem, Taylor series, integration and differential equations. Finally, we will use this to show some practical applications of multiplicative calculus in Biomedical image analysis [2], using multivariable differential equations of matrix functions.

Contents

1 Introduction

Calculus is a broad field of mathematics based on studying change of functions, comparing infinitesimal changes of different variables and using it to obtain various properties about functions. This is done using the derivative, a quantity that makes us able to relate the change in one variable to that of another. How we conventionally do this is with the derivative $f'(x) := \lim_{a\to x} \frac{f(a)-f(x)}{a-x} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$, finding the quantity that multiplies dx, to get dy in essence, where dx,dy represents the additive change $x_2 - x_1$,$y_2 - y_1$ in the given variable taken to infinitesimals. With this derivative, all of calculus is constructed, including the tangent line, mean value theorem, Taylor series, integration and all the fields built upon it. However, the way we described the derivative before defining it suggest that this can be done in more than one way. As an example, we consider $dy - dx$ instead of dy/dx being our derivative, meaning it is the quantity that one adds to dx to get dy. However, this is a bad example as for any continuous function f we have $(df - dx)(x) = \lim_{h\to 0} f(x+h) - f(x) - h = f(x+0) - f(x) - 0 = 0$, making the quantity trivial. To have a differently defined derivative be not-trivial, i.e. non-constant, we need to have that simply evaluating the general expression with the limit leads to an invalid expression, since the difference between two terms that are equal at the limit is trivial when evaluated. For example, when we evaluate the conventional derivative at the limit we get $\frac{f(a)-f(x)}{a-x}\Big|_{a=x} = \frac{f(x)-f(x)}{x-x} = \frac{0}{0}$, which is undefined within normal arithmatic. Note that in the case that a non-trivial derivative is invalid for all $f \in C(\mathbb{R}), x \in \mathbb{R}$, we will not count it as a valid derivative either, since this too gives no interesting results. An example for such an expression is $\frac{1}{dy-dx}$, which always equals $\frac{1}{0}$.

There are many ways to construct a derivative with these principles in mind. In this paper we will discuss different derivative alternatives based not on changing the equation between the terms dx and dy, but changing the terms dx and dy themselves. This means changing what we mean with change in a variable. To start, consider the two numbers $x_1, x_2 \in \mathbb{R}$. When we talk about the difference between x_1 and x_2, we usually refer to the additive difference $x_2 - x_1$. However, there are more ways we can compare two quantities, notably the multiplicative difference also known as the ratio between x_1 and x_2 being x_2/x_1. This too is a valid way to measure change, as we have that $x_2 > x_1 \implies x_2/x_1 > 1$ and $x_2 < x_1 \implies x_2/x_1 < 1$ assuming x_1 is positive, similar to how $x_2 > x_1 \implies x_2 - x_1 > 0$. Note that this does have the downside that we need to be careful with the domain of the variables, since the ratio does not exist when $x_1 = 0$ and for $x_1 < 0$ the inequalities in the above implications are reversed. Next we have the natural property of preserving multiplication, since $\frac{a_2 b_2}{a_1 b_1} = \frac{a_2}{a_1}\frac{b_2}{b_1}$ similar to how the regular difference preserves addition. As for something the additive and multiplicative difference have in common is that when $x_1 = x_2$, we get the constants 0 and 1 respectively, coinciding with what we said about trivial and non-trivial derivatives. The last thing to note is that we can actually write multiplication in terms of addition, or division in terms of subtraction, by doing the following: $x_2/x_1 = e^{\ln(x_2/x_1)} = e^{\ln x_2 - \ln x_1}$. Writing this in infinitesimal form with the notation qx for the quotient difference in x, we can rewrite it as $qx = e^{d(\ln x)}$. This decomposition will come of much use in the new forms of calculus we will construct, as the terms dx and dy have already been studied in great extent, especially when of the form dy/dx. Furthermore, in this paper we will many times come to encounter the transformation $f \to \exp \circ f \circ \ln$ in which f may be a function, a derivative or something else, which always comes down to the underlying idea of turning multiplication into addition.

With this in mind, we will be defining many alternative derivatives, largely based on turning addition into multiplication. We will observe their properties, redefine the many concepts in calculus and in the end find practical applications for the first introduced derivative.

2 The multiplicative derivative

2.1 Constructing a new derivative

Let f be a real function. Conceptually, the derivative is about asking how much change in $y = f(x)$ happens for a given change in x. We know this as being defined in the following way:

$$f'(x) := \lim_{a \to x} \frac{f(a) - f(x)}{a - x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

However, there is another natural way we can measure change between two variables, the quotient. For this new derivative we will be asking by how much $y = f(x)$ *gets multiplied by* for a given addivive change in x. The first thing to note is how exponential functions compare to this notion.

Let $f(x) = ab^x$ for $a \in \mathbb{R}$ and $b > 0$. Then for $h \in \mathbb{R}$ we have

$$f(x+h) = ab^{x+h} = ab^x b^h = f(x) \cdot b^h$$

We can see that, independent of x, each additive change h in x multiplies the function's value $f(x)$ by b^h, so multiplies it 'h times'. Therefore, the multiplicative derivative of this function must be b, and with that idea we can construct our definition.

$$f^*(x) := \lim_{a \to x} \sqrt[a-x]{\frac{f(a)}{f(x)}} = \lim_{h \to 0} \sqrt[h]{\frac{f(x+h)}{f(x)}}$$

We will refer to it as the multiplicative derivative or the *-derivative, and will denote it by $f^* = \sqrt[dx]{qf} = qf^{\frac{1}{dx}}$ or $^* = \sqrt[dx]{q} = q^{\frac{1}{dx}}$, where the q stands for quotient.

Note that the root is there to take care of the power term we seen before:

$$(ab^x)^* = \lim_{h\to 0} \sqrt[h]{\frac{ab^{x+h}}{ab^x}} = \lim_{h\to 0} \sqrt[h]{b^h} = b$$

Let us now try this on a different function:

$$(x)^* = \lim_{h\to 0} \sqrt[h]{\frac{x+h}{x}} = \lim_{h\to 0} \left(1+\frac{h}{x}\right)^{\frac{1}{h}} \overset{h\to hx}{=} \lim_{h\to 0} (1+h)^{\frac{1}{hx}} = e^{\frac{1}{x}}$$

Doing this calculation for every elementary function would be cumbersome. However, with some simple computations we can rewrite the definition to get a formula that relates the multiplicative derivative to the conventional derivative.

$$\begin{aligned} f^*(x) = \lim_{h\to 0} \left[\frac{f(x+h)}{f(x)}\right]^{\frac{1}{h}} &= \lim_{h\to 0} e^{\ln\left(\left[\frac{f(x+h)}{f(x)}\right]^{\frac{1}{h}}\right)} \\ &= \lim_{h\to 0} e^{\frac{1}{h}\ln\left(\frac{f(x+h)}{f(x)}\right)} \\ &= \lim_{h\to 0} e^{\frac{\ln(f(x+h))-\ln(f(x))}{h}} \\ &= e^{(\ln\circ f)'(x)} \\ &= e^{\frac{f'(x)}{f(x)}} \end{aligned}$$

This identity will allow us to use all our knowledge from conventional calculus to multiplicative calculus. The second to last equality also lets us interpret the multiplicative derivative in the following commutative diagram:

$$\begin{array}{ccc} f(x) & \xrightarrow{\ln} & \ln(f(x)) \\ \Big\downarrow{*} & & \Big\downarrow{'} \\ f^*(x) & \xleftarrow[\exp]{} & \frac{f'(x)}{f(x)} \end{array}$$

Another thing to note is that one can rearrange the equation to get $f'(x) = f(x)\ln(f^*(x))$. This leads to the following theorem.

Theorem 1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a real function and let $x \in \mathbb{R}$. Then*

$$f \text{ is differentiable at } x \text{ and } f(x) \neq 0 \implies f \text{ is } {}^*\text{-differentiable at } x$$
$$f \text{ is } {}^*\text{-differentiable at } x \implies f \text{ is differentiable at } x$$

Note that $f^*(x) = e^{\frac{f'(x)}{f(x)}} > 0$ always holds true, meaning that $f(x)\ln(f^*(x))$ is always well-defined. Because of this requirement for $f(x) \neq 0$, it is easier to consider just positive functions, which we will from now on do. For functions with negative values, the equation $f^*(x) = e^{(\ln(f(x))'}$ must instead by written as $e^{(\ln|f(x)|)'}$, as logarithms are not defined for negative numbers. This still fits with the other ways to write the multiplicative derivative.

2.2 Multiplicative differentiation rules

Now that we know the different ways of how the multiplicative derivative is defined, let us analyze how it acts on functions. Let $a > 0$ and g, h be real *-differentiable functions. Let $a \odot b := \exp(\ln(a)\ln(b))$.

#	$f(x)$	$f^*(x)$	alternative
1	a	1	
2	a^x	a	
3	x	$e^{\frac{1}{x}}$	
4	x^a	$e^{\frac{a}{x}}$	
5	$\ln(x)$	$e^{\frac{1}{x\ln(x)}}$	
6	$\sin(x)$	$e^{\frac{1}{\tan(x)}}$	
7	$\cos(x)$	$e^{-\tan(x)}$	
8	$\tan(x)$	$e^{\frac{2}{\sin(2x)}}$	
9	$a \cdot g(x)$	$g^*(x)$	
10	$g(x) \cdot h(x)$	$g^*(x) \cdot h^*(x)$	
11	$g(x) + h(x)$	$e^{\frac{g'(x)+h'(x)}{g(x)+h(x)}}$	$f^*(x)^{\frac{f(x)}{f(x)+g(x)}} g^*(x)^{\frac{g(x)}{f(x)+g(x)}}$
12	$g(x)^a$	$g^*(x)^a$	
13	$a^{g(x)}$	$a^{g'(x)}$	$a^{g(x)\ln(g^*(x))}$
14	$g(x)^{h(x)}$	$g^*(x)^{h(x)} g(x)^{h'(x)}$	$g^*(x)^{h(x)} g(x)^{h(x)\ln(h^*(x))}$
15	$g(h(x))$	$g^*(h(x))^{h'(x)}$	$g^*(h(x))^{h(x)\ln(h^*(x))}$
16	$g(x) \odot h(x)$	$(g^*(x) \odot h(x)) \cdot (g(x) \odot h^*(x))$	

One of the most important properties of the multiplicative derivative is rule (10), as it shows that multiplication is preserved, just like how the addition is preserved under regular differentiation. Because of this, constant multiples of functions (9) do not change its multiplicative derivative. The rule for addition (11) however is much less simple, being written as $f^*(x)^{\frac{f(x)}{f(x)+g(x)}} g^*(x)^{\frac{g(x)}{f(x)+g(x)}}$ when put only in terms of *-derivatives. As for the rule for exponentiation (14), it is also simpler than the one for the regular derivative, giving rise to rule (12) telling us constant exponentiation is preserved and rule (13) telling us exponentiated functions in some sense turn the multiplicative derivative into the regular derivative.

Then we have the chain rule (15), which instead of multiplying with the inner function's respective derivative, we exponentiate with the regular derivative. This can again be expressed purely in terms of *-derivatives, but this formulation is easier for us to work with. Lastly there is one operation, $\odot$, that according to rule (16) behaves in a very similar manner to the product rule of the regular derivative. This operation is seen as the natural extension of what comes after addition and multiplication that is associative and commutative. In light of this, rule (16) makes sense given rule (10).

2.3 The tangent exponential

For the regular derivative, we naturally come to the notion of the tangent line. Using the fact that linear functions have a constant derivative and setting the constrain of $l(a) = f(a)$ and $l'(a) = f'(a)$ for a chosen point $a \in \mathbb{R}$, we get the tangent line at a:

$$l(x) = f'(a)(x - a) + f(a)$$

Just like the regular derivative, the multiplicative derivative also gives rise to an approximation, but instead of a linear function one uses an exponential function. Hence we suggest the name 'tangent exponential'. has the so called tangent exponential, a name I have coined myself. As we seen before, any function $e(x) = cb^x$ has a constant *-derivative of b. For a chosen $a \in \mathbb{R}$ and *-differentiable function f, set $e(a) = f(a)$ and $e^*(a) = f^*(a)$. Then

$$f^*(a) = e^*(a) = b, \quad f(a) = e(a) = cf^*(a)^a \implies c = f(a)f^*(a)^{-a}$$

This gives us the formula for the tangent exponential at a:

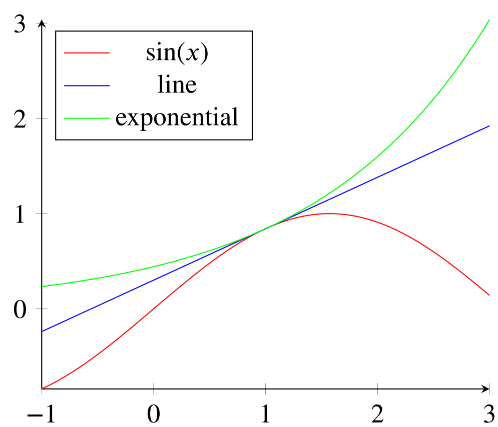
$$e(x) = f(a)f^*(a)^{x-a}$$

There are quite some similarities to be found with the two formulas. Instead of multiplying the derivation constant by $x - a$, it is here exponentiated, and instead of adding $f(a)$ to the quantity, it is multiplied to it.

Another observation is that the tangent line of the tangent exponential is the same as the tangent line of f and the same holds the other way around, assuming all approximations are centered at a and $f(a) \neq 0$. This is because

$$\begin{cases} l(a) = f(a) = e(a) \\ l^*(a) = \exp(l'(a)/l(a)) = \exp(f'(a)/f(a)) = f^*(a) = e^*(a) \\ l'(a) = f'(a) = f(a)\ln(f^*(a)) = e(a)\ln(e^*(a)) = e'(a) \end{cases}$$

With that, let us look at an example of the tangent exponential and tangent line applied to the sine function at $x = 1$.



3 The quotientive derivative

3.1 A derivative based solely on multiplication

We have seen the multiplicative derivative, which looks at the quotient difference in y, but still uses the additive difference in x. We will now take a look at what happens when we consider the quotient difference in both x and y. The question that surrounds this new derivative will be, how much $y = f(x)$ gets multiplied by for a given multiplication in x. Let us now consider power functions, as they fit nicely with this notion.

Let $f(x) = ax^b$ with $a, b \in \mathbb{R}$. For $m \in \mathbb{R}$ we have

$$f(mx) = a(mx)^b = am^b x^b = f(x) \cdot m^b$$

As we can see, independent of x, each multiplicative change in x multiplies the function's value $f(x)$ by m^b, meaning it multiplied by the quantity m the set amount of times b. So in this sense, our new derivative must have the value b, which brings us to our definition:

$$f^\circ(x) := \lim_{a \to x} \log_{\frac{a}{x}} \left(\frac{f(a)}{f(x)} \right) = \lim_{m \to 1} \log_m \left(\frac{f(mx)}{f(x)} \right)$$

We will refer to this derivative as the quotientive derivative or the $\circ$-derivative, and will denote it by $f^\circ = \log_{qx} qf$ or $\circ = \log_{qx} q$, using similar notation as previously.

Note that the logarithm is there to take out the exponent:

$$(ax^b)^\circ = \lim_{m \to 1} \log_m \left(\frac{a(mx)^b}{ax^b} \right) = \lim_{m \to 1} \log_m \left(m^b \right) = b$$

Let us now look at a different function:

$$\begin{aligned}
(e^x)^\circ = \lim_{m \to 1} \log_m \left(\frac{e^{mx}}{e^x} \right) &= \lim_{m \to 1} \log_m \left(e^{(m-1)x} \right) \\
&= \lim_{m \to 1} \frac{(m-1)x}{\ln(m)} \\
&\overset{h=\ln(m)}{=} \lim_{h \to 0} \frac{e^h - e^0}{h} x \\
&= e^0 x = x
\end{aligned}$$

Again, this formula takes a while to calculate for each function. However, the quotientive derivative also can be transformed to be expressed in terms of the regular derivative.

$$
\begin{aligned}
f^\circ(x) = \lim_{a\to x} \log_{\frac{a}{x}} \left(\frac{f(a)}{f(x)}\right) &= \lim_{a\to x} \frac{\ln\left(\frac{f(a)}{f(x)}\right)}{\ln\left(\frac{a}{x}\right)} \\
&= \lim_{a\to x} \frac{\ln(f(a)) - \ln(f(x))}{\ln(a) - \ln(x)} \\
&= \lim_{a\to x} \frac{\ln(f(a)) - \ln(f(x))}{a - x} \frac{a - x}{\ln(a) - \ln(x)} \\
&= \frac{(\ln \circ f)'(x)}{\ln'(x)} \\
&= x \frac{f'(x)}{f(x)}
\end{aligned}
$$

With this formula we can again bring all our knowledge from standard calculus to quotientive calculus. However, this time we cannot write it as a sequence of functions. We can also now write $f'(x)$ in terms of f° using the formula $f'(x) = \frac{f(x)}{x} f^\circ(x)$. Now we have the following theorem.

Theorem 2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a real function and let $x \in \mathbb{R}$. Then*

$$
\begin{aligned}
f \text{ is differentiable at } x \text{ and } f(x) \neq 0 &\implies f \text{ is }^\circ\text{-differentiable at } x \\
f \text{ is }^\circ\text{-differentiable at } x \text{ and } x \neq 0 &\implies f \text{ is differentiable at } x
\end{aligned}
$$

As we can see, for us to write $f^\circ(x)$ in terms of $f'(x)$, we again need $f(x)$ to be non-zero. Like with the multiplicative derivative, we will therefore only use positive functions f in the context of quotientive calculus, even if our definition works with negative values as well. Unlike the multiplicative derivative, the second implication requires x not to be zero, bringing restrictions onto our domain. However, we will not take it much into account, as we do not write $f'(x)$ in terms of $f^\circ(x)$ often, in many cases it is possible to evaluate the $^\circ$-derivative at $x = 0$ and since working with functions undefined at $x = 0$ is not an uncommon thing in mathematics either way. However, if we do want a function to hold both properties, we need $f(x), x \neq 0$.

3.2 Quotientive differentiation rules

We will now look at the table of the how different functions act under quotientive differentiation, making much use of the different ways to write it. Let $a > 0$ and let g, h be real $\circ$-differentiable functions.

#	$f(x)$	$f^\circ(x)$	alternative
1	a	0	
2	x^a	a	
3	e^x	x	
4	a^x	$\ln(a)x$	
5	$\ln(x)$	$\frac{1}{\ln(x)}$	
6	$\sin(x)$	$\frac{x}{\tan(x)}$	
7	$\cos(x)$	$-x\tan(x)$	
8	$\tan(x)$	$\frac{2x}{\sin(2x)}$	
9	$a \cdot g(x)$	$g^\circ(x)$	
10	$g(x) \cdot h(x)$	$g^\circ(x) + h^\circ(x)$	
11	$g(x) + h(x)$	$x\frac{f'(x)+g'(x)}{f(x)+g(x)}$	$\frac{g(x)}{g(x)+h(x)}g^\circ(x) + \frac{h(x)}{g(x)+h(x)}h^\circ(x)$
12	$g(x)^a$	$ag^\circ(x)$	
13	$a^{g(x)}$	$\ln(a)xg'(x)$	$\ln(a)g(x)g^\circ(x)$
14	$g(x)^{h(x)}$	$h(x)g^\circ(x) + xh'(x)\ln(g(x))$	$h(x)(g^\circ(x) + h^\circ(x)\ln(g(x)))$
15	$g(h(x))$	$g^\circ(h(x))h^\circ(x)$	
16	$g(x) \odot h(x)$	$g^\circ(x)\ln(h(x)) + h^\circ(x)\ln(g(x))$	$\ln\left(g(x)^{h^\circ(x)}h(x)^{g^\circ(x)}\right)$

The first thing to note is rule (10), as it states that the quotientive derivative turns multiplication turns into addition. This and rule (12) directly follow from the definition using the properties of the logarithm. As constants have a $\circ$-derivative of zero, constant multiples of the derivative (9) do not change the quotientive derivative, just like with the multiplicative derivative. Next we have the addition rule (11), which does not have a simple way to write it, but does have some resemblance to the regular product rule. As for the exponentiation rule (14), it is unfortunately not quite simple, although for one of the two being constant (12,13) the formula looks much simpler. Lastly, we have the chain rule (15), which has the outer function's $\circ$-derivative that is this time multiplied by the regular derivative of the inner function. Rule (16) also is no simple expression.

3.3 The tangent power function

Like the tangent line and the tangent exponential that we have looked at earlier, we can define something alike for the quotientive derivative called the tangent power function. As we know, any function $p(x) = cx^b$ has a constant $^\circ$-derivative of b. For a chosen $a \in \mathbb{R}$ and $^\circ$-differentiable function f, set $p(a) = f(a)$ and $p^\circ(a) = f^\circ(a)$. Then

$$f^\circ(a) = p^\circ(a) = b, \quad f(a) = p(a) = ca^{f^\circ(a)} \implies c = f(a)a^{-f^\circ(a)}$$

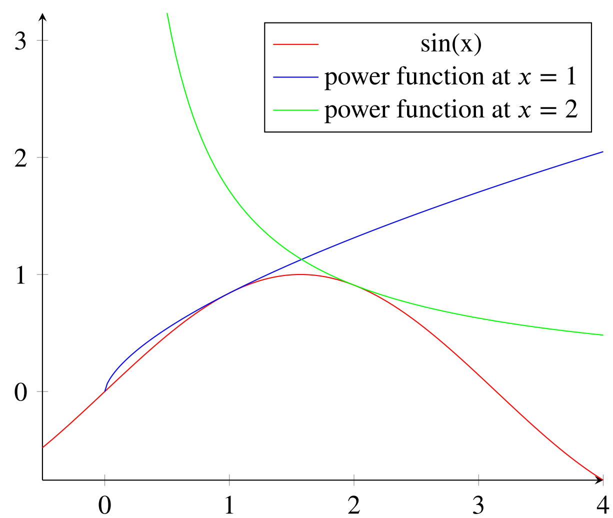
Thus, we get the following equation for the tangent power function at a:

$$p(x) = f(a)\left(\frac{x}{a}\right)^{f^\circ(a)}$$

First thing to note is that, instead of the usual $x - a$ term, we have the quotient $\frac{x}{a}$. However, it does share with the *-derivative that the term $f(a)$ is multiplied with the rest. This is most likely because the *-derivative also has the multiplicative difference in y, while only the $^\circ$-derivative has the multiplicative difference in x.

Like before, it is easy to show that the tangent power function of the tangent line is the tangent line of f and that the same holds the other way around, given all are at a and the different derivatives are well defined at a.

With that, we can take a look at example with the sine function, drawing the tangent power function at $x = 1$ and $x = 2$. For $x = 1$, we have the following.



4 The anti-multiplicative derivative

4.1 The last derivative of four

We have been looking at different combinations of derivatives that view the x or y axis either additively or multiplicatively. Now we have one combination left, which is looking at how multiplicative change in x affects additive change in y. This is similar in concept as the multiplicative derivative, only that x and y are reversed. To understand this concept more, let us look at the following family of functions:

Let $f(x) = \log_b(x) + a$ with $a \in \mathbb{R}$ and $1 \neq b > 0$. For $m \in \mathbb{R}$, we have

$$f(mx) = \log_b(mx) + a = \log_b(m) + \log_b(x) + a = f(x) + \log_b(m)$$

As we can see, independent of x, each multiplicative change in x adds $\log_b(m)$ to the function's value $f(x)$, meaning it adds how much m fits into the fixed quantity b multiplicatively. In this sense, the new derivative must here be equal to b, with which we can construct our derivative:

$$f^{\square}(x) := \lim_{a \to x} \sqrt[f(a)-f(x)]{\frac{a}{x}} = \lim_{m \to 1} \sqrt[f(mx)-f(x)]{m}$$

We will refer to this derivative as the anti-multiplicative derivative or the $^{\square}$-derivative, and will denote it by $f^{\square} = \sqrt[df]{qx} = qx^{\frac{1}{df}}$ or $^{\square} = \sqrt[d]{qx} = qx^{\frac{1}{d}}$. The reason we call it the anti-multiplicative derivative is because of the following identity using the multiplicative derivative and a bijective function f:

$$f^{\square}(x) = \lim_{a \to x} \sqrt[f(a)-f(x)]{\frac{f^{-1}(f(a))}{f^{-1}(f(x))}} = (f^{-1})^{*}(f(x))$$

Note that the root in the definition is there to take out the logarithm, for $a, b > 0$:

$$(\log_b(x) + a)^{\square} = \lim_{m \to 1} m^{\frac{1}{\log_b(mx)+a-\log_b(x)-a}} = \lim_{m \to 1} m^{\frac{1}{\log_b(m)}} = \lim_{m \to 1} m^{\log_m(b)} = b$$

Let us try out the anti-multiplicative derivative for a different function:

$$(x)^{\square} = \lim_{m \to 1} m^{\frac{1}{mx-x}} = \lim_{m \to 1} m^{\frac{1}{(m-1)x}} \overset{h=m-1}{=} \lim_{h \to 0} (1+h)^{\frac{1}{hx}} = e^{\frac{1}{x}}$$

Once more, we will look for an easier formula to express the anti-multiplicative derivative in terms of the regular derivative.

$$
\begin{aligned}
f^{\square}(x) = \lim_{a\to x} \left(\frac{a}{x}\right)^{\frac{1}{f(a)-f(x)}} &= \lim_{a\to x} e^{\ln\left(\left(\frac{a}{x}\right)^{\frac{1}{f(a)-f(x)}}\right)} \\
&= \lim_{a\to x} e^{\frac{\ln(a)-\ln(b)}{f(a)-f(x)}} \\
&= \lim_{a\to x} e^{\frac{\ln(a)-\ln(x)}{a-x}\frac{a-x}{f(a)-f(x)}} \\
&= e^{\frac{\ln'(x)}{f'(x)}} \\
&= e^{\frac{1}{xf'(x)}}
\end{aligned}
$$

This formula can now be used to figure out all the differentiation rules for the $\square$-derivative using what we know about the regular derivative. This again is not a sequence of elementary functions, but we can use the inverse identity to express it as $f^{\square} = \exp\circ\frac{d}{dx}(\ln\circ f^{-1})\circ f$. Rewriting the formula to express $f'(x)$ in terms of $f^{\square}(x)$ gives us $f'(x) = \frac{1}{x\ln(f^{\square}(x))}$. This gives us the following theorem:

Theorem 3. *Let $f : \mathbb{R} \to \mathbb{R}$ be a real function and let $x \in \mathbb{R}$. Then*

$$
\begin{aligned}
f \text{ is differentiable at } x \text{ and } f'(x), x \neq 0 &\implies f \text{ is } {}^{\square}\text{-differentiable at } x \\
f \text{ is } {}^{\square}\text{-differentiable at } x \text{ and } x \neq 0 &\implies f \text{ is differentiable at } x
\end{aligned}
$$

The first thing to note about this is that in both implications, we need x needs to be non-zero. Indeed, when we put $x = 0$ into our original definition, we see that the $\square$-derivative cannot be computed, meaning our domain can at most be $\mathbb{R}\setminus\{0\}$. For similar reasons as with the quotientive derivative, we will not limit our functions to $x > 0$. The other condition for writing $f^{\square}(x)$ in terms of $f'(x)$ is that the derivative must be non-zero. This will not affect our domain or codomain for a general function f, but does tell us that, unlike the other discussed derivatives, the anti-multiplicative derivative is undefined at stationary points. One should note here that since there is no restriction on $f(x)$, there is no need to restrict the codomain of $f(x)$, meaning $\square$-differentiable functions can be negative or zero.

4.2 Anti-multiplicative differentiation rules

Let us now look at how the anti-multiplicative derivative acts on different elementary functions and operations between functions. Let $a > 0$ and let g, h be real $^{\square}$-differentiable functions.

#	$f(x)$	$f^{\square}(x)$	alternative
1	a	undefined	
2	$\log_a(x)$	a	
5	x	$e^{\frac{1}{x}}$	
4	x^a	$e^{\frac{1}{ax^a}}$	
5	b^x	$e^{\frac{1}{\ln(b)xb^x}}$	
6	$\sin(x)$	$e^{\frac{1}{x\cos(x)}}$	
7	$\cos(x)$	$e^{-\frac{1}{x\sin(x)}}$	
8	$\tan(x)$	$e^{\frac{\cos^2(x)}{x}}$	
9	$a + g(x)$	$g^{\square}(x)$	
10	$a \cdot g(x)$	$g^{\square}(x)^{\frac{1}{a}}$	
11	$g(x)^a$	$g^{\square}(x)^{\frac{1}{ag(x)^{a-1}}}$	
12	$a^{g(x)}$	$g^{\square}(x)^{\frac{1}{\ln(a)a^{g(x)}}}$	
13	$g(x) + h(x)$	$e^{\frac{1}{x(g'(x)+h'(x))}}$	$e^{\frac{\ln(g^{\square}(x))\ln(h^{\square}(x))}{\ln(g^{\square}(x)h^{\square}(x))}}$
14	$g(x) \cdot h(x)$	$e^{\frac{1}{x(g'(x)h(x)+g(x)h'(x))}}$	$e^{\frac{\ln(g^{\square}(x))\ln(h^{\square}(x))}{\ln\left(g^{\square}(x)^{g(x)}h^{\square}(x)^{h(x)}\right)}}$
15	$g(h(x))$	$h^{\square}(x)^{\frac{1}{g'(h(x))}}$	$g^{\square}(h(x))^{h(x)\ln(h^{\square}(x))}$

As we can see here, there are not as many rules that have simple expressions, compared to the others we have looked at. The anti-multiplicative derivative has no simple rule for addition (13) nor does it have one for multiplication (14). The simplest of general rules is addition of a constant (9), as these vanish in $^{\square}$-differentiation, similar to the regular derivative. As for constant multiplication (10), the constant surprisingly returns as an inverted exponent. One thing to note about its chain rule (15) is that the alternative expression is reminiscent of the multiplicative derivative's chain rule, while the first way to write it shows us how rules (11) and (12) make sense. The nicest function to work with the $^{\square}$-derivative is of course the logarithm, as it simply returns the base's value (2). As for the worst aspect of the $^{\square}$-derivative, constants do not have a well defined $^{\square}$-derivative, meaning we cannot use it in the rules (8)-(14) to get any results. One thing to note is that some of these equations can be written slightly simpler when making use of the operators $\odot$ and $\oslash$, defined in chapters 2.2 and 7 respectively.

4.3 The tangent logarithm

The anti-multiplicative derivative also has an equivalent of the tangent line named the tangent logarithm. For this we will use the fact that any function of the form $l(x) = \log_b(x) + c$ has a constant $^\square$-derivative of b. Now for a chosen $a \in \mathbb{R} \setminus \{0\}$ and $^\square$-differentiable function f, we set $l(a) = f(a)$ and $l^\square(a) = f^\square(a)$. Then

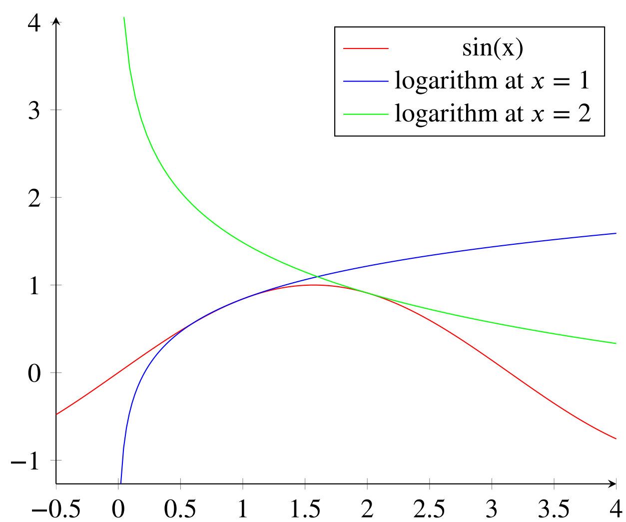
$$f^\square(a) = l^\square(a) = b, \quad f(a) = l(a) = \log_{f^\square(a)}(a) + c \implies c = f(a) - \log_{f^\square(a)}(a)$$

Thus, we get the following equation for the tangent logarithm at a:

$$l(x) = \log_{f^\square(a)}\left(\frac{x}{a}\right) + f(a)$$

Just like the *-derivative, we again have the term $\frac{x}{a}$. However unlike with the *- and $^\circ$-derivative, the term $f(a)$ is added onto the rest of the equation instead of multiplied, which more resembles the behavior of the regular derivative. The pattern that we can notice here is that derivatives defined by an additive difference in x subtract a from x, while ones with multiplicative difference in x divide x by a. In a similar sense, derivatives defined by an additive difference in y add $f(a)$ to the rest of the equation, while ones with multiplicative difference in y multiply $f(a)$ to the rest of the equation. In the next part we will explore the other ways in which the four kinds of derivatives compare.

At last, let us look at what the tangent logarithm of the sine function looks like as an example, at the points $x = 1$ and $x = 2$.



5 Comparing the calculi

We now have four kinds of calculus, created from the notion of taking the regular derivative and replacing additive differences with multiplicative differences. The following shows all four definitions along side one another, classified using this notion.

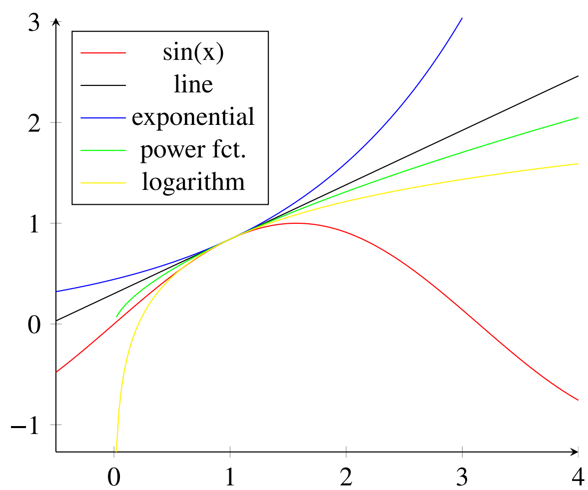
	additive in y	multiplicative in y
additive in x	regular derivative $\lim_{a\to x} \frac{f(a)-f(x)}{a-x}$	multiplicative derivative $\lim_{a\to x} \sqrt[a-x]{\frac{f(a)}{f(x)}}$
multiplicative in x	anti-multiplicative derivative $\lim_{a\to x} \sqrt[f(a)-f(x)]{\frac{a}{x}}$	quotientive derivative $\lim_{a\to x} \log_{\frac{a}{x}}\left(\frac{f(a)}{f(x)}\right)$

Here the ones that are additive in x use $\lim_{h\to 0}$, as it reduces $a-x=h$, while the ones that are multiplicative in x use $\lim_{m\to 1}$, as it reduces $\frac{a}{x}=m$. Generally, these definitions are of the form $\lim_{a\to x} r(d_1(f(a), f(x)), d_2(a,x))$ where r, d_1, d_2 are functions of the form $S \subseteq \mathbb{R}^2 \to \mathbb{R}$ with $d_1, d_2 \in \{-, /\}$.

As we have discussed in Chapter 4.3, there are quite a few similarities when it comes to the different kinds of tangential functions at $a \in \mathbb{R}$, a term not to be confused with the tangent function tan. Let us now put them along side each other, using the reference above to determine which calculus we are referring to.

	additive in y	multiplicative in y
additive in x	tangent line $f'(a)(x-a)+f(a)$	tangent exponential $f(a)f^*(a)^{x-a}$
multiplicative in x	tangent logarithm $\log_{f^{\square}(a)}\left(\frac{x}{a}\right)+f(a)$	tangent power function $f(a)\left(\frac{x}{a}\right)^{f^{\circ}(a)}$

We can see that the tangential functions all follow a distinct pattern, based on which axis is viewed additively or multiplicatively and which type of function is constant under its respective derivative. Applying all the tangential function to sin at $a=1$ gives us:



Let us now turn to ways to write the derivatives in terms of one another. We have already looked at ways in which each can be written in terms of the regular derivative and the other way around. Using those formulas, we can figure out all combinations:

write\as	$f'(x)$	$f^*(x)$	$f^\circ(x)$	$f^\square(x)$
$f'(x)$	$f'(x)$	$f(x)\ln(f^*(x))$	$f(x)\frac{f^\circ(x)}{x}$	$\frac{1}{x\ln(f^\square(x))}$
$f^*(x)$	$e^{\frac{f'(x)}{f(x)}}$	$f^*(x)$	$e^{\frac{f^\circ(x)}{x}}$	$e^{\frac{1}{xf(x)\ln(f^\square(x))}}$
$f^\circ(x)$	$x\frac{f'(x)}{f(x)}$	$x\ln(f^*(x))$	$f^\circ(x)$	$\frac{1}{f(x)\ln(f^\square(x))}$
$f^\square(x)$	$e^{\frac{1}{xf'(x)}}$	$e^{\frac{1}{xf(x)\ln(f^*(x))}}$	$e^{\frac{1}{f(x)f^\circ(x)}}$	$f^\square(x)$

The first observation is that all of these equations are written using the relevant derivative together with x and $f(x)$, using only the operations $\cdot$, $/$, exp and ln. Looking closer, there seems to be a pattern in these equations when it comes to containing the term x or $f(x)$. Whenever one changes the calculus from being additive to multiplicative in x, or the other way around, the term x appears in the equation, while if it does not change, the term is not included. In the same way, whenever one changes the calculus from being additive to multiplicative in y, or the other way around, the term $f(x)$ appears in the equation between derivatives, while if it does not change, the term is not included. For example, the *-derivative is additive in x, the $^\circ$-derivative is multiplicative in x and both are multiplicative in y. That is why $f^*(x) = e^{\frac{f^\circ(x)}{x}}$ includes the term x while not including $f(x)$. The likely reason for this is that including the term x or $y = f(x)$ in such an equation fundamentally changes the nature of how the derivative treats the respective axis, which makes sense as the terms directly correspond to the value of each axis.

6 Generalization using arbitrary bijections

We have seen that that the *-derivative can be expressed in the following way: $f^* = \exp \circ (\ln \circ f)'$. As we have seen throughout the chapters, the exponential function exp and its inverse ln come up in quite a lot of equations. In this part we will look at what happens when we replace exp with an arbitrary bijective differentiable function $\varphi : U \to V$ where $U, V \subset \mathbb{R}$ are open sets, and also replace ln with φ^{-1}. Let us start with generalizing the multiplicative derivative:

$$f^*_\varphi := \varphi \circ (\varphi^{-1} \circ f)'$$
$$f^*_\varphi(x) = \varphi\left(f'(x) \cdot (\varphi^{-1})'(f(x)\right) = \varphi\left(\frac{f'(x)}{\varphi'(\varphi^{-1}(f(x)))}\right)$$

We can also write the regular derivative in terms of this derivative, using the equation $f'(x) = \varphi'(\varphi^{-1}(f(x)))\varphi^{-1}(f^*_\varphi(x))$. As we can see, if $\varphi = \exp$, then $f^*_\varphi = f^*$, where we use that $\exp' \circ \exp^{-1} = \exp \circ \ln = \mathrm{id}$. Next, if $f = \varphi$ we get that $f^*_\varphi(x) = \varphi(1)$ and if $f = c \in \mathbb{R}$, then $f^*_\varphi = \varphi(0)$, both of which can be found to hold through either formula. The regular derivative is also part of this family of derivatives, as $\varphi = \mathrm{id}$ gives us $f^*_\varphi = f'$. Lastly, as for some non-trivial examples:

$$\varphi(x) = x^3 \, \forall x \in \mathbb{R} \text{ gives } f^*_\varphi(x) = \frac{f'(x)^3}{3f(x)^2}$$
$$\varphi(x) = \frac{1}{x} \, \forall x \neq 0 \text{ gives } f^*_\varphi(x) = -\frac{f(x)^2}{f'(x)}$$
$$\varphi(x) = ax + b \, \forall x \in \mathbb{R} \text{ gives } f^*_\varphi(x) = f'(x) + b$$
$$\varphi(x) = \ln(x) \, \forall x > 0 \text{ gives } f^*_\varphi(x) = \ln(f'(x)) + f(x)$$
$$\varphi(x) = \sin(x) \, \forall x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \text{ gives } f^*_\varphi(x) = \sin\left(\frac{f'(x)}{\sqrt{1 - f(x)^2}}\right)$$
$$\varphi(x) = \tan(x) \, \forall x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \text{ gives } f^*_\varphi(x) = \tan\left(\frac{f'(x)}{1 + f(x)^2}\right)$$

One thing what all of these have in common (with the exception for φ linear) is that they include the term $f(x)$. However, the $^\circ$- and $^\square$-derivative both include the term x in their expression in terms of the regular derivative. These derivatives cannot be of the form f^*_φ for some φ, because when you fix f and x, $\varphi'(\varphi^{-1}(f(x)))$ is purely in terms of $f(x)$ which is different from x and $f'(x)$ is also a different term from x, meaning that $f^*_\varphi(x) = \varphi\left(\frac{f'(x)}{\varphi'(\varphi^{-1}(f(x)))}\right)$ must also not include the term x. Using our knowledge from before in chapter 5, we can conclude that for non-linear functions φ, the derivative $(\cdot)^*_\varphi$ changes the nature from the regular derivative on how it treats the y axis, but does not change the way it treats the x axis.

We will now look at how we can make a derivative based on a bijection φ reminiscent of the anti-multiplicative derivative, which changes how it treats the x axis but not the y axis. Using the fact that $f^{\square} = (f^{-1})^{*} \circ f$, we will define this new derivative in terms of f^{*}_{φ}:

$$f^{\square}_{\varphi} := (f^{-1})^{*}_{\varphi} \circ f$$

$$\begin{aligned} f^{\square}_{\varphi}(x) = \varphi\left(\frac{(f^{-1})'(f(x))}{\varphi'(\varphi^{-1}(f^{-1}(f(x))))}\right) &= \varphi\left(\frac{1}{\varphi'(\varphi^{-1}(x))f'(f^{-1}(f(x)))}\right) \\ &= \varphi\left(\frac{1}{\varphi'(\varphi^{-1}(x))f'(x)}\right) \end{aligned}$$

We will now use the following alternate definition that does not require f to be invertible:

$$f^{\square}_{\varphi} = \varphi \circ \frac{(\varphi^{-1})'}{f'}$$

Again we arrive at that if $\varphi = \exp$, then $f^{\square}_{\varphi} = f^{\square}$. Next, if $f = \varphi^{-1}$, then $f^{\square}_{\varphi}(x) = \varphi(1)$ and if $f = c \in \mathbb{R}$, then $f^{\square}_{\varphi}(x)$ is undefined. This time having $\varphi = \mathrm{id}$ gives us that $f^{\square}_{\varphi} = \frac{1}{f'}$. As for some other examples, $\varphi = \ln$ gives us $f^{\square}_{\varphi}(x) = x - \ln(f'(x))$ and for $c \in \mathbb{R}$ we have $(f + c)^{\square}_{\varphi} = f^{\square}_{\varphi}$.

Using similar logic as before, $f^{\square}_{\varphi}$ is purely in terms of x and $f'(x)$, meaning that these derivatives change the nature from the regular derivative on how it treats the x axis, but not the y axis. We will now look at the generalization of the quotientive derivative, this time by simply replacing exp with φ again, as the way we derived the previous generalization was equivalent to replacing exp with φ. Therefore we have

$$f^{\circ}_{\varphi} := \frac{(\varphi^{-1} \circ f)'}{(\varphi^{-1})'}$$

$$f^{\circ}_{\varphi}(x) = \frac{(\varphi^{-1})'(f(x))f'(x)}{(\varphi^{-1})'(x)} = \frac{\varphi'(\varphi^{-1}(x))f'(x)}{\varphi'(\varphi^{-1}(f(x)))}$$

Once again, if $\varphi = \exp$ then $f^{\circ}_{\varphi} = f^{\circ}$. Next if $f = \varphi$, then $f^{\circ}_{\varphi}(x) = \varphi'(\varphi^{-1}(x))$, if $f = c \in \mathbb{R}$, then $f^{\circ}_{\varphi}(x) = 0$ and if $f = \mathrm{id}$, then $f^{\circ}_{\varphi}(x) = 1$. Lastly, for $\varphi = \mathrm{id}$ we have $f^{\circ}_{\varphi} = f'$. As for some other examples, $\varphi = \ln$ gives us $f^{\circ}_{\varphi}(x) = e^{f(x)-x}f'(x)$ and for fixed points $f(c) = c$, we have $f^{\circ}_{\varphi}(c) = f'(c)$.

7 The bigeometric derivative

We will now look at a derivative similar to the ones we have defined in chapter 2 to 4, but this time not based on some underlying idea of multiplication. We will define the bigeometric derivative is defined as follows:

$$f^{\star}(x) := \lim_{h\to 0}\left(\frac{f((h+1)x)}{f(x)}\right)^{\frac{1}{h}} = \lim_{m\to 1}\left(\frac{f(mx)}{f(x)}\right)^{\frac{1}{\ln(m)}}$$

Note that the second equality of limits holds because $\lim_{m\to 1}\frac{m-1}{\ln(m)} = 1$. The definition and name of this derivative comes from [5] and is based on slightly changing the multiplicative derivative, which they call the geometric derivative. We can rewrite this to get the following:

$$\begin{aligned} f^{\star}(x) = \lim_{m\to 1}\left(\frac{f(mx)}{f(x)}\right)^{\frac{1}{\ln(m)}} &= \lim_{m\to 1} e^{\frac{\ln(f(mx))-\ln(f(x))}{\ln(m)}} \\ &\overset{a=mx}{=} \lim_{a\to x} e^{\frac{\ln(f(a))-\ln(f(x))}{\ln(a)-\ln(x)}} \\ &= e^{\frac{(\ln\circ f)'(x)}{\ln'(x)}} \\ &= e^{x\frac{f'(x)}{f(x)}} \end{aligned}$$

Another way to define the bigeometric derivative is by looking at the regular derivative and replacing addition with multiplication and multiplication with the operation $\odot$. Note that division will be replaced with $\oslash$ defined by $a \oslash b = \exp(\ln(a)/\ln(b))$. Doing this gives

$$\lim_{a\to x}\frac{f(a)}{f(x)}\oslash\frac{a}{x} = \lim_{a\to x}\exp\left(\frac{\ln(f(a)/f(x))}{\ln(a/x)}\right) = \lim_{a\to x}\exp\left(\frac{\ln(f(a))-\ln(f(x))}{\ln(a)-\ln(x)}\right) = f^{\star}(x)$$

where the last equality comes from when when deriving the formula $f^{\star}(x) = e^{x\frac{f'(x)}{f(x)}}$.

As we can see, the bigeometric derivative is similar to the *-derivative as they have similar definitions, but also from that $f^{\star}(x) = f^*(x)^x$. The $^\circ$-derivative is quite similar too, as we have that $f^{\star}(x) = e^{f^\circ(x)}$, as well as knowing both change the nature of how the x and y axis are treated. With this, we have the following rules:

#	$f(x)$	$f^{\star}(x)$
1	a	1
2	x^a	e^a
3	a^x	a^x
4	$\ln(x)$	$e^{\frac{1}{\ln(x)}}$
5	$x \odot b$	b
6	$a \cdot g(x)$	$g^{\star}(x)$
7	$g(x) \cdot h(x)$	$g^{\star}(x) \cdot h^{\star}(x)$
8	$g(x)^a$	$g^{\star}(x)^a$
9	$a^{g(x)}$	$(a \odot g^{\star}(x))^{g(x)}$
10	$g(h(x))$	$g^{\star}(h(x)) \odot h^{\star}(x)$
11	$g(x) \odot h(x)$	$g^{\star}(x) \odot h(x) \cdot g(x) \odot h^{\star}(x)$

Note that $a > 0$, g, h are real *-differentiable functions and $a \odot b = \exp(\ln(a)\ln(b))$. As we can see, a lot of rules are the same as the ones from the *-derivative, like the rules (1), (6) to (9) and (11). Next, we note that there are many ways in which the operation $\odot$ works well with the bigeometric derivative. For example, rule (10) is like the regular chain rule, but instead of multiplication between the terms we have $\odot$. Another example is rule (5) which shows us that the kind of functions that are turned into a constant under this calculus are of the form $x \odot b = x^{\ln(b)}$, similar to how the regular derivative has $(a \cdot x)' = a$. One more thing that holds similarity to the regular derivative is the rule (3), as both have e^x unchanging under either derivatives. The only difference then is that for the regular derivative, the unchanging functions are ae^x, while for the bigeometric derivative they are of the form e^{ax}.

As for the tangential function, it will have to be of the form $m(x) = c \cdot b \odot x$. Note that $\odot$ comes before $\cdot$ in the order of operations. Now for a chosen $a \in \mathbb{R}$ and function f we set $m(a) = f(a)$ and $m^{\star}(a) = f^{\star}(a)$, giving us

$$f^{\star}(a) = m^{\star}(a) = b, \quad f(a) = m(a) = c \cdot f^{\star}(a) \odot a \implies c = \frac{f(a)}{f^{\star}(a) \odot a}$$

As $\frac{b \odot x}{b \odot a} = \frac{x^{\ln(b)}}{a^{\ln(b)}} = \left(\frac{x}{a}\right)^{\ln(b)} = b \odot \frac{x}{a}$, we can conclude that

$$m(x) = f(a) \cdot f^{\star}(a) \odot \frac{x}{a}$$

However, as $\ln(f^{\star}(x)) = f^{\circ}(x)$, we see that $m(x) = f(a) \cdot \left(\frac{x}{a}\right)^{f^{\circ}(a)}$, which is also the tangent power function from the $^{\circ}$-derivative. This makes sense, as both derivatives turn power functions into constants, only that they differ by exponentiation.

8 Exploring multiplicative calculus

8.1 The Mean Value theorem

To use what we know from classical calculus, we will first note that for a positive function f we have that

$$f'(x) = 0 \iff f^*(x) = 1$$

This can be seen directly from the formula $f^*(x) = e^{\frac{f'(x)}{f(x)}}$. Let us now turn to the following theorem.

Theorem 4. ***-Rolle's Theorem:** *Let f be positive and continuous on $[a, b]$ and *-differentiable on (a, b). If $f(a) = f(b)$, then there exists $c \in (a, b)$ such that*

$$f^*(c) = 1$$

As Theorem 1 tells us that f is also differentiable, we can use Rolle's Theorem (the classical one) to get a number $c \in (a, b)$ such that $f'(c) = 0$, meaning $f^*(c) = 1$. With this we can turn to the important theorem.

Theorem 5. ***-Mean Value Theorem:** *Let f be positive and continuous on $[a, b]$ and *-differentiable on (a, b). There exists $c \in (a, b)$ such that*

$$f^*(c) = \left(\frac{f(b)}{f(a)}\right)^{\frac{1}{b-a}}$$

Similar to the regular Mean Value Theorem, we will prove this by constructing a function to which we can apply *-Rolle's Theorem. Consider the following function.

$$F(x) := f(a)\left(\frac{f(b)}{f(a)}\right)^{\frac{x-a}{b-a}}$$

This function satisfies $F(a) = f(a)$ and $F(b) = f(b)$. Next, consider the function $G(x) = \frac{F(x)}{f(x)}$. Using that $G(a) = G(b) = 1$ and the properties of f, we can apply *-Rolle's Theorem to G to get that there exists $c \in (a, b)$ such that $G^*(c) = 1$. As multiplication is preserved under *-differentiation, we can rearrange the equation to get that $F^*(c) = f^*(c)$. At last, all we need now is to apply rule (2) and (9) for the *-derivative to get that $F^*(c)$ is equal to the constant we are looking for.

One thing to note about this proof is that $F(x)$ is the unique exponential function that coincides with f at a and b, in the same way that $\frac{f(b)-f(a)}{b-a}(x - a) + f(a)$ is the unique line with this property. Another way to think about the value $f^*(c)$ is that we can rearrange the formula to get $f^*(c)^{b-a} f(a) = f(b)$. In this sense, $b-a$ is the interval length and $f^*(c)$ is the growth factor that turns $f(a)$ into $f(b)$.

We can use this same method to find the $^\circ$-Mean Value Theorem, but not the $^\square$-Mean Value Theorem. This is because we know that $f'(x) = 0 \iff f^\circ(x) = 0$, while $f'(x) = 0 \implies f^\square(x)$ is undefined. From the equivalency we get

Theorem 6. *$\circ$-**Rolle's Theorem:** Let f be continuous on $[a, b]$ and $\circ$-differentiable on (a, b). If $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f^\circ(c) = 0$.*

With this we get our equivalent Mean Value Theorem for the $\circ$-derivative.

Theorem 7. *$\circ$-**Mean Value Theorem:** Let f be continuous on $[a, b]$ and $\circ$-differentiable on (a, b). There exists $c \in (a, b)$ such that*

$$f^\circ(c) = \log_{\frac{b}{a}}\left(\frac{f(b)}{f(a)}\right)$$

We start with the unique power function F that satisfies $F(a) = f(a)$, $F(b) = f(b)$.

$$F(x) := f(a)\left(\frac{x}{a}\right)^{\log_{\frac{b}{a}}\left(\frac{f(b)}{f(a)}\right)}$$

Next, we define $G(x) = \frac{F(x)}{f(x)}$ as it has the property $G(a) = G(b) = 1$, meaning $\circ$-Rolle's Theorem gives us a number $c \in \mathbb{R}$ with $G^\circ(c) = 0$. Using rule (10) we get that $G^\circ(c) = F^\circ(c) - f^\circ(c) = 0$, meaning $F^\circ(c) = f^\circ(c)$. Lastly, applying the $\circ$-derivative to F gives us the constant we are looking for.

8.2 The Taylor series

As we know, analytical functions can be expressed as the following sum, for x around some point $a \in \mathbb{R}$:

$$f(x) = \sum_{n=0}^{\infty} f^{(n)}(a)\frac{(x-a)^n}{n!}$$

This holds true for most elementary functions. One way to think about this series is that it works to match all the derivatives (repeated derivation) with the function f at a. To start, we observe the fact that for $n, k \in \mathbb{N}$

$$\left.\left(\frac{(x-a)^k}{k!}\right)^{(n)}\right|_{x=a} = \delta_k^n = \begin{cases} 1 & \text{if } n = k \\ 0 & \text{if } n \neq k \end{cases}$$

For $n = k$ this comes from repeated application of $\frac{d}{dx}(x-a)^k = k(x-a)^{k-1}$, as after k times it is equal to $k!$. Next, for $n < k$ we have some the term $(x-a)^{k-n}$ in our formula, which equals 0 when evaluating $x = a$. Lastly, for $n > k$ we can use the fact that after $n - k$ differentiations we get 1, which after another differentiation equals 0. Now we can see that

$$\left.\left(\sum_{k=0}^{\infty} f^{(k)}(a)\frac{(x-a)^k}{k!}\right)^{(n)}\right|_{x=a} = \sum_{k=0}^{\infty} f^{(k)}(a)\left.\left(\frac{(x-a)^k}{k!}\right)^{(n)}\right|_{x=a}$$
$$= \sum_{k=0}^{\infty} f^{(k)}(a)\cdot\delta_k^n$$
$$= f^{(n)}(a) + \sum_{k\neq n} 0 = f^{(n)}(a)$$

We will now consider the following term, using the notation $(\cdot)^{*(n)}$ for applying the *-derivative n times.

$$e_k(x) := f^{*(k)}(a)^{\frac{(x-a)^k}{k!}}$$

Using the fact that $(b^{g(x)})^* = b^{g'(x)}$ also means that $(b^{g(x)})^{*(n)} = b^{g^{(n)}(x)}$, taking the *-derivative n times and evaluating $x = a$ gets us

$$e_k^{*(n)}(a) = \left.\left(f^{*(k)}(a)^{\frac{(x-a)^k}{k!}}\right)^{*(n)}\right|_{x=a} = f^{*(k)}(a)^{\left.\left(\frac{(x-a)^k}{k!}\right)^{(n)}\right|_{x=a}}$$
$$= f^{*(k)}(a)^{\delta_k^n} = \begin{cases} f^{*(n)}(a) & \text{if } n = k \\ 1 & \text{if } n \neq k \end{cases}$$

This is quite similar to how our terms in the Taylor series act with respect to regular differentiation, except here the term is 1 if $n \neq k$ instead of 0. Now, using the fact that multiplication is preserved under *-differentiation, we arrive at the *Taylor product*:

$$\prod_{n=0}^{\infty} e_n(x) = \prod_{n=0}^{\infty} f^{*(n)}(a)^{\frac{(x-a)^n}{n!}}$$

This is the function that we are looking for, because

$$\left.\left(\prod_{k=0}^{\infty} e_k(x)\right)^{*(n)}\right|_{x=a} = \prod_{k=0}^{\infty} e_k^{*(n)}(a) = f^{*(n)}(a)\cdot\prod_{k\neq n} 1 = f^{*(n)}(a)$$

To show that this is equal to $f(x)$, we will use the fact that

$$f^{*(n)} = e^{(\ln\circ f^{*(n-1)})'} = e^{(\ln\circ e^{(\ln\circ f^{*(n-2)})'})'} = e^{(\ln\circ f^{*(n-2)})''} = \cdots = e^{(\ln\circ f)^{(n)}}$$

Plugging this into the Taylor product gives

$$\prod_{n=0}^{\infty} f^{*(n)}(a)^{\frac{(x-a)^n}{n!}} = \prod_{n=0}^{\infty} e^{(\ln\circ f)^{(n)}(a)\frac{(x-a)^n}{n!}} = \exp\left(\sum_{n=0}^{\infty}(\ln\circ f)^{(n)}(a)\frac{(x-a)^n}{n!}\right) = e^{\ln(f(x))} = f(x)$$

Here we used the fact that exponentials turn sums into products and that $\ln\circ f$ is analytic at a, which will be the necessary requirement for the Taylor product to exist.

8.3 Integration

We will be using the following notation for the multiplicative integral of a function f.

$$*\int f(x)^{dx} = F(x)$$

Here, F is understood to have the property $F^*(x) = f(x)$. The reason dx is taken to the power instead of multiplied like with the normal integral is because $\sqrt[dx]{qy} = f(x)$ can be rewritten as $qy = f(x)^{dx}$, similar to how $\frac{dy}{dx} = f(x)$ gives us $dy = f(x)\,dx$. Viewing the *-integral as the *-antiderivative, we can find how to express it using regular calculus.

$$\begin{aligned} f(x) = F^*(x) = e^{(\ln\circ F)'(x)} &\implies \ln(f(x)) = (\ln\circ F)'(x) \\ &\implies \int \ln(f(x))\,dx = \ln(F(x)) \\ &\implies F(x) = *\int f(x)^{dx} = e^{\int \ln(f(x))\,dx} \end{aligned}$$

This gives us a way to figure out all the integration rules for multiplicative calculus. The first thing to note is how the general *-antiderivative looks like. If $F' = \ln\circ f$, then $*\int f(x)^{dx} = e^{\int \ln(f(x))\,dx} = e^{F(x)+C} = e^C \cdot e^{F(x)}$ for $C \in \mathbb{R}$. In other words, if $F^* = f$ then the general form is $*\int f(x)^{dx} = C \cdot F(x)$ for $C > 0$. This can also be seen by the $*$-differentiation rule (9) saying constant multiples do not change the multiplicative derivative, just like how adding constants does not change the regular derivative. With that we can look at the following table. Note that we will only be using positive functions, so $a > 0$ and g, h are positive functions.

	$f(x)$	$*\int f(x)\,dx$
1	a	Ca^x
2	x^a	$Cx^{ax}e^{-ax}$
3	a^x	$Ca^{\frac{x^2}{2}}$
4	$\ln(x)$	undefined
5	$g(x)h(x)$	$CG(x)H(x)$
6	$g(x)^a$	$CG(x)^a$
7	$a^{g(x)}$	$Ca^{\int g(x)\,dx}$

As mentioned before, all results have an unknown constant multiplied with it to account for the different functions that $*$-differentiate to the given function. The first rule tells us that constants turn into exponential functions, which comes from the fact that the multiplicative derivative was designed to turn exponential functions into constants. Next, rule (2) gives us a hint of what rule (7) is, which states that *-integrating the exponential of a function results in the exponential of the regular integral of our function. This relates to the *-differentiation rule (13) stating that $\left(a^{g(x)}\right)^* = a^{g'(x)}$. The most important rule however is the multiplication rule (5), as multiplication too is preserved under *-integration. Another action that is preserved is raising a function to a constant power, according to rule (6), as this too is preserved under *-differentiation. One more thing to note is that according to rule (4), logarythmic functions do not have any *-antiderivative, as the integral $\int \ln(\ln(x))\,dx$ cannot be solved.

Next let us look at what the substitution rule is for the multiplicative integral. Using the *-chain rule, we have that

$$f(g(x)) = *\int f^*(g(x))^{g'(x)\,dx} \overset{u=g(x)}{=} *\int f^*(u)^{du}$$

As we chose $u = g(x)$, taking the conventional derivative gives us $du = g'(x)\,dx$, which gets us the result above. Indeed, as dx is exponentiated, the way we do the substitution rule in multiplicative calculus is essentially the same as we do in conventional calculus, except that here we do make use of the derivative of another calculus. There is the only way of doing the substitution method, as what we need is to replace dx with du for both to be a *-integral. This can only be done with the derivative comparing additive differences (d) in both u and x, which is the regular derivative $\frac{du}{dx}$.

As for 'integration by parts', there is an equivalent in multiplicative calculus, but instead of multiplication it uses the $\odot$ operation. Using the rule for *-differentiating $f(x) \odot g(x)$, we get

$$f(x) \odot g(x) = *\int \left(f^*(x) \odot g(x) \cdot f(x) \odot g^*(x)\right)^{dx}$$

Rewriting this using rule (5) gives us the equivalent of integration by parts.

$$*\int \left(f(x) \odot g^*(x)\right)^{dx} = \frac{f(x) \odot g(x)}{*\int (f^*(x) \odot g(x))^{dx}}$$

Other calculi also have their own integrals, but we will not give as much detail this time as we did for the multiplicative integral. For notation, we will use the logic used in previous parts, where we will solve the equations $\log_{qx} qy = f(x)$ and $\sqrt[dy]{qx} = g(x)$ for qy and dy respectively, putting our result as the notation coming after the integral. What we get is the following notation.

$$\circ\!\!\int qx^{f(x)} = F(x) \quad \text{and} \quad \square\!\!\int \log_{g(x)} qx = G(x)$$

where $F^\circ = f$ and $G^\square = g$. This notation may not be the simplest to read, but it is the most truthful when it pertains to the meaning of the term qx. Using the formula relating these calculi to regular calculus and solving for the function, we get

$$\circ\!\!\int qx^{f(x)} = e^{\int \frac{f(x)}{x}\,dx} \qquad \square\!\!\int \log_{g(x)} qx = \int \frac{1}{x\ln(g(x))}\,dx$$

The substitution rule for these two is interesting, as to go from qx to qu, we must use the $^\circ$-derivative in the following way. Setting $u = g(x)$, we get $\log_{qx} qu = g^\circ(x)$ meaning $qu = qx^{g^\circ(x)}$. Verifying that this coincides with the chain rules of each derivative is left as an exercise to the reader.

8.4 Definite integration and the fundamental theorem of calculus

As for the definite multiplicative integral of a positive continuous function f, we will formally define it in the following way.

$$*\int_a^b f(x)^{dx} := \lim_{n\to\infty} \prod_{i=1}^{n} f(c_i)^{x_i - x_{i-1}}$$

where for each n, we choose an increasing sequence of points $(x_i)_{i=0,\ldots,n}$ where $x_0 = a, x_n = b$ and choose $c_i \in [x_{i-1}, x_i]$. Note that this limit exists because $f(c_i)^{x_i - x_{i-1}}$ is always positive and because f is continuous. This makes sense to choose as

$$\begin{aligned} *\int_a^b f(x)^{dx} = \exp\left(\ln\left(\lim_{n\to\infty}\prod_{i=1}^{n} f(c_i)^{x_i-x_{i-1}}\right)\right) &= \exp\left(\lim_{n\to\infty}\sum_{i=1}^{n} \ln(f(c_i))(x_i - x_{i-1})\right) \\ &= \exp\left(\int_a^b \ln(f(x))\,dx\right) \end{aligned}$$

With this we will now show the fundamental theorem of *-calculus.

Theorem 8. ***Fundamental theorem of *-calculus:*** *Let f be a positive *-differentiable function and let $a, b \in \mathbb{R}$. Then*

$$*\int_a^b f^*(x)^{dx} = \frac{f(b)}{f(a)} \quad \textit{and} \quad \sqrt[dx]{q}\ *\int_a^x f(t)^{dt} = f(x)$$

We can prove both of these theorems using the transformation into regular integrals and applying the fundamental theorems of regular calculus to the function $\ln \circ f$.

$$\sqrt[dx]{q}\ *\int_a^x f(t)^{dt} = e^{\frac{d}{dx}\int_a^x \ln(f(t))\,dt} = e^{\ln(f(x))} = f(x)$$

$$*\int_a^b f^*(x)^{dx} = e^{\int_a^b \ln(f^*(x))\,dx} = e^{\int_a^b (\ln\circ f)'\,dx)} = e^{\ln(f(b)) - \ln(f(a))} = \frac{f(b)}{f(a)}$$

8.5 Generalization to higher dimensions

Let us start with the partial multiplicative derivative. Let $f : \mathbb{R}^n \to \mathbb{R}$ be positive. For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $1 \le i \le n$ we have

$$f^*_{x_i}(x_1, \dots, x_n) := \lim_{h\to 0} \sqrt[h]{\frac{f(x_1, \dots, x_i + h, \dots, x_n)}{f(x_1, \dots, x_n)}}$$

We can then rewrite this in terms of the regular derivative, the same way as we have done before.

$$f^*_{x_i}(x) = e^{\frac{f_{x_i}(x)}{f(x)}}$$

A function f now is *-differentiable if all partial *-derivatives exist. Similarly, the function $f : \mathbb{R}^n \to \mathbb{R}^m$ is *-differentiable if all partial *-derivatives $f^*_{x_i} := (f^*_{1,x_i}, \dots, f^*_{m,x_i})$ exist for $1 \le i \le n$, where f_j is the j-th component of f.

Next we will look at different class of functions: $A : \mathbb{R} \to \mathbb{R}^{n\times n}$. As these are matrices when applied at some value, we can add them and multiply them the way we do in linear algebra. We can even exponentiate these matrices using the formula $e^M := \sum_{k=0}^{\infty} \frac{M^k}{k!}$ for $M \in \mathbb{R}^{n\times n}$. We will now define the multiplicative derivative of a function $A : \mathbb{R} \to \mathbb{R}^{n\times n}$ in the following way.

$$A^*(t) := e^{A'(t)A^{-1}(t)}$$

Here, $A'(t) := (a'_{ij}(t))$ and $A^{-1}(t)$ is the inverse of $A(t)$. As we can see, not all matrix functions have a well defined *-derivative, as not all matrix functions A are differentiable. But besides needing a_{ij} to be differentiable for all i, j, we also have the condition that $A(t)$ must be invertible for all t. This condition is equivalent to the operator ln being well defined, the inverse of exp, which will make us able to write the *-derivative as $A^*(t) = e^{(\ln\circ A)'(t)}$. Furthermore, being invertible is also equivalent to another condition, that being having all eigenvalues be non-zero. Since the function is differentiable, it is also continuous, meaning the eigenvalues change continuously when t changes. Therefore, we cannot have an eigenvalue of a matrix function A change sign, as that would require it to equal 0 for some t, making $A(t)$ not invertible. Knowing this, we will now add the requirement for A to be positive definite, for similar reasons that we set *-differentiable functions $f : \mathbb{R} \to \mathbb{R}$ to be positive. It is not a necessary condition, but it makes the math easier while not having us miss any important results, given that the negative numbers are not much different to the positive numbers aside from a sign.

There are some articles that instead define the multiplicative derivative as $A'(t)A^{-1}(t)$, which gets rid of the exponential. We however choose this because it coincides with the *-derivative we have defined at the start of the paper. Another thing to note is that, since matrices generally do not commute, the terms $A'(t)A^{-1}(t)$, $A^{-1}(t)A'(t)$ and $(\ln\circ A)'(t)$ are generally not equal for $n > 1$. Therefore we simply choose for $A'(t)A^{-1}(t)$ to be the term in the exponent.

Now we can also define the multiplicative integral of a matrix function. Let $A : \mathbb{R} \to \mathbb{R}^{n\times n}$ be a matrix function with $A(t)$ being positive definite for all $t \in \mathbb{R}$. Then

$$*\int_a^b A(t)^{dt} := \lim_{n\to\infty} \prod_{i=1}^{m} A(c_i)^{x_i - x_{i-1}}$$

where for each m we have an increasing sequence $(x_i)_{i=0,\ldots,m}$ with $x_0 = a, x_m = b$ and numbers $c_i \in [x_i - x_{i-1}]$. Note that for $M \in \mathbb{R}^{n\times n}$ positive definite and $r > 0$, we define $M^r := e^{r \ln M}$. Also note that since matrices generally do not commute, we define the product as $\prod_{i=1}^{m} M_i := M_1 M_2 \cdots M_m$ for $M_i \in \mathbb{R}^{n\times n}$.

9 Multiplicative differential equations

9.1 The ordinary case

Now that we know about integrals in different calculi, we can use them to solve differential equations. We will start with the following, with f being a continuous and positive function.

$$y^* = f(x) \implies y = C \, *\!\int f(x)^{dx} = Ce^{\int \ln(f(x))\, dx}$$

Note that solving this is equivalent to solving this ordinary differential equation $y' = \ln(f(x)) \cdot y$. Next, let us look at the equation describing functions that do not change under the *-derivative.

$$\begin{aligned} y^* = y &\implies \frac{y'}{y} = \ln y \\ &\implies \frac{y'}{y \ln y} = 1 \\ &\implies \int \frac{1}{y \ln y}\, dy = \int dx \\ &\overset{u=\ln y}{\implies} \int \frac{1}{u}\, du = \ln |u| = x + C \\ &\implies u = \ln y = \pm e^{x+C} = Ae^x \\ &\implies y = e^{Ae^x} = B^{e^x} \end{aligned}$$

for $A, C \in \mathbb{R}, B > 0$. Our results coincides with things we know about exponentiation like $\left(a^{g(x)}\right)^* = a^{g'(x)}$ and $(g(x)^a)^* = g^*(x)^a$ for $a > 0$ and g positive and *-differentiable.

In a similar manner we can compute

$$
\begin{aligned}
y^{\circ} = y &\implies y = -\frac{1}{\ln|x| + C_1} = -\frac{1}{\ln|C_2 x|} \\
y^{\square} = y &\implies y = \frac{\ln x + C_1}{W\left(\frac{\ln x + C_1}{e}\right)} \text{ where } W(ae^a) := a \\
y^{\star} = y &\implies y = e^{C_1 x} = C_2^x \\
y^{*}_{\varphi} = y &\implies y = \varphi\left(C_1 e^x\right)
\end{aligned}
$$

for $C_1 \in \mathbb{R}, C_2 > 0$. As we can see, some calculi have simpler functions that are preserved under its derivative than others, with $^{\square}$-calculus needing non-elementary functions to solve the equation.

Let us now turn to second degree equations. For $A, B, C > 0$ we have

$$
\begin{cases}
y^{**} = A \\
y^{*}(0) = B \\
y(0) = C
\end{cases}
\implies y = \exp\left(\frac{1}{2}ax^2 + bx + c\right) = A^{\frac{1}{2}x^2} B^x C
$$

where $a = \ln A, b = \ln B, c = \ln C \in \mathbb{R}$. This can be shown by simply applying the multiplicative integral twice and then plugging in the initial conditions. One thing to note is the fact that the differential equation $y'' = a, y'(0) = b, y = c$ has the same solution as the above, but without the exponentiation.

In regular calculus there is a way to define the sine function using the following initial value problem.

$$
y'' = -y, \quad y(0) = 0, \quad y'(0) = 1
$$

Replacing the derivative with the *-derivative gives us no solution, as *-differentiable functions cannot be zero at any point. Furthermore, if we want y to be positive, the equation cannot be solved as y^{**} is positive while $-y$ is negative. Instead, we will modify the *-initial value problem in a way that more fits the world of multiplication.

$$
y^{**} = y^{-1}, \quad y(0) = 1, \quad y^{*}(0) = e
$$

Here I turned additive inverse $(-)$ into multiplicative inverse $(\cdot^{-1})$ of y, and changed the initial conditions by exponentiating them. To solve this, we will first solve the following first degree equation. For $k \in \mathbb{C}$, we have

$$
\begin{aligned}
y^{*} = y^k &\implies \frac{y'}{y} = k \ln y \\
&\implies \int \frac{1}{y \ln y}\, dy = \int k\, dx \\
&\implies \ln|\ln y| = kx + C_1 \\
&\implies y = e^{Ce^{kx}} \quad \text{for } C \in \mathbb{R}.
\end{aligned}
$$

Plugging this y into out equation gives us

$$y^{**} = (y^k)^* = (y^*)^k = y^{k^2} = y^{-1}$$

This gives us that $k^2 = -1$, meaning $k = \pm 1$. As multiplication is preserved under the *-derivative, we can simply multiply the two solutions to get the following general solution.

$$y = e^{C_1 e^{ix}} \cdot e^{C_2 e^{-ix}} = e^{C_1 e^{ix} + C_2 e^{-ix}}$$

where $C_1, C_2 \in \mathbb{R}$. To verify, we can do the following calculation.

$$y^{**} = \left[e^{C_1 e^{ix} + C_2 e^{-ix}}\right]^{**} = e^{\left[C_1 e^{ix} + C_2 e^{-ix}\right]''} = e^{-C_1 e^{ix} - C_2 e^{-ix}} = y^{-1}$$

Next, we can plug in the initial values to find C_1 and C_2.

$$\begin{aligned} y(0) = e^{C_1 + C_2} = 1 &\implies C_1 = -C_2 \\ y^*(0) = e^{C_1 i - C_2 i} = e &\implies C_2 = -\frac{1}{2i} \\ &\implies C_1 = \frac{1}{2i} \end{aligned}$$

Thus, the solution to the initial value problem.

$$y = e^{\frac{1}{2i} e^{ix} - \frac{1}{2i} e^{-ix}} = e^{\frac{e^{ix} - e^{-ix}}{2i}} = e^{\sin x}$$

Thus, the multiplicative calculus version of the sine function is the exponential of the sign function. Like before, we have found a way to turn an initial value problem into a *-initial value problem, and getting the exponentiated result as solution. We can formalize and prove this in the following way.

Consider the following kind of initial value problem of degree $n > 0$.

$$y^{(n)} + k_{n-1}(x) y^{(n-1)} + \cdots + k_1(x) y' + k_0(x) y = r(x), \quad y(x_0) = a_0, \quad \ldots \quad y^{(n)}(x_0) = a_n$$

where $r, k_0, \ldots, k_{n-1} : \mathbb{R} \to \mathbb{R}$ continuous, $x_0 \in \mathbb{R}$ and $a_0, \ldots, a_n > 0$. To turn this into a multiplicative initial value problem, we will the variable $\hat{y} = e^y$. Now we get the following.

$$\begin{aligned} &\iff (\ln \hat{y})^{(n)} + k_{n-1}(x)(\ln \hat{y})^{(n-1)} + \cdots + k_1(x)(\ln \hat{y})' + k_0(x) \ln \hat{y} = r(x) \\ &\iff \ln \hat{y}^{*(n)} + k_{n-1}(x) \ln \hat{y}^{*(n-1)} + \cdots + k_1(x) \ln \hat{y}^* + k_0(x) \ln \hat{y} = r(x) \\ &\iff \hat{y}^{*(n)} \cdot \left(\hat{y}^{*(n-1)}\right)^{k_{n-1}(x)} \cdots \left(\hat{y}^*\right)^{k_1(x)} \cdot \hat{y}^{k_0(x)} = e^{r(x)} \\ &\quad y(x_0) = a_0, \ldots, y^{(n)}(x_0) = a_n \quad \iff \quad \hat{y}(x_0) = e^{a_0}, \ldots, \hat{y}^{*(n)}(x_0) = e^{a_n} \end{aligned}$$

Here, we substitute $y = \ln \hat{y}$ into the differential equation, use identity $f^{*(k)} = e^{(\ln \circ f)^{(k)}}$ and lastly exponentiate all the equations. Finally, we have that if $y = g(x)$ is the solution to the initial value problem, then $\hat{y} = e^{g(x)}$ is the solution to the *-initial value problem that we derived to be equivalent. This method is also how the *-initial value problems discussed earlier are related to certain regular initial value problems. As for another example, consider $y^* = y$ with $y(0) = y_0 > 0$. We have shown before that the general solution is $y = e^{Ce^x}$, which we can solve for C with the initial condition to get $y = e^{\ln(y_0)e^x} = y_0^{e^x}$. We can also arrive at the solution by looking at the problem $y' = y$ with $y(0) = y_0$, rewriting the equation to $y' - y = 0$, substituting $y \to \ln y$ to get $(\ln y)' - \ln y = 0$ and then exponentiating to end up with $y^* y^{-1} = 1$ gives us the *-differential equation. Since the solution to the regular problem is $y = y_0 e^x$ and the initial condition also turns from y_0 to $\ln y_0$, we get the solution to the multiplicative problem is indeed the exponentiation of $\ln(y_0)e^x$.

9.2 Applications in higher dimensions

In Chapter 8.5 we have defined the multiplicative derivative and integral of a continuous function $A : \mathbb{R} \to \mathbb{R}^{n\times n}$ with $A(t)$ differentiable and positive definite for all $t \in \mathbb{R}$. We will now look at an application of multiplicative calculus in biomedical image analysis described in [2], which makes use of multiplicative differential matrix equations.

To start we will look at an initial value problem that is multivariable in the domain, but single variable in the codomain. With $u : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^+$ being *-differentiable and f continuous, consider the following.

$$\begin{cases} u_t^* = \Delta^* u \\ u(x,0) = f(x) \quad \text{for } x \in \mathbb{R} \end{cases}$$

where $\Delta^* = \exp \circ \Delta \circ \ln$, with $\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$. Rewriting this we get

$$\Delta^* u = u^*_{x_1 x_1} \cdots u^*_{x_n x_n}$$

We can solve the initial value problem by relating it to the known problem of the heat equation. Consider $y_t' = \Delta y$ with $y(x,0) = g(x)\,\forall x \in \mathbb{R}^n$. From theory, we know that the solution to this is $(\Phi_t * g)(x)$ for $x \in \mathbb{R}^n$, where $\Phi_t(x) := \frac{1}{\sqrt{4\pi t}^n} \exp\left(-\frac{||x||^2}{4t}\right)$ and $*$ is convolution between two functions. Letting $y = \ln u$ and exponentiating the equation gets us our original problem.

$$u_t^* = \exp(\ln u)_t' = \exp \Delta \ln u = \Delta^* u, \quad \ln u(x,0) = g(x) = \ln f(x)$$

Thus, the solution to our *-initial value problem is

$$u(x,t) = \exp(\Phi_t * \ln f)(x)$$

With this we will move to something from physics. Consider the variables $v = (v^1, \dots, v^n)$ and $x = (x^1, \dots, x^n)$ in $\mathbb{R}^n$ as velocity and position respectively and let L be the velocity gradient tensor defined by

$$L^\alpha_\beta := \frac{\partial v^\alpha}{\partial x^\beta} \text{ for } \alpha, \beta = 1, \dots, n$$

Viewing $\dot{x}^\alpha = \frac{\partial x^\alpha}{\partial t}$, we note that $d\dot{x}^\alpha = dv^\alpha$. Applying the chain rule to this and writing with Einstein summation, it follows that

$$d\dot{x}^\alpha = dv^\alpha \frac{\partial v^\alpha}{\partial x^\beta} dx^\beta = L^\alpha_\beta dx^\beta$$

Next, consider the variable $X \in \mathbb{R}^n$ as the position of the material at the starting point t_0, while x is considered position of the same material at a point $t \geq t_0$, dependent on X. To relate the two, we use the so called deformation tensor field F, defined by

$$F^\alpha_i = \frac{\partial x^\alpha}{\partial X^i} \text{ for } \alpha, i = 1, \dots, n$$

.

Combining all that gets us the following differential equation between matrices.

$$\dot{F} = \left(\frac{\partial}{\partial t}\frac{\partial x^\alpha}{\partial X^i}\right)_{\alpha,i} = \left(\frac{\partial \dot{x}^\alpha}{\partial X^i}\right)_{\alpha,i} = \left(L^\alpha_\beta \frac{\partial x^\beta}{\partial X^i}\right)_{\alpha,i} = \left(L^\alpha_\beta F^\beta_i\right)_{\alpha,i} = LF$$

To solve for F, we can treat the equation $\dot{F} = LF$ as a differential matrix equation for the variable t, giving it the initial value condition of $F(t_0) = I$, since at $t = t_0$ we have $x = X$ by definition, making the partial derivatives $\frac{\partial x^\alpha}{\partial X^i} = \delta^\alpha_i$. When we solve $\dot{F} = LF$ with the initial condition $F(t_0) = I$, we get

$$F(t) = *\!\!\int_{t_0}^{t} \exp(L(\tau))^{d\tau}$$

As we can see, we could also have phrased the problem in the context of multiplicative calculus.

$$F^* = \exp(L), \quad F(t_0) = I$$

The first thing to note about the solution is that if $L(t) = L_0 \in \mathbb{R}^{n\times n}$ is constant, then $F(t) = \exp\big((t - t_0)L_0\big)$ is our solution. However, in practice this is rarely the case, so this is generally not a good approximation. Next thing to note is that if $\text{tr}L = \text{div}v = 0$, then taking the determinant gives

$$\det F(t) = *\!\!\int_{t_0}^{t} \det \exp(L(\tau))^{d\tau} = *\!\!\int_{t_0}^{t} \exp(\text{tr}L(\tau))^{d\tau} = *\!\!\int_{t_0}^{t} \exp(0)^{d\tau} = 1$$

where we use the continuity of det and that $\det AB = \det A \det B$ for exchanging the integral with det, and that $\det \exp(A) = \exp(\text{tr}A)$, for $A, B \in \mathbb{R}^{n\times n}$. Thus, a divergence free velocity field F preserves volumes.

Going back to the solution of our multiplicative initial value problem of matrices, we can use this in practice to compute F with a discrete version of the matrix *-integral we defined in Chapter 8.5, using regular intervals for some chosen m to make $x_i - x_{i-1}$ constant. This problem can then be applied to Lagrangian strain analysis of Myocardium, which studies muscle movements in parts of the heart, by finding F using L and then calculating the Lagrangian strain tensor field $E = \frac{1}{2}(F^T F - I)$. The article going into more detail about this [2] in fact applies this method with the multiplicative integral on real 2-dimensional datasets.

We will now look at a different application, one that will make use of the heat equation. To start we will define the so called diffusion tensor image $X : \mathbb{R}^n \to \mathbb{S}_n^+$, where $\mathbb{S}_n^+$ is the set of symmetric positive definite matrices in $\mathbb{R}^{n\times n}$. Since these matrices are invertible, there is a function $X^{\text{inv}} : \mathbb{R}^n \to \mathbb{S}_n^+$ such that $(XX^{\text{inv}})(x) = (X^{\text{inv}}X)(x) = I$ for $x \in \mathbb{R}^n$. The set of these kinds of functions is $C^\omega(\mathbb{R}^n, \mathbb{S}_n^+)$, i.e. the set of analytical functions $X : \mathbb{R}^n \to \mathbb{S}_n^+$.

Now we will introduce the time variable to X using the following blurring operator:

$$\mathcal{F} : C^\omega(\mathbb{R}^n, \mathbb{S}_n^+) \times \mathbb{R}_{\geq 0} \to C^\omega(\mathbb{R}^n, \mathbb{S}_n^+), (X, t) \mapsto \mathcal{F}(X, t)$$

Logically, for $X \in C^\omega(\mathbb{R}^n, \mathbb{S}_n^+), t \geq 0$ we will define $\mathcal{F}(X, 0) = X$ and for short hand notation write $\mathcal{F}(X, t) \equiv X_t$. Next, consider the set $\mathbb{S}_n$ of symmetric matrices in $\mathbb{R}^{n\times n}$. Taking the matrix exponential of a symmetric matrix grants us a symmetric matrix that is also positive definite, meaning $\exp : \mathbb{S}_n \to \mathbb{S}_n^+$ is well defined and so is the unique inverse $\ln : \mathbb{S}_n^+ \to \mathbb{S}_n$, making them both bijective.

There is an important property that $\mathcal{F}$ can have called the closing property, which states that for (X, t) in the domain,

$$\mathcal{F}(X, t)^{\text{inv}} = \mathcal{F}(X^{\text{inv}}, t)$$

This property is important because it means that the blurring operator essentially preserves the inverse, making $\mathcal{F}(X, t)$ and $\mathcal{F}(X^{\text{inv}}, t)$ inverses for any $t \geq 0$, which is a useful thing to have in the context of multiplicative calculus. One such operator turns out to be an expression reminiscent to what we have seen earlier:

$$\mathcal{F}(X, t) = \exp(\Phi_t * \ln X)$$

Note that Φ_t is as defined earlier and the convolution is with respect to a vector, returning a matrix. We can show that the property is preserved in the following way.

$$\mathcal{F}(X, t)^{\text{inv}} = \exp(\Phi_t * \ln X)^{\text{inv}} = \exp(-\Phi_t * \ln X) = \exp(\Phi_t * \ln X^{\text{inv}}) = \mathcal{F}(X^{\text{inv}}, t)$$

This comes from the fact that $(\exp A)^{\text{inv}} = \exp(-A)$ and $\ln B^{\text{inv}} = -\ln B$ for $A \in \mathbb{S}_n, B \in \mathbb{S}_n^+$, which can easily be shown by definition. As we can see, $X_t = \exp(\Phi_t * \ln X)$ is quite similar to the solution we found to the multiplicative heat equation from earlier, only here being expressed in terms of matrix functions.

For this purpose we will use an alternative definition for the matrix derivative from what we used in Chapter 8.5, namely $X^* := \exp(\ln X)'$ as opposed to $\exp(X'X^{\text{inv}})$ for $n = 1$. Similarly, we can define the derivative of X w.r.t. x_i as $X_i^* = \partial_i^* X := \exp(\partial_i^* \ln X)$. With that, we define the multiplicative derivative of X_t with respect to x^i as

$$\partial_i^* X_t := \exp(\partial_i \Phi_t * \ln X)$$

With that we can write the heat equation that has $U = X_t$ as the solution.

$$\partial_t^* U = \partial_{11}^* U \cdots \partial_{nn}^* U, \quad U(x, 0) = X(x) \; \forall x \in \mathbb{R}^n$$

The way we can use this for practical purposes is that with a given data set $X : W \subset \mathbb{R}^n \to \mathbb{S}_+^n$, we add random small pertubations to the matrices to get $\tilde{X}$, and then apply the blurring operator $\mathcal{F}$ coming from the multiplicative heat equation of matrices by some chosen value $t > 0$. Note that $W \subset \mathbb{R}^n$ is usually a finite grid of points, for example $W = \{0, \dots, k\}^n$. The result of this, $\tilde{X}_t$, is a regularized version of $\tilde{X}$, giving us something likely reminiscent of the original X. In a sense, this is similar to how computers turn images into noise and then use algorithms to recover the image, to save data. The paper [2] also provides an example of this application for $n = 2$, showing X, $\tilde{X}$ and $\tilde{X}_t$ with the matrices visualized as ellipses based on its eigenvectors and eigenvalues. They then put down fixed points and found the shortest path between them when viewing X as a velocity field to travel through. There, $\tilde{X}$ took very different paths to X, while $\tilde{X}_t$ took ones more similar to X, being somewhat shorter compared to X.

10 Conclusion and Discussion

When we look at all the derivatives we have discussed, comparing their respective differentiation rules, tangent lines, integrals and more, the overall most useful derivative remains to be the regular derivative $\frac{d}{dx}$. This is because it preserves addition, has a simple chain rule, does not add restrictions on the domain or codomain, gives the approximation at a point by a simple line, does not require the use of other derivatives to keep the formulas concise and more. However, this does not mean that it is the best option in all aspects. The multiplicative derivative has the advantage that it preserves multiplication, has an alternative Taylor Series and has been shown to have applications which make it preferable to use over the regular derivative. Next, the quotientive derivative has multiplication turn into addition, has a chain rule reminiscent to the regular derivative and has the an integral substitution rule making no use of other derivatives. As for the anti-multiplicative derivative, there are no clear advantages to using it aside from its relation to the multiplicative derivative. Lastly we have the bigeometric derivative, which preserves multiplication and behaves well with the $\odot$ operator. Overall, the multiplicative derivative is seen as the most preferred alternative to the derivative, as x remains additive like the regular derivative and since $\exp\left(\frac{d}{dx} \ln \circ f\right)$ is a simple formula to move between the two derivatives.

There are many more ways to define the derivative that may hold all kinds of advantages. For example we can take some bijective real function φ to generate new derivatives as described in Chapter 7, finding out what properties they hold. Another interesting option is to use the operator $\oslash_n$ defined by $a \oslash_n b = \exp^n\left(\ln^n(a)/\ln^n(b)\right)$ for $n \in \mathbb{Z}$, where $\oslash_1 = \oslash$, $\oslash_0 = /$ and $\oslash_{-1} = -$, based on the commutative and distributive operator $\odot_n$ defined with multiplication instead. We can then construct derivatives of the form $\lim_{a \to x} \left(f(a) \oslash_{n-1} f(x)\right) \oslash_n \left(a \oslash_{n-1} x\right)$, which gives the regular derivative for $n = 0$, the bigeometric derivative for $n = 1$, and for other n cannot be expressed using dx or qx, instead using that $x_2 \oslash_n x_1$ gives us a new way to quantify change for each n. Lastly, there is the option to write each of our found derivatives as a discrete derivative which instead of taking the limit, takes it at a specific point, for example $f(x+1) - f(x)$ being a discrete regular derivative.

Aside from looking at different derivatives, we can also explore the ones we already have more deeply. To start, we could look at systems of differential equations like the Lotka-Voltra equations, rewriting them to more suit the *-derivative or whichever derivative we are using. As for matrix *-differential equations, there remains a lot which we could further develop [3], as we have already seen it having applications in Biomedical image analysis [2]. Another thing we could do is expand our definition to the complex numbers, figuring out what properties emerge, finding the *-Cauchy-Riemann conditions and finding an equivalent to line integrals to rederive many results from complex analysis [4]. There are also parts of multiplicative calculus that we have not yet expressed in another calculus, like the Jacobian, directional derivatives and surface integrals. Next, in statistics we often take the logarithmic derivative $\ln(\cdot)'$ of expressions in order to turn multiplication into addition, which the quotientive derivative does as well. Lastly there is the important task of finding practical applications to our alternative calculi in the fields discussed above.

The study of alternative calculi is a largely underdeveloped subject, as many of its uses can be rewritten in regular calculus, making it more difficult to see when it is worth investigating whether using an alternative derivative is worth the effort of rewriting already established theory. However, we believe that there many places in which using an alternative calculus will be more useful, as for theoretical use we can make use of advantageous properties like preserving multiplication, and for practical uses we can make use of instances where calculating some result using the definition of some alternative derivative or integral is faster to compute than to use the definitions of regular calculus. We thus encourage further research to be done in developing the theory and finding applications for the subject of alternative calculi.

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