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Essential Self-Adjointness Of Singular Magnetic Laplace Beltrami Operator On Singular Riemannian Manifolds

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Contents

Acknowledgements **3**

Page

Abstract **5**

1 Introduction **6**
1.1 Relevance of our work 6
1.2 Our Problem 8
1.3 Remaining open problems 9

2 Preliminaries **10**
2.1 Functional Analysis 10
2.1.1 Unbounded Operators 10
2.1.2 Essential Self-Adjointness 12
2.1.3 Sobolev Spaces 13
2.1.4 Quadratic Forms 14
2.2 Riemmanian Geometry 15
2.2.1 Riemannian manifolds and Metrics 15
2.2.2 Volume Form 16
2.2.3 Derivatives, Covariant Derivatives and Connections 17
2.2.4 Fundamental Theorems 19
2.3 Laplace-Beltrami 20

3 Setup and Main Results **23**
3.1 Hodge-Helmholtz Decomposition 23
3.2 Setup 25
3.3 Main Results 26

4 Intermediate Results **28**

5 Magnetic Singularity **30**

6 Metric Singularity **36**

7 Overlapping Singularities **42**

Bibliography **47**

Abstract

Essential self-adjointness is a fundamental global property of a differential operator which guarantees that the quantum system has a unique time evolution without requiring boundary conditions. Physically this is denoted as quantum confinement, and it is still an active area of research. This thesis examines the quantum geometric magnetic confinement problem on a singular Riemannian manifold. Specifically, we analyze the conditions required to achieve essential self-adjointness for a magnetic Laplace-Beltrami operator in the presence of singular metric and magnetic boundaries where the corresponding geometric quantities are degenerate.

To determine the criteria for confinement, we examine two distinct cases: first, when the metric and magnetic boundaries are separated by a strictly positive distance, and second, when the two boundaries completely coincide. Our analysis depends on a major assumption by transforming the magnetic potential into a divergence-free form using the Hodge-Helmholtz decomposition theorem. Lower bounds on the spectral norm of the magnetic field singularity alongside the effective potentials generated by the metric smooth measure, establish criteria for confinement. These criteria define sufficient bounds where the singular magnetic field or the metric dominates the system, ensuring that quantum particle are confined within the manifold.

1 Introduction

1.1 Relevance of our work

Our problem originates from the study of Schroedinger equation to describe motions of electrons. In a quantum system, the time evolution of a wave function $\psi \in L^2(\mathbb{R}^m)$ is governed by the Schroedinger equation

$$i\partial_t \psi = H\psi \quad \text{for} \quad \psi \in L^2(\mathbb{R}^m),$$

where $H = \Delta + V$ the Schroedinger operator under the existence of 'electric' potential $V \in L^2(\mathbb{R}^m)$. Hermann Weyl realized that a mathematical treatment of such system requires analysis of the domains of these differential operators. More precisely, Weyl investigated whether an unbounded operator admits a unique self-adjoint realization, providing the base of modern theory of operator extensions.

Mathematically this is known as essential self-adjointness. The essential self-adjointness of an operator, is precisely that it admits a unique self adjoint extension, without imposing boundary conditions and giving a precise description of the domain of the extended operator. Physically this means no interaction with the outside environment. Different boundary conditions lead to different extensions and different domains of each extension. For unbounded operators, as H such extensions may fail. Self-adjointness and symmetry is also not always guaranteed, however when this property holds true the domain of the adjoint operator is much larger than the original.

We will introduce a simple example of the above using the Laplace operator. The Laplace operator was introduced by Pierre-Simon Laplace while investigating the motion of planets in the solar system. The operator describes various physical phenomena and quantum mechanics.

In the Euclidean case for domains $\Omega \subset \mathbb{R}^m$ with coordinates $(x_1, \cdots, x_m)$ the Laplacian is given by

$$\Delta u = \sum_{i=1}^{m} \frac{\partial^2 u}{\partial x_i^2},$$

for $u : \mathbb{R}^m \to \mathbb{C}$ at least twice differentiable. Let e.g. $\Omega := (0,1)^2 \subset \mathbb{R}^2$, the Laplacian is

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},$$

assuming here the domain $\mathcal{D}(\Delta) = C_c^\infty(\Omega)$. One can easily show by double integration by parts that our operator Δ is symmetric in $L^2(\Omega)$, that is for any $u, v \in C_c^\infty(\Omega)$

$$\langle \Delta u, v \rangle = \langle u, \Delta v \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product. Our operator here is not self-adjoint since the domain of the adjoint operator is larger. However, the closure of $C_c^\infty(\Omega)$ is dense subspace of $\mathcal{D}(\Delta^*) := H^2(\Omega)$ with Δ^* the adjoint found in [1, Chapter 2.2,2.5] and $H^2(\Omega)$ as in Definition 2.11. In this case if we impose two different boundary conditions, e.g.

$$u(x,y) = 0 \quad \text{for} \quad (x,y) \in \partial\Omega := \{x = 0, x = 1, y = 0, y = 1\}$$

or

$$\partial u(x,y) = 0 \quad \text{for} \quad (x,y) \in \partial\Omega := \{x = 0, x = 1, y = 0, y = 1\}$$

we can obtain two different self-adjoint extensions of the original operator, hence Δ on $L^2(\Omega)$ is not essentially self adjoint. If we consider Δ in $L^2(\mathbb{R}^2)$ and take the closure of $C_c^\infty(\mathbb{R}^2)$ in $\mathbb{R}^2$ with the L^2-norm as in [1, Example 3.3] implies essential self-adjointness.

From physical prespective, the uniqueness translates to absolute quantum confinement. In this work our primary focus is on the geometric and magnetic confinement. For this reason, we restrict our study to the case where electric potential is absent. This does not limit the relevance of our results, because our system is described in the setting where, geometric and magnetic structures force confinement. One can easily extend our results to further confinement, where $V \neq 0$ by combining it with established results in [2].

Expanding beyond classical Euclidean spaces to display singular potentials the mathematical critetia become more complex. The foundational criteria for magnetic Schroedinger operators with singular potentials in Euclidean domains were established by Kato and Simon [3],[4]. Herbert Leinfelder and Christian G. Simader [5] completed their foundation providing alternative proof for essential self-adjointness of magnetic Schroedinger introducing cut-off arguments and detailed constrains regarding the divergence and the underlying magnetic potential.

For bounded Euclidean domains without external electric potentials, Yves Colin de Verdiere and with Francoise Truc [6] proved the essential self-adjointness of magnetic type Schrodinger operators. They utilized Agmon-type Estimates, a localization method developed Gheorghe Nenciu and Irina Nenciu [7],[8] for bounded domains of $\mathbb{R}^m$ which turned out to be applicable in different structures like Riemannian manifolds. They relied on conditions as the semi-boundedness of the associated quadratic form and the explosion rate of the spectral norm to the associated magnetic field, depending on the distance from the magnetic boundary. Shubin [9] subsequently adapted cut-off techniques to establish related results on non-compact, complete Riemannian manifolds.

On the other side, for non-magnetic Schrodinger type operators especially in the Riemannian case, many results were developed the previous decades. Our work bridges these developments. We combine the frameworks of singular magnetic potential from [6] with geometric confinement strategies developed in [2] for non complete Riemmanian manifolds. Curvature criteria for essential self-adjointeness for the Riemannian case together with results for the spectrum of the operator that were established in [2], can be adapted to magnetic type operators. This is an interesting challenge, since the magnetic field alters curvature based criteria and the adaptation is non trivial. It will allow to quantify whether the magnetic field compete or cooperate against geometric barriers to ensure quantum confinement.

We will illustrate a simple example for a further understanding of magnetic type operators. For a charged particle in a magnetic field , the Schroedinger operator associated with the magnetic vector potential $A : \mathbb{R}^3 \to \mathbb{R}^3$ is

$$H = (i\nabla + A)^2,$$

with core domain $C_c^\infty(\mathbb{R}^3)$. For f, g in the core domain we have

$$\langle f, Hg\rangle = \langle (i\nabla + A)f, (i\nabla + A)g\rangle,$$

which shows that H is symmetric as an operator in $C_c^\infty(\mathbb{R}^3)$. Expanding the squares gives

$$H = -\Delta + 2iA\cdot\nabla + (\nabla\cdot A) + A^2.$$

By [1, Example 7.10.] where the constrains of [5] were used, one can conclude by the boundedness of the terms involving the magnetic potential A that H is essentially self-adjoint in $C_c^\infty(\mathbb{R}^3)$.

1.2 Our Problem

We consider a Riemannian manifold (M,g) of an arbitrary dimension m with a magnetic field B. The main purpose of this is work to study essential self-adjointness of magnetic Laplace-Beltrami operator Δ_ω^A in $L^2(M)$, where the strength of the magnetic field tends to infinity as we approach the magnetic boundary and for which the metric also blows up as we approach the metric boundary. The magnetic $\Sigma_A \subset M$ and metric $\Sigma_g \subset M$ boundaries are the regions where the associated magnetic potential A and metric g are degenerate.

A magnetic potential $A \in \Omega^1(M)$ is a real valued 1-form. We assume that A is the divergence free magnetic potential to our associated magnetic field $B = dA \in \Omega^2(M)$ and that is smooth on $M \setminus \Sigma_A$. The assumption of divergence free magnetic potential is not a restrictive attempt but requires certain constrains that will be discussed in Section 3.1.

Our manifold is equipped with a measure ω that is smooth on $M \setminus \Sigma_g$. We always assume that our manifold $\hat{M} = M \setminus (\Sigma_A \cup \Sigma_g)$ consists of compact connected components, metric and magnetic boundaries.

We are checking if there are unique self adjoint extensions of our Laplace-Beltrami operator Δ_ω^A depending on the rate of explosion of the metric g and the magnetic potential A in two distinct cases. Firstly in the case where the two singularities are distinct, that is $\exists\, c > 0$ such that $\forall p \in \Sigma_g$ and $\forall q \in \Sigma_A$ we have that $d_g(p,q) \geq c$ to check how the two singularities affect the self adjoint extension studied independently, and lastly when the two of them coincide which is $\Sigma_g = \Sigma_A$.

We will denote by $\delta : \hat{M} \to [0,+\infty)$ and $\rho : \hat{M} \to [0,+\infty)$ the distance from the metric Σ_g and magnetic Σ_A boundary respectively given by

$$\delta(p) := \inf\{d_g(p,q) | q \in \Sigma_g\} \qquad \rho(p) := \inf\{d_g(p,q) | q \in \Sigma_A\}$$

which satisfy the following condition

$$\begin{aligned} \exists : \varepsilon_1, \varepsilon_2 > 0 \quad & \text{such that } \delta \text{ is } C^2 \text{ on } M_{\varepsilon_1} := \{0 < \delta \leq \varepsilon_1\} \\ & \text{and that } \rho \text{ is } C^2 \text{ on } M_{\varepsilon_2} := \{0 < \rho \leq \varepsilon_2\}. \end{aligned} \tag{C}$$

Under this assumption we obtain that $\exists C^1$ diffeomorphism, $M_{\varepsilon_1} \cong (0,\varepsilon_1] \times X_{\varepsilon_1}$ where M_{ε_1} is a tubular neighborhood of Σ_g and $X_\varepsilon = \{\delta = \varepsilon\}$ a C^2 embedded hypersurface with $\delta(t,x) = t$ with (t,x) the coordinates of the identification above. Similarly for the distance function ρ with condition (C) also imposed on the tubular neighborhood of Σ_A but in this case we do not obtain the above diffeomorphism for the tubular neighborhood M_{ε_2}.

We will prove in this work, that when the the spectral norm of the associated magnetic field explodes with a rate of $(1+\varepsilon_2/)\rho^2$ and the effective potential, a geometric-magnetic quantity depending on the metric and the magnetic potential, with a rate of $1/\delta^2 - \lambda/\delta$ then we achieve a unique self adjoint extensions of Δ_ω^A for the case when the singularities are distinct. Finally, for the coincident case a rate of explosion $C_a/\delta^a + C_b/\delta^b - \lambda/\delta$ under certain restrictions for the constants a,b,C_a,C_b is necessary to achieve the same result.

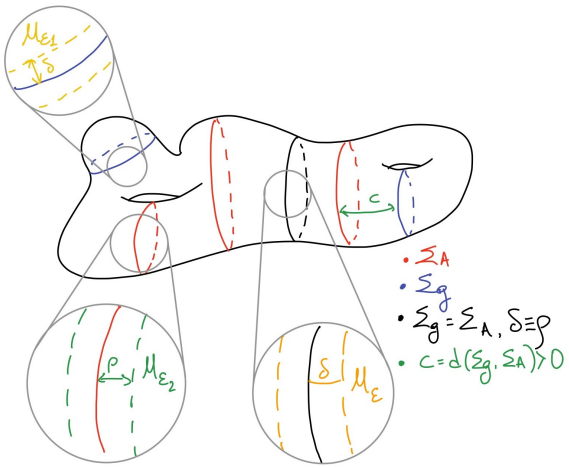

Figure 1: Structure with the Singularities

1.3 Remaining open problems

There are several open questions which are not answered in our work.

- The main open question is, which are the sufficient conditions to achieve essential self-adjointness in the case where the two singularities are transversal, that is $\Sigma_g \perp \Sigma_A$. The main issue is the definition of the tubular neighborhood close to the crossing points of $\Sigma_A \cap \Sigma_g$ where it is unclear how the tubular neighborhood should be constructed.
- The current result is specific for Almost Riemannian surface, it will be also interesting to study the same magnetic type operator on general non-complete Almost Riemannian structures [10] and for magnetic sub-Laplacians [11],[12].
- Another important question is under which conditions the Hodge decomposition for the tangent bundle holds, allowing us to obtain a divergence free magnetic potential and how the main results and theorems should change in the case we have a non divergence free magnetic potential. Which additional bounds should be implicated?

2 Preliminaries

The following section contains all the prerequisites that the reader should be familiar with, for the understanding of the rest of our work. The necessary preliminaries can differ from person to person, thus the interested reader could find additional information of the content in the material used to prepare this chapter [13],[14],[1],[15],[16],[17],[18],[19],[20].

2.1 Functional Analysis

Many of the operators in physical phenomena are unbounded operators, a prime example of this is the Laplace operator that we study in this current problem. In this subsection, all the necessities will be introduced such that the reader can recall all the prerequisites needed, regarding Functional Analysis and Unbounded Operators. The following is based on [1] and [14]

2.1.1 Unbounded Operators

One might have seen several definitions of operators in various mathematical spaces, thus we make sure to introduce a definition on Hilbert spaces in which we work.

Definition 2.1. *An operator T on a Hilbert space $\mathcal{H}$ is a linear map $T : \mathcal{D}(T) \to \mathcal{H}$, with $\mathcal{D}(T)$ called the domain of T which is a dense subspace of $\mathcal{H}$*

When a linear operator is not bounded will be called unbounded.

Definition 2.2. *An operator $T : \mathcal{D}(T) \to \mathcal{H}$ is bounded if*

$$\sup_{v \in \mathcal{D}(T)\setminus\{0\}} \frac{\|Tv\|}{\|v\|} < \infty,$$

and unbounded otherwise.

For bounded operators one can show since $\mathcal{D}(T)$ is a dense subspace of $\mathcal{H}$ the operator will admit unique continuous extension to the whole Hilbert. For unbounded operators such extensions do not always exists. The following Theorem is necessary to the process of finding such extensions.

Theorem 2.1 (Riesz Representation Theorem)**.** *Let $F \in \mathcal{H}'$ where $\mathcal{H}' := \{F : \mathcal{H} \to \mathbb{R} | \text{ linear}\}$ denotes the dual space of our Hilbert space, then there exists unique $v \in \mathcal{H}$ such that*

$$F(u) = \langle v, u \rangle$$

for all $u \in \mathcal{H}$ and $\|F\| = \|v\|$.

Definition 2.3 (Adjoint of Bounded Operator)**.** *Given $u \in \mathcal{H}$ and T bounded operator, with Riesz Representation theorem allowing to define a unique w_u such that*

$$\langle u, Tv \rangle = \langle w_u, v \rangle$$

for all $v \in \mathcal{H}$. The adjoint T^* of T is the map $T^* : u \mapsto w_u$ satisfying

$$\langle u, Tv \rangle = \langle T^* u, v \rangle$$

for all $u \in \mathcal{H}$.

To define adjoint of unbounded operator and apply Riesz Representation theorem, one should consider that $\langle v, T(\cdot)\rangle$ should be bounded on $\mathcal{D}(T)$ which is not true for all $v \in \mathcal{H}$

Definition 2.4. *The adjoint of an unbounded operator $T : \mathcal{D}(T) \to \mathcal{H}$ is the unique linear map T^* defined by*

$$\langle v, Tu\rangle = \langle T^*v, u\rangle$$

for all $u \in \mathcal{D}(T)$ and $v \in \mathcal{D}(T^*)$, with the domain of the adjoint given by

$$\mathcal{D}(T^*) := \{v \in \mathcal{H} \mid \exists\, w \text{ such that } \langle Tu, v\rangle = \langle u, w\rangle,\ \forall u \in \mathcal{D}(T)\},$$

with $\langle Tu, v\rangle$ bounded on $\mathcal{D}(T)$.

It can happen that the two $T = T^*$ are equal on some subspace $\mathcal{D}'$ contained in both $\mathcal{D}(T)$ and $\mathcal{D}(T^*)$, but $\mathcal{D}(T) \neq \mathcal{D}(T^*)$.

Definition 2.5. *An operator T is self adjoint if $T = T^*$ as unbounded operators.*

Thus an operator is called self adjoint if for all $u \in \mathcal{D}(T)$ and $v \in \mathcal{D}(T^*)$

$$\langle v, Tu\rangle = \langle Tv, u\rangle$$

where the two domains in this case coincide $\mathcal{D}(T) = \mathcal{D}(T^*)$, so sometimes we need to extend a domain to obtain a self-adjoint operator. To define self adjoint extensions, one should make sense about extensions and closures of operators.

Definition 2.6. *An operator S is said to be an extension of T if,*

$$\mathcal{D}(T) \subset \mathcal{D}(S) \quad \text{and} \quad S|_{\mathcal{D}(T)} = T$$

The notation used for the above definition is $T \subset S$. By the definition of the adjoint $T \subset S$ implies $S^* \subset T^*$. To show this, let $v \in \mathcal{D}(S^*)$. By the definition of the adjoint $\exists\, S^*v$ such that for all $u \in \mathcal{D}(S)$

$$\langle Su, v\rangle = \langle u, S^*v\rangle$$

Since $\mathcal{D}(T) \subset \mathcal{D}(S)$ and $Tu = Su$ for all $u \in \mathcal{D}(T)$, we get that for all $u \in \mathcal{D}(T)$

$$\langle Tu, v\rangle = \langle u, S^*v\rangle$$

which implies by Riesz that $u \mapsto \langle Tu, v\rangle$ is bounded. This implies that $v \in \mathcal{D}(T^*)$ and $T^*v = S^*v$ which concludes $S^* \subset T^*$. Everything is set to define closed and closable operators.

Definition 2.7 (Closed operators)**.** *An operator is closed if its graph $\Gamma(T) := \{(u, Tu) \mid u \in \mathcal{D}(T)\} \subset \mathcal{H} \times \mathcal{H}$ is closed as a subspace of $\mathcal{H} \times \mathcal{H}$.*

Here the phrase T is closed is exactly the sequential. Let $\{u_n\} \subseteq \mathcal{D}(T)$ a sequence, if both $\{u_n\}$ and $\{Tu_n\}$ are convergent in $\mathcal{H}$ then $\lim u_n \in \mathcal{D}(T)$.

$$T(\lim_{n\to\infty} u_n) = \lim_{n\to\infty} Tu_n.$$

.

Definition 2.8 (Closable Operators)**.** *An unbounded operator T is closable if $\overline{\Gamma(T)}$ is the graph of an operator, where the closure $\bar{T}$ is defined as*

$$\Gamma(\overline{T}) = \overline{\Gamma(T)}.$$

Obviously $T \subset \bar{T}$.

Note that if an unbounded operator admits a closed extension, then it is closable. Closed implies boundedness [1, Theorem 3.15] and closability makes sense only for unbounded operators.

2.1.2 Essential Self-Adjointness

To give a better highlight of what an essentially self adjoint operator is we start with the following example similar to the one introduced for the Laplace operator.
On $L^2([0,1])$ consider the operator $T = i\frac{d}{dx}$ with domain,

$$\mathcal{D}(T) := \{f \in AC[0,1] \mid f(0) = f(1) = 0,\ f' \in L^2([0,1])\}.$$

For $f, g \in \mathcal{D}(T)$ integration by parts gives

$$\langle Tf, g\rangle = i\int_0^1 f'\overline{g}dx = i\, f\overline{g}|_0^1 - i\int_0^1 f\overline{g}'dx = \langle f, Tg\rangle$$

because the boundary terms vanishes. However the domain of the adjoint $\mathcal{D}(T^*) := \{f \in AC[0,1] \mid f' \in L^2([0,1])\}$ is larger than the one of the operator. We can consider two different extension by giving the following boundary conditions. One is $T_1 = \overline{T}$ such that

$$\mathcal{D}(T_1) := \{f \in AC[0,1] \mid f(0) = f(1)\ , f' \in L^2([0,1])\},$$

and the other $T_2 = \overline{T}$ with domain

$$\mathcal{D}(T_2) := \{f \in AC[0,1] \mid f(0) = -f(1)\ , f' \in L^2[0,1]\}.$$

As shown in [1, Example 3.21] both T_1 and T_2 are self adjoint operators and thus T is not essentially self adjoint.
We are interested to study when our operators admit a unique self adjoint extension.

Definition 2.9 (Essentially Self-Adjoint Operator)**.** *A symmetric operator T is essentially self-adjoint if it is closable with its closure satisfying*

$$\bar{T} = T^*.$$

One easily can see that T is essentially self adjoint if and only if $\bar{T} = T^*$, meaning that the closure is the unique self-adjoint extension. Since computing domains of adjoint operators can be quite hard, it is useful to look instead for more practical criteria as the following.

Theorem 2.2. *Suppose T is a symmetric operator, and let $z \in \mathbb{C}$ strictly complex. The following are equivalent.*

1. *T is essentially self-adjoint ;*
2. *Both $T^* - z$ and $T^* - \bar{z}$ are injective ;*
3. *Both $T - z$ and $T - \bar{z}$ have dense range.*

2.1.3 Sobolev Spaces

Let $\Omega \subset \mathbb{R}^m$ and consider the non-homogeneous Laplace equation, subjected to Dirichlet boundary conditions

$$\Delta u = f\ , \quad u = 0 \text{ on } \partial\Omega.$$

Recall that our Laplacian was defined as

$$\Delta u = \sum_{i=1}^{m} \frac{\partial^2 u}{\partial x_i^2}.$$

For a function u to be a classical solution, it must be twice continuous differentiable, meaning $u \in C^2(\overline{\Omega})$. However C_c^∞ which is considered as the core domain for many problems, is not a complete space which for many classical problems is a necessary property.

To provide a broader class of solutions we shift to $L^2(\Omega)$ of square integrable functions, equipped with the inner product

$$\langle u, v\rangle_{L^2(\Omega)} = \int_\Omega u\bar{v}dx.$$

The challenge here is that L^2 functions are not always differentiable in the classical sense. To resolve this, derivatives in the sense of distributions should be introduced, leading to weak solutions as in [20, Chapter 2.1].

By equipping L^2 with these weak derivatives and weak norm, we construct Sobolev spaces. Later, using Riesz Representation Theorem 2.1, we will define the Laplace-Beltrami operator. The core domain of this operator consists smooth functions, which form a dense subspace of Sobolev spaces.

For $\Omega \subset \mathbb{R}^m$, we will denote by $L^1_{loc}(\Omega)$ the space of locally compact functions of $L^1(\Omega)$.

Definition 2.10 (Weak Derivatives)**.** *For an index $\alpha \in \mathbb{N}^m$ and $u, v \in L^1_{loc}(\Omega)$ with α being a multi-index, we say that v is the α-th derivative of u denoted $D^a u$, if*

$$\int_\Omega u\, D^\alpha \phi\, dx^m = (-1)^{|\alpha|} \int_\Omega v\, \phi\, dx^m$$

for all $\phi \in C_0^\infty(\Omega)$, with

$$D^a \phi = \frac{\partial^a \phi}{\partial^{\alpha_1} x_1 \cdots \partial^{\alpha_m} x_m}$$

with $\alpha = \alpha_1 + \cdots + \alpha_m$ and $(x_1, \cdots, x_m) \in \Omega \subseteq \mathbb{R}^m$

The notion of weak derivation and classical derivation of order m coincide in the case the weak derivative of a function $u \in L^1_{loc}(\Omega)$ of order m exists and it is continuous.

Definition 2.11. *The Sobolev Space on a open subset $\Omega \subset \mathbb{R}^m$ associated to L^2 of order $n \in \mathbb{N}$ is defined by*

$$H^n(\Omega) = W^{n,2}(\Omega) := \{u \in L^2(\Omega) \mid D^\alpha u \in L^2(\Omega)\ for\ |a| \leq n\}.$$

This space is equipped with the inner product

$$\langle u, v\rangle_{W^{n,2}} := \sum_{|\alpha| \leq n} \langle D^\alpha u, D^\alpha v\rangle_{L^2},$$

and the weak norm,

$$\|u\|^2_{W^{n,2}} := \sum_{|\alpha|\leq n} \|D^\alpha u\|^2_{L^2}.$$

One can notice that $W^{n,2}$ are Hilbert spaces, since L^2 spaces are, but endowed with the weak topology associated to the weak norm. We use Sobolev spaces as an alternate of smooth functions and the following theorems will give a desired density result of smooth functions in Sobolev spaces.

Theorem 2.3 (Sobolev Embedding)**.** *For $\Omega \subset \mathbb{R}^m$, if $n > k + m/2$ then a function in $W^{n,2}(\Omega)$ admits a representative in $C^k(\Omega)$.*

Theorem 2.4 (Rellich-Kondrachov)**.** *For $\Omega \subset \mathbb{R}^m$ the space $W_c^{1,2}(\Omega)$ is compactly embedded in $L_c^2(\Omega)$. If also Ω has a Lipschitz boundary then $W^{1,2}(\Omega)$ is compactly embedded in $L^2(\Omega)$.*

Theorem 2.5 (Meyers-Serrin)**.** *For $\Omega \subset \mathbb{R}^m$, $W^{n,2}(\Omega)$ is the completion of the smooth functions with bounded weak norm:*

$$W^{n,2}(\Omega) := \overline{\{u \in C^\infty(\Omega) : \ \|u\|_{W^{n,2}}. < \infty\}}$$

We obtain by the combination of Theorem 2.5 and Theorem 2.4 that $\overline{C^\infty(\Omega)} = W^{n,2}(\Omega)$. The proofs of the above theorems are in [1],[20]

2.1.4 Quadratic Forms

A convenient way to represent second order operators and make sense of their weak norm is with the use of their quadratic forms.

Definition 2.12. *A quadratic form is a map $Q : \mathcal{D}(Q) \times \mathcal{D}(Q) \to \mathbb{C}$, where $\mathcal{D}(Q)$ is the largest dense subset of our Hilbert space $\mathcal{H}$, such that is conjugate linear in the second entry and linear in the first. If $Q(u,v) = \overline{Q(v,u)}$ for $u, v \in \mathcal{D}(Q)$ then Q is symmetric.*
If $Q(u,u) \geq 0$ then the quadratic form is positive and if $Q(u,u) \geq -M\|u\|^2$ then is called semi-bounded, where $\|\cdot\|$ the associated norm of the Hilbert space

Note that when the quadratic form is semi-bounded, then is automatically symmetric in $\mathcal{H}$.

Definition 2.13. *Let $T : \mathcal{D}(T) \to \mathcal{H}$ unbounded operator. The quadratic form Q associated to T is defined by*

$$Q(u,v) = \langle Tu, v\rangle$$

for all $u, v \in \mathcal{D}(Q)$ satisfying Definition 2.12. The domain $\mathcal{D}(Q)$ is the largest dense subspace of $\mathcal{H}$ such that $\langle Tu, u\rangle$ is well defined.

A quadratic form Q associated to an operator T is closed if the operator is closed as well. The same results hold true also for closable operators.

Theorem 2.6. *If Q is closed semibounded quadratic form, then Q is the quadratic form of a unique self-adjoint operator T.*

For semibounded operators it is always possible to obtain a closure using their associate quadratic forms.

Theorem 2.7 (Friedrichs Extension)**.** *Let T semibounded symmetric operator and $Q(u,v) = \langle Tu, v\rangle$. Then Q is closable quadratic form and its closure $\overline{Q}$ is the quadratic form of a unique self-adjoint operator $\overline{T}$, where $\overline{T}$ is a positive extension of T. Moreover $\overline{T}$, is the only self-adjoint extension of T whose domain is contained in the form domain of $\overline{Q}$.*

If $\overline{T}$ as in the theorem, then T is essentially self adjoint. We can state now the Essential Self-Adjointness Criterion.

Theorem 2.8 (Criterion for Essential Self-Adjointness)**.** *Let T symmetric, closable operator on a Hilbert space $\mathcal{H}$ and consider $Q(\phi,\psi) = \langle \phi, T\psi \rangle$ the associated closable quadratic form. Then T is essentially self-adjoint if and only if $dim[\mathrm{Ker}(\lambda I - T^*)] = 0$ in $\mathcal{H}$ for $\lambda \in \mathbb{C}$ with $\mathcal{D}(\overline{Q}) = \mathcal{D}(\overline{T})$*

We can restrict to $\lambda \in \mathbb{R}$ since the spectrum $\sigma(T) := \{\lambda \in \mathbb{C} \mid T - \lambda I \text{ is invertible}\}$ of symmetric closabel operator is real.

Agmon-type estimate method that we will develop on the following sections used for the proof of Theorem 5.1 and Theorem 6.2, proves that if our symmetric operator with associate semi-bounded quadratic form has a weak solution then this solution is trivial. This is equivalent that the kernel in Theorem 2.8 is trivial. In general Agmon type estimates are mathematical bounds to prove exponential decay of eigenfunctions for differential operators [21].

2.2 Riemmanian Geometry

The abstract definition of Riemannian manifold was introduced by Whitney in 1936. Riemann introduced the Riemmanian metrics in domains of Euclidean Space. Riemannian manifolds are objects that locally are Euclidean endowed with an inner product on their tangent space. In general, Riemannian manifolds are curved spaces that locally "look" flat. Everything from this chapter is obtained from [15] and [16]

2.2.1 Riemannian manifolds and Metrics

Definition 2.14 (Riemannian Manifold)**.** *A Riemannian manifold (M,g) consists of a smooth manifold M, and a Riemannian metric on M.*

Definition 2.15 (Riemannian Metric)**.** *A Riemannian metric on a smooth manifold M is a smooth covariant 2-tensor field $g \in \mathcal{T}^2(M)$ whose value g_p at each $p \in M$ is an inner product on T_pM, that is, for any $X,Y,Z \in T_pM$ and $a,b \in \mathbb{R}$,*

1. $g(X,Y) = g(Y,X)$;
2. $g(aX+bY,Z) = a\,g(X,Y) + b\,g(Y,Z)$;
3. $g(X,X) \geq 0$ *with equality if and only if* $X \equiv 0$.

If $(x^1,\cdots,x^m)$ a local coordinated system of a Riemannian manifold (M,g) of dimension m, on a open subset $U \subseteq M$, then

$$g = g_{ij}\,dx^i dx^j$$

with m^2 smooth functions $g_{ij} : M \to \mathbb{R}$ for $i,j = 1,\cdots,m$, which form a matrix value function (g_{ij}).

A vector field $X : M \to TM := \cup_{p\in M} T_pM$ is the map that assigns to each point p a single tangent vector in T_pM. For the local coordinates as above, vector fields are given by $\partial_i = \frac{\partial}{\partial x^i}$. The metric value function is characterized by, $g_{ij}(p) = \langle \partial_i|_p, \partial_j|_p \rangle$, with $\langle \cdot,\cdot \rangle$ the Euclidean inner product of the tangent space. Similarly for any local frame $(E_1,\cdots,E_m)$ on an open $U \subseteq M$ with $(\varepsilon^1,\cdots,\varepsilon^m)$ the co-frame

$$g = g_{ij}\varepsilon^i\varepsilon^j$$

where $g_{ij}(p) = \langle E_i|_p, E_j|_p \rangle$. Frames and co-frames on an open $p \in U \subset M$ are basis of T_pM and T_p^*M respectively.

2.2.2 Volume Form

In Euclidean space, the geometry allows us to compute areas, space, volumes and more. To compute the volume of parallelepiped spanned by m vectors $\{v_1, \cdots, v_m\}$, consider $\{e_1, \cdots, e_m\}$ canonical basis in $\mathbb{R}^m$ then the volume form is,

$$vol(v_1, \cdots, v_m) = det[g(v_i, e_j)] = det([v_1, \cdots, v_m][e_1, \cdots, e_m]^T) = det[v_1, \cdots, v_m]$$

In a same way, in a oriented Riemannian manifold (M, g) we can define

$$vol_g(v_1, \cdots, v_m) = det[g(v_i, e_j)]$$

with $\{e_1, \cdots, e_m\}$ positively oriented basis. The following proposition summarizes the above using the wedge product notation.

Proposition 2.1 (Volume Form)**.** *Let (M, g) oriented Riemannian manifold with or without a boundary. There is a unique m-form vol_g on M, called the Riemannian volume form, given by one of the three equivalent following properties:*

- *if $(\varepsilon^1, \cdots, \varepsilon^m)$ is any local orthonormal co-frame of T^*M then,*

$$vol_g = \varepsilon^1 \wedge \cdots \wedge \varepsilon^m,$$

- *if $(E_1, \cdots, E_m)$ is any orthonormal local frame of TM then,*

$$vol_g(E_1, \cdots, E_m) = 1,$$

- *if $(x^1, \cdots, x^m)$ are any local coordinates on M, then*

$$vol_g = \sqrt{\det(g_{ij})}\, dx^1 \wedge \cdots \wedge dx^m.$$

Volume forms allows us to compute integrals. The volume of M is defined by

$$Vol(M) = \int_M vol_g.$$

To avoid dealing with orientation we can instead use densities. These are multilinear maps $\kappa : \otimes^k \Gamma(M) \to \mathbb{R}$ that transforms according to the rule

$$\kappa \circ T^n = |det\ T| \kappa$$

For any linear automorphism of the domain of κ

Proposition 2.2 (Riemannian Density)**.** *If (M,g) Riemannian manifold, then there exists a unique smooth positive density κ on M, with the property than for any local orthonormal frame $(E_1, \cdots, E_m)$ of TM*

$$\kappa(E_1, \cdots, E_m) = 1.$$

2.2.3 Derivatives, Covariant Derivatives and Connections

Our main goal is to define derivation on sections of the tangent bundle, but also functions along a specific direction on the manifold. When $M = \mathbb{R}^m$, let $X : \mathbb{R}^m \to \mathbb{R}^m$ a vector field in $\mathbb{R}^m$ and V a vector in $\mathbb{R}^m$. We want to analyze the derivative of X at p in the direction V, that is

$$\lim_{t\to 0} \frac{X(p+tV) - X(p)}{t}.$$

In the Euclidean world this works because everytihng is identified with $\mathbb{R}^m$. On a manifold the terms $X(p+tV) \in T_{p+tV}M$ and $X(p) \in T_pM$ are in different spaces, thus the difference of the two in the limit above in general lie in two different vector spaces and in order to subtract these two an initial identification of the underlying spaces is necessary.

In Riemannian manifolds, we need to introduce a notion of derivatives of vector fields given a vector bundle. If we have a function $f : M \to \mathbb{R}$, the differential $df : TM \to \mathbb{R}$ measures how the function changes. Locally this is given by $df = \partial_i(f)dx^i$.

By the precense of g, the gradient of the function ∇f is defined by $g(X, \nabla f) = \nabla_X f = df(X)$ for all $X \in \mathfrak{X}(M)$.

We will introduce Lie derivatives which are the directional derivative along a specific vector field of a function or another vector field.

Definition 2.16. *Let X, Y an arbitrary vector fileds in out tangent bundle TM and $f : M \to \mathbb{R}$. Then the Lie derivative of f along the vector field $X \in \mathfrak{X}(M)$ is $L_X(f) = X(f) = df(X)$ and the Lie derivative of Y along X is $L_X Y = [X, Y]$ where $[\cdot, \cdot]$ denotes the Lie bracket.*

On a Riemannian manifold (M, g, ω), with ω smooth volume, the divergence of a vector field $X \in \mathfrak{X}(M)$ is described by

$$L_X \omega = (div_\omega X)\omega$$

For Lie derivatives, vector fields should be described locally.

It is necessary to establish the framework of covariant differentiation and connections.

Definition 2.17 (Connection)**.** *Let (M, g) a Riemannian manifold and $\mathfrak{X}(M)$ the space of smooth vector fields. A connection on TM is a map*

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

written as $(X, Y) \mapsto \nabla_X Y$ satisfying,

1. *$\nabla_X Y$ is linear over $C^\infty(M)$ in X: for $f_1, f_2 \in C^\infty(M)$ and $X_1, X_2 \in \mathfrak{X}(M)$*

$$\nabla_{f_1 X_1 + f_2 X_2} Y = f_1 \nabla_{x_1} Y + f_2 \nabla_{X_2} Y;$$

2. *$\nabla_X Y$ is linear over $\mathbb{R}$ in Y: for $a_1, a_2 \in \mathbb{R}$ and $Y_1, Y_2 \in \mathfrak{X}(M)$*

$$\nabla_X (a_1 Y_1 + a_2 Y_2) = a_1 \nabla_X Y_1 + a_2 \nabla_X Y_2;$$

3. *∇ satisfies the product rule: for $f \in C^\infty(M)$*

$$\nabla_X (fY) = f \nabla_X Y + X(f)Y.$$

For local frame (E_i) on $U \subseteq M$, for any choice of the indices we can expand $\nabla_{E_i} E_j$ in terms of the local frame as

$$\nabla_{E_i} E_j = \Gamma_{ij}^k E_k.$$

These m^3 smooth functions $\Gamma_{ij}^k : U \to \mathbb{R}$ are called connection coefficients of ∇. Thus for any vector fields $X, Y \in \mathfrak{X}(M)$ locally given as $X = X^i E_i$, $Y = Y^j E_j$, one has

$$\nabla_X Y = (X(Y^k) + X^i Y^j \Gamma_{ij}^k) E_k.$$

Connections can give rise to a way of differentiating vector fields along curves. Let $\gamma : I \to M$ a smooth curve, a smooth vector filed along γ is a continuous map $X : I \to TM$ such that $X(t) \in T_{\gamma(t)}M$ for every $t \in I$. Let $\mathfrak{X}(\gamma)$ denote the vector space of such smooth vector fields.

A common example of the above is simply the speed vector $\gamma'(t) \in T_{\gamma(t)}M$. We will say that $X(t)$ is extensible if there exists $\tilde{X}(t)$ smooth vector field that contains the image of γ and $\tilde{X} = X \circ \gamma$

Proposition 2.3 (Derivation along a curve)**.** *Let (M, g) a smooth manifold and ∇ a connection in TM. For each $\gamma : I \to M$ the connection determines a unique operator*

$$D_t : \mathfrak{X}(\gamma) \to \mathfrak{X}(\gamma),$$

called covariant derivative along γ, satisfying

1. *Linear over $\mathbb{R}$: for $a, b \in \mathbb{R}, X, Y \in \mathfrak{X}(\gamma)$*

$$D_t(aX + bY) = aD_t(X) + bD_t(Y);$$

2. *Product rule: for $f \in C^\infty(I)$*

$$D_t(fX) = f'V + fD_t(X);$$

3. *if $V \in \mathfrak{X}(\gamma)$ extensible, then for every extension $\tilde{V}$ of X*

$$D_t(X) = \nabla_{\gamma'(t)} \tilde{X}.$$

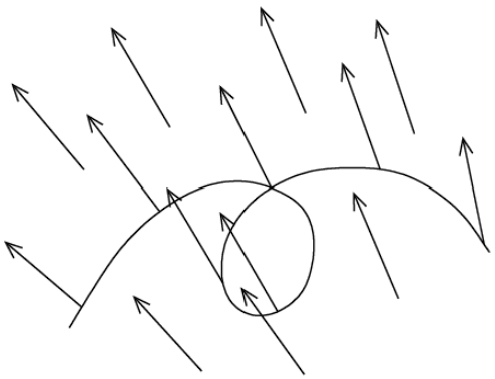

Figure 2: Extensible Vector Field along a curve

Now we can define a geodesic on our manifold M.

Definition 2.18 (Geodesics)**.** *Let (M,g) Riemannian manifold and ∇ connection on TM. For every smooth curve $\gamma : I \to M$ we define its acceleration to be the vector field $D_t\gamma'$ along γ as in Proposition 2.3. Such a smooth curve γ is called geodesic if $D_t\gamma' \equiv 0$*

Geodesics locally have as components, $\gamma(t) = (x^1(t), \cdots, x^m(t))$ and indeed γ is a geodesic if it satisfies

$$\ddot{x}^k(t) + \dot{x}^i(t)\dot{x}^j(t)\Gamma^k_{ij}(x(t)) = 0.$$

By Picard-Lindelöf theorem for ODE one can prove existence and uniqueness of such geodesics with suitable initial conditions.

Geodesics are curves that locally minimize length between two points of our space. In the case of our Riemannian manifold lengths of geodesics that are given by $\text{length}(\gamma) = \int_0^1 |\gamma(t)'|\, dt$ are equal to the distance of the two points $d_g(p,q) = \inf\{\text{length}(\gamma) \mid \gamma : [0,1] \to M\ , \gamma(0) = p\ , \gamma(1) = q\}$.

2.2.4 Fundamental Theorems

Every ingredient is now established for introducing the two fundamental theorems of Riemannian Geometry.

Theorem 2.9 (Fundamental Theorem of Riemannian Geometry)**.** *Let (M,g) a Riemannian manifold, and let ∇ be a unique connection on TM called the Levi-Civita connection satisfying the following properties,*

1. *∇ is compatible with g, that is for $X,Y,Z \in \mathfrak{X}(M)$*

$$\nabla_X g(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z);$$

2. *g is parallel with respect to ∇ that is $\nabla g \equiv 0$;*
3. *∇ is torsion free, that is, for $X,Y \in \mathfrak{X}(M)$*

$$\nabla_X Y - \nabla_Y X = [X,Y];$$

4. *In term of local smooth frames (E_i), the connection coefficients of ∇ satisfy*

$$\Gamma^l_{ki}\, g_{lj} + \Gamma^l_{kj}\, g_{il} = E_k(g_{ij});$$

5. *If $X,Y \in \mathfrak{X}(M)$ along any smooth curve γ, then*

$$\frac{d}{dt} g(X,Y) = g(D_t X, Y) + g(X, D_t Y);$$

There is an important connection between metric spaces, Riemannian manifolds and geodesic flow given by the Hopf-Rinow Theorem.

Theorem 2.10 (Hopf-Rinow)**.** *A Riemannian manifold (M,g) is complete as a metric space if and only if it is geodesically complete that is, given two distinct points $p,q \in M$ there exists a minimizing geodesic segment γ from p to q*

2.3 Laplace-Beltrami

The extension of the Laplace operator from the Euclidean case to the Riemannian case is called Laplace-Beltrami operator. The main difference is that the divergence that depends on the measure.

The operators that will be defined have core domain $C_c^\infty(M)$ which is a dense subspace of $W_c^{n,2}(M)$ as seen in Theorem 2.5.

The Laplace-Beltrami operator on a Riemannian manifold (M,g,ω) with the Levi-Civvita connection ∇ is given for $u \in C_c^\infty(M)$ by

$$\Delta_\omega u = div_\omega(\nabla u)$$

locally described as

$$\Delta_\omega u = -\frac{1}{\kappa}\sum_{l,j=1}^{m}\frac{\partial}{\partial x_l}(g^{lj}\kappa\frac{\partial u}{\partial_j}),$$

with the associated quadratic form

$$Q(u) = \int_M g(\nabla u, \nabla u)\, d\omega.$$

Volume ω here is related to κ as $\omega = \kappa\, vol_g$, with $vol_g = dx^1 \wedge \cdots \wedge dx^m$ and $\kappa = \sqrt{g} = \sqrt{det(g_{ij})}$.

We will make it explicit using the weak formulation. It is a consequence of uniqueness of a functional in Theorem 2.1 and the correspondence with the quadratic form as in Definition 2.13. The Laplace-Beltrami operator given hence by the relation, for $u, v \in C_c^\infty(M)$

$$\langle \Delta_\omega u, v\rangle_{L^2(M)} = \langle \nabla u, \nabla v\rangle_{L^2(TM)},$$

where

$$\langle f, g\rangle_{L^2(M)} = \int_M f\bar{g}\, d\omega \quad \text{for } f, g \in C_c^\infty(M),$$

and

$$\langle X, Y\rangle_{L^2(TM)} = \int_M g(X,Y)\, d\omega \quad \text{for } X, Y \in \mathfrak{X}(M).$$

To show the relation above we start by

$$\langle \nabla u, \nabla v\rangle_{L^2(TM)} = \int_M g(du, dv)\, d\omega.$$

Integrating by parts and removing the boundary terms, by the choice of domain we get

$$\int_M g(du,dv)\, d\omega = -\int_M -div_\omega(du)\bar{v}\, d\omega,$$

leading to

$$\langle \nabla u, \nabla v\rangle_{L^2(TM)} = \int_M \Delta_\omega \bar{v}\, d\omega = \langle \Delta_\omega u, v\rangle_{L^2(M)}.$$

Now we have to show for the magnetic case before defining the magnetic Laplace-Beltrami that the deformed connection $\nabla^A = \nabla - iA$ defines indeed a connection on M. Let A the 1-form defining the magnetic potential on TM and $X, Y, X_1, X_2, Y_1, Y_2 \in \mathfrak{X}(M)$, $h, h_1, h_2 \in C^\infty(M)$, $a, b \in \mathbb{R}$ then

- $\nabla^A_X Y$ is C^∞ linear in X

$$\nabla^A_{X_1h_1+X_2h_2}Y = \nabla_{X_1h_1+X_2h_2}Y - i(h_1X_1+h_2X_2)A(Y)$$

where

$$\nabla_{X_1h_1+X_2h_2}Y = h_1\nabla_{X_1}Y + h_2\nabla_{X_2}Y$$

since ∇ is the Levi-Civita connection, thus the previous equation after expanding the brackets yields C^∞ linearity.

- $\nabla^A_X Y$ is $\mathbb{R}$ linear in Y

$$\nabla^A_X(aY_1+bY_2) = \nabla_X(aY_1+bY_2) - iXA(aY_1+bY_2)$$

both ∇ and A are $\mathbb{R}$ linear hence ∇^A is.

- ∇^A satisfies the product rule

$$\nabla^A_X(hY) = \nabla_X(hY) - iXA(hY)$$

where both ∇ satisfies the product rule as a connection and A as a 1-form by definition.

We will define the magnetic Laplace-Beltrami operator using as before the weak formulation

$$\langle \Delta^A_\omega u, v\rangle_{L^2(M)} = \langle \nabla^A u, \nabla^A v\rangle_{L^2(TM)}.$$

For any $u, v \in C^\infty_c(M)$,

$$\begin{aligned}\langle \nabla^A u, \nabla^A v\rangle_{L^2(TM)} &= \langle (d-iA)u, (d-iA)v\rangle_{L^2(TM)}\\ &= \langle du, dv\rangle_{L^2(TM)} - \langle du, iAv\rangle_{L^2(TM)} - \langle iAu, dv\rangle_{L^2(TM)} + \langle iAu, iAv\rangle_{L^2(TM)}\\ &= \int_M g(du,dv)\,d\omega - \int_M g(du,iAv)\,d\omega - \int_M g(iAu,dv)\,d\omega + \int_M g(iAu,iAv)\,d\omega,\end{aligned}$$

where after an integration by parts the boundary terms vanish and

$$\int_M g(iAu,dv)\,d\omega = \langle iAu, dv\rangle_{L^2(TM)} = \langle d^*(iAu), v\rangle_{L^2(M)}.$$

The formal adjoint of the d using divergence theorem is given by

$$\begin{aligned}\int_M d^*(iAu)\bar{v}\,d\omega &= \int_M -i\,div_\omega(A)u\bar{v}\,d\omega - \int_M g(du,iAv)\,d\omega\\ &= -\langle i\,div_\omega(A)u, v\rangle_{L^2(M)} - \langle du, iAv\rangle_{L^2(TM)}.\end{aligned}$$

Thus we get

$$\langle \Delta^A_\omega u, v\rangle_{L^2(M)} = \langle \Delta_\omega u, v\rangle_{L^2(M)} - 2\langle du, iAv\rangle_{L^2(TM)} - \langle idiv_\omega(A)u, v\rangle_{L^2(M)} + \langle Au, Av\rangle_{L^2(TM)}.$$

The local expression of our operator can be derived with a similar computation:

$$\langle \nabla^A u, \nabla^A v\rangle_{L^2(TM)} = \int_M g(\nabla^A u, \nabla^A v)\,d\omega = \int_M g(du-iAv, dv-iAv)\,d\omega$$

$$= \int_M g(du,dv)\,d\omega - \int_M g(du,iAv)\,d\omega - \int_M g(iAu,dv)\,d\omega + \int_M g(iAu,iAv)\,d\omega$$
$$= \int_M g_{lj}\partial_l u\,\partial_j\bar{v}\,d\omega - \int_M ig_{lj}A_l u\,\partial_j\bar{v}\,d\omega - \int_M g_{lj}\partial_l u\,\bar{i}\bar{A}_j\bar{v}\,d\omega + \int_M g_{lj}A_l u\bar{A}_j\bar{v}\,d\omega$$
$$= -i\int_M \partial_j(g_{lj}\partial_l u)\bar{v}\,d\omega + \int_M ig_{lj}\partial_j(A_l)u\bar{v}\,d\omega + \int_M ig_{lj}A_l\partial_j(u)\bar{v}\,d\omega$$
$$+ \int_M ig_{lj}\partial_l u\bar{A}_j\bar{v}\,d\omega + \int_M g_{lj}A_l u\bar{A}_j\bar{v}\,d\omega.$$

Thus locally we get the following expression

$$\Delta^A_\omega u = -\partial_j(g_{lj}\partial_l u) + ig_{lj}\partial_j(A_l)u + ig_{lj}A_l\partial_j u + ig_{lj}\partial_l u\bar{A}_j + g_{lj}A_i\bar{A}_j u.$$

The quadratic form associated to our operator is given for any $u \in C^\infty_c(M)$ by

$$Q_A(u) = \int_M \|\nabla^A u\|^2_g\,d\omega = \int_M g(\nabla^A u,\nabla^A u)\,d\omega$$
$$= \int_M g(du,du)\,d\omega - 2\int_M g(du,iA)\bar{u}\,d\omega - \int_M i\,div_\omega(A)|u|^2 d\omega + \int_M g(A,A)|u|^2\,d\omega.$$

The quadratic form satisfies the properties of Definition 2.12. The core form domain $C^\infty_c(M)$ of the quadratic form is a dense subspace of $W^{n,2}(M)$. Friedrichs extension Theorem 2.7 has as a consequence that $\mathcal{D}(\Delta^A_\omega) = \{u \in \mathcal{D}(Q_A)\big|\Delta^A_\omega u \in L^2(M)\}$. Now that all the prerequisites are set, we can give a more explicit description of the setup, with stating and proving the main results.

3 Setup and Main Results

Recall that our operator is given for any $u \in C_c^\infty(M)$

$$\Delta_\omega^A(u) = \Delta_\omega(u) + V_{eff}^A$$

where

$$V_{eff}^A = -2\langle du, iA\rangle_{L^2(TM)} - i\,div_\omega(A)u + \langle A, A\rangle_{L^2(TM)}u.$$

3.1 Hodge-Helmholtz Decomposition

We assume $div_\omega(A) = 0$ to prove our results. This will be achieved by gauge freedom $A \mapsto A + dF$ and Hodge-Helmholtz decomposition. We use mainly [22],[23] in this section. Gauge freedom comes from [6, Proposition 2.13] stating

Proposition 3.1. *Let (M,g) a smooth Riemannian manifold with volume ω. Consider the magnetic Laplace-Beltrami Δ_ω^A as before with $A' = A + dF$ for $F \in C_c^\infty(M)$. Then if Δ_ω^A is essentially self-adjoint, $\Delta_\omega^{A'}$ is essentially self-adjoint.*

Now we should show in few steps how we construct the divergence free magnetic potential, but the following construction is for general k-forms.

On (M,g) Riemannian manifold of dimension m with volume ω the definition of the right Hodge star $* : \Omega^k(M) \to \Omega^{m-k}(M)$ and its inverse the left Hodge star $*^{-1} : \Omega^{m-k}(M) \to \Omega^k(M)$ for (m-k)-forms satisfying

$$\int_M \eta \wedge *\zeta = \int_M \langle \eta, \zeta\rangle_g \, d\omega$$

where $\langle .,. \rangle_g$ is a bilinear oriented inner product on $\Omega^k(M)$ defined by g, making sense for L^2 k-forms. These are k-forms such that the norm

$$\|\zeta\|_{L^2(\Omega^k)} = \int_M \langle \zeta, \zeta\rangle \, d\omega,$$

is finite. The Hodge star helps us to define the adjoint of

$$d : \Omega^k(M) \to \Omega^{k+1}(M),$$

as the map

$$d^* : \Omega^k(M) \to \Omega^{k-1}(M)$$

given by the relation $d^* = *d*$. Now d together with d^* satisfy

$$\langle d\zeta, \eta\rangle_{L^2(\Omega^{k+1})} = \langle \zeta, d^*\eta\rangle_{L^2(\Omega^k)}$$

and by applying Stokes Theorem for differential forms one can get

$$\int_M d(\zeta \wedge *\eta)\, d\omega = 0$$

only if our manifold has no boundary. In the case of a manifold with any type of boundary we demand that our forms vanish at the boundary.

These give rise to the operator $\Delta = (d^*d + dd^*)$, called the Hodge Laplace-Beltrami operator which is self-adjoint by [24, Corollary 2.1.4].

Lemma 3.1. *For any $\zeta \in \Omega^k(M)$ holds*

$$\Delta\zeta = 0 \Leftrightarrow d\zeta = 0 \text{ and } d^*\zeta = 0.$$

Such ζ in the Lemma is called a harmonic k-form.

Definition 3.1. *The elements of*

$$\mathcal{H}^k(M) := \{\zeta \in \Omega^k(M) \mid \Delta\zeta = 0\} = \ker(\Delta_k)$$

are called harmonic k-forms. We define

$$\mathcal{H}^*(M) := \oplus_{k=0}^{m} \mathcal{H}^k(M).$$

Every harmonic function $f \in C^\infty(M)$ on a compact oriented manidold is constant. When the Hodge Laplacian is defined on $C_c^\infty(M)$ then it matches the Laplace-Beltrami operator [24, Lemma 2.0.8]. We can now state the Hodge-Helmholtz decomposition theorem for L^2 k-forms.

Theorem 3.1 (Hodge-Helmholtz decomposition)**.** *For any $0 \leq k \leq m$ the space $\mathcal{H}^k(M)$ is of finite dimension and there is a direct L^2 decomposition*

$$\Omega^k(M) = d(\Omega^{k-1}(M)) \oplus d^*(\Omega^{k+1}) \oplus \mathcal{H}^k(\Omega).$$

Moreover for any $\zeta \in \Omega^k(M)$

$$\exists\, \eta \in \Omega^k(M) \text{ such that } \Delta\eta = \zeta \Leftrightarrow \zeta \in (\mathcal{H}^k)^\perp.$$

We apply the result to 1-forms. The respective decomposition of $L^2(T^*M)$ is given by

- $L^2(T^*M) = L^2(\Omega^1(M)) = \mathcal{E} \oplus \mathcal{C} \oplus \mathcal{H}^1$
- $\mathcal{E} = \overline{\{df : f \in C^\infty(M)\}}$
- $\mathcal{C} = \overline{\{d^*B : B \in \Omega^2(M)\}}$
- $\mathcal{H}^1$ finite dimensional space of harmonic 1-forms.

where we take the $\mathcal{E}, \mathcal{C}$ as closures.

We start with potential A_0 such that $dA_0 = B$ with $A_0 \in L^2(T^*M)$. $A_0 \in \mathcal{E}$ is exact and we project A_0 to its complement with respect to the descomposition. Since $\mathcal{H}^1$ contains 1-forms that satisfy $d^*A = 0$ and $dA = 0$, with every element of $\mathcal{C}$ is such that $d^*(d^*B) = 0$ then any element $A \in \mathcal{C} \oplus \mathcal{H}$ satisfy $d^*A = 0$. Note only that in this case of 1-forms d^* is given as the divergence when there is no boundary. In our problem we will assume that all the 1-forms vanish at the boundaries.

Now consider $A' \in \mathcal{C} \oplus \mathcal{H}$ such that

$$\langle A', \phi\rangle_{L^2(TM)} = \langle A_0, \phi\rangle_{L^2(TM)} \quad \forall \phi \in \mathcal{C} \oplus \mathcal{H},$$

which is well defined due to the above orthogonal decomposition. Thus $A_0 - A' \in \mathcal{E}$ implying $A_0 - A' = dF$. Then $A' = A_0 + dF$ satisfies $dA' = B$ and $d^*A' = 0$. Here F satisfies $\Delta F = d^*(A')$. By $d^*(A') = 0$ we get $d^*(dF) = -d^*(A_0)$ where d^*d is self-adjoint and positive. To prove that the equation admits a unique weak solution in $W_c^{1,2}(M \setminus (\Sigma_g \cup \Sigma_A))$ we need the following theorem [23, Theorem 3.2].

Theorem 3.2 (Lax-Milgram theorem)**.** *Assume that the quadratic form Q associated to d^*d is semi-bounded from below by $Q(v) \geq -M\|v\|_{L^2(M)}$ for $M > 0$. Then there exists a unique weak solution $u \in W^{1,2}(M)$ of $Q(u) = f$ with f continuous function. Moreover $\|u\|_{L^2(M)} \leq \frac{1}{M}\|f\|_{L^2(TM)}$.*

Elliptic regularity guarantees smoothness regardless the boundaries appearing [1, Theorem A.20],[25]. So for $F \in W_c^{1,2}(M \setminus (\Sigma_g \cup \Sigma_A))$ such that

$$\int_M g(dF, d\psi)\, d\omega = -\int_M g(A_0, d\psi)\, d\omega, \quad \forall \psi \in C_c^\infty(M).$$

We get that $\exists\,!\, F$ that solves the above which leads to $A' = A_0 + dF$ satisfies $d^*A' = 0$.

We can assume thus a divergence free magnetic potential and define

$$V_{eff}^A u = -2\langle du, iA\rangle_{L^2(TM)} + \langle A, A\rangle_{L^2(TM)} u. \tag{1}$$

3.2 Setup

We consider a Riemannian manifold (M, g) with a smooth volume ω of an arbitrary dimension m and a magnetic field B. In this setup the connection is the deformed connection given by $\nabla^A = \nabla - iA$ with domain $\mathcal{D}(\nabla^A) = C^\infty(M \setminus (\Sigma_A \cup \Sigma_g)$. The magnetic field $B = dA \in \Omega^2(M)$ of the associated divergence free potential. Locally

$$B = \sum_{1 \leq j < k \leq m} B_{jk} dx^j \wedge dx^k \quad \text{with} \quad B_{jk} = \partial_j(A_k) - \partial_k(A_j),$$

because the connection of our manifold on a local neighborhood is of the form $\nabla_j^A = \partial_j - iA_j$ with $\partial_j = \partial/\partial x^j$ the canonical Euclidean vector fields. Note that this is a consequence of magnetic potential locally satisfying

$$A = \sum_{j=1}^m A_j\, dx^j \quad \text{for} \quad A_j \in C_c^\infty(M \setminus \Sigma_A)$$

The measure volume ω consists of a smooth density κ on $M \setminus \Sigma_g$ with respect to the Lebesque measure $dx = dx^1 \cdots dx^m$ locally and coincides with the canonical measure vol_g induced by the metric g

$$\kappa = \sqrt{g} = \sqrt{det(g_{jk})}.$$

Recall that $(\hat{M}, \hat{d}_g)$ is the induced metric space by g on $\hat{M} = M \setminus (\Sigma_A \cup \Sigma_g)$ and that (M, d_g) is the completion of $(\hat{M}, \hat{d}_g)$, that is 'filling' the missing 'points at infinity' with $\partial M = \Sigma_A \cup \Sigma_g$.

We want essential self-adjointness of our Laplace-Beltrami operator $\Delta_\omega^A = div_\omega \circ (\nabla^A)$ depending by the rates of explosion, on Σ_g and Σ_A in the two cases, with condition (C) satisfied at all times.

A necessary condition is that the directional functions have limit as we approach the boundaries, which demands that our manifold is regular and hence satisfies the definitions below.

Definition 3.2 (Directional functions)**.** *We define the directional functions $n_{jk}(x) = \frac{B_{jk}(x)}{|B|_{sp}}$ with the spectral norm defined as*

$$|B|_{sp} = B_{12} + B_{34} + \cdots + B_{2\bar{m}-1,2\bar{m}}, \quad \bar{m} = [\frac{m}{2}] \tag{2}$$

The definition of the spectral norm makes sense only locally so we use it on the tubular neighborhoods of the singularities.

Definition 3.3. *The Riemmanian manifold (M,g) is called regular if*

- Σ_A,Σ_g *are compact*
- $\hat{M}$ *is quasi-isometric that is,* $\forall x_0 \in (\Sigma_A \cup \Sigma_g)$ *there exists a neighborhood* U *of* x_0 *and a smooth diffeomorphism* $F : U \to V$ *such that for* $U,V \subset M$ *open,* $\forall \varepsilon > 0$ *and* $\forall x,y \in U$ *holds that*

$$(1-\varepsilon)d(x,y) \leq d(F(x),F(y)) \leq (1+\varepsilon)d(x,y).$$

We will assume that Σ_A,Σ_g are compact so on M_{ε_2} and M_{ε_1} respectively the directional functions $n_{jk}(x) = B_{jk}(x)/|B|_{sp}$ are continuous. This is for $x_0 \in \Sigma_A \cup \Sigma_g$ such that $d_g(x,x_0) \leq \varepsilon$ the limit $\lim_{x\to x_0} n_{jk}(x)$ exists.

We assume without loss of generality that $B_{12} \geq \cdots \geq B_{2\bar{m}-1,2\bar{m}} \geq 0$ by continuity of n_{jk} hence the components have constant sign.

The components of the magnetic filed satisfy $[\nabla_j^A, \nabla_k^A] = -iB_{jk}$. Indeed by a direct computation

$$\begin{aligned}
[\nabla_j^A, \nabla_k^A]u &= \nabla_j^A(\nabla_k^A(u)) - \nabla_k^A(\nabla_j^A(u)) = \nabla_j^A(\partial_k(u) - iA_k(u)) - \nabla_k^A(\partial_j(u) - iA_j(u)) \\
&= (\partial_j - iA_j)(\partial_k(u) - iA_k(u)) - (\partial_k - iA_k)(\partial_j(u) - iA_j(u)) \\
&= \partial_j\partial_k(u) - i\partial_j(A_k(u)) - iA_k\partial_j(u) - iA_j\partial_k(u) - A_jA_k(u) \\
-[\partial_k\partial_j(u) - i\partial_k(A_j(u)) - iA_j\partial_k(u) - iA_k\partial_j(u) - A_kA_j(u)] &= -iB_{jk}u, \quad \forall u \in C_c^\infty(M). \qquad (3)
\end{aligned}$$

3.3 Main Results

In this subsection the main results will be stated. The first two theorems belong to the cases where the two singular sets Σ_A and Σ_g are distinct and the last one when $\Sigma_A = \Sigma_g$ as described in Section 1.2.

Note that for the proof of the below, IMS localization formula [6, Theorem 4.10]

$$Q_A(u) = \sum_{l=0}^{q} Q_A(\phi_l u) - \int_M (\sum_{l=0}^{q} |\nabla\phi_l|^2)|u|^2 \, d\omega \qquad (4)$$

for $u \in C_c^\infty(M)$ and ϕ_l smooth test functions is necessary. This gives a partition of unity subordinate to some open cover.

Recall that $\delta(p) := \inf\{d_g(p,q) | q \in \Sigma_g\}$ and $\rho(p) := \inf\{d_g(p,q) | q \in \Sigma_A\}$ satisfying condition (C) given explicitely as in Section 1.2.

Theorem 3.3. *We assume that* Σ_A *and* Σ_g *are compact with condiciton* (C) *hold at all times and that* $\exists\, c \in \mathbb{R}, \lambda \geq 0$ *and* η_1, η_2 *as in Theorems 5.3,6.1 respectively such that* $\forall u \in C_c^\infty(M \setminus (\Sigma_g \cup \Sigma_A))$ *we have that,*

$$Q_A(u) - \int_{M_{\eta_1}} \frac{|u|^2}{\rho^2} \, d\omega \geq c\|u\|^2$$

and

$$Q_A(u) - \int_{M_{\eta_2}} (\frac{1}{\delta^2} - \frac{\lambda}{\delta})|u|^2 \, d\omega \geq c\|u\|^2$$

Then Δ_ω^A is essentially self-adjoint in each connected component of $M \setminus (\Sigma_A \cup \Sigma_g)$.

Proof. The proof of this Theorem follows from 2.8 and from the Agmon-type estimate results for the magnetic singularity 5.1 with Theorem 3.4 for the tubular neighborhood M_{ε_2} and the metric singularity 6.2 with Theorem 3.5 for the tubular neighborhood M_{ε_1}. □

Theorem 3.4. *Let (M,g) Riemannian manifold with magnetic field $B = dA$. Assume that*

$$|B|_{sp} \geq (1+\varepsilon_2)/\rho^2$$

near Σ_A, for each $x_0 \in \Sigma_A$ the direction $n(x)$ of B has limit as $x \to x_0$ and g is regular that is $\kappa = \sqrt{det(g_{ij})} \neq 0$. Then Δ_ω^A with domain $\mathcal{D}(\Delta_\omega^A) = C_c^\infty(M \setminus \Sigma_A)$ is essentially self adjoint in $L^2(M)$.

Theorem 3.5. *Let (M,g) Riemannian manifold which is regular satisfying condition* (C) *for $\varepsilon_1 > 0$. Assume there exists $\lambda \geq 0$ such that,*

$$V_{eff} \geq \frac{3}{4\delta^2} - \frac{\lambda}{\delta} \quad for \quad \delta \leq \varepsilon_1,$$

and that A is regular on Σ_g, that is $A = \sum_{j=1}^m A_j \, dx^j$ has smooth components on Σ_g. Then Δ_ω^A with domain $\mathcal{D}(\Delta_\omega^A) = C_c^\infty(M \setminus \Sigma_g)$ is essentially self adjoint in $L^2(M)$

Theorem 3.6. *Let (M,g) Riemannian manifold which is regular and satisfying condition* (C) *for $\varepsilon > 0$ at all times. Assume there exists $\lambda \geq 0$ such that*

$$V_{eff}^A \geq \frac{C_b}{\delta^b} \quad and \quad V_{eff}^\omega \geq \frac{C_a}{\delta^a} - \frac{\lambda}{\delta}$$

Then the operator Δ_ω^A with domain $\mathcal{D}(\Delta_\omega^A) = C_c^\infty(M \setminus \Sigma)$ is essentially self adjoint in $L^2(M)$, if

- *For $a > 2, b > 2$ let $C_a, C_b > 0$;*
- *For $a \geq 2, 0 \leq b \leq 1$ let $C_a \geq \frac{3}{4}$;*
- *For $b \geq 2, 0 \leq a \leq 1$ let $C_b \geq \frac{3}{4}$,*

4 Intermediate Results

In this section we will establish the two following crucial results. We will make clear the form of the distance functions δ and ρ in their respective tubular neighborhoods, together with some properties that they satisfy and a necessary identity to prove the Agmon-type estimate results later.

Lemma 4.1. *Assuming condition* (C) *on M_{ε_1} and M_{ε_2} then the following conditions hold:*

1. $|\nabla\delta| \leq 1$;
2. *Integral curves of $\nabla\delta$ are smooth geodesics, where $\gamma : I \to M \setminus \Sigma_g$ integral curve if $\gamma'(t) = \nabla(\delta(\gamma(t)))$;*
3. *Let $X_\varepsilon = \{\delta = \varepsilon\}$, the map $\varphi : (0,\varepsilon] \times X_\varepsilon \to M_\varepsilon$ defined by the flow of $\nabla\delta$*

$$\varphi(t,x) = e^{(t-\varepsilon)\nabla\delta(x)}$$

   is a C^1 diffeomorphism such that $\delta(\varphi(t,x)) = t$;
4. *For any smooth ω, Δ^A_ω and all its derivatives along $\nabla\delta$ are smooth in M_{ε_1};*
5. $|\nabla\rho| \leq 1$

Proof. The proof is obtained by [6, Proposition 2.4] and [2, Lemma 2.1] □

Lemma 4.2. *Let $v \in L^2(M)$ weak solution of $(\Delta^A_\omega - E)v = 0$ for $E \in \mathbb{R}$ and $f \in C^\infty_c(M)$ test function then*

$$\langle fv, (\Delta^A_\omega - E)v\rangle_{L^2(M)} = \langle v, |df|^2_g v\rangle_{L^2(M)}.$$

Proof. Our operator had the form,

$$\langle \Delta^A_\omega u, v\rangle_{L^2(M)} = \langle \Delta_\omega u, v\rangle_{L^2(M)} - 2\langle du, iAv\rangle_{L^2(TM)} - \langle idiv_\omega(A)u, v\rangle_{L^2(M)} + \langle Au, Av\rangle_{L^2(TM)}.$$

and $(\Delta^A_\omega - E)v = 0$ weak solution means $\forall\phi \in C^\infty_c(M)$ test function we have

$$\langle \phi, (\Delta^A_\omega - E)v\rangle_{L^2(M)} = 0 \Rightarrow \langle \phi, \Delta^A_\omega v\rangle_{L^2(M)} = \langle \phi, Ev\rangle_{L^2(M)}. \tag{5}$$

We will apply the above for the weak solution on the test function $\phi = f^2 v$ for $f \in C^\infty_c(M)$ Lipschitz and $v \in C^\infty_c(M)$ to get

$$0 = \langle f^2 v, (\Delta^A_\omega - E)v\rangle_{L^2(M)} = \langle fv, (\Delta^A_\omega - E)fv\rangle_{L^2(M)} - \langle fv, [\Delta^A_\omega, f]v\rangle_{L^2(M)},$$

with

$$[\Delta^A_\omega, f]\phi = -f\Delta^A_\omega(\phi) + \Delta^A_\omega(f\phi) \quad \text{for } \phi \in C^\infty_c(M).$$

which gives

$$\langle fv, (\Delta^A_\omega - E)fv\rangle_{L^2(M)} = \langle fv, [\Delta^A_\omega, f]v\rangle_{L^2(M)} = \langle v, f[\Delta^A_\omega, f]v\rangle_{L^2(M)} = \frac{1}{2}\langle v, [f, [\Delta^A_\omega, f]]v\rangle_{L^2(M)}.$$

We left now to show the principal symbol identity,

$$[[\Delta^A_\omega, f], f] = -2|df|^2_g$$

and by the antisymmetric property of the commutator bracket we will conclude.
We compute the commutator bracket for $f \in C_c^\infty(M)$

$$[\Delta_\omega^A, f]\phi = [-div_g(d), f]\phi - [i\,div_g(A), f]\phi - 2[g(d, iA), f]\phi + [g(A,A), f]\phi.$$

For any $\phi \in C_c^\infty(M)$, $\Delta_\omega^A(\phi)$ is given by,

$$\begin{aligned} &-div_g(d\phi) - div_g(A\phi) - 2g(d\phi, iA) + |A|_g^2\phi \\ &= -div_g(d\phi) - i\phi\, div_g(A) - 3g(d\phi, iA) + |A|_g^2\phi \end{aligned}$$

thus computing the above Lie bracket on a test function $\phi \in C_c^\infty(M)$ we get

$$\begin{aligned} &[\Delta_\omega^A, f]\phi = -f\Delta_\omega^A(\phi) + \Delta_\omega^A(f\phi) \\ &= -f\Delta_\omega^A(\phi) - div_g(d(f\phi)) - if\phi\, div_g(A) - 3g(d(f\phi), iA) + |A|_g^2 f\phi \\ &= -f\Delta_\omega(\phi) + if\phi\, div_g(A\phi) + 3fg(d\phi, iA) - |A|_g^2 f\phi + f\Delta_\omega(\phi) \\ &+\phi\Delta_\omega(f) - 2g(df, d\phi) - 3fg(d\phi, iA) - 3\phi g(df, iA) + |A|_g^2 f\phi \end{aligned}$$

which after cancellations leaves us with

$$\phi\Delta_\omega(f) - 3\phi\, g(df, iA) - 2g(df, d\phi).$$

Thus defining $D = [\Delta_\omega^A, f]$ we get

$$D\phi = [\Delta_\omega^A, f]\phi = \phi\Delta_\omega(f) - 3\phi\, g(df, iA) - 2g(df, d\phi)$$

Now we compute

$$[[\Delta_\omega^A, f], f] = [D, f]\phi = D(f\phi) - fD(\phi).$$

Since

$$D(f\phi) = (\Delta_\omega f)(f\phi) - 2g(df, d(f\phi)) - 3g(df, iA)f\phi,$$

we get

$$\begin{aligned} &[D, f]\phi = (\Delta_\omega f)(f\phi) - 2g(df, d(f\phi)) - 3g(df, iA)f\phi \\ &-f(\Delta_\omega f)\phi + 2g(df, d\phi)f + 3g(df, iA)f\phi \\ &= -2g(df, df)\phi - 2g(df, d\phi)f + 2g(df, d\phi)f = -2g(df, df)\phi = -2|df|_g^2 \end{aligned}$$

concluding the proof. □

5 Magnetic Singularity

We prove Theorem 3.4 by deriving the Agmon-type estimate result and semi-boundedness of the associated quadratic.

Theorem 5.1 (Agmon-type Estimate)**.** *Assume* (C) *holds. Let* $v \in L^2(M)$ *weak solution of* $(\Delta_\omega^A - E)v = 0$ *for some* $E \in \mathbb{R}$*. Assume that* $\exists\, c > 0$ *and* $\eta_1 \geq 1$ *finite such that* $\forall u \in C_c^\infty(M \setminus (\Sigma_g \cup \Sigma_A))$*,*

$$\langle u, (\Delta_\omega^A - E)u\rangle_{L^2(M)} - \int_{M_{\eta_1}} \frac{|u|^2}{\rho^2}\, d\omega \geq c\|u\|^2, \tag{6}$$

then $v \equiv 0$*.*

Proof. Considering $0 < \rho_1 \leq \frac{\eta_1}{2}$ and $R_1 > \eta_1$. Using Lemma 4.2 for $f_1 v$ test function with $f_1 : M \to \mathbb{R}$ Lipschitz function and $v \in C_c^\infty(M)$ with $supp\, f_1 \subset \overline{M_{R_1+\eta_1} \setminus M_{\rho_1}}$ where $M_{R_1+\eta_1} := \{0 < \rho \leq R_1 + \eta_1\}$ and $M_{\rho_1} := \{0 < \rho \leq \rho_1\}$. We will show that $v \equiv 0$. Define

$$F_1(x) = \begin{cases} 0, & x \leq \rho_1 \quad and \quad x \geq R_1 + \eta_1 \\ 2(x - \rho_1), & \rho_1 \leq x \leq 2\rho_1 \\ x, & 2\rho_1 \leq x \leq \eta_1 \\ \eta_1, & \eta_1 \leq x \leq R_1 \\ R_1 + \eta_1 - x, & R_1 \leq x \leq R_1 + \eta_1 \end{cases}$$

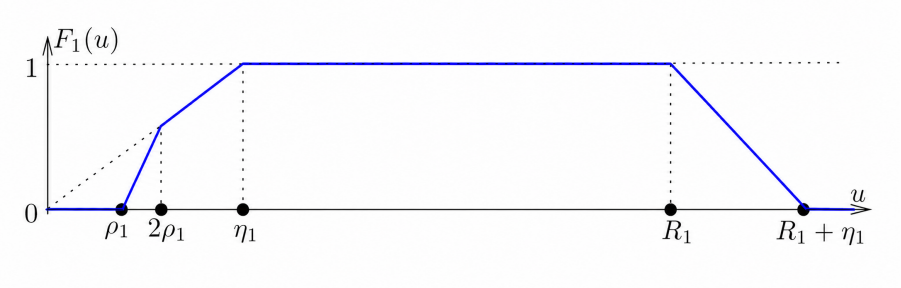

Figure 3: Cut-off Function F_1

We choose f_1 such that $f_1(x) = F_1(\rho(x))$ and by the Lemma 4.2

$$\langle f_1 v, (\Delta_\omega^A - E) f_1 v\rangle_{L^2(M)} = \langle v, |df_1|_g^2 v\rangle_{L^2(M)} \tag{7}$$

for any $x \in M_{\eta_1}$. By Lemma 4.1 we get

$$|df_1|_g = F_1'(\rho(x)) \cdot |\nabla\rho| \leq F_1'(\rho(x)). \tag{8}$$

Using (7),(6) for the test function $f_1 v \in C_c^\infty(M)$ we get

$$c\|f_1 v\|^2 \leq \langle v, |df((x))|^2 v\rangle_{L^2(TM)} - \int_{M_{\eta_1}} f_1^2 \frac{|v|^2}{\rho^2} d\omega. \tag{9}$$

Also (8) implies $|df_1(x)|^2 \leq F_1'(\rho(x))^2$ almost everywhere in the weak sense. We use that $F_1(\rho(x))$ is piecewise smooth and ρ is C^2 on M_{η_1} by condition (C) which give existence of df_1 satisfying

$$|df_1|^2 \leq \begin{cases} 0, & \rho \leq \rho_1 \quad \rho \geq R_1 + \eta_1 \\ 4, & \rho_1 \leq \rho \leq 2\rho_1 \\ 1, & 2\rho_1 \leq \rho \leq \eta_1 \\ 0, & \eta_1 \leq \rho \leq R_1 \\ 1, & R_1 \leq \rho \leq R_1 + \eta_1. \end{cases}$$

Thus we get

$$\langle v, |df_1|_g^2 v\rangle_{L^2(M)} \leq \int_{M_{2\rho_1}\setminus M_{\rho_1}} 4|v|^2 d\omega + \int_{M_{\eta_1}\setminus M_{2\rho_1}} |v|^2 d\omega + \int_{MR_1+\eta_1\setminus M_{R_1}} |v^2| d\omega. \tag{10}$$

We want to estimate $\int_{M_{\eta_1}} f_1^2 \frac{|v|^2}{\rho^2}\, d\omega$:

- for $\rho \leq \rho_1$, $f_1 = 0$,
- for $\rho_1 \leq \rho \leq 2\rho_1$, $f_1 \geq 2\rho_1 \Rightarrow \frac{|f_1 v|^2}{\rho^2} \geq 4|v|^2$,
- for $2\rho_1 \leq \rho \leq \eta_1$, $f_1 = \rho \Rightarrow \frac{|f_1 v|^2}{\rho^2} = |v|^2$,

which leads to

$$\int_{M_{\eta_1}} \frac{|f_1 v|^2}{\rho^2} d\omega \geq 4 \int_{M_{2\rho_1}\setminus M_{\rho_1}} |v|^2 d\omega + \int_{M\eta_1\setminus M_{2\rho_1}} |v|^2 d\omega. \tag{11}$$

Thus (6),(11) combined give

$$\langle f_1 v, (\Delta_\omega^A - E) f_1 v\rangle_{L^2(M)} \geq 4 \int_{M_{2\rho_1}\setminus M_{\rho_1}} |v^2| d\omega + \int_{M_{\eta_1}\setminus M_{2\rho_1}} |v|^2 d\omega + c\|f_1 v\|^2. \tag{12}$$

Concluding we combine (7),(10),(12) together with $4\int_{M_{2\rho_1}\setminus M_{\rho_1}} |v^2| d\omega \geq 0$ to get

$$\int_{M_{\eta_1}\setminus M_{2\rho_1}} |v|^2 d\omega + c\|f_1 v\|^2 \leq 4 \int_{M_{2\rho_1}\setminus M_{\rho_1}} |v|^2 d\omega + \int_{M_{\eta_1}\setminus M_{2\rho_1}} |v|^2 d\omega + \int_{M_{R_1+\eta_1}\setminus M_{R_1}} |v|^2 d\omega$$

$$\Rightarrow c\|f_1 v\|^2 \leq 4 \int_{M_{2\rho_1}\setminus M_{\rho_1}} |v| 2 d\omega + \int_{M_{R_1+\eta_1}\setminus M_{R_1}} |v|^2 d\omega.$$

Sending $\rho_1 \to 0$ and $R_1 \to +\infty$ we get that $c\|f_1 v\|^2 \leq 0$ implying $f_1 v \equiv 0$. But for $\rho_1 \to 0, R_1 \to +\infty$ we get that $f_1 \to \eta_1 > 0$ point-wise almost everywhere which gives that $v \equiv 0$. □

Recall, locally on M_{ε_2} the magnetic components satisfy (3). This identity together with the local expansion

$$Q_A(u) = \int_{M\setminus(\Sigma_A\cup\Sigma_g)} \|\nabla^A u\|_g^2 d\omega = \int_{M\setminus(\Sigma_A\cup\Sigma_g)} \sum_{j,k} g_{jk} \nabla_j^A u \nabla_k^A \bar{u} d\omega$$

for $u \in C_c^\infty(M \setminus (\Sigma_A \cup \Sigma_g))$, lead to the following result.

Lemma 5.1. *For any $u \in C_c^\infty(M \setminus (\Sigma_g \cup \Sigma_A))$ we have that,*

$$Q_A(u) \geq |\langle B_{12}u, u\rangle_{L^2(M\setminus(\Sigma_g\cup\Sigma_A))}| + |\langle B_{34}u, u\rangle_{L^2(M\setminus(\Sigma_g\cup\Sigma_A))}| + \cdots + |\langle B_{2\bar{m}-1,2\bar{m}}u, u\rangle_{L^2(M\setminus(\Sigma_g\cup\Sigma_A))}|$$

Proof. Let $\tilde{M} = M \setminus (\Sigma_g \cup \Sigma_A)$. For any indices $j,k \in \{1,\cdots 2\bar{m}\}$ we get

$$|\langle B_{jk}u,u\rangle_{L^2(\tilde{M})}| = |\langle [\nabla_j^A,\nabla_k^A]u,u\rangle_{L^2(\tilde{M})}|.$$

Computing

$$\langle \nabla_j^A\nabla_k^A u,u\rangle_{L^2(\tilde{M})} = \int_{\tilde{M}} \nabla_j^A\nabla_k^A(u)\bar{u}\,d\omega = \int_{\tilde{M}} [(\partial_j - iA_j)(\nabla_k^A u)]\bar{u}\,d\omega$$

$$= \int_{\tilde{M}} (\partial_j(\nabla_k^A u))\bar{u}\,d\omega - i\int_{\tilde{M}} A_j(\nabla_k^A u)\bar{u}\,d\omega = \int_{\tilde{M}} (\nabla_k^A u)\partial_j(\bar{u})d\omega - i\int_{\tilde{M}} A_j(\nabla_k^A)\bar{u}d\omega$$

$$= \int_{\tilde{M}} -(\nabla_k^A u)(\nabla_j^A\bar{u})d\omega = -\langle \nabla_k^A u,\nabla_j^A u\rangle_{L^2(T\tilde{M})}$$

and observing that

$$-i\langle B_{jk}u,u\rangle_{L^2(\tilde{M})} = \langle [\nabla_j^A,\nabla_k^A]u,u\rangle_{L^2(\tilde{M})} = \langle \nabla_j^A u,\nabla_k^A u\rangle_{L^2(T\tilde{M})} - \langle \nabla_k^A u,\nabla_j^A u\rangle_{L^2(T\tilde{M})},$$

we conclude

$$|\langle \nabla_j^A u,\nabla_k^A u\rangle_{L^2(T\tilde{M})} - \langle \nabla_k^A u,\nabla_j^A u\rangle_{L^2(T\tilde{M})}| \le |\langle \nabla_j^A u,\nabla_k^A u\rangle_{L^2(T\tilde{M})}| + |\langle \nabla_k^A u,\nabla_j^A u\rangle_{L^2(\tilde{M})}|$$

$$= 2|\langle \nabla_j^A u,\nabla_k^A u\rangle_{L^2(T\tilde{M})}| \le 2\|\nabla_j^A u\|\cdot\|\nabla_k^A u\| \le \|\nabla_j^A u\|^2 + \|\nabla_k^A u\|^2.$$

This implies

$$|\langle B_{jk}u,u\rangle_{L^2(\tilde{M})}| \le \int_{\tilde{M}} (\|\nabla_j^A u\|_g^2 + \|\nabla_k^A u\|_g^2)d\omega = \langle \nabla_j^A u,\nabla_k^A u\rangle_{L^2(T\tilde{M})}$$

and by summing all over the pairs indices we get

$$|\langle B_{12}u,u\rangle_{L^2(\tilde{M})}| + |\langle B_{34}u,u\rangle_{L^2(\tilde{M})}| + \cdots + |\langle B_{2\bar{m}-1,2\bar{m}}u,u\rangle_{L^2(\tilde{M})}|$$

$$\le \sum_{j,k}\langle \nabla_j^A u,\nabla_k^A u\rangle_{L^2(TM)} = \int_{\tilde{M}} \sum_{j,k} g_{jk}\nabla_j^A u\nabla_k^A\bar{u}d\omega.$$

and we conclude. □

Semi-boundedness of the Quadratic form is given by the following.

Theorem 5.2. *Assume that (M,d) the induced metric space from (M,g). Let $x_0 \in \Sigma_A$ and we assume that $B(x)$ does not vanish in M_{ε_2} and all the directional functions $n_{jk}(x)$ are continuous around x_0. Consider also A such that $B = dA$ defined locally as before near x_0 then for any $\varepsilon_2 > 0$ there exists open neighborhood U of x_0 small enough such that $\forall u \in C_c^\infty(U \cap M_{\varepsilon_2})$ it holds,*

$$Q_A(u) \ge (1-\varepsilon_2)\int_{U\cap M_{\varepsilon_2}} |B|_{sp}|u(x)|^2 d\omega$$

Proof. We can assume that such U exists by continuity of the directional functions n_{jk} and regularity of the boundary 3.3. At each $x \in U$ we can write $B(x) = \sum_{1\le j<k\le\bar{m}} B_{jk}dx^j \wedge dx^k$ with the spectral norm by Definition 3.2

$$|B(x)|_{sp} = B_{12}(x) + \cdots + B_{2\bar{m}-1,2\bar{m}}(x), \quad B_{12} \ge \cdots \ge B_{2\bar{m}-1,2\bar{m}} \ge 0$$

due to continuity of the directional functions, $n_{jk}(x) = B_{jk}(x)/|B|_{sp}$ which extend continuously to x_0. In particular locally at $x_0 \in U$ we can choose coordinates such that

$$n_{2j-1,2j}(x_0) \geq \quad \sum_{i=1}^{\bar{m}} n_{2j-1,2j}(x_0) = 1.$$

By the continuity assumption to U we get that $\forall \varepsilon' > 0$ and $\forall x \in U$ we obtain $|n(x) - n(x_0)| \leq \varepsilon'$. In particular $\sum_{j=1}^{\bar{m}} n_{2j-1,2j}(x) \geq 1 - \varepsilon'$ with $n_{2j-1,2j}(x) > 0 \quad \forall\, j \in \{1, \cdots, \bar{m}\}$. Applying the previous Lemma 5.1 $\forall u \in C_c^\infty(U \cap M_\varepsilon)$,

$$Q_A(u) \geq \sum_{j=1}^{\bar{m}} |\langle B_{2j-1,2j} u, u\rangle_{L^2(M)}|$$

where

$$|\langle B_{2j-1,2j} u, u\rangle_{L^2(U \cup M_{\varepsilon_2})}| = \int_{U \cap M_{\varepsilon_2}} B_{2j-1,2j}(x)|u(x)|^2 d\omega.$$

Since for all $x \in U$ it is true that $B_{2j-1,2j}(x) = |B(x)|_{sp}\, n_{2j-1,2j}(x) > 0$ we get

$$Q_A(u) \geq \sum_{j=1}^{\bar{m}} \int_{U \cap M_{\varepsilon_2}} B_{2j-1,2j}(x)|u(x)|^2 d\omega = \int_{U \cap M_{\varepsilon_2}} \sum_{j=1}^{\bar{m}} B_{2j-1,2j}(x)|u(x)|^2 d\omega$$

$$= \int_{U \cap M_{\varepsilon_2}} \sum_{j=1}^{\bar{m}} |B(x)|_{sp}\, n_{2j-1,2j}(x)|u(x)|^2 d\omega \geq (1-\varepsilon') \int_{U \cap M_{\varepsilon_2}} |B(x)|_{sp}|u(x)|^2 d\omega$$

Choosing $\varepsilon' = \varepsilon_2$ we conclude the result. □

We generalize Theorem 5.2 such that Theorem 5.1 is applicable.

Theorem 5.3. *Assume that Σ_A,Σ_g are compact. Also that $B = dA$ does not vanish near Σ_A and that $n_{jk}(x)$ are regular at the boundaries. Then $\forall \varepsilon_2 > 0, \exists C_{\varepsilon_2}, \eta_1 > 0$ such that $\forall u \in C_c^\infty(M \setminus (\Sigma_A \cup \Sigma_g))$ it holds*

$$Q_A(u) \geq (1-\varepsilon_2) \int_{M_{\eta_1}} |B|_{sp}|u|^2 d\omega - C_{\varepsilon_2}\|u\|^2$$

Proof. Since Σ_A is compact and $n_{jk} = B_{jk}/|B|_{sp}$ are regular for any $x \in U$ where U open neighborhood of $x_0 \in \Sigma_A$. We can cover the compact $\bar{M}_{\varepsilon_2}$ with an open cover $\{U_1, \cdots, U_N\}$ finite such that,

- On each $\{U_l\}_{l=1}^N$ the direction of B is constant so the previous Theorem 5.2 is applicable
- $\Sigma_A \subset \cup_{l=1}^N U_l = M_{\varepsilon_2}$.

Previous Theorem 5.2 applies for each U_l and any $\varepsilon_2 > 0$ is such that $\forall u \in C_c^\infty(U_l)$

$$Q_A(u) \geq (1-\varepsilon_2) \int_{U_l} |B(x)|_{sp}|u(x)|^2 d\omega.$$

We also consider an open set $U_0 \subset M_{\eta_1} \setminus \Sigma_A$, which precisely is $M_{\eta_1} \setminus \overline{M_{\varepsilon_2}}$ such that $Q_A(u) > 0$ for any $u \in C_c^\infty(U_0)$.

Assume that U_0 is bounded away from Σ_A and also away from the metric boundary. This is essential because we want a partition of unity and apply the IMS formula (4) where we want to cover all M_{η_1}.

We get that $\{U_0,\cdots,U_N\}$ covers M_{η_1} with $\eta_1 \le c-\varepsilon_1$ or $\eta_1 \ge c+\varepsilon_1$ where $c = d_g(\Sigma_g,\Sigma_A)$. Hence subordinate to the cover we choose a partition of unity $\{\phi_0,\cdots,\phi_N\}$ such that, $\phi_j \in C_c^\infty(U_l)$ for all $j = 0,1,\cdots,N$ and $\sum_{j=0}^N \phi_j^2(x) = 1$ for $x \in M$ with bounded gradients $\sup_M |\nabla^A \phi_j|_g < +\infty$.

The IMS formula (4) gives

$$Q_A(u) = \sum_{j=0}^N Q_A(\phi_j u) - \int_{M_{\eta_1}} (\sum_{j=0}^N |\nabla^A \phi_j|^2)|u|^2 d\omega.$$

By $\sup_M |\nabla^A \phi_j|_g < +\infty$ we set $L = \sup_{M_{\eta_1}} (\sum_{j=1}^N |\nabla^A \phi_j|^2) < +\infty$ then

$$Q_A(u) \ge \sum_{j=0}^N Q_A(\phi_j u) - L\|u\|^2.$$

For $j = 1,\cdots,N$ since $\phi_j u \in C_c^\infty(u_j)$ we apply the previous Theorem (5.2) with $\varepsilon' = \frac{\varepsilon_2}{2}$ thus we get

$$Q_A(u) \ge (1-\frac{\varepsilon_2}{2}) \int_{U_j} |B(x)|_{sp} |\phi_j(x)u(x)|^2 d\omega.$$

For the case $j = 0$ we had that $Q_A(u) \ge 0$ thus altogether

$$\sum_{j=0}^N Q_A(\phi_j u) \ge (1-\frac{\varepsilon_2}{2}) \sum_{j=1}^N \int_{U_j} |B(x)|_{sp} |\phi_j u|^2 d\omega.$$

We want to estimate now

$$I = \int_{M_{\eta_1}} |B(x)|_{sp} |u|^2 d\omega \quad \text{since} \quad \sum_{j=0}^N \phi_j^2 = 1.$$

We have that

$$I = \sum_{j=0}^N \int_{M_{\eta_1}} |B(x)|_{sp} \phi_j^2 |u|^2 d\omega = \int_{U_0} |B(x)|_{sp} \phi_0^2 |u|^2 d\omega + \sum_{j=1}^N \int_{U_j} |B(x)|_{sp} |\phi_j|^2 |u|^2 d\omega$$

$$\le \int_{U_0} |B(x)|_{sp} \phi_0^2 |u|^2 d\omega + \frac{1}{(1-\frac{\varepsilon_2}{2})} \sum_{j=1}^N Q_A(\phi_j u).$$

Let $M_0 = \sup_{x\in U_0} |B(x)|_{sp} < +\infty$ then $\int_{U_0} |B(x)|_{sp} \phi_0^2 |u|^2 d\omega \le M_0 \|u\|^2$. Thus

$$\sum_{j=0}^N \int_{M_{\eta_1}} |B(x)|_{sp} |\phi_j|^2 |u|^2 d\omega \le \frac{1}{1-\frac{\varepsilon_2}{2}} \sum_{j=1}^N Q_A(\phi_j u) + M_0 \|u\|^2.$$

Rearranging, and combining with $Q_A(\phi_0 u) \ge 0$

$$\sum_{j=0}^N Q_A(\phi_j u) \ge (1-\frac{\varepsilon_2}{2}) \sum_{j=0}^N \int_{M_{\eta_1}} |B(x)|_{sp} |\phi_j|^2 |u|^2 d\omega - (1-\frac{\varepsilon_2}{2}) M_0 \|u\|^2,$$

yielding

$$Q_A(u) \ge \sum_{j=0}^N Q_A(\phi_j u) + L\|u\|^2 \ge (1-\frac{\varepsilon_2}{2}) \sum_{j=0}^N \int_{M_{\eta_1}} |B(x)|_{sp} |\phi_j|^2 |u|^2 d\omega - [(1-\frac{\varepsilon_2}{2}) M_0 + L] \|u\|^2.$$

Setting $C_{\varepsilon_2} = (1-\frac{\varepsilon_2}{2})M_0 + L$ and summing over the partition yields the desired result

$$Q_A(u) \geq (1-\frac{\varepsilon_2}{2})\int_{M_{\eta_1}} |B(x)|_{sp}|u|^2 d\omega - C_{\varepsilon_2}\|u\|^2.$$

□

Theorem 3.4 can be trivially proven by using all the results of this section.

Theorem (3.4)**.** *Let (M,g) Riemannian manifold with magnetic field $B = dA$ and connection given by $\nabla^A = d - iA$. Assume also that*

$$|B|_{sp} \geq \frac{1+\varepsilon_2}{\rho^2}$$

near Σ_A, for each $x_0 \in \Sigma_A$ the direction $n(x)$ of B has limit as $x \to x_0$ and g is regular that is $\kappa = \sqrt{det(g_{ij})} \neq 0$. Then Δ^A_ω with domain $\mathcal{D}(\Delta^A_\omega) = C^\infty_c(M \setminus \Sigma_A)$ is essentially self adjoint in $L^2(M)$.

Proof. By Theorem 5.3 $\forall \varepsilon_2 > 0$ there exists $C_{\varepsilon_2} \in \mathbb{R}$ such that,

$$Q_A(u) \geq (1-\varepsilon_2)\int_{M_{\eta_1}} |B(x)|_{sp}|u|^2 d\omega - C_{\varepsilon_2}\|u\|^2$$

which implies that,

$$Q_A(u) \geq (1-\varepsilon_2^2)\int_{M_{\eta_1}} \frac{|u|^2}{\rho^2} d\omega - C_{\varepsilon_2}\|u\|^2$$

and we conclude by Theorem 2.8 together with Theorem 5.1. □

6 Metric Singularity

Recall, on M_{ε_1} of the metric singularity Σ_g our volume is $\omega = \sqrt{g}\,dx$ and $M_{\varepsilon_1} \simeq (0,\varepsilon_1] \times X_{\varepsilon_1}$ satisfying the same condition (C), we allowed to consider C^1 diffeomorphism as assumed in Lemma 4.1. That is $X_{\varepsilon_1} = \{\delta = \varepsilon_1\}$ is C^2-embedded hypersurface with $\delta(t,x) = t \in (0,\varepsilon_1] \times X_{\varepsilon_1}$. This is a consequence of the measure decomposition $\omega = \kappa(t,x)\,dtd\mu$ where $\kappa = \sqrt{g}$ with dt measure on $(0,\varepsilon_1]$ and $d\mu$ of X_{ε_1} smooth as described in Section 1.2.

Proposition 6.1. *Under Hodge-Helmholtz Decomposition 3.1 the effective potential* (1) *on* $M_{\varepsilon_1} \simeq (0,\varepsilon_1] \times X_{\varepsilon_1}$ *is*

$$V^A_{eff}\,u = \langle A, A\rangle_{L^2(TM)} u \quad \forall u \in C^\infty_c(M).$$

Proof. On $M_{\varepsilon_1} \simeq (0,\varepsilon_1] \times X_{\varepsilon_1}$ with coordinates (t,x) the volume is given by

$$\omega = \kappa(t,x) dt\, d\mu$$

The effective potential (1) is descirbed as

$$V^A_{eff}\,u = -2((\partial_t u)\overline{A_t} + \sum_{i,j} h_{ij}(\partial_i u)\overline{A_j}) + (A_t^2 + \sum_{i,j} h_{ij}A_i\overline{A_j})u,$$

with $\kappa(t,x) = \sqrt{det\,h(t,x)}$. Metric and the magnetic potential is given as a consequence of Gauss Lemma [15, Theorem 6.9]

$$g = dt^2 + h_{ij}(x)dx^i dx^j, \quad A = A_t dt + \sum_i^{m-1} A_i dx^i.$$

By the divergence free condition of our magnetic potential, if we consider the dual $\tilde{A}$ of the magnetic potential, as in Section 3.1

$$\tilde{A} = A_t\partial_t + \sum_{i,j} h_{ij}A_i\partial_j, \quad \text{for } 1 \le i < j \le m-1,$$

and compute the divergence

$$div_\omega(\tilde{A}) = \frac{1}{\kappa}\partial_t(\kappa\overline{A_t}) + \frac{1}{\kappa}\sum_{i<j}\partial_i(\kappa h_{ij}\overline{A_j}) = 0 \quad \Rightarrow \quad \partial_t(\kappa\overline{A_t}) + \sum_{i<j}\partial_i(\kappa h_{ij}\overline{A_j}) = 0.$$

Integrating our effective potential on M_{ε_1} and using Fubini's Theorem

$$\int_{M_{\varepsilon_1}} V^A_{eff} u\, d\omega = \int_0^{\varepsilon_1} dt \int_{X_{\varepsilon_1}} [-2(\partial_t u)\overline{A_t}\kappa - 2\sum_{i,j}(h_{ij}(\partial_i u)\overline{A_j}\kappa) + (A_t^2)u\kappa + \sum_{i,j}(h_{ij}A_i\overline{A_j}u\kappa)]d\mu.$$

the term after integrating by parts for $u \in C^\infty_c(M)$

$$\int_{X_{\varepsilon_1}} \sum_{i<j} h_{ij}\partial_i(u)\overline{A_j}\kappa d\mu = \int_{X_{\varepsilon_1}} \sum_{i,j}\partial_i(u)(\kappa h_{ij}\overline{A_j})d\mu = -\int_{X_{\varepsilon_1}} \sum_{i,j} u\partial_i(h_{ij}\kappa\overline{A_j})d\mu$$

Combining this on X_{ε_1}

$$\int_{X_{\varepsilon_1}} V^A_{eff} u\,\kappa\, d\mu = \int_{X_{\varepsilon_1}} -2(\partial_t u)\overline{A_t}\kappa + \sum_{i,j} 2u\,\partial_i(h_{ij}\kappa\overline{A_j}) + (A_t^2 + \sum_{i,j} h_{ij}A_i\overline{A_j})u\kappa\, d\mu.$$

We use the divergence free condition

$$-\partial_t(\kappa A_t) = \sum_{i,j} \partial_i(h_{ij}\kappa\overline{A_j}),$$

to get

$$\int_{X_{\varepsilon_1}} V^A_{eff} u\, d\mu = \int_{X_{\varepsilon_1}} [-2\partial_t u\overline{A_t}\kappa - 2u\partial_t(\kappa\overline{A_t}) + (A_t^2 + A_x^2)u\kappa] d\mu.$$

We set $A_x^2 = \sum_{i,j} h_{ij}A_i\overline{A_j}$ and observe that $-2\partial_t u\overline{A_t}\kappa - 2u\partial_t(\kappa\overline{A_t}) = -2\partial_t(u\overline{A_t}\kappa)$ which eventually leads to

$$\int_{M_{\varepsilon_1}} V^A_{eff} u d\omega = \int_0^{\varepsilon_1} dt \int_{X_{\varepsilon_1}} [-2\partial_t(u\overline{A_t}\kappa) + (A_t^2 + A_x^2)u\kappa] d\mu.$$

Using again Fubini's

$$\int_{M_{\varepsilon_1}} V^A_{eff} u\, d\omega = \int_{X_{\varepsilon_1}} d\mu \int_0^{\varepsilon_1} [-2\partial_t(u\overline{A_t}\kappa) + (A_t^2 + A_x^2)u\kappa] dt.$$

Since $u \in C_c^\infty(M)$, $u(0) = u(\varepsilon_1) = 0$ gives

$$\int_0^{\varepsilon_1} -2\partial_t(u\overline{A_t}\kappa) dt = 0.$$

and conclude

$$\int_{M_{\varepsilon_1}} V^A_{eff} u\, d\omega = \int_0^{\varepsilon_1} \int_{X_{\varepsilon_1}} (A_t^2 + A_x^2) u\, \kappa\, d\mu\, dt$$

for the desired result. □

Recall

$$Q_A(u) = \int_M g(\nabla^A u, \nabla^A u) d\omega,$$

but restricted now to M_ε and using Proposition (6.1) takes the form

$$Q_A(u)\big|_{M_{\varepsilon_1}} = \int_{M_{\varepsilon_1}} (g(du,du) + g(A,A)|u|^2)\, d\omega.$$

However together with the identification of the tubular $M_{\varepsilon_1} \simeq (0,\varepsilon_1] \times X_{\varepsilon_1}$, if the density is given as $\kappa = e^{2\vartheta}$ then the Laplace-Beltrami satisfies

$$\Delta_\omega = \Delta + g(\nabla\vartheta, \nabla\cdot),$$

where Δ the Laplace-Beltrami operator for the volume vol_g. The extra term of the effective potential appearing in this tubular neighborhood depending on the metric and measure through the identification is described as

$$V^\omega_{eff} = (\frac{\Delta_\omega \delta}{2})^2 + (\frac{\Delta_\omega \delta}{2})'.$$

with the second term denoting the usual derivative. Using [2, Proposition 2.2]

$$V^\omega_{eff} = (\partial_t\vartheta)^2 + \partial_t^2\vartheta.$$

with ϑ smooth on t and continuous on x. This is a consequence of the properties given by Lemma 4.1. By defining $V_{eff} = V^\omega_{eff} + V^A_{eff}$ we get the following.

Theorem 6.1. *Let (M,g) our Riemannian manifold satisfying the condition* (C) *for some $\varepsilon_1 > 0$ and assume that $\exists \lambda \geq 0$ such that on M_{ε_1},*

$$V_{eff} \geq \frac{3}{4\delta^2} - \frac{\lambda}{\delta}$$

then $\exists \eta_2 \leq \frac{1}{\lambda}$ and $c \in \mathbb{R}$ such that,

$$Q_A(u) \geq \int_{M_{\eta_2}} (\frac{1}{\delta^2} - \frac{\lambda}{\delta})|u|^2 d\omega + c\|u\|^2, \quad \forall u \in C_c^\infty(M \setminus (\Sigma_A \cup \Sigma_g)).$$

Proof. First for $u \in C_c^\infty(M_{\varepsilon_1})$ and $\eta_2 = \varepsilon_1$ to prove our lower bound on $M_{\varepsilon_1} \simeq (0,\varepsilon_1] \times X_{\varepsilon_1}$. Recall,

$$V_{eff}^A = \langle A, A \rangle_{L^2(TM)} = \int_{M_{\varepsilon_1}} g(A,A) d\omega$$

and hence our quadratic form

$$Q_A(u)|_{M_{\varepsilon_1}} = \int_{M_{\varepsilon_1}} g(du,du) + (A_t^2 + A_x^2)|u|^2 d\omega$$

$$= \int_{X_{\varepsilon_1}} \int_0^{\varepsilon_1} (g(du,du) + (A_t^2 + A_x^2)|u|^2)\kappa \, dt d\mu.$$

Using now Lemma 4.1 our gradient on M_{ε_1} is $\nabla = \partial_t$ we obtain that

$$g(du,du) \geq |\partial_t u|^2 \quad \textit{and} \quad V_{eff}^A = \langle A, A \rangle_{L^2(TM)} \geq |A_t|^2,$$

leading to

$$Q_A(u) \geq \int_{X_{\varepsilon_1}} \int_0^{\varepsilon_1} (|\partial_t u|^2 + |A_t|^2 |u|^2)\kappa \, dt d\mu.$$

Consider $T : L^2(M, d\omega) \to L^2(M, dt \, d\mu)$ unitary transformation defined by $Tu = e^{\vartheta} u$. Unitary transformations preserve the spectrum and self-adjointness [3],[4]. Setting $v = Tu$ and integrating by parts yields

$$Q_A(u) \geq \int_{M_{\varepsilon_1}} ((|\partial_t u|^2 + |A_t|^2 |u|^2)) d\omega = \int_{M_{\varepsilon_1}} (|\partial_t v|^2 + ((\partial_t \vartheta)^2 + \partial_t^2 \vartheta + |A_t|^2)|v|^2) dt \, d\mu.$$

By [18] we get the 1-D Hardy inequality for any $g_1 \in C_c^\infty((0,\varepsilon])$

$$\int_0^\varepsilon |g_1'(s)|^2 ds \geq \frac{1}{4} \int_0^\varepsilon \frac{|g_1(s)|^2}{s^2} ds.$$

When combined with the assumption for $V_{eff} \geq \frac{3}{4\delta^2} - \frac{\lambda}{\delta}$ and $|A_t|^2 \geq 0$ gives

$$Q_A(u) \geq \int_{M_{\varepsilon_1}} (|\partial_t v|^2 + (V_{eff}^\omega + |A_t|^2)|v|^2) dt \, d\mu \geq \int_{X_{\varepsilon_1}} \int_0^{\varepsilon_1} (\frac{1}{4\delta^2} + \frac{3}{4\delta^2} - \frac{\lambda}{\delta})|u|^2 \, dt \, d\mu$$

$$= \int_{X_{\varepsilon_1}} \int_0^{\varepsilon_1} (\frac{1}{\delta^2} - \frac{\lambda}{\delta})|u|^2 dt \, d\mu \tag{13}$$

and conclude with $\eta_2 = \varepsilon_1$ and $c = 0$.

We need to generalize on M_{η_2} with $\eta_2 \leq c - \varepsilon_1$, $c + \varepsilon_1 \leq \eta_2$ where $c = d_g(\Sigma_A, \Sigma_g)$. Consider $u \in C_c^\infty(M \setminus (\Sigma_g \cup \Sigma_A))$ and consider χ_1, χ_2 smooth on $[0, +\eta_2)$ such that,

- $0 \leq \chi_i \leq 1$ for $i = 1,2$,
- $\chi_1 \equiv 1$ on $[0, \varepsilon_1/2]$ and $\chi_1 \equiv 0$ on $[\varepsilon_1, +\eta_2)$,
- $\chi_2 \equiv 0$ on $[0, \varepsilon_1/2]$ and $\chi_2 \equiv 1$ on $[\varepsilon_1, +\eta_2)$,
- $\chi_1^2 + \chi_2^2 \equiv 1$.

We compose the above function with the distance function δ to obtain $\phi_i : M_{\eta_2} \to \mathbb{R}$ such that $\phi_i = \chi \circ \delta$ with the following properties.

- $\phi_1 \equiv 1$ on $M_{\frac{\varepsilon_1}{2}} \Rightarrow M_{\frac{\varepsilon_1}{2}} \subseteq supp(\phi_1) \subseteq M_{\varepsilon_1}$,
- $0 \leq \phi_1 \leq 1$ and $\phi_1^2 + \phi_2^2 = 1$.

We can consider such χ_i, ϕ_i not affected by Σ_A from the choice of η_2. Note that on $M_{\eta_2} \setminus M_{\varepsilon_1}$ that $\phi_2 \equiv 0$ and $\phi_1 \equiv 1$ which eventually implies that $\nabla\phi_1 \equiv 0$ since $|\nabla\delta| \leq 1$

$$c_1 = \sup_M \sum_{i=1}^{2} |\nabla\phi_i|^2 \leq \sup_{[0,\varepsilon]} \sum_{i=1}^{2} |\chi_i'|^2 < +\infty.$$

Noting $supp(\phi_2 u) \subseteq M_{\eta_2} \setminus M_{\frac{\varepsilon_1}{2}}$ we get $Q_A(\phi_2 u) \geq 0$ because

$$Q_A(\phi_2 u) = \int_{M_{\eta_2}} (g(\nabla\phi_2 u, \nabla\phi_2 u) + g(A,A)|\phi_2 u|^2)\, d\omega.$$

where each term in the integral is bounded from below by the positive definite property of g. Recalling Lemma 4.2 we are allowed to apply the IMS (4) formula,

$$Q_A(u) = \sum_{i=1}^{2} Q_A(\phi_i u) - \sum_{i=1}^{2} \int_{M_{\eta_2}} |\nabla\phi_i|^2 |u|^2\, d\omega \geq Q_A(\phi_1 u) - c_1 \|u\|^2$$

and applying (13) for $\eta_2 = \varepsilon_1$ and $c = 0$ we get,

$$Q_A(u) \geq \int_{M_{\varepsilon_1}} (\frac{1}{\delta^2} - \frac{\lambda}{\delta})|\phi_1 u|^2\, d\omega - c_1 \|u\|^2$$

for $\eta_2 = \min\{\frac{\varepsilon_1}{2}, \frac{1}{\lambda}, c - \varepsilon_2\}$ we get that,

$$Q_A(u) \geq \int_{M_{\eta_2}} (\frac{1}{\delta^2} - \frac{1}{\lambda})|u|^2\, d\omega - \int_{M_{\varepsilon_1} \setminus M_{\eta_2}} (\frac{1}{\delta^2} - \frac{\lambda}{\delta})|\phi_1 u|^2\, d\omega - c_1 \|u\|^2$$

$$\geq \int_{M_{\eta_2}} (\frac{1}{\delta^2} - \frac{\lambda}{\delta})|u|^2\, d\omega - (c_1 + \sup_{\eta_2 \leq \delta \leq \varepsilon_1} |\frac{1}{\delta^2} - \frac{1}{\lambda}|)\|u\|^2$$

where we set $c = c_1 + \sup_{\eta_2 \leq \delta \leq \varepsilon_1} |\frac{1}{\delta^2} - \frac{1}{\lambda}|$ and conclude. □

Final result of this section is the analogues of the Agmon-type estimate Theorem 5.3 but for the metric singularity region.

However the cut-off function in this case differs,when we compare the two Figures 3 and 4

Theorem 6.2 (Agmon-type Estimate)**.** *Assume that* $\exists\, \lambda \geq 0,\ \eta_2 \leq \frac{1}{\lambda}$ *and* $c \in \mathbb{R}$ *such that,*

$$Q_A(u) \geq \int_{M_{\eta_2}} (\frac{1}{\delta^2} - \frac{\lambda}{\delta})|u|^2 d\omega + c\|u\|^2,\ \forall u \in C_c^\infty(M \setminus (\Sigma_A \cup \Sigma_g))$$

then $\forall E < c$ *the only weak solution* $\psi \in L^2(M)$ *for* $(\Delta_\omega^A - E)\psi = 0$ *is* $\psi \equiv 0$*.*

Proof. First recall the identity proven in Lemma 4.2 where we got that

$$\langle f_2 v, (\Delta_\omega^A - E) f_2 v\rangle_{L^2(M)} = \langle v, |df_2|_g^2 v\rangle_{L^2(M)}.$$

This can be written equivalently as

$$Q_A(f_2 v) = E\|f_2 v\|^2 + \langle v, |df_2|_g^2 v\rangle_{L^2(M)}.$$

Consider $f_2 : M \to \mathbb{R}$ Lipschitz with compact support with $supp(f_2) \subset \overline{M_{\eta_2} - M_\zeta}$ for some $\zeta > 0$, η_2 as in Theorem 6.1 and let v weak solution of $(\Delta_\omega^A - E)v = 0$ for $E < c$.

By Theorem 6.1 we claim

$$(c - E)\|f_2 v\|^2 \leq \langle v, |df_2|_g^2 v\rangle_{L^2(M)} - \int_{M_{\eta_2}} (\frac{1}{\delta^2} - \frac{\lambda}{\delta})|f_2 v|^2 d\omega.$$

If f_2 is compactly supported, then $f_2 v \in C_c^\infty(M \setminus (\Sigma_A \cup \Sigma_g))$ and hence claim holds true. In general the product of $f_2 v$ is not always smooth but if f is Lipschitz we obtain differentiability almost everywhere. We set

$$f_2(x) = \begin{cases} F_2(\delta(x)), & 0 < \delta \leq \eta_2 \\ 1, & \delta > \eta_2 \end{cases}$$

where F_2 is Lipschitz and differentiable almost everywhere. By Lemma 4.1 $|\nabla\delta| \leq 1$ almost everywhere on M hence on M_{η_2} we have that

$$|\nabla f_2| = |F_2'(\delta)||\nabla\delta| \leq |F_2'(\delta)|,$$

thus our claim yields

$$(c - E)\|f_2 v\|^2 \leq \int_{M_{\eta_2}} [(F_2'(\delta))^2 - (\frac{1}{\delta^2} - \frac{\lambda}{\delta})F(\delta)^2]|v|^2 d\omega.$$

Let $0 < 2\zeta < \eta_2$, and we choose F_2 for $\tau \in [2\zeta, \eta_2]$ to be solution of

$$F_2'(\tau) = \sqrt{\frac{1}{\tau^2} - \frac{\lambda}{\tau}} F_2(\tau), \quad F_2(\eta_2) = 1,$$

zero on $[0, \zeta]$ and linear everywhere else on $[\zeta, 2\zeta]$. By $\eta_2 \leq \frac{1}{\lambda}$ the solution of the above ODE is well defined. f_2 is Lipschitz hence differentiable almost everywhere with compact support which implies that on $[\zeta, 2\zeta]$ we get $F_2' \leq N$ for some finite constant N.

If $\lambda = 0$ the claim is trivial because we want $F_2'(\tau) = \frac{1}{\tau} F_2(\tau)$ which implies that

$$F(\tau) = \frac{\tau}{\eta_2} \quad \Rightarrow \quad F(2\zeta) = \frac{2\zeta}{\eta_2}.$$

For $\lambda > 0$ and $\tau \in [\zeta, 2\zeta]$ by linearity that we need

$$F_2(\tau) = \frac{\tau - \zeta}{\zeta} F_2(2\zeta), \quad F_2'(\tau) = \frac{F_2(2\zeta)}{\zeta}.$$

The solution is given by on $[2\zeta, \eta_2]$,

$$F_2(\tau) = c(\lambda, \eta_2) \frac{1 - \sqrt{1-\lambda\tau}}{1 + \sqrt{1+\lambda\tau}} e^{2\sqrt{1-\lambda\tau}}, \quad \tau \in [2\zeta, \eta_2]$$

for some constant $c(\lambda, \eta_2)$ such that $F_2(\eta_2) = 1$. Thus on $[\zeta, 2\zeta]$ we get that,

$$F_2(2\zeta) = \frac{1}{2} c(\lambda, \eta_2) e^2 \lambda\zeta + o(\zeta)$$

which implies boundedness of F_2' on $[\zeta, 2\zeta]$ and therefore

$$(c - E)\|f_2 v\|^2 \leq N \int_{\zeta \leq \delta \leq 2\zeta} |v|^2 d\omega.$$

Sending $\zeta \to 0$ and from the fact that $f_2 \to 1$ pointwise almost everywhere with $c < E$,

$$\|f_2 v\|^2 = 0 \Rightarrow v \equiv 0$$

and we conclude by setting $\psi = f_2 v$. □

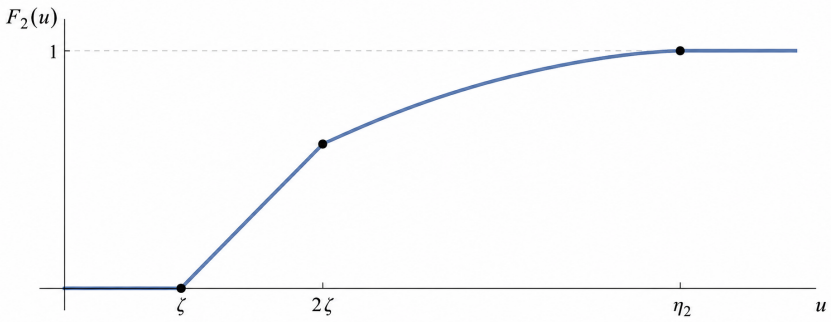

Figure 4: Cut-off Function F_2

Theorem 3.5 can now be trivially proven

Theorem (3.5)**.** *Let (M, g) Riemmanian manifold which is regular satisfying condition* (C) *for $\varepsilon_1 > 0$. Assume there exists $\lambda \geq 0$ such that*

$$V_{eff} \geq \frac{3}{4\delta^2} - \frac{\lambda}{\delta},$$

and that A is regular on Σ_g, that is $A = \sum_{j=1}^m A_j \, dx^j$ has smooth components on Σ_g. Then Δ_ω^A with domain $\mathcal{D}(\Delta_\omega^A) = C_c^\infty(M \setminus \Sigma_A)$ is essentially self adjoint in $L^2(M)$

Proof. By Theorem 2.8 together with Theorem 6.2 we obtain the desired result. □

7 Overlapping Singularities

We will assume from now that $\Sigma_g = \Sigma_A = \Sigma$ are the only singularity appearing on our manifold (M,g). We will consider tubular neighborhood $M_\varepsilon \simeq (0,\varepsilon] \times X_\varepsilon$ with the distance function $\delta = \rho$ satisfying the condition (C) at all times.
Recall that using Proposition 6.1 and the divergence free condition that the effective potential depended on the magnetic field A is

$$V^A_{eff}\, u = \langle A, A\rangle_{L^2(TM)} u \quad \forall u \in C^\infty_c(M),$$

hence the quadratic form is

$$Q_A(u) = \int_M (g(du,du) + g(A,A)|u|^2) d\omega$$

where $d\omega = \kappa\, dt d\mu$ in $M_\varepsilon \simeq (0,\varepsilon] \times X_\varepsilon$. Again in this tubular neighborhood density is given as $\kappa = e^{2\vartheta}$ and the effective potential dependent on the measure is described as

$$V^\omega_{eff} = (\partial_t \vartheta)^2 + \partial_t^2 \vartheta$$

with ϑ smooth on t and continuous on x. By giving the appropriate bounds on the potentials above we achieve semi-boundedness of the quadratic form.

Theorem 7.1. *Let (M,g) our Riemannian manifold satisfying condition* (C) *for some $\varepsilon > 0$ and assume that $\exists \lambda \geq 0$ such that on M_ε*

$$V^\omega_{eff} \geq \frac{C_a}{\delta^a} - \frac{\lambda}{\delta} \quad \text{and} \quad V^A_{eff} \geq \frac{C_b}{\delta^b} \tag{14}$$

then there exists η satisfying

- *for $\lambda = 0$*

$$\frac{C_a}{\eta^a} + \frac{C_b}{\eta^b} + \frac{1}{4\eta^2} \geq 0 \tag{15}$$

- *for $\lambda > 0$*

$$\frac{C_a}{\eta^{a-1}} + \frac{C_b}{\eta^{b-1}} + \frac{1}{4\eta} \geq \lambda \tag{16}$$

for some finite constants $C_a, C_b > 0$ with $\max\{a,b\} \geq 2$. Under this assumptions we obtain

$$Q_A(u) \geq \int_{M_\eta} (\frac{C_a}{\delta^a} + \frac{C_b}{\delta^b} + \frac{1}{4\delta^2} - \frac{\lambda}{\delta})|u|^2 d\omega + \tilde{c}\|u\|^2, \quad \forall u \in C^\infty_c(M \setminus \Sigma).$$

Proof. The proof is obtained exactly as in Theorem 6.1 by using the bounds (14). □

Note that the case $a = b = 2$ is the case in Section 6 with the constants satisfying $C_a + C_b + \frac{1}{4} \geq 1$. We will give the Agmon-type estimate result for the case $a,b > 2$ and $\max\{a,b\} \geq 2$ where not both a,b are at least 2 simultaneously.

Theorem 7.2. *Assume that* $\exists\, \lambda \geq 0$ *and* η *satisfying conditions* (15),(16) *and* $\tilde{c} \in \mathbb{R}$ *such that*

$$Q_A(u) \geq \int_{M_\eta} (\frac{C_a}{\delta^a} + \frac{C_b}{\delta^b} + \frac{1}{4\delta^2} - \frac{\lambda}{\delta})|u|^2 d\omega + \tilde{c}\|u\|^2, \quad \forall u \in C_c^\infty(M \setminus \Sigma).$$

Then $\forall E < \tilde{c}$ *the only weak solution* $\psi \in L^2(M)$ *for* $(\Delta_\omega^A - E)\psi = 0$ *is* $\psi \equiv 0$.

Proof. Recall by Lemma 4.2 we have

$$\langle fv, (\Delta_\omega^A - E) fv \rangle_{L^2(M)} = \langle v, |df|_g^2 v \rangle_{L^2(M)},$$

which is written equivalently as

$$Q_A(fv) = E\|fv\|^2 + \langle v, |df|_g^2 v \rangle_{L^2(M)}.$$

Consider $f : M \to \mathbb{R}$ Lipschitz with compact support with $\mathrm{supp}(f) \subset \overline{M_\eta \setminus M_\zeta}$ for some $\zeta > 0$. Let v weak solution of $(\Delta_\omega^A - E)v = 0$ for $E < \tilde{c}$. By Theorem 7.1 we claim

$$(\tilde{c} - E)\|fv\|^2 \leq \langle v, |df|_g^2 v \rangle_{L^2(M)} - \int_{M_\eta} (\frac{C_a}{\delta^a} + \frac{C_b}{\delta^b} + \frac{1}{4\delta^2} - \frac{\lambda}{\delta})|v|^2 d\omega. \tag{17}$$

For f compactly supported we have that $fv \in C_c^\infty(M \setminus \Sigma)$ and hence claim holds true. In general this is not true but by Lipschitz assumption f is differentiable almost everywhere. Set

$$f = \begin{cases} F(\delta(x)), & 0 < \delta \leq \eta \\ 1, & \delta > \eta \end{cases}$$

with F Lipschitz and differentiable almost everywhere. Using $|\nabla \delta| \leq 1$ by Lemma 4.1 on M_ε we have

$$|\nabla f| = |F'(\delta)||\nabla \delta| \leq |F'(\delta)|,$$

thus the claim (17) gives

$$(\tilde{c} - E)\|fv\|^2 \leq \int_{M_{\eta_2}} [(F'(\delta))^2 - (\frac{C_a}{\delta^a} + \frac{C_b}{\delta^b} + \frac{1}{4\delta^2} - \frac{\lambda}{\delta})F(\delta)^2]|v|^2 d\omega. \tag{18}$$

We set F equal 0 on $[0, \zeta]$, linear in $[\zeta, 2\zeta]$ and on $[2\zeta, \eta]$ satisfying

$$F'(\tau) = \sqrt{\frac{C_a}{\tau^a} + \frac{C_b}{\tau^b} + \frac{1}{4\tau^2} - \frac{\lambda}{\tau}} F(\tau), \quad F(\eta) = 1. \tag{19}$$

We need the conditions (15),(16) to achieve well definiteness of the ODE (19) and that $F' \leq N$ for some finite constant in $[\zeta, 2\zeta]$.

The general solution since (19) is separable ODE can be described

$$F(\tau) = \exp(-\int_\tau^\eta \sqrt{\left(\frac{C_a}{\delta^a} + \frac{C_b}{\delta^b} + \frac{1}{4\delta^2} - \frac{\lambda}{\delta}\right)} ds). \tag{20}$$

By linearity on $[\zeta, 2\zeta]$ we get

$$F(\tau) = \frac{\tau - \zeta}{\zeta} F(2\zeta) \quad \Rightarrow \quad F'(\tau) = \frac{F(2\zeta)}{\zeta}. \tag{21}$$

Thus to establish boundedness as $\zeta \to 0$ we need to show that at least $F(2\zeta) \leq \zeta N$.

Case 1: For $a,b>2$ we set $\max\{a,b\}=p>2$. By the choice of $C_a,C_b>0$ for $s\in(0,\eta]$ there exists $C_p>0$ such that

$$\sqrt{R(s)}=\sqrt{\left(\frac{C_a}{s^a}+\frac{C_b}{s^b}+\frac{1}{4s^2}-\frac{\lambda}{s}\right)}\geq\frac{C_p}{s^{\frac{p}{2}}}. \tag{22}$$

For both cases $\lambda=0$ and $\lambda\neq 0$ satisfying (15) and (16) respectively the term $\frac{1}{s^{\frac{p}{2}}}$ overpowers the rest. Thus

$$\int_{2\zeta}^{\eta}\sqrt{R(s)}ds\geq\int_{2\zeta}^{\eta}\frac{C_p}{s^{\frac{p}{2}}}ds=\frac{C_p}{\frac{p}{2}-1}\left(\frac{1}{(2\zeta)^{\frac{p}{2}-1}}-\frac{1}{\eta^{\frac{p}{2}-1}}\right). \tag{23}$$

By substituting (23) to (20) yields

$$F(2\zeta)\leq\exp\left(-\frac{C_p}{\frac{p}{2}-1}(2\zeta)^{1-\frac{p}{2}}\right)\exp\left(\frac{C_p}{\frac{p}{2}-1}\eta^{1-\frac{p}{2}}\right). \tag{24}$$

Let $k_1=\frac{C_p\cdot 2^{1-\frac{p}{2}}}{\frac{p}{2}-1}$ and $k_2=\exp\left(\frac{C_p}{\frac{p}{2}-1}\eta^{1-\frac{p}{2}}\right)$ then (21) satisfies

$$F'(\tau)\leq\frac{k_2\ \exp(k_1\zeta^{1-\frac{p}{2}})}{\zeta}. \tag{25}$$

Since $p>2$ we have that the exponential decay tends to zero faster than the linear term ζ as $\zeta\to 0$ which gives the desired boundedness of $F'(\tau)$.
Case 2: Let $\max\{a,b\}=2$ and we assume without loss of generality that $a=2$ and $b<2$. The exact same result holds for the inverse by swapping the indices a and b.
For this particular case for $s\in(0,\eta]$

$$\sqrt{R(s)}=\sqrt{\frac{C_a}{s^2}+\frac{1}{4s^2}+\frac{C_b}{s}-\frac{\lambda}{s}},$$

thus the term $\frac{1}{s^2}$ dominates the rest. We define $K^2=C_a+\frac{1}{4}$.
Subcase 1: For $\lambda=0$

$$F'(\tau)=\frac{K}{\tau}F(\tau)$$

and

$$\int_{2\zeta}^{\eta}\frac{K}{s}ds=\ln(\frac{\eta}{2\zeta})^K.$$

Hence

$$F(2\zeta)=\exp(-\ln(\frac{\eta}{2\zeta})^K)=(\frac{2\zeta}{\eta})^K,$$

which eventually implies

$$F'(\tau)=\frac{2^K}{\eta^K}\zeta^{K-1}.$$

To achieve boundedness we need $K-1\geq 0$, which leads to the optimal constant condition $C_a\geq\frac{3}{4}$ and thus boundedness of $F'(\tau)$ as $\zeta\to 0$.
Subcase 2: For $\lambda>0$

$$F'(\tau)=\sqrt{\frac{K^2}{\tau^2}-\frac{\lambda}{\tau}}F(\tau),$$

where we must evaluate

$$\int_{\tau}^{\eta} \sqrt{\frac{K^2}{s^2} - \frac{\lambda}{s}} ds.$$

Let $u = \sqrt{K^2 - \lambda s}$ then $s = \frac{K^2 - u^2}{\lambda}$ and $ds = -2\frac{u}{\lambda}\, du$. We get

$$\int \frac{\lambda}{K^2 - u^2} u \left(-\frac{2u}{\lambda}\right) du = 2\int \frac{u^2}{u^2 - K^2} du = 2u + K \ln \left|\frac{K-u}{K+u}\right|.$$

Substituting u back and applying the bounds on the general solution (20) yields

$$F(\tau) = c(\lambda, \eta) \left(\frac{K - \sqrt{K^2 - \lambda\tau}}{K + \sqrt{K^2 - \lambda\tau}}\right)^K \exp(2\sqrt{K^2 - \lambda\tau}).$$

To evaluate $F(2\zeta)$ we find the leading term of the numerator as $\zeta \to 0$. Using Taylor Series expansion for the square root we obtain

$$K - \sqrt{K^2 - 2\lambda\zeta} = K - K\sqrt{1 - \frac{2\lambda\zeta}{K^2}} = \frac{\lambda\zeta}{K} + o(\zeta).$$

The denominator converges to $2K$ therefore

$$F(2\zeta) = c(\lambda, \eta) e^{2K} \left(\frac{\lambda}{2K^2}\right)^K \zeta^K + o(\zeta^K),$$

which eventually gives

$$F'(\tau) = \frac{F(2\zeta)}{\zeta} = c(\lambda, \eta) e^{2K} \left(\frac{\lambda}{2K^2}\right)^K \zeta^{K-1} + o(\zeta^{K-1}).$$

Hence to achieve boundedness as $\zeta \to 0$ we need $K - 1 \geq 0$ which implies $C_a \geq \frac{3}{4}$.

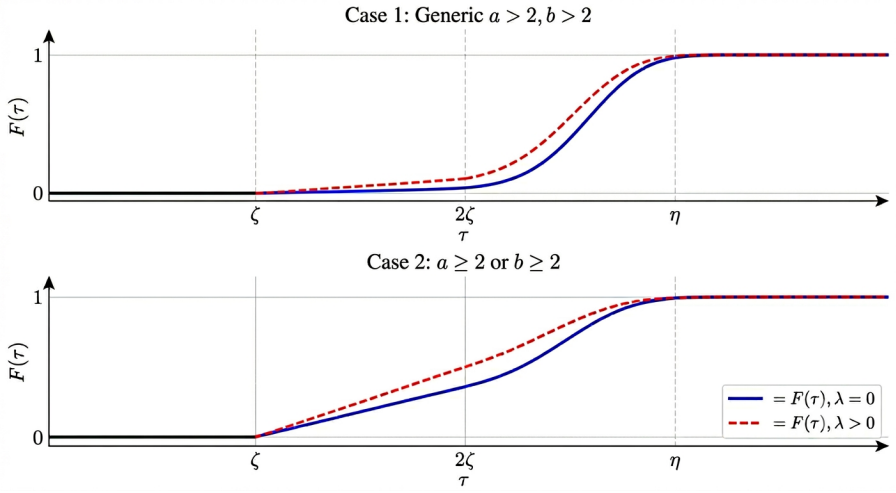

Figure 5: Cut-off Functions for Cases 1 and 2

Using both cases we achieve boundedness of F' in $[\zeta, 2\zeta]$ thus (18) leads to

$$(\tilde{c} - E)\|fv\|^2 \leq \int_{\zeta \leq \delta \leq 2\zeta} N|v|^2 d\omega$$

Sending $\zeta \to 0$, f is converging pointwise almost everywhere to 1 and by the choice $\tilde{c} < E$ we obtain $v \equiv 0$. Define $\psi = fv$ and conclude. □

Recall the main theorem 3.6 of this section is given as

Theorem (3.6)**.** *Let (M, g) Riemannian manifold which is regular and satisfying condition* (C) *for $\varepsilon > 0$ at all times. Assume there exists $\lambda \geq 0$ such that*

$$V_{eff}^{A} \geq \frac{C_b}{\delta^b} \quad \textit{and} \quad V_{eff}^{\omega} \geq \frac{C_a}{\delta^a} - \frac{\lambda}{\delta},$$

then operator Δ_ω^A with domain $\mathcal{D}(\Delta_\omega^A) = C_c^\infty(M \setminus \Sigma)$ is essentially self adjoint in $L^2(M)$, if

- *For $a > 2$,$b > 2$ let C_a,$C_b > 0$;*
- *For $a \geq 2$,$0 \leq b \leq 1$ let $C_a \geq \frac{3}{4}$;*
- *For $b \geq 2$,$0 \leq a \leq 1$ let $C_b \geq \frac{3}{4}$.*

Proof. Using Theorem 2.8 together with the Agmon-type estimate Theorem 7.2 we conclude the proof. □

Bibliography

[1] D. Borthwick, *Spectral Theory: Basic Concepts and Applications*, vol. 284 of *Graduate Texts in Mathematics*. Cham: Springer International Publishing, 2020.

[2] D. Prandi, L. Rizzi, and M. Seri, "Quantum confinement on non-complete Riemannian manifolds," *Journal of Spectral Theory*, vol. 8, pp. 1221–1280, jul 2018. arXiv:1609.01724.

[3] T. Kato, "Remarks on schrödinger operators with vector potentials," *Integral Equations and Operator Theory*, vol. 1, p. 103–113, Mar. 1978.

[4] B. Simon, "Kato's inequality and the comparison of semigroups," *Journal of Functional Analysis*, vol. 32, p. 97–101, Apr. 1979.

[5] H. Leinfelder and C. G. Simader, "Schrödinger operators with singular magnetic vector potentials," *Mathematische Zeitschrift*, vol. 176, no. 1, pp. 1–19, 1981.

[6] Y. Colin de Verdière and F. Truc, "Confining quantum particles with a purely magnetic field," *Annales de l'Institut Fourier*, vol. 60, no. 7, pp. 2333–2356, 2010. arXiv:0903.0803.

[7] G. Nenciu and I. Nenciu, "On confining potentials and essential self-adjointness for Schrödinger operators on bounded domains in $\mathbb{R}^n$," *Annales Henri Poincaré*, vol. 10, pp. 377–394, may 2009. arXiv:0811.2982.

[8] G. Nenciu and I. Nenciu, "On essential self-adjointness for magnetic Schroedinger and Pauli operators on the unit disc in $\mathbb{R}^2$," *Letters in Mathematical Physics*, vol. 98, pp. 207–223, nov 2011. arXiv:1003.3099.

[9] M. Shubin, "Essential self-adjointness for semi-bounded magnetic Schrödinger operators on non-compact manifolds," *Journal of Functional Analysis*, vol. 186, pp. 92–116, oct 2001. arXiv:math/0007019.

[10] U. Boscain and C. Laurent, "The laplace-beltrami operator in almost-riemannian geometry," *Annales de l'Institut Fourier*, vol. 63, no. 5, p. 1739–1770, 2013.

[11] M. Gordina and T. Laetsch, "Sub-Laplacians on sub-Riemannian manifolds," *Potential Analysis*, vol. 44, no. 4, pp. 811–837, 2016. arXiv:1412.0155.

[12] V. Franceschi, D. Prandi, and L. Rizzi, "On the essential self-adjointness of sub-Laplacians," *Potential Analysis*, vol. 53, no. 1, pp. 89–112, 2020. arXiv:1708.09626.

[13] M. Reed and B. Simon, *Functional analysis*, vol. 1 of *Methods of Modern Mathematical Physics*. New York: Academic Press, 1972.

[14] M. Reed and B. Simon, *Methods of Modern Mathematical Physics. II. Fourier Analysis, Self-adjointness*, vol. 2. New York: Academic Press, 1975.

[15] J. M. Lee, *Introduction to Riemannian Manifolds*, vol. 176 of *Graduate Texts in Mathematics*. Cham: Springer International Publishing, 2018.

[16] P. Petersen, *Riemannian Geometry*, vol. 171 of *Graduate Texts in Mathematics*. Cham: Springer, 3rd ed., 2016.

[17] J. Jost, *Riemannian Geometry and Geometric Analysis*. Berlin, Heidelberg: Springer-Verlag Berlin Heidelberg, 4th ed., 2005.

[18] A. M. Hinz, *Topics from spectral theory of differential operators*, vol. 340, p. 1–50. Providence, Rhode Island: American Mathematical Society, 2004.

[19] R. Sjamaar, "Manifolds and differential forms." Lecture notes, Cornell University. Available at https://pi.math.cornell.edu/~sjamaar/manifolds/manifold.pdf, 2017.

[20] M. Levitin, D. Mangoubi, and I. Polterovich, *Topics in Spectral Geometry*, vol. 237 of *Graduate Studies in Mathematics*. Providence, Rhode Island: American Mathematical Society, 2023.

[21] S. Agmon, *Lectures on exponential decay of solutions of second-order elliptic equations: bounds on eigenfunctions of N-body Schrödinger operators*, vol. 29 of *Mathematical Notes*. Princeton, N.J.: Princeton University Press, 1982.

[22] M. Gromov, "Kähler hyperbolicity and L^2-Hodge theory," *Journal of Differential Geometry*, vol. 33, pp. 263–290, jan 1991.

[23] F. Clément and V. Martin, "The Lax–Milgram theorem. A detailed proof to be formalized in Coq," *HAL preprint*, 2021. hal-03134114.

[24] N. Nowaczyk, "The Hodge Decomposition." Expository article. Available at https://nikno.de/wp-content/uploads/2016/07/hodge.pdf, January 2010.

[25] X. Fernández-Real and X. Ros-Oton, *Regularity Theory for Elliptic PDE*, vol. 28 of *Zurich Lectures in Advanced Mathematics*. EMS Press, 1 ed., dec 2022.