Crazy Things in $\mathbb{R}$

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There once was a bright man and wise
whose sets were a peculiar size
    no length it is true
    and dense points were few
but more points than $\mathbb{Q}$ inside.

Institute for Mathematics
and Computing Science

RuG

Bachelor thesis

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December 2007

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Chapter 1

Introduction

One might be tempted to think that the real line $\mathbb{R}$ is a rather simple set, and one about which not much remains to be discovered. However, we soon find out that matters are not as simple as that. The continuum remains a mysterious entity, even to this day. What does one do, for example, when confronted with two subsets of $\mathbb{R}$ and asked which of them is larger?

There are many possibilities, for a start one might count the elements in each and then see which contains more elements. This even goes quite well when considering sets with an infinite number of points, using an injective function we are able to compare the number of elements in one set with the number in the other.

On the other hand, we might find this method a bit restrictive, many sets are put on an equal footing with each other that intuitively seem very different. For example, we find that the set of integers $\mathbb{Z}$ and the set of rational numbers $\mathbb{Q}$ contain exactly the same number of elements. In some cases such as this one, looking at the distribution of the sets in $\mathbb{R}$ gives us a better idea of how the sets are different.

Of course, one could do some simple measuring, one could simply put a ruler next to $\mathbb{R}$ and see how long a set is if you squish all the points in it together. A function which allows us to determine the length of a set in such a manner is called a measure. In this way we are able to distinguish between, for example, intervals of different length, something nothing up till now has done.

Thus all sets we want to describe must be looked at from many points of view before anything can be said concerning how 'large' that set is. We have numerous examples of a set being immense from the point of view of topology, yet tiny with respect to measure, or vice versa. We need not look far for interesting sets, even some very simple ones have many surprising things happening in them, and are well worth our time studying.

After having been loaded with several different ways of looking at sets, we might start wondering where it will all end. Do any of these definitions have anything to do with each other? We certainly cannot equate any of them, as numerous examples show. This question brings us the essence of Erdös's Duality theorem. This metatheorem manages to give a surprising result concerning two completely different definitions of 'smallness'.

Chapter 2

Leading Examples of Real Sets

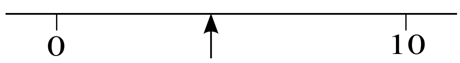
Several examples will be recurrent throughout the text to clarify definitions and concepts. To give some basic properties of these sets, and to avoid repetition, we shall start by treating them in some detail.

2.1 Integers and Nonintegers, Rationals and Irrationals, Algebraic and Transcendental Numbers

The set of integers is the set of (positive and negative) whole numbers. It is denoted by $\mathbb{Z}$. We shall let $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ and $\mathbb{Z}^- = \{-1, -2, -3, \ldots\}$. We shall let the set $\{0, 1, 2, 3, \ldots\}$, the natural numbers, be denoted by $\mathbb{N}$. The set of rational numbers, denoted by $\mathbb{Q}$, is the set of all real numbers which can be represented in the form $\frac{a}{b}$, where $a \in \mathbb{Z}$ and $b \in \mathbb{Z}^+$.

A real number z is called *algebraic* if it satisfies an equation of the form $a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0$, $a_i \in \mathbb{Z}$ and not all equal to zero. The smallest integer n such that z satisfies an equation of degree n is called the *degree* of z. For example, $\sqrt{3}$ has degree 2 as it satisfies the equation $x^2 - 3 = 0$. Any rational number $\frac{p}{q}$ is of degree 1 by virtue of satisfying $qx - p = 0$. Denoting the set of algebraic numbers by $\mathbb{A}$, clearly $\mathbb{Q} \subset \mathbb{A}$. Any real number which is not algebraic is called *transcendental*.

There are many ways of representing real numbers. One of the silliest and least useful ways might be:



This however tells us none of the things we would like to know. For instance, is this number an integer? Or a rational? Is there a way of distinguishing all the rational numbers from the irrationals in some easy way? These questions, and others, start making more sense when we instead attempt to represent real numbers in their base 10 nonterminating decimal forms, which we shall discuss in detail in appendix A.

2.2 Middle Third and Middle-α Cantor Sets

Using a deceptively simple method we are able to construct a group of extremely interesting sets called Cantor sets. The class ‘Cantor set’ contains a very broad array of sets, we shall consider only a very small subset, which we shall call the $\mathbb{C}_\alpha$ Cantor sets. There are many such sets, as we shall see later on. We shall however keep matters simple by first constructing the easiest of such sets, called the Middle Third Cantor set, or Triadic Cantor Dust.

Constructing the Middle Third Cantor set

The following algorithm is used in the construction of the Middle Third Cantor set, which we denote by $\mathbb{C}_{\frac{1}{3}}$:

step 1 Let $C_0 = [0, 1]$ and let $i = 0$.

step 2 Construct C_{i+1} by $C_{i+1} = \frac{1}{3}C_i \cup (\frac{2}{3} + \frac{1}{3}C_i)$ (this is equivalent to removing the middle third open set from C_i).

step 3 Increase i by 1 and go to step 2.

The set $\mathbb{C}_{\frac{1}{3}}$ is constructed with

$$\mathbb{C}_{\frac{1}{3}} = \bigcap_{i=0}^{\infty} C_i$$

This algorithm is nonterminating, and each step doubles the number of intervals. In C_i, there are 2^i intervals of length $\frac{1}{3^i}$. The total length of the set C_i is therefore $\left(\frac{2}{3}\right)^i$, which tends to 0 as $i \to \infty$. Figure 2.1 shows what the construction of $\mathbb{C}_{\frac{1}{3}}$ looks like.

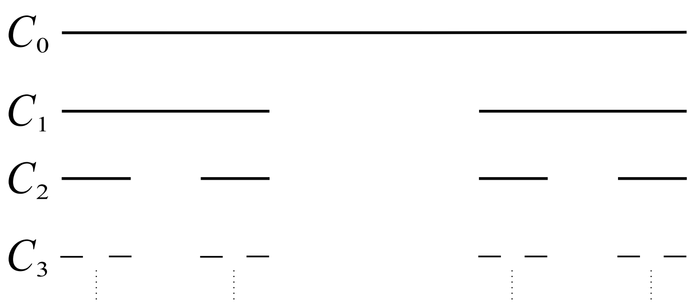


Figure 2.1: Construction of the Cantor Middle Third set

The Points Contained in $\mathbb{C}_{\frac{1}{3}}$

Although it seems to be so at first sight, the set $\mathbb{C}_{\frac{1}{3}}$ is definitely not a simple collection of endpoints of the intervals in C_i ($i \in \mathbb{N}$). To demonstrate this, we shall identify a point between 0 and 1 which is not the endpoint of one of the intervals, but is an element of $\mathbb{C}_{\frac{1}{3}}$.

To do so, we must first note down the real numbers between 0 and 1 in a different way, namely in their ternary form. Consider as an example the point $\frac{1}{5}$. If it lies in $[0, \frac{1}{3}]$ the first ternary place will be 0; if it lies in $(\frac{1}{3}, \frac{2}{3})$ the first ternary place will be 1; if it lies in $[\frac{2}{3}, 1]$ the first ternary place will be 2. Of course, $\frac{1}{5} \in [0, \frac{1}{3}]$ thus the first ternary place will be 0. We now see whether $\frac{1}{5}$ lies in $[0, \frac{1}{9}], (\frac{1}{9}, \frac{2}{9})$ or $[\frac{2}{9}, \frac{3}{9}]$ to determine the second ternary place. We continue in this manner, always letting the middle interval be open on both sides when determining the next decimal.

intervals to choose from	$\frac{1}{5}$ is in	thus the ternary place is
$[0, \frac{1}{3}](\frac{1}{3}, \frac{2}{3})[\frac{2}{3}, 1]$	$[0, \frac{1}{3}]$	0
$[0, \frac{1}{9}](\frac{1}{9}, \frac{2}{9})[\frac{2}{9}, \frac{3}{9}]$	$(\frac{1}{9}, \frac{2}{9})$	1
$(\frac{3}{27}, \frac{4}{27}](\frac{4}{27}, \frac{5}{27})[\frac{5}{27}, \frac{6}{27})$	$[\frac{5}{27}, \frac{6}{27})$	2
$[\frac{15}{81}, \frac{16}{81}](\frac{16}{81}, \frac{17}{81})[\frac{17}{81} . \frac{18}{81}]$	$(\frac{16}{81}, \frac{17}{81})$	1
$(\frac{48}{243}, \frac{49}{243}](\frac{49}{243}, \frac{50}{243})[\frac{50}{243}, \frac{51}{243})$	$(\frac{48}{243}, \frac{49}{243}]$	0

Table 2.1: Construction of the base 3 representation of $\frac{1}{5}$

Thus the representation of $\frac{1}{5}$ becomes $0.01210\ldots$. The ternary expansion of several other rational points are:

rational	ternary expansion
$\frac{1}{3}$	0.02222...
$\frac{2}{3}$	0.20000...
$\frac{1}{2}$	0.11111...
$\frac{4}{9}$	0.10222...

Table 2.2: The ternary expansion of several rational numbers

Any points which have a 1 anywhere in their ternary expansions are in an interval which is removed during the construction of $\mathbb{C}_{\frac{1}{3}}$. Any points which have no 1 in their expansions are never removed. Thus $\mathbb{C}_{\frac{1}{3}}$ contains precisely those points which have a base 3 expansion which uses only the digits 0 and 2.

We shall now investigate the representation of $\frac{1}{4}$. In ternary form, the number 4 is noted as $(11)_3$ and 1 as $(1)_3$. If we divide $(1)_3$ by $(11)_3$, we obtain the ternary representation of $\frac{1}{4}$:

$$
\begin{array}{r|l}
 & 0.0202\ldots \\
11 & 1.00 \\
- & \;\;22 \\
 & \;\;100 \\
- & \;\;\;\;22 \\
 & \;\;\;\;\;\;1 \\
 & \;\;\;\;\;\;\vdots
\end{array}
$$

The representation of $\frac{1}{4}$ is thus the repeating ternary expansion $0.02020202\ldots$. This, as we can see, contains no 1, thus $\frac{1}{4}$ is never removed during the construction of $\mathbb{C}_{\frac{1}{3}}$. Thus for every i, $\frac{1}{4} \in C_i$, so $\frac{1}{4} \in \bigcap_{i=0}^{\infty} C_i = \mathbb{C}_{\frac{1}{3}}$. We can also easily see that $\frac{1}{4}$ is not the endpoint of any interval, as any such endpoint is of the form $\frac{k}{3^n}$, where k and n are both nonnegative integers. Thus $\frac{1}{4}$ is a point in $\mathbb{C}_{\frac{1}{3}}$ which is not an endpoint of any interval.

Constructing other Cantor Sets

We need not remove a third from each interval during construction, we can remove any part we feel like. If $0 < \alpha < 1$, by removing the middle α from each interval we form $\mathbb{C}_\alpha$, the Middle α Cantor set. Figure 2.2 shows the so-called Cantor curtain, which shows what each of these sets looks like.

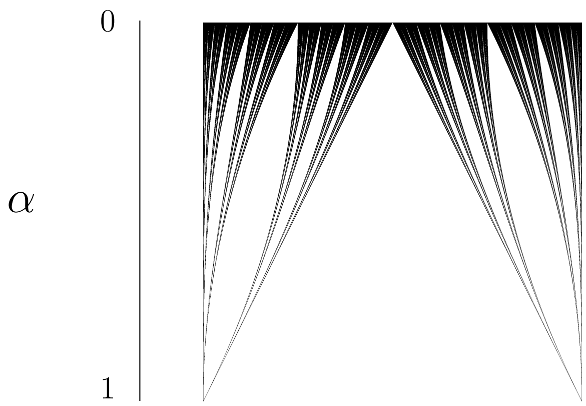


Figure 2.2: Cantor Curtain

Properties of all $\mathbb{C}_\alpha$ Cantor sets

There are many properties which all these Cantor sets have in common. We shall now mention a few of these which are important in intuitively understanding the nature of the Cantor set.

Theorem 1 ([6]). *Any Cantor set* $\mathbb{C}_\alpha$ $(0 < \alpha < 1)$ *as defined above has the following properties.*

1. $\mathbb{C}_\alpha$ *is closed.*

2. $\mathbb{C}_\alpha$ *is totally disconnected (this means that every sufficiently small neighbourhood of every point in the Cantor set is a empty. In* $\mathbb{R}$ *this means that no* $\mathbb{C}_\alpha$ *contains an interval).*

3. $\mathbb{C}_\alpha$ *is perfect (this means that it contains no isolated points).*

Proof.

1. The first statement is the easiest to prove: because $\mathbb{C}_\alpha$ Cantor sets are intersections of closed sets, they are themselves closed.

2. Consider the Cantor set $\mathbb{C}_\alpha$, we need to prove that this set contains no intervals. Assume the contrary, that $\mathbb{C}_\alpha$ contains an interval (δ) of length δ. We will now investigate the length of the intervals in each stage of the construction of $\mathbb{C}_\alpha$. Let $\mathbb{C}_\alpha = \bigcap_{i=0}^{\infty} C_i$ where each set C_i is a stage in the construction of $\mathbb{C}_\alpha$.

C_0 contains just one interval of length 1. C_1 contains two intervals, the length of each of which is half what is left over after α has been removed, thus the length of intervals in C_1 is $\frac{1}{2}(1-\alpha)$. In C_2, a further middle α has been removed from each interval. Each new interval is now half the length of what is left over after removing α of the previous interval:

$$\begin{aligned}\frac{1}{2}\left[\frac{1}{2}(1-\alpha)-\frac{1}{2}\alpha(1-\alpha)\right] &= \left(\frac{1}{2}\right)^2(1-\alpha)[1-\alpha] \\ &= \left(\frac{1}{2}\right)^2(1-\alpha)^2\end{aligned}$$

Inductively, we find that C_i contains intervals of length

$$\left(\frac{1}{2}\right)^i(1-\alpha)^i = \left(\frac{1}{2}-\frac{1}{2}\alpha\right)^i$$

Now, choose $i \in \mathbb{N}$ such that $(\frac{1}{2}-\frac{1}{2}\alpha)^i < \delta$. This means that no matter what the length δ is, we can always find C_i such that the intervals in C_i are smaller than δ. This means that (δ) cannot be contained in C_i. However, $\mathbb{C}_\alpha = \bigcap_{i=0}^{\infty} C_i$, thus (δ) cannot be contained in $\mathbb{C}_\alpha$, and $\mathbb{C}_\alpha$ contains no intervals.

3. Consider now the third statement that $\mathbb{C}_\alpha$ Cantor sets are perfect. This means that if $a \in \mathbb{C}_\alpha$, then for any $\varepsilon > 0$, the interval $(a-\varepsilon, a+\varepsilon)$ contains points of $\mathbb{C}_\alpha$ other than a. For simplicity's sake I shall prove this property for $\mathbb{C}_{\frac{1}{3}}$ only, but it can be shown for all other $\mathbb{C}_\alpha$ as well. Consider the point $a \in \mathbb{C}_{\frac{1}{3}}$, which has as a property that it has a ternary representation containing only 0 and 2. Given $\varepsilon > 0$, determine n such that the number $a_n = 0.0\ldots020\ldots$ which has a zero at every ternary place except at position n where it has a 2, is smaller than ε. If a has a 2 at position n, then the number $a - a_n$ is a point which is closer than ε to a. It will be in $\mathbb{C}_{\frac{1}{3}}$, because it has a ternary expansion using only 0's and 2's. If a has a 0 at position n, then the number $a + a_n$ will be closer than ε to a, and contained in $\mathbb{C}_\alpha$. □

We can go even further than this. The great Dutch mathematician L.E.J. Brouwer (1881-1966) proved that all compact metric spaces that are perfect and totally disconnected are homeomorphic, which gives a topological characterisation of the general Cantor set. For details, see [9], page 35.

2.3 Liouville Numbers

The next set we will look at is the set of Liouville numbers, which shall be denoted by $\mathbb{L}$. A real number z is in $\mathbb{L}$ if it is irrational and if for each positive integer n there exist integers p and q such that

$$\left|z - \frac{p}{q}\right| < \frac{1}{q^n} \text{ and } q > 1$$

$\mathbb{L}$ is the set of irrational numbers which are 'very close' to rational numbers. It is difficult to see from this definition that $\mathbb{L}$ is nonempty, or that $\mathbb{L} \neq \mathbb{R} \setminus \mathbb{Q}$. Let us first address this first issue.

Examples of Liouville Numbers

Using the numbers in $(0, 1)$, we shall construct infinitely many Liouville numbers. Let $a \in (0, 1)$ be any real number and consider its decimal expansion $a = \sum_{k=1}^{\infty} \frac{a_k}{10^k} = 0.a_1a_2a_3a_4\ldots$ with $a_k \in \{0, 1, \ldots, 9\}$. Now let

$$z = \sum_{k=1}^{\infty} \frac{a_k}{10^{k!}} = 0.a_1a_2000a_3\underbrace{0\ldots0}_{17}a_40\ldots$$

We claim that $z \in \mathbb{L}$. What we need to do now is for every n to determine p and q such that $|z - \frac{p}{q}| < \frac{1}{q^n}$. Let $q = 10^{n!}$, then

$$\begin{aligned} \left|z - \frac{p}{q}\right| &= \left|\sum_{k=1}^{\infty} \frac{a_k}{10^{k!}} - \frac{p}{10^{n!}}\right| \\ \frac{1}{q^n} &= \frac{1}{10^{n \cdot n!}} = 0.\underbrace{0\ldots0}_{n \cdot n!-1}1 \end{aligned}$$

Let $p = \sum_{k=1}^{n} \frac{a_k 10^{n!}}{10^{k!}}$, then:

$$\begin{aligned} \left|z - \frac{p}{q}\right| &= \left|\sum_{k=1}^{\infty} \frac{a_k}{10^{k!}} - \sum_{k=1}^{n} \frac{a_k}{10^{k!}}\right| \\ &= \left|\sum_{k=n+1}^{\infty} \frac{a_k}{10^{k!}}\right| = 0.\underbrace{0\ldots0}_{(n+1)!-1}a_{n+1}0\ldots \end{aligned}$$

The first 1 after the decimal point in $\frac{1}{q^n}$ comes at position $n \cdot n!$. The first nonzero after the decimal point in $|z - \frac{p}{q}|$ comes at position $(n + 1)!$, or later if $a_{n+1} = 0$. Of course,

$(n+1)! > n \cdot n!$, thus $|z - \frac{p}{q}| < \frac{1}{q^n}$ for these choices of p and q. Thus $z = \sum_{k=1}^{\infty} \frac{a_k}{10^{k!}}$ is a Liouville number.

We would now like to name an irrational number which is not Liouville to show that $\mathbb{L} \neq \mathbb{R} \setminus \mathbb{Q}$. To do this, we must first analyse more closely the nature of the numbers contained in $\mathbb{L}$.

The Transcendentality of Liouville Numbers

To help us prove that $\mathbb{L}$ contains only transcendental numbers we need first to prove the following lemma concerning algebraic numbers:

Lemma 1 (Liouville [1]). *For any algebraic number z of degree $n > 1$ there exists a positive integer M such that*

$$\left| z - \frac{p}{q} \right| > \frac{1}{Mq^n}$$

for all integers p and q, $q > 0$.

Proof. Let $f(x)$ be a polynomial of degree n with integer coefficients such that $f(z) = 0$. Let M be a positive integer such that $|f'(x)| \leq M$ when $|z - x| \leq 1$. By the mean value theorem:

$$|f(x)| = |f(z) - f(x)| \leq M|z - x| \text{ whenever } |z - x| \leq 1$$

Consider two integers p, q, with $q > 0$. When $|z - \frac{p}{q}| > 1$, we see easily that $|z - \frac{p}{q}| > \frac{1}{Mq^n}$, as $\frac{1}{Mq^n} \leq 1$. We now consider the case when $|z - \frac{p}{q}| \leq 1$. Then we see that $|f(\frac{p}{q})| \leq M|z - \frac{p}{q}|$, and also

$$q^n \left| f\left(\frac{p}{q}\right) \right| \leq Mq^n \left| z - \frac{p}{q} \right|$$

Can the equation $f(x) = 0$ have a rational root? Assume it does, then $f(\frac{a}{b}) = 0$. This means however, that we can factor out $(x - \frac{a}{b})$ from $f(x)$: $f(x) = (x - \frac{a}{b})g(x)$, where g is of degree $n - 1$, g has rational coefficients and $g(z) = 0$. However, this cannot happen because the degree of z is n. Thus f has no rational root.

Thus $f(\frac{p}{q}) \neq 0$, and, even more, $|q^n f(\frac{p}{q})|$ is an integer:

$$\begin{aligned} \left| q^n f\left(\frac{p}{q}\right) \right| &= \left| q^n \left(a_0 + a_1 \frac{p}{q} + a_2 \frac{p^2}{q^2} + \cdots + a_n \frac{p^n}{q^n} \right) \right| \\ &= \left| a_0 q^n + a_1 p q^{n-1} + a_n p^2 q^{n-2} + \cdots + a_n p^n \right| \in \mathbb{Z} \text{ as } p, q \in \mathbb{Z} \end{aligned}$$

Thus $|q^n f(\frac{p}{q})| \geq 1$, and from this we get:

$$\begin{aligned} Mq^n \left| z - \frac{p}{q} \right| &\geq 1 \\ \left| z - \frac{p}{q} \right| &\geq \frac{1}{Mq^n} \end{aligned}$$

Because z is irrational (the degree of z is greater than 1), the left member will be irrational. The right member will, however, be rational. Thus equality cannot hold, and we conclude:

$$\left| z - \frac{p}{q} \right| > \frac{1}{Mq^n}$$ □

We are now ready to show that Liouville numbers are transcendental.

Theorem 2 ([1]). *All Liouville numbers are transcendental.*

Proof. Suppose some $z \in \mathbb{L}$ is algebraic of degree n. We first observe that we must have $n > 1$, else z would be rational which is forbidden by the definition of $\mathbb{L}$. We now have two inequalities which both hold:

$$\text{There exists } M \in \mathbb{Z}^+ \text{ such that } \left| z - \frac{p}{q} \right| > \frac{1}{Mq^n} \quad \forall p, q \in \mathbb{Z}, q > 0$$

$$\text{For each } m \in \mathbb{Z}^+ \text{ there exist } p, q \in \mathbb{Z} \text{ such that } \left| z - \frac{p}{q} \right| < \frac{1}{q^m} \quad q > 1$$

Choose an integer k such that $2^k \geq 2^n M$. Then $p, q \in \mathbb{Z}$ exist such that:

$$\left| z - \frac{p}{q} \right| < \frac{1}{q^k}$$

and also

$$\frac{1}{Mq^n} < \left| z - \frac{p}{q} \right|$$

thus

$$\begin{aligned} \frac{1}{Mq^n} &< \frac{1}{q^k} \\ q^k &< Mq^n \end{aligned}$$

hence

$$M > q^{k-n} \geq 2^{k-n} \geq M$$

This is a contradiction, thus $\mathbb{A} \cap \mathbb{L} = \emptyset$ and $\mathbb{L}$ contains only transcendental numbers. □

The theorem above means we can easily give numerous examples of irrational numbers which are not Liouville numbers, such as $\sqrt{2}$, $\sqrt{2} + \sqrt{3}$, $\sqrt[3]{2}$, et cetera. Thus it is evident that $\mathbb{L}$ is not empty, but neither does it comprise all of the irrational numbers.

Chapter 3

Cardinality

We now leave our descriptions of various sets to proceed to our first method of comparing them: cardinality. By setting up injective maps between two sets, we can reach some surprising conclusions.

3.1 Countability

When presented with two sets of objects and asked which is the larger, most people would resort to the oldest method to be discussed in this paper, namely counting. When dealing with most of daily life, this has always served us well, as real objects tend to come in finite amounts. When presented with the sets $\{1, 2\}$ and $\{1, 2, 3\}$, all are agreed that the first set has 2 elements, the second 3, and that 3 is larger than 2 so the second set must be the larger.

The number of elements in a finite set is called the *cardinality* of the set. The cardinality of a set A will be denoted by $|A|$. For example, the cardinality of the set $\{1, 2, 3\}$, denoted $|\{1, 2, 3\}|$, is 3.

We now want to do away with the part in the process just described that assigns a value to the number of elements of each set. The way in which we can then see which set is the larger is by setting up an injective map between the sets. If there exists an injective map from a set A to a set B, it means that every element in A can be assigned a unique element from B. For this to be possible, B must contain at least as many elements as A. An example will clarify this idea:

$$\begin{array}{lccc} \{1,2\} & 1 & 2 & \\ & \downarrow & \downarrow & \\ \{1,2,3\} & 1 & 2 & 3 \end{array}$$

There exists an injective map from $\{1, 2\}$ into $\{1, 2, 3\}$, thus $\{1, 2\}$ contains at least as many elements as $\{1, 2\}$. However, there does not exist an injective function from $\{1, 2, 3\}$

into $\{1,2\}$, thus clearly $\{1,2\}$ does not contain at least as many elements as $\{1,2,3\}$:

$$\begin{array}{cccc} \{1,2\} & 1 & 2 & ? \\ & \uparrow & \uparrow & \uparrow \\ \{1,2,3\} & 1 & 2 & 3 \end{array}$$

If we can set up a bijection f between two sets A and B, this of course means that we have two injective functions; $f : A \to B$ and $f^{-1} : B \to A$. Thus both $|A| \leq |B|$ and $|B| \leq |A|$, and we must conclude $|A| = |B|$, by virtue of the Cantor-Bernstein theorem [10].

$$\begin{array}{ccccc} \{1,2,3,4\} & 1 & 2 & 3 & 4 \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ \{\spadesuit, \heartsuit, \clubsuit, \diamondsuit\} & \spadesuit & \heartsuit & \clubsuit & \diamondsuit \end{array}$$

Comparing Infinite Sets

We will now ask ourselves whether this way of thinking can be applied to sets which contain an infinite number of elements, such as intervals or the set of rational numbers. Consider the set $\mathbb{N}$ of natural numbers. Are there any sets which have the same cardinality as $\mathbb{N}$? Let us consider $\mathbb{Z}$, the integers. It looks as if it contains twice as many elements as $\mathbb{N}$, yet observe the following:

$$\begin{array}{ccccccc} \mathbb{N} & 1 & 2 & 3 & 4 & 5 & \cdots \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ \mathbb{Z} & 0 & 1 & -1 & 2 & -2 & \cdots \end{array}$$

$$f : \mathbb{N} \to \mathbb{Z}, \quad f(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ even} \\ \frac{1-x}{2} & \text{if } x \text{ odd} \end{cases}$$

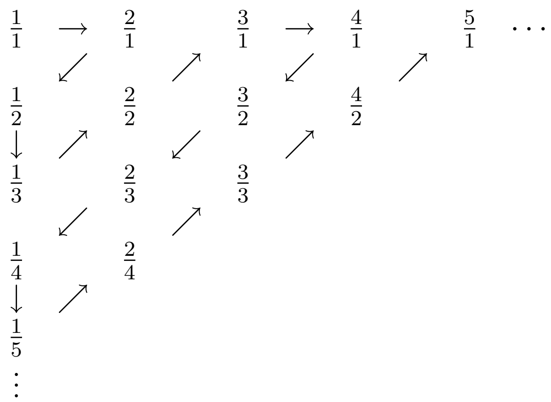
The function f is the bijection we seek. Thus $\mathbb{N}$ and $\mathbb{Z}$ have the same cardinality, which we call $\aleph_0$ (pronounced aleph zero) for reasons which will become clear later. Another term we use to describe a set with the same cardinality as $\mathbb{N}$ is *countable*.

Cantor's First Diagonal Procedure

What happens when we look at a set that looks even larger than $\mathbb{Z}$? If $\mathbb{Z}$ seems to contain twice as many points as $\mathbb{N}$, the set $\mathbb{Q}$ seems to have an infinite number of times more – between every two points of $\mathbb{N}$ there are an infinite number of rational numbers. For simplicity, we shall only consider those rational numbers which are larger than 0, the set $\mathbb{Q}^+$. We first rearrange these rational numbers in a grid-like structure as follows:

	1	2	3	4	5	$\cdots$
1	$\frac{1}{1}$	$\frac{2}{1}$	$\frac{3}{1}$	$\frac{4}{1}$	$\frac{5}{1}$	
2	$\frac{1}{2}$	$\frac{2}{2}$	$\frac{3}{2}$	$\frac{4}{2}$	$\frac{5}{2}$	
3	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{3}{3}$	$\frac{4}{3}$	$\frac{5}{3}$	
4	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$	$\frac{4}{4}$	$\frac{5}{4}$	
5	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{5}{5}$	
$\vdots$						

Every rational number appears in this grid, each one even appears more than once. We can now make a list of all rational numbers by using Cantor's first diagonal procedure:



When making our list of rational numbers, we leave out those we have listed previously. This list we can use to set up the following bijection between $\mathbb{N}$ and $\mathbb{Q}^+$:

$\mathbb{N}$	1	2	3	4	5	6	7	8	$\cdots$
	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	
$\mathbb{Q}^+$	1	2	$\frac{1}{2}$	$\frac{1}{3}$	3	4	$\frac{3}{2}$	$\frac{2}{3}$	$\cdots$

Thus also $\mathbb{Q}^+$ has the same number of elements as $\mathbb{N}$, and is countable. This method in fact demonstrates that any set which is the product of two countable sets is itself countable. We simply rearrange the elements of the two sets in a grid and move diagonally through it.

From this it follows inductively that the product of any finite number of countable sets is countable. Even more, the union of any countable number of countable sets is countable.

Consider the set $A = \bigcup_{i=0}^{\infty} A_i$, where each of the sets A_i is countable. Let the elements of A_i be denoted by $A_i^1, A_i^2, A_i^3, A_i^4, \ldots$. The elements of A can now be arranged in a grid-like structure, and Cantor's first diagonal procedure can still be applied.

$$\begin{array}{ccccccccc}
A_1^1 & \rightarrow & A_1^2 & & A_1^3 & \rightarrow & A_1^4 & \cdots \\
 & \swarrow & & \nearrow & & \swarrow & & \\
A_2^1 & & A_2^2 & & A_2^3 & & & \\
\downarrow & \nearrow & & \swarrow & & & & \\
A_3^1 & & A_3^2 & & & & & \\
 & \swarrow & & & & & & \\
A_4^1 & & & & & & & \\
\vdots & & & & & & &
\end{array}$$

Figure 3.1: The countable union of countable sets is countable

As an example of which sets we can prove to be countable, let us consider the following theorem.

Theorem 3. *The set $\mathbb{A}$ of algebraic numbers is countable.*

Proof. Any algebraic number is the root of some equation $a_0 + a_1 z + \cdots + a_n z^n = 0$ with $a_i \in \mathbb{Z}$. If $a_n \neq 0$, the degree of the polynomial is n. Each polynomial has a finite number of real roots, thus all we need to do to prove this theorem is to show that there is a countable number of polynomials.

There are a countable number of polynomials of degree n, which follows from the fact that the product of a finite number of countable sets is countable. We thus have $\aleph_0$ polynomials of degree n, and $\aleph_0$ possible degrees. This means that there are a countable number of polynomials, and thus a countable number of algebraic numbers. □

We may now start to doubt whether there is any infinity larger than $\aleph_0$, whether 'uncountability' exists at all.

3.2 Uncountability

In the previous section we looked at the integers and the rationals. Let us now turn our attention to the interval in the continuum. Any interval also contains an infinite number of points, does this mean that an interval is countable? As an example, take the interval $(0, 1)$, and assume that we have found a bijection between it and $\mathbb{N}$. Recall the unique representation of real numbers discussed in appendix A: no infinite succession of 0's is allowed to occur in the decimal representation. We write down the list of all real numbers in $(0, 1)$ as follows, and thus construct a bijection:

$$\begin{array}{ccc} 1 & \leftrightarrow & 0.a_{11}a_{12}a_{13}a_{14}\dots \\ 2 & \leftrightarrow & 0.a_{21}a_{22}a_{23}a_{24}\dots \\ 3 & \leftrightarrow & 0.a_{31}a_{32}a_{33}a_{34}\dots \\ 4 & \leftrightarrow & 0.a_{41}a_{42}a_{43}a_{44}\dots \\ \vdots & & \vdots \end{array}$$

We shall now construct a real number not in the list using Cantor's second diagonal procedure. This will prove that the interval $(0, 1)$ is uncountable, as we have assumed that we have made a list containing *all* points in $(0, 1)$. Take first the real number constructed by taking the elements a_{ii}

$$0.a_{11}a_{22}a_{33}a_{44}\dots$$

Now modify it in the following manner: if $a_{ii} = 1$, then replace it with 2, if $a_{ii} \neq 1$, then replace it with 1. We thus obtain a new number, and call it

$$0.b_1b_2b_3b_4\dots$$

This number has the property that $b_i = 2$ if $a_{ii} = 1$ and $b_i = 1$ if $a_{ii} \neq 1$. It has neither an unending succession of zeroes nor of nines, thus it lies somewhere between 0 and 1. This means that it must occur on the list somewhere, say at position d. However, if $a_{dd} = 1$, then $b_d = 2$ and if $a_{dd} \neq 1$ then $b_d = 1$, so $0.b_1b_2b_3b_4\dots$ does not appear at position d, or at any other position. Thus the interval $(0, 1)$ contains more points than $\mathbb{N}$, and hence it is an uncountable set.

The Uncountability of $\mathbb{R}$

We have now discovered the fascinating fact that there are different kinds of infinity, and that some are larger than others. Is there one which is larger than the number of points in $(0, 1)$? Before we try finding such an infinity in $\mathbb{R}$, let us first see how many points $\mathbb{R}$ itself contains. Consider the following function:

$$f : (0, 1) \to \mathbb{R} \qquad f(x) = \begin{cases} \frac{2x-1}{2x} & \text{if } x \leq \frac{1}{2} \\ \frac{1-2x}{2x-2} & \text{if } x > \frac{1}{2} \end{cases}$$

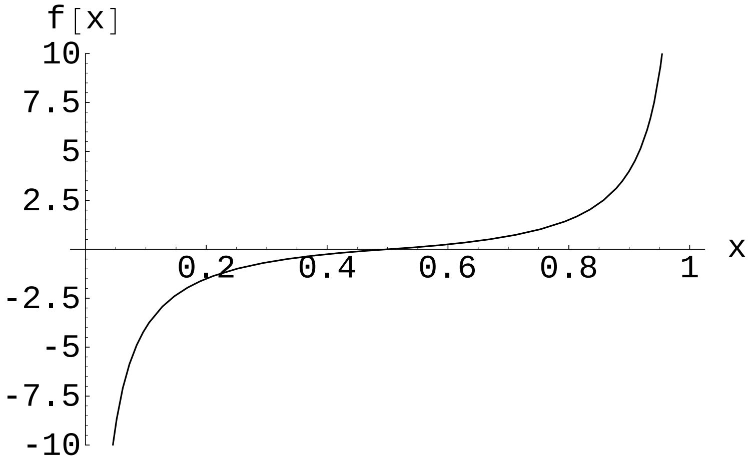


Figure 3.2: A bijection between $(0, 1)$ and $\mathbb{R}$

This is a bijective function, as the graph clearly shows, thus the interval $(0, 1)$ and $\mathbb{R}$ contain the same number of points. We shall call this cardinality c, to signify the continuum. There are many bijective functions we could have used to show this, the most common being based on the tangent function.

This leaves us with the question whether there is then an infinity beyond c: are there an infinite number of infinities or have we now reached the greatest possible cardinality? Also, what is the relation between c and $\aleph_0$?

3.3 The Power Set

The first question we might ask ourselves is, given a (finite or infinite) set, can we use a general method to construct an even larger set? This seems a simple matter when we consider a finite set, for example $\{1, 2, 3\}$. Simply add an element which was not yet in the set to it to obtain a larger set: $\{0, 1, 2, 3\}$. However, we have seen that this does not work when we are dealing with an infinite set. Consider the set $\{1, 2, 3, \ldots\}$ which has cardinality $\aleph_0$. Add to it an element and we obtain $\{0, 1, 2, 3, \ldots\}$, which still has cardinality $\aleph_0$. Making the set twice as large will not make it larger either, as we saw when we considered $\mathbb{Z}$.

Consider now the following definition:

Definition 1. *The power set of a set A, denoted by $P(A)$, is the set of all subsets of A.*

The power set of $\{1, 2\}$ would be $\{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$, for example. Perhaps this definition will help us to construct larger sets from smaller ones. It certainly seems to work when we try a finite set. To find out whether $|P(A)| > |A|$, observe first that $P(A)$ cannot contain fewer elements than A, after all, for all $a \in A$, the singleton $\{a\} \in P(A)$, so $P(A)$ has at least as many elements as A. All we need to do now is to show that no set has the same cardinality as its power set, and we are done.

Theorem 4 (Cantor [2])**.** *No set is of the same cardinality as its power set:* $|P(A)| > |A|$.

Proof. Consider a set A and its power set $P(A)$. Assume by contradiction that A and $P(A)$ are of the same cardinality, and let f be a bijective mapping from A to $P(A)$. Consider the subset $S \subset A$ such that

$$S = \{a \mid a \notin f(a)\}$$

We now prove that S does not belong to the range of f, thus that for no element $y \in A$ we have $f(y) = S$. Assume the contrary, let $y \in A$ such that $f(y) = S$. Either $y \in S$ or $y \notin S$. If $y \in S$, then $y \notin f(y)$ but $f(y) = S$, so $y \notin S$ and we have a contradiction. If, on the contrary, $y \notin S$, then $y \in f(y)$ but $f(y) = S$, so $y \in S$ and we have again a contradiction. Thus there is no y such that $f(y) = S$.

Thus we have found an element of $P(A)$ which is not reached by the function f which we had supposed to be bijective, thus $P(A)$ must be of a larger cardinality than A. □

The above theorem proves that whether A is of a finite or infinite cardinality, $P(A)$ will always be of a larger cardinality. This also proves the existence of an infinite number of infinities, since we can use the function P as often as we like. We can even compute how many elements are in $P(A)$, given that we know how many elements there are in A.

The Relationship in Size Between $\mathbb{R}$ and $\mathbb{N}$

The first thing we would like to do is, given the number of elements in some set A, to make a statement concerning the exact number of elements in $P(A)$. To do this, we first need a definition.

Definition 2. *Let A, B be two sets. Then*

$$B^A = \{f \mid f : A \to B \text{ is a map}\}$$

We can now examine the set $\{0,1\}^A$, and come to the conclusion that it has precisely the same number of elements as $P(A)$.

Theorem 5 ([2])**.** $\{0,1\}^A$ *is of the same cardinality as* $P(A)$.

Proof. What we need to do is to construct a bijection from $P(A)$ to the set of all maps from A to $\{0,1\}$.

Let S be a subset of A, and let f_S be defined as

$$f_S : A \to \{0,1\}$$

$$f_S(x) = \begin{cases} 1 \text{ if } x \in S \\ 0 \text{ if } x \in A \setminus S \end{cases}$$

This gives us the bijection we seek, thus $|P(A)| = |\{0,1\}^A|$. This is usually noted as

$$|P(A)| = 2^{|A|}$$ □

Remark: The number of elements in $P(A)$, given that $|A| = n$, can be computed by using Newton's Binomium. For if the number of ways of choosing k elements from n is $\binom{n}{k}$, then $\sum_{k=1}^{n} \binom{n}{k} = \sum_{k=1}^{n} \binom{n}{k} 1^k 1^{n-k} = (1+1)^n = 2^n$ gives us the number of elements in $P(A)$.

We already know that $c > \aleph_0$, however, using the above we can now come to a more precise conclusion than this.

Theorem 6 ([2])**.** $|P(\mathbb{N})| = c$.

Proof. We know $|P(A)| = 2^{|A|}$, thus $|P(\mathbb{N})| = 2^{|\mathbb{N}|}$. An element from $\{0,1\}^{\mathbb{N}}$ is an infinite string of 0's and 1's, such as 010101 . . .. We now associate with each such a string a decimal in base 2 representation.

$$\begin{aligned} 010011010101\ldots &\leftrightarrow 0.010011010101\ldots \\ 010101010101\ldots &\leftrightarrow 0.010101010101\ldots \\ 111111111111\ldots &\leftrightarrow 0.111111111111\ldots \end{aligned}$$

In such a manner we have created a bijection between the elements of $\{0,1\}^{\mathbb{N}}$ and the real numbers between 0 and 1 (given in binary representation). This last set has cardinality c, as we saw earlier, thus $|\{0,1\}^{\mathbb{N}}| = c$, and also $|P(\mathbb{N})| = c$. We have thus shown that

$$2^{\aleph_0} = c$$ □

We have thus shown that $2^{|\mathbb{N}|} = |\mathbb{R}|$, and $\mathbb{R}$ is of the same cardinality as the set of all subsets of $\mathbb{N}$.

Remark: The next cardinal after $\aleph_0$ is denoted by $\aleph_1$. If q is a cardinal such that $\aleph_0 \leq q \leq \aleph_1$, then either $q = \aleph_0$ or $q = \aleph_1$. The assumption that $c = \aleph_1$ is called the Continuum Hypothesis.

3.4 Application to Leading Examples

1. We have already seen that $\mathbb{Q}$, $\mathbb{Z}$ and $\mathbb{A}$ are all countable. $\mathbb{R}\backslash\mathbb{Z}$ is uncountable, as it is simply a countable union of intervals. $\mathbb{R}\setminus\mathbb{Q}$ and $\mathbb{R}\setminus\mathbb{A}$ must also be uncountable, for if they were not $\mathbb{R}=\mathbb{Q}\cup(\mathbb{R}\backslash\mathbb{Q})=\mathbb{A}\cup(\mathbb{R}\backslash\mathbb{A})$ would be a union of two countable sets, which yields a countable set, as we saw in section 3.1 on page 21. This is impossible, as $\mathbb{R}$ is uncountable.

2. The Middle Third Cantor set, as we have seen, consists of those real numbers which use only 0 and 2 in their ternary representations, such as

$$0.022002020\ldots$$
$$0.222222222\ldots$$
$$0.020202020\ldots$$

Divide all these numbers by 2 to obtain:

$$0.011001010\ldots$$
$$0.111111111\ldots$$
$$0.010101010\ldots$$

We obtain the set of all real numbers between 0 and 1 expressed in binary form, which is an uncountable set. The Middle Third Cantor set is therefore of the same cardinality as the interval [0,1]:

$$|\mathbb{C}_{\frac{1}{3}}| = c$$

In a similar fashion, we can show all $\mathbb{C}_\alpha$ Cantor sets to be of cardinality c.

3. The set $\mathbb{L}$ of Liouville numbers cannot be of greater cardinality than c, since it is contained in $\mathbb{R}$. Thus if we can construct an injective function from (0,1) to $\mathbb{L}$, we will have shown that $\mathbb{L}$ is of a cardinality at least as large as $\mathbb{R}$. Then we will have proved that $\mathbb{L}$ is of the same cardinality as $\mathbb{R}$.

Recall the construction of an infinite number of Liouville numbers described on page 15. Let $f:(0,1)\to\mathbb{L}$ be the injective function which maps a real number $a=0.a_1a_2a_3a_4\ldots$ to the number $0.a_10a_200a_3000000a_40\ldots$. This is an injective function, thus

$$|\mathbb{L}| = c$$

To summarize the results of this section, we compile a table of cardinalities of the leading examples.

set	cardinality
$\mathbb{Z}, \mathbb{Q}, \mathbb{A}$	$\aleph_0$
$\mathbb{R} \setminus \mathbb{Z}, \mathbb{R} \setminus \mathbb{Q}, \mathbb{R} \setminus \mathbb{A}$	c
$\mathbb{C}_\alpha$	c
$\mathbb{L}$	c

Table 3.1: Summary of the cardinalities of the Leading Examples

Chapter 4

Topology

In this section, some basic knowledge of topology and analysis is assumed, all of which can be found in [10].

The Difference Between Integers and Rationals

Thus far we have not been as productive as we would like. We have shown that when we look at $\mathbb{Z}$ and $\mathbb{Q}$ from the point of view of cardinality, that they are exactly the same, that is, they contain exactly the same number of points. We are not satisfied by this result. When we look at $\mathbb{R}$, we see a difference – between every two integers there are an infinite number of rationals. This must be a difference in distribution at the very least. The question then of course arises whether we can describe this mathematically.

The difference lies in the following property. Observe the set of intervals in $\mathbb{R}$. Every single interval contains some point from $\mathbb{Q}$. However, this is not true for $\mathbb{Z}$: the interval $(\frac{1}{4}, \frac{1}{2})$ for instance contains no integer.

Definition 3.

1. *A set $A \subset \mathbb{R}$ is said to be* dense in the interval *I if it has a nonempty intersection with every open subinterval of I. (This means that $I \subset cl(A)$, where $cl(A)$ denotes the closure of A.) A is* dense *if it is dense in the line* $\mathbb{R}$.

2. *A set $A \subset \mathbb{R}$ is* nowhere dense *if it is dense in no interval of* $\mathbb{R}$. *(We can also say a set is nowhere dense if every interval has a subinterval contained in the complement of A.)*

4.1 Large and Small Sets Seen in a Topological Manner

Can we use 'dense' as a description of largeness? The problem is that it would not be very meaningful: although $\mathbb{Q}$ is countable and $\mathbb{R} \setminus \mathbb{Q}$ is not, they are both dense in $\mathbb{R}$. Is there some further distinction we can make? In 1899 René Baire formulated the following definitions:

Definition 4. *A subset A of $\mathbb{R}$*[1] *is said to be* meagre *if it can be represented as a countable union of nowhere dense sets: $A = \bigcup_{n=1}^{\infty} A_n$ where A_i is nowhere dense for each i. A set B that is the complement of a meagre set is said to be* residual*: $B = \mathbb{R} \setminus A$ for A meagre.*[2]*.*

The difference between meagre and residual sets will become more clear when we look at examples in the next section. Is there any evidence to offer now to see that we should confine meagre sets to the realm of 'small' sets? Firstly, we can show that the complement of any meagre set can never be very small: the complement of any meagre set on the line is dense. This can be reformulated as in theorem 7.

Theorem 7 (Baire [1])**.** *Any residual subset of $\mathbb{R}$ is dense.*

Proof. Let $A = \bigcup_{n=0}^{\infty} A_n$ be a meagre set, with A_n nowhere dense. Then $\mathbb{R} \setminus A$ is residual. For any interval I_0, we can find a closed subinterval I_1 of $I_0 \setminus A_1$. Let I_2 be a closed subinterval of $I_1 \setminus A_2$. We can continue choosing closed subintervals in this manner to obtain a nested sequence $\{I_n\}$ of closed intervals. Because the I_n are closed, $\bigcap_{n=0}^{\infty} I_n$ is nonempty. This follows from the fact that $\mathbb{R}$ is complete, for proof see page 186 of [10]. Thus every interval I contains a point from the complement of A and $\mathbb{R} \setminus A$ is dense, as desired.

All we need to do now is to specify how to choose such intervals so as to avoid using the axiom of choice, which we do not like to use unless absolutely necessary. For more details see appendix E and [7]. The set of intervals with rational endpoints is a countable set, thus we can arrange it into a list L. Now for each choice of I_n we can choose the first element in L which satisfies all the requirements. □

Residual sets are the 'large' sets. The following theorem shows that they are even so large that intersections of them remain large, while 'small' sets are so small that unions of them remain small.

Theorem 8 ([1])**.**

1. *A countable union of meagre sets is meagre.*
2. *In $\mathbb{R}$, the intersection of any countable family of residual sets is residual.*

[1] or any other topological Baire space (see page 41 of [1])

[2] The terms originally used by Baire were 'first category' for meagre and 'second category' for residual.

Proof. We start with the first statement. We have already shown that a countable union of countable sets is countable on page 21. Using precisely the same method, we can also show that a countable union of sets made up of a countable number of nowhere dense sets must also be composed of a countable number of nowhere dense sets, and we are done.

We can now use de Morgan's law to see that the second statement follows from the first. Indeed, let F_i be meagre sets. Then:

$$\begin{aligned} F &= \bigcup_{i=0}^{\infty} F_i \quad \text{is meagre} \\ \bigcap_{i=0}^{\infty} \mathbb{R} \setminus F_i &= \mathbb{R} \setminus \left(\bigcup_{i=0}^{\infty} F_i \right) \\ &= \mathbb{R} \setminus F \end{aligned}$$

Thus a countable intersection of residual sets is itself residual. □

The Difference Between Rationals and Irrationals

Consider $\mathbb{Q}$, the set of rational numbers, which is a dense countable set. It is meagre as it is a countable union of singletons, which are nowhere dense. $\mathbb{R} \setminus \mathbb{Q}$ is also dense: in every interval $I \subset \mathbb{R}$ there is at least one point from $\mathbb{R} \setminus \mathbb{Q}$ as well as from $\mathbb{Q}$. We have also seen that $\mathbb{R} \setminus \mathbb{Q}$ is uncountable. We now have the tools to express a difference in a topological manner as well.

We can easily see that $\mathbb{R} \setminus \mathbb{Q}$ is residual. Let $\mathbb{R} \setminus \mathbb{Q} = \bigcap_{q \in \mathbb{Q}} \mathbb{R} \setminus \{q\}$, and then use theorem 8 to see that $\mathbb{R} \setminus \mathbb{Q}$ must be residual. Of course, this means that $\mathbb{Q}$ is meagre.

As a final demonstration of the smallness of even dense meagre sets, observe the sets $\mathbb{Q}$ and $\pi + \mathbb{Q}$ (the set obtained by translating $\mathbb{Q}$ over π), two dense meagre sets. All elements in $\mathbb{Q}$ are rational, all those in $\pi + \mathbb{Q}$ are irrational, thus $\mathbb{Q} \cap (\pi + \mathbb{Q}) = \emptyset$, and we can see that 'dense' is not nearly as good a description of topological largeness as 'residual' is.

We can immediately identify many residual sets in $\mathbb{R}$: the dense open sets. Assume that B is dense and open. B dense implies that every interval contains at least one point in B. B open implies that some open neighbourhood of this point is also contained in B. Thus every interval contains some subinterval in B, which implies that $\mathbb{R} \setminus B$ is nowhere dense. Thus B is residual.

We shall now move on to our leading examples.

4.2 Leading Examples Revisited

1. We can see immediately that all countable sets are meagre. Let $A = \bigcup_{n=1}^{\infty}\{a_n\}$ be any countable set, then we can see that it is meagre because all singletons are nowhere dense.

2. Cantor sets are nowhere dense. We shall prove this only for the Cantor Middle Third set $\mathbb{C}_{\frac{1}{3}}$. What we need to show is that every interval I has a subinterval contained in the complement of $\mathbb{C}_{\frac{1}{3}}$. It is sufficient to assume I is closed, for if $[a, b]$ contains a subinterval in $\mathbb{R} \setminus \mathbb{C}_{\frac{1}{3}}$, then so does (a, b).

   Take any interval $[a, b] \subset [0, 1]$, and recall that $\mathbb{C}_{\frac{1}{3}} = \bigcap_{n=i}^{\infty} C_i$. Choose $k, i \in \mathbb{N}$ such that $\frac{k}{3^i} \in (a, b)$. This point can be one of two things:

   (a) It could be the endpoint of some interval in C_i, in which case either $\left[\frac{k-1}{3^i}, \frac{k}{3^i}\right]$ or $\left[\frac{k}{3^i}, \frac{k+1}{3^i}\right]$ is in C_i, but not both. Thus either $\left(a, \frac{k}{3^i}\right)$ or $\left(\frac{k}{3^i}, b\right)$ is a subinterval of $[a, b]$ in $\mathbb{R} \setminus C_i \subset \mathbb{R} \setminus \mathbb{C}_{\frac{1}{3}}$.

   (b) If $\frac{k}{3^i}$ is not the endpoint of some interval in C_i, then it was removed during the construction of some C_j, $j < i$. In this case, because $\mathbb{R} \setminus C_i$ is an open set, there is an $\epsilon > 0$ such that $\left(\frac{k}{3^i} - \varepsilon, \frac{k}{3^i} + \varepsilon\right) \subset \mathbb{R} \setminus C_i \subset \mathbb{R} \setminus \mathbb{C}_{\frac{1}{3}}$

   Thus $\mathbb{C}_{\frac{1}{3}}$ is nowhere dense, which means that it is also meagre.

3. We shall now show that the set $\mathbb{L}$ of Liouville numbers is residual. By definition

$$\begin{aligned} \mathbb{L} &= (\mathbb{R} \setminus \mathbb{Q}) \cap \bigcap_{n=1}^{\infty} G_n \\ \text{where} \quad G_n &= \bigcup_{q=2}^{\infty} \bigcup_{p=-\infty}^{\infty} (\frac{p}{q} - \frac{1}{q^n}, \frac{p}{q} + \frac{1}{q^n}) \end{aligned}$$

   G_n thus is a union of open intervals. Also, G_n contains all of $\mathbb{Q}$, thus G_n is dense as well as open, and must be a residual set. We now observe the complement of $\mathbb{L}$:

$$\mathbb{R} \setminus \mathbb{L} = \mathbb{Q} \cup \bigcup_{n=1}^{\infty} (\mathbb{R} \setminus G_n)$$

   Because G_n is residual, $\mathbb{R} \setminus G_n$ must be meagre. A countable union of meagre sets is meagre, thus $\mathbb{R} \setminus \mathbb{L}$ is meagre. This means however that $\mathbb{L}$ is a residual set.

To summarize the results from this section, we once again compile a table, this time of the topological size of the leading examples.

set	topological size
$\mathbb{Z}, \mathbb{Q}, \mathbb{A}$	meagre
$\mathbb{R} \setminus \mathbb{Z}, \mathbb{R} \setminus \mathbb{Q}, \mathbb{R} \setminus \mathbb{A}$	residual
$\mathbb{C}_\alpha$	meagre
$\mathbb{L}$	residual

Table 4.1: Summary of the topological sizes of the Leading Examples

For an illustration involving the concept of 'meagre', see appendix C.

Chapter 5

Lebesgue Measure

We shall now move on to a completely different approach. We have considered counting and topology as valid ways of looking at the sizes of sets, let us now consider measuring. We assume some basic knowledge, all of which can be found in [13].

5.1 The Definition of a Measure

We can use length as a valid way of comparing the sizes of different intervals: surely the one with the largest length should be considered the larger. This notion of length we now want to extend to a larger class of subsets of $\mathbb{R}$, namely the so-called class of (Lebesgue) measurable sets.

We want all open sets and closed sets to be measurable, thus the class of measurable sets should be closed with respect to taking complements and countable unions. In appendix G we explain how to obtain from this the σ-algebra of (Lebesgue) measurable sets.

We now want to find a function, a *measure*, which assigns a 'length' to every set in the σ-algebra of measurable sets. To do so, we start by defining exactly what a measure is.

Definition 5 ([11])**.** *Let S be the σ-algebra of measurable sets in $\mathbb{R}$. A* measure *on S is a map $m : S \to [0, \infty]$ such that*

1. $m(\emptyset) = 0$
2. *m is countably additive, i.e. if $A_1, A_2, A_3, \ldots \in S$ then $m(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} m(A_i)$*

We shall be using a very specific measure function, namely the Lebesgue measure, which we shall now define.

Definition 6 (Lebesgue measure [1])**.** *Let S be the σ-algebra of measurable sets in $\mathbb{R}$. Let I_i be a open, half-open or closed interval in $\mathbb{R}$ for each i. Then the* Lebesgue measure λ *is defined as*

$$\lambda : S \to [0, \infty]$$

$$\lambda(A) \quad = \quad \inf \Big\{ \sum_{i=1}^{\infty} |I_i| \mid A \subset \bigcup_{i=1}^{\infty} I_i \Big\}$$

In effect, we cover the set A with intervals having a certain total length. We then keep taking smaller such covers until we can go no smaller. Because the smallest interval we can cover some interval I with is I itself, the Lebesgue measure gives us $\lambda(I) = |I|$.

Large and Small Sets in a Measure-Theoretical Manner

Just like in the previous chapter, we would like to define now which sets constitute the 'large' sets, and which the 'small'. We start with the small sets.

Definition 7. *Sets of measure* 0 *in* $\mathbb{R}$ *are called* nullsets*.*

This is very intuitive, and also the large sets are unsurprising.

Definition 8. *Consider the interval I. Any set $A \subset I$ such that $\lambda(A) = |I|$ is said to be* of full measure in I*. A set is of* full measure *if it is of full measure in* $\mathbb{R}$*.*

One of the advantages of the measure approach is the fact that there are an infinite number of gradations between nullsets and full measure sets. This we did not have when looking at sets in a topological manner.

Remark: We are now in a position to say exactly which sets constitute the σ-algebra of Lebesgue measurable sets. This is exactly the σ-algebra generated by the nullsets together with the open sets. See appendix G for more details.

Some Properties of the Lebesgue measure

We shall now look explicitly at two properties of the Lebesgue measure, which shall be useful later on.

1. If $A \subseteq B$, then $\lambda(A) \leq \lambda(B)$. This is a natural property, because any sequence $\{I_n\}$ of intervals that covers B must by necessity also cover A, thus the measure of A cannot be larger than that of B.

2. Lebesgue measure has the property of *countable subadditivity*. This means that if $A = \bigcup_{i=0}^{\infty} A_i$, then $\lambda(A) \leq \sum_{i=0}^{\infty} \lambda(A_i)$. Every A_i has a value assigned to it by Lebesgue measure, following the definition we get

$$\lambda(A_i) = \inf\Big\{ \sum_{j=0}^{\infty} I_i^j \mid A_i \subset \bigcup_{j=0}^{\infty} I_i^j \Big\}$$

where the I_i^j may be open, half-open or closed intervals. This statement implies the following, which says we can get very close to this infimum with one particular set of intervals $\{I_i^j\}$:

for any $\varepsilon > 0$, there is a sequence $\{I_i^j\}$ that covers A_i such that $\sum_{j=0}^{\infty} |I_i^j| \leq \lambda(A_i) + \frac{\varepsilon}{2^i}$.

Since $A = \bigcup_{i=0}^{\infty} A_i$, we now see that $A \subset \bigcup_{i,j=0}^{\infty} I_i^j$ and thus

$$\begin{aligned}
\lambda(A) &\leq \lambda\Big(\bigcup_{i,j=0}^{\infty} I_i^j \Big) \\
&\leq \sum_{i,j=0}^{\infty} |I_{ij}| \\
&\leq \sum_{i=0}^{\infty} \Big(\lambda(A_i) + \frac{\varepsilon}{2^i} \Big) \\
&= \varepsilon + \sum_{i=0}^{\infty} \lambda(A_i)
\end{aligned}$$

We can now let $\varepsilon \to 0$ to obtain:

$$\lambda(A) \leq \sum_{i=1}^{\infty} \lambda(A_i)$$

which is what we wanted to get.

Theorem 8 in the previous chapter gave some information on the large and small sets topologically speaking. This we can reformulate for large and small sets measure-theoretically speaking.

Theorem 9 ([1])**.**

1. *The union of any countable set of nullsets is a nullset.*
2. *The intersection of any countable set of full measure sets is of full measure.*

Proof. The second statement follows from the first by using de Morgan's laws, as in theorem 8.

To prove the first statement, let $A = \bigcup_i A_i$, where A_i is a nullset for each i. The property of countable subadditivity gives us that $\lambda(A) \leq \sum_i \lambda(A_i)$.
For every i, $\lambda(A_i) = 0$, thus $\lambda(A) \leq 0$. The Lebesgue measure can never be less than 0, thus $\lambda(A) = 0$, and A is a nullset. □

5.2 Leading Examples Revisited

1. First of all, all countable sets are nullsets. Indeed, let $A = \{a_0, a_1, a_2, a_3, \ldots\}$ be a countable set. Around every point a_i we now place the interval $(a_i - \frac{\varepsilon}{2^i}, a_i + \frac{\varepsilon}{2^i})$, with $\varepsilon > 0$. We now observe the union of all these intervals:

$$\lambda(A) \leq \lambda\Big(\bigcup_{i=0}^{\infty}\Big(a_i - \frac{\varepsilon}{2^i}, a_i + \frac{\varepsilon}{2^i}\Big)\Big) \leq \sum_{i=0}^{\infty} \frac{2\varepsilon}{2^i} = 2\varepsilon$$

   We can now let $\varepsilon \downarrow 0$ to obtain the following:

$$\lambda(A) \leq 2\varepsilon \to 0$$

   Thus, because measure is a positive function, $\lambda(A) = 0$ for any countable set A. Hence $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{A}$ are all nullsets.

2. $\mathbb{R} \setminus \mathbb{Q}$, $\mathbb{R} \setminus \mathbb{Z}$ and $\mathbb{R} \setminus \mathbb{A}$ all have full measure. It is actually true that for any nullset A, the complement $\mathbb{R} \setminus A$ is a full measure set. To prove this we first notice that $\lambda(\mathbb{R} \setminus A) \leq \lambda(\mathbb{R})$, else it could not be contained in $\mathbb{R}$. We need now only show the other inequality.

   First, we note the fact that $\mathbb{R} = A \cup (\mathbb{R} \setminus A)$. Using the property of countable subadditivity, we conclude $\lambda(\mathbb{R}) \leq \lambda(A) + \lambda(\mathbb{R} \setminus A)$. However, we have $\lambda(A) = 0$, thus $\lambda(\mathbb{R}) \leq \lambda(\mathbb{R} \setminus A)$, and we have our second inequality. We can therefore conclude that $\lambda(\mathbb{R}) = \lambda(\mathbb{R} \setminus A)$ for any nullset A.

3. The $\mathbb{C}_\alpha$ Cantor sets are nullsets, which is (surprisingly perhaps) easily seen. We know that $\mathbb{C}_\alpha = \bigcap_{i=0}^{\infty} C_i$ where C_i consists of 2^i disjoint intervals of length $(\frac{1}{2})^i(1-\alpha)^i$. Thus $\lambda(\mathbb{C}_\alpha) \leq 2^i(\frac{1}{2})^i(1-\alpha)^i = (1-\alpha)^i$, and by letting $i \to \infty$ we find $\lambda(\mathbb{C}_\alpha) \leq 0$. This means that $\mathbb{C}_\alpha$ is a nullset for any $0 < \alpha \leq 1$.

4. We once again consider $\mathbb{L}$. It will take us a bit more work, but what we will prove is that $\mathbb{L} \cap (-m, m)$ is a nullset for every integer m. We then use the fact that

$\mathbb{L} = \bigcup_{m\in\mathbb{Z}} \mathbb{L} \cap (-m, m)$ and theorem 9 to show that $\mathbb{L}$ is a nullset.

To show that $\mathbb{L}\cap(-m, m)$ is a nullset, we first remark that according to the definition, $\mathbb{L}$ is the intersection of the irrational numbers and an intersection of sets G_n such that

$$G_n = \bigcup_{q=2}^{\infty} \bigcup_{p=-\infty}^{\infty} \left(\frac{p}{q} - \frac{1}{q^n}, \frac{p}{q} + \frac{1}{q^n} \right)$$

In symbols, this gives us

$$\mathbb{L} = (\mathbb{R} \setminus \mathbb{Q}) \cap \bigcap_{n=1}^{\infty} G_n$$

Thus for any n, $\mathbb{L} \subset G_n$. We now go on to analyse what kind of points are in $\mathbb{L} \cap (-m, m)$. To do this, we want to split each G_n up into smaller, more manageable sets. We thus introduce

$$G_{nq} = \bigcup_{p=-\infty}^{\infty} \left(\frac{p}{q} - \frac{1}{q^n}, \frac{p}{q} + \frac{1}{q^n} \right)$$

and of course we have

$$G_n = \bigcup_{q=2}^{\infty} G_{nq}$$

Thus we now see

$$\begin{aligned} \mathbb{L} \cap (-m, m) &\subseteq G_n \cap (-m, m) \\ &= \bigcup_{q=2}^{\infty} [G_{nq} \cap (-m, m)] \\ &\subseteq \bigcup_{q=2}^{\infty} \bigcup_{p=-mq}^{mq} \left(\frac{p}{q} - \frac{1}{q^n}, \frac{p}{q} + \frac{1}{q^n} \right) \end{aligned}$$

This shows us that $\mathbb{L}\cap(-m, m)$ can be covered by the countable collection of intervals shown above. We now want to find an upper bound for the length of the union of these intervals.

$$\begin{aligned}
\left|\bigcup_{q=2}^{\infty}\bigcup_{p=-mq}^{mq}\left(\frac{p}{q}-\frac{1}{q^n},\frac{p}{q}+\frac{1}{q^n}\right)\right| &\leq \sum_{q=2}^{\infty}\sum_{p=-mq}^{mq}\frac{2}{q^n} \\
&= \sum_{q=2}^{\infty}(2mq+1)(\frac{2}{q^n}) \\
&= \sum_{q=2}^{\infty}(4mq+2)(\frac{1}{q^n}) \\
&\leq \sum_{q=2}^{\infty}(4mq+q)(\frac{1}{q^n}) \\
&= (4m+1)\sum_{q=2}^{\infty}\frac{1}{q^{n-1}}
\end{aligned}$$

Recall that $\mathbb{L} = (\mathbb{R}\setminus\mathbb{Q})\cap\bigcap_{n=1}^{\infty}G_n$, where $\{G_n\}$ is a nested sequence. Thus we might just as well say $\mathbb{L} = (\mathbb{R}\setminus\mathbb{Q})\cap\bigcap_{n=3}^{\infty}G_n$, and use $n \geq 3$ instead of $n \geq 1$. We need this to find an upper bound for $\sum_{q=2}^{\infty}\frac{1}{q^{n-1}}$, for the function $f(q) = \frac{1}{q^{n-1}}$ and its integral become monotonically decreasing from $n = 3$ onwards. Observe the graph for $f(q) = \frac{1}{q^{n-1}}, \quad n \geq 3$:

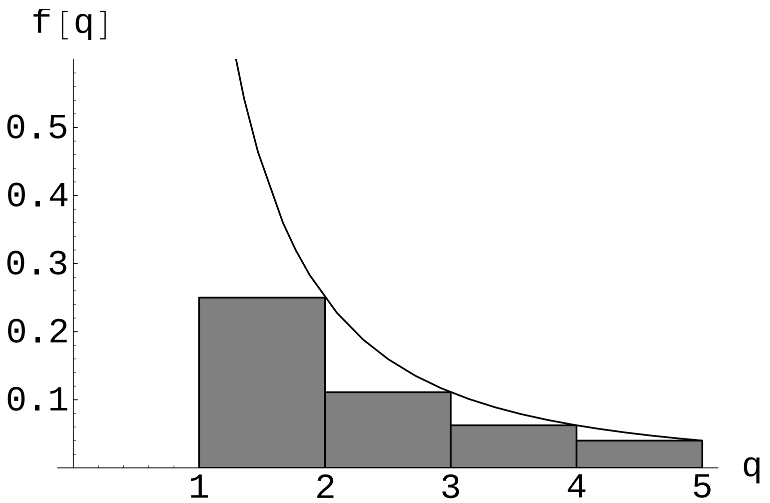


Figure 5.1: One of the functions $f(q) = \frac{1}{q^{n-1}}$

The shaded boxes represent the sum $\sum_{q=2}^{\infty}\frac{1}{q^{n-1}}$. The area under the curve is of course larger thus

$$\begin{aligned}(4m+1)\sum_{q=2}^{\infty}\frac{1}{q^{n-1}} &\leq (4m+1)\int_1^{\infty}\frac{\mathrm{d}q}{q^{n-1}} \\ &= (4m+1)\left[\frac{1}{2-n}q^{2-n}\right]_1^{\infty} \\ &= \frac{4m+1}{2-n}(0-1) = \frac{4m+1}{n-2}\end{aligned}$$

We now see that $\lambda(\mathbb{L}\cap(-m,m)) \leq \frac{4m+1}{n-2}$, and send $n \to \infty$ to obtain

$$\lambda\left(\mathbb{L}\cap(-m,m)\right) \leq 0$$

Thus $\mathbb{L}\cap(-m,m)$ is a nullset for every integer m. $\mathbb{L}$ must then be a nullset in $\mathbb{R}$.

To summarize the results from this chapter, we once again compile a table.

set	measure-theoretical size
$\mathbb{Z}, \mathbb{Q}, \mathbb{A}$	nullset
$\mathbb{R}\setminus\mathbb{Z}, \mathbb{R}\setminus\mathbb{Q}, \mathbb{R}\setminus\mathbb{A}$	full measure set
$\mathbb{C}_\alpha$	nullset
$\mathbb{L}$	nullset

Table 5.1: Summary of the measure-theoretical sizes of the Leading Examples

Chapter 6

Some Crazy Results

We have seen several interesting properties and examples, but as yet no ‘craziness’. Let us take the time in this chapter to look at some more ‘crazy’ results.

We first remark that none of the definitions of ‘largeness’ given up till now are equivalent. This is illustrated clearly if we assemble all results concerning the leading examples into one table. We now see that whether a set is large or small according to one definition says nothing about what size it is according to another definition.

set	cardinality	topological size	measure-theoretical size
$\mathbb{Z}, \mathbb{Q}, \mathbb{A}$	$\aleph_0$	meagre	nullset
$\mathbb{R} \setminus \mathbb{Z}, \mathbb{R} \setminus \mathbb{Q}, \mathbb{R} \setminus \mathbb{A}$	c	residual	full measure set
$\mathbb{C}_\alpha$	c	meagre	nullset
$\mathbb{L}$	c	residual	nullset

Table 6.1: Sizes of the Leading Examples given several definitions

6.1 Neighbourhoods of $\mathbb{Q}$

One might be tempted to think that, because $\mathbb{Q}$ is dense in $\mathbb{R}$, the only possible neighbourhood of $\mathbb{Q}$ is $\mathbb{R}$ itself. However, we shall see that this is not so. Let $\mathbb{Q} = \{q_1, q_2, q_3, \ldots\}$ be a list of the rational numbers, for example obtained by Cantor’s first diagonal procedure. Let $I_{ij} = (q_i - \frac{1}{2^{1+i+j}}, q_i + \frac{1}{2^{1+i+j}})$, and take

$$G_j = \bigcup_{i=1}^{\infty} I_{ij}$$

For every j, G_j is an open neighbourhood of $\mathbb{Q}$. It is hence a dense and open set, which we have proved to be residual on page 31. Let us now investigate the Lebesgue measure of G_j.

$$\begin{aligned}\lambda(G_j) &\leq \sum_{i=1}^{\infty} |I_{ij}| \\ &= \sum_{i=1}^{\infty} \frac{1}{2^{i+j}} \\ &= \frac{1}{2^j}\end{aligned}$$

Thus we have found that $\mathbb{Q}$ *has neighbourhoods of arbitrary small measure in* $\mathbb{R}$*. Hence* $\mathbb{R} \setminus \mathbb{Q}$ *contains closed sets of arbitrarily large measure.*

6.2 A Strange Decomposition of $\mathbb{R}$

Consider the set $\mathbb{L}$ of Liouville numbers. We have seen that it is a residual nullset, thus it is topologically large but measure-theoretically small.

There is a special set to be found in the complement of $\mathbb{L}$ called the set of Diophantine numbers, which we denote by $\mathbb{D}$. Now

$$\mathbb{D} = \mathbb{R} \setminus (\mathbb{L} \cup \mathbb{Q})$$

What is the size of $\mathbb{D}$? Firstly, it must be of full measure because both $\mathbb{L}$ and $\mathbb{Q}$ are nullsets. Secondly, it is meagre, for it is in the complement of a residual set.

We have now discovered a rather surprising fact. Observe that $\mathbb{D}, \mathbb{L}$ and $\mathbb{Q}$ are all disjoint. Together they make up $\mathbb{R}$:

$$\mathbb{R} = \mathbb{D} \cup \mathbb{L} \cup \mathbb{Q}$$

Thus $\mathbb{R}$*, itself a huge set in both senses of the word, can be decomposed into three extremely small sets: a meagre set, a nullset and a countable set (both meagre and measure* 0*).*

Chapter 7

Duality

The previous chapter showed us once and for all that if a set is large or small in the topological sense, this will tell us nothing about its size measure-theoretically speaking (and vice versa). It would seem from this that there is no relationship at all between topology and measure theory. This, however, turns out to be a false view of how things are, as we discover when we study Erdös's Duality theorem.

7.1 Erdös's theorem

What we may have noticed is that whatever is true for meagre sets is also true for nullsets, that is to say whenever we have proved a theorem about meagre sets, the same theorem has proved to be true for nullsets. This is the essence of Erdös's Duality theorem, which can be stated as follows:

Theorem 10 (Erdös [1]). *Assuming the Continuum Hypothesis*[1]*, there exists a one-to-one mapping f of the line onto itself such that $f = f^{-1}$ and such that $f(E)$ is a nullset if and only if E is meagre (It follows that $f(E)$ is meagre if and only if E is a nullset).*

The importance of this theorem is that it implies the following duality principle, which is a metatheorem.

Principle 1 (Duality Principle [1]). *Let P be any proposition involving solely the notions of measure zero, meagreness and notions of pure set theory. Let $P*$ be the proposition obtained from P by interchanging the terms 'nullset' and 'meagre set' wherever they appear. Then each of the propositions P and $P*$ implies the other, assuming the Continuum Hypothesis.*

[1]See appendix E for details on the Continuum Hypothesis.

7.2 Supporting Evidence for the Duality Principle

To make the idea of the Duality theorem more palatable, we shall now offer some evidence in its defence, before proving the theorem itself in the next section. The following dual propositions illustrate the working of the Duality theorem.

Proposition 1. *The union of any countable family of meagre sets is meagre. The union of any countable family of nullsets is a nullset.*

Proof. The first statement is proved in theorem 8. The second in theorem 9. □

Thus the union of any countable family of small sets is small, and we have shown our first dual proposition.

Proposition 2. *The complement of any meagre set on the line is dense. The complement of any nullset on the line is dense.*

Proof. The first statement is proved in theorem 7.

To prove the second statement, let N be a nullset, and consider its complement $A = \mathbb{R} \setminus N$. Because N is a nullset, A must be of full measure. Assume A is not dense in some interval I. Then $\lambda(A \cap I) = |I|$ but for any interval in I, there is some subinterval contained in the complement of A. Consider the interval I itself, there is an interval $J \subset I$ with $J \subset \mathbb{R} \setminus A$. However, this means that $A \cap I = A \cap (I \setminus J)$, thus $\lambda(A \cap I) = \lambda(A \cap (I \setminus J)) \leq |I \setminus J| < |I|$. This gives a contradiction, since we had assumed that $A \cap I$ was of full measure in I. Thus A must be dense in every interval, and hence in $\mathbb{R}$ itself. □

We have now shown that the complement of any small set is dense. We refer to appendix F for the real hard work – proving Erdös's theorem.

Chapter 8

Extended Duality

We can now ask ourselves the obvious question – can the idea of Erdös's duality theorem be extended to include other properties besides those of measure zero and meagreness? A Polish mathematician called Edward Marczewski (originally named Szpilrajn) was the first to consider the property of measurability as a possible candidate for such an extended principle of duality. See appendix G for details on measurability.

8.1 A Topological equivalent of Measurability

The class of measurable sets is the σ-algebra generated by the nullsets together with the open sets. To find a topological equivalent of the property of measurability, perhaps it is a good idea to have a look at the class of sets we obtain when generating a σ-algebra using the open sets and the small sets topologically speaking, i.e. meagre sets. What we obtain when we do this is a class of sets which all have the property of Baire, which is defined as follows:

Definition 9. *A set A has the property of Baire if and only if it can be represented in the form $A = F \triangle Q$, where F is closed and Q is meagre.* $[A \triangle B = (A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)]$

The first way in which we can show this to be a plausible equivalent is if we consider the Bernstein set B, which is a set which contains a point from every uncountable closed subset of $\mathbb{R}$. See appendix G for a description of its construction and a proof of the fact that it is unmeasurable. We can prove that B also lacks the property of Baire. Let B have the property of Baire, then $B = F \triangle Q$ where F is closed and Q is meagre. Recall that $B = F \triangle Q = (F \setminus Q) \cup (Q \setminus F)$. Then $F \setminus Q$ is countable, since every uncountable closed set contains a point from $\mathbb{R} \setminus B$. Thus B is the union of a countable set and a meagre set. The same reasoning applies to $\mathbb{R} \setminus B$, it must also be the union of a countable set and a meagre set. This would mean that $\mathbb{R} = B \cup \mathbb{R} \setminus B$ is the union of four meagre sets, which is meagre. This is impossible, we have shown before that $\mathbb{R}$ is residual. Thus B and

$\mathbb{R} \setminus B$ cannot both have the property of Baire. Both are Bernstein sets, so out of a pair of Bernstein sets at least one lacks the property of Baire.

This is not very hard evidence, so let us now go on to consider some propositions which seem to support the idea of extended duality.

8.2 Supporting Evidence for the Extended Principle of Duality

Proposition 3. *Any set with positive Lebesgue measure has a non-measurable subset, assuming the axiom of choice. Any residual set has a subset that lacks the property of Baire, assuming the axiom of choice.*

Proof. Let $B, \mathbb{R} \setminus B$ be a pair of Bernstein sets. $A \cap B$ and $A \cap \mathbb{R} \setminus B$ cannot both be measurable, which we can prove in the same way as we proved in appendix G that not both B and $\mathbb{R} \setminus B$ can be measurable. In the same manner, $A \cap B$ and $A \cap \mathbb{R} \setminus B$ cannot both have the property of Baire. □

Thus any sufficiently large set has a ‘difficult’ subset, one to which we can assign no size.

Proposition 4. *If A is measurable, then $\mathbb{R} \setminus A$ is measurable. If A has the property of Baire, then $\mathbb{R} \setminus A$ has the property of Baire.*

Proof. To prove the first statement, let $F \subset A \subset G$, F closed, G open. Then

$$\mathbb{R} \setminus F \supset \mathbb{R} \setminus A \supset \mathbb{R} \setminus G$$

$\mathbb{R} \setminus G$ open, $\mathbb{R} \setminus F$ closed, and $\lambda(G \setminus F) = \lambda((\mathbb{R} \setminus F) \setminus (\mathbb{R} \setminus G)) < \varepsilon$.

To prove the second statement, let $A = G \triangle P$ where G is open and P is meagre. Then $\mathbb{R} \setminus A = \mathbb{R} \setminus (G \triangle P)$. Consider the interior of $\mathbb{R} \setminus G$, $\text{int}(\mathbb{R} \setminus G)$. Then $N = (\mathbb{R} \setminus G) \setminus \text{int}(\mathbb{R} \setminus G)$ is nowhere dense, and $Q = N \triangle P$ is meagre. Thus $\mathbb{R} \setminus A = \text{int}(\mathbb{R} \setminus G) \triangle Q$, int $(\mathbb{R} \setminus G)$ open, Q meagre, so it has the property of Baire. □

This proposition shows that in either definition, if we can assign a ‘size’ to a set A, then we can also do so for the set $\mathbb{R} \setminus A$.

8.3 Impossibility of Extended Duality

Let us proceed by stating another metatheorem, the extended principle of duality:

Principle 2. *(Extended Principle of Duality) Let P be any proposition involving solely the notions of measure zero, measurability, meagreness, property of Baire and notions of pure set theory. Let $P*$ be the proposition obtained from P by interchanging the terms 'nullset' and 'meagre set', and 'measurable set', and 'set with property of Baire' wherever they appear. Then each of the propositions P and $P*$ implies the other, assuming the Continuum Hypothesis.*

Now this principle looks very beautiful. Which is unfortunate, because it is not true. One pair of propositions provide a counterexample:

Proposition 5. *Let $E_{i,j}$ be double sequence of measurable sets such that $E_{i,j} \supset E_{i,j+1}$ for all positive integers i and j, and such that $\bigcap_j E_{i,j}$ is a nullset for each i. Then there exists a sequence of mappings $n_k(i)$ of the set of positive integers into itself such that $\bigcap_k \bigcup_i E_{i,n_k(i)}$ is a nullset.*

Proof. Let $I_k = [-k, k]$. Because $\bigcap_j E_{i,j}$ is a nullset for each i, and $E_{i,j}$ is a nested sequence such that $E_{i,j+1} \subset E_{i,j}$, at some point $E_{i,j}$ must approach being a nullset if j is large enough. Thus for each i and k there is a positive integer $n_k(i)$ such that the measure of $E_{i,n_k(i)}$ becomes very small in the interval I_k:

$$\lambda(E_{i,n_k(i)} \cap I_k) < \frac{1}{k2^i}$$

Hence

$$\lambda\Big(\bigcup_i E_{i,n_k(i)} \cap I_k\Big) < \frac{1}{k}$$

We now let $E = \bigcap_k \bigcup_i E_{i,n_k(i)}$. For any finite interval I we must have $I \subset I_k$ for some sufficiently large k. Thus:

$$(E \cap I) \subset \bigcup_i E_{i,n_k(i)} \cap I_k$$

And so $\lambda(E \cap I) < \frac{1}{k}$ for all sufficiently large k. Thus $E \cap I$ is a nullset for every interval I, which implies that E is a nullset, which is the result we wanted. □

Proposition 6. *Let $E_{i,j}$ be double sequence of sets having the property of Baire such that $E_{i,j} \supset E_{i,j+1}$ for all positive integers i and j, and such that $\bigcap_j E_{i,j}$ is meagre for each i. Then there exists a sequence of mappings $n_k(i)$ of the set of positive integers into itself such that $\bigcap_k \bigcup_i E_{i,n_k(i)}$ is meagre.*

This proposition is false, as demonstrated by the following example. Let $\{r_i\}$ be a list of all the rationals, and let $E_{i,j} = (r_i - \frac{1}{j}, r_i + \frac{1}{j})$. For any surjective mapping n, $\bigcup_i E_{i,n(i)}$ is a dense open set (a residual set). We know from the chapter on topology that the intersection of any number of residual sets is itself residual. Thus for any sequence of mappings

$n_k(i)$, the set $\bigcap_k \bigcup_i E_{i,n_k(i)}$, being an intersection of residual sets, is residual. If we assume extended duality to hold, we would expect it to be meagre.

Thus we must close this chapter with a failure to extend the Principle of Duality in this direction. Hopefully other properties will some day be included in the Principle of Duality, but for now we must leave duality altogether.

Chapter 9

Concluding Remarks

One of the great achievements of mathematics is the gaining of sophisticated and subtle results from seemingly unpromising beginnings. Let us take counting for instance. Nothing could be more mundane or easy. Yet by thinking about what it means to count, Georg Cantor was able to supply us with some wonderful theory about the nature of infinity. We are now able to compare two different infinite sets, and to determine which of them contains the more points. In this theory, we can see the first inkling that the continuum is not a straightforward and easy set, containing, as it does, as many points as the power set of $\mathbb{N}$.

The procedure above yielded some interesting results, so we were tempted to explore further ways of examining the size of subsets of $\mathbb{R}$. The tiniest rudiments of topology and measure theory were enough to be able to classify some of the most interesting sets there are to be found in $\mathbb{R}$, for example $\mathbb{C}_\alpha$ Cantor sets and the set of Liouville numbers.

We also took the time to examine some so-called 'crazy things'. First we were forced to admit that our different definitions of size had nothing to do with each other, and then to concede this point when we examined Erdös's Principle of Duality. It was a great pity that the suggested extension by Marczewski did not hold, let us hope that someday someone will be able to add to the Principle.

Erdös's Principle of Duality simply asks to be extended, more methods of comparing the sizes of sets long to be created, and most probably many properties of the continuum lie undiscovered as yet. I think we have seen evidence enough that the real line $\mathbb{R}$, the continuum, is worthy of any time or energy we may choose to expend in studying it.

Appendix A

Representation of Real Numbers

A real number r is completely determined by describing the set of all numbers smaller than r. Thus each real number r can be described as a set A_r of all other real numbers such that $a \in A_r$ if and only if $a < r$. This result can be refined even further by showing that every real number r can be described as a set A_r of all terminating decimals such that $a \in A_r$ if and only if $a < r$ [2].

Decimal Representation of Real Numbers

Let us now describe a positive real number with a nonterminating decimal representation. A *terminating decimal* is a rational number of which the denominator is of the form 10^n where $n = 0, 1, 2, 3, \ldots$. Some examples are:

rational number	terminating decimal representation
$\frac{123}{1000}$	0.123
$\frac{123}{100000}$	0.00123
$\frac{9999999}{10000000}$	0.9999999
$\frac{3}{1}$	3

Table A.1: Examples of terminating decimals

A *nonterminating decimal* is one which is not of the form described above.

Consider the positive real number r and the set A_r of terminating decimals t such that $t < r$. A_r is bounded above and has no largest element. Let $A_r^{(0)} \subset A_r$ be the subset containing those elements with largest integer part p. This will be the part in our representation of r before the decimal point. Of course, $A_r^{(0)}$ has no maximal element, thus we can now consider the subset $A_r^{(1)} \subset A_r^{(0)}$, which consists of those elements of $A_r^{(0)}$ which have the largest digit a_1 in the first decimal place. Thus far we have

$$p.a_1$$

Once again, we note that $A_r^{(1)}$ has no largest element, and define $A_r^{(2)} \subset A_r^{(1)}$ to be the set of those elements in $A_r^{(1)}$ which have the largest digit a_2 in the second decimal place. We continue this process ad infinitum, and obtain the representation

$$p.a_1a_2a_3 \dots a_n \dots$$

This representation of the real number r will be unique. Also, and more surprisingly, the representation can contain no infinite succession of zeroes. This is because at each stage of the construction, there are terminating decimals smaller than r but greater than the terminating decimal produced thus far.

Consider as an example the real number we usually denote by '5'. The set A_5 will contain many numbers, including

$$A_5 = \{1.11, 2.22, 3.33, 4.44, 4.45, 4.46, \dots\}$$

The set $A_5^{(0)}$ will be:

$$A_5^{(0)} = \{4.1, 4.2, 4.3, 4.4, \dots, 4.9999, 4.99999, \dots\}$$

And so, according to our construction, the integer part of '5' will be 4. We continue:

$$\begin{aligned} A_5^{(1)} &= \{4.90, 4.91, 4.92, \dots, 4.99, 4.999, \dots\} \\ A_5^{(2)} &= \{4.990, 4.991, 4.992, \dots, 4.999, 4.9999, \dots\} \end{aligned}$$

And so we see that in our unique representation the number usually denoted by '5' will now be represented as

$$4.9999999999999\dots$$

Thus all numbers in the set of integers (denoted $\mathbb{Z}$) can be recognized by the fact that in their decimal representations the only digit occurring after the decimal point is 9.

Decimal Representation of Rational Numbers

We know that some numbers, called the set of rational numbers, can be represented as $\frac{a}{b}$, where $a, b \in \mathbb{Z}$. However, it is difficult to compare them with irrationals when they are given in this form, as irrationals are defined by the very fact that they cannot be so represented. We thus write the rationals as nonterminating decimals. Observe the following:

$$\begin{aligned} \frac{1}{3} &= 0.3333\ldots \\ \frac{1}{7} &= 0.142857\,142857\ldots \\ \frac{1}{2} &= 0.49999\ldots \end{aligned}$$

All these rational numbers end with a repeating pattern. We can prove that this is the general case.

Theorem 11. *All real numbers ending in repeating patterns are rationals. All rationals end in a repeating pattern.*

Proof. Assume we have a decimal of the form $p = \ell.b_1 \ldots b_m a_1 \ldots a_n a_1 \ldots a_n \ldots$, where m, n are natural numbers, ℓ is an integer and a_i, b_i are digits. We first wish to isolate the repeating part:

$$\begin{aligned} 10^m p - \ell b_1 \ldots b_m &= \ell b_1 \ldots b_m.a_1 \ldots a_n a_1 \ldots a_n \ldots - \ell b_1 \ldots b_m \\ &= 0.a_1 \ldots a_n a_1 \ldots a_n \ldots \end{aligned}$$

We now observe the following

$$\begin{aligned} 0.a_1 \ldots a_n a_1 \ldots a_n \ldots \times 10^n &= a_1 \ldots a_n.a_1 \ldots a_n a_1 \ldots a_n \ldots \\ &= a_1 \ldots a_n + 0.a_1 \ldots a_n a_1 \ldots a_n \ldots \\ 0.a_1 \ldots a_n a_1 \ldots a_n \ldots (10^n - 1) &= a_1 \ldots a_n \end{aligned}$$

We can now describe the repeating part as a rational

$$0.a_1 \ldots a_n a_1 \ldots a_n \ldots = \frac{a_1 \ldots a_n}{10^n - 1}$$

This means

$$\begin{aligned} p &= \ell + 0.b_1 \ldots b_m + 0.a_1 \ldots a_n a_1 \ldots a_n \ldots \times 10^{-m} \\ &= \ell + \frac{b_1 \ldots b_m}{10^m} + \frac{a_1 \ldots a_n}{(10^n - 1)10^m} \end{aligned}$$

Thus p is a sum of three rational numbers, so p itself must be rational.

Now we must show that $\frac{p}{q} \in \mathbb{Q}$ ends in a repeating pattern. Observe the following long division:

$$\begin{array}{rlll} & 0.\ s_1 & s_2 & s_3\ldots \\ q \,\big) & p.\ \ 0 & & \\ - & \ \ q \cdot s_1 & & \\ & \quad r_1 & 0 & \\ - & & q \cdot s_2 & \\ & & r_2 & \\ & & \vdots & \end{array}$$

The remainder r_i must be smaller than q. Thus r_i0 must be smaller than $10 \cdot q$. This means that there is only a finite number of things that r_i0 can be. If we repeat this division an infinite number of times, we must at some point have $r_t0 = r_s0$ for $t \neq s$. This then means $r_{t+1}0 = r_{s+1}0$, and so on, thus we have the decimal expansion ending in a repeating pattern. □

Irrationals such as π and $\sqrt{2}$ have no repeating tail.

$$\pi = 3.14159265\ldots$$
$$\sqrt{2} = 1.41421356\ldots$$

As far back as the ancient Greeks, it was known that $\sqrt{2}$, or for that matter $\sqrt{p}$ where p is prime, is not a rational number. We shall now treat Euclid's proof of this fact [8]: assume $\sqrt{p} = \frac{m}{n} \in \mathbb{Q}$ and m and n have no common prime factors. Then $p = \frac{m^2}{n^2}$, and $m^2 = pn^2$. This means however, that one of the prime factors of m is p, thus $m = pa$. Filling in gives $p^2a^2 = pn^2$, thus $n^2 = pa^2$. This means that n also must have p as a prime factor. However, we had simplified $\frac{m}{n}$ such that m and n had no prime factors in common. Thus $\sqrt{p}$ is not a rational number when p is a prime number.

Appendix B

Ordinal Numbers

Ordinal numbers are essentially equivalence classes of well ordered sets. Like von Neumann, however, we shall let ordinal numbers be particular elements of equivalence classes instead of the equivalence classes themselves.

B.1 Basic Definitions

The first ordinal number is 0, the number of elements of $\emptyset$. We now build all finite ordinals. An ordinal α is the set of its predecessors:

$$\begin{array}{rcl} 0 & = & \emptyset \\ 1 & = & \{0\} \\ 2 & = & \{0,1\} \\ 3 & = & \{0,1,2\} \\ \vdots & & \vdots \\ \alpha & = & \{\beta \mid \beta < \alpha\} \end{array}$$

The first infinite ordinal, called ω, is equal to $\{0, 1, 2, \ldots\} = \mathbb{N}$. We now construct ordinals beyond ω by the same method as before. Some examples include

$$\begin{array}{rcl} \omega & = & \{0,1,2,\ldots\} \\ \omega+1 & = & \{0,1,2,\ldots,\omega\} \\ \omega+2 & = & \{0,1,2,\ldots,\omega,\omega+1\} \\ \omega+3 & = & \{0,1,2,\ldots,\omega,\omega+1,\omega+2\} \end{array}$$

In this way a natural order is created in the class of ordinal numbers: $\alpha < \beta$ if and only if $\alpha \in \beta$.

There are two different kinds of ordinal number: successors and limit ordinals. These are defined as follows

Definition 10. *An ordinal number is a* successor *if it is the immediate successor of some other ordinal. It is a* limit ordinal *if it is neither 0 nor a successor.*

Thus all finite ordinals such as 5 or 8 are successors. Ordinals such as $\omega + 1$ and $\omega + 2$ are also successors. ω, on the other hand, is a limit ordinal because it has no immediate predecessor.

B.2 Transfinite Induction

What makes ordinal numbers especially important is their use in transfinite induction.

Theorem 12 (Transfinite Induction [5])**.** *Assume that $\Phi(\alpha)$ is a proposition for any ordinal α. If for all α we have that*

$$\text{if } \Phi(\beta) \text{ holds for all } \beta < \alpha \text{ then } \Phi(\alpha) \text{ holds}$$

then $\Phi(\alpha)$ holds for all ordinals α.

Proof. Suppose that for some α we have $\neg\Phi(\alpha)$. We use Schema's theorem (see [5], page 37) which states that if there exists α such that $\Phi(\alpha)$ holds, then there is a least ordinal t for which $\Phi(t)$ holds. Thus $\neg\Phi(t)$ holds for some smallest ordinal t. Thus we have for all $\beta < t$ that $\Phi(\beta)$ holds. By assumption, this should mean that $\Phi(t)$ holds, which gives a contradiction. Thus $\Phi(\alpha)$ is true for all ordinals α. □

Appendix C

The Banach-Mazur Game

To see a case wherein the concept of meagreness might be useful, we observe the following game devised by the Polish mathematician S. Mazur. We start with an interval I_0 and two players (A) and (B). Player (A) is dealt a subset A of I_0, (B) receives the complement $B = I_0 \setminus A$. The game then begins: player (A) chooses a closed interval $I_1 \subset I_0$. Player (B) then chooses a closed interval $I_2 \subset I_1$. In this manner the game continues, players alternately choosing closed intervals. An 'interval' consisting of only one point is not allowed. In such a manner, a nested sequence of closed intervals $\{I_n\}$ is determinded, player (A) determining the I_n with odd index, player (B) those with even index. If the set $\bigcap_{n=0}^{\infty} I_n$ has at least one point in common with A, then player (A) wins, otherwise player (B) wins. The completeness of $\mathbb{R}$ ensures that $\bigcap_{n=0}^{\infty} I_n$ is always a nonempty set.

A strategy for a player is a rule which determines what that player should do under every circumstance. Let us consider player (B), at his nth move, he knows only the sets A and B, and what intervals $I_0, \ldots, I_{2n-1}$ have been chosen. A strategy for (B) would be a sequence of closed-interval-valued functions $f_n(I_0, I_1, \ldots, I_{2n-1})$ which have the property that $f_n(I_0, I_1, \ldots, I_{2n-1}) \subset I_{2n-1}$, and the functions f_n must be defined for all intervals that satisfy $I_0 \supset I_1 \supset \ldots \supset I_{2n-1}$. We want $f_n(I_0, I_1, \ldots, I_{2n-1}) = I_{2n}$, thus f_n specifies (B)'s nth choice. If this is to be a winning strategy, we must have $\bigcap_{n=0}^{\infty} I_n \subset B$ for every nested sequence $\{I_n\}$ satisfying $f_n(I_0, I_1, \ldots, I_{2n-1}) = I_{2n}$.

Now we use the concept of meagreness to make the following statement: meagre sets are so small that (B) has a winning strategy if and only if A is meagre. Thus, even though (A) chooses the playing field because he can choose the first interval, it is of no avail if A is a meagre set, he will inevitably lose. This statement of course requires a proof, so let us restate it in theorem form.

Theorem 13 ([1])**.** (B) *has a winning strategy if and only if* A *is meagre.*

Proof. It is easy to see that (A) will lose if A is meagre. Let $A = \bigcup_{n=1}^{\infty} A_n$, with A_n nowhere dense. (B) must then choose $I_{2n} \subset I_{2n-1} \setminus A_n$ for every n. Whatever (A) does, every interval he chooses will contain many points from B which is, after all, a residual set. Thus (B)'s strategy is valid and he will win the game. It is more difficult to see that if (B) has a winning strategy then A must be meagre.

Let player (B) have the winning strategy $f_1, f_2, f_3, \ldots$. We need to show that B contains a residual set, for then A would be contained in a meagre set. Define a sequence $J_i (i \geq 1)$ of closed intervals in I_0^0 (the interior of I_0) such that

i) $K_i = f_1(I_0, J_i)$ are pairwise disjoint

ii) the union $\bigcup_{i=1}^{\infty} K_i^0$ is dense in I_0

We can do this by letting S be the sequence of all closed intervals with rational end-points in $I_0 \setminus K_1$. Having defined $J_1, \ldots J_i$, let J_{i+1} be the first term of S contained in $I_0 \setminus K_1 \setminus K_2 \setminus \ldots \setminus K_i$. These sets J_i are all possible first choices for (A), although they are by no means the only possible choices. This does not matter, the rest of this proof shows that considering just a few of the possible choices (A) might make suffices.

For each i, let J_{ij} $(j = 1, 2, \ldots)$ be a sequence of closed intervals in K_i^0 such that

i) $K_{ij} = f_2(I_0, J_i, K_i, J_{ij})$ disjoint

ii) the union $\bigcup_{j=1}^{\infty} K_{ij}^0$ is dense in K_i

This means of course that $\bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} K_{ij}^0$ is dense in I_0.
We now proceed inductively to obtain two families of closed intervals $J_{i_1 i_2 \ldots i_n}$ and $K_{i_1 i_2 \ldots i_n}$ $(i_k, n \in \mathbb{N})$.

i) $K_{i_1 i_2 \ldots 1_n} = f_n(I_0, J_i, K_i, J_{i_1 i_2}, K_{i_1 i_2}, \ldots, J_{i_1 \ldots i_n})$ disjoint

ii) $\bigcup_{i_n} \bigcup_{i_{n-1}} \ldots \bigcup_{i_1} K_{i_1 \ldots i_n}^0$ is dense in I_0

iii) $J_{i_1 \ldots i_n} \subset K_{i_1 \ldots i_n}$

Consider now an arbitrary sequence of positive integers i_n, and define

$$I_{2n-1} = J_{i_1 \ldots i_n} \qquad I_{2n} = K_{i_1 \ldots i_n}$$

The nested sequence $\{I_n\}$ is a possible play of the game. By hypothesis, $\bigcap_{n=1}^{\infty} I_n \subset B$.

Let $G_n = \bigcup_{i_1 \ldots i_n} K_{i_1 \ldots i_n}^0$ and let $E = \bigcap_{n=1}^{\infty} G_n$. For each $x \in E$, we now want to show that $x \in B$. For each $x \in E$, there exists a unique infinite sequence $i_1, i_2, i_3, \ldots$ such that $x \in K_{i_1 \ldots i_n}$ for each n, thus $x \in \bigcap_{n=1}^{\infty} I_{2n}$. However, $I_{2n} \subset I_{2n-1}$, so $x \in \bigcap_{n=1}^{\infty} I_n$. By assumption, (B) had a winning strategy, so $x \in \bigcap_{n=1}^{\infty} I_n \subset B$. This means that $E \subset B$.

We know that $A = I_0 \setminus B \subset I_0 \setminus E = \bigcup_{n=1}^{\infty}(I_0 \setminus G_n)$. G_n is a dense open set, for it is the union of a number of disjoint open intervals, whose union is dense. We saw on page 31 that dense open sets are residual, thus $I_0 \setminus G_n$ is meagre for each n. Thus $\bigcup_{n=1}^{\infty}(I_0 \setminus G_n)$ is meagre, and A is contained in a meagre set. It must then itself be meagre. □

This theorem can be applied to a possible winning strategy for (A) as well: (A) has a winning strategy if and only if $I_1 \cap B$ is meagre for some interval $I_1 \subset I_0$. It is possible for neither A to be meagre, nor B to be meagre in some interval. In this case, the game is undecided. However, it remains interesting to see exactly how small a meagre set really is – that one cannot possibly win the Banach-Mazur game if one has been dealt a meagre set.

Appendix D

Comparison of the $\mathbb{C}_\alpha$ Cantor sets

Observe once again the Cantor curtain on page 13. We have shown that all $\mathbb{C}_\alpha$ Cantor sets are nowhere dense nullsets, thus both on topological and measure-theoretic level we have not found any difference between them. Yet when we see the Cantor curtain all these sets look different, and some definitely look larger than others. Perhaps some notion of dimension will help us to describe this difference.

D.1 Topological and Fractal Dimension

There are broadly speaking two classes of dimension definitions: *topological* and *fractal* dimension. A topological dimension is easiest to understand, and we shall consider the covering dimension as an example. From basic geometry, we know that points have dimension 0, lines dimension 1, planes dimension 2 et cetera. The following definition of the covering dimension brings about this distinction:

Definition 11 ([3])**.** *A set is* 0-dimensional *if it can be covered by arbitrarily small disjoint open sets. A set is considered* n-dimensional *if it can be covered by small open sets that intersect just* $n + 1$ *at a time.*

The following illustration clarifies this definition. Consider a line, and cover it with small open sets:

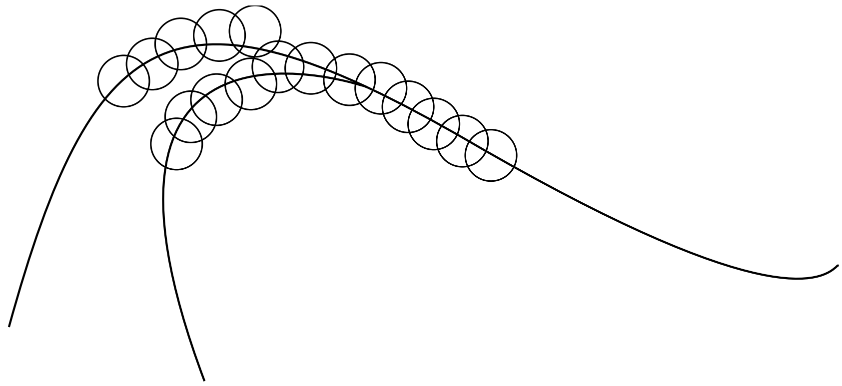


Figure D.1: An illustration of the Covering Dimension of a line

The small open sets overlap just 2 at a time, thus this set is 1-dimensional. This overlapping property is defined as the order of a family of sets:

Definition 12 ([3])**.** *The* order *of a family of sets is $\leq n$ iff any $n+2$ of the sets have empty intersection. It has order n iff it has order $\leq n$ but does not have order $\leq n-1$.*

The question we will now ask ourselves is whether the covering dimension will help us to distinguish between different $\mathbb{C}_\alpha$ Cantor sets. Unfortunately, this definition will only give integer values as the dimension of a nonempty set. As $\mathbb{R}$ itself is 1-dimensional, that means that the dimension of every Cantor set can be only 0 or 1. This will obviously not give us the distinction we seek, and indeed we find the covering dimension of any Cantor set $\mathbb{C}_\alpha$ is 0. I shall demonstrate this using the set $\mathbb{C}_{\frac{1}{3}}$; the result follows in exactly the same way for any set $\mathbb{C}_\alpha$.

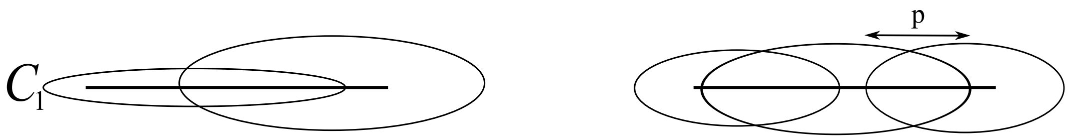
Before proving that $\mathbb{C}_{\frac{1}{3}}$ is a 0-dimensional set, we shall give a slightly modified and more accurate definition of the covering dimension which we shall use.

Definition 13 ([3])**.**

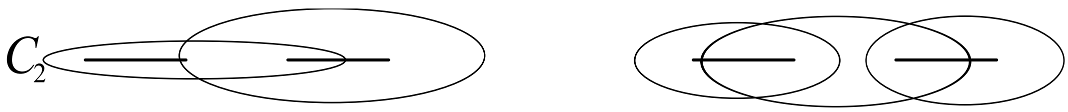
1. *Let P and Q be two open covers of a metric space S. Q is a* refinement *of P iff for every $q \in Q$ there is $p \in P$ with $q \subset p$.*

2. *Let $n \geq -1$ be an integer, and S a metric space. S has* covering dimension *$\leq n$ iff every finite open cover of S has an open refinement with order $\leq n$. The covering dimension,* $\mathrm{Cov}(S)$ *is n iff the covering dimension is $\leq n$ but not $\leq n-1$.*

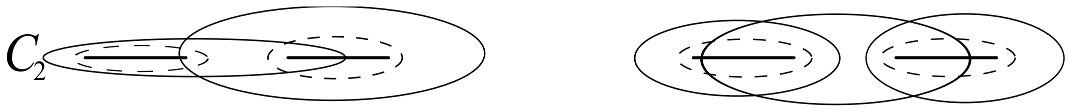
Consider now $\mathbb{C}_{\frac{1}{3}} = \bigcap_{n=0}^{\infty} C_n$. Assume we have a finite open cover O. We now claim that whatever this is, it is possible to refine it into an open cover P such that all elements of P are disjoint. Assume the smallest overlap between two different sets of O is p, thus either $\lambda(O_1 \cap O_2) \geq p$ or $\lambda(O_1 \cap O_2) = 0$ for all $O_1, O_2 \in O$.



If all the sets $O_i \in O$ are pairwise disjoint, we are done. If not, we choose n in such a manner that every interval in C_n is contained in O_i for some $O_i \in O$. We might not need to do so, but choosing n such that the length of the intervals in C_n is smaller than p will definitely accomplish this.



It is now a simple matter to take a small enough open neighbourhood around each of the intervals in C_n such that there is no overlap between them, but each neighbourhood lies in its entirety in one of the sets of O. This collection of neighbourhoods is the refinement P.



Thus, because of the existence of this disjoint refinement, $\text{Cov}(\mathbb{C}_{\frac{1}{3}}) \leq 0$. It cannot be less than 0, because this set is not empty. Thus $\text{Cov}(\mathbb{C}_{\frac{1}{3}}) = 0$. In exactly the same manner we can show that $\text{Cov}(\mathbb{C}_\alpha) = 0$ for all $0 < \alpha < 1$.

D.2 Similarity Dimension

Having found the covering dimension inadequate to our purpose, we turn to the realm of fractal dimensions. These allow noninteger-dimensional sets. One of the easiest fractal dimensions is the similarity dimension. This definition is based on the concept of self-similarity.

Self-similarity

Self-similarity is the idea that a part, if enlarged, equals the whole. Consider for example $\mathbb{C}_{\frac{1}{3}}$, the Cantor Middle Third set. If we blow up the left third of this set, we get a perfect replica of the entire set:

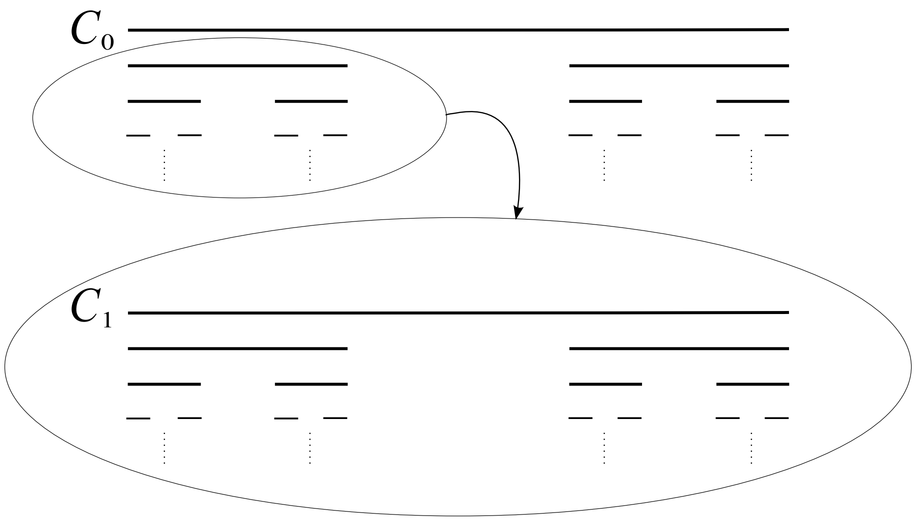


Figure D.2: The self-similar nature of $\mathbb{C}_{\frac{1}{3}}$

To describe this self-similarity, we must first go through several definitions. A *ratio list* is a finite list of positive numbers $(r_1, r_2, \ldots, r_n)$. A function $f : S \to T$ is a *similarity* if there is a positive number r such that $|f(x) - f(y)| = r|x - y|$. The smallest such number r is called the *ratio*. An *iterated function system realizing a ratio list* $(r_1, r_2, \ldots, r_n)$ in a metric space S is a list $(f_1, f_2, \ldots, f_n)$ of functions where $f_i : S \to S$ is a similarity with ratio r_i. A nonempty compact set $K \subset S$ is an *invariant set* for the iterated function system $(f_1, f_2, \ldots, f_n)$ if and only if $K = f_1(K) \cup f_2(K) \cup \ldots \cup f_n(K)$.

The set $\mathbb{C}_{\frac{1}{3}}$ is an invariant set for an iterated function system realizing the ratio list $(\frac{1}{3}, \frac{1}{3})$, because $\mathbb{C}_{\frac{1}{3}}$ consists of two sets which are each $\frac{1}{3}$ of the length of $\mathbb{C}_{\frac{1}{3}}$ itself, and which are identical to $\mathbb{C}_{\frac{1}{3}}$ if we were to triple each in size.

Defining the similarity dimension

We now have the necessary tools to move on to a definition of the similarity dimension.

Definition 14 ([3]). *The* similarity dimension *of a set K, $Sim(K)$, is the positive number s such that $r_1^s + r_2^s + \cdots + r_n^s = 1$ where K is the invariant set of an interated function system realizing the ratio list $(r_1, \ldots, r_n)$.*

As an illustration of what this means, let us consider the ratio list $(r_1, r_2, \ldots, r_n)$ with iterated function system $(f_1, f_2, \ldots, f_n)$ and invariant set K. Assume that all r_i are equal to r, then the similarity dimension of K can be computed by

$$nr^{\text{Sim}(K)} = 1$$

$$\mathrm{Sim}(K) = \frac{\log \frac{1}{n}}{\log r} = -\frac{\log n}{\log r}$$

Assume that r is always less than 1. As n increases, the number of parts that look like the whole, $\mathrm{Sim}(K)$ gently increases, but not dramatically. Therefore the more parts look like the whole the larger the similarity dimension. As the part that is similar to the whole increases, i.e. as r increases towards 1, the dimension increases. Thus the bigger the part that looks like the whole the larger the dimension.

We shall now move on to determine $\mathrm{Sim}(\mathbb{C}_\alpha)$ for $0 < \alpha < 1$. Consider the Cantor set $\mathbb{C}_\alpha$. Its ratio list is $(\frac{1}{2}(1-\alpha), \frac{1}{2}(1-\alpha))$, thus $\mathrm{Sim}(\mathbb{C}_\alpha)$ can be calculated by

$$\begin{aligned} 2(\frac{1}{2}(1-\alpha))^{\mathrm{Sim}(\mathbb{C}_\alpha)} &= 1 \\ \mathrm{Sim}(\mathbb{C}_\alpha) &= \frac{\log \frac{1}{2}}{\log(\frac{1}{2}(1-\alpha))} \\ &= \frac{-\log 2}{\log(\frac{2}{1-\alpha})} \\ &= \frac{\log 2}{\log 2 - \log(1-\alpha)} \end{aligned}$$

The similarity dimension of several $\mathbb{C}_\alpha$ are given below. As expected, $\mathrm{Sim}(\mathbb{C}_\alpha)$ decreases as α, the part removed during construction, increases.

α	$\mathrm{Sim}(\mathbb{C}_\alpha)$
0	1
$\frac{1}{6}$	0.7917
$\frac{2}{6}$	0.6309
$\frac{3}{6}$	0.5
$\frac{4}{6}$	0.3869
$\frac{5}{6}$	0.2789
1	0

Table D.1: Several values of $\mathrm{Sim}(\mathbb{C}_\alpha)$

D.3 A Mathematical Poem

Cantor's Craziness

There once was a bright man and wise
whose sets were a peculiar size
    no length it is true
    and dense points were few
but more points than $\mathbb{Q}$ inside.

Appendix E

The Axiom of Choice

The Axiom of Choice at first sight seems very innocent:

Axiom 1 (Axiom of Choice [7])**.** *Let* Λ *be some family of indices. Given* $F = \{S_\lambda\}_{\lambda \in \Lambda}$, *there exists* $f : \Lambda \to F$ *such that* $f(\lambda) \in S_\lambda$.

This function f is called a *choice function* on F. What this axiom says is actually just that given a family of nonempty sets, we can choose one element from each set.

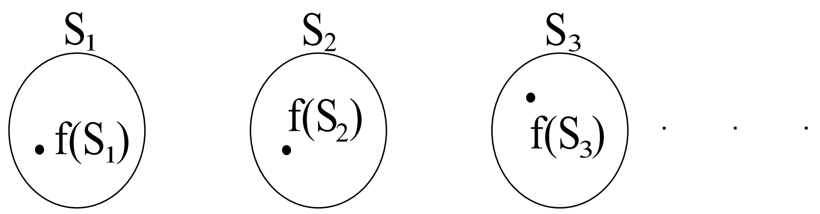


Figure E.1: The Axiom of Choice

E.1 An Equivalent Statement to the Axiom of Choice

One equivalent statement used in the text is the well-ordering principle.

Definition 15. *A* well-ordering *of a set* S *is a relation* $<$ *such that*

1. $x \not< x$
2. $x < y$ *and* $y < z$ *implies* $x < z$
3. *every nonempty* $X \subset S$ *has a least element in the ordering* $<$.

The well-ordering principle can now be stated as

Principle 3. *Every set can be well-ordered.*

This is a surprising thing to assume when we consider such sets as $\mathbb{R}$, it would seem that these sets have no 'least' element. Yet this intuitively 'wrong' statement is equivalent with the intuitively 'right' statement.

Proof. Assume first the Axiom of Choice. Consider the set S, we will show that we can well-order this set. Consider the choice function $f : F \to S$, where F is the set of all subsets of S. Using the Axiom of Choice, we find the element $s_1 = f(S) \in S$. We continue as follows:

$$\begin{array}{cc} s_1 & f(S) \\ s_2 & f(S \setminus \{s_1\}) \\ s_3 & f(S \setminus \{s_1, s_2\}) \\ \vdots & \vdots \\ s_n & f(S \setminus \{s_i \mid i < n\}) \end{array}$$

This gives us the well-ordering we are looking for.

We now assume the well-ordering principle, showing the Axiom of Choice is now a very simple matter. We need to find a choice function f on some set S. Any subset $S_0 \subset S$ has a minimal element according to the well-ordering principle. We now simply let $f(S_0)$ be equal to this minimal element for any subset. This gives us a choice function. □

E.2 Relations with (Generalized) Continuum Hypothesis

The Continuum Hypothesis is the supposition that the 'next' cardinality after $\aleph_0$ is c, the continuum. We shall now formulate it in a more formal manner.

Hypothesis 1 (Continuum Hypothesis [7])**.** *If q is a cardinal such that $\aleph_0 \leq q \leq 2^{\aleph_0}$ then either $q = \aleph_0$ or $q = 2^{\aleph_0}$.*

This is usually stated as $\aleph_1 = 2^{\aleph_0}$.

The Generalized Continuum Hypothesis is, as its name implies, a generalization of the Continuum Hypothesis.

Hypothesis 2 (Generalized Continuum Hypothesis [7])**.** *For all infinite cardinals p and q if $p \leq q \leq 2^p$ then either $q = p$ or $q = 2^p$.*

This is usually stated as $\aleph_{n+1} = 2^{\aleph_n}$.

The Continuum Hypothesis is independent of the Axiom of Choice, but the Generalized Continuum Hypothesis is not. In fact,

Theorem 14 ([12])**.** *The Generalized Continuum Hypothesis implies the Axiom of Choice.*

For full details see [12]. The idea of the proof is to show that every set can be well-ordered. Because the Well-Ordering Principle is equivalent to the Axiom of Choice, we are then done.

Appendix F

Proof of Erdös's Duality Theorem

Before proving the theorem of Erdös, we need a lemma.

Lemma 2 ([1]). *Let X be a set of cardinality $\aleph_1$ (the second smallest infinite cardinal), and let K be a class of subsets of X with the following properties:*

1. *K is a σ-ideal (K contains arbitrary subsets and countable unions of its members).*
2. *the union of K is X.*
3. *K has a subclass G of power $\leq \aleph_1$ with the property that each member of K is contained in some member of G.*
4. *the complement of each member of K contains a set of power $\aleph_1$ that belongs to K.*

Then X can be decomposed into $\aleph_1$ disjoint sets X_α, each of power $\aleph_1$ such that a subset E of X belongs to K if and only if E is contained in a countable union of the sets X_α.

Proof. This proof is easiest to follow if we break it up into three steps.

step 1 The important thing we need to remember now is that we can use ordinal numbers as indices for elements in a set which may not be countable. For example, the set $A = \{\alpha \mid 0 \leq \alpha < \Omega\}$ is the set of all ordinals smaller than Ω, the first ordinal with $\aleph_1$ predecessors. This set can be used to index a set with $\aleph_1$ elements.

We do this now with the sets in the generating class G, we are able to index them with indices from set A as G contains at most $\aleph_1$ elements.

step 2 We now construct disjoint sets K_α, which will help us during our final construction in the next step of the sets X_α we were seeking. A set K_α consists of the generating set G_α, with all overlaps with 'previous' generating sets removed:

$$K_\alpha = G_\alpha \setminus \bigcup_{\beta<\alpha} G_\beta$$

We now make a list of indices $B = \{\alpha \in A \mid K_\alpha \text{ is uncountable}\}$. We can show that there are an uncountable number of these uncountable sets K_α.

Assume there are just a countable number of uncountable K_α's. Property 1 implies that all thes K_α are members of the class K, as they are simply subsets of members of K. Also, since $\bigcup_{\alpha\in B} K_\alpha$ is a countable union of members of K, it too is a member of K.
Property 4 now implies that the complement $(\bigcup_{\alpha\in B} K_\alpha)^\complement$ contains some set of power $\aleph_1$ that belongs to K, and we call this set K'.
Property 3 now implies that this set $K' \subset G_\gamma$ for some γ. Even more, $K' \subset G_\gamma \setminus \bigcup_{\alpha\in B} K_\alpha$. Thus $G_\gamma \setminus \bigcup_{\alpha\in B} K_\alpha$ is an uncountable set, as it contains the uncountable set K'. Remember now that all K_α are countable if $\alpha \notin B$, and that γ has a countable number of predecessors. A countable union of countable sets is countable, and if we remove a countable set from an uncountable one, it remains uncountable. We can therefore remove all countable K_α such that $\alpha < \gamma$ from $G_\gamma \setminus \bigcup_{\alpha\in B} K_\alpha$ without reducing its cardinality:

$$G_\gamma \setminus \bigcup_{\alpha\in B} K_\alpha \setminus \bigcup_{\alpha\notin B, \alpha<\gamma} K_\alpha \qquad \text{is uncountable.}$$

$$\begin{aligned} G_\gamma \setminus \bigcup_{\alpha\in B} K_\alpha \setminus \bigcup_{\alpha\notin B, \alpha<\gamma} K_\alpha &\subset G_\gamma \setminus \bigcup_{\alpha<\gamma} K_\alpha \\ &= G_\gamma \setminus \bigcup_{\alpha<\gamma} G_\alpha \\ &= K_\gamma \end{aligned}$$

Thus K_γ is an uncountable set, but we cannot have $\gamma \in B$, for then we would have $D \setminus K_\gamma = K_\gamma$ for some set D, which is impossible. This is a contradiction, so there must be an uncountable number of uncountable sets K_α. Thus B contains an uncountable number of indices.

Summary: In this step, we constructed disjoint sets K_α using the sets of the generating class. We made a set of indices $B = \{\alpha \in A \mid K_\alpha \text{ is uncountable}\}$ and showed that this set is uncountable.

step 3 The sets X_α we are looking or can now be constructed as follows. Consider two consecutive indices i, j in B. Each set X_α consists of a union of the generating sets up to index j, with the union up to index i removed. As an example, consider the set $B = \{1, 5, 8, 77, \ldots, \omega, \omega + 7, \ldots\}$. The sets X_α are then:

$$\begin{aligned} X_1 &= G_1 \\ X_2 &= \bigcup_{i=1}^{5} G_i \setminus G_1 \\ X_3 &= \bigcup_{i=1}^{8} G_i \setminus \bigcup_{i=1}^{5} G_i \\ &\vdots \\ X_{\omega+7} &= \bigcup_{i=1}^{\omega+7} G_i \setminus \bigcup_{i=1}^{\omega} G_i \\ &\vdots \end{aligned}$$

Because A and B are both uncountable, we can construct a bijective function $\Phi : A \to B$ which sends the first ordinal in A to the first ordinal in B, the second to the second etc. This will help us in our notation for the general case:

$$X_\alpha = \bigcup_{\beta \leq \Phi(\alpha)} G_\beta \setminus \bigcup_{\beta < \alpha} G_{\Phi(\beta)}$$

All we need to do now is to check that these sets X_α have all the properties we wanted them to have. Firstly, these sets are disjoint, and there are $\aleph_1$ of them because our set B is uncountable.
Secondly, we see they are also uncountable. Using our example, we see

$$K_5 = G_5 \setminus \bigcup_{i=1}^{4} G_i \subset X_2$$

and more generally

$$K_{\Phi(\alpha)} = G_{\Phi(\alpha)} \setminus \bigcup_{\beta < \Phi(\alpha)} G_\beta \subset X_\alpha$$

This we have for every X_α, it contains the uncountable set $K_{\Phi(\alpha)}$ meaning that X_α is uncountable.

Now for the last property. For any $\beta \in A$, we have $\beta < \Phi(\alpha)$ for some $\alpha \in A$ and thus

$$\begin{aligned} G_\beta &\subset \bigcup_{\gamma \leq \beta} G_\gamma \\ &\subset \bigcup_{\gamma \leq \Phi(\alpha)} G_\gamma \\ &= \bigcup_{\gamma \leq \alpha} X_\gamma \end{aligned}$$

Thus every set in the generating class G is a subset of some X_α. This means the last property is also satisfied, as according to property 3 every member of K is contained in some member of G. □

We now proceed to prove Erdös's theorem.

Theorem 15 (Erdös [1])**.** *Assuming the Continuum Hypothesis, there exists a one-to-one mapping f of the line onto itself such that $f = f^{-1}$ and such that $f(E)$ is a nullset if and only if E is meagre (It follows that $f(E)$ is meagre if and only if E is a nullset).*

Proof. Let K be the class of sets meagre. The generating class in this case is the class consisting of a countable union of closed nullsets.
Let L be the class of nullsets, the generating class in this case is the class consisting of the countable intersections of open sets which yield nullsets.

Both K and L are classes of subsets of $\mathbb{R}$ which satisfy properties 1 to 4 of the previous lemma. Property 3 is satisfied if we assume that $c = \aleph_1$, that is, the Continuum Hypothesis.

We now want to construct a function $f : P(\mathbb{R}) \to P(\mathbb{R})$ such that $f(E) \in L$ if and only if $E \in K$, and $f = f^{-1}$. This will prove this theorem.

Recall that Ω is the first ordinal with an uncountable number of predecessors. Let $X_\alpha(0 \leq \alpha < \Omega)$ be a decomposition of $\mathbb{R}$ corresponding to K, as constructed in the proof of the previous lemma. Thus $E \in K$ iff E is contained in a countable union of the sets X_α. Recall the decomposition of $\mathbb{R}$ into two sets A and B such that A is meagre and B is a nullset, which we constructed in chapter 7. We can assume that A belongs to the generating class of K, it cannot be countable because then $\mathbb{R}$ would be the union of two nullsets. Thus $A = X_0$.
Let $Y_\alpha(0 \leq \alpha < \Omega)$ be a decomposition of $\mathbb{R}$ corresponding to L, with $Y_0 = B$. Then:

$$A = \bigcup_{0<\alpha<\Omega} Y_\alpha \text{ and } B = \bigcup_{0<\alpha<\Omega} X_\alpha$$

The sets X_α and Y_α, for $0 < \alpha < \Omega$, both decompose $\mathbb{R}$ into sets of power $\aleph_1$. For each $0 < \alpha < \Omega$, let f_α be a one-to-one mapping of X_α onto Y_α. We can now define:

$$f = \begin{cases} f_\alpha & \text{on } X_\alpha \\ f_\alpha^{-1} & \text{on } Y_\alpha \end{cases}$$

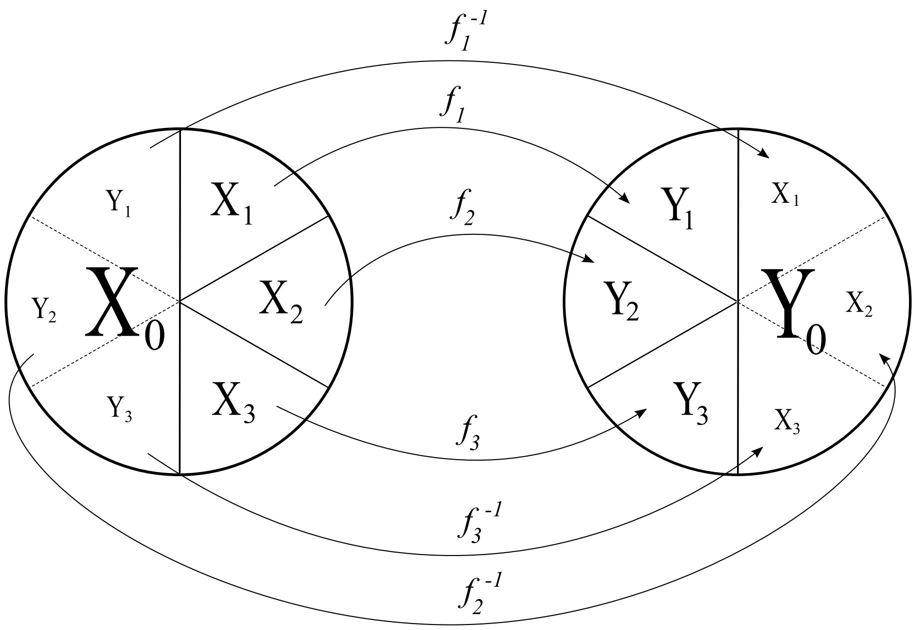


Figure F.1: The function f

This we can do because X_α and Y_α are disjoint sets for each α. Now f is a one-to-one mapping of $\mathbb{R}$ onto itself, $f = f^{-1}$ and $f(X_\alpha) = Y_\alpha$ for all $0 < \alpha < \Omega$. We also have

$$X_0 = \bigcup_{0<\alpha<\Omega} Y_\alpha \text{ and } Y_0 = \bigcup_{0<\alpha<\Omega} X_\alpha$$

thus we also have $f(X_0) = Y_0$. Thus $f(X_\alpha) = Y_\alpha$ for all $0 \leq \alpha < \Omega$. From the previous lemma we now get that:

$$\text{a subset } f(E) \in L$$
$$\Leftrightarrow$$
$$f(E) \text{ is contained in a countable union of the sets } Y_\alpha$$
$$\Leftrightarrow$$
$$E \text{ is contained in a countable union of the sets } X_\alpha$$
$$\Leftrightarrow$$
$$E \in K$$

Thus $f(E)$ is a nullset if and only if E is meagre. □

Appendix G

Measurability

Measurability is precisely what it seems to be – a set is said to be measurable if we are able to allocate to it a (finite or infinite) number as its measure. This can be done if we can approach it closely from both inside the set and outside. A formal definition is:

Definition 16. *A set A is Lebesgue measurable if for each $\varepsilon > 0$ there exists a closed set F and an open set G such that $F \subset A \subset G$ and $\lambda(G \setminus F) < \varepsilon$.*

Examples of measurable sets are easily found. Most sets one instantly thinks of are measurable. Take for example the interval $[a, b]$. The closed set G we might take to be $[a+\frac{\varepsilon}{4}, b-\frac{\varepsilon}{4}]$, and we could let the open set F be $(a - \frac{\varepsilon}{8}, b + \frac{\varepsilon}{8})$. Then $\lambda(G \setminus F) = \frac{3}{4}\varepsilon < \varepsilon$.

G.1 An Example of an Unmeasurable Set: the Bernstein Set

Finding an example of an unmeasurable set is a more difficult undertaking. To do this we need to use a statement which is equivalent to the axiom of choice. The Bernstein set is a set B of real numbers such that both B and $\mathbb{R} \setminus B$ meet every uncountable closed subset of the line. Consider the set F of uncountable closed subsets of the line. By the well-ordering principle, we can index this set by the ordinal numbers less that Ω, where Ω is the first ordinal with an uncountable number of predecessors, thus $F = \{F_\alpha \mid \alpha < \Omega\}$. We also assume that each member of F has itself been well-ordered. Let p_1, q_1 be the first two members of F_1. Let p_2, q_2 be the first two members of F_2 which are different from both p_1 and q_1. For $1 < \alpha < \Omega$, if p_β, q_β have been defined for all $\beta < \alpha$, let p_α and q_α be the first two elements of $F_\alpha \setminus \bigcup_{\beta<\alpha}\{p_\beta, q_\beta\}$. This set is non-empty for all $\alpha < \Omega$ because F_α is uncountable but $\bigcup_{\beta<\alpha}\{p_\beta, q_\beta\}$ is countable, thus p_α and q_α are defined for all $\alpha < \Omega$. Let $B = \{p_\alpha \mid \alpha < \Omega\}$. Because $p_\alpha \in B \cap F_\alpha$ and $q_\alpha \in \mathbb{R} \setminus B \cap F_\alpha$ for every $\alpha < \Omega$, the set B has the property that both it and its complement meet every uncountable closed set. Such a set is called a *Bernstein set*.

We shall now proceed to prove that either it of its complement must be unmeasurable.

Let F be a closed subset of B. It must be countable, else it would contain a point from $\mathbb{R} \setminus B$. This means however that $\lambda(F) = 0$, as we saw in example 1 in section 5.2. We can apply exactly the same reasoning to $\mathbb{R} \setminus B$ – any closed subset of $\mathbb{R} \setminus B$ must be a nullset. This means however that the measure of both B and $\mathbb{R} \setminus B$ can be at most 0. For if B is measurable, we must have a series of open sets G_i with $B \subset G_i$, $\lambda(G_i \setminus F) \to 0$, thus $\lambda(G_i) \to 0$. Using the same argument with $\mathbb{R} \setminus B$, we must conclude that if both B and $\mathbb{R} \setminus B$ are measurable, then $\lambda(B) = \lambda(\mathbb{R} \setminus B) = 0$, thus $\lambda(\mathbb{R}) = \lambda(B \cup \mathbb{R} \setminus B) = 0$. This is, however, an absurd conclusion, thus either B, $\mathbb{R} \setminus B$ or both must be unmeasurable.

G.2 The σ-algebra of Measurable Sets

Let us now consider what sets make up the measurable sets. When we look at the definition of measurability, we see that a set A is measurable if $F \subset A \subset G$, F open, G closed, and $\lambda(G \setminus F) = 0$. This suggests the idea that to get a measurable set what we need to do is to take an open set and to it we can add any set which does not increase the measure, i.e. a nullset. This is exactly what theorem 16 says. Before looking at it, we first need a definition.

Definition 17 ([11]). *A* σ-algebra S *on* $\mathbb{R}$ *is a collection of subsets of* $\mathbb{R}$ *such that*

1. $\mathbb{R} \in S$
2. *whenever* $A \in S$, *then* $\mathbb{R} \setminus A \in S$
3. *let* $A_1, A_2, A_3 \ldots \in S$, *then* $\bigcup_{i=1}^{\infty} A_i \in S$

We now move on to the theorem.

Theorem 16 ([1]). *The class* S *of measurable sets is the* σ*-algebra generated by the open sets with the nullsets.*

Proof. We will show that a set A is measurable if and only if it can be represented as a countable intersection of open sets minus a nullset.

Assume first that A is measurable, then for each n there is F_n closed and G_n open such that $F_n \subset A \subset G_n$ and $\lambda(G_n \setminus F_n) < \frac{1}{n}$. Let $E = \bigcup_{n=1}^{\infty} F_n = \bigcap_{n=1}^{\infty}(\mathbb{R} \setminus \mathbb{R} \setminus F_n)$ and $N = A \setminus E$. Then N is a nullset since $N \subset G_n \setminus F_n$ and $\lambda(N) \leq \frac{1}{n}$ for every n, and E is a countable intersection of open sets.(see [1], p. 13)

Conversely, both nullsets and open sets are measurable, thus a combination as described in this theorem will also be measurable. □

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