University of Groningen

Master's Thesis

# Contact process on the 1-dimensional disk percolation model

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**Abstract.** We study the contact process on the random disk graph defined on the integer lattice $\mathbb{Z}$. We assume that the radius of influence follows a geometric distribution on $(1, 2, \ldots)$ with parameter $p$. In this process, the infection either survives indefinitely on the graph or eventually dies out. It is well known that the system undergoes a phase transition at a critical value.

In this thesis, we show that the contact process on the random disk graph has the same critical value for both the annealed and quenched processes. We then consider the supercritical regime and condition on survival to obtain our first main result: a shape theorem in one dimension. Although the process is not translation invariant, we prove that a shape theorem still holds under the survival conditioning. We also prove that the speed of the rightmost infected site satisfies an appropriate central limit theorem. Finally, we show that, at the critical value, the process dies out almost surely.

# 1 Introduction

The contact process on $\mathbb{Z}^d$ is a continuous-time Markov process that is widely used in probability theory to model the spread of infectious diseases. It was introduced by Harris in 1974 [9]. The contact process is classical example of an interacting particle system [15] [14], consisting of a configuration of infected and healthy sites. Infected sites recover independently at rate 1, while they infect each neighboring site at rate $\lambda$. A fundamental result for this model is the existence of a phase transition and a corresponding shape theorem [5].

In this thesis, we study a variant of the contact process on a random disk graph. The disk graph model was introduced in [7]. and is constructed by adding edges to a given underlying graph according to a distance-dependent rule. In the present work, the radius of influence associated with each vertex is assumed to be a random variable following a geometric distribution on $(1, 2, \ldots)$ with parameter $p$. We restrict attention to the one-dimensional lattice $\mathbb{Z}$.

This extension introduces a long-range interactions between vertices, and may therefore provide a more realistic framework for modeling the spread of infectious diseases, where individuals exhibit varying ranges of contact.

The main object of interest in this work is the behaviour of the rightmost infected site and the growth speed. Related quantities have also been studied in the context of the rumour model on random disk graphs [17].

n Section 2, we introduce the contact process on the random disk graph on $\mathbb{Z}$. To the best of our knowledge, this is the first study of the contact process on this class of random graphs. We define the model precisely and establish its basic properties.

In Section 3, we state the main results of this thesis, which concern the existence of a phase transition, the rightmost edge speed, and a shape theorem.

We first prove the existence of a finite critical value $\lambda_c < \infty$, and show that the annealed and quenched critical values coincide. The proof is given in Section 5. In the supercritical regime, we condition on survival and establish a shape theorem. Since the model lacks translation invariance, classical arguments based

on Kingman's Subadditive Ergodic Theorem [11] cannot be applied directly. Instead, we construct an appropriate renewal structure adapted from [17].

In Section 6, this renewal structure is used to derive a central limit theorem for the position of the rightmost infected site [12].

Finally, in Section 7, we develop a renormalization argument, inspired by techniques originally introduced for oriented percolation and related models[14] [15] [4]. This allows us to obtain bounds on crossing probabilities and to show that the process dies out almost surely at criticality. Due to the presence of long-range interactions in the disk graph, a generalized oriented percolation framework is required to handle larger spatial jumps, following the approach in[10]

# 2 Model Description

## 2.1 Disk Graph

To introduce the disk graph model, we begin with some standard graph-theoretic terminology.

Let $G = (V, E)$ be an infinite, locally finite, strongly connected directed graph. Here, $V$ denotes the set of vertices, and

$$E \subseteq \{(u, v) \in V \times V : u \neq v\}$$

denotes the set of directed edges. If $(u, v) \in E$, we write $u \sim v$ and say that $v$ is a child of $u$. We define the set of children of a vertex $u \in V$ by

$$\mathrm{ch}(u) := \{v \in V : (u, v) \in E\}.$$

The graph is said to be *locally finite* if $|\mathrm{ch}(v)| < \infty$ for every $v \in V$. A directed path in $G$ is a sequence of vertices $v_0, v_1, \ldots, v_n$ such that $v_i \sim v_{i+1}$ for all $i$. The path is finite if the sequence is finite and infinite otherwise.

The graph is *strongly connected* if, for every pair of vertices $u, v \in V$, there exist directed paths from $u$ to $v$ and from $v$ to $u$.

With this terminology, we can now define the disk graph. On $\mathbb{Z}$, we define the graph $G_p = (\mathbb{Z}, E(p))$ to be an infinite, locally finite, strongly connected random graph. The edges are determined by the parameter $p \in [0, 1]$ and a collection of independent and identically distributed random variables $\{R_v : v \in \mathbb{Z}\}$, where each $R_v$ has a geometric distribution on $\{1, 2, \ldots\}$,

$$\mathbb{P}(R_v = k) = (1 - p)p^{k-1}, \qquad k = 1, 2, \ldots.$$

For each $v \in \mathbb{Z}$, we add directed edges $(v, u)$ for every $u \in \mathbb{Z}$ satisfying $|v - u| \leq R_v$. Thus,

$$E(p) := \{(v, u) : u, v \in \mathbb{Z},\ |v - u| \leq R_v\}.$$

Each vertex $v$ is assigned a random radius $R_v$, which determines its outgoing edges. Consequently, $G_p = (\mathbb{Z}, E(p))$ is an infinite, locally finite, strongly connected random graph with the collection of radii $\{R_v\}$.

Since $R_v \geq 1$ almost surely, every vertex is connected to its nearest neighbours. Hence, the disk graph on $\mathbb{Z}$ is almost surely strongly connected.

We now define the rumour propagation process on the random disk graph $G_p = (\mathbb{Z}, E(p))$. This process was introduced inc[7].

Let $x \in \mathbb{Z}$ be the initial vertex where the rumour originates.

We define the sequence of sets $\{A_n\}_{n \geq 0}$, where $A_n \subseteq \mathbb{Z}$ represents the set of vertices that have heard the rumour by time $n$. In addition, let $\{\bar{A}_n\}_{n \geq 0}$ denote the set of vertices that hear the rumour for the first time at time $n$.

The process is defined recursively as follows:

$$A_0 = \{x\}, \qquad \bar{A}_0 = \{x\}.$$

For $n \geq 1$,

$$A_n = A_{n-1} \cup \bigcup_{u \in \bar{A}_{n-1}} \{v \in \mathbb{Z} : (u, v) \in E(p)\},$$

and

$$\bar{A}_n = A_n \setminus A_{n-1}.$$

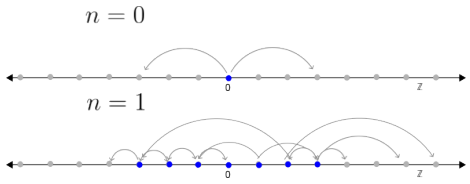


Figure 1: Rumour model for step 1

Thus, at each time step, only the vertices that first received the rumour at time $n-1$, namely those in $\bar{A}_{n-1}$, propagate the rumour to all vertices within their respective radii.

By construction, the sequence $\{A_n\}$ is increasing:

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots.$$

Moreover, since each $R_v$ is almost surely finite and the initial set is finite, it follows by induction that $A_n$ is finite for every $n \geq 0$. In fact, for each $n$,

$$A_n = [A_n^-, A_n^+] \cap \mathbb{Z},$$

for some integers $A_n^- \leq A_n^+$.

We define the rightmost point reached by the rumour at time $n$ by

$$A_n^+ := \sup A_n,$$

and similarly the leftmost point by

$$A_n^- := \inf A_n.$$

Using this process, we obtain several results from [17]. The first concerns the asymptotic speed of the rightmost edge.

**Theorem 2.1.** *Let the radius of influence be given by $\{R_v\}$. Then there exists a deterministic constant $\mu \geq 1$ such that*

$$\lim_{n \to \infty} \frac{A_n^+}{n} = \mu, \qquad \textit{almost surely.}$$

Another important result is a central limit theorem for the position of the rightmost edge.

**Theorem 2.2.** *Fix $p \in [0,1]$, and let the radius of influence be given by $\{R_v\}$. Then*

$$\frac{A_n^+ - n\mu}{\sqrt{n}} \xrightarrow{\mathcal{L}} N(0, \psi^2),$$

*where $\xrightarrow{\mathcal{L}}$ denotes convergence in distribution.*

These results will play an important role in the study of the combined model developed in the following chapters.

## 2.2 Contact Process

Let $G = (V, E)$ be a locally finite directed graph with an infinite vertex set. The contact process (CP) is a stochastic model for the spread of an infectious disease, where each vertex represents an individual that can be either infected (state 1) or healthy (state 0).

Infected individuals infect each of their neighbours after an exponentially distributed waiting time with rate $\lambda \geq 0$. Consequently, if an infected individual has $m$ neighbours, the total rate at which infection events occur is $m\lambda$. Infected individuals recover independently after an exponentially distributed waiting time with rate 1. The parameter $\lambda$ is called the *infection rate*.

Throughout this section, we write $x \sim y$ if there is a directed edge from $x$ to $y$. The neighbourhood of a vertex this is introduced as in the disk graph Section 2.1, this is equivalently given by

$$\mathrm{ch}(x) = \{y \in \mathbb{Z} : (x, y) \in E\}.$$

Let $\Sigma = \{0, 1\}$. A configuration $\xi \in \Sigma^V$ specifies the state of every vertex in $V$. The contact process is a continuous-time Markov chain $(\xi_t)_{t \geq 0}$ with state space $S = \Sigma^V$. For each $x \in V$, the value $\xi_t(x) \in \{0, 1\}$ denotes the state of vertex $x$ at time $t$.

Throughout this thesis, we will abuse notation slightly by identifying a configuration $\xi \in S$ with the set of infected vertices, $\{x \in V : \xi(x) = 1\}$.

The dynamics of the contact process are described using local maps. Let $m : S \to S$ be such a map. We define its domain of dependence by

$$D(m) := \{x \in V : \exists\, \xi \in S \text{ such that } \xi(x) \neq m(\xi)(x)\},$$

that is, the set of vertices whose states may be changed by the map $m$.

For each $x \in V$, we define the set of $m$-relevant sites by

$$R_x(m) := \left\{y \in V : \exists\, \xi \in S \text{ such that } m(\xi)(x) \neq m(\xi^{\,1-y\to y})(x)\right\},$$

where $\xi^{\,1-y\to y}$ denotes the configuration obtained from $\xi$ by flipping the state of the vertex $y$.

A map $m : S \to S$ is called *local* if both sizes $D(m)$ and $R_x(m)$ are finite for every $x \in V$.

Each local map is associated with an independent Poisson process with rate $r_m$. The corresponding map is applied whenever its Poisson process has an arrival.

For each $x \in V$, the recovery map, which has rate 1, is defined by

$$\xi^{x\to 0}(z) = \begin{cases} 0, & z = x, \\ \xi(z), & \text{otherwise.} \end{cases}$$

For each $x \in V$ and $y \in \mathrm{ch}(x)$, the infection map, which has rate $\lambda$, is given by

$$\xi^{\,y\vee x\to x}(z) = \begin{cases} \xi(x) \vee \xi(y), & z = x, \\ \xi(z), & \text{otherwise.} \end{cases}$$

The process $(\xi_t^A)_{t\geq 0}$ is the contact process with initial configuration $\xi_0^A = A \subseteq V$. For $t \geq 0$, the configuration $\xi_t^A$ is obtained by applying all local maps whose Poisson processes have arrivals before time $t$, in chronological order.

The generator of the contact process is

$$\Omega f(\xi) = \sum_{x\in V} \big(f(\xi^{x\to 0}) - f(\xi)\big) + \lambda \sum_{x\in V} \sum_{y\in \mathrm{ch}(x)} \left(f(\xi^{\,y\vee x\to x}) - f(\xi)\right),$$

for cylinder functions $f : S \to \mathbb{R}$.

A graphical representation of the contact process is constructed on $V\times[0,\infty)$. For every vertex $x \in V$, recovery marks are placed at the arrival times of the Poisson process associated with the recovery map $\xi^{x\to 0}$. Likewise, for every $x \in V$ and $y \in \mathrm{ch}(x)$, infection arrows from $(y,t)$ to $(x,t)$ are placed at the arrival times of the Poisson process associated with the infection map $\xi^{\,y\vee x\to x}$.

An infection path from $(x,s)$ to $(y,t)$, where $s \leq t$, is a càdlàg function $\gamma : [s,t] \to V$ satisfying

1. $\gamma(s) = x$ and $\gamma(t) = y$;
2. $(\gamma(v), v)$ is not a recovery mark for every $v \in [s,t]$;
3. whenever $\gamma(v^-) \neq \gamma(v)$, there is an infection arrow from $(\gamma(v^-), v)$ to $(\gamma(v), v)$.

Figure 2.2 illustrates an example of this graphical representation.

We write

$$(x,s) \to (y,t)$$

if such an infection path exists. For subsets $A, B \subseteq V$, we write

$$(A,s) \to (B,t)$$

if there exist $x \in A$ and $y \in B$ such that

$$(x,s) \to (y,t).$$

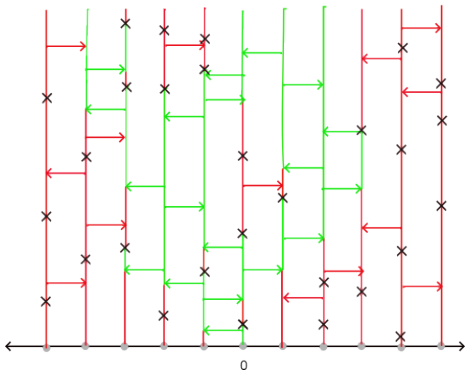


Figure 2: Graphical representation of the contact process on a subset of the integer lattice. The green sites represent infected vertices, while the red sites represent healthy vertices. Infection spreads along the indicated arrows, and recovery events are marked by crosses.

Finally, the configuration at time $t$ is given by

$$\xi_t^A = \{x \in V : (A, 0) \to (x, t)\}.$$

We write $(x, s) \to (y, t)$ if such a path exists, and for $A, B \subset V$, $(A, s) \to (B, t)$ if there exist $x \in A$ and $y \in B$ such that $(x, s) \to (y, t)$.

The configuration at time $t$ is given by

$$\xi_t^A = \{x \in V : (A, 0) \to (x, t)\}.$$

The contact process is *attractive*, that is,

$$\xi_t^B \subseteq \xi_t^A \qquad \text{for all } B \subseteq A \subseteq V,$$

and *additive*,

$$\xi_t^{A \cup B} = \xi_t^A \cup \xi_t^B, \qquad \text{for all } A, B \subseteq V.$$

To establish attractiveness, we first define monotone maps. A map $m : S \to S$ is said to be *monotone* if, for every pair of configurations $\xi_1, \xi_2 \in S$ satisfying

$$\xi_1 \leq \xi_2,$$

we have

$$m(\xi_1) \leq m(\xi_2).$$

Since the contact process is generated by a countable collection of local maps that are all monotone, together with their associated Poisson processes, it follows that the contact process itself is monotone (or attractive).

Now let $(\xi_t)_{t \geq 0}$ and $(\eta_t)_{t \geq 0}$ be contact processes with infection rates $\lambda$ and $\lambda + c$, respectively, where $c > 0$. We couple the two processes using the graphical

representation described above. Both processes use the same recovery marks (rate 1) and the same type 1 infection arrows (rate $\lambda$). In addition, the process $(\eta_t)$ uses an independent collection of type 2 infection arrows with rate $c$.

Thus, $(\eta_t)$ uses both types of infection arrows, whereas $(\xi_t)$ uses only the type 1 arrows. By attractiveness, the coupling satisfies

$$\xi_t \subseteq \eta_t, \qquad \forall\, t \geq 0,$$

almost surely.

Let $\mathbb{P}_\lambda$ and $\mathbb{P}_{\lambda+c}$ denote the probability measures corresponding to the contact processes with infection rates $\lambda$ and $\lambda+c$, respectively. Since every infection path available to $(\xi_t)$ is also available to $(\eta_t)$, survival of $(\xi_t)$ implies survival of $(\eta_t)$. Consequently, $\mathbb{P}_\lambda\left(\xi_t \neq \varnothing\ \forall t \geq 0\right) \leq \mathbb{P}_{\lambda+c}\left(\eta_t \neq \varnothing\ \forall t \geq 0\right)$.

Hence, the survival probability is a non-decreasing function of the infection rate $\lambda$.

The critical infection rate is defined by

$$\lambda_c(G) = \inf\left\{\lambda : \mathbb{P}_\lambda\left(\xi_t^A \neq \varnothing\ \forall t \geq 0\right) > 0\right\},$$

where $A \subseteq V$ is any finite non-empty initial configuration.

Using $\lambda_c(G)$, we distinguish three regimes. Subcritical regime: If $\lambda < \lambda_c(G)$, then $\mathbb{P}_\lambda\left(\xi_t^A \neq \varnothing\ \forall t \geq 0\right) = 0$, so the contact process dies out almost surely.

The critical regime: When $\lambda = \lambda_c(G)$, the process is at the threshold between almost sure extinction and survival with positive probability.

Supercritical regime: If $\lambda > \lambda_c(G)$, then $\mathbb{P}_\lambda\left(\xi_t^A \neq \varnothing\ \forall t \geq 0\right) > 0$, so the process survives with positive probability.

The study of these phase transitions is one of the central topics in percolation theory and interacting particle systems.

## 2.3 Contact Process Results on the Integer Line

In the contact process on $\mathbb{Z}$, we study the rightmost infected site, starting from a single infection at the origin at time 0. We define the rightmost edge by $r_t := \sup \xi_t$. Conditioning on the event that the process survives, we can study the position of the rightmost edge as time evolves.

Suppose that $\lambda > \lambda_c$, so that the process is in the supercritical regime, and condition on survival of the process, where survival means that the contact process never dies out for any time $t > 0$. We can then study the speed of the rightmost edge. The asymptotic speed of the rightmost edge on $\mathbb{Z}$ is known to grow linearly.

**Theorem 2.3** ([14]). *Under $\mathbb{P}_\lambda$, almost surely,*

$$\lim_{t\to\infty} \frac{r_t}{t} = \mu(\lambda),$$

*where*

$$\mu(\lambda) := \lim_{t\to\infty} \frac{\mathbb{E}_\lambda[r_t]}{t} = \inf_{t>0} \frac{\mathbb{E}_\lambda[r_t]}{t} \in [-\infty, \lambda].$$

We have already introduced the critical infection rate. From earlier work, analytical bounds are known: $\lambda_c \leq 1.942$ by [20], and $\lambda_c \geq 1.5517$ by [19]. Using numerical simulation methods, these bounds have been significantly improved, yielding $1.6488 < \lambda_c < 1.6489$ for the contact process on $\mathbb{Z}$. These bounds were obtained through numerical simulations in [16].

## 2.4 Contact Process on the Disk Graph

For the disk graph introduced earlier, we define the contact process using directed infection paths. Recall that

$$\mathrm{ch}(x) = \{y \in \mathbb{Z} : |x - y| \leq R_x\},$$

where $R_x$ is the random radius of influence associated with vertex $x$ in the graph $G_p$.

The construction of the contact process is essentially the same as that of the classical contact process, except that the infection paths are directed. Consequently, an edge $x \sim y$ may exist while the reverse edge $y \sim x$ does not.

Our goal is to establish the same linear growth result as for the classical contact process and the rumour process. In fact, we expect the asymptotic growth speed to be larger than that of the nearest-neighbour contact process on $\mathbb{Z}$. Intuitively, this is because the random radii of influence allow infections to spread over longer distances, enabling the infection front to advance more rapidly than in the nearest-neighbour model.

Several interesting questions remain open. In particular, it would be natural to investigate whether the rightmost infected site satisfies a central limit theorem analogous to that of the classical contact process. More generally, it would be interesting to determine which other asymptotic properties of the classical contact process continue to hold for the contact process on the directed disk graph.

# 3 Results

For the classical contact process, it is well known that there exists a critical infection rate $\lambda_c$ such that, if $\lambda > \lambda_c$, the process survives with positive probability. That is, with positive probability there exists an infinite infection path.

For the contact process on the disk graph, we define an analogous critical infection rate, denoted by $\lambda_c(G_p)$. To this end, fix $p \in [0,1]$, and let $G_p$ be a random disk graph defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P}_p)$, where $\Omega = \mathbb{N}^{\mathbb{Z}}$ is the sample space consisting of all possible realizations of the radii of influence. A realization is written as $\omega = (\ldots, \omega_{-1}, \omega_0, \omega_1, \ldots)$, where $\omega_i \in \mathbb{N}$ represents the value of $R_i$, the radius of influence of vertex $i$.

The $\sigma$-algebra $\mathcal{F}$ is taken to be the product $\sigma$-algebra generated by cylinder sets. Intuitively, a cylinder set corresponds to specifying the values of the radii of influence for only finitely many vertices while leaving the remaining coordinates unrestricted. The probability measure $\mathbb{P}_p$ is the corresponding product measure induced by the i.i.d. distribution of the random variables $\{R_v\}_{v \in \mathbb{Z}}$.

We now introduce the following lemma. Let $p \in [0,1]$ be fixed. The contact process on the annealed random graph has critical infection rate $\lambda_c(p)$ if and only if there exists a realization $g_p \in G_p$ on which the contact process survives with positive probability. Consequently, the annealed and quenched contact processes have the same critical infection rate.

**Lemma 3.1.** *Let $G_p$ be a random disk graph, let $\lambda > \lambda_c$, and let $g_p \in G_p$ be a realization of the graph. Let $(\xi_t)_{t \geq 0}$ denote the contact process on $G_p$ with infection rate $\lambda$, started from the origin infected at time 0. Then*

$$\mathbb{P}\Big(\xi_t^{\{0\}} \neq \varnothing \ \forall\, t > 0\Big) > 0 \tag{1}$$

*if and only if*

$$\mathbb{P}\Big(\xi_t^{\{0\}} \neq \varnothing \ \forall\, t > 0 \,\Big|\, g_p\Big) > 0, \tag{2}$$

*almost surely.*

We now investigate the existence of the critical infection rate $\lambda_c(G_p)$. To distinguish between the subcritical and supercritical regimes, it is necessary to show that this critical value is finite. Otherwise, there would be no supercritical contact process on $G_p$. The following result establishes an upper bound on $\lambda_c(G_p)$, thereby proving that the critical infection rate is finite.

**Theorem 3.2.**

$$\lambda_c(G_p) \leq \lambda_c(\mathbb{Z}).$$

Since the critical infection rate is finite, we can distinguish between the subcritical, critical, and supercritical regimes. This also allows us to study the speed of the rightmost infected site. Recall that we define $r_t(G_p) := \sup\{x : x \in \xi_t^{\{0\}}\}$, the position of the rightmost infected site at time $t > 0$, conditioned on survival.

**Theorem 3.3.** *Fix $p \in [0,1]$, and let $G_p$ be a random disk graph on which the contact process is defined. If $\lambda > \lambda_c(G_p)$, then, conditioned on survival, there exists a constant $\mu_p(\lambda)$, depending on $p$ and $\lambda$, such that*

$$\lim_{t\to\infty} \frac{r_t(G_p)}{t} = \mu_p(\lambda), \qquad \textit{almost surely.}$$

In addition to the law of large numbers above, we will show that the rightmost infected site satisfies a central limit theorem under appropriate moment assumptions, analogous to the rumour process introduced earlier.

**Theorem 3.4.** *Suppose that $\mathbb{E}[R_0^4] < \infty$ and $\lambda > \lambda_c(G_p)$. Then*

$$\frac{r_t(G_p) - t\mu_p(\lambda)}{\sqrt{t}} \xrightarrow{\mathcal{L}} N(0, \psi^2),$$

*where $\xrightarrow{\mathcal{L}}$ denotes convergence in distribution.*

We know that in the subcritical regime, $\lambda < \lambda_c(G_p)$, the contact process dies out almost surely. At the critical value $\lambda = \lambda_c(G_p)$, however, the behaviour of the process is not immediately clear. In particular, it is not known a priori whether the process survives with positive probability or dies out almost surely. The following theorem resolves this question by showing that the contact process dies out almost surely at the critical infection rate.

**Theorem 3.5.** *For every $p \in [0,1]$,*

$$\mu_p(\lambda_c(p)) = 0.$$

This result implies that, at the critical infection rate, the infection front has zero asymptotic speed. Consequently, the process cannot exhibit linear growth, which is consistent with the fact that the contact process dies out almost surely.

In the next section, we introduce the preliminary results that will be needed for the proofs of these theorems.

# 4 Preliminaries

In this section, we introduce several preliminary results that are needed to proof the results. Most of these preliminaries are already established in the literature and will therefore not be reproved in this thesis, with the exception of the Harris-FGK theorem.

The first one is a result about Martingale theory. Martingales are often used to model situations in which the expected value of a future observation, given all previous information, is equal to the present value. This can be interpretend as the mathematical formulation of a fair game.

**Theorem 4.1.** *[13][Theorem 14.3.4]*

*Let $(\Sigma, \mathcal{F}, \mathbb{P})$ be a probability space and let $(X_n)$ for all $n \in \mathbb{N}$ be a martingale, sub-martingale also super-martingale with respect to an increasing sequence of $\sigma$-algebras*

$$\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \dots$$

*Define*

$$\mathcal{F} = \sigma(\bigcup_{n=1}^{\infty} \mathcal{F}_n)$$

*If $\mathbb{E}[|X|] < \infty$ then*

$$\mathbb{E}[X|\mathcal{F}_n] \xrightarrow[a.s.]{n\to\infty} \mathbb{E}(X|\mathcal{F})$$

The next statement concerns a result for growth models, which we denote by $\theta_t^A$, where $A$ is the initial set of infected vertices (or 1's). These models describe processes whose growth is asymptotically linear in time.

We first define what it means for a process to be a growth model. The required conditions are:

1. The state $\emptyset$ is a trap. That is, if $\theta_t = \emptyset$, then $\theta_s = \emptyset, \qquad \forall\, s \geq t.$
2. The system is attractive.
3. The system is local. For every $L < \infty$, if

   $$A \cap \{x : |x| < L\} = \emptyset,$$

   then

   $$P\big(1 \in \theta_t^A \cap \{x : |x| < L\}\big)\, rightarrow 0, \qquad t \to 0.$$

These conditions are satisfied by the contact process on the disk graph $(\xi_t)_{t\geq 0}$.

1. If the process dies out, then no infected vertices remain and the infection cannot spread any further. Hence $\emptyset$ is a trap.
2. Attractiveness was established in Section 2.2 for the contact process.

3. An infection can only spread to neighboring vertices. In the graphical representation, infections are transmitted along infection arrows generated by independent Poisson processes. Hence, the probability that an infection arrow appears during a time interval of length $t$ is $O(t)$, and therefore tends to zero as $t \to 0$. If $x$ is not adjacent to an infected vertex, then this probability is zero.

Hence, the contact process on the disk graph is a growth model. We can now establish suitable upper and lower bounds and apply the shape theorem. We will use this result to show that the contact process $(\xi_t^0)$ on the disk graph satisfies the hypotheses of the theorem.

Define $\tau := \inf\{t : \xi_t^0 = \emptyset\}$, the extinction time, and $t(x) := \inf\{t : x \in \xi_t^0\}$, the first infection time of $x$.

**Theorem 4.2** ([5, Theorem 1])**.** *Let $(\theta_t^0)$ be a permanent on-site growth process. Suppose there exist constants $C_1, C_2, C_3 > 0$ such that*

$$P(t < \tau < \infty) \le C_2 e^{-C_1 t}, \qquad t \ge 0, \tag{3}$$

*and*

$$P(t(x) \ge t, \tau = \infty) \le C_2 e^{-C_1 t}, \qquad t \ge 0,\ |x| \le C_3 t. \tag{4}$$

*Then there exists a interval $U \subset \mathbb{R}$ such that, for every $\varepsilon > 0$,*

$$(1-\varepsilon)U \subseteq \frac{\theta_t^0}{t} \subseteq (1+\varepsilon)U \quad \textit{eventually}, \tag{5}$$

*almost surely on the event $\{\tau = \infty\}$.*
*If, in addition,*

$$P\big(x \notin \xi_t^0,\, \xi_s^0 \ne \emptyset\ \forall s > 0\big) \le C_2 e^{-C_1 t}, \qquad t \ge 0,\ |x| \le C_3 t, \tag{6}$$

*then, conditioned on survival, the contact process grows linearly. More precisely,*

$$(1-\varepsilon)U \subseteq \frac{\theta_t^0}{t} \subseteq (1+\varepsilon)U \quad \textit{eventually}, \tag{7}$$

*almost surely on the event $\{\xi_t^0 \ne \emptyset$ for all $t \ge 0\}$.*

The next statement is a classical result in percolation theory, known as the Harris–FKG inequality. This inequality plays a fundamental role in many arguments in percolation theory. Since it will be used later. To introduce this claim we first have to introduce increasing event A as, If the event A is increasing then if we have $x \in A$ and then have $y$ is monotone such that $y \ge x$, then $y \in A$, So if we talk about x as a set of open edges , then y having more open edges then this set will be increasing in the set of open paths so we get this ,

**Theorem 4.3.** *For every $A, B$ that are increasing events on $\{0,1\}^\Lambda$ with $|\Lambda| < \infty$ we have*

$$\mathbb{P}(A \cap B) \geq \mathbb{P}(A) \cdot \mathbb{P}(B)$$

After this we need to introduce the oriented percolation below ,

## 4.1 Two-Dimensional Oriented Percolation and Results

In this section, we introduce the oriented percolation model. In particular, we will use the 1-independent oriented percolation model and couple it with the contact process on the disk graph through a suitable construction. This coupling will allow us to prove that the contact process dies out at the critical infection rate.

Throughout this section, we will use several results from [4]. We begin by defining the two-dimensional oriented percolation model.

Let

$$\mathcal{L} := \{(x,m) \in \mathbb{Z}^2 : x + m \text{ is even}\},$$

and

$$\overrightarrow{E} := \{((x,m),(x \pm 1, m+1)) : (x,m) \in \mathbb{Z}^2\}.$$

The graph $(\mathcal{L}, \overrightarrow{E})$ is called the two-dimensional oriented lattice. Fix $p \in [0,1]$. Each oriented edge in $\overrightarrow{E}$ is declared to be open with probability $p$ and closed with probability $1-p$, independently of all other edges.

Let $\Sigma = \{0,1\}^{\overrightarrow{E}}$ denote the set of all edge configurations. For each ,

$$\omega(e) = \begin{cases} 1, & \text{if } e \text{ is open,} \\ 0, & \text{if } e \text{ is closed.} \end{cases}$$

An *open path* is a sequence of distinct vertices $(x_0, m_0), \ldots, (x_k, m_k)$ such that $((x_i, m_i), (x_{i+1}, m_{i+1})) \in \overrightarrow{E}$ is an open edge for every $i = 0, \ldots, k-1$. The length of the path is $k$.

We write

$$(x,m) \to (y,n) := \{ \text{ There exist a open path from } (x,m) \text{ to } (y,n)\}$$

For any vertex $(x,m) \in \mathcal{L}$, the cluster of $(x,m)$ is defined by

$$\mathcal{C}^{x,m} := \{(y,n) \in \mathcal{L} : (x,m) \to (y,n)\}.$$

Throughout this thesis, we consider oriented percolation started from the origin $o = (0,0)$. Let $\mathcal{C}^o$ denote the cluster containing the origin. We say that *percolation occurs* if the cluster of the origin is infinite, that is,

$$|\mathcal{C}^o| = \infty.$$

This condition is equivalent to the existence of an infinite oriented open path starting from the origin.

We define

$$\theta(p) := \mathbb{P}_p(|\mathcal{C}^o| = \infty),$$

which is the probability that there exists an infinite open path starting from the origin.

Let $\mathbb{P}_p$ denote the probability measure on $\Omega = \{0,1\}^{\vec{E}}$, under which each edge is independently open with probability $p$ and closed with probability $1-p$. Thus, $\mathbb{P}_p$ is a Bernoulli product measure.

To couple the measures $\mathbb{P}_p$ for different values of $p$, we use the standard monotone coupling. If $p \leq q$, then every edge that is open in the configuration with parameter $p$ is also open in the configuration with parameter $q$, possibly together with additional open edges. Consequently, percolation is an increasing event, and therefore

$$\theta(p) \leq \theta(q).$$

Hence, $\theta(p)$ is a non-decreasing function of $p$. Since $\theta(0) = 0$ and $\theta(1) = 1$, there exists a phase transition. We define the critical parameter by

$$p_c = \inf\{p : \theta(p) > 0\}.$$

The model is said to be in the *supercritical regime* whenever $p > p_c$.

In the literature there are also know bounds for this critical value we have a upperbound given by $p_c < 0.647$ [2] and the lower bound given by $p_c \geq 0,636893$ [3]

# 5 Proof of lemma 3.1 and Theorem 3.2

For the classical contact process, it is well known that there exists a critical parameter $\lambda_c$ such that if $\lambda > \lambda_c$, the process survives with positive probability. That is, there exists an infinite infection path with positive probability.

For the contact process on the disk graph, we define an analogous critical parameter, denoted by $\lambda_c(G_p)$. To this end, fix $p \in [0,1]$, and let $G_p$ be a random disk graph defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P}_p)$ defined earlier.

We will prove the Lemma 3.1

*Proof.* We first prove the implication (2)⇒ (1). Fix any $p \in [0,1]$, and assume that there exists a realization $g_p'$ such that $\mathbb{P}\left(\xi_t^0 \neq \emptyset \ \forall t > 0 \,\middle|\, g_p'\right) > 0$.

Since this conditional survival probability is strictly positive for at least one realization of the graph, we may average over all graph realizations with respect to the law of the random graph. By the law of total probability,

$$\int_{G_p} \mathbb{P}\left(\xi_t^0 \neq \emptyset \ \forall t > 0 \,\middle|\, g_p\right) \, \mathbb{P}_p(dG_p) = \mathbb{P}\left(\xi_t^0 \neq \emptyset \ \forall t > 0\right) > 0.$$

We now prove the reverse implication. For a fixed realization $g_p \in G_p$, define

$$f(g_p) := \sup_{v \in \mathbb{Z}} \mathbb{P}\left(\xi_t^v \neq \emptyset \ \forall t > 0\right),$$

where $(\xi_t^v)_{t \geq 0}$ denotes the contact process on the realization $g_p$ started from the vertex $v$. We regard $f(G_p)$ as a random variable and let

$$\rho := \mathbb{P}\left(\xi_t^0 \neq \emptyset \ \forall t > 0\right).$$

It suffices to show that

$$f(G_p) \geq \rho \qquad \text{almost surely.} \tag{3}$$

For each $n \in \mathbb{N}_0$, let $\mathcal{A}_n$ denote the set of vertices within graph distance $n$ of the origin, and let $\mathcal{F}_n$ be the $\sigma$-algebra generated by the graph structure inside $\mathcal{A}_n$. Furthermore, let $\mathcal{F}$ denote the $\sigma$-algebra generated by the entire graph.

Since

$$\mathcal{A}_n \subseteq \mathcal{A}_{n+1},$$

the filtration $(\mathcal{F}_n)_{n \geq 0}$ is increasing. Moreover, $f(G_p) \in [0,1]$, so it is bounded. Therefore, the process

$$\left(\mathbb{E}\left[f(G_p) \mid \mathcal{F}_n\right]\right)_{n \geq 0}$$

is a bounded martingale. By the Martingale Convergence Theorem,

$$\lim_{n \to \infty} \mathbb{E}\left[f(G_p) \mid \mathcal{F}_n\right] = \mathbb{E}\left[f(G_p) \mid \mathcal{F}\right] = f(G_p) \quad \text{almost surely.}$$

Next, observe that for every $n$,

$$\mathbb{E}\left[f(G_p) \mid \mathcal{F}_n\right] \geq \rho.$$

Indeed, conditional on $\mathcal{F}_n$, the unexplored part of the graph is an independent random extension having the same distribution as the original graph. Consequently, the maximal survival probability from a vertex in the unexplored region is at least $\rho$.

Taking the limit as $n \to \infty$ yields

$$f(G_p) \geq \rho \qquad \text{almost surely,}$$

which proves (3) and completes the proof. □

The previous lemma shows that the critical parameter $\lambda_c$ is the same in both the quenched setting, where the contact process evolves on a fixed realization of the graph, and the annealed setting, where the graph is first sampled randomly and the contact process is subsequently run on that realization.

It remains to prove 3.2. We do so by constructing a coupling using the graphical representation introduced in Section 2.2.

*Proof.* Let $(\xi_t^0)_{t\geq 0}$ denote the contact process on the random disk graph $G_p$, and let $(\eta_t^0)_{t\geq 0}$ denote the classical contact process on $\mathbb{Z}$.

We construct both processes on the same graphical representation. For every vertex $x \in \mathbb{Z}$, place recovery marks according to independent Poisson processes of rate 1. We then introduce two types of infection arrows.

Type 1, For every pair of nearest neighbours $x, y \in \mathbb{Z}$ satisfying $|x - y| = 1$, place infection arrows from $x$ to $y$ according to independent Poisson processes of rate $\lambda$.

Type 2, For every pair $x, y \in \mathbb{Z}$ satisfying

$$1 < |x - y| \leq R_x,$$

place infection arrows from $x$ to $y$ according to independent Poisson processes of rate $\lambda$.

The process $(\xi_t^0)_{t\geq 0}$ uses both type 1 and type 2 infection arrows, whereas $(\eta_t^0)_{t\geq 0}$ uses only the type 1 arrows. Consequently, every infection path available to $\eta_t^0$ is also available to $\xi_t^0$. Hence,

$$\eta_t^0 \subseteq \xi_t^0, \qquad t \geq 0.$$

It follows that

$$\mathbb{P}\left(\eta_t^0 \neq \emptyset \ \forall t \geq 0\right) \leq \mathbb{P}\left(\xi_t^0 \neq \emptyset \ \forall t \geq 0\right).$$

Therefore,

$$\lambda_c(\mathbb{Z}) \geq \lambda_c(G_p).$$

Since $\lambda_c(\mathbb{Z}) < \infty$, we conclude that

$$\lambda_c(G_p) < \infty.$$

□

# 6 Rightmost edge

In this section we use the method explained in [17, Section 4] to study the asymptotic behavior of the rightmost speed and to prove a central limit theorem for the rightmost edge. Before stating the main results, we introduce some notation. For this section we will assume that the $\lambda > \lambda_c(G_p)$ such that we are in the supercritical regime.

## 6.1 Overshoot Random Variable

In the disk graph paper [18], we consider the overshoot random variable. This describes the rightmost point that can be reached in one step if all negative integers already know the rumor. In other words, if all sites in $\mathbb{Z}^-$ already know the rumour, we want to determine the rightmost site that can hear about the rumour in the next step.

We define

$$O := \sup\{x + R_x : x \in \mathbb{Z}^-\}. \tag{4}$$

With this definition we can write some properties.

**Proposition 6.1.** *Let $R_x$ a geometric distribution on $\{1, 2, \ldots\}$. We then have*

1. *$O$ is nonnegative*
2. *$O$ is finite a.s.*
3. *$\mathbb{E}(O) < \infty$*
4. *$\mathbb{E}(O^2) < \infty$*
5. *There exist constant $C_0, C_1 > 0$ such that*

$$\mathbb{P}(O > n) \leq C_0 e^{-C_1 n} \text{ for all } n \in \mathbb{N}$$

*Proof.* (1).Since $R_x$ is geometrically distributed, we have $R_x \geq 1$. In particular, for $x = -1$ this implies $O \geq 0$, and hence $O$ is nonnegative.

(2) we show that $O$ is finite almost surely. To do this, we use the Borel–Cantelli lemma. It is suffices to show that

$$\sum_{n=1}^{\infty} \mathbb{P}(O \geq n) < \infty,$$

if this hold, then

$$\mathbb{P}\left(\limsup_{n\to\infty}\{O \geq n\}\right) = 0,$$

which implies that $\mathbb{P}(O = \infty) = 0$.

We estimate

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathbb{P}(O \geq n) &= \sum_{n=0}^{\infty} \sum_{j=n}^{\infty} \mathbb{P}(O = j) \leq \sum_{n=0}^{\infty} \sum_{j=n}^{\infty} \sum_{i=1}^{\infty} \mathbb{P}(R_{-i} = i + j) \\
&= \sum_{n=0}^{\infty} \sum_{j=n}^{\infty} \sum_{i=1}^{\infty} p^{i+j-1}(1-p) = \sum_{n=0}^{\infty} \sum_{j=n}^{\infty} p^j \sum_{i=1}^{\infty} p^{i-1}(1-p) \\
&= \sum_{j=0}^{\infty} \sum_{n=0}^{j} p^j = \sum_{j=0}^{\infty} (j+1)p^j \\
&= \sum_{j=0}^{\infty} (j+1)p^j = \frac{1}{(1-p)^2} < \infty
\end{aligned}$$

In the third line, we used that $R_{-i}$ has a geometric distribution with parameter $p$. In the fourth line, the inner sum is equal to 1, since it is the sum of the probability mass function over its entire support.

Hence, $O$ is finite almost surely.

(3) We now use the result from part (2) to show that the expectation of $O$ is finite. We have

$$\mathbb{E}(O) = \sum_{j=1}^{\infty} j\, \mathbb{P}(O = j) = \sum_{j=1}^{\infty} \mathbb{P}(O \geq j) < \infty.$$

The second equality follows from the triangular rearrangement of the series, where each probability $\mathbb{P}(O = j)$ is counted exactly $j$ times.

(4) Here we use a more general result. Since geometric random variables have a finite $(4+\varepsilon)$-th moment, that is, $\mathbb{E}(R_0^{4+\varepsilon}) < \infty$, we may apply Markov's inequality to obtain

$$\begin{aligned}
\mathbb{E}(O^2) &= \sum_{n=1}^{\infty} \mathbb{P}(O^2 \geq n) \\
&= \sum_{n=1}^{\infty} \mathbb{P}\Big( \bigcup_{i \in \mathbb{Z}^+} R_{-i} \geq N + i \Big) \\
&\leq \sum_{n=1}^{\infty} \sum_{i=0}^{\infty} \mathbb{P}(R_{-i} \geq \sqrt{N+i}) \\
&\leq \sum_{n=1}^{\infty} \sum_{i=0}^{\infty} \mathbb{E}(R_{-i}^{4+\varepsilon}/(n+i)^{(4+\varepsilon)/2} \\
&= \mathbb{E}(R_0^{4+\varepsilon} \sum_{n=1}^{\infty} \sum_{i=0}^{\infty} 1/(N+i)^{4+\varepsilon/2} \\
&= \mathbb{E}(R_0^{4+\varepsilon} \sum_{m=1}^{\infty} (m+1)/m^{(4+\varepsilon)/2} < \infty
\end{aligned}$$

This completes the proof.

(5) we show that the overshoot has an exponentially decaying tail. Since $R_0$ is geometrically distributed there exist a $C_0, C_1 > 0$ such that,

$$\mathbb{P}(R_0 > n) = p^n \text{ for all } n$$

Using the definition of $O$, we obtain

$$\mathbb{P}(O > n) = \mathbb{P}(\bigcup_{i \in \mathbb{Z}^+} R_{-i} > n + i) \leq \sum_{i=0}^{\infty} \mathbb{P}(R_{-1} > n + i) = \sum_{i=0}^{\infty} p^{n+i} = \frac{p^n}{1-p}$$

This value can be rewritten as $\frac{1}{1-p} e^{n \ln p}$ were $p \in [0, 1]$ so $\ln(p) < 0$ so we get exponential decay. □

We also define the right overshoot of a subset $A \subset \mathbb{Z}$ as

$$O_{A,+} := \max\{(x + R_x) - \sup(A) : x \in A\} \quad \text{a.s.} \tag{5}$$

## 6.2 The speed of the rightmost edge

We have already defined the rightmost edge for both the contact process and the rumour process, denoted by $r_t$ and $A_n^+$, respectively. In this section, we extend this notion to the contact process on the disk percolation model.

Recall that $A_n^+$ describes the rightmost point reached after $n$ steps in the rumour process. More precisely, we define

$$A_n^+ := \sup\{x \in \mathbb{Z} : x \in \mathcal{A}_n\}, \quad n \in \mathbb{N}.$$

Thus, $A_n^+$ represents the furthest point that can be reached after $n$ steps in the rumour process.

To define the rightmost edge for the contact process on a disk graph, we first specify the process on a random graph. Let $G_p$ be the random disk graph with parameter $p$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ as defined earlier.

$$r_t(G_p) := \sup\{x \in \mathbb{Z} : \xi_t^0(x) = 1\}.$$

which gives the rightmost infected site at time $t$, starting from the initial configuration $\xi^0$.

With this notation in place, we can begin the renewal construction that will be used to prove Theorem3.3.

## 6.3 Renewal process

In this section we introduce a renewal process on the disk graph. This process is not directly related to the contact process at first, but it provides a way to construct i.i.d. random boxes through which the process must pass.

We base this construction on the fireworks process on $\mathbb{N}$ as described in [18]. Since this renewal process does not work directly for the classical contact

process, we introduce a new process that dominates the contact process on the disk graph.

We consider the process $(\eta_t)_{t\geq 0}$, which is obtained from the contact process by removing recovery maps. We can define the rightmost edge as $\bar{r}_t(G_p)$ This process can be coupled with the contact process such that

$$\xi_t^0 \subseteq \eta_t^0$$

for any graph $G_p$. In particular,

$$r_t(G_p) \leq \bar{r}_t(G_p).$$

A major difficulty, compared with the fireworks process on $\mathbb{N}$, is that the infection on $\mathbb{Z}$ can spread in both directions. In particular, the infection may first spread to the left, where a vertex with a large radius of influence can produce a large jump to the right. Such a jump originating from negative vertices may significantly affect the asymptotic speed of the rightmost edge. This makes the analysis considerably more involved than in the one-sided fireworks model. To overcome this we will create a renewal model that needs the first time to make it an easier model.

$$\sigma := \inf\{n \geq 1 : O((-\infty, -n], +) \leq n\}. \tag{6}$$

If $\sigma = n$ then it means that there cannot exist an edge from $(-\infty, -n]$ to $[n, \infty)$. The value $\sigma$ is a random variable. This allows us to ignore the part of the process on $(-\infty, -n]$ once the infection has moved beyond $n$, and we may view the process as starting from $-n+1$ on $\mathbb{N}$.

**Corollary 6.2.** *The random variable $\sigma$ is finite almost surely.*

*Proof.* For $n \geq 1$, consider the event

$$A_n^+ := \{R_{-n} \geq 2n\}.$$

Then

$$\{\sigma = \infty\} \subseteq \limsup_{n\to\infty} A_n^+. \tag{7}$$

Under the assumption that $\mathbb{E}(R_0^{1+\varepsilon}) < \infty$ this hold because we have a geometric distribution starting at 1, which with any fixed $p \in [0, 1]$ has finite moments, for some $\varepsilon > 0$ Using Markov's inequality we obtain

$$\sum_{n=1}^{\infty} \mathbb{P}(A_n^+) = \sum_{n=1}^{\infty} \mathbb{P}(R_0 \geq 2n) \leq \mathbb{E}(R_0^{1+\varepsilon}) \sum_{n=1}^{\infty} \frac{1}{(2n)^{1+\varepsilon}} < \infty. \tag{8}$$

Therefore, the result follows from the Borel–Cantelli lemma. □

The variable $\sigma$ can be interpreted as the first vertex $-n$ such that everything to its left does not affect the rightmost edge anymore. Hence, we can restart the process from this point.

We define

$$\tau_0 := \sigma, \tag{9}$$

and for $j \geq 1$,

$$\tau_j := \inf\{n > \tau_{j-1} : A_n^+ - A_{n-1}^+ = 1\}. \tag{10}$$

Here, $\tau_j$ is the time at which the rumour process advances by exactly one step to the right from its previous position. This means that for the process to progress further, it must infect the vertices $A_{\tau_j}^+$ for all $j \geq 1$.

Thus, the intervals $[A_{\tau_j}^+, A_{\tau_{j+1}}^+]$ can be seen as boxes that the process must cross. The behavior of the process inside each box is independent of the behavior in other boxes.

We define the passage of time

$$\zeta(\eta^0, n) := \inf\{\ell : (0,0) \to (n,\ell)\ \ell \geq t\}. \tag{11}$$

We now study the increments

$$\zeta(\eta^0, A_{\tau_{j+1}}^+) - \zeta(\eta^0, A_{\tau_j}^+).$$

**Proposition 6.3.** *Let $\eta$ be the contact process without recovery marks, and let $G_p$ be the random disk graph whose radii are given by $R_0$ i.i.d. Then the following hold.*

1. *There exist constants $C_0, C_1 > 0$ such that, for every $t > 0$,*

$$\mathbb{P}\Big(\zeta(\eta^0, A_{\tau_{j+1}}^+) - \zeta(\eta^0, A_{\tau_j}^+) > t \,\Big|\, \zeta(\eta^0, A_{\tau_{j-1}}^+)\Big) \leq C_0 e^{-C_1 t}.$$

2. *The increment*

$$A_{\tau_{j+1}}^+ - A_{\tau_j}^+$$

*has finite expectation.*

*Proof.* We first prove (1). We wish to estimate the probability that the path

$$(A_{\tau_j}^+, s) \longrightarrow (A_{\tau_{j+1}}^+, \ell)$$

takes longer than $t$, that is, $\ell - s > t$.

To obtain an upper bound, we consider a dominating process. Suppose that, whenever a vertex $x$ becomes infected, the first infection event from $x$ infects all of its children simultaneously. Consequently, each infected vertex produces offspring according to a Poisson process with rate $\lambda|\mathrm{ch}(x)|$, where $\mathrm{ch}(x)$ denotes the set of children of $x$.

By the graphical representation, this modified process dominates the $\eta$ contact process, since every infection that occurs in the original process also occurs in the modified one.

Along an optimal path from $A_{\tau_j}^+$ to $A_{\tau_{j+1}}^+$, let $m_i = \max\{|\mathrm{ch}(x)| : x \in \bar{A}_i\}$, denote the largest number of children available at the $i$-th step. Therefore,

$$\mathbb{P}\Big(\zeta(\eta^0, A_{\tau_{j+1}}^+) - \zeta(\eta^0, A_{\tau_j}^+) > t \,\Big|\, \zeta(\eta^0, A_{\tau_1}^+), \ldots, \zeta(\eta^0, A_{\tau_{j-1}}^+)\Big)$$

is bounded above by

$$\sum_{i=0}^{\tau_{j+1}-\tau_j-1} \mathbb{P}(N(t)=i)\,,$$

where $N(t)$ is a Poisson random variable with parameter

$$\lambda t \sum_{l=0}^{i} m_l.$$

Hence,

$$\sum_{i=0}^{\tau_{j+1}-\tau_j-1} \frac{\left(\lambda t \sum_{l=0}^{i} m_l\right)^i}{i!} e^{-\lambda t \sum_{l=0}^{i} m_l} \leq C_0 e^{-C_1 t},$$

for suitable constants $C_0, C_1 > 0$. The exponential decay follows from the tail estimate for the Poisson distribution were we have $\zeta(\eta^0, A^+_{\tau_{j+1}}) - \zeta(\eta^0, A^+_{\tau_j}) > t$ the bound works.

We now prove (2). Observe that

$$A^+_{\tau_{j+1}} - A^+_{\tau_j} = \sum_{m=\tau_j+1}^{\tau_{j+1}} \left(A^+_m - A^+_{m-1}\right),$$

which is a telescoping sum.

Furthermore,

$$A^+_m - A^+_{m-1} = O^{([A^+_{m-1}, A^+_m), +)},$$

where $O^{(\cdot,+)}$ denotes the corresponding overshoot random variable.

These increments are independent and can be stochastically dominated by an i.i.d. sequence of overshoot random variables $(O_i)_{i\geq 1}$. Since the overshoot variable has finite expectation by 6.1 and the time steps $\tau_{j+1} - \tau_j$ has finite expactation so by Walds inequality ,

$$\mathbb{E}[\sum_{i=0}^{\bar{\tau}} O_i] < \infty,$$

it follows that

$$\mathbb{E}\Big[A^+_{\tau_{j+1}} - A^+_{\tau_j}\Big] < \infty.$$

This completes the proof. □

**Proposition 6.4.** *The collection*

$$\left\{\left(\zeta(\eta^0, A^+_{\tau_{j+1}}) - \zeta(\eta^0, A^+_{\tau_j}),\ A^+_{\tau_{j+1}} - A^+_{\tau_j}\right) : j \geq 1\right\}$$

*forms a sequence of i.i.d. random vectors taking values in* $\mathbb{R}_{>0} \times \mathbb{N}$.

*Proof.* At each renewal time, the process restarts from a single active vertex. Moreover, the unexplored part of the graph consists of vertices whose radii of influence are independent and have the same distribution as in the original model. Consequently, the evolution after every renewal time has the same law as the original rumour process on $\mathbb{Z}^+$ started from a single active vertex at the origin. we get that for all boxes the probability of $A_n^+ = 1$ has an i.i.d distribution starting for a single infected person at time 0.

It remains to show that these increment vectors are identically distributed.

To this end, consider the rumour process restricted to $\mathbb{Z}_+$, with the origin as the only active vertex at time zero. Let $(A_n^r)_{n\geq 0}$ and $(\bar{A}_n^r)_{n\geq 0}$ denote the analogues of $(A_n)_{n\geq 0}$ and $(\bar{A}_n)_{n\geq 0}$ for the process on $\mathbb{Z}_+$.

Let

$$\zeta = \inf\{n \geq 0 : |\bar{A}_n^r| = 1\},$$

that is, the first time at which there is exactly one active vertex. Let $r_\zeta^+$ denote this unique active vertex at time $\zeta$, and let $\zeta(\eta_0, r_\zeta^+)$ denote the corresponding passage time.

By the renewal construction, after each renewal time the future evolution of the process depends only on previously unexplored vertices. Since these vertices have independent radii of influence with the same distribution as the original process, the process restarts from the renewal point with the same law as the rumour process on $\mathbb{Z}_+$ started from the origin.

Consequently, each increment vector between successive renewal times has the same distribution as $\big(\zeta,\ \zeta(\eta_0, r_\zeta^+)\big)$, and, in particular,

$$\zeta(\eta_0, A^+_{\tau_{j+1}}) - \zeta(\eta_0, A^+_{\tau_j}) \stackrel{d}{=} \zeta(\eta_0, r_\zeta^+).$$

Hence, the increment vectors between successive renewal times are independent and identically distributed. □

## 6.4 proof of Theorem 3.3

In section we will finish the proof using our renewal construction and provide a precise understanding of the limit too. For this we go back to our process $(\eta_t^0)_{t\geq o}$ a contact process on a random disk graph with a fixed $p$ were the recovery maps are taken out

**Proposition 6.5.** *We have for a $(\eta_t^0)_{t\geq 0}$ process as defined before,*

$$\lim_{t\to\infty} \frac{r_t(G_P)}{t} \leq \lim_{t\to\infty} \frac{\bar{r}_t}{t} = \frac{\mathbb{E}(A^+_{\tau_2} - A^+_{\tau_1})}{\mathbb{E}(\zeta(\eta^0, A^+_{\tau_{j+1}}) - \zeta(\eta^0, A^+_{\tau_j}))} \tag{12}$$

*a.s*

*Proof.* From Propostition 6.4 we can write the right hand side of (12) were $k \to \infty$ so we get

$$\lim_{k\to\infty} \frac{\frac{1}{k}(\sum_{j=1}^k A^+_{\tau_{j+1}} - A^+_{\tau_j})}{\frac{1}{k}(\sum_{j=1}^k \zeta(\eta^0, A^+_{\tau_{j+1}}) - \zeta(\eta^0, A^+_{\tau_j}))} \tag{13}$$

This sum can be rewritten with the telescoping sum to get this

$$\frac{\mathbb{E}(A^+_{\tau_2} - A^+_{\tau_2})}{\mathbb{E}(\zeta(\eta^0, A_{\tau_2}) - \zeta(\eta^0, A_{\tau_1}))} = \lim_{k\to\infty} \frac{A_{\tau_k} - A_{\tau_1}}{\zeta(\eta^0, A_{\tau_k}) - \zeta(\eta^0, A_{\tau_1})} \tag{14}$$

Let n be large enough such that $\zeta(\eta^0, A^+_{\tau_1}) < n$. Let $k(n) \in \mathbb{N}$ be such $\zeta(\eta^0, A_{\tau_{k(n)}}) \leq n < \zeta(\eta^0, A_{\tau_{k(n)+1}})$.From now on we go back to just use k to denote k(n) to make it easier. Then $A^+_{\tau_k} - A^+_{\tau_1} \leq A^+_n - A^+_1 < A^+_{\tau_{k+1}} - A^+_{\tau_1}$. Since we have

$$\lim_{k\to\infty} \frac{A^+_{\tau_{k+1}} - A^+_{\tau_1}}{\zeta(\eta^0, A^+_{\tau_{k+1}}) - \zeta(\eta^0, A_{\tau_1})} = \lim_{k\to\infty} \frac{A_{\tau_k} - A^+_{\tau_1}}{\zeta(\eta^0, A_{\tau_k}) - \zeta(\eta^0, A_{\tau_1})} \tag{15}$$

Hence if at the limit this means that the for n is the same,

$$\lim_{n\to\infty} \frac{A^+_n - A^+_1}{n - \zeta(\eta^0, A^+_{\tau_1)}} = \lim_{k\to\infty} \frac{A^+_{\tau_{k+1}} - A^+_{\tau_1}}{\zeta(\eta^0, A^+_{\tau_{k+1}}) - \zeta(\eta^0, A^+_{\tau_1})} \tag{16}$$

So from the equation 14 and 16 we get,

$$\frac{\mathbb{E}(A^+_{\tau_2} - A^+_{\tau_1})}{\mathbb{E}(\zeta(\eta^0, A^+_{\tau_2}) - \zeta(\eta^0, A^+_{\tau_1}))} = \lim_{n\to\infty} \frac{A^+_n - A^+_1}{n - \zeta(\eta^0, A^+_{\tau_1})} = \lim_{n\to\infty} \frac{A^+_n}{n}. \tag{17}$$

□

Now with this result above we can conclude that we know the contact process on $\mathbb{Z}$ and the contact process on the disk. we know that $r_t(\mathbb{Z}) \leq r_t(G_p)$ this hold with this claim we can conclude that

$$\lim_{t\to\infty} \frac{r_t(\mathbb{Z})}{t} \leq \lim_{t\to\infty} \frac{r_t(\mathbb{G}_\text{ı})}{t} \leq \lim_{t\to\infty} \frac{\bar{r}_t(G_p)}{t}$$

With The two outsides processes having a know rightmostedge speed that is linearly we can have then this

$$\mu(\lambda) \leq \frac{r_t(G_p)}{t} \leq \mu_p(\lambda)$$

with this we can use 4.2 and see that we can finally conclude that we get linearly growth.

## 6.5 central limit theorem for the rightmost vetex

In this section we show that there is a central limit theorem for the position of the rightmost vertex in a supercritical contact process on disk graph . The proof is based on the argument in [12]. We require the assumption that $\mathbb{E}[R_0^{4+\varepsilon}] < \infty$ for some $\varepsilon > 0$. This assumption holds for our model. Under this assumption we also have that the renewal process increments $A_{\tau_{j+1}} - A_{\tau_j}$ have finite second moments. We can now prove Result 3.4, where the notation $\xrightarrow{\mathcal{L}}$ denotes convergence in distribution.

*Proof.* Choose $t$ sufficiently large such that $\zeta(\xi_0^0, A_{\tau_0}) < t$. Let $k = k(t) \in \mathbb{N}$ be such that

$$\zeta(\xi_0^0, A_{\tau_k}) \leq t < \zeta(\xi_0^0, A_{\tau_{k+1}}). \tag{18}$$

For convenience we write $k$ instead of $k(t)$. By construction,

$$A^+_{\tau_k} \leq r_t(G_p) \leq A^+_{\tau_{k+1}}. \tag{19}$$

Hence,

$$\frac{A^+_{\tau_k} - \mu t}{\sqrt{t}} \leq \frac{r_t(G_p) - \mu t}{\sqrt{t}} \leq \frac{A^+_{\tau_{k+1}} - \mu t}{\sqrt{t}}.$$

From 6.3 we know that

$$\frac{\zeta(\xi_0^0, A_{\tau_{j+1}}) - \zeta(\xi_0^0, A_{\tau_j})}{\sqrt{t}} \xrightarrow{\mathbb{P}} 0, \tag{20}$$

as $t \to \infty$, for every $j \geq 1$. Moreover,

$$\frac{A^+_{\tau_{k+1}} - A^+_{\tau_k}}{\sqrt{t}} \xrightarrow{\mathbb{P}} 0. \tag{21}$$

It therefore remains to show that

$$\frac{r_t(G_p) - \mu t}{\sqrt{t}} - \frac{A^+_{\tau_{k+1}} - \mu\zeta(\xi_0^0, A_{\tau_{k+1}})}{\sqrt{t}} \xrightarrow{\mathbb{P}} 0.$$

Observe that

$$r_t(G_p) - \mu t = \big(r_t(G_p) - A^+_{\tau_{k+1}}\big) - \big(\mu\zeta(\xi_0^0, A_{\tau_{k+1}}) - A^+_{\tau_{k+1}}\big) + \mu\big(\zeta(\xi_0^0, A_{\tau_{k+1}}) - t\big).$$

We show that each of the three terms on the right-hand side, divided by $\sqrt{t}$, converges to zero in probability.

For the first term,

$$r_t(G_p) - A^+_{\tau_{k+1}},$$

this follows immediately from (19) together with (21).

For the third term,

$$\mu\big(\zeta(\xi_0^0, A_{\tau_{k+1}}) - t\big),$$

the choice of $k$ in (18), together with (20), implies that

$$\frac{\mu(\zeta(\xi_0^0, A_{\tau_{k+1}}) - t)}{\sqrt{t}} \xrightarrow{\mathbb{P}} 0.$$

Finally, for the second term we use the almost sure convergence

$$\mu = \lim_{k\to\infty} \frac{A^+_{\tau_k}}{\zeta(\xi_0^0, A_{\tau_k})},$$

which implies that

$$\frac{\mu\zeta(\xi_0^0, A_{\tau_{k+1}}) - A^+_{\tau_{k+1}}}{\sqrt{t}} \xrightarrow{\mathbb{P}} 0.$$

Consequently,

$$\frac{(r_t(G_p) - A^+_{\tau_{k+1}}) - (\mu\zeta(\xi_0^0, A_{\tau_{k+1}}) - A^+_{\tau_{k+1}}) + \mu(\zeta(\xi_0^0, A_{\tau_{k+1}}) - t)}{\sqrt{t}} \xrightarrow{\mathbb{P}} 0.$$

Since

$$\frac{A^+_{\tau_{k+1}} - \mu t}{\sqrt{t}} = \frac{A^+_{\tau_{k+1}} - \mu\zeta(\xi_0^0, A_{\tau_{k+1}})}{\sqrt{t}} + \frac{\mu(\zeta(\xi_0^0, A_{\tau_{k+1}}) - t)}{\sqrt{t}},$$

the second term converges to zero in probability, while the first term satisfies the central limit theorem,

$$\frac{A^+_{\tau_{k+1}} - \mu\zeta(\xi_0^0, A_{\tau_{k+1}})}{\sqrt{\zeta(\xi_0^0, A_{\tau_{k+1}})}} \xrightarrow{\mathcal{L}} N(0, \psi^2).$$

Since $\zeta(\xi_0^0, A_{\tau_{k+1}})/t \to 1$ in probability,

$$\frac{A^+_{\tau_{k+1}} - \mu t}{\sqrt{t}} \xrightarrow{\mathcal{L}} N(0, \psi^2).$$

Combining this with the previous convergence in probability proves that

$$\frac{r_t(G_p) - \mu t}{\sqrt{t}} \xrightarrow{\mathcal{L}} N(0, \psi^2),$$

which completes the proof. □

# 7 Renormalization

In this section, we introduce a renormalization argument that was originally developed to obtain improved upper bounds for the two-dimensional oriented percolation model [1]. We adapt this argument to prove that the contact process on the disk graph, in the critical regime, dies out for every fixed parameter $p \in [0,1]$.

More specifically, the renormalization argument was originally developed to compare supercritical two-dimensional oriented percolation (2DOP) with a 1-dependent oriented percolation model whose parameter can be made arbitrarily close to 1.

A 1-dependent oriented percolation model on the graph $(\mathcal{L}, \overrightarrow{E})$ is a percolation model in which, for any $k \geq 2$ and edges $e_1, \ldots, e_k \in \overrightarrow{E}$, the events

$$\{e_1 \text{ is open}\}, \ldots, \{e_k \text{ is open}\}$$

are independent whenever the edges do not share any endpoints. Thus, in a 1-dependent oriented percolation model, dependencies may exist only between edges that share a common endpoint. Such models possess a critical value $p_c^{\text{1-dep}}$, which satisfies $p_c^{\text{1-dep}} \leq 1 - \frac{1}{97}$. Throughout this section, we assume that $p > p_c^{\text{1-dep}}$, so that $p$ is sufficiently close to 1.

The same renormalization construction also applies to the contact process on a disk graph, yielding the existence of a rightmost edge speed for every $p \in [0,1]$. We begin by describing the renormalization construction informally for our model before proving that it is valid.

Fix $p \in [0,1]$ and $\lambda > \lambda_c$. Then there exist a speed $\mu := \mu_p(\lambda) > 0$. given by theorem 3.3 Choose a $\varepsilon > 0$. We define a tilted box $B(\varepsilon L, L, \mu)$ to be a box of height $L \in \mathbb{R}$ width $\varepsilon L$ and tilted in the direction $\mu$, as in the figure below

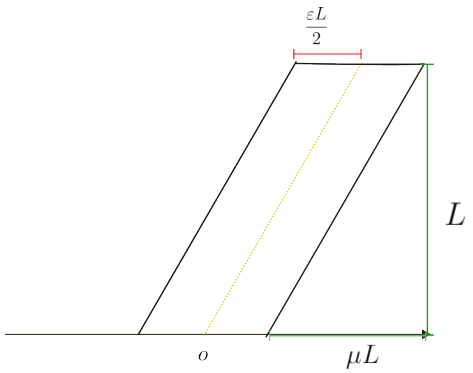


Figure 3: The tilted box with length L and width $\varepsilon L$ tilt given by $\mu$

We say that $B(\varepsilon L, L, \mu)$ is crossed from bottom to top if there exists an infection path that starts at a vertex on the bottom boundary of the box and

ends at a vertex on top of the box, while remaining entirely inside the tilted box.

Next we place infinitely many translated copies of $B(\varepsilon L, L, \mu)$ and $B(\varepsilon L, L, -\mu)$ on space-time diagram, in the following way.

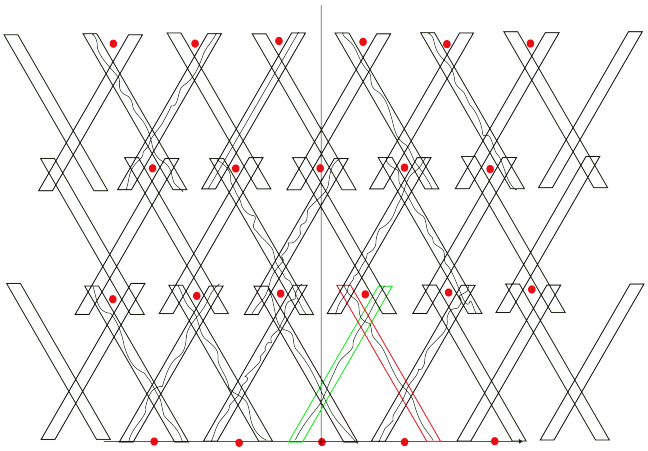

Figure 4: Renormalization process with red dots being oriented percolation with the boxes defined the green box as $B^+_{(m,n)}$ with the red box $B^-_{m,n}$

Let's take two constants $c_1$, and $c_2$ depending on $L, \varepsilon$, and $\mu$ and define the lattice of red dots on this figure as,

$$(c_1, c_2) \cdot \mathcal{L} = \{(c_1 m, c_2 n) : (m, n) \in \mathcal{L}\}.$$

For each $(m, n) \in \mathcal{L}$ denote the boxes positioned at $(c_1 m, c_2 n)$ and tilted to the left and right by naming them, $B^-_{(m,n)}$ and $B^+_{(m,n)}$. respectively. We also define the lower halves of these boxes, denoted by $\underline{B}^-_{(m,n)}$ and $\underline{B}^+_{(m,n)}$, which are tilted boxes with height $L/2$ positioned above $(c_1 m, c_2 n)$. We now define a 1-independent percolation process on $(\mathcal{L}', \vec{E}')$.This is an oriented percolation model whose vertices correspond to the stretched out lattice points with $(m, n) \in \mathcal{L}'$.The resulting model will have different edge-opening probability which will be specified below.

$$\omega'(\langle (m,n), (m+1, n+1)\rangle) = \begin{cases} 1, & \text{if both } B^+_{(m,n)} \text{ and } \underline{B}^-_{(m,n)} \text{ are crossed from bottom to top,} \\ 0, & \text{otherwise} \end{cases}$$

$$\omega'(\langle (m,n), (m-1, n+1)\rangle) = \begin{cases} 1, & \text{if both } B^-_{(m,n)} \text{ and } \underline{B}^+_{(m,n)} \text{ are crossed from bottom to top,} \\ 0, & \text{otherwise.} \end{cases}$$

To make this renormalization work, we need to verify the following four properties.

1. For any $\delta > 0$ we can choose L large enough, such that

$$\mathbb{P}_{p,\lambda}(B(\varepsilon L, L, \mu) \text{ is crossed from bottom to top}) > 1 - \delta \tag{22}$$

With this we also have by symmetries that

$$\mathbb{P}_{p,\lambda}(B(\varepsilon L, L, -\mu) \text{ is crossed from bottom to top}) > 1 - \delta$$

and also we get

$$\mathbb{P}_{p,\lambda}(\underline{B}(\varepsilon L, L, \mu) \text{ is crossed from bottom to top}) > 1 - \delta$$

2. The percolation process defined above on $(\mathcal{L}', \vec{E}')$ is 1-independent and each edge $e \in \vec{E}'$ is open with probability $> 1 - 2\delta$.

3. If there exist an infinite cluster on $(\mathcal{L}', \vec{E}')$, then there is an infinite cluster on $(\mathcal{L}, \vec{E})$.

4. The intersection of the boxes allows an infection directed path to be formed from either side. We denote these boxes by $B^+_{(m,n)}$ and $B^-_{(m,n)}$. Let $\gamma = (a_0, a_1, \ldots, a_m)$ and $\gamma' = (a'_0, a'_1, \ldots, a'_n)$ be open paths crossing $B^+_{(m,n)}$ and $B^-_{(m,n)}$, respectively. Then, with positive probability, there exists an infection path connecting $a_0$ to $a'_n$.

Given that 1 and 4 hold, we can easily see that 2 follows , we need 4 is necessary because one box may contain a path that crosses the entire box, while the second box contains a path that only crosses half of its box. In this case, the two paths need not intersect, so we cannot guarantee that the coupling is successful. Since the event that an edge is open is determined by the existence of an appropriate connecting path, the absence of such an intersection prevents us from establishing the required coupling.Indeed, for any edge $e = \langle (m,n), (M+1, n+1) \rangle \in \vec{E}'$, the probability of e is open is

$$\begin{aligned}\mathbb{P}_{\lambda,p}(e \text{ is open}) =& \mathbb{P}_{\lambda,p}(B^+_{(m,n)} \text{ and } \underline{B}^-_{(m,n)} \text{is crossed from bottom to top}) \\ \geq& (1-\delta)^2 > 1 - 2\delta\end{aligned}$$

in the lines above we use that the crossing of $B^+_{(m,n)}$ from bottom to top and $\underline{B}^-_{(m,n)}$ is crossed from bottom to top can be seen as increasing events on which we can apply the Harris-FKG inequality as stated in 4.3. This result holds because $\{B^+_{(m,n)}$ is crossed from bottom to top$\}$ and $\{B^-_{(m,n)}$ is crossed from bottom to top$\}$ are increasing events on $\{0,1\}^V$ where $V = (B^+_{(m,n)} \cup B^-_{(m,n)})$.

A similar argument holds for every edge of the form

$$e = \langle (m,n), (M+1, n+1) \rangle \in \vec{E}'$$

. It is also straightforward to verify that every infinite open path on $(\mathcal{L}', \vec{E}')$ gives rise to a corresponding infinite open path on $(\mathcal{L}, \vec{E})$. This immediately implies property 3. Consequently it remains only to prove properties 1 and 4 to claim the renormalization coupling.

To begin, we require the following proposition.

**Proposition 7.1.** *Fix any $p \in [0,1]$ and any $\lambda > \lambda_c(p)$, $\mu = \mu_p(\lambda) > 0$, for every $\varepsilon > 0$ and $\delta > 0$ we can choose $L$ large enough such that*

$$\mathbb{P}_{\lambda,p}(B(\varepsilon L, L, \mu) \text{ is crossed from bottom to top}) > 1 - \delta$$

For any $\delta > 0$ we may therefore choose $(L)$ such that the proposition holds. The renormalization construction then yields a 1-independent oriented percolation model with $\mathbb{P}(e \text{ open }) > 1 - 2\delta$ for all $e \in \vec{E}$. We will need the following statement in the proof of the proposition 7.1 above.

**Proposition 7.2.** *Fix any $p \in [0,1]$ then for any $\lambda > \lambda_c(p)$, $b > \mu$ and $N \in \mathbb{R}$ there exist constants $c, c', C > 0$ such that for all $t \geq 0$*

$$\mathbb{P}_{\lambda,p}(r_t > bt + N) \leq Ce^{-ct - c'N}$$

*Proof.* Since $\mu(\lambda, p) = \lim_{t\to\infty} r_t/t < b$ there exist $\tau_i$ of the renewal time for some $i \in \mathbb{N}$ large enough such that $r_{\tau_i} - b\zeta(\xi, A^+_{\tau_i}) < 0$. We fix such a renewal time $\tau_i$ defined as we did in the section 2.3 rightmost-edge speed. Define $S_i := r_{\tau_i} - b_i\zeta(\xi, A^+_{\tau_i})$ for $i \in \mathbb{Z}^+$. As introduced earlier in this paper let $(\eta^0_t)_{t\geq 0}$ denote the contact process on a disk graph without recovery map. As shown in the section 2.3 on the rightmost edge speed, the rightmost edge of the original process is almost surely dominated by the rightmost edge of $\eta$. Hence $r_{\tau_i} \leq r^{\eta}_{\tau_i}$ $\mathbb{P}_\lambda$ a.s. Therefore

$$\mathbb{E}(e^{sS_1}) \leq \mathbb{E}(e^{s(\bar{r}_{\tau_i} - b\zeta(\eta^0_t, A^+_{\tau_i})}) = \mathbb{E}(e^{s\bar{r}_{\tau_i}})\mathbb{E}(e^{sb\zeta(\eta^0_t, A^+_{\tau_i})})$$

We can rewrite the inside sums,

$$\bar{r}_{\tau_i} = \sum_{l=1}^{\tau_i} \bar{r}_{\tau_l} - \bar{r}_{\tau_{l-1}}$$

$$\zeta(\eta^0_t, A^+_{\tau_i}) = \sum_{l=1}^{i} \zeta(\eta^{'} A^+_{\tau_l}) - \zeta(\eta, A^+_{\tau_{l-1}})$$

Because these sums are independent and have the same expected value we can sum over all values by multiplying by $i$

$$\begin{aligned}\mathbb{E}(e^{sS_1}) \leq& \mathbb{E}(e^{si(\bar{r}_{\tau_i} - \bar{r}_{\tau_{i-1}})})\mathbb{E}(e^{bsi(\zeta(\eta^{'} A^+_{\tau_2}) - \zeta(\eta, A^+_{\tau_1}))}) \\ =& (\mathbb{E}(e^{s(\bar{r}_{\tau_i} - \bar{r}_{\tau_{i-1}})})\mathbb{E}(e^{bs(\zeta(\eta^{'} A^+_{\tau_2}) - \zeta(\eta, A^+_{\tau_1})}))^i\end{aligned}$$

Since $\zeta((\eta\cdot A^+_{\tau_2}) - \zeta(\eta, A^+_{\tau_1}))$ has exponentially decaying tail by propositiom 6.3, and get $\bar{r}_{\tau_i} - \bar{r}_{i-1}$ can be dominated by a sum of overshoot random variables as this

$$\sum_{\tau_1+1}^{\tau_{i+1}} O_i$$

as shown in the proof of Proposition 6.3, by Proposition 6.1 property 5 implies that these increments also have exponentially tails. We can rewrite the sum as

$$\mathbb{E}(e^{s(\bar{r}_{\tau_i} - \bar{r}_{\tau_{i-1}})}) \leq \mathbb{E}(e^{s\sum_{\tau_{i+1}}^{\tau_{i+1}} O}).$$

Consequently we can get $\mathbb{E}(e^{sS_1}) < \infty$ so we fix an $s > 0$ sufficiently small enough such that $\mathbb{E}(e^{sS_1}) < 1$. Hence, there exist constants $c_1 > 0, c_2 > 0$ such that $\mathbb{E}(e^{sb(\zeta(\eta\cdot A^+_{\tau_2}) - \zeta(\eta, A^+_{\tau_1})}) \leq e^{-c_1}$ and $\mathbb{E}(e^{sb(r_{\tau_i} - r_{\tau_{i-1}})}) \leq e^{-c_2}$. So if we take the probability given in the proposition,

$$\begin{aligned}
\mathbb{P}_{\lambda,p}(r_{\tau_i} > b\tau_i + N) =& \mathbb{P}_{\lambda,p}(e^{r_{\tau_i} - b\tau_i} > e^N) \\
=& \mathbb{P}_{\lambda,p}(e^{S_i} > e^N) \\
\leq& \mathbb{E}(e^{sS_i})^n e^{-sN} \\
\leq& e^{-i(c_1+c_2)} e^{-sN} \\
\leq& e^{-(i/T(c_1+c_2))T} e^{-sN},
\end{aligned}$$

Thus, the desired statement holds for every renewal time $\tau_j$ . To extend the estimate for all $t \geq 0$ note that, $t = \tau_j + r$ where $0 \leq r \in \tau_{j+1} - \tau_j$. Since $\mathbb{P}_{\lambda,p}(r_t > bt + N)$ is non-increasing in t, there exist a constant $C$ such that,

$$\mathbb{P}_{\lambda,p}(r_{n\tau_i} > bn\tau_i + N) \leq Ce^{-ct - c'N}$$

□

which proves the proposition.

We are now ready to prove Proposition 7.1.

*Proof.* Consider the tilted box $B(\varepsilon L, L, \mu)$ introduced in the previous section. By translation invariance, we may assume that the box is centered at the origin.

Fix $\varepsilon > 0$ and $\delta > 0$. Since $\lim_{t\to\infty} r_t/t \to \mu$ $\mathbb{P}_{\lambda,p}$- a.s. we can choose L sufficiently large so that

$$\mathbb{P}_{\lambda,p}(r_L \in ((\mu - \frac{\varepsilon}{4})L, (\mu + \frac{\varepsilon}{4})L)) > 1 - \frac{\delta}{2}$$

This guarantees with probability at least $1 - \frac{\delta}{2}$ the existence of an infection path from $(-\infty, 0] \times \{0\}$ to $((\mu - \frac{\varepsilon}{4})L, (\mu + \frac{\varepsilon}{4})L) \times \{L\}$.It remains to show that this path stays entirely contained inside the box $B(\varepsilon L, L, \mu)$.We therefore estimate the probability that the path exits the box through either the left or the right boundary before reaching the top.

We first consider the probability that the path leaves the box through the left boundary. Suppose that the infection path either starts on the left side of the box or exits through the left boundary at some point. Let $(\hat{x}, \hat{t})$ denote the last space-time point of the path lying on the left boundary of the box, and set

$$b := \mu + \frac{\varepsilon}{4},$$

With this, we require the average slope of a path from $\left(-\frac{\varepsilon L}{2}, 0\right)$ to $\left(\left(\mu - \frac{\varepsilon}{4}\right) L, L\right)$, which is the average slope required for a left-to-right crossing.

Suppose that the path passes through the point $(\hat{x}, \hat{t})$. Since the path has already spent $\hat{t}$ units of time before reaching $(\hat{x}, \hat{t})$, it must compensate during the remaining time interval $L - \hat{t}$. Consequently, to reach the target interval $\left[\left(\mu - \frac{\varepsilon}{4}\right) L, \left(\mu + \frac{\varepsilon}{4}\right) L\right] \times \{L\}$, the remaining segment of the path must make at least

$$b(L - \hat{t}) + \frac{\varepsilon}{4}\hat{t}$$

steps to the right.

We now estimate the probability of this event.

$\text{Path}_{\text{left}}$:= { there is a path from $(-\infty, 0] \times \{0\}$ to $(\mu + \frac{\varepsilon}{4})L) \times \{L\}$ that starts on the left side of the box/leaves the box to the left at some point } Then

$$\mathbb{P}_{\lambda,p}(\text{Path}_{\text{left}}) \leq \int_{t=0}^{L} \mathbb{P}_{\lambda,p}(\text{Path}_{\text{left}} | \hat{t} = t) dt$$

Moreover,

$$\mathbb{P}_{\lambda,p}(\text{Path}_{\text{left}} | \hat{t} = t) \leq \mathbb{P}_{\lambda,p}(r_{L-t} > b(L-t) + \frac{\varepsilon}{4}t) \leq Ce^{-c(L-t)-c'\frac{\varepsilon}{4}t}.$$

Where the final inequality follows from Proposition 7.2

$$\begin{aligned}
\mathbb{P}_{\lambda,p}(\text{Path}_{\text{left}}) &\leq \int_{t=0}^{L} Ce^{-c(L-t)-c'\frac{\varepsilon}{4}t} dt \\
&\leq \int_{t=0}^{L} Ce^{-\tilde{c}(L-t)-c'\frac{\varepsilon}{4}t} dt \qquad (\tilde{c} < c \wedge c'\frac{\varepsilon}{4}) \\
&= Ce^{\tilde{c}L}(\frac{\varepsilon}{4} - c')^{-1}(1 - e^{-L(\varepsilon/4-\tilde{c})})
\end{aligned}$$

Therefore we can choose L sufficiently large enough so that $\mathbb{P}_{\lambda,p}(\text{Path}_{\text{left}}) < \frac{\delta}{4}$.

For the argument of the right-hand side, assume that the path reaches $((\mu - \frac{\varepsilon}{4})L, (\mu + \frac{\varepsilon}{4})L) \times \{L\}$ and leaves the box through the right boundary. Denote by $(\hat{m}, \hat{n})$ the first vertex along this path that lies on the right side of the box. Then the segment of the path between $(-\infty, 0] \times \{0\}$ and $(\hat{m}, \hat{n})$ has average slope at least

$$\tilde{b}(\tilde{n}) := \mu + \frac{\varepsilon L}{4\tilde{n}} > \mu.$$

Observe that $\tilde{b}(\tilde{n})$ is decreasing in $\tilde{n}$. Define the event $\text{Path}_{\text{right}}$:= { there is a path from $(-\infty, 0] \times \{0\}$ to $((\mu - \frac{\varepsilon}{4})L, (\mu + \frac{\varepsilon}{4})L) \times \{L\}$ that leaves the box to the right at some point }

Then,

$$\mathbb{P}_{\lambda,p}(\text{Path}_{\text{right}}) \leq \int_{y=0}^{L} \mathbb{P}_{\lambda,p}(\text{Path}_{\text{right}}|\tilde{n} = y)dy.$$

Observe that

$$\mathbb{P}_{\lambda,p}(\text{Path}_{\text{right}}|\tilde{n} = y) \leq \mathbb{P}_{\lambda,p}(r_y > \tilde{b}(y)y) \leq Ce^{-c(\mu y + \frac{\varepsilon L}{4})}.$$

Using the previous estimate, we obtain

$$\begin{aligned}\mathbb{P}_{\lambda,p}(\text{Path}_{\text{right}}) \leq & \int_{t=0}^{L} Ce^{-c(\mu y + \frac{\varepsilon L}{4})}dy \\ =& C(\frac{e^{\frac{\varepsilon Lc}{4}}}{\mu c} - \frac{e^{-L\mu c - \frac{\varepsilon Lc}{4}}}{ac}).\end{aligned}$$

Hence, L be chosen sufficiently large sop that $\mathbb{P}_{\lambda,p}(\text{Path}_{\text{right}}) < \frac{\delta}{4}$.

We can therefore conclude that

$$\mathbb{P}_{\lambda,p}(B(\varepsilon L, L, \mu)\text{is crossed from bottom to top })$$

$$> \mathbb{P}_{\lambda,p}(r_L \in ((\mu - \frac{\varepsilon}{4})L, (\mu + \frac{\varepsilon}{4})L)) - \mathbb{P}_{\lambda,p}(\text{Path}_{\text{right}}) - \mathbb{P}_{\lambda,p}(\text{Path}_{\text{left}}) > 1 - \delta$$

□

For the last state 4 we need to show ,

**Proposition 7.3.** *Let $\gamma = (a_0, \ldots, a_m)$ and $\gamma' = (a'_0, \ldots, a'_n)$ be open crossing paths of $B^+_{(m,n)}$ and $B^-_{(m,n)}$, respectively. Then there exists a constant $c > 0$, independent of $L$, such that*

$$P_{\lambda,p}\left(a_0 \xrightarrow{\gamma \cup \gamma'} a'_n\right) \geq c.$$

*In particular, whenever both box-crossing events occur, the two crossing paths can be connected with positive probability.*

*Proof.* It is necessary to ensure that whenever both box-crossing events occur, there exists an infection path connecting the bottom of $B^-_{m,n}$ to the upper part of $B^+_{m,n}$. In the disk graph, however, two infection paths do not necessarily intersect, since infections may "jump over" one another. Consequently, the existence of two crossing paths does not automatically imply that they are connected. We therefore show that such a connection can be created with strictly positive probability.

To establish this, we follow the ideas of [10, 8]. We enlarge the graphical representation by coupling the contact process of infection rate $\lambda$ with a second contact process of infection rate $\lambda + l$, where $l > 0$ is arbitrarily small. The

additional infection arrows introduced by this coupling are independent of the original graphical representation and are used to bridge the gap between the two crossing paths. We modify the construction in [8] slightly. Rather than requiring the two paths to meet at a single crossing point, it is sufficient to create a small region in which every vertex becomes infected, thereby providing a connection between the paths.

For every $n > 0$, there exist $t > 0$ and $v \in \mathbb{Z}$ such that

$$P_{\lambda,p}\big(\xi_t^0 \supset v + B_n\big) > 0, \tag{23}$$

where $B_n = \big([-n, n] \times [0, R)\big) \cap \mathbb{Z}^2$. Indeed, since the graphical representation is generated by finitely many Poisson processes on any finite space-time region, there exists a finite realization in which every vertex of the translated box $v + B_n$ becomes infected before time $t$. As this realization requires only finitely many infection and recovery events, it occurs with strictly positive probability. Throughout the remainder of the proof, we fix $t > 0$ sufficiently small so that condition (1) holds.

Let $\gamma$ and $\gamma'$ be open crossing paths of $B(\varepsilon L, L, \mu)$ and $B(\varepsilon L, L, -\mu)$, respectively.

Using the coupled graphical representation introduced above, we add independent infection arrows corresponding to the contact process of infection rate $\lambda + l$. These additional arrows are confined to the intersection

$$B(\varepsilon L, L, \mu) \cap B(\varepsilon L, L, -\mu),$$

where the two crossing paths pass within distance $O(t)$ of one another. Each additional infection arrow appears independently with probability $\varepsilon(l) > 0$, where $\varepsilon(l)$ depends only on $l$. Denote by $\pi$ the collection of infection paths generated by these additional arrows.

To show that $\gamma$ and $\gamma'$ can be connected, consider the thickened path

$$\widehat{\gamma} = \bigcup_{a \in \gamma} \big(a + B_n + (v, t)\big),$$

which is obtained by translating a copy of the finite box $B_n$ to every vertex of $\gamma$. Since the two crossing paths intersect geometrically, the thickened path $\widehat{\gamma}$ intersects $\gamma'$. Consequently, there exists $a \in \gamma$ such that

$$(a + B_n + (v, t)) \cap \gamma' \neq \varnothing.$$

Suppose that every vertex of the translated box $a + B_n + (v, t)$ becomes infected and remains infected for a time $t$.. By the local occupation event established 23, this occurs with strictly positive probability. In this case the additional infection paths in $\pi$ provide a connection between $\gamma$ and $\gamma'$. Hence,

$$P_{\lambda,p}\left(a_0 \overset{\gamma \cup \gamma' \cup \pi}{\longrightarrow} a_n'\right) \geq \varepsilon(l)^{\,O(t^2)} > 0,$$

where the exponent $O(t)$ corresponds to the number of vertices in the finite connecting region.

Therefore, the probability that a renormalized site is open can be made arbitrarily close to one. Indeed, by choosing $\delta > 0$ sufficiently small and $L$ sufficiently large, each box is crossed with probability at least $1 - 2\delta$, while the connecting event occurs with probability $\varepsilon(l)^{O(t)} > 0$. Consequently, the renormalized oriented percolation process is supercritical, so the coupling construction is valid. “‘

□

Now the important question is if we are in the critical value of $\lambda_c(G_p)$ do we have a right-most edge speed and those it survives, this is the statement of Theorem 3.5

*of Theorem 3.5.* We argue by contradiction. Suppose that $\mu(\lambda_c^p) > 0$.Then the renormalization argument can be carried out at the critical value $\lambda_c(G_p)$. For some $\varepsilon > 0$ and $\delta > 0$ small enough we choose $L$ sufficiently large so that $B(\varepsilon L, L, \mu(\lambda_c(G_p)))$ is crossed from bottom to top with probability greater than $1 - \delta$.

Since the box is finite we may choose a $\lambda < \lambda_c(G_P)$ were $\lambda$ arbitrary close to $\lambda_c(p)$ such that,

$$\mathbb{P}_{\lambda,p}(B(\varepsilon L, L, \mu(\lambda, p)) \text{ is crossed from bottom to top } ) > 1 - \delta$$

For this choice of $\varepsilon$ and $\delta$, the renormalization argument implies that the contact process on the disk graph survives with positive probability. This yields positive survival probability for some subcritical parameter $\lambda < \lambda_c(p)$,contradicting the definition of the critical value $\lambda_c(p)$. □

# 8 conclusion

In this thesis, we studied the contact process on the disk graph of $\mathbb{Z}$ with geometric interaction radii starting from 1. This model is motivated by the spread of infectious diseases, where individuals may have inhomogeneous ranges of contact.

We proved a law of large numbers for the position of the rightmost infected site, showing that the rightmost edge advances with an almost sure deterministic speed under appropriate scaling. A key step in the analysis was to establish that the critical parameter $\lambda_c(G_p)$ is finite; without this property, the construction of the supercritical regime and the derivation of the linear growth results would not be possible here we use the results we know from rumour process.

In the next result, we assume that the rightmost edge speed exist and we can construct a renormalization argument inspired by techniques from oriented percolation. Using this construction, we further show that at the critical value $\lambda = \lambda_c(G_p)$, the process dies out almost surely.

Several natural questions remain open. First, while an upper bound is immediate from comparison with the classical contact process on $\mathbb{Z}$, namely

$$\lambda_c(G_p) \leq \lambda_c(\mathbb{Z}),$$

a matching lower bound appears to be significantly more difficult. In particular, a simple comparison with a Galton–Watson branching process does not yield a sharp bound.

A second open problem concerns quantitative extinction probabilities in the supercritical regime. More precisely, for $\lambda > \lambda_c(G_p)$, one expects that if the process is started from an initially infected box $[-N, N]$, then the probability of extinction decays exponentially in $N$. To formulate this, let

$$\tau^B := \inf\{t \geq 0 : \xi_t^B = \emptyset\}$$

denote the extinction time of the process started from the initial infected set $B$. One would like to prove that there exist constants $C, c > 0$ such that

$$\mathbb{P}\big(\tau^{[-N,N]} < \infty\big) \leq Ce^{-2cN}, \quad \forall N \in \mathbb{N}.$$

Finally, a central open question is whether the shape theorem can be extended to higher dimensions. The techniques developed in this thesis rely strongly on the one-dimensional structure, and they do not appear to extend directly to $\mathbb{Z}^d$ for $d \geq 2$. Developing methods to overcome this limitation remains an interesting direction for future research.

# References


[1] P. Balister, B. Bollobas, and A. Stacey. "Upper bounds for the critical probability of oriented percolation in two dimensions". In: *Proceedings of the Royal Society of London. Series A: Mathematical and Physical Sciences* (1993).

[2] B. Bollabás and A. Stacey. "Approximate Upper Bounds for the Critical Probability of Oriented Percolation in Two Dimensions Based on Rapidly Mixing Markov Chains". In: *Journal of Applied Probability* (1997).

[3] O. Couronné. "An Update on Lower Bounds for the Critical Values of Oriented Percolation Models". In: *Journal of Statistical Physics* (2025).

[4] R. Durret. "Oriented percolation in two dimensions". In: *The Annals of Probability* (1984).

[5] R. Durrett and D. Griffeath. "Contact Processes in Several Dimensions". In: *Wahrscheinlichkeitstheorie* (1982).

[6] R. Durrett and R.H. Schonman. "Stochastic growth models". In: *Percolation theory and ergodic theory of infinite particle systems* (1984).

[7] P.M. Rodríguez E. Lebensztayn. "The disk-percolation model on graphs". In: *Elsevier* (2008).

[8] O. Garet and R. Marchand. "Asomptotic shape for the contact process in random enviroinment". In: *The annals of applied probability* (2012).

[9] T. Harris. "Contact Interactions on a Lattice". In: *The Annals of Probability* (1974).

[10] I. Hartarsky and R. Szabó. "Generalised Oriented Site Percolation". In: *Markov processes and Related Fields* (2021).

[11] J. Kingman. "Subadditive Ergodic Theory". In: *The Annals of Probability* (1973).

[12] T. Kuczek. "The central limit theorem for the right edge of supercritical oriented percolation." In: *The Annals of Probability* (1989).

[13] R. Leadbetter, S. Cambanis, and V. Pipiras. *A Basic Course in Measure and Probability: Theory for Applications*. Cambridge University Press, 2014.

[14] T. Ligget. *Interacting Particle systems*. Springer, 1985.

[15] T. Ligget. *Stochastic interacting systems : contact, voter, and exclusion processes*. Springer, 1999.

[16] W. Merbis, c. de Mulatier, and P. Corboz. "Efficient simulations of epidemic models with tensor networks: Application to the one-dimensional susceptible-infected-susceptible model". In: *Phys. Rev. E* (2023).

[17] R. Rishideep and S. Kumarjit. "How fast do rumours spread?" In: *Journal of statistical Physics* (2024).

[18] V. Vargas S. Gallo N. L. Garcia and Rodríguez P. M. "Rumor processes on $\mathbb{N}$ and discrete renewal processes". In: *Journal of statistical Physics* (2014).

[19] A. Sudbury. "Rigorous lower bounds for the critical infection rate in the diffusive contact process". In: *Journal of Applied Probability* (2001).

[20] T. Ligget. "Improved Upper Bounds for the Contact Process Critical Value". In: *Annals of Probability* (1995).