Boundary Triplets and Canonical Systems of Differential Equations

H.L. Wietsma

A Master Thesis
Submitted to the Faculty of Mathematics and Natural Sciences
University of Groningen
Groningen, The Netherlands
August 2007

Primairy supervisors:
Prof. H.S.V. De Snoo
Dr. J. Behrndt

Secondary supervisor:
Prof. A.J. van der Schaft

Preface and acknowledgements

This report, a partial fulfilment for the requirements for the degree of master of science, is an illustration of the theory of boundary triplets. This theory was presented in a seminar between April 2006 and April 2007 during a visit of Jussi Behrndt (TU Berlin, Germany).

First and foremost I would like to express my gratitude to my supervisor Henk de Snoo for providing the interesting subject of this thesis and for giving me the opportunity to get involved in doing research. In particular, by giving small sugestions and lots of articles, perhaps a bit frustrating a first, my understanding of mathematics has grown immensly.

In June 2007 the department of mathematics kindly allowed me to visit Jussi Behrndt in Berlin, who generously made time free to aid me. Thanks to his assistence the formal correctness of this thesis was improved significantly.

Finally, I would like to thank Ineke Kruizinga for taking care of the administrative tasks related to this thesis.

Oldeboorn, August 2007,
Hendrik Luit Wietsma

Contents

Preface and acknowledgements **iii**

Contents **v**

1 Introduction **1**

2 Preliminairy results **3**

2.1 Notation 3
2.2 Boundary triplets and associated operators 5
2.3 The Kreĭn-Naimark formula 8
2.4 Spectral theory 13
2.5 Intermediate extensions 14
2.6 Finite-dimensional graph restrictions 18

3 General results concerning a system of differential equations **20**

3.1 A system of differential equations 20
3.2 Canonical systems of differential equations 23
3.3 Hamiltonian canonical systems 26
3.4 Linear relations associated with a canonical system 27

4 Extensions of regular canonical systems **33**

4.1 Regular canonical systems 34
4.2 Regular Hamiltonian canonical systems 39
4.3 Intermediate extensions of regular Hamiltonian canonical systems . . 42
4.4 Finite-dimensional graph restrictions of regular systems 47
4.5 Regular Hamiltonian canonical systems with an interface condition . 54

5 Examples **60**

A Existence and uniqueness proof **64**

Bibliography **70**

Chapter 1: Introduction

This report is concerned with boundary value problems on a compact interval $[a, b]$ for canonical sytems of the form

$$Jf(t)' - [H(t) + \lambda\Delta(t)]\, f(t) = \Delta(t)g(t),$$

where the $n \times n$ matrix J satisfies $J^* = J^{-1} = -J$, the $n \times n$ matrix functions H and Δ are absolutely integrable on $[a, b]$, H satifies $H^* = H$ and $\Delta \geq 0$. Furthermore it is assumed that a certain definiteness condition holds. Then in the Hilbert space $L^2_\Delta((a, b))$ (made up of equivalence classes) a minimal and a maximal relation (multivalued operator) T_{min} and T_{max} associated with the above differential equation are introduced, such that T_{min} is symmetric and $T_{max} = T^*_{min}$. The minimal relation is in general not densely defined; it may happen that the relation T_{min} is indeed multivalued. A treatment of such systems in terms of relations goes back to B.C. Orcutt [31]; for different treatments see for instance [20], [24], [30], and [34].

Orcutt has associated with the system of differential equations a class of self-adjoint boundary value problems of the form

$$Ay(a) + By(b) = 0$$

where A and B are $n \times n$ matrices satisfying

$$A(iJ)A^* \geq B(iJ)B^*, \quad \operatorname{ran} \begin{pmatrix} A & B \end{pmatrix} = \mathbb{C}^n.$$

These boundary conditions were allowed to be depending on the eigenvalue parameter by H. Langer and B. Textorius [26]. In fact, by means of an associated Q-function and the Kreĭn-Naimark formula they associated an $n \times n$ matrix-valued spectral function to the boundary value problem. These ideas were also considered in a similar way for singular systems on a halfline in [15] and for singular 2×2 systems in [21], [35] and [36]. A further generalization of the work of Langer and Textorius was given in [14] where also nonstandard boundary conditions (depending on the eigenvalue parameter) were allowed. Nonstandard boundary conditions are 'boundary conditions' which may involve interior points as in interface conditions and in Stieltjes type boundary conditions. A systematic treatment of such nonstandard boundary-value problems goes back to E.A. Coddington [3] and was continued, for instance, in [4], [5], [6], [14] and [17], for a large class of formally self-adjoint differential equations or systems. For a related, more abstract, framework, see also [27], [28], [29] and [37].

The present objective is to treat the case of nonstandard boundary conditions for canonical systems of differential equations from the point of view of extension theory via boundary triplets; cf. [19]. For singular Sturm-Liouville equations this was already done in [22] and [23]. Boundary triplets were extensively studied by V.A. Derkach and M.M. Malamud, see [8] and [9] (for the more general notion of boundary relation, see [10], [11], [12] and [13]). Boundary triplets provide a flexible way to describe (abstract) boundary value problems. Moreover, there is a uniquely defined Weyl function (generalizing the Titchmarsh-Weyl coefficient from singular Sturm-Liouville equations) which shows up in the classical Kreĭn-Naimark formula describing all self-adjoint realizations. This Weyl function is a uniformly strict Nevanlinna functions, which uniquely determines the corresponding boundary value problems (see [25], [9], and the corresponding realizations for boundary relations in [12] and [2]).

The treatment of nonstandard boundary conditions is particularly elegant when boundary triplets are used. In general one has an 'ordinary' boundary value problem with a symmetric operator or relation S and a self-adjoint extension A, determined by a Weyl function $M(\lambda)$. In order to obtain nonstandard boundary conditions one takes a symmetric restriction S_1 of A (which need not be an extension of S), which produces a Weyl function $M_1(\lambda)$. The interplay between $M(\lambda)$ and $M_1(\lambda)$ gives rise to spectral matrices associated with the nonstandard boundary value problems.

The contents of this report are as follows; in Chapter 2 we introduce notation and abstract extension theory of closed symmetric relations in terms of boundary triplets. In Chapter 3 we will study the canonical system of differential equations in detail. In particular, we will show that the maximal and minimal relation, $T_{\max}$ and $T_{\min}$, are each others adjoint and that $T_{\min}$ is symmetric. These facts allows us to apply the theory of Chapter 2 to canonical systems of differential equations, the obtained results are listed in Chapter 5. In Chapter 6 we give a number of examples of (systems of) differential equations which can be interpreted as canonical systems of differential equations. Finally, to make the report more or less selfcontained, an appendix has been added concerning the existence and uniqueness of solutions of canonical systems of differential equations.

Chapter 2: Preliminairy results

2.1 Notation

Let $\mathcal{G}$ and $\mathcal{H}$ be Hilbert spaces. By $B(\mathcal{G},\mathcal{H})$ we denote the space of bounded linear operator from $\mathcal{G}$ into $\mathcal{H}$, if $\mathcal{H}=\mathcal{G}$ we will use the notation $B(\mathcal{G})$. A (linear) relation S from $\mathcal{G}$ into $\mathcal{H}$ is a (linear) subspace of the Cartesian product $\mathcal{G}\times\mathcal{H}$, if $\mathcal{H}=\mathcal{G}$ we will call S a relation in $\mathcal{G}$. If S is a relation in a finite-dimensional Hilbert space $\mathcal{G}$ the dimension of S as a subspace of $\mathcal{G}^2$ will be called the dimension of S. For two relations S_1 and S_2 from $\mathcal{G}$ into $\mathcal{H}$, their sum is defined as

$$S_1+S_2=\{\{f,g_1+g_2\}:\{f,g_1\}\in S_1 \quad \text{and} \quad \{f,g_2\}\in S_2\}$$

and their componentwise sum as

$$S_1\hat{+}S_2=\{\{f_1+f_2,g_1+g_2\}:\{f_1,g_1\}\in S_1 \quad \text{and} \quad \{f_2,g_2\}\in S_2\}.$$

If the componentwise sum $S_1\hat{+}S_2$ is direct, i.e. if $S_1\cap S_2=\{0,0\}$, we shall write $S_1\dot{\hat{+}}S_2$.

The inner product of a Hilbert space $\mathcal{G}$ will be denoted by $(\cdot,\cdot)_{\mathcal{G}}$. For $f=\begin{pmatrix}f_1 & \dots & f_m\end{pmatrix}$ where $f_i\in\mathcal{G}$, $1\le i\le m$, and $g=\begin{pmatrix}g_1 & \dots & g_n\end{pmatrix}$ where $g_i\in\mathcal{G}$, $1\le i\le n$, we define

$$(f,g)_{\mathcal{G}}=\begin{pmatrix}(f_1,g_1)_{\mathcal{G}} & \dots & (f_m,g_1)_{\mathcal{G}}\\ \vdots & \ddots & \vdots\\ (f_1,g_n)_{\mathcal{G}} & \dots & (f_m,g_n)_{\mathcal{G}}\end{pmatrix}.$$

By $<\cdot,\cdot>_{\mathcal{G}^2}$ we denote the indefinite inner product on $\mathcal{G}^2$ defined by

$$<\{f,g\},\{h,k\}>_{\mathcal{G}^2}=i\left[(f,k)_{\mathcal{G}}-(g,h)_{\mathcal{G}}\right],\quad \{f,g\},\{h,k\}\in\mathcal{G}^2.$$

Note that $<\{\cdot,\cdot\},\{\cdot,\cdot\}>_{\mathcal{H}^2}$, is continuous in each of its entries, because the inner product $(\cdot,\cdot)_{\mathcal{G}}$ is continuous in its entries.

Using this indefinite inner product on $\mathcal{G}^2$ we define the adjoint of a relation S in $\mathcal{G}$, denoted by S^*, as

$$S^*=\{\{f,g\}\in\mathcal{G}^2:<\{f,g\},\{h,k\}>_{\mathcal{G}^2}=0 \quad \text{for all } \{h,k\}\in S\}.$$

With this definition of S^* we call a relation S symmetric (or self-adjoint) if $S\subseteq S^*$ (or $S=S^*$). Furthermore, we call a relation S in the space $\mathcal{G}$ *dissipative* (or *accumulative*) if $\mathrm{Im}\,(g,f)\le 0$ (or $\mathrm{Im}\,(g,f)\ge 0$) for all elements $\{f,g\}\in S$. A dissipative (or accumulative) relation S is called maximal dissipative (or maximal accumulative) if it does not admit a proper dissipative (accumulative) extension in the space $\mathcal{G}$.

Furthermore, for a relation S we will use the notation $\mathfrak{N}_\lambda(S) = \ker\,(S-\lambda)$ and $\widehat{\mathfrak{N}}_\lambda(S) = \{\{f, \lambda f\} : f \in \mathfrak{N}_\lambda(S)\}$. Using this notation we define the deficiency index for a relation S, denoted by $n_\lambda(S)$, as $n_\lambda(S) = \dim(\mathfrak{N}_\lambda(S^*))$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$. For closed symmetric relations the deficiency index is constant on $\mathbb{C}_+$ and $\mathbb{C}_-$, see [1]. Therefore the deficiency index of a closed symmetric relation S is characterized by two numbers $n_+(S) = n_i(S)$ and $n_-(S) = n_{\bar{i}}(S)$, which are called the defect numbers.

By $(\cdot,\cdot)$ we denote the usual inner product on $\mathbb{C}^n$, i.e. for $f, g \in \mathbb{C}^n$ $(f,g) = g^* f$. On $\mathbb{C}^n$ we will use the following norms

$$||f||_\infty = \max_{1\le i\le n} |f_i|, \quad |f| = \sum_{i=1}^n |f_i| \quad \text{and} \quad ||f|| = (f^*f)^{\frac{1}{2}} = \left(\sum_{i=1}^n |f_i|^2\right)^{\frac{1}{2}}, \quad f \in \mathbb{C}^n.$$

The space of $\mathbb{C}^{m\times n}$-valued functions on (a,b) will be denoted by $\mathcal{F}^{m\times n}((a,b)))$ and we will use the notation $\mathcal{F}^n((a,b)) = \mathcal{F}^{n\times 1}((a,b))$. By $\mathrm{AC}^{m\times n}([a,b])$ we denote the space of absolutely continuous elements of $\mathcal{F}^{m\times n}([a,b])$. Here $f \in \mathcal{F}^{m\times n}([a,b])$ is absolutely continuous on $[a,b]$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\sum_{i=1}^j |f(y_i) - f(x_i)| < \epsilon,$$

for every collection of disjoint segments $\{[x_1,y_1],\ldots,[x_j,y_j]\}$, $j \in \mathbb{N}$, of $[a,b]$, which satisfies

$$\sum_{i=1}^j |y_i - x_i| < \delta.$$

Or, equivalently, $f \in \mathcal{F}^{m\times n}([a,b])$ is absolutely continuous if there exists an $\mathbb{C}^{m\times n}$-valued function $g \in L^1([a,b])$ such that

$$f(x) = f(a) + \int_a^x g(t)\mathrm{d}t,$$

i.e. f' exists a.e. on $[a,b]$, see [18]. Furthermore, $f \in \mathrm{AC}^{m\times n}_{\mathrm{loc}}(\iota)$, ι a bounded interval in $\mathbb{R}$, if $f \in \mathrm{AC}^{m\times n}([\alpha,\beta])$ for every $[\alpha,\beta] \subseteq \iota$ and we will use the notation $\mathrm{AC}^n_{\mathrm{loc}}(\iota) = \mathrm{AC}^{n\times 1}_{\mathrm{loc}}(\iota)$.

Finally, the space of locally absolutely integrable $\mathbb{C}^{m\times n}$-valued functions on (a,b), denoted by $L^1_{\mathrm{loc}}((a,b))$, consists of all $f \in \mathcal{F}^{m\times n}((a,b))$ which satisfy

$$\int_\alpha^\beta |f(t)|\mathrm{d}t < \infty,$$

for every $[\alpha,\beta] \subset (a,b)$.

2.2 Boundary triplets and associated operators

In this section we will introduce boundary triplets, γ-fields and Weyl functions. These concepts will be used to investigate closed symmetric and self-adjoint extensions of a closed symmetric relation A in a Hilbert space $\mathfrak{H}$. All statements in this section orginate from [1].

Definition 2.2.1. *Let A be a closed symmetric relation in the Hilbert space $\mathfrak{H}$. Then $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is called a boundary triplet for A^* if*

(i) *$\mathcal{G}$ is a Hilbert space;*

(ii) *The mappings $\Gamma_0, \Gamma_1 : A^* \to \mathcal{G}$ are such that $\Gamma = \begin{pmatrix}\Gamma_0 \\ \Gamma_1\end{pmatrix} : A^* \to \mathcal{G}^2$ is surjective;*

(iii) *Γ satisfies the Green's identity*

$$< \{f, g\}, \{h, k\} >_{\mathfrak{H}^2} = < \Gamma\{f, g\}, \Gamma\{h, k\} >_{\mathcal{G}^2}$$

for all $\{f, g\}, \{h, k\} \in A^*$.

Note that Γ satisfies the Green's identity if and only if

$$(g, h)_{\mathfrak{H}} - (f, k)_{\mathfrak{H}} = (\Gamma_1\{f, g\}, \Gamma_0\{h, k\})_{\mathcal{G}} - (\Gamma_0\{f, g\}, \Gamma_1\{h, k\})_{\mathcal{G}}$$

for every $\{f, g\}, \{h, k\} \in A^*$.

The following proposition shows how boundary triplets can be used to reduce the problem of finding extensions of the closed symmetric relation A in the "big" space $\mathfrak{H}$ to a problem in the "smaller" boundary space $\mathcal{G}$.

Proposition 2.2.2. *Let A be a closed symmetric relation in the Hilbert space $\mathfrak{H}$ and let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*. Then Γ induces via*

$$\Theta \mapsto A_\Theta := \{\{f, g\} \in A^* : \Gamma\{f, g\} \in \Theta\}$$

a bijective correspondance between the set of all closed linear relations in $\mathcal{G}$ and the set of closed extensions of A which are restrictions of A^*. Moreover, $(A_\Theta)^* = A_{\Theta^*}$ and therefore A_Θ is symmetric (or self-adjoint) if and only if Θ is symmetric (or self-adjoint). In particular $\ker \Gamma_0$ *and* $\ker \Gamma_1$ *are self-adjoint.*

If a closed symmetric relation A has equal defect numbers, then there exists a boundary triplet for A^*. Therefore Proposition 2.2.2 implies that a closed symmetric relation with equal defect numbers has self-adjoint extensions. The following lemma shows that the converse is also true.

Lemma 2.2.3. *Let A be a closed symmetric relation in the Hilbert space $\mathfrak{H}$ and let $\tilde{A}$ be a self-adjoint extension of A in $\mathfrak{H}$. Then the decomposition*

$$A^* = \tilde{A} \hat{\dot{+}} \widehat{\mathfrak{N}}_\lambda(A^*)$$

holds for all $\lambda \in \rho(\tilde{A})$.

The following result shows when surjective mappings Γ_0 and Γ_1 between Hilbert spaces are boundary mappings.

Theorem 2.2.4. *Let $\mathfrak{H}$ and $\mathcal{G}$ be Hilbert spaces and let T be a linear relation in $\mathfrak{H}$. Assume that $\Gamma = \begin{pmatrix} \Gamma_0 \\ \Gamma_1 \end{pmatrix} : T \to \mathcal{G}^2$ is a linear mapping such that the conditions*

(i) $\ker \Gamma_0$ *contains a self-adjoint relation,*

(ii) $\operatorname{ran} \Gamma = \mathcal{G}^2$,

(iii) $< \{f, g\}, \{h, k\} >_{\mathfrak{H}^2} = < \Gamma\{f, g\}, \Gamma\{h, k\} >_{\mathcal{G}^2}$, *for all $\{f, g\}, \{h, k\} \in T$,*

are satisfied. Then $A := \ker \Gamma$ is a closed symmetric relation in $\mathfrak{H}$ and $A^* = T$. Moreover, $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is a boundary triplet for A^*.

Associated with a boundary triplet $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ are the γ-field, denoted by γ_λ, and Weyl function, denoted by $M(\lambda)$. The following proposition contains their definition and elementary properties.

Proposition 2.2.5. *Let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for a closed symmetric relation A in the Hilbert space $\mathfrak{H}$. Then the γ-field γ_λ and Weyl function $M(\lambda)$ are defined as*

$$\gamma_\lambda = \pi_1 (\Gamma_0|_{\widehat{\mathfrak{N}}_\lambda(A^*)})^{-1} \quad \text{and} \quad M(\lambda) = \Gamma_1 (\Gamma_0|_{\widehat{\mathfrak{N}}_\lambda(A^*)})^{-1}, \quad \lambda \in \rho(A_0),$$

where π_1 is the projection defined by $\pi_1 : \{f, g\} \mapsto f$ and $A_0 = \ker \Gamma_0$. Furthermore, we define $\hat{\gamma}_\lambda$ as $\hat{\gamma}_\lambda = (\Gamma_0|_{\widehat{\mathfrak{N}}_\lambda(A^*)})^{-1}$. The γ-field and Weyl function have the following properties

(i) $\gamma_\lambda \in B(\mathcal{G}, \mathfrak{H})$ *for every $\lambda \in \rho(A_0)$;*

(ii) For all $\lambda, \mu \in \rho(A_0)$ the operators γ_λ and γ_μ are connected via

$$\gamma_\lambda = (I_{\mathfrak{H}} + (\lambda - \mu)(A_0 - \lambda)^{-1})\gamma_\mu;$$

(iii) For all $\lambda \in \rho(A_0)$ the operator $\gamma_\lambda^* \in B(\mathfrak{H}, \mathcal{G})$ satisfies

$$\gamma_\lambda^* h = \Gamma_1 \{(A_0 - \bar{\lambda})^{-1} h, (I_{\mathfrak{H}} + \bar{\lambda}(A_0 - \bar{\lambda})^{-1} h\}, \quad h \in \mathfrak{H};$$

(iv) $M(\lambda) \in B(\mathcal{G})$ *for every $\lambda \in \rho(A_0)$;*

(v) $M(\lambda)\Gamma_0\{f_\lambda, \lambda f_\lambda\} = \Gamma_1\{f_\lambda, \lambda f_\lambda\}$ *for every $f_\lambda \in \mathfrak{N}_\lambda(A^*)$ and $\lambda \in \rho(A_0)$;*

(vi) *$\lambda \mapsto M(\lambda)$ is holomorphic on $\rho(A_0)$;*

(vii) *For all $\lambda, \mu \in \rho(A_0)$ the relation*

$$M(\lambda) - M(\mu)^* = (\lambda - \bar{\mu})\gamma_\mu^* \gamma_\lambda$$

holds. In particular $M(\lambda)^* = M(\bar{\lambda})$ and $\frac{\operatorname{Im} M(\lambda)}{\operatorname{Im} \lambda}$ is uniformly positive for $\lambda \in \rho(A_0)$.

Note that the γ-field and Weyl function are well-defined functions by Lemma 2.2.3 with $\tilde{A} = A_0$. Since a function $Q(\lambda)$ is a (uniform strict) Nevanlinna function if $Q(\bar{\lambda}) = Q(\lambda)^*$, $\lambda \mapsto Q(\lambda)$ is holomorphic on $\mathbb{C} \setminus \mathbb{R}$ and $\frac{\operatorname{Im} M(\lambda)}{\operatorname{Im} \lambda}$ is (uniformly positive) nonnegative, the above proposition shows that the Weyl function $M(\lambda)$ is an uniformly strict Nevanlinna function. The converse also holds, i.e. each uniformly strict Nevanlinna function can be realized as the Weyl funtion of a boundary triplet, see [2].

Using the γ-field and Weyl function we are able to give an explicit expression for the resolvent of any extension A_Θ of A.

Proposition 2.2.6. *For any closed linear relation Θ in $\mathcal{G}$ let A_Θ be defined as in Proposition 2.2.2. Then, with $A_0 = \ker \Gamma_0$,*

$$(A_\Theta - \lambda)^{-1} = (A_0 - \lambda)^{-1} - \gamma_\lambda (M(\lambda) + \Theta)^{-1} \gamma_{\bar{\lambda}}^*,$$

for $\lambda \in \rho(A_0)$.

Boundary triplets for closed symmetric relations are non-unique. The following proposition shows which transformations conserve boundary mappings and how the γ-field and Weyl function change under the transformation.

Proposition 2.2.7. *Let A be a closed symmetric relation in the Hilbert space $\mathfrak{H}$ and let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* with associated γ-field γ_λ and Weyl function $M(\lambda)$. Then the following implications hold*

(i) *If $\tilde{\mathcal{G}}$ is a Hilbert space and*

$$W = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}$$

an operator from $\mathcal{G}^2$ onto $\tilde{\mathcal{G}}^2$, which satisfies

$$W^* \begin{pmatrix} 0 & -iI_{\tilde{\mathcal{G}}} \\ iI_{\tilde{\mathcal{G}}} & 0 \end{pmatrix} W = \begin{pmatrix} 0 & -iI_{\mathcal{G}} \\ iI_{\mathcal{G}} & 0 \end{pmatrix}.$$

Then $\{\tilde{\mathcal{G}}, \tilde{\Gamma}_0, \tilde{\Gamma}_1\}$, where

$$\begin{pmatrix} \tilde{\Gamma}_0 \\ \tilde{\Gamma}_1 \end{pmatrix} := W \begin{pmatrix} \Gamma_0 \\ \Gamma_1 \end{pmatrix},$$

is another boundary triplet for A^*. Furthermore, the γ-field and Weyl function associated with $\{\tilde{\mathcal{G}}, \tilde{\Gamma}_0, \tilde{\Gamma}_1\}$, $\lambda \in \rho(\ker \Gamma_0) \cap \rho(\ker \tilde{\Gamma}_0)$, are

$$\tilde{\gamma}_\lambda = \gamma_\lambda (W_{11} + W_{12} M(\lambda))^{-1}$$

and

$$\tilde{M}(\lambda) = (W_{21} + W_{22} M(\lambda))(W_{11} + W_{12} M(\lambda))^{-1}.$$

(ii) *Each two boundary triplets $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ and $\{\tilde{\mathcal{G}}, \tilde{\Gamma}_0, \tilde{\Gamma}_1\}$ for A^* are related by the surjective operator W defined as*

$$W = \{\{\Gamma\{f,g\}, \tilde{\Gamma}\{f,g\}\} : \{f,g\} \in A^*\}.$$

2.3 The Kreĭn-Naimark formula

In the previous section we have seen that all self-adjoint extensions of the closed symmetric relation A in a Hilbert space $\mathfrak{H}$ are characterized by self-adjoint relations in the boundary space, see Proposition 2.2.2. In this section we will look at the families of extensions of A induced by self-adjoint exit-space extensions of A. Here a self-adjoint exit space extension is a self-adjoint relation $\tilde{A}$ in the space $\mathfrak{H} \oplus \mathfrak{K}$, where the exit space $\mathfrak{K}$ is a Hilbert space, if $\tilde{A}$ restricted to $\mathfrak{H}$ is an extension of A. In particular, we will show that each of these families of extensions can be identified with a function of a certain class.

An exit space extension will be called minimal if there is no nontrivial decomposition of $\mathfrak{H} \oplus \mathfrak{K}$ into $\mathfrak{H}_1 \oplus \mathfrak{H}_2$, such that $\tilde{A} \cap \mathfrak{H}_1^2$ is self-adjoint. Finally, self-adjoint relations in $\mathfrak{H}$ which are extension of A will henceforth be called canonical self-adjoint extensions of A to dinstinguish them from self-adjoint exit-space extension.

For a self-adjoint (exit space) extension $\tilde{A}$ the corresponding Štraus family $T(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, of extension of A is defined as

$$\begin{aligned} T(\lambda) &= \{\{\pi_{\mathfrak{H}} f, \pi_{\mathfrak{H}} g\} : \{f,g\} \in \tilde{A}, g - \lambda f \in \mathfrak{H}\}, \\ T(\infty) &= \{\{\pi_{\mathfrak{H}} f, \pi_{\mathfrak{H}} g\} : \{f,g\} \in \tilde{A}, f \in \mathfrak{H}\}. \end{aligned}$$

The following proposition gives an equivalent characterization for Štraus extensions, see [16].

Proposition 2.3.1. *The family of relations $T(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, in the Hilbert space $\mathfrak{H}$ corresponding to a self-adjoint extension $\tilde{A}$ in $\mathfrak{H} \oplus \mathfrak{K}$ of the closed symmetric relation A in $\mathfrak{H}$ is called a Štraus family if and only if*

(i) *$T(\lambda)$ is maximal accumulative (maximal dissipative) for $\lambda \in \mathbb{C}_+$ ($\lambda \in \mathbb{C}_-$);*

(ii) *$T(\bar{\lambda}) = T(\lambda)^*$ for $\lambda \in \mathbb{C} \setminus \mathbb{R}$;*

(iii) *The $B(\mathfrak{H})$-valued function $\lambda \mapsto (T(\lambda) + \lambda)^{-1}$ is holomorphic on $\mathbb{C} \setminus \mathbb{R}$.*

Between the minimal self-adjoint extensions and Štraus families there exist (up to unitary equivalence) an one-to-one correspondence, see [16].

Another object related to self-adjoint extensions of closed symmetric relations is the generalized resolvent. With $\tilde{A}$ a self-adjoint extension of A, the generalized resolvent of A corresponding to $\tilde{A}$ is defined as

$$R(\lambda) = \pi_{\mathfrak{H}}(\tilde{A} - \lambda)^{-1}|_{\mathfrak{H}}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

where $\pi_{\mathfrak{H}} : \mathfrak{H} \oplus \mathfrak{K} \to \mathfrak{H}$ is the projection defined by $\pi_{\mathfrak{H}} : \{f, g\} \mapsto f$. The following proposition gives an equivalent characterization for generalized resolvents, see [16].

Proposition 2.3.2. *The $B(\mathfrak{H})$-valued function $\lambda \mapsto R(\lambda)$ is a generalized resolvent of the closed symmetric relation A in the Hilbert space $\mathfrak{H}$ corresponding to a self-adjoint extension $\tilde{A}$ in $\mathfrak{H} \oplus \mathfrak{K}$ if and only if*

(i) $(A - \lambda)^{-1} \subset R(\lambda)$*;*

(ii) *For all* $\lambda \in \mathbb{C} \setminus \mathbb{R}$
$$\frac{R(\lambda) - R(\lambda)^*}{\lambda - \bar{\lambda}} - R(\lambda)^* R(\lambda) \geq 0;$$

(iii) $R(\bar{\lambda}) = R(\lambda)^*$*;*

(iv) *The $B(\mathfrak{H})$-valued function $\lambda \mapsto (R(\lambda) + \lambda)^{-1}$ is holomorphic on $\mathbb{C} \setminus \mathbb{R}$.*

The correspondence between Štraus families, corresponding to minimal self-adjoint extensions of A, and generalized resolvents of A, corresponding to minimal self-adjoint extensions of A, given by

$$R(\lambda) = (T(\lambda) - \lambda)^{-1}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \tag{2.1}$$

is easily seen to be one-to-one.

Generalized resolvents can also be characterized by Nevanlinna families via the Kreĭn-Naimark formula, see [1].

Theorem 2.3.3. *Let A be a closed symmetric relation in the Hilbert space $\mathfrak{H}$ and let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* with associated γ-field γ_λ and Weyl function $M(\lambda)$. Then, with $A_0 = \ker \Gamma_0$, the Kreĭn-Naimark formula*

$$R(\lambda) = (A_0 - \lambda)^{-1} - \gamma_\lambda \left(M(\lambda) + \tau(\lambda)\right)^{-1} \gamma_{\bar{\lambda}}^*, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \tag{2.2}$$

establishes an one-to-one correspondence between the generalized resolvents $R(\lambda)$ of A and Nevanlinna families $\tau(\lambda)$ in $\mathcal{G}$.

Here the definition of Nevanlinna families, which is a generalization of Nevanlinna functions, is given below, see [2].

Definition 2.3.4. *A family $Q(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, of linear relations in the Hilbert space $\mathcal{G}$ is said to be a Nevanlinna family in $\mathcal{G}$ if*

(i) $Q(\lambda)$ *is maximal dissipative (maximal accumulative) for $\lambda \in \mathbb{C}_+$ ($\lambda \in \mathbb{C}_-$);*

(ii) $Q(\bar{\lambda}) = Q(\lambda)^*$*;*

(iii) *For some, and hence for all, $\nu \in \mathbb{C}_+$ ($\nu \in \mathbb{C}_-$) the $B(\mathcal{G})$-valued function $\lambda \mapsto (Q(\lambda) + \nu)^{-1}$ is holomorphic on $\mathbb{C}_+$ ($\mathbb{C}_-$).*

Let $Q(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, be a Nevanlinna family in $\mathcal{G}$ and let $\mu \in \mathbb{C}_+$, then the $B(\mathcal{G})$-valued functions

$$\lambda \mapsto A(\lambda) = \begin{cases} (Q(\lambda) + \mu)^{-1}, & \lambda \in \mathbb{C}_+, \\ (Q(\lambda) + \bar{\mu})^{-1}, & \lambda \in \mathbb{C}_-, \end{cases} \tag{2.3}$$

and

$$\lambda \mapsto B(\lambda) = \begin{cases} I - \mu(Q(\lambda) + \mu)^{-1}, & \lambda \in \mathbb{C}_+, \\ I - \bar{\mu}(Q(\lambda) + \bar{\mu})^{-1}, & \lambda \in \mathbb{C}_-, \end{cases} \tag{2.4}$$

are well-defined functions by (iii) of the above definition. Furthermore, the $B(\mathcal{G})$-valued functions $A(\lambda)$ and $B(\lambda)$ are *symmetric*, meaning that $A(\lambda) = A(\bar{\lambda})^*$ and $B(\lambda) = B(\bar{\lambda})^*$ for $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Using the above defined functions $Q(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, can be written as

$$Q(\lambda) = \{\{A(\lambda)g, B(\lambda)g\} : g \in \mathcal{G}\}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

We conclude that a Nevanlinna family can be represented by functions $A(\lambda)$ and $B(\lambda)$ which satify $A(\lambda) = A(\bar{\lambda})^*$ and $B(\lambda) = B(\bar{\lambda})^*$. This gives cause for the following definition, see [2].

Definition 2.3.5. *A pair $\{\Phi(\lambda), \Psi(\lambda)\}$ of $B(\mathcal{G})$-valued functions is said to be a Nevanlinna pair in the Hilbert space $\mathcal{G}$ if*

(i) $(\mathrm{Im}\,\lambda)\mathrm{Im}\,(\Psi(\lambda)\Phi(\lambda)^*) \geq 0$ *for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$;*

(ii) $\Psi(\lambda)\Phi(\bar{\lambda})^* = \Phi(\lambda)\Psi(\bar{\lambda})^*$*;*

(iii) $(\Psi(\lambda) + \nu\Phi(\lambda))^{-1} \in B(\mathcal{G})$ *for $\lambda, \nu \in \mathbb{C}_\pm$.*

For a Nevanlinna family $Q(\lambda)$ $\{-B(\lambda), A(\lambda)\}$, where $A(\lambda)$ and $B(\lambda)$ are given by (2.3) and (2.4), is a symmetric Nevanlinna pair.

Conversely, for a Nevanlinna pair $\{\Phi(\lambda), \Psi(\lambda)\}$ $Q(\lambda)$ defined as

$$Q(\lambda) = \{\{-\Psi(\bar{\lambda})^* g, \Phi(\bar{\lambda})^* g\} : g \in \mathcal{G}\}$$

is a Nevanlinna family. In particular, if the Nevanlinna pair $\{\Phi(\lambda), \Psi(\lambda)\}$ is symmetric, i.e. $\Phi(\lambda)$ and $\Psi(\lambda)$ are symmetric, $Q(\lambda)$ is given by

$$Q(\lambda) = \{\{-\Psi(\lambda)g, \Phi(\lambda)g\} : g \in \mathcal{G}\}.$$

Using the above results we will rewrite the term $(M(\lambda)+\tau(\lambda))^{-1}$, occuring in the Kreĭn-Naimark formula, see (2.2), using a Nevanlinna pair representation of $\tau(\lambda)$. I.e. with $\tau(\lambda) = \{\{-B(\lambda)g, A(\lambda)g\} : g \in \mathcal{G}\}$

$$M(\lambda)+\tau(\lambda) = \{\{-B(\lambda)g, [A(\lambda) - M(\lambda)B(\lambda)]\, g\} : g \in \mathcal{G}\},$$

from which it follows that

$$\begin{aligned} (M(\lambda)+\tau(\lambda))^{-1} &= \{\{[M(\lambda)B(\lambda) - A(\lambda)]\, g, B(\lambda)g\} : g \in \mathcal{G}\} \\ &= B(\lambda)\,(M(\lambda)B(\lambda) - A(\lambda))^{-1}. \end{aligned} \tag{2.5}$$

Here the last equality holds for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$, see [25].

Since the generalized resolvents are in one-to-one correspondence with Štraus families via (2.1), the resolvent in the Kreĭn-Naimark formula is the resolvent of a Štraus family. The following proposition gives a characterization of that Štraus family.

Proposition 2.3.6. *Let $T(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, be the Štraus family corresponding to the generalized resolvent in* (2.2)*, then*

$$T(\lambda) = \{\{f,g\} \in A^* : \Gamma\{f,g\} \in -\tau(\lambda)\}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

Proof. In this proof $\{-B(\lambda), A(\lambda)\}$ is any Nevanlinna pair representation of the Nevanlinna family $\tau(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$. By (2.2) and (2.5)

$$(T(\lambda)-\lambda)^{-1} = \{\{h, (A_0-\lambda)^{-1}h - \gamma_\lambda B(\lambda)(M(\lambda)B(\lambda) - A(\lambda))^{-1}\gamma_{\bar\lambda}^* h\} : h \in \mathfrak{H}\}.$$

Therefore with $X(\lambda) = (M(\lambda)B(\lambda) - A(\lambda))^{-1}\gamma_{\bar\lambda}^*$

$$T(\lambda) - \lambda = \{\{(A_0-\lambda)^{-1}h - \gamma_\lambda B(\lambda)X(\lambda)h, h\} : h \in \mathfrak{H}\}$$

from which it follows that $T(\lambda)$ is given by

$$T(\lambda) = \{\{(A_0-\lambda)^{-1}h - \gamma_\lambda B(\lambda)X(\lambda)h, (I_{\mathfrak{H}} + \lambda(A_0-\lambda)^{-1})h - \lambda\gamma_\lambda B(\lambda)X(\lambda)h\} : h \in \mathfrak{H}\}.$$

Now apply the boundary mappings to an element in $T(\lambda)$. With $h \in \mathfrak{H}$ we have that

$$\begin{aligned} &\Gamma\{(A_0-\lambda)^{-1}h - \gamma_\lambda B(\lambda)X(\lambda)h, (I_{\mathfrak{H}} + \lambda(A_0-\lambda)^{-1})h - \lambda\gamma_\lambda B(\lambda)X(\lambda)h\} \\ =&\Gamma\{(A_0-\lambda)^{-1}h, (I_{\mathfrak{H}} + \lambda(A_0-\lambda)^{-1})h\} + \Gamma\hat\gamma_\lambda B(\lambda)X(\lambda)h \\ =&\{0, \gamma_{\bar\lambda}^* h\} + \{B(\lambda)X(\lambda)h, M(\lambda)B(\lambda)X(\lambda)h\} = \{B(\lambda)X(\lambda)h, A(\lambda)X(\lambda)h\}. \end{aligned}$$

Here in the first step we used the linearity of the boundary mappings on A^*. In the second step we used the fact that $\{(A_0-\lambda)^{-1}h, (I_{\mathfrak{H}} + \lambda(A_0-\lambda)^{-1})h\} \in A_0$ and Proposition 2.2.5 (iii) for the first term and Proposition 2.2.5 (v) combined with the definition of $\hat\gamma_\lambda$, see also Proposition 2.2.5, for the second term. □

As a consequence of the above proposition define $A_{-\tau(\lambda)}$ as

$$A_{-\tau(\lambda)} = \{\{f,g\} \in A^* : \Gamma\{f,g\} \in -\tau(\lambda)\}. \tag{2.6}$$

Note that this notation is an extension of the notation in Proposition 2.2.2 to Nevanlinna families. Then using this notation and Proposition 2.3.6 the Kreĭn-Naimark formula can be written as

$$(A_{-\tau(\lambda)} - \lambda)^{-1} = (A_0 - \lambda)^{-1} - \gamma_\lambda \left(M(\lambda) + \tau(\lambda)\right)^{-1} \gamma_{\bar{\lambda}}^*, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{2.7}$$

Using the (Nevanlinna) pair respresentation of $\tau(\lambda)$ and the following lemma, see [1], we will determine an alternative expression for $A_{-\tau(\lambda)}$.

Lemma 2.3.7. *Let A be a relation in a Hilbert space, then*

$$\ker A^* = (\operatorname{ran} A)^\perp.$$

Proposition 2.3.8. *Assume that $\tau(\lambda) = \{\{-B(\lambda)g, A(\lambda)g\} : g \in \mathcal{G}\}$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, then $A_{\tau(\lambda)^*}$ can be written as*

$$A_{\tau(\lambda)^*} = \{\{f,g\} \in A^* : \begin{pmatrix} A^*(\lambda) & B^*(\lambda) \end{pmatrix} \Gamma\{f,g\} = 0\}$$

for $\lambda \in \mathbb{C} \setminus \mathbb{R}$. In particular, if $\tau(\lambda)$ is a Nevanlinna family and $\{A(\lambda), B(\lambda)\}$ is a symmetric Nevanlinna pair representation, then $A_{-\tau(\lambda)}$ can be written as

$$A_{-\tau(\lambda)} = \{\{f,g\} \in A^* : \begin{pmatrix} A(\lambda) & -B(\lambda) \end{pmatrix} \Gamma\{f,g\} = 0\}$$

for $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

Proof. By definition

$$A_{\tau(\lambda)^*} = \{\{f,g\} \in A^* : \Gamma\{f,g\} \in \tau(\lambda)^*\},$$

thus to prove the first part of the proposition we need to show that $\Gamma\{f,g\} \in \tau(\lambda)^*$ if and only if $\Gamma\{f,g\} \in \ker \begin{pmatrix} A(\lambda)^* & B(\lambda)^* \end{pmatrix}$.

By definition $\Gamma\{f,g\} \in \tau(\lambda)^*$ if and only if $< \{\Gamma_0\{f,g\}, \Gamma_1\{f,g\}\}, \{h,k\} >_{\mathcal{G}^2} = 0$ for every $\{h,k\} \in \tau(\lambda)$. Or, equivalently, $\Gamma\{f,g\} \in \tau(\lambda)^*$ if and only if $< \{\Gamma_0\{f,g\}, \Gamma_1\{f,g\}\}, \{-B(\lambda)g, A(\lambda)g\} >_{\mathcal{G}^2} = 0$ for every $g \in \mathcal{G}$. Thus $\Gamma\{f,g\} \in \tau(\lambda)^*$ if and only if

$$i\Big[(\{\Gamma_0\{f,g\}, A(\lambda)g)_{\mathcal{G}} + (\{\Gamma_1\{f,g\}, B(\lambda)g)_{\mathcal{G}}\Big] = i\left(\Gamma\{f,g\}, \begin{pmatrix} A(\lambda) \\ B(\lambda) \end{pmatrix} g\right)_{\mathcal{G}^2} = 0$$

for every $g \in \mathcal{G}$. From the above calculation it follows that $\Gamma\{f,g\} \in \tau(\lambda)^*$ if and only if $\Gamma\{f,g\} \in \operatorname{ran} \begin{pmatrix} A(\lambda) \\ B(\lambda) \end{pmatrix}^\perp$, which proves the first statement of the proposition by Lemma 2.3.7.

If $\tau(\lambda)$ is a Nevanlinna family and $\{A(\lambda), B(\lambda)\}$ is a symmetric Nevanlinna pair representation, then using the first part of the proposition

$$\begin{aligned} A_{-\tau(\lambda)} = A_{-\tau(\bar\lambda)^*} &= \{\{f,g\} \in A^* : \begin{pmatrix} A^*(\bar\lambda) & -B^*(\bar\lambda) \end{pmatrix} \Gamma\{f,g\}\} \\ &= \{\{f,g\} \in A^* : \begin{pmatrix} A(\lambda) & -B(\lambda) \end{pmatrix} \Gamma\{f,g\}\}, \end{aligned}$$

which proves the second part of the proposition. □

2.4 Spectral theory

Given a closed symmetric relation A with boundary triplet $\{\mathbb{C}^n, \Gamma_0, \Gamma_1\}$ the Kreĭn-Naimark formula, see (2.7), allows us to determine the structure of the resolvent of any extension $A_{-\tau(\lambda)}$, see (2.6), of A. In this section we will show how we can use the information on the resolvent of $A_{-\tau(\lambda)}$ to obtain information on the extension $A_{-\tau(\lambda)}$ itself.

For any self-adjoint relation $\tilde{A}$ in a Hilbert space $\mathfrak{G}$ $\mathrm{mul}\,(\tilde{A}) = \mathrm{dom}\,(\tilde{A})^\perp$. Therefore with $\mathfrak{G}_0 = \mathrm{mul}\,(\tilde{A})$ we can decompose $\tilde{A}$ into a self-adjoint operator and a multi-valued part as $\tilde{A} = \tilde{A}_o \oplus \tilde{A}_\infty$, where $\tilde{A}_o = \pi_{\mathfrak{G}_0^\perp} \tilde{A}|_{\mathfrak{G}_0^\perp}$ and $\tilde{A}_\infty = 0_{\mathfrak{G}_0} \times \mathrm{mul}\,(\tilde{A})$. For the self-adjoint operator $\tilde{A}_o$ we can use the spectral theorem, see [32].

Theorem 2.4.1. *Let S be a self-adjoint operator in $\mathcal{H}$, then there exists a family $E(l)$, $l \in \mathbb{R}$, of orthogonal projection operators on $\mathcal{H}$, called the resolution of the identity, satisfying*

(i) *$E(l)$ is increasing, i.e. $E(l_2) - E(l_1)$ is nonnegative for $l_1 \le l_2$;*

(ii) *$E(l)$ is right-continuous, i.e. $E(l)u \to E(l)_0 u$ as $l_0 < l \to l_0$, $\quad u \in \mathcal{H}$;*

(iii) $E(l) \to \begin{cases} 0 & \text{as } l \to -\infty, \\ I_{\mathcal{H}} & \text{as } l \to \infty; \end{cases}$

(iv) *$p(S) = \int_{\mathbb{R}} p(l)\, dE(l)$ for any polynomial $p(l)$.*

With $E_o(l)$, $l \in \mathbb{R}$, the resolution of the identity associated with $\tilde{A}_0$ it follows from the boundedness of $l \mapsto (l-\lambda)^{-1}$, $l \in \mathbb{R}$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$, that

$$(\tilde{A}_o - \lambda)^{-1} = \int_{\mathbb{R}} \frac{\mathrm{d}E_o(l)}{l - \lambda}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

but then

$$(\tilde{A} - \lambda)^{-1} = \int_{\mathbb{R}} \frac{\mathrm{d}E(l)}{l - \lambda}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

where

$$E(l)u = E_o(l)\pi_{\mathfrak{G}_0^\perp} u \oplus 0_{\mathfrak{G}_0}.$$

If we assume that the self-adjoint relation $\tilde{A}$ in the space $\mathfrak{G} = \mathfrak{H} \oplus \mathfrak{K}$ is an extension of a closed symmetric relation A in the space $\mathfrak{H}$, then, see the previous section, the compression of the resolvent of $\tilde{A}$ is a generalized resolvent, denoted by $R(\lambda)$, of A, thus

$$R(\lambda) = \pi_{\mathfrak{H}}(\tilde{A} - \lambda)^{-1}|_{\mathfrak{H}} = \int_{\mathbb{R}} \frac{\pi_{\mathfrak{H}} \mathrm{d}E(l)|_{\mathfrak{H}}}{l - \lambda}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

The family $F(l) = \pi_{\mathfrak{H}} E(l)|_{\mathfrak{H}}$, $l \in \mathbb{R}$, is a so-called generalized resolution of the identity. Conversely, given the resolvent of a closed symmetric relation the corresponding generalized resolution of the identity can be obtained via the Stieltjes-Livšic inversion formula, see [6]. For continuity points ν, μ of F the Stieltjes-Livšic inversion formula says that

$$([F(\nu) - F(\mu)]\, u, u)_{\mathfrak{H}} = \lim_{\epsilon \to 0} \frac{1}{\pi} \int_{\mu}^{\nu} \mathrm{Im}\,(R(\xi + i\epsilon) u, u)_{\mathfrak{H}} \mathrm{d}\xi, \quad u \in \mathfrak{H}. \tag{2.8}$$

If we now assume that $R(\lambda)$ has the following special form

$$R(\lambda) = G_{\lambda} + Y_{\lambda} Q(\lambda) Y_{\bar{\lambda}}^{*}, \tag{2.9}$$

where $Q(\lambda) \in B(\mathbb{C}^n)$ is a Nevanlinna function, $G_{\lambda} \in B(\mathfrak{H})$ is entire in λ and satisfies $G_{\bar{\lambda}} = G_{\lambda}^{*}$. Finally, $Y_{\lambda} \in B(\mathbb{C}^n, \mathfrak{H})$ is analytic on $\mathbb{C} \setminus \mathbb{R}$ and $Y_{\lambda} c \in \mathfrak{N}_{\lambda}(A^{*})$ for all $c \in \mathbb{C}^n$. Then the Stieltjes-Livšic inversion formula implies the following result, see [6].

Theorem 2.4.2. *Under the assumptions made in this section the following statements hold*

(i) $[F(\nu) - F(\mu)]\, f = \int_{\mu}^{\nu} Y_{\lambda} d\rho(\lambda) Y_{\lambda}^{*} f$ *for all* $f \in \mathfrak{H}$*;*

(ii) $Y_{\bar{\lambda}}^{*} f \in L_{\rho}^{2}$ *for all* $f \in \mathfrak{H}$*;*

(iii) $F(\infty) f = \int_{\mathbb{R}} Y_{\lambda} d\rho(\lambda) Y_{\lambda}^{*} f$*, where the integral converges in norm in* $\mathfrak{H}$*.*

Here the operator valued function $\rho(\lambda)$ defined as

$$\rho(\lambda) = \lim_{\epsilon \to 0+} \frac{1}{\pi} \int_{0}^{\lambda} \mathrm{Im}\, Q(\xi + i\epsilon) \mathrm{d}\xi$$

is non-decreasing and of bounded variation on any finite subinterval of $\mathbb{R}$. Note that the operator $F(\infty)$ is given by

$$F(\infty) : (\mathfrak{H} \cap \mathfrak{G}_0^{\perp}) \oplus (\mathfrak{H} \cap \mathfrak{G}_0) \to (\mathfrak{H} \cap \mathfrak{G}_0^{\perp}) \oplus (\mathfrak{H} \cap \mathfrak{G}_0), \quad \{f_1, f_2\} \mapsto \{f_1, 0\}.$$

Therefore Theorem 2.4.2 (iii) gives a decomposition for $f \in \mathfrak{H} \cap \mathfrak{G}_0^{\perp}$.

2.5 Intermediate extensions

Proposition 2.2.2 shows that all closed symmetric (or self-adjoint) extensions A_{Θ} of a closed symmetric relation A with boundary triplet $\{\mathbb{C}^n, \Gamma_0, \Gamma_1\}$ are characterized by closed symmetric (or self-adjoint) relations Θ in the boundary space $\mathbb{C}^n$. In this section we will determine a boundary triplet for the adjoint $A_{\Theta^{*}}$ of the closed extension A_Θ of A.

To determine the boundary triplet we will first give a characterization of the symmetric (or self-adjoint) relations Θ in the boundary space. Thereafter we will use a specific representation of Θ to construct an auxilliary boundary triplet for A^*. In the final step we will use this auxilliary boundary triplet to construct a boundary triplet for A_Θ.

Proposition 2.5.1. *Let A be a closed symmetric relation and let Θ be defined as $\Theta = \{\{-D^*g, C^*g\}, g \in \mathbb{C}^n\}$ for matrices C and D in $\mathbb{C}^n$. Then the following statements hold*

(i) Θ is symmetric if and only if $CD^* = DC^*$;

(ii) Θ is self-adjoint if and only if Θ is symmetric and $\operatorname{ran}\begin{pmatrix} C & D\end{pmatrix} = \mathbb{C}^n$.

Moreover, if Θ is self-adjoint its dimension is n.

Proof. Recall that with Θ as in statement of the proposition $\Theta^* = \ker\begin{pmatrix} C & D\end{pmatrix}$, see Proposition 2.3.8. Let $\{f, g\} \in \Theta$, then $\{f, g\} = \{-D^*x, C^*x\}$, $x \in \mathbb{C}^n$ and

$$\begin{pmatrix} C & D\end{pmatrix}\begin{pmatrix} -D^*x \\ C^*x\end{pmatrix} = \left[-CD^* + DC^*\right]x,$$

which shows that $\Theta \subset \Theta^*$ if and only if $CD^* = DC^*$.

Since we have already proven (i) to prove (ii) we only need to show that a symmetric relation Θ is sel-adjoint if and only if $\operatorname{ran}\begin{pmatrix} C & D\end{pmatrix} = \mathbb{C}^n$.

Assume that $\operatorname{ran}\begin{pmatrix} C & D\end{pmatrix} = \mathbb{C}^n$, then on the one hand the dimension of $\ker\begin{pmatrix} C & D\end{pmatrix}$, i.e. Θ^*, is n. On the other hand using Lemma 2.3.7 and the stated assumption we have that $\ker\begin{pmatrix} C & D\end{pmatrix}^* = \operatorname{ran}\begin{pmatrix} C & D\end{pmatrix}^\perp = \{0\}$. But if $\ker\begin{pmatrix} C & D\end{pmatrix}^* = \{0\}$, then also $\ker C^* \cap \ker D^* = \{0\}$, i.e. Θ has dimension n. Thus Θ and Θ^* have equal dimension and since we assumed that $\Theta \subset \Theta^*$, we conclude that $\Theta = \Theta^*$.

Next we prove the converse implication, i.e. we prove that if Θ is self-adjoint then $\operatorname{ran}\begin{pmatrix} C & D\end{pmatrix} = \mathbb{C}^n$. Assume the contrary, i.e. let $0 \neq x \in \operatorname{ran}\begin{pmatrix} C & D\end{pmatrix}^\perp$, then by the previous reasoning $x \in \ker C^* \cap \ker D^*$, from which it follows that the dimension of Θ is strictly smaller than n. On the other hand the dimension of $\ker\begin{pmatrix} C & D\end{pmatrix}$, i.e. Θ^*, is clearly strictly bigger than n, because $\operatorname{ran}\begin{pmatrix} C & D\end{pmatrix} \subsetneq \mathbb{C}^n$, a contradiction which proves (ii). $\square$

For any invertible operator T in $\mathbb{C}^n$ the symmetric relation Θ in $\mathbb{C}^n$ can be written as

$$\begin{aligned}\Theta &= \{\{-D^*g, C^*g\}, g \in \mathbb{C}^n\} = \{\{-D^*T^*g, C^*T^*g\}, g \in \mathbb{C}^n\} \\ &= \{\{-(TD)^*g, (TC)^*g\}, g \in \mathbb{C}^n\}.\end{aligned}$$

Therefore it is no restriction to assume that the representation $\{\{-D^*g, C^*g\}, g \in \mathbb{C}^n\}$ of Θ is such that

$$\begin{pmatrix} C & D \end{pmatrix} = \begin{pmatrix} \tilde{C} & \tilde{D} \\ 0 & 0 \end{pmatrix},$$

where $\begin{pmatrix} \tilde{C} & \tilde{D} \end{pmatrix}$ has full rank and its row are orthonormal. In the remainder of this section we will assume that Θ is in this specific form.

If the dimension of Θ, denoted by k, is strictly smaller than n, then to the above matrix there corresponds a closed symmetric extension A_Θ of A by Proposition 2.2.2. Let $e^*_{\alpha_1}, \dots, \mathrm{e}^*_{\alpha_{n-k}}$, $e_{\alpha_1} \in \mathbb{C}^{2n}$ and $\alpha_i \in \{0, \dots, n\}$, be orthogonal to the rows of $\begin{pmatrix} C & D \end{pmatrix}$, such e_{α_1} exist because the dimension of Θ is smaller than n, and define

$$\begin{pmatrix} \breve{C} & \breve{D} \end{pmatrix} = \begin{pmatrix} e^*_{\alpha_1} \\ \vdots \\ e^*_{\alpha_{n-k}} \end{pmatrix}.$$

Then

$$\begin{pmatrix} \hat{C} & \hat{D} \end{pmatrix} = \begin{pmatrix} \tilde{C} & \tilde{D} \\ \breve{C} & \breve{D} \end{pmatrix}$$

is such that $\hat{\Theta} = \{\{-\hat{D}^*g, \hat{C}^*g\}, g \in \mathbb{C}^n\}$ is self-adjoint and the rows of $\begin{pmatrix} \hat{C} & \hat{D} \end{pmatrix}$ are orthonormal. Now define the matrix W as

$$W = \begin{pmatrix} \hat{C} & \hat{D} \\ -\hat{D} & \hat{C} \end{pmatrix},$$

then

$$WW^* = \begin{pmatrix} \hat{C} & \hat{D} \\ -\hat{D} & \hat{C} \end{pmatrix} \begin{pmatrix} \hat{C}^* & -\hat{D}^* \\ \hat{D}^* & \hat{C}^* \end{pmatrix} = \begin{pmatrix} \hat{C}\hat{C}^* + \hat{D}\hat{D}^* & -\hat{C}\hat{D}^* + \hat{D}\hat{C}^* \\ -\hat{D}\hat{C}^* + \hat{C}\hat{D}^* & \hat{D}\hat{D}^* + \hat{C}\hat{C}^* \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ 0 & I_n \end{pmatrix},$$

which shows that W is unitairy, because W is a finite dimensional linear operator. Moreover,

$$\begin{aligned} W \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} W^* &= \begin{pmatrix} \hat{C} & \hat{D} \\ -\hat{D} & \hat{C} \end{pmatrix} \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \begin{pmatrix} \hat{C}^* & -\hat{D}^* \\ \hat{D}^* & \hat{C}^* \end{pmatrix} = \begin{pmatrix} \hat{D} & -\hat{C} \\ \hat{C} & \hat{D} \end{pmatrix} \begin{pmatrix} \hat{C}^* & -\hat{D}^* \\ \hat{D}^* & \hat{C}^* \end{pmatrix} \\ &= \begin{pmatrix} \hat{D}\hat{C}^* - \hat{C}\hat{D}^* & -\hat{D}\hat{D}^* - \hat{C}\hat{C}^* \\ \hat{C}\hat{C}^* + \hat{D}\hat{D}^* & -\hat{C}\hat{D}^* + \hat{D}\hat{C}^* \end{pmatrix} = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}. \end{aligned}$$

Therefore

$$\begin{pmatrix} C & D \end{pmatrix} \Gamma\{f, g\} = \begin{pmatrix} \tilde{C} & \tilde{D} \\ 0 & 0 \end{pmatrix} W^*W\Gamma\{f, g\} = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix} \Gamma_W\{f, g\}, \tag{2.10}$$

where Γ_W is a boundary mapping for A^* by Proposition 2.2.7. Its associated γ-field and Weyl function will be denoted by $\gamma_{\lambda,W}$ and $M_W(\lambda)$. We recall that by Proposition 2.3.8 we can write A_{Θ^*} as

$$A_{\Theta^*} = \{\{f,g\} \in A^* : \begin{pmatrix} C & D \end{pmatrix} \Gamma\{f,g\} = 0\}.$$

If we combine the above formula for A_{Θ^*} with (2.10) we obtain the following characterization of A_{Θ^*}

$$A_{\Theta^*} = \{\{f,g\} \in A^* : [\Gamma_W\{f,g\}]_i = 0,\ 1 \le i \le k\}. \tag{2.11}$$

Using the above representation for A_{Θ^*} straight-forward calculations show that A_Θ is given by

$$A_\Theta = \{\{f,g\} \in A^* : [\Gamma_W\{f,g\}]_i = 0,\ 1 \le i \le n,\ n+k < i \le 2n\}. \tag{2.12}$$

Finally, define the $n \times (n-k)$ matrix R as

$$R = \begin{pmatrix} e_{k+1} & \dots & e_n \end{pmatrix},$$

then $R^*R = I_{n-k}$. Then with the introduced notation we have the following result.

Proposition 2.5.2. *Let A be a closed symmetric relation with defect numbers (n,n) and let Θ be a symmetric relation in $\mathbb{C}^n$ of dimension k. Furthermore, let R, Γ_W, $\gamma_{\lambda,W}$ and $M_W(\lambda)$ be as above. Then A_Θ is a closed symmetric relation with defect numbers $(n-k,n-k)$ and its adjoint is A_{Θ^*}. A boundary triplet $\{n-k, \Gamma_{\Theta,0}, \Gamma_{\Theta,1}\}$ for A_{Θ^*} is given by*

(2.13)

$$\Gamma_{\Theta,0}\{f,g\} := R^*\Gamma_{W,0}\{f,g\} \quad \textit{and} \quad \Gamma_{\Theta,1}\{f,g\} := R^*\Gamma_{W,1}\{f,g\}, \quad \{f,g\} \in A_{\Theta^*}.$$

Furthermore, $A_{\Theta,0} := \ker \Gamma_{\Theta,0} = \ker \Gamma_{W,0}$ is a self-adjoint extension of A_Θ. The γ-field and Weyl function associated with the boundary mappings in (2.13) are

$$\gamma_{\lambda,\Theta} = \gamma_{\lambda,W} R \quad \textit{and} \quad M_\Theta(\lambda) = R^* M_W(\lambda) R,$$

for $\lambda \in \rho(A_{\Theta,0})$.

Proof. To prove the first part of this statement we use Theorem 2.2.4. Since $A_{\Theta,0} = \ker \Gamma_{W,0}$, $A_{\Theta,0}$ is self-adjoint by Proposition 2.2.2. The surjectivity of Γ_Θ is a direct consequence of the surjectivity of Γ_W combined with the fact that $\operatorname{ran} R^* = \mathbb{C}^{n-k}$. Finally, note that for any $\{f,g\} \in A_{\Theta^*}$ and $z \in \mathbb{C}^n$ $(\Gamma_{W,0}\{f,g\}, z) = (R^*\Gamma_{W,0}\{f,g\}, R^*z\}) = (\Gamma_{\Theta,0}\{f,g\}, R^*z\})$, because $[\Gamma_{W,0}\{f,g\}]_i = 0$, $1 \le i \le k$, for $\{f,g\} \in A_{\Theta^*}$, see (2.11). With this observation the Green's identity for Γ_Θ is a direct consequence of the Green's identity for Γ_W.

Finally, we show that $\gamma_{\lambda,\Theta}$ and $M_\Theta(\lambda)$ are the γ-field and Weyl function associated with the boundary mappings (2.13). Let $g \in \mathbb{C}^{n-k}$, then

$$\Gamma_{W,0}\hat{\gamma}_{\lambda,\Theta} g = \Gamma_{W,0}\hat{\gamma}_{\lambda,W} R g = Rg = \begin{pmatrix} 0 \\ g \end{pmatrix},$$

which shows that $\gamma_{\lambda,\Theta} g \in \operatorname{dom}(A_{\Theta^*})$, see (2.11). Multiplying the above result from the left with R^* shows that $\Gamma_{\Theta,0}\hat{\gamma}_{\lambda,\Theta} g = g$ for all $g \in \mathbb{C}^{n-k}$. Combined these two result prove that $\gamma_{\lambda,\Theta}$ is the γ-field. $M_\Theta(\lambda)$ is by definition, see Proposition 2.2.5, given by

$$\begin{aligned} M_\Theta(\lambda) &= \{\{\Gamma_{\Theta,0}\hat{\gamma}_{\lambda,\Theta} g, \Gamma_{\Theta,1}\hat{\gamma}_{\lambda,\Theta} g\} : g \in \mathbb{C}^{n-k}\} \\ &= \{\{g, R^*\Gamma_{W,1}\hat{\gamma}_{\lambda,W} Rg\} : g \in \mathbb{C}^{n-k}\} \\ &= \{\{g, R^* M(\lambda) Rg\} : g \in \mathbb{C}^{n-k}\} = R^* M(\lambda) R. \end{aligned}$$

□

2.6 Finite-dimensional graph restrictions

In this section we will investigate finite-dimensional graph restrictions S of a given a closed symmetric relation A in $\mathfrak{H}$ with finite and equal defect numbers, in particular we will determine a boundary tiplet for S^*. Here a relation S is a finite-dimensional graph restriction of A if there exist a finite-dimensional subspace Z of $\mathfrak{H}^2$ such that $S = A \cap Z^*$.

To determine a boundary triplet for S^* we will start by determining the structure of S^*. Thereafter we will show that it is no restriction to assume that A is self-adjoint and Z symmetric. Finally, using the specific structure of S^* and Z we will be able to write down a boundary triplet for S^*.

Proposition 2.6.1. *Let A be a closed symmetric relation in the Hilbert space $\mathfrak{H}$ and Z be a finite-dimensional subspace of $\mathfrak{H}^2$ such that $A^* \cap Z = \{0, 0\}$. Then the adjoint of $S = A \cap Z^*$ is given by*

$$S^* = A^* \dot{\hat{+}} Z.$$

Proof. Note that we only need to prove that $A^* \hat{+} Z = S^*$, because by assumption A^* and Z are disjoint as sets. We start by proving that $A^* \hat{+} Z \subseteq S^*$. Let $\{f, g\} \in A^* \hat{+} Z$, i.e. $\{f, g\} = \{f_0, g_0\} + \{\sigma, \tau\}$, where $\{f_0, g_0\} \in A^*$ and $\{\sigma, \tau\} \in Z$. For an arbitrary element $\{h, k\}$ of $S = A \cap Z^*$ we have that

$$(g, h) - (f, k) = (g_0 + \tau, h) - (f_0 + \sigma, k) = [(g_0, h) - (f_0, k)] + [(\tau, h) - (\sigma, k)] = 0,$$

which proves that $A^* \hat{+} Z \subseteq S^*$.

Next we will prove that $S^* \subseteq A^* \hat{+} Z$ by proving the equivalent inclusion $(A^* \hat{+} Z)^* \subseteq S = A \cap Z^*$. These two statements are equivalent because S and $A^* \hat{+} Z$ are closed. Let $\{f, g\} \in (A^* \hat{+} Z)^*$ and let $\{h, k\}$ be an arbitrary element of $A^* \hat{+} Z$ be arbitrary. Then $\{h, k\} = \{h_0, k_0\} + \{\sigma, \tau\}$, where $\{h_0, k_0\} \in A^*$ and $\{\sigma, \tau\} \in Z$ and we have that

$$0 = (g, h) - (f, k) = (g, h_0 + \sigma) - (f, k_0 + \tau) = [(g, h_0) - (f, k_0)] + [(g, \sigma) - (f, \tau)].$$

Since the above equality holds for all $\{h,k\} \in A^*\widehat{+}Z$, $\{h_0,k_0\} = \{0,0\}$ shows that $\{f,g\} \in Z^*$ and $\{\sigma,\tau\} = \{0,0\}$ shows that $\{f,g\} \in A^{**} = A$. We conclude that $\{f,g\} \in A \cap Z^* = S$, which proves the inclusion $S^* \subseteq A^*\widehat{+}Z$ and thereby the proposition. □

Henceforth we will only consider finite-dimensional graph restrictions of self-adjoint relations. This is no restriction as we will proceed to show. A closed symmetric relation A with equal and finite defect numbers has self-adjoint extensions, see Proposition 2.2.2, let $\tilde{A}$ be one such a self-adjoint extension. By Lemma 2.2.3 $A^* = \tilde{A}\dot{\widehat{+}}\widehat{\mathfrak{N}}_\lambda(A^*)$ for each $\lambda \in \rho(\tilde{A})$, thus if we investigate graph restrictions of A by a finite-dimensional subspace Z, we can equivalently investigate the graph restriction of $\tilde{A}$ by $Z \cup \widehat{\mathfrak{N}}_\lambda(A^*)$, $\lambda \in \rho(\tilde{A})$.

Two finite-dimensional subspace Z and $\tilde{Z}$ of $\mathfrak{H}^2$ are called equivalent (with respect to $\tilde{A}$) if

$$\tilde{A} \cap Z = \{0,0\} = \tilde{A} \cap \tilde{Z} \quad \text{and} \quad \tilde{A}\widehat{+}Z = \tilde{A}\widehat{+}\tilde{Z}.$$

The following result shows that it is no restriction to assume that Z is symmetric, see [22].

Lemma 2.6.2. *Let $\tilde{A}$ be a self-adjoint relation in a Hilbert space $\mathfrak{H}$ and let Z be a n-dimensional subspace of $\mathfrak{H}^2$ satisfying $\tilde{A} \cap Z = \{0,0\}$. Then there exists a n-dimensional symmetric subspace $\tilde{Z}$ of $\mathfrak{H}^2$, which is equivalent to Z.*

For a n-dimensional symmetric subspace Z of $\mathfrak{H}^2$, $\tilde{A} \cap Z = \{0,0\}$, fix a basis $\{\{\varphi_1,\psi_1\},\dots,\{\varphi_n,\psi_n\}\}$. Then each element of $\tilde{A}\widehat{+}Z$ can uniquely be written as

$$\{f,g\} = \{f_0,g_0\} + \{\varphi,\psi\}c, \quad \{f_0,g_0\} \in \tilde{A} \quad \text{and} \quad c \in \mathbb{C}^n.$$

Here $\varphi = \begin{pmatrix}\varphi_1 & \dots & \varphi_n\end{pmatrix}$, $\psi = \begin{pmatrix}\psi_1 & \dots & \psi_n\end{pmatrix}$ and $\{\varphi,\psi\}c = \{\varphi c, \psi c\}$. The following theorem gives a boundary triplet for $\tilde{A}\dot{\widehat{+}}Z$ and its associated γ-field and Weyl function, see [22].

Theorem 2.6.3. *Let $\tilde{A}$ be a self-adjoint relation in a Hilbert space $\mathfrak{H}$. Assume that Z is as above, then $S = \tilde{A} \cap Z^*$ is a closed symmetric relation with defect numbers (n,n) and $S^* = \tilde{A}\dot{\widehat{+}}Z$. A boundary triplet $\{\mathbb{C}^n, \Gamma_0, \Gamma_1\}$ for S^* is given by*

$$\Gamma_0\{f,g\} = c \quad \textit{and} \quad \Gamma_1\{f,g\} = < \{f_0,g_0\}, \{\varphi,\psi\} >_{\mathfrak{H}^2}.$$

Here $\{f,g\} = \{f_0,g_0\} + \{\varphi,\psi\}c \in S^*$, where $\{f_0,g_0\} \in \tilde{A}$ and $\{\varphi,\psi\}c \in Z$, $c \in \mathbb{C}^n$. Futhermore, $\ker \Gamma_0 = \tilde{A}$, $\ker \Gamma_1 = S\widehat{+}Z$ *and the γ-field and Weyl function associated with the above boundary triplet are*

$$\gamma_\lambda = \varphi + (\tilde{A} - \lambda)^{-1}(\lambda\varphi - \psi)$$

and

$$M(\lambda) = (\gamma_\lambda(\cdot), \bar{\lambda}\varphi - \psi)_{\mathfrak{H}}$$

for $\lambda \in \rho(\tilde{A})$. Here $\gamma_\lambda(\cdot) = \begin{pmatrix}\gamma_\lambda e_1 & \dots & \gamma_\lambda e_n\end{pmatrix}$.

Chapter 3: General results concerning a system of differential equations

3.1 A system of differential equations

Consider the following system of differential equations in $\mathbb{C}^n$

$$Jf'(t) - [H(t) + \lambda\Delta(t)]\, f(t) = \Delta(t)g(t), \quad t \in (a,b), \tag{3.1}$$

where $-\infty \leq a < b \leq \infty$. Assume that the above system of differential equations satisfies the following conditions

(C1) J is a $n \times n$ matrix such that $J^* = J^{-1} = -J$;

(C2) $H(t)$ and $\Delta(t)$ are locally absolutely integrable $\mathbb{C}^{n\times n}$-valued functions on (a,b) such that $H(t) = H(t)^*$ and $\Delta(t) = \Delta(t)^*$ a.e. on (a,b).

The endpoint a (or b) is called regular if a (or b) is finite and Δ and H are absolutely integrable up to a (or b). An endpoint which is not regular is called singular.

The following result will be fundamental for our investigation, a proof of the statement can be found in the appendix, see [7].

Theorem 3.1.1. *Let $g \in \mathcal{F}^n((a,b))$ be such that $\Delta g \in L^1_{loc}((a,b))$ and let ξ be an analytic function with values in $\mathbb{C}^n$. Then the initial value problem*

$$Jf'(t) - [\lambda\Delta(t) + H(t)]\, f(t) = \Delta(t)g(t), \quad f(\tau) = \xi(\lambda), \quad \lambda \in \mathbb{C} \quad \textit{and} \quad t,\tau \in (a,b),$$

has an unique locally absolutely continuous solution, which depends analytically on λ. If the system is regular at a (or b) and Δg is absolutely integrable up to a (or b), then $\tau = a$ (or $\tau = b$) is allowed and the solution is locally absolutely continuous on $[a,b)$ (or $(a,b]$).

Corollary 3.1.2. *For each $c \in (a,b)$ there exists an unique linear operator $Y_\lambda : \mathbb{C}^n \to AC^n_{loc}((a,b))$, $z \mapsto Y_\lambda(\cdot)z$, such that*

(i) For all $z \in \mathbb{C}^n$ $Y_\lambda(\cdot)z$ is the unique locally absolutely continuous solution of

$$Jf'(t) = [H(t) + \lambda\Delta(t)]\, f(t), \quad f(c) = z \quad \textit{and} \quad t \in (a,b); \tag{3.2}$$

(ii) $Y_\lambda(\cdot)z$ is entire in λ for all $z \in \mathbb{C}^n$.

$Y_\lambda(\cdot) \in AC^{n\times n}_{loc}((a,b))$, the matrix representation of Y_λ, will be called the fundamental matrix of (3.1). If the endpoint a (or b) is regular the operator Y_λ exist for $c \in [a,b)$ (or $c \in (a,b]$). Finally, we define $\hat{Y}_\lambda$ by $\hat{Y}_\lambda z = \{Y_\lambda z, \lambda Y_\lambda z\}$ for $z \in \mathbb{C}^n$.

The fundamental matrix $Y_\lambda(\cdot)$ has the following elementary properties.

Lemma 3.1.3. *Let $Y_\lambda(\cdot)$ be the fundamental matrix of* (3.1)*, see Corollary 3.1.2. Then the following statements hold*

(i) $Y_l(t)^*JY_\lambda(t) - J = (\lambda - \bar{l})\int_c^t Y_l(s)^*\Delta(s)Y_\lambda(s)ds, \quad l,\lambda \in \mathbb{C}$ *and* $t \in (a,b)$*;*

(ii) $Y_{\bar{\lambda}}(t)^*JY_\lambda(t) = J = Y_\lambda(t)JY_{\bar{\lambda}}(t)^*, \quad \lambda \in \mathbb{C}$ *and* $t \in (a,b)$*;*

(iii) $Y_\lambda(t)$ *is invertible for* $\lambda \in \mathbb{C}$ *and* $t \in (a,b)$*.*

Proof. To prove (i) note that by definition $Y_\lambda(\cdot)$ satisfies

$$JY_\lambda'(s) = [H(s) + \lambda\Delta(s)]\,Y_\lambda(s), \quad s \in (a,b), \tag{3.3}$$

with initial condition $Y_\lambda(c) = I_n$. Therefore using partial integration for absolutely continuous functions, see [18], we have that

$$\begin{aligned}
&\lambda\int_c^t Y_l(s)^*\Delta(s)Y_\lambda(s)\mathrm{d}s = \int_c^t Y_l(s)^*JY_\lambda'(s)\mathrm{d}s - \int_c^t Y_l(s)^*H(s)Y_\lambda(s)\mathrm{d}s \\
=&[Y_l(s)^*JY_\lambda(s)]_c^t - \int_c^t Y_l'(s)^*JY_\lambda(s)\mathrm{d}s - \int_c^t Y_l(s)^*H(s)^*Y_\lambda(s)\mathrm{d}s \\
=&Y_l(t)^*JY_\lambda(t) - J + \int_c^t \left[JY_l'(s) - H(s)Y_l(s)\right]^* Y_\lambda(s)\mathrm{d}s \\
=&Y_l(t)^*JY_\lambda(t) - J + \int_c^t \left[l\Delta(s)Y_l(s)\right]^* Y_\lambda(s)\mathrm{d}s \\
=&Y_l(t)^*JY_\lambda(t) - J + \bar{l}\int_c^t Y_l(s)^*\Delta(s)Y_\lambda(s)\mathrm{d}s,
\end{aligned}$$

which proves (i). The first equality of (ii) follows directly from (i) by taking l as $\bar{\lambda}$. That equality indicates that $Y_\lambda(t)$ has a trivial kernel for every $t \in (a,b)$, because J is invertible. We conclude that $Y_\lambda(t)$, $t \in (a,b)$, is invertible proving (iii). The second equality of (ii) is obtained by multiplying the first equality of (ii) by $Y_\lambda(t)J$ from the left by which, after rearrangement, the following expression is obtained

$$\left[Y_\lambda(t)JY_{\bar{\lambda}}(t)^* - I_n\right]JY_\lambda(t) = 0.$$

Since J and $Y_\lambda(t)$, $t \in (a,b)$, are invertible the above equality proves the second equality of (ii). ☐

The fundamental matrix $Y_\lambda(\cdot)$ can be used to make the solutions of the system of differential equations (3.1) explicit.

Proposition 3.1.4. *Let $Y_\lambda(\cdot)$ be the fundamental matrix. Then for all $g \in \mathcal{F}^n((a,b))$ for which*

$$(G_\lambda g)(t) = \frac{Y_\lambda(t)J}{2}\int_a^b \operatorname{sgn}(s-t)Y_{\bar{\lambda}}(s)^*\Delta(s)g(s)\,ds$$

exists for all $t \in (a,b)$ and all $\lambda \in \mathbb{C}$, the following statements hold *(i)* $(G_\lambda g)(t)$ *is a locally absolutely continuous solution of*

$$Jf'(t) - [\lambda\Delta(t) + H(t)]\, f(t) = \Delta(t)g(t), \quad t \in (a,b),$$

which is entire in λ*;*

(ii) The unique locally absolutely continuous solution of the initial value problem

$$Jf'(t) - [\lambda\Delta(t) + H(t)]\, f(t) = \Delta(t)g(t), \quad f(\tau) = \xi, \quad t, \tau \in (a,b),$$

which is entire in λ*, is given by*

$$f(t) = Y_\lambda(t)c_\lambda + (G_\lambda g)(t), \tag{3.4}$$

where

$$c_\lambda = Y_\lambda(\tau)^{-1}\xi - \int_a^b \frac{\operatorname{sgn}(s-\tau)}{2} J Y_{\bar\lambda}(s)^* \Delta(s) g(s)\, ds. \tag{3.5}$$

Proof. We start by proving that $(G_\lambda g)(\cdot)$ is a solution of the inhomogeneous differential equation (3.1). Since $(G_\lambda g)(t)$ exists for all $t \in (a,b)$ we have that

$$\int_a^b \operatorname{sgn}(s-t) Y_{\bar\lambda}(s)\Delta(s)g(s)\mathrm{d}s$$

exist for all $t \in (a,b)$, i.e. the above function is locally absolutely continuous on (a,b). Since $\frac{Y_\lambda(\cdot)J}{2}$ is also locally absolutely continuous on (a,b), see Corollary 3.1.2, their product $G_\lambda g$ is locally absolutely continuous on (a,b). Therefore the product rule for absolutely continuous functions, see [18], implies that

$$\begin{aligned}
J\left[(G_\lambda g)(t)\right]' &= J\left[Y_\lambda(t)\frac{J}{2}\int_t^b Y_{\bar\lambda}(s)^*\Delta(s)g(s)\mathrm{d}s - Y_\lambda(t)\frac{J}{2}\int_a^t Y_{\bar\lambda}(s)^*\Delta(s)g(s)\mathrm{d}s\right]' \\
&= JY_\lambda'(t)\frac{J}{2}\int_t^b Y_{\bar\lambda}(s)^*\Delta(s)g(s)\mathrm{d}s - JY_\lambda'(t)\frac{J}{2}\int_a^t JY_{\bar\lambda}(s)^*\Delta(s)g(s)\mathrm{d}s \\
&\quad - JY_\lambda(t)\frac{J}{2}Y_{\bar\lambda}(t)^*\Delta(t)g(t) - JY_\lambda(t)\frac{J}{2}Y_{\bar\lambda}(t)^*\Delta(t)g(t) \\
&= JY_\lambda'(t)\frac{J}{2}\int_a^b \operatorname{sgn}(s-t)Y_{\bar\lambda}(s)^*\Delta(s)g(s)\mathrm{d}s - JY_\lambda(t)JY_{\bar\lambda}(t)^*\Delta(t)g(t) \\
&= \left[H(t) + \lambda\Delta(t)\right] Y_\lambda(t)\frac{J}{2}\int_a^b \operatorname{sgn}(s-t)Y_{\bar\lambda}(s)^*\Delta(s)g(s)\mathrm{d}s + \Delta(t)g(t) \\
&= \left[H(t) + \lambda\Delta(t)\right](G_\lambda g)(t) + \Delta(t)g(t).
\end{aligned}$$

In the above calculations (ii) of Lemma 3.1.3 was used. Here as well as in (ii) the solution is entire in λ by Theorem 3.1.1.

To prove (ii) we first note that c_λ is well-defined by (iii) of Lemma 3.1.3. Since $Y_\lambda(\cdot)c_\lambda$ is a solution of (3.2) for all λ, it follows from (i) that (3.4) is a solution of (3.1). Finally, we show that f as defined in (3.4) satisfies the stated initial condition;

$$\begin{aligned} f(\tau) &= Y_\lambda(\tau)\left[Y_\lambda(\tau)^{-1}\xi - \int_a^b \frac{\operatorname{sgn}(s-\tau)}{2} J Y_{\bar\lambda}(s,)^* \Delta(s) g(s)\mathrm{d}s\right] + (G_\lambda g)(\tau) \\ &= \xi - Y_\lambda(\tau)\int_a^b \frac{\operatorname{sgn}(s-\tau)}{2} J Y_{\bar\lambda}(s)^* \Delta(s) g(s)\mathrm{d}s + (G_\lambda g)(\tau) = \xi. \end{aligned}$$

The uniqueness of the solution f is a consequence of Theorem 3.1.1, because f as in (3.4) is locally absolutely continuous on (a,b) by (i) combined with Corollary 3.1.2. □

3.2 Canonical systems of differential equations

If in addition to the conditions (C1) and (C2) the system of differential equations in (3.1) is assumed to satisfy the following two conditions

(C3) $\Delta(t) \geq 0$ a.e. on (a,b);

(C4) The system is definite, which means that if $f \in \mathrm{AC}^n_{\mathrm{loc}}((a,b))$ is such that $Jf'(t) = H(t)f(t)$ a.e. on (a,b) and $\Delta(t)f(t) = 0$ a.e. on (a,b), then $f(t) = 0$ on (a,b).

Then the system of differential equations is called a canonical system of differential equations or, abriviated, a canonical system. The non-negativity of Δ allows us to define the following inner product

$$(f,g)_{\mathfrak{H}} = \int_a^b \tilde g(t)^* \Delta(t) \tilde f(t)\mathrm{d}t, \quad \tilde f \in f, \tilde g \in g, \tag{3.6}$$

on the space of equivalence classes consisting of $\mathbb{C}^n$-valued measurable functions on (a,b). Here two functions f_1 and f_2 are in the same equivalence class if $\Delta(f_1 - f_2) = 0$ a.e. on (a,b). The space of equivalence classes which have finite norm with respect to the norm induced by the inner product $(\cdot,\cdot)_{\mathfrak{H}}$ is denoted by $\mathfrak{H} = L^2_\Delta((a,b))$.

A canonical system can always be reduced to a canonical system where $H = 0$, see [26]. Namely by the substitution $f(t) = Y_0(t)\hat f(t)$ and $g(t) = Y_0(t)\hat g(t)$, because then

$$Jf'(t) = J\left[Y_0(t)\hat f(t)\right]' = JY_0'(t)\hat f(t) + JY_0(t)\hat f'(t) = H(t)Y_0(t) + [Y_0(t)^*]^{-1} J\hat f'(t),$$

where we used (ii) of Lemma 3.1.3, and

$$[H(t) + \lambda\Delta(t)]\, f(t) + \Delta(t)g(t) = [H(t) + \lambda\Delta(t)]\, Y_0(t)\hat f(t) + \Delta(t)Y_0(t)\hat g(t).$$

Using the above two result in (3.1), we obtain

$$[Y_0(t)^*]^{-1} J\hat f'(t) = \lambda\Delta(t)Y_0(t)\hat f(t) + \Delta(t)Y_0(t)\hat g(t),$$

or equivalently

$$J\hat{f}'(t) = \lambda Y_0(t)^*\Delta(t)Y_0(t)\hat{f}(t) + Y_0(t)^*\Delta(t)Y_0(t)\hat{g}(t) = \lambda\hat{\Delta}(t)\hat{f}(t) + \hat{\Delta}(t)\hat{g}(t). \tag{3.7}$$

Since $\hat{\Delta} = Y_0^*\Delta Y_0$ is nonnegative a.e. on (a,b) the above system of differential equations can be considered in the space $\hat{\mathfrak{H}} = L^2_{\hat{\Delta}}((a,b))$. The mapping Y_0 is a unitairy mapping between $\hat{\mathfrak{H}}$ and $\mathfrak{H}$, because for $\hat{f} \in \hat{\mathfrak{H}}$

$$\begin{aligned}||Y_0\hat{f}||^2_{\mathfrak{H}} &= \int_a^b \left[Y_0(t)\hat{f}(t)\right]^* \Delta(t) \left[Y_0(t)\hat{f}(t)\right] \mathrm{d}t \\ &= \int_a^b \hat{f}(t)^* \left[Y_0(t)^*\Delta(t)Y_0(t)\right] \hat{f}(t)\mathrm{d}t = \int_a^b \hat{f}(t)^*\hat{\Delta}(t)\hat{f}(t)\mathrm{d}t = ||\hat{f}||^2_{\hat{\mathfrak{H}}},\end{aligned}$$

which shows that $Y_0\hat{f} \in \mathfrak{H}$. Conversely, let $f \in \mathfrak{H}$ and define $\hat{f} = Y_0^{-1}f$, then this function is well-defined by Lemma 3.1.3 (iii) and by the above calculation $||\hat{f}||_{\hat{\mathfrak{H}}} = ||f||_{\mathfrak{H}}$, i.e. $\hat{f} \in \hat{\mathfrak{H}}$. This proves that Y_0 is an unitairy mapping.

Note that if we study a system (3.1) where $H = 0$, then the system is definite if any constant vector $z \in \mathbb{C}^n$ for which $\Delta z = 0$ a.e. on (a,b) is zero.

Example 3.2.1. With $a < c < d < b$ consider the canonical system where Δ is positive on $c < t < d$ and zero on $(a,c] \cup [d,b)$. If $f \in \mathrm{AC}^n_{\mathrm{loc}}((a,b))$ is a solution of $Jf' = Hf$ with $\Delta f = 0$ on (a,b), then certainly $\Delta f = 0$ on (c,d), which by the invertibility of Δ on $c < t < d$ means that $f = 0$ on (c,d). But then $f \in \mathrm{AC}^n_{\mathrm{loc}}((a,b))$ is a solution of $Jf' = Hf$ on the interval (a,c) with $f(c) = 0$, therefore $f = 0$ on (a,c) by Theorem 3.1.1. The same arguments show that $f = 0$ on (d,b), thus $f = 0$ on (a,b) and we conclude that the system is definite.

Using the norm induced by the inner product (3.6) equivalent characterizations of the definiteness property can be given, see [24].

Proposition 3.2.2. *The following statements are equivalent to the definiteness property*

(i) *If $f \in AC^n_{loc}((a,b))$ satisfies $Jf'(t) = [H(t) + \lambda\Delta(t)]\, f(t)$ a.e. on (a,b) and $||f||_{\mathfrak{H}} = 0$, then $f(t) = 0$ on (a,b);*

(ii) *There exists a closed interval $[\alpha,\beta] \subseteq (a,b)$ such that if $f \in AC^n_{loc}((a,b))$ satisfies $Jf'(t) = H(t)f(t)$ on a.e. (a,b) and $\int_\alpha^\beta f(t)^*\Delta(t)f(t)dt = 0$, then $f(t) = 0$ on (a,b).*

Proof. Let $f \in \mathrm{AC}^n_{\mathrm{loc}}((a,b))$ satisfy $Jf' = [H + \lambda\Delta]\, f$ a.e. on (a,b) and $||f||^2_{\mathfrak{H}} = 0$. Then by the latter condition $\Delta f = 0$ a.e. on (a,b). Thus $f \in \mathrm{AC}^n_{\mathrm{loc}}((a,b))$ is such that $Jf' = Hf$ and $\Delta f = 0$ a.e. on (a,b), which by the definition of definiteness implies that $f = 0$ on (a,b).

Next we show that (i) implies (ii). Introduce for each compact subinterval ι of (a,b) the set

$$d(\iota) = \{z \in \mathbb{C}^n : ||z|| = 1, \quad \int_\iota [Y_0(t)z]^* \, \Delta(t) \, [Y_0(t)z] \, \mathrm{d}t = 0\}.$$

Then $d(\iota)$ is a compact subset of the unit ball in $\mathbb{C}^n$, because the inner product is continuous, and $\iota_1 \subseteq \iota_2$ implies that $d(\iota_1) \subseteq d(\iota_2)$, because Δ is nonnegative.

Choose an increasing sequence of compact intervals $(\iota_n)_{n\geq 0}$ such that $\cup_{n\geq 0}\iota_n = (a,b)$, then $(d(\iota_n))_{n\geq 0}$ is a decreasing sequence of compact intervals and $\cap_{n\geq 0} d(\iota_n) = \emptyset$. Because otherwise there would exist a $z \neq 0$ such that

$$\int_{\iota_n} [Y_0(t)z]^* \, \Delta(t) \, [Y_0(t)z] \, \mathrm{d}t = 0,$$

for every n and hence $||Y_0 z||_{\mathfrak{H}} = 0$. Since $Y_0(\cdot)z$ is a solution of $Jf' = Hf$ on (a,b) this would imply that $Y_0(\cdot)z = 0$ on (a,b) by (i). But $Y_0(\cdot)z = 0$ on (a,b) implies that $z = 0$, because $Y_0(\cdot)$ is invertible on (a,b), a contradiction. Therefore $\cap_{n\geq 0} d(\iota_n) = \emptyset$ and hence there exists a set $[\alpha,\beta]$ such that $d([\alpha,\beta]) = \emptyset$. This $[\alpha,\beta]$ satisfies the requirements.

For suppose that $f \in \mathrm{AC}^n_{\mathrm{loc}}((a,b))$ is a solution of $Jf' = Hf$ on (a,b) which satisfies $\int_\alpha^\beta f(t)^* \Delta(t) f(t) \mathrm{d}t = 0$. Then by Corollary 3.1.2 the former condition implies that $f = dY_0 z$ on (a,b), with $d \in \mathbb{R}$, $z \in \mathbb{C}^n$ and $||z|| = 1$. Now the latter condition implies that

$$d^2 \int_\alpha^\beta [Y_0(t)z]^* \, \Delta(t) \, [Y_0(t)z] \, \mathrm{d}t = 0.$$

If $d = 0$, then $f = 0$. Suppose $d \neq 0$, then by above equation

$$\int_\alpha^\beta [Y_0(t)z]^* \, \Delta(t) \, [Y_0(t)z] \, \mathrm{d}t = 0,$$

i.e. $z \in d([\alpha,\beta])$, but this set is empty, a contradiction. Therefore d must be zero, from which it follows that $f = 0$.

Now we prove that (ii) implies definiteness. Let $f \in \mathrm{AC}^n_{\mathrm{loc}}((a,b))$ satisfy $Jf' = [H + \lambda\Delta] f$ on (a,b) with $\Delta f = 0$ a.e. on (a,b). Then certainly $Jf' = Hf$ a.e. on (a,b) and $\Delta f = 0$ a.e. on $[\alpha,\beta]$, therefore by (ii) $f = 0$ on (a,b). □

A consequence of characterization (ii) of definiteness is that if the system (3.1) is definite on (a,b) and α,β are as in (ii) of the above proposition, then the system (3.1) is also definite on any interval $(a',b') \supset [\alpha,\beta]$.

In the above proposition not every interval $[\alpha,\beta]$ suffices, take for instance in Example 3.2.1, $[\alpha,\beta] \in (a,c]$. For suppose that f is an absolutely continuous solution of $Jf' = Hf$ on $[\alpha,\beta]$ and $\Delta f = 0$ on $[\alpha,\beta]$. Then since Δ is identically equal to zero on $[\alpha,\beta] \subset (a,c]$, the latter condition does not give a restriction on f on $[\alpha,\beta]$.

We conclude that f can be any absolutely continuous solution of $Jf' = Hf$ on $[\alpha, \beta]$ and that differential equation has nontrivial solutions on $[\alpha, \beta]$ by Theorem 3.1.1, thus f need not be trivial on the whole interval. This example shows that definiteness as defined in (C4) is weaker than the definiteness condition in [20].

As a consequence of Proposition 3.2.2 we will prove another elementary property of the fundamental matrix $Y_\lambda(\cdot)$.

Corollary 3.2.3. *With $[\alpha, \beta]$ as in Proposition 3.2.2 let Y_λ be such that the associated fundamental matrix $Y_\lambda(\cdot)$ satisfies $Y_\lambda(c) = I_n$ for $c \in (a, \alpha]$ (or $[a, \alpha]$ if a is regular). Then the following statements hold*

(i) The operator Υ_λ defined as

$$\Upsilon_\lambda : \mathbb{C}^n \to \mathbb{C}^n \quad \Upsilon_\lambda : z \mapsto \int_c^b Y_\lambda^*(t)\Delta(t)Y_\lambda dt\, z,$$

is positive for $\lambda \in \mathbb{C}$;

(ii) The operators $\frac{Y_\lambda(b)^* J Y_\lambda(b) - J}{\lambda - \bar{\lambda}}$ and $\frac{Y_\lambda(b) J Y_\lambda(b)^* - J}{\lambda - \bar{\lambda}}$ are positive for $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

Proof. Since Δ is nonnegative it is clear that Υ_λ is nonnegative. Assume that there exists a g such that $g^*\Upsilon_\lambda g = 0$, then clearly $\Delta Y_\lambda g = 0$ a.e. on (a, b). We conclude that $Y_\lambda g$ is a solution of $Jf' = Hf$ such that $\Delta Y_\lambda g = 0$ a.e. on (a, b), therefore by the definiteness condition $Y_\lambda g = 0$ on (a, b) from which it follows that $g = 0$ by Lemma 3.1.3 (iii).

The first statement of (ii) is a direct consequence of (i) combined with Lemma 3.1.3, while the second statement of (ii) is a consequence of the first statement of (ii) combined with Lemma 3.1.3 (ii). □

3.3 Hamiltonian canonical systems

A special class of canonical systems are the Hamiltonian systems, which are canonical systems of dimension $2m$, $m \in \mathbb{N}$, with J given by

$$J = \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix}.$$

For Hamiltonian systems the fundamental matrix $Y_\lambda(\cdot)$ is decomposed in correspondance with J as

$$Y_\lambda(\cdot) = \begin{pmatrix} Y_\lambda^{11}(\cdot) & Y_\lambda^{12}(\cdot) \\ Y_\lambda^{21}(\cdot) & Y_\lambda^{22}(\cdot) \end{pmatrix}, \tag{3.8}$$

where $Y_\lambda^{ij}(\cdot) \in \mathcal{F}^{m \times m}((a, b))$, $1 \leq i, j \leq 2$. Furthermore, for $f \in \mathcal{F}^{2m}((a, b))$ introduce the notation f_1 and f_2 for the elements of $\mathcal{F}^m((a, b))$ which satisfy

$$f(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}.$$

If statement (iii) of Lemma 3.1.3 is written out in full using the decomposition (3.8) for $Y_\lambda(\cdot)$, then the following equalities are obtained for $t \in (a,b)$;

$$
\begin{aligned}
&Y_{\bar\lambda}^{21}(t)^* Y_\lambda^{11}(t) - Y_{\bar\lambda}^{11}(t)^* Y_\lambda^{21}(t) = 0,\\
&Y_{\bar\lambda}^{21}(t)^* Y_\lambda^{12}(t) - Y_{\bar\lambda}^{11}(t)^* Y_\lambda^{22}(t) = -I_m,\\
&Y_{\bar\lambda}^{22}(t)^* Y_\lambda^{11}(t) - Y_{\bar\lambda}^{12}(t)^* Y_\lambda^{21}(t) = I_m,\\
&Y_{\bar\lambda}^{22}(t)^* Y_\lambda^{12}(t) - Y_{\bar\lambda}^{12}(t)^* Y_\lambda^{22}(t) = 0,\\
&Y_\lambda^{12}(t) Y_{\bar\lambda}^{11}(t)^* - Y_\lambda^{11}(t) Y_{\bar\lambda}^{12}(t)^* = 0,\\
&Y_\lambda^{12}(t) Y_{\bar\lambda}^{21}(t)^* - Y_\lambda^{11}(t) Y_{\bar\lambda}^{22}(t)^* = -I_m,\\
&Y_\lambda^{22}(t) Y_{\bar\lambda}^{11}(t)^* - Y_\lambda^{21}(t) Y_{\bar\lambda}^{12}(t)^* = I_m,\\
&Y_\lambda^{22}(t) Y_{\bar\lambda}^{21}(t)^* - Y_\lambda^{21}(t) Y_{\bar\lambda}^{22}(t)^* = 0.
\end{aligned}
\tag{3.9}
$$

Another useful result, which is a consequence of the Lemma 3.1.3 and the definiteness property, is the following result.

Lemma 3.3.1. *Let $[\alpha,\beta]$ be as in Proposition 3.2.2. If the fundamental matrix $Y_\lambda(\cdot)$ satisfies the initial condition $Y_\lambda(c) = I_{2m}$ for $c \in (a,\alpha]$ (or $c \in [a,\alpha]$ if a is regular), then for all λ and $d \in [\beta,b)$ (or $d \in [\beta,b]$ if b is regular) $Y_\lambda^{ij}(d)$, $1 \le i,j \le 2$, is invertible. Conversely, if $Y_\lambda(c) = I_{2m}$ for $c \in [\beta,b)$ (or $c \in [\beta,b]$ is b is regular), then $Y_\lambda^{ij}(d)$, $1 \le i,j \le 2$, is invertible for all λ and $d \in (a,\alpha]$ (or $d \in [a,\alpha]$ if a is regular).*

Proof. We prove the statement for $Y_\lambda^{11}(d)$, the proofs for the other submatrices of $Y_\lambda(d)$ are similar. Let $Y_\lambda(\cdot)$ satisfy $Y_\lambda(c) = I_{2m}$ for $c \in (a,\alpha]$ and let $z_\lambda \in \ker Y_\lambda^{11}(d)$, $d \in [\beta,b)$. Define $\hat z_\lambda = \begin{pmatrix} z_\lambda & 0\end{pmatrix}^T$, then by Lemma 3.1.3

$$
\hat z_\lambda^* \left[Y_\lambda(d)^* J Y_\lambda(d) - J\right] \hat z_\lambda = (\lambda - \bar\lambda) \int_c^d \left[Y_\lambda(s)\hat z_\lambda\right]^* \Delta(s) \left[Y_\lambda(s)\hat z_\lambda\right] \mathrm{d}s.
\tag{3.10}
$$

The lefthand side of the above equality is equal to

$$
z_\lambda^* \left[Y_{\bar\lambda}^{21}(d)^* Y_\lambda^{11}(d) - Y_{\bar\lambda}^{11}(d)^* Y_\lambda^{21}(d)\right] z_\lambda,
$$

which is zero by assumption. Therefore (3.10) implies that $||Y_\lambda \hat z_\lambda||^2_{L^2_\Delta((c,d))} = 0$, because $(c,d) \supseteq [\alpha,\beta]$ it follows from Proposition 3.2.2 that $Y_\lambda(\cdot)\hat z_\lambda = 0$ on (a,b). Finally, by the invertibility of $Y_\lambda(s)$, $s \in (a,b)$, z_λ must be zero, which proves the invertiblity of $Y_\lambda^{11}(d)$. The converse result is proven in the same manner and the proof can easily be extended if an endpoint is regular. ☐

3.4 Linear relations associated with a canonical system

In the space $\mathfrak{H}$ introduce the relation $T_{\max}$ by

$$
\begin{aligned}
T_{\max} = \Big\{ &\{f,g\} \in \mathfrak{H}^2 : \exists \tilde f \in f,\ \exists \tilde g \in g,\ \text{s.t.}\\
&\qquad \tilde f \in AC^m_{\mathrm{loc}}((a,b)) \text{ and } J\tilde f' - H\tilde f = \Delta \tilde g \text{ a.e. on } (a,b) \Big\}.
\end{aligned}
\tag{3.11}
$$

If $\{f,g\} \in T_{\max}$, then, as a consequence of the definiteness condition (C4), the equivalence class f contains precisely one $\tilde{f} \in \mathrm{AC}^n_{\mathrm{loc}}((a,b))$ such that $J\tilde{f}' - H\tilde{f} = \Delta\tilde{g}$ for a $\tilde{g} \in g$. For suppose that $\tilde{f}_1 \in \mathrm{AC}^n_{\mathrm{loc}}((a,b))$ is such that $J\tilde{f}_1' - H\tilde{f}_1 = \Delta\tilde{g}_1$ for a $\tilde{g}_1 \in g$, with $\tilde{f}$, $\tilde{f}_1$ in the same equivalence class and $\tilde{g}$, $\tilde{g}_1$ in the same equivalence class. Then by definition of the norm on $\mathfrak{H}$ $\Delta(\tilde{f} - \tilde{f}_1) = 0 = \Delta(\tilde{g} - \tilde{g}_1)$ a.e. on (a,b). Therefore $F = \tilde{f} - \tilde{f}_1 \in \mathrm{AC}^n_{\mathrm{loc}}((a,b))$ satisfies $JF' - HF = 0 = \Delta F$ a.e. on (a,b), which implies that $F = 0$ on (a,b) by the definiteness property. Therefore, if no confusion can arise, equivalence classes will be identified with their representatives.

Lemma 3.4.1. *The limits*

$$\lim_{t\downarrow a} h(t)^* J f(t) \quad \textit{and} \quad \lim_{t\uparrow b} h(t)^* J f(t)$$

exist and the Lagrange identity

$$(g,h)_{\mathfrak{H}} - (f,k)_{\mathfrak{H}} = \lim_{t\uparrow b} h(t)^* J f(t) - \lim_{t\downarrow a} h(t)^* J f(t) \tag{3.12}$$

holds for all $\{f,g\}, \{h,k\} \in T_{max}$.

Proof. With χ_ι the characteristic function on ι and $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ sequences in (a,b) converging to a and b, we have that

$$\begin{aligned}
(g,h)_{\mathfrak{H}} - (f,k)_{\mathfrak{H}} &= \int_a^b [h(t)^*\Delta(t)g(t) - k(t)^*\Delta(t)f(t)]\,\mathrm{d}t \\
&= \int_a^b \Big[h(t)^*\left[Jf'(t) - H(t)f(t)\right] - \left[Jh'(t) - H(t)h(t)\right]^* f(t)\Big]\mathrm{d}t \\
&= \int_a^b \left[h(t)^*Jf'(t) + h'(t)^*Jf(t)\right]\mathrm{d}t = \int_a^b \lim_{n\to\infty} \chi_{[a_n,b_n]}[h(t)^*Jf(t)]'\mathrm{d}t \\
&= \lim_{n\to\infty}\int_{a_n}^{b_n} [h(t)^*Jf(t)]'\mathrm{d}t = \lim_{n\to\infty}[h(t)^*Jf(t)]_{a_n}^{b_n}
\end{aligned} \tag{3.13}$$

for $\{f,g\}, \{h,k\} \in T_{\max}$. In the fifth step integration and taking the limit can be interchanged by the dominated convergence theorem. The above calculation shows that (3.12) holds if we can show that the two limits exist. For $c \in (a,b)$

$$\begin{aligned}
&||g||_{\mathfrak{H}}||h||_{\mathfrak{H}} + ||f||_{\mathfrak{H}}||k||_{\mathfrak{H}} \geq ||\chi_{[c,b]}g||_{\mathfrak{H}}||h||_{\mathfrak{H}} + ||\chi_{[c,b]}f||_{\mathfrak{H}}||k||_{\mathfrak{H}} \\
\geq& \left|\int_c^b h(t)^*\Delta(t)g(t)\mathrm{d}t\right| + \left|\int_c^b k(t)^*\Delta(t)f(t)\mathrm{d}t\right| \\
\geq& \left|\int_c^b [h(t)^*\Delta(t)g(t) - k(t)^*\Delta(t)f(t)]\,\mathrm{d}t\right| = \left|\lim_{n\to\infty}[h(t)^*Jf(t)]_c^{b_n}\right| \\
=& \left|\lim_{n\to\infty} h(b_n)^*Jf(b_n) - h(c)^*Jf(c)\right|.
\end{aligned}$$

Here in the second step we used the Cauchy-Schwartz inequality, in the fourth step we used the calculations preformed in (3.13) with a replaced by c and in the final step we used the fact that we can assume f and h to be locally absolutely continuous on (a,b). The above calculation shows that the limit at b exists. Using similar arguments the existence of the limit at a can be shown. $\square$

Define the relation T_0 and $T_{\min}$ as

$$T_0 = \{\{f, g\} \in T_{\max} : f \text{ has compact support}\}$$

and

$$T_{\min} = \overline{T_0}. \tag{3.14}$$

Then we will show that $T_{\min}$ and $T_{\max}$, defined in (3.11), are each others adjoint. The proof will proceed in multiple steps, see [31]. First we will prove a proposition in which properties of $T_{\max}$ and $T_{\min}$ are listed in case a and b are regular. Using this proposition we will prove that $T_{\min}$ and $T_{\max}$ are each others adjoint if the endpoints are regular.

Proposition 3.4.2. *If the endpoints a and b are regular, then the following statements hold*

(i) $\hat{Y}_\lambda z \in T_{max}$ *for all* $z \in \mathbb{C}^n$ *and* $\lambda \in \mathbb{C}$;

(ii) $\{G_\lambda g, g + \lambda G_\lambda g\} \in T_{max}$ *for all* $g \in \mathfrak{H}$ *and* $\lambda \in \mathbb{C}$;

(iii) The mapping $\Upsilon : \{f, g\} \in T_{max} \mapsto \begin{pmatrix} f(a) \\ f(b) \end{pmatrix}$ *is surjective;*

(iv) T_{max} *is closed;*

(v) $T_{min} = \{\{f, g\} \in T_{max} : f(a) = f(b) = 0\}$;

(vi) $\operatorname{ran}(T_{min}) = (\mathfrak{N}_0(T_{max}) \cap \mathfrak{H})^\perp$;

(vii) $\operatorname{ran}(T_{min})^\perp = \mathfrak{N}_0(T_{max}) \cap \mathfrak{H}$.

Proof. By Corollary 3.1.2 $Y_\lambda z$ is absolutely continuous on $[a, b]$, therefore

$$\begin{aligned} ||Y_\lambda z||_{\mathfrak{H}}^2 &= \int_a^b z^* Y_\lambda(t)^* \Delta(t) Y_\lambda(t) z \mathrm{d}t = \int_a^b \left| z^* Y_\lambda(t)^* \Delta(t) Y_\lambda(t) z \right| \mathrm{d}t \\ &\leq ||Y_\lambda z||_\infty^2 \int_a^b |\Delta(t)| \mathrm{d}t. \end{aligned}$$

Since (a, b) is regular $\Delta \in L^1([a, b])$, therefore the above calculation shows that $Y_\lambda z \in \mathfrak{H}$ from which it follows that $\hat{Y}_\lambda z \in T_{\max}$.

To prove that $\{G_\lambda g, g\} \in T_{\max}$ for $g \in \mathfrak{H}$ we need to show that $(G_\lambda g)(t)$ exists for every $t \in (a,b)$ and that $G_\lambda g \in \mathfrak{H}$. We start by proving the former condition;

$$\begin{aligned}
|(G_\lambda g)(t)| &= \left|\frac{Y_\lambda(t)J}{2}\int_a^b \mathrm{sgn}\,(s-t)Y_{\bar\lambda}(s)^*\Delta(s)g(s)\mathrm{d}s\right| \\
&\le \left|\left|\frac{Y_\lambda(t)J}{2}\right|\right|_\infty \left|\int_a^b Y_{\bar\lambda}(s)^*\Delta(s)^{\frac{1}{2}}\Delta(s)^{\frac{1}{2}}g(s)\mathrm{sgn}\,(s-t)\mathrm{d}s\right| \\
&\le \left|\left|\frac{Y_\lambda(t)J}{2}\right|\right|_\infty \left(\int_a^b Y_{\bar\lambda}(s)^*\Delta(s)^{\frac{1}{2}}\Delta(s)^{\frac{1}{2}}Y_{\bar\lambda}(s)\mathrm{d}s\right)^{\frac{1}{2}} \\
&\times \left(\int_a^b \mathrm{sgn}\,(s-t)g(s)^*\Delta(s)^{\frac{1}{2}}\Delta(s)^{\frac{1}{2}}g(s)\mathrm{sgn}\,(s-t)\mathrm{d}s\right)^{\frac{1}{2}} \\
&\le \left|\left|\frac{Y_\lambda(t)J}{2}\right|\right|_\infty \left(||Y_\lambda||_\infty^2\int_a^b |\Delta(s)|\mathrm{d}s\right)^{\frac{1}{2}} ||g||_{\mathfrak{H}}.
\end{aligned}$$

Here $\Delta^{\frac{1}{2}}$ exists because Δ is nonnegative and in the third step we applied the Chauchy-Schwartz inequality to each column of $\Delta(s)^{\frac{1}{2}}Y_{\bar\lambda}(s)$ and $\mathrm{sgn}\,(s-t)\Delta(s)^{\frac{1}{2}}g(s)$. Since a and b are regular $\Delta \in L^1([a,b])$ and $Y_\lambda(\cdot)J \in \mathrm{AC}^{n\times n}([a,b])$, see Corollary 3.1.2. These results combined with the fact that $g \in \mathfrak{H}$ prove that $(G_\lambda g)(t)$ exists for every $t \in (a,b)$.

Since $G_\lambda g$ is bounded, see the above calculation, we have that

$$\begin{aligned}
||G_\lambda g||_{\mathfrak{H}}^2 &= \int_a^b \left[(G_\lambda g)(t)\right]^*\Delta(t)\left[(G_\lambda g)(t)\right]\mathrm{d}t = \int_a^b\left|\left[(G_\lambda g)(t)\right]^*\Delta(t)\left[(G_\lambda g)(t)\right]\right|\mathrm{d}t \\
&\le ||G_\lambda g||_\infty \int_a^b |\Delta(t)|\mathrm{d}t,
\end{aligned}$$

which proves that $G_\lambda g \in \mathfrak{H}$, because $\Delta \in L^1([a,b])$ by the regularity of a and b.

Next we prove (iii) by constructing an element $\{f,g\} \in T_{\max}$ such that $f(a) = \xi$ and $f(b) = \eta$, for arbitrary $\xi, \eta \in \mathbb{C}^n$. Define g as

$$g(t) = Y_0(t)\left[\int_a^b Y_0^*(s)\Delta(s)Y_0(s)\mathrm{d}s\right]^{-1} J^{-1}\left[\xi - Y_0(b)^{-1}\eta\right],$$

then g is well-defined by Corollary 3.2.3 and $g \in \mathfrak{H}$ by (i). With this g let f be the solution of $Jf' - Hf = \Delta g$ with initial condition $f(a) = \xi$, then Proposition 3.1.4 implies that f is given by

$$f(t) = Y_0(t)\left[\xi + \frac{J}{2}\int_a^b \mathrm{sgn}\,(s-t)Y_0(s)^*\Delta(s)g(s)\mathrm{d}s - \frac{J}{2}\int_a^b Y_0(s)^*\Delta(s)g(s)\mathrm{d}s\right].$$

Note that $\{f,g\} \in T_{\max}$ by (i) and (ii). Now $f(b)$ is given by

$$f(b) = Y_0(b)\left[\xi - J\int_a^b Y_0(s)^*\Delta(s)g(s)\mathrm{d}s\right] = Y_0(b)\left[\xi - \left[\xi - Y_0(b)^{-1}\eta\right]\right] = \eta,$$

which proves (iii).

To prove (iv) let $(\{f_n, g_n\})_{n\geq 0}$ be a sequence in $T_{\max}$ converging to a limit $\{f, g\} \in \mathfrak{H}^2$. Note that the limit exists because $\mathfrak{H}$ is a Hilbert space. Since $\operatorname{ran}(T_{\max}) = \mathfrak{H}$ by (ii), there exists an $u \in \mathfrak{H}$ such that $\{u, g\} \in T_{\max}$. Then $(\{f_n - u, g_n - g\})_{n\geq 0}$ is a sequence in $T_{\max}$ converging to the limit $\{f - u, 0\}$. By (i) that limit is in $T_{\max}$ proving the closedness of $T_{\max}$.

For (v) let $(\{f_n, g_n\})_{n\geq 0}$ be a sequence in T_0 converging to a limit $\{f, g\}$, which is an element of $T_{\max}$ by (iv). Now apply Lemma 3.4.1 to $\{f, g\}$ and an arbitrary element $\{h, k\}$ of $T_{\max}$, then by the continuity of $(\cdot,\cdot)_{\mathfrak{H}}$

$$0 = \lim_{n\to\infty} [(g_n, h)_{\mathfrak{H}} - (f_n, k)_{\mathfrak{H}}] = (g, h)_{\mathfrak{H}} - (f, k)_{\mathfrak{H}} = \lim_{t\uparrow b} h(t)^* Jf(t) - \lim_{t\downarrow a} h(t)^* Jf(t).$$

By (iii) we know that h can be chosen such that $h(a) = \xi$ and $h(b) = \eta$, for arbitrary $\xi, \eta \in \mathbb{C}^n$, because J is invertible (iv) follows now from the above equality.

To prove (vii) let $g \in (\mathfrak{N}_0(T_{\max}) \cap \mathfrak{H})^{\perp}$, then

(3.15)
$$\begin{aligned} f(t) &= -\frac{1}{2} Y_0(t) \int_a^b JY_0(s)^* \Delta(s) g(s) \mathrm{d}s + \frac{1}{2} \int_a^b \operatorname{sgn}(s-t) Y_0(t) JY_0(s)^* \Delta(s) g(s) \mathrm{d}s \\ &= -Y_0(t) \int_a^t JY_0(s)^* \Delta(s) g(s) \mathrm{d}s, \end{aligned}$$

is such that $f(a) = 0$ and $\{f, g\} \in T_{\max}$, see (i), (ii) and Proposition 3.1.4. Since $g \in (\mathfrak{N}_0(T_{\max}) \cap \mathfrak{H})^{\perp}$

$$z^* \int_a^b Y_0(s)^* \Delta(s) g(s) \mathrm{d}s = (g, Y_0 z)_{\mathfrak{H}} = 0$$

for each $z \in \mathbb{C}^n$. Therefore

$$\int_a^b Y_0(s)^* \Delta(s) g(s) \mathrm{d}s = 0$$

and it follows from (3.15) that $f(b) = 0$, i.e. $\{f, g\} \in T_{\min}$.

If on the other hand $\{f, g\} \in T_{\min}$, then f is given by (3.15), see Proposition 3.1.4, with the additionally condition that $f(b) = 0$. The condition $f(b) = 0$ is equivalent with the condition

$$\int_a^b Y_0(s)^* \Delta(s) g(s) \mathrm{d}s = 0,$$

because J and $Y_0(b)$ are invertible. Premultiplying the above equation by z^*, $z \in \mathbb{C}^n$, the conclusion is that $(g, Y_0 z)_{\mathfrak{H}} = 0$ for each $z \in \mathbb{C}^n$, i.e. $g \in (\mathfrak{N}_0(T_{\max}) \cap \mathfrak{H})^{\perp}$.

For statement (vii) take the orthogonal complement in $\mathfrak{H}$ of statement (vi), and use the fact that $(\mathfrak{N}_0(T_{\max}) \cap \mathfrak{H})$ is closed, because it is a finite-dimensional space. $\square$

Theorem 3.4.3. *If the endpoints a and b are regular, then following statements hold*

(i) $T_{max} = T_{min}{}^*$,

(ii) $T_{min} = T_{max}{}^*$,

(iii) The relation T_{min} is symmetric.

Proof. We start by proving the inclusion $T_{\text{max}} \subset T_{\text{min}}{}^*$. Let $\{f, g\} \in T_{\text{min}}$ and $\{h, k\} \in T_{\text{max}}$, then by Lemma 3.4.1 $(g, h)_{\mathfrak{H}} = (f, k)_{\mathfrak{H}}$, which proves the inclusion.

Next we prove the converse inclusion; $T_{\text{min}}{}^* \subseteq T_{\text{max}}$. Let $\{f, g\} \in T_{\text{min}}{}^*$, then there exist an element $f_0 \in \mathfrak{H}$ such that $\{f_0, g\} \in T_{\text{max}}$ by Proposition 3.4.2. Let $\{h, k\}$ be an arbitrary element of T_{min}, then $(g, h)_{\mathfrak{H}} = (f, k)_{\mathfrak{H}}$ while on the other hand

$$(g, h)_{\mathfrak{H}} = (f_0, k)_{\mathfrak{H}} + (g, h)_{\mathfrak{H}} - (f_0, k)_{\mathfrak{H}} = (f_0, k)_{\mathfrak{H}} + h(b)^* J f_0(b) - h(a)^* J f_0(a) = (f_0, k)_{\mathfrak{H}}.$$

If the above two equalities are combined, we conclude that $(f - f_0, k) = 0$ for all $k \in \operatorname{ran}(T_{\text{min}}) = \mathfrak{H} \ominus \mathcal{N}_0(T_{\text{max}})$, see Proposition 3.4.2, consequently $f - f_0 = Y_0 c$, $c \in \mathbb{C}^n$. But then $f = f_0 + Y_0 c$, which is an element of T_{max} by Proposition 3.4.2. Therefore the inclusion $T_{\text{min}}{}^* \subseteq T_{\text{max}}$ holds and combined with the earlier inclusion this proves (i).

Statement (ii) follows from (i) by taking the adjoint of that result and using the closedness of T_{min}. Finally, by (i)

$$T_{\text{min}} \subseteq T_{\text{max}} = T_{\text{min}}{}^*,$$

which proves (iii). ☐

Chapter 4: Extensions of regular canonical systems

Since we have shown that $T_{\min}$ is a closed symmetric relation and that $T_{\max}$ is its adjoint, we can apply the theory of Chapter 2 to canonical systems on a compact interval $[a, b]$. In particular, using the Kreĭn-Naimark formula, we will be able to give an expression for the resolvent of every generalized resolvent of $T_{\min}$ corresponding to a self-adjoint (exit space) extension $\tilde{A}$. This explicit formula for the resolvent makes it possible to use spectral methods to investigate the self-adjoint extensions of $T_{\min}$, see the section on spectral theory.

Before proceeding to the calculations we will first show what the meaning is of self-adjoint (exit-space) extensions, or finite dimensional graph restrictions, of $T_{\min}$ in terms of the canonical system of differential equations. For a self-adjoint relation $\Theta = \{\{-B^*g, A^*g\}, g \in \mathbb{C}^n\}$ the self-adjoint extension A_Θ, see Proposition 2.2.2, is given by

$$A_\Theta = \{\{f, g\} \in T_{\max} : A\Gamma_0\{f, g\} + B\Gamma_1\{f, g\} = 0\},$$

where Γ_0 and Γ_1 are boundary mappings for $T_{\max}$. Therefore $\{f, g\} \in A_\Theta$ if

$$\begin{aligned} Jf'(t) - H(t)f(t) &= \Delta(t)g(t), \quad t \in (a, b), \\ A\Gamma_0\{f, g\} + B\Gamma_1\{f, g\} &= 0. \end{aligned}$$

Since boundary mappings corresponding to $T_{\max}$ will only contain information about the functions in the domain of $T_{\max}$ at the endpoints, compare Definition 2.2.1 and Lemma 3.4.1, the above result shows that canonical self-adjoint extensions of $T_{\min}$ correspond to self-adjoint boundary value problems for the canonical system. Now it is not difficult to see the family of Štrauss extensions of $T_{\min}$, corresponding to a self-adjoint exit space extension of $T_{\min}$, corresponds to a canonical system with λ-dependent boundary conditions. I.e $\{f, g\} \in A_{-\tau(\lambda)}$, where $\tau(\lambda)$ is a Nevanlinna family with Nevanlinna pair respresentation $\{-B(\lambda), A(\lambda)\}$ see (2.6), if

$$\begin{aligned} Jf'(t) - H(t)f(t) &= \Delta(t)g(t), \quad t \in (a, b), \\ A(\lambda)\Gamma_0\{f, g\} - B(\lambda)\Gamma_1\{f, g\} &= 0. \end{aligned}$$

Finally, finite-dimensional graph restrictions allow us to apply the theory of boundarty triplets to closed symmetric relations $S \subset T_{\min}$. In terms of canonical differential equations, this means that using finite-dimensional graph restrictions we can study the canonical system (3.1) with more general boundary conditions. As an example consider the relation T, where $\{f,g\} \in T$ if

$$\begin{aligned} Jf'(t) - H(t)f(t) &= \Delta(t)g(t), \quad t \in (a,b), \\ A\Gamma_0\{f,g\} - B\Gamma_1\{f,g\} + \int_a^b \sigma(t)^*\Delta(t)f(t)\mathrm{d}t &= 0, \\ \int_a^b \tau(t)^*\Delta(t)g(t)\mathrm{d}t &= 0. \end{aligned}$$

Here $\sigma = \begin{pmatrix}\sigma_1 & \dots & \sigma_n\end{pmatrix} \in \mathcal{F}^{n\times n} \cap \mathfrak{H}$ and $\tau = \begin{pmatrix}\tau_1 & \dots & \tau_m\end{pmatrix} \in \mathcal{F}^{n\times m} \cap \mathfrak{H}$. Then $T_{\min} \nsubseteq T \nsubseteq T_{\max}$, but with Z defined as $Z = \{\{0,\sigma_1\},\dots,\{0,\sigma_n\},\{\tau_1,0\},\dots,\{\tau_m,0\}\}$ $T_{\min} \cap Z \subseteq T \subseteq T_{\max} \hat{\dotplus} Z$. Thus using finite-dimensional graph restrictions T can be seen as an extension of $T_{\min} \cap Z$ and as such we can, for instance, determine its resolvent using Propostion 2.2.6.

In each of the following sections we will give explicit expressions for the boundary triplet, γ-field, Weyl function and the Kreĭn-Naimark formula. Here in the first section we will study the regular canonical system and in the second section we will additionally assume that the system is Hamiltonian. In the third section we will look at intermediate extensions of Hamiltonian canonical systems and in the fourth section we will look at finite-dimensional graph restrictions of $T_{\min}$. In the fifth and last section we will look at interface conditions for Hamiltonian canonical systems.

4.1 Regular canonical systems

In this section we assume that both endpoints of (a,b) are regular, i.e. (a,b) is a finite interval and $H, \Delta \in L^1((a,b))$. In this section Y_λ will be chosen such that the associated fundamental matrix satisfies the condition $Y_\lambda(a) = I_n$, see Corollary 3.1.2. Recall that $T_{\max}$ is given by (3.11) and $T_{\min}$ is given in Proposition 3.4.2 (iv).

Theorem 4.1.1. *T_{min} is a closed symmetric relation with defect numbers $n_+(T_{min}) = n_-(T_{min}) = n$. The adjoint relation ${T_{min}}^*$ is T_{max}. A boundary triplet $\{\mathbb{C}^n, \Gamma_0, \Gamma_1\}$ for T_{max} is given by*

$$\Gamma_0\{f,g\} = \frac{1}{\sqrt{2}}(f(a) + f(b)), \quad \Gamma_1\{f,g\} = \frac{J}{\sqrt{2}}(f(b) - f(a)), \quad \{f,g\} \in T_{max}.$$

Furthermore, $A_0 := \ker \Gamma_0$ is a self-adjoint extension of T_{min}.

Proof. Theorem 3.4.3 shows that $T_{\min}$ and $T_{\max}$ are closed relations, that $T_{\min}$ is symmetric and that ${T_{\min}}^* = T_{\max}$. The self-adjointness of A_0 is a consequence of Proposition 2.2.2 and the fact that the defect numbers are n follows from Proposition 3.4.2 (i). Therefore we only need to show that Γ satisfies the Green's identity and that Γ is surjective. With $\{f,g\},\{h,k\} \in T_{\max}$

$$\begin{aligned}&(\Gamma_1\{f,g\},\Gamma_0\{h,k\}) - (\Gamma_0\{f,g\},\Gamma_1\{h,k\})\\ =&\frac{1}{2}\Big[[(h(a)+h(b)]^* J(f(b)-f(a)) - [J(h(b)-h(a))]^* (f(a)+f(b))\Big]\\ =&\frac{1}{2}\left[(h(a)+h(b))^* J(f(b)-f(a)) + (h(b)-h(a))^* J(f(a)+f(b))\right]\\ =&h(b)^* Jf(b) - h(a)^* Jf(a),\end{aligned}$$

which shows that Green's identity holds by Lemma 3.4.1. Since J is invertible the surjectivity of Γ follows from Proposition 3.4.2 (iii). $\square$

Proposition 4.1.2. *The γ-field and Weyl function for the boundary triplet in Theorem 4.1.1 are given by*

$$\gamma_\lambda = \sqrt{2}Y_\lambda(I_n + Y_\lambda(b))^{-1}$$

and

$$M(\lambda) = J^{-1}(I_n - Y_\lambda(b))(I_n + Y_\lambda(b))^{-1},$$

for $\lambda \in \rho(A_0)$.

Proof. First we show that γ_λ and $M(\lambda)$ are well-defined operators for $\lambda \in \rho(A_0)$ by showing that $I_n + Y_\lambda(b)$ is invertible. If $g \in \mathbb{C}^n$ is such that $(I_n + Y_\lambda(b))g = 0$, then $\hat{Y}_\lambda g \in A_0 \cap \widehat{\mathfrak{N}}_\lambda(T_{\max})$. But for $\lambda \in \rho(A_0)$, A_0 and $\widehat{\mathfrak{N}}_\lambda(T_{\max})$ have a trivial intersection, see Lemma 2.2.3, therefore $g = 0$ and we conclude that $(I_n + Y_\lambda(b))^{-1}$ is an operator for $\lambda \in \rho(A_0)$.

Let $g \in \mathbb{C}^n$, then

$$\Gamma_0\hat{\gamma}_\lambda g = \Gamma_0\sqrt{2}\hat{Y}_\lambda(t)(I_n + Y_\lambda(b))^{-1}g = \frac{\sqrt{2}}{\sqrt{2}}(Y_\lambda(a) + Y_\lambda(b))(I_n + Y_\lambda(b))^{-1}g = g,$$

where we used that by assumption $Y_\lambda(a) = I_n$. Since $\gamma_\lambda g \in \mathfrak{N}_\lambda(T_{\max})$ and $\gamma_\lambda g \in \mathfrak{H}$ by Proposition 3.4.2 (i), the above equality shows that γ_λ is the γ-field for the boundary triplet in Theorem 4.1.1. By definition $M(\lambda)$ is given by

$$\begin{aligned}M(\lambda) &= \{\{\Gamma_0\hat{Y}_\lambda g, \Gamma_1\hat{Y}_\lambda g\} : g \in \mathbb{C}^n\}\\ &= \{\{\frac{1}{\sqrt{2}}(Y_\lambda(a) + Y_\lambda(b))g, \frac{J}{\sqrt{2}}(Y_\lambda(b) - Y_\lambda(a))g\} : g \in \mathbb{C}^n\}\\ &= J^{-1}(I_n - Y_\lambda(b))(I_n + Y_\lambda(b))^{-1},\end{aligned}$$

where we used that $-J = J^{-1}$. $\square$

Recall that closed symmetric (or self-adjoint) extension of $T_{\min}$ are characterized by closed symmetric (or self-adjoint) relations in the boundary space, see Proposition 2.2.2. More generally, we investigate extensions $T_{\min,-\tau(\lambda)}$, see (2.6), of the closed symmetric relation $T_{\min}$ characterized by Nevanlinna families $\tau(\lambda)$ in the boundary space via their resolvents using the Kreĭn-Naimark formula, see (2.7).

To obtain an explicit expression for the resolvent of $T_{\min,-\tau(\lambda)}$ we will need to determine the resolvent of A_0. Therefore we will first determine the adjoints of the $B(\mathbb{C}^n,\mathfrak{H})$-valued functions Y_λ and γ_λ.

Lemma 4.1.3. *With $\xi_\lambda(\cdot)$ the $n\times n$ matrix associated with a $B(\mathbb{C}^n,\mathfrak{H})$-valued function $\lambda\mapsto\xi_\lambda$, i.e. the i^{th} column of $\xi_\lambda(\cdot)$ is equal to $\xi_\lambda e_i$, $\xi_\lambda^* g$ is given by*

$$\xi_\lambda^* g = (g(\cdot),\xi_\lambda(\cdot))_{\mathfrak{H}},$$

for $g\in\mathfrak{H}$.

Proof. Let $z\in\mathbb{C}^n$ and $g\in\mathfrak{H}$, then

$$\begin{aligned}(z,\xi_\lambda^* g) = (\xi_\lambda z, g)_{\mathfrak{H}} &= \int_a^b g(t)^*\Delta(t)\xi_\lambda(t)z\mathrm{d}t = \left(\left(\int_a^b g(t)^*\Delta(t)\xi_\lambda(t)\mathrm{d}t\right)^*\right)^* z\\ &= \left(\int_a^b \xi_\lambda(t)^*\Delta(t)g(t)\mathrm{d}t\right)^* z = (z,(g(\cdot),\xi_\lambda(\cdot))_{\mathfrak{H}}).\end{aligned}$$

Since $z\in\mathbb{C}^n$ is arbitrary the above equality proves the lemma. □

Lemma 4.1.4. *For $\lambda\in\rho(A_0)$ the resolvent of A_0 is given by*

$$(A_0-\lambda)^{-1} = G_\lambda - Y_\lambda J\frac{M(\lambda)}{2}JY_{\bar\lambda}^*,$$

where G_λ is entire in λ.

Proof. The general solution of the canonical system (3.1), see Proposition 3.1.4, is given by

$$(H_\lambda g)(t) = (G_\lambda g)(t) + Y_\lambda(t)d_\lambda(g), \tag{4.1}$$

where d_λ is a $B(\mathfrak{H},\mathbb{C}^n)$-valued function on $\rho(A_0)$. To obtain an expression for the resolvent of A_0, d_λ has to chosen such that $\Gamma_0\{H_\lambda g, g\}=0$, which gives the following condition on d_λ if we apply Γ_0 to (4.1)

$$\begin{aligned}0=\Gamma_0\{H_\lambda g,g\} = \Gamma_0\{Y_\lambda d_\lambda(g)+G_\lambda g, g\} &= \frac{-1}{\sqrt{2}}[(G_\lambda g)(a)+G_\lambda g)(b)]\\ +\frac{1}{\sqrt{2}}(I_n+Y_\lambda(b))d_\lambda(g) &= \frac{-1}{\sqrt{2}}(I_n-Y_\lambda(b))\frac{J}{2}(g(\cdot),Y_{\bar\lambda}(\cdot))_{\mathfrak{H}} + \frac{1}{\sqrt{2}}(I_n+Y_\lambda(b))d_\lambda(g).\end{aligned}$$

The above result implies, using Lemma 4.1.3, that

$$\begin{aligned}d_\lambda(g) &= -(I_n+Y_\lambda(b))^{-1}(I_n-Y_\lambda(b))\frac{J}{2}Y_{\bar\lambda}^* g = -J\frac{J^{-1}}{2}(I_n-Y_\lambda(b))(I_n+Y_\lambda(b))^{-1}JY_{\bar\lambda}^* g\\ &= -J\frac{M(\lambda)}{2}JY_{\bar\lambda}^* g,\end{aligned}$$

where we used Proposition 4.1.2. Now (4.1) combined with the above formula for d_λ proves the lemma, because G_λ is entire in λ by Proposition 3.1.4. □

The resolvent of $T_{\min,-\tau(\lambda)}$, see (2.6), can be written in the following way, see [26].

Theorem 4.1.5. *The resolvent of an extension $T_{min,-\tau(\lambda)}$, see* (2.6), *of T_{min} is given by*

$$(T_{min,-\tau(\lambda)} - \lambda)^{-1} = G_\lambda + Y_\lambda \Omega(\lambda) Y_{\bar{\lambda}}^*,$$

for $\lambda \in \rho(T_{min,-\tau(\lambda)}) \cap \rho(A_0)$. Here G_λ is entire in λ and, with $\{A(\lambda), B(\lambda)\}$ a symmetric Nevanlinna pair representation of the Nevanlinna family $\tau(\lambda)$, $\Omega(\lambda)$ is given by

$$\Omega(\lambda) = \frac{1}{2} \left[B(\lambda)D(\lambda) - A(\lambda)C(\lambda)\right]^{-1} \left[B(\lambda)JC(\lambda) + A(\lambda)JD(\lambda)\right] J,$$

where

$$C(\lambda) = (I_n + Y_\lambda(b)) \quad \textit{and} \quad D(\lambda) = J^{-1}(I_n - Y_\lambda(b))$$

are such that $M(\lambda) = \{\{C(\lambda)g, D(\lambda)g\} : g \in \mathbb{C}^n\}$. Furthermore, $\Omega(\lambda)$ is a Nevanlinna function and satisfies

(4.2)
$$\frac{\Omega(\lambda) - \Omega(\mu)^*}{\lambda - \bar{\mu}} = \left[B(\lambda)D(\lambda) - A(\lambda)C(\lambda)\right]^{-1} \Bigg[2\frac{B(\lambda)A(\mu)^* - A(\lambda)B(\mu)^*}{\lambda - \bar{\mu}}$$
$$+ \left[A(\lambda) - B(\lambda)J\right] \frac{Y_\lambda(b)JY_\mu(b)^* - J}{\lambda - \bar{\mu}} \left[A(\mu) - B(\mu)J\right]^*\Bigg] \left[B(\mu)D(\mu) - A(\mu)C(\mu)\right]^{-*}.$$

Proof. Since $M(\lambda)$ and $\tau(\lambda)$ are Nevanlinna families $-(M(\lambda) + \tau(\lambda))^{-1}$ is also a Nevanlinna family. In particular $-(M(\lambda) + \tau(\lambda))^{-1} = -(M(\bar{\lambda}) + \tau(\bar{\lambda}))^{-*}$. If in the preceding identity we use formula (2.5) for $-(M(\lambda) + \tau(\lambda))^{-1}$, then we obtain the following result

$$\begin{aligned} &- B(\lambda)(M(\lambda)B(\lambda) - A(\lambda))^{-1} = \left[-B(\bar{\lambda})(M(\bar{\lambda})B(\bar{\lambda}) - A(\bar{\lambda}))^{-1}\right]^* \\ &= - (B(\bar{\lambda})^* M(\bar{\lambda})^* - A(\bar{\lambda})^*)^{-1} B(\bar{\lambda})^* = -(B(\lambda)M(\lambda) - A(\lambda))^{-1} B(\lambda), \end{aligned}$$

where we used the symmetricity of $A(\lambda)$ and $B(\lambda)$ and the fact that $M(\lambda)$ is a Nevanlinna function. If we use the above expression in the Kreĭn-Naimark formula, see (2.7), we obtain the following expression for the resolvent of $T_{\min,-\tau(\lambda)}$

$$\begin{aligned} (T_{\min,-\tau(\lambda)} - \lambda)^{-1} = (A_0 - \lambda)^{-1} - 2Y_\lambda(I_n + Y_\lambda(b))^{-1}(B(\lambda)M(\lambda) - A(\lambda))^{-1} \\ \times B(\lambda)(I_n + Y_{\bar{\lambda}}(b))^{-*} Y_{\bar{\lambda}}^*. \end{aligned} \tag{4.3}$$

The next step is to determine the term $(I_n + Y_{\bar{\lambda}}(b))^{-*}$ occuring in the above formula;

$$\begin{aligned} (I_n + Y_{\bar{\lambda}})^{-*} &= (I_n + Y_{\bar{\lambda}}(b)^*)^{-1} J^{-1} J = (J + JY_{\bar{\lambda}}(b)^*)^{-1} J = (J + Y_\lambda(b)^{-1}J)^{-1} J \\ &= J^{-1}(I_n + Y_\lambda(b)^{-1})^{-1} J = J^{-1} Y_\lambda(b)(I_n + Y_\lambda(b))^{-1} J, \end{aligned}$$

where we used Lemma 3.1.3. Now use the above result and the expression for the resolvent of A_0 as given by Lemma 4.1.4 in (4.3) to write (4.3) as

$$(T_{\min,-\tau(\lambda)} - \lambda)^{-1} = G_\lambda + Y_\lambda \Omega(\lambda) Y_{\bar\lambda}^*, \tag{4.4}$$

where $\Omega(\lambda)$ is given by

$$\Omega(\lambda) = -J\frac{M(\lambda)}{2}J - 2(I_n+Y_\lambda(b))^{-1}(B(\lambda)M(\lambda)-A(\lambda))^{-1}B(\lambda)J^{-1}Y_\lambda(b)(I_n+Y_\lambda(b))^{-1}J. \tag{4.5}$$

To rewrite the above expression for $\Omega(\lambda)$ start by rewriting $JM(\lambda)J$;

$$\begin{aligned} JM(\lambda)J &= (I_n + Y_\lambda(b))^{-1}(B(\lambda)M(\lambda) - A(\lambda))^{-1}(B(\lambda)M(\lambda) - A(\lambda))(I_n + Y_\lambda(b))JM(\lambda)J \\ &= (I_n + Y_\lambda(b))^{-1}(B(\lambda)M(\lambda) - A(\lambda))^{-1}B(\lambda)J^{-1}(I_n - Y_\lambda(b))^2(I_n + Y_\lambda(b))^{-1}J \\ &\quad - (I_n + Y_\lambda(b))^{-1}(B(\lambda)M(\lambda) - A(\lambda))^{-1}A(\lambda)(I_n - Y_\lambda(b))J. \end{aligned}$$

If we use the above formula for $JM(\lambda)J$ in (4.5) we have that

$$\begin{aligned} \Omega(\lambda) &= \frac{1}{2}(I_n + Y_\lambda(b))^{-1}(B(\lambda)M(\lambda) - A(\lambda))^{-1} \\ &\quad \times \Big[A(\lambda)(I_n - Y_\lambda(b)) - B(\lambda)J^{-1}\left[(I_n - Y_\lambda(b))^2 + 4Y_\lambda(b)\right](I_n + Y_\lambda(b))^{-1}\Big]J \\ &= \frac{1}{2}\Big[B(\lambda)M(\lambda)(I_n + Y_\lambda(b)) - A(\lambda)(I_n + Y_\lambda(b))\Big]^{-1} \\ &\quad \times \Big[A(\lambda)(I_n - Y_\lambda(b)) - B(\lambda)J^{-1}(I_n + Y_\lambda(b))^2(I_n + Y_\lambda(b))^{-1}\Big]J \\ &= \frac{1}{2}\Big[B(\lambda)J^{-1}(I_n - Y_\lambda(b)) - A(\lambda)(I_n + Y_\lambda(b))\Big]^{-1} \\ &\quad \times \Big[A(\lambda)JJ^{-1}(I_n - Y_\lambda(b)) + B(\lambda)J(I_n + Y_\lambda(b))\Big]J. \end{aligned}$$

This proves the first part of the statement. Finally, we will prove that $\Omega(\lambda)$ is a Nevanlinna function and that (4.2) holds. Since for $\lambda \in \mathbb{C} \setminus \mathbb{R}$ $M(\lambda) = M(\bar\lambda)^*$ and $\tau(\lambda) = \tau(\bar\lambda)^*$, (4.5) implies that $\Omega(\lambda)$ has this property as well. The same formula allows us to conclude that $\Omega(\lambda)$ is holomorphic on $\mathbb{C} \setminus \mathbb{R}$, because $M(\lambda)$ and $(\tau(\lambda) + M(\lambda))^{-1}$ are holomorphic on $\mathbb{C} \setminus \mathbb{R}$. The nonnegativity of $\frac{\operatorname{Im}\Omega(\lambda)}{\operatorname{Im}\lambda}$ is a direct consequence of (4.2), with $\mu = \bar\lambda$, using Definition 2.3.5 and Corollary 3.2.3. Thus we only need to prove formula (4.2).

With the notation $Z(\lambda) = [B(\lambda)D(\lambda) - A(\lambda)C(\lambda)]^{-1}$, where $C(\lambda)$ and $D(\lambda)$ are as in the statement of the theorem, we have that

$$
\begin{aligned}
\Omega(\lambda) - \Omega(\mu)^* &= \frac{1}{2}\Big[Z(\lambda)\left[B(\lambda)JC(\lambda)J + A(\lambda)JD(\lambda)J\right] \\
&- \left[JC(\mu)^*JB(\mu)^* + JD(\mu)^*JA(\mu)^*\right] Z(\mu)^*\Big] \\
&= \frac{1}{2}Z(\lambda)\Big[\left[B(\lambda)JC(\lambda)J + A(\lambda)JD(\lambda)J\right]\left[D(\mu)^*B(\mu)^* - C(\mu)^*A(\mu)^*\right] \\
&- \left[B(\lambda)D(\lambda) - A(\lambda)C(\lambda)\right]\left[JC(\mu)^*JB(\mu)^* + JD(\mu)^*JA(\mu)^*\right]\Big] Z(\mu)^* \\
&= \frac{1}{2}Z(\lambda)\Big[A(\lambda)Z_1(\lambda,\mu)A(\mu)^* + A(\lambda)Z_2(\lambda,\mu)JB(\mu)^* \\
&- B(\lambda)JZ_3(\lambda,\mu)A(\mu)^* + B(\lambda)JZ_4(\lambda,\mu)JB(\mu)^*\Big] Z(\mu)^*,
\end{aligned} \tag{4.6}
$$

where

$$
\begin{aligned}
Z_1(\lambda,\mu) &= C(\lambda)JD(\mu)^*J - JD(\lambda)JC(\mu)^*, \\
Z_2(\lambda,\mu) &= JD(\lambda)JD(\mu)^*J^{-1} + C(\lambda)JC(\mu)^*, \\
Z_3(\lambda,\mu) &= J^{-1}D(\lambda)JD(\mu)^*J + C(\lambda)JC(\mu)^*, \\
Z_4(\lambda,\mu) &= C(\lambda)JD(\mu)^*J^{-1} - J^{-1}D(\lambda)JC(\mu)^*.
\end{aligned}
$$

Some straight-forward calculations show that

$$
\begin{aligned}
Z_1(\lambda,\mu) &= 2\left[Y_\lambda(b)JY_\mu(b)^* - J\right], \\
Z_2(\lambda,\mu) &= 2\left[Y_\lambda(b)JY_\mu(b)^* + J\right] = 2\left[Y_\lambda(b)JY_\mu(b)^* - J\right] + 4J, \\
Z_3(\lambda,\mu) &= 2\left[Y_\lambda(b)JY_\mu(b)^* + J\right] = 2\left[Y_\lambda(b)JY_\mu(b)^* - J\right] + 4J, \\
Z_4(\lambda,\mu) &= -2\left[Y_\lambda(b)JY_\mu(b)^* - J\right].
\end{aligned}
$$

If we use the above results in (4.6) we have that

$$
\begin{aligned}
\Omega(\lambda) - \Omega(\mu)^* = Z(\lambda)\left[A(\lambda) - B(\lambda)J\right]&\Big[Y_\lambda(b)JY_\mu(b)^* - J\Big]\left[A(\mu) - B(\mu)J\right]^* Z(\mu)^* \\
&+ 2Z(\lambda)\Big[B(\lambda)A(\mu)^* - A(\lambda)B(\mu)^*\Big] Z(\mu)^*,
\end{aligned}
$$

which proves (4.2). □

4.2 Regular Hamiltonian canonical systems

In this section we assume that both endpoints of (a,b) are regular, i.e. (a,b) is a finite interval and $H,\ \Delta \in L^1((a,b))$. Since the endpoints are regular Y_λ can be chosen such that the associated fundamental matrix $Y_\lambda(\cdot)$ satisfies the initial condition $Y_\lambda(c) = I_{2m}$ for $c \in [a,b]$, see Corollary 3.1.2. Unless specified otherwise, $Y_\lambda(\cdot)$ will satisfy the initial condition $Y_\lambda(a) = I_{2m}$ in this section.

Theorem 4.2.1. *T_{min} is a closed symmetric relation in $\mathfrak{H}$ with defect numbers $n_+(T_{min}) = n_-(T_{min}) = 2m$. The adjoint relation $T_{min}{}^*$ is the maximal relation T_{max}. A boundary triplet $\{\mathbb{C}^{2m}, \Gamma_0, \Gamma_1\}$ for T_{max} is given by*

$$\Gamma_0\{f,g\} := \begin{pmatrix} f_1(a) \\ f_1(b) \end{pmatrix} \quad \text{and} \quad \Gamma_1\{f,g\} := \begin{pmatrix} f_2(a) \\ -f_2(b) \end{pmatrix}, \quad \{f,g\} \in T_{max}.$$

Furthermore, $A_0 := \ker \Gamma_0$ is a self-adjoint extension of T_{min}.

Proof. All statements except for the fact that Γ is surjective and satisfies Green's identity follow from Theorem 4.1.1. Statement (iii) of Proposition 3.4.2 shows that Γ is surjective. For the latter condition let $\{f,g\}, \{h,k\} \in T_{\max}$, then

$$\begin{aligned} &(g,h)_{\mathfrak{H}} - (f,k)_{\mathfrak{H}} = h_2(b)^* f_1(b) - h_1(b)^* f_2(b) - h_2(a)^* f_1(a) + h_1(a)^* f_2(a) \\ &= \begin{pmatrix} h_1(a) \\ h_1(b) \end{pmatrix}^* \begin{pmatrix} f_2(a) \\ -f_2(b) \end{pmatrix} - \begin{pmatrix} h_2(a) \\ -h_2(b) \end{pmatrix}^* \begin{pmatrix} f_1(a) \\ f_1(b) \end{pmatrix} \\ &= (\Gamma_1\{f,g\}, \Gamma_0\{h,k\}) - (\Gamma_0\{f,g\}, \Gamma_1\{h,k\}), \end{aligned}$$

which combined with Lemma 3.4.1 proves Green's identity. □

Proposition 4.2.2. *The γ-field associated with the boundary triplet defined in Theorem 4.2.1 is*

$$\gamma_\lambda = Y_\lambda \begin{pmatrix} I_m & 0 \\ -Y_\lambda^{12}(b)^{-1} Y_\lambda^{11}(b) & Y_\lambda^{12}(b)^{-1} \end{pmatrix},$$

for $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and its Weyl function $M(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, is given by

$$\begin{aligned} M(\lambda) &= \begin{pmatrix} 0 & I_m \\ -Y_\lambda^{21}(b) & -Y_\lambda^{22}(b) \end{pmatrix} \begin{pmatrix} I_m & 0 \\ Y_\lambda^{11}(b) & Y_\lambda^{12}(b) \end{pmatrix}^{-1} \\ &= \begin{pmatrix} -Y_\lambda^{12}(b)^{-1} Y_\lambda^{11}(b) & Y_\lambda^{12}(b)^{-1} \\ -Y_\lambda^{21}(b) + Y_\lambda^{22}(b) Y_\lambda^{12}(b)^{-1} Y_\lambda^{11}(b) & -Y_\lambda^{22}(b) Y_\lambda^{12}(b)^{-1} \end{pmatrix}. \end{aligned}$$

Proof. Note first that

$$\begin{pmatrix} I_m & 0 \\ Y_\lambda^{11}(b) & Y_\lambda^{12}(b) \end{pmatrix}^{-1} = \begin{pmatrix} I_m & 0 \\ -Y_\lambda^{12}(b)^{-1} Y_\lambda^{11}(b) & Y_\lambda^{12}(b)^{-1} \end{pmatrix}, \tag{4.7}$$

where the last matrix exists for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$ by Lemma 3.3.1. Let $g \in \mathbb{C}^{2m}$, then

$$\begin{aligned} \Gamma_0 \hat{\gamma}_\lambda g &= \Gamma_0 \hat{Y}_\lambda(t) \begin{pmatrix} I_m & 0 \\ -Y_\lambda^{12}(b)^{-1} Y_\lambda^{11}(b) & Y_\lambda^{12}(b)^{-1} \end{pmatrix} g \\ &= \begin{pmatrix} [Y_\lambda(a)]_1 \\ [Y_\lambda(b)]_1 \end{pmatrix} \begin{pmatrix} I_m & 0 \\ -Y_\lambda^{12}(b)^{-1} Y_\lambda^{11}(b) & Y_\lambda^{12}(b)^{-1} \end{pmatrix} g \\ &= \begin{pmatrix} I_m & 0 \\ Y_\lambda^{11}(b) & Y_\lambda^{12}(b) \end{pmatrix} \begin{pmatrix} I_m & 0 \\ -Y_\lambda^{12}(b)^{-1} Y_\lambda^{11}(b) & Y_\lambda^{12}(b)^{-1} \end{pmatrix} g = g. \end{aligned}$$

since it is clear that $\gamma_\lambda g \in \mathfrak{N}_\lambda(T_{\max})$, γ_λ is the γ-field. By definition the Weyl function is given by

$$\begin{aligned}
M(\lambda) &= \{\{\Gamma_0 \hat{Y}_\lambda(t) g, \Gamma_1 \hat{Y}_\lambda(t) g\} : g \in \mathbb{C}^{2m}\} \\
&= \{\{\begin{pmatrix} Y_\lambda^{11}(a) & Y_\lambda^{12}(a) \\ Y_\lambda^{11}(b) & Y_\lambda^{12}(b) \end{pmatrix} g, \begin{pmatrix} Y_\lambda^{21}(a) & Y_\lambda^{22}(a) \\ -Y_\lambda^{21}(b) & -Y_\lambda^{22}(b) \end{pmatrix} g\} : g \in \mathbb{C}^{2m}\} \\
&= \begin{pmatrix} 0 & I_m \\ -Y_\lambda^{21}(b) & -Y_\lambda^{22}(b) \end{pmatrix} \begin{pmatrix} I_m & 0 \\ Y_\lambda^{11}(b) & Y_\lambda^{12}(b) \end{pmatrix}^{-1}.
\end{aligned}$$

Multiplicating out the above result using (4.7) gives the second formula for $M(\lambda)$. □

Next we investigate the extensions of the closed symmetric relation $T_{\min}$, characterized by Nevanlinna families $\tau(\lambda)$ in the space $\mathbb{C}^{2m}$. The extensions, denoted by $T_{\min,-\tau(\lambda)}$, are investigated via their resolvents using the Kreĭn-Naimark formula (2.7), with γ_λ and $M(\lambda)$ as the γ-field and Weyl function respectively. To obtain an explicit expression for the resolvent of $T_{\min,-\tau(\lambda)}$ we will first determine the resolvent of A_0.

Lemma 4.2.3. *For $\lambda \in \rho(A_0)$ the resolvent of A_0 is given by*

$$(A_0 - \lambda)^{-1} = G_\lambda + \frac{\gamma_\lambda}{2} \begin{pmatrix} 0 & Y_{\bar{\lambda}}^{12}(b)^* \\ Y_\lambda^{12}(b) & 0 \end{pmatrix} \gamma_{\bar{\lambda}}^*,$$

where G_λ is entire in λ.

Proof. The general solution of the canonical system (3.1), see Proposition 3.1.4, is given by

$$(H_\lambda g)(t) = (G_\lambda g)(t) + \gamma_\lambda(t) d_\lambda(g). \tag{4.8}$$

Here d_λ is a $B(\mathfrak{H}, \mathbb{C}^{2m})$-valued function on $\rho(A_0)$ and G_λ is entire in λ. To obtain an expression for the resolvent of A_0, conditions on d_λ have to be found such that $\Gamma_0\{H_\lambda g, g\} = 0$, which gives the following condition on d_λ if we apply Γ_0 to (4.8)

(4.9)

$$\begin{aligned}
0 = \Gamma_0\{H_\lambda g, g\} &= \begin{pmatrix} [\gamma_\lambda(a) d_\lambda(g)]_1 \\ [\gamma_\lambda(b) d_\lambda(g)]_1 \end{pmatrix} - \begin{pmatrix} [(G_\lambda g)(a)]_1 \\ [(G_\lambda g)(b)]_1 \end{pmatrix} \\
&= d_\lambda(g) - \begin{pmatrix} \left[\frac{1}{2} J(g(\cdot), Y_{\bar{\lambda}}(\cdot))_{\mathfrak{H}}\right]_1 \\ \left[-\frac{1}{2} Y_\lambda(b) J(g(\cdot), Y_{\bar{\lambda}}(\cdot))_{\mathfrak{H}}\right]_1 \end{pmatrix} = d_\lambda(g) - \frac{1}{2} \begin{pmatrix} -I_m & 0 \\ Y_\lambda^{11}(b) & Y_\lambda^{12}(b) \end{pmatrix} J Y_{\bar{\lambda}}^* g \\
&= d_\lambda(g) - \frac{1}{2} \begin{pmatrix} 0 & I_m \\ Y_\lambda^{12}(b) & -Y_\lambda^{11}(b) \end{pmatrix} Y_{\bar{\lambda}}^* g.
\end{aligned}$$

Combining the above result with (4.8) we see the resolvent of A_0 is given by

$$\begin{aligned}
(A_0 - \lambda)^{-1} &= G_\lambda + \frac{\gamma_\lambda}{2} \begin{pmatrix} 0 & I_m \\ Y_\lambda^{12}(b) & -Y_\lambda^{11}(b) \end{pmatrix} Y_{\bar{\lambda}}^* \\
&= G_\lambda + \frac{\gamma_\lambda}{2} \begin{pmatrix} 0 & I_m \\ Y_\lambda^{12}(b) & -Y_\lambda^{11}(b) \end{pmatrix} \begin{pmatrix} I_m & 0 \\ Y_{\bar{\lambda}}^{11}(b) & Y_{\bar{\lambda}}^{12}(b) \end{pmatrix}^* \gamma_{\bar{\lambda}}^*.
\end{aligned}$$

If the matrix product is multiplied out and (3.9) is used we obtain the specific form of the resolvent of A_0 as in the statement of the lemma. □

Theorem 4.2.4. *The resolvent of an extension $T_{min,-\tau(\lambda)}$ of T_{min} is given by*

$$(T_{min,-\tau(\lambda)} - \lambda)^{-1} = G_\lambda - \gamma(t,\lambda)\Omega(\lambda)\Lambda(\lambda)\,\gamma_{\bar\lambda}^*$$

for $\lambda \in \rho(T_{min,-\tau(\lambda)}) \cap \rho(A_0)$. Here G_λ is entire in λ and, with $\{A(\lambda), B(\lambda)\}$ a symmetric Nevanlinna pair representation of the Nevanlinna family $\tau(\lambda)$, $\Omega(\lambda)$ is given by

$$\Omega(\lambda) = \frac{1}{2}\begin{pmatrix} 0 & Y_{\bar\lambda}^{12}(b)^* \\ Y_\lambda^{12}(b) & 0 \end{pmatrix} A(\lambda) + \frac{1}{2}\begin{pmatrix} I_m & Y_\lambda^{22}(b) \\ Y_\lambda^{11}(b) & I_m \end{pmatrix} B(\lambda)$$

and $\Lambda(\lambda)$ is defined as

$$\Lambda(\lambda) = (M(\lambda)B(\lambda) - A(\lambda))^{-1}.$$

Proof. Using Lemma 4.2.3 in the Kreĭn-Naimark formula, see (2.7), the indicated form of the resolvent is obtained. $\Omega(\lambda)$ is given by

$$\begin{aligned}
&\Omega(\lambda) - \frac{1}{2}\begin{pmatrix} 0 & Y_{\bar\lambda}^{12}(b)^* \\ Y_\lambda^{12}(b) & 0 \end{pmatrix} A(\lambda) = \left[-\frac{1}{2}\begin{pmatrix} 0 & Y_{\bar\lambda}^{12}(b)^* \\ Y_\lambda^{12}(b) & 0 \end{pmatrix} M(\lambda) + I_m\right] B(\lambda) \\
&= \frac{1}{2}\begin{pmatrix} 2I_m + Y_{\bar\lambda}^{12}(b)^* Y_\lambda^{21}(b) - Y_{\bar\lambda}^{12}(b)^* Y_\lambda^{22}(b) Y_\lambda^{12}(b)^{-1} Y_\lambda^{11}(b) & Y_{\bar\lambda}^{12}(b)^* Y_\lambda^{22}(b) Y_\lambda^{12}(b)^{-1} \\ Y_\lambda^{11}(b) & 2I_m - I_m \end{pmatrix} B(\lambda) \\
&= \frac{1}{2}\begin{pmatrix} 2I_m + Y_{\bar\lambda}^{12}(b)^* Y_\lambda^{21}(b) - Y_{\bar\lambda}^{22}(b)^* Y_\lambda^{11}(b) & Y_\lambda^{22}(b) \\ Y_\lambda^{11}(b) & I_m \end{pmatrix} B(\lambda) \\
&= \frac{1}{2}\begin{pmatrix} I_m & Y_\lambda^{22}(b) \\ Y_\lambda^{11}(b) & I_m \end{pmatrix} B(\lambda)
\end{aligned}$$

In the calculations use was made of the relations (3.9). □

4.3 Intermediate extensions of regular Hamiltonian canonical systems

In this section we look at specific closed symmetric extensions of the closed symmetric extension $T_{\min}$ in the regular case. Hereby we assume that $\{\mathbb{C}^{2m}, \Gamma_0, \Gamma_1\}$ is any boundary triplet for $T_{\max}$ with associated Weyl function $M(\lambda)$ and γ-field γ_λ. Unless specified otherwise, Y_λ is such that the associated fundamental matrix $Y_\lambda(\cdot)$ satisfies the initial condition $Y_\lambda(a) = I_{2m}$ in this section.

Divide the set $\{1, \dots, 2m\}$ into four sets; $\alpha = \{\alpha_1, \dots, \alpha_k\}$, $\alpha^c = \{\alpha_{k+1}, \dots, \alpha_l\}$, $\beta = \{\beta_1, \dots, \beta_q\}$ and $\beta^c = \{\beta_{q+1}, \dots, \beta_r\}$. For a subset $\delta = \{\delta_1, \dots, \delta_k\}$ of $\{1, \dots, 2m\}$ define the $2m \times k$ matrix E_δ as

$$E_\delta = \begin{pmatrix} e_{\delta_1} & \dots & e_{\delta_k} \end{pmatrix}.$$

Then with the $4m \times (k+q)$ matrix R defined as

$$R = \begin{pmatrix} E_\alpha & 0 \\ 0 & E_\beta \end{pmatrix},$$

which satisfies by construction $R^*R = I_{k+q}$, we will investigate the closed extension T of $T_{\min}$ defined as

$$T = \{\{f,g\} \in T_{\max} : R^*\Gamma\{f,g\} = 0\}.$$

To determine its boundary mappings, γ-field and Weyl function we will use Proposition 2.5.2, for which we will first determine the mapping W. The $4m \times 4m$ matrix W^* is easily seen, see the reasoning after Proposition 2.5.1, to be the matrix

$$W^* = \begin{pmatrix} R & \begin{matrix} E_{\alpha^c} & 0 & 0 & -E_\beta & 0 & -E_{\beta^c} \\ 0 & E_{\beta^c} & E_\alpha & 0 & E_{\alpha^c} & 0 \end{matrix} \end{pmatrix}. \tag{4.10}$$

Proposition 4.3.1. *Let α, α^c, β and β^c be as above, then the relation $\dot{A}$ defined as*

$$\dot{A} = \{\{f,g\} \in T_{max} : E^*_{\alpha+\alpha^c+\beta^c}\Gamma_0\{f,g\} = 0 \quad \textit{and} \quad E^*_{\beta+\beta^c+\alpha^c}\Gamma_1\{f,g\} = 0\},$$

is a closed symmetric relation in $\mathfrak{H}$. $\dot{A}$ has defect numbers $(2m-(k+q), 2m-(k+q))$ and the adjoint relation $\dot{A}^*$ is

$$\dot{A}^* = \{\{f,g\} \in T_{max} : E^*_\alpha\Gamma_0\{f,g\} = 0 \quad \textit{and} \quad E^*_\beta\Gamma_1\{f,g\} = 0\},$$

A boundary triplet $\{\mathbb{C}^{2m-(k+q)}, \dot{\Gamma}_0, \dot{\Gamma}_1\}$ is given by

$$\dot{\Gamma}_0\{f,g\} := \begin{pmatrix} E^*_{\alpha_c}\Gamma_0\{f,g\} \\ E^*_{\beta_c}\Gamma_1\{f,g\} \end{pmatrix} \quad \textit{and} \quad \dot{\Gamma}_1\{f,g\} := \begin{pmatrix} E^*_{\alpha_c}\Gamma_1\{f,g\} \\ -E^*_{\beta_c}\Gamma_0\{f,g\} \end{pmatrix}, \quad \{f,g\} \in \dot{A}^*.$$

Furthermore, $\dot{A}_0 := \ker \dot{\Gamma}_0$ is a self-adjoint extension of $\dot{A}$. The γ-field and Weyl function associated with the above boundary mappings are

$$\dot{\gamma}_\lambda = \gamma_\lambda \begin{pmatrix} E^*_\alpha \\ E^*_\beta M(\lambda) \\ E^*_{\alpha^c} \\ E^*_{\beta^c} M(\lambda) \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ I_{2m-(k+q)} \end{pmatrix}$$

and

$$\dot{M}(\lambda) = \begin{pmatrix} 0 & I_{2m-(k+q)} \end{pmatrix} \begin{pmatrix} E^*_\alpha M(\lambda) \\ -E^*_\beta \\ E^*_{\alpha^c} M(\lambda) \\ -E^*_{\beta^c} \end{pmatrix} \begin{pmatrix} E^*_\alpha \\ E^*_\beta M(\lambda) \\ E^*_{\alpha^c} \\ E^*_{\beta^c} M(\lambda) \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ I_{2m-(k+q)} \end{pmatrix},$$

for $\lambda \in \rho(\dot{A}_0)$. Here γ_λ and $M(\lambda)$ are the γ-field associated with a boundary triplet $\{\mathbb{C}^{2m}, \Gamma_0, \Gamma_1\}$ for T_{max}.

Proof. All statements follow from Proposition 2.5.2 using Proposition 2.2.7 with W^* as in (4.10). □

Next we investigate the extensions of the closed symmetric relation $\dot{A}$, characterized by Nevanlinna families $\dot{\tau}(\lambda)$ in the space $\mathbb{C}^{2m-(k+q)}$. In this case we assume that the boundary triplet for $T_{\max}$ is as defined in Theorem 4.2.1 with γ-field and Weyl function as in Proposition 4.2.2. The extensions, denoted by $\dot{A}_{-\dot{\tau}(\lambda)}$, are investigated via their resolvents using the Kreĭn-Naimark formula (2.7), with $\dot{\gamma}_\lambda$ and $\dot{M}(\lambda)$ as the γ-field and Weyl function respectively.

To obtain an explicit expression for the Kreĭn-Naimark formula the resolvent of $\dot{A}_0$ needs to be determined. The general solution of the canonical system (3.1), see Proposition 3.1.4, is given by

$$(H_\lambda g)(t) = (G_\lambda g)(t) + Y_\lambda(t) d_\lambda(g). \tag{4.11}$$

Here d_λ is a $B(\mathfrak{H}, \mathbb{C}^{2m})$-valued function on $\rho(\dot{A}_0)$ and G_λ is entire in λ. Recall from Proposition 4.3.1 that $H_\lambda g \in \operatorname{dom}(\dot{A}_0)$ if $E^*_{\alpha+\alpha^c}\Gamma_0\{H_\lambda g, g\} = 0$ and $E^*_{\beta+\beta^c}\Gamma_1\{H_\lambda g, g\} = 0$. Since

$$\begin{pmatrix} [(G_\lambda g)(a)]_1 \\ [(G_\lambda g)(b)]_1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & I_m \\ Y_\lambda^{12}(b) & -Y_\lambda^{11}(b) \end{pmatrix} Y^*_{\bar{\lambda}} g,$$

see (4.9), and

$$\begin{aligned} \begin{pmatrix} [(G_\lambda g)(a)]_2 \\ -[(G_\lambda g)(b)]_2 \end{pmatrix} &= \begin{pmatrix} \left[\frac{1}{2}J(g(\cdot), Y_{\bar{\lambda}}(\cdot))_{\mathfrak{H}}\right]_2 \\ -\left[-\frac{1}{2}Y_\lambda(b)J(g(\cdot), Y_{\bar{\lambda}}(\cdot))_{\mathfrak{H}}\right]_2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & I_m \\ Y_\lambda^{21}(b) & Y_\lambda^{22}(b) \end{pmatrix} J Y^*_{\bar{\lambda}} g \\ &= \frac{1}{2}\begin{pmatrix} I_m & 0 \\ Y_\lambda^{22}(b) & -Y_\lambda^{21}(b) \end{pmatrix} Y^*_{\bar{\lambda}} g, \end{aligned}$$

the conditions $E^*_{\alpha+\alpha^c}\Gamma_0\{H_\lambda g, g\} = 0$ and $E^*_{\beta+\beta^c}\Gamma_1\{H_\lambda g, g\} = 0$ imply that d_λ has to satisfy the following conditions

$$\begin{aligned} 0 &= E^*_{\alpha+\alpha^c}\left[\frac{1}{2}\begin{pmatrix} 0 & -I_m \\ -Y_\lambda^{12}(b) & Y_\lambda^{11}(b) \end{pmatrix} Y^*_{\bar{\lambda}} g + \begin{pmatrix} I_m & 0 \\ Y_\lambda^{11}(b) & Y_\lambda^{12}(b) \end{pmatrix} d_\lambda(g)\right], \\ 0 &= E^*_{\beta+\beta^c}\left[\frac{1}{2}\begin{pmatrix} I_m & 0 \\ Y_\lambda^{22}(b) & -Y_\lambda^{21}(b) \end{pmatrix} Y^*_{\bar{\lambda}} g + \begin{pmatrix} 0 & I_m \\ -Y_\lambda^{21}(b) & -Y_\lambda^{22}(b) \end{pmatrix} d_\lambda(g)\right]. \end{aligned}$$

From these equations $d_\lambda(g)$ can be uniquely solved, because the matrix in front of d_λ will contain $\begin{pmatrix} I_m & 0 \end{pmatrix}$ and m rows of $\begin{pmatrix} I_m & 0 \\ 0 & -I_m \end{pmatrix} Y_\lambda(b)$. These rows are linearly independent among themselves by the invertibility of $Y_\lambda(b)$ and are linearly independent of the rows of $\begin{pmatrix} I_m & 0 \end{pmatrix}$, because $Y_\lambda^{12}(b)$ and $Y_\lambda^{22}(b)$ are invertible, see Lemma 3.3.1. We write $d_\lambda = Q(\lambda)Y^*_{\bar{\lambda}}$ and with this notation the resolvent of A_0 is given by

$$(A_0 - \lambda)^{-1} = G_\lambda + Y_\lambda Q(\lambda) Y^*_{\bar{\lambda}}. \tag{4.12}$$

Therefore the Kreĭn-Naimark formula (2.7), with $\dot{\tau}(\lambda) = \{\{-\dot{B}(\lambda)g, \dot{A}(\lambda)g\} : g \in \mathbb{C}^{2m-(k+q)}\}$, has the following form.

Theorem 4.3.2. *The resolvent of an extension* $\dot{A}_{-\dot{\tau}(\lambda)}$ *of* $\dot{A}$ *is given by*

$$(\dot{A}_{-\dot{\tau}(\lambda)} - \lambda)^{-1} = G_\lambda + Y_\lambda \dot{\Omega}(\lambda) Y_{\bar{\lambda}}^*,$$

for $\lambda \in \rho(\dot{A}_{-\dot{\tau}(\lambda)}) \cap \rho(\dot{A}_0)$. *Here* G_λ *is entire in* λ *and, with* $\{\dot{A}(\lambda), \dot{B}(\lambda)\}$ *a symmetric Nevanlinna pair representation of the Nevanlinna family* $\dot{\tau}(\lambda)$, $\dot{\Omega}(\lambda)$ *is given by*

$$\dot{\Omega}(\lambda) = -\begin{pmatrix} I_m & 0 \\ -Y_\lambda^{12}(b)^{-1}Y_\lambda^{11}(b) & Y_\lambda^{12}(b)^{-1} \end{pmatrix} \begin{pmatrix} E_\alpha^* \\ E_\beta^* M(\lambda) \\ E_{\alpha^c}^* \\ E_{\beta^c}^* M(\lambda) \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ 0 & \dot{B}(\lambda)\dot{\Lambda}(\lambda) \end{pmatrix}$$
$$\times \begin{pmatrix} E_\alpha & M(\lambda)E_\beta & E_{\alpha^c} & M(\lambda)E_{\beta^c} \end{pmatrix}^{-1} \begin{pmatrix} I_m & 0 \\ -Y_\lambda^{12}(b)^{-1}Y_\lambda^{11}(b) & Y_\lambda^{12}(b)^{-1} \end{pmatrix}^* + Q(\lambda),$$

where $\dot{\Lambda}(\lambda)$ *is defined as*

$$\dot{\Lambda}(\lambda) = (\dot{B}(\lambda)\dot{M}(\lambda) - \dot{A}(\lambda))^{-1}.$$

Proof. Combining Kreĭn-Naimark formula (2.7) with (4.12) the indicated result is obtained. □

As an example of the above theory we will consider the closed symmetric relation $\ddot{A}$ defined as

$$\ddot{A} = \{\{f, g\} \in T_{\max} : f(a) = 0 \quad \text{and} \quad f_1(b) = 0\}.$$

By Proposition 4.3.1 we have the following result.

Corollary 4.3.3. *Let* $\ddot{A}$ *be defined as above, then* $\ddot{A}$ *is a closed symmetric relation in* $\mathfrak{H}$ *with defect numbers* $n_+(\ddot{A}) = n_-(\ddot{A}) = m$. *The adjoint relation* $\ddot{A}^*$ *is given by*

$$\ddot{A}^* = \{\{f, g\} \in T_{max} : f_1(b) = 0\}.$$

A boundary triplet $\{\mathbb{C}^m, \ddot{\Gamma}_0, \ddot{\Gamma}_1\}$ *for* $\ddot{A}^*$ *is given by*

$$\ddot{\Gamma}_0\{f, g\} := f_1(a) \quad \text{and} \quad \ddot{\Gamma}_1\{f, g\} := f_2(a), \quad \{f, g\} \in \ddot{A}^*.$$

Furthermore, $\ddot{A}_0 := \ker \ddot{\Gamma}_0 = A_0$, *see Theorem 4.2.1, is a self-adjoint extension of* $\ddot{A}$ *and the* γ*-field and Weyl function associated with the above boundary triplet are*

$$\ddot{\gamma}_\lambda = Y_\lambda \begin{pmatrix} I_m \\ -Y_\lambda^{12}(b)^{-1}Y_\lambda^{11}(b) \end{pmatrix}$$

and

$$\ddot{M}(\lambda) = -Y_\lambda^{12}(b)^{-1}Y_\lambda^{11}(b),$$

for $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

Proof. Set $\alpha = \{m+1, \ldots, 2m\}$, $\beta = \{\}$, $\alpha^c = \{1, \ldots, m\}$ and $\beta^c = \{\}$, then apply Proposition 4.3.1. $\square$

Theorem 4.3.4. *The resolvent of $A_{-\ddot{\tau}(\lambda)}$ is given by*

$$(A_{-\ddot{\tau}(\lambda)} - \lambda)^{-1} = H_\lambda - Y_{\lambda,-\ddot{\tau}(\lambda)} \ddot{\Lambda}(\lambda) \begin{pmatrix} I_m & \ddot{M}(\lambda) \end{pmatrix} Y_{\bar{\lambda}}^*,$$

for $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Here, with $\{\ddot{A}(\lambda), \ddot{B}(\lambda)\}$ a symmetric Nevalinna pair representation of the Nevanlinna family $\ddot{\tau}(\lambda)$, $Y_{\lambda,-\ddot{\tau}(\lambda)}$ is defined as

$$Y_{\lambda,-\ddot{\tau}(\lambda)} = Y_\lambda \begin{pmatrix} \ddot{B}(\lambda) \\ \ddot{A}(\lambda) \end{pmatrix},$$

and $\ddot{\Lambda}(\lambda)$ is defined as

$$\ddot{\Lambda}(\lambda) = (\ddot{M}(\lambda)\ddot{B}(\lambda) - \ddot{A}(\lambda))^{-1}.$$

Finally, H_λ defined as

$$H_\lambda = G_\lambda - \frac{1}{2} Y_\lambda J Y_{\bar{\lambda}}^*$$

is entire in λ.

Proof. To prove the statement we will start by using Theorem 4.3.2, therefore we need to determine $\dot{\Omega}(\lambda)$. A straight-forward calculation shows that

$$\dot{\Omega}(\lambda) - Q(\lambda) = - \begin{pmatrix} \ddot{B}(\lambda)\ddot{\Lambda}(\lambda) & \ddot{B}(\lambda)\ddot{\Lambda}(\lambda)\ddot{M}(\lambda) \\ \ddot{M}(\lambda)\ddot{B}(\lambda)\ddot{\Lambda}(\lambda) & \ddot{M}(\lambda)\ddot{B}(\lambda)\ddot{\Lambda}(\lambda)\ddot{M}(\lambda) \end{pmatrix}. \tag{4.13}$$

The next step is to determine $Q(\lambda)$, therefore we need to solve $Q(\lambda)$ from

$$0 = \frac{1}{2} \begin{pmatrix} 0 & -I_m \\ -Y_\lambda^{12}(b) & Y_\lambda^{11}(b) \end{pmatrix} + \begin{pmatrix} I_m & 0 \\ Y_\lambda^{11}(b) & Y_\lambda^{12}(b) \end{pmatrix} Q(\lambda),$$

see the discussion above Theorem 4.3.2. We conclude that $Q(\lambda)$ is given by

$$\begin{aligned} Q(\lambda) &= \frac{-1}{2} \begin{pmatrix} I_m & 0 \\ Y_\lambda^{11}(b) & Y_\lambda^{12}(b) \end{pmatrix}^{-1} \begin{pmatrix} 0 & -I_m \\ -Y_\lambda^{12}(b) & Y_\lambda^{11}(b) \end{pmatrix} \\ &= \frac{-1}{2} \begin{pmatrix} I_m & 0 \\ -Y_\lambda^{12}(b)^{-1} Y_\lambda^{11}(b) & Y_\lambda^{12}(b)^{-1} \end{pmatrix} \begin{pmatrix} 0 & -I_m \\ -Y_\lambda^{12}(b) & Y_\lambda^{11}(b) \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 0 & I_m \\ I_m & -2Y_\lambda^{12}(b)^{-1} Y_\lambda^{11}(b) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & I_m \\ I_m & 2\ddot{M}(\lambda) \end{pmatrix}, \end{aligned}$$

see (4.7) and Corollary 4.3.3. If we combine this result with (4.13) we obtain the following formula for $\dot{\Omega}(\lambda)$;

$$\begin{aligned} \dot{\Omega}(\lambda) &= - \begin{pmatrix} \ddot{B}(\lambda)\ddot{\Lambda}(\lambda) & \ddot{B}(\lambda)\ddot{\Lambda}(\lambda)\ddot{M}(\lambda) \\ \ddot{M}(\lambda)\ddot{B}(\lambda)\ddot{\Lambda}(\lambda) & \ddot{M}(\lambda)\ddot{B}(\lambda)\ddot{\Lambda}(\lambda)\ddot{M}(\lambda) \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & I_m \\ I_m & 2\ddot{M}(\lambda) \end{pmatrix} \\ &= - \begin{pmatrix} \ddot{B}(\lambda)\ddot{\Lambda}(\lambda) & \ddot{B}(\lambda)\ddot{\Lambda}(\lambda)\ddot{M}(\lambda) \\ \ddot{A}(\lambda)\ddot{\Lambda}(\lambda) & \ddot{A}(\lambda)\ddot{\Lambda}(\lambda)\ddot{M}(\lambda) \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix} \\ &= - \begin{pmatrix} \ddot{B}(\lambda) \\ \ddot{A}(\lambda) \end{pmatrix} \ddot{\Lambda}(\lambda) \begin{pmatrix} I_m & \ddot{M}(\lambda) \end{pmatrix} - \frac{1}{2} J, \end{aligned}$$

which proves the statement, because G_λ and Y_λ are entire in λ. $\square$

Note that for each $c \in \mathbb{C}^m$ $Y_{\lambda,-\ddot{\tau}(\lambda)}c$ is a solution of (3.2) such that $\ddot{\Gamma}Y_{\lambda,-\ddot{\tau}(\lambda)}c \in -\ddot{\tau}(\lambda)$ and that, conversely, each solution f of (3.2) such that $\ddot{\Gamma}f \in -\ddot{\tau}(\lambda)$ is $Y_{\lambda,-\ddot{\tau}(\lambda)}c$ for some $c \in \mathbb{C}^m$, because the defect number of $\ddot{A}$ are (m, m), see Corollary 4.3.3.

4.4 Finite-dimensional graph restrictions of regular systems

In this section, unless mentioned otherwise, Y_λ is such that the associated fundamental matrix $Y_\lambda(\cdot)$ satisfies $Y_\lambda(a) = I_n$.

With $\{\mathbb{C}^n, \Gamma_0, \Gamma_1\}$ the boundary triplet for $T_{\max}$ as defined in Theorem 4.2.1, fix the self-adjoint extension A_Θ of $T_{\max}$ by the self-adjoint mapping $\Theta = \{\{-B^*g, A^*g\} : g \in \mathbb{C}^n\}$, see Proposition 2.2.2. Using the self-adjoint mapping Θ we will transform the old boundary mappings Γ_0 and Γ_1 for $T_{\max}$ into new boundary mappings $\Gamma_{\Theta,0}$ and $\Gamma_{\Theta,1}$ such that A_Θ coincides with the kernel of $\Gamma_{\Theta,0}$.

Lemma 4.4.1. *Let $\Theta = \{\{-B^*g, A^*g\} : g \in \mathbb{C}^n\}$ be a self-adjoint relation in $\mathbb{C}^n$, then $(AA^* + BB^*)^{\frac{-1}{2}}$ exists.*

Proof. Since $(AA^* + BB^*)$ is a positive mapping, $(AA^* + BB^*)^{-1}$, if it exists, is also a positive mapping. We conclude that $(AA^* + BB^*)^{\frac{-1}{2}}$ is a well-defined mapping if $(AA^* + BB^*)$ is invertible.

Assume the contrary and let $0 \neq g \in \ker\,(AA^* + BB^*)$, then $g \in \ker\,A^* \cap \ker\,B^*$. Therefore the dimension of Θ is strictly smaller than n, which is in contradiction with Proposition 2.5.1. □

Lemma 4.4.2. *Let $\Theta = \{\{-B^*g, A^*g\} : g \in \mathbb{C}^n\}$ be a self-adjoint relation, then the mappings*

$$\Gamma_{\Theta,0} = (AA^* + BB^*)^{\frac{-1}{2}}\,[A\Gamma_0 + B\Gamma_1] \quad \text{and} \quad \Gamma_{\Theta,1} = (AA^* + BB^*)^{\frac{-1}{2}}\,[-B\Gamma_0 + A\Gamma_1]$$

are boundary mappings for T_{max} such that $A_\Theta = \ker\,\Gamma_{\Theta,0}$. Furthermore, with γ_λ and $M(\lambda)$ the γ-field and Weyl function associated with the boundary triplet $\{\mathbb{C}^n, \Gamma_0, \Gamma_1\}$ for T_{max}, see Proposition 4.2.2, the γ-field and Weyl function associated with the boundary triplet $\{\mathbb{C}^n, \Gamma_{0,\Theta}, \Gamma_{1,\Theta}\}$ are

$$\gamma_{\lambda,\Theta} = \gamma_\lambda(A + BM(\lambda))^{-1}(AA^* + BB^*)^{\frac{1}{2}}$$

and

$$M_\Theta(\lambda) = (AA^* + BB^*)^{\frac{-1}{2}}(AM(\lambda) - B)(A + BM(\lambda))^{-1}(AA^* + BB^*)^{\frac{1}{2}},$$

for $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

Proof. By Proposition 2.3.8 it is clear that $A_\Theta = \ker\,\Gamma_{\Theta,0}$. The remainder of the lemma will be proven using Proposition 2.2.7 with the mapping W defined as

$$W = (AA^* + BB^*)^{\frac{-1}{2}} \begin{pmatrix} A & B \\ -B & A \end{pmatrix},$$

which is a well-defined mapping by Lemma 4.4.1. Since

$$\begin{aligned} WW^* &= (AA^*+BB^*)^{\frac{-1}{2}} \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \begin{pmatrix} A^* & -B^* \\ B^* & A^* \end{pmatrix} (AA^*+BB^*)^{\frac{-1}{2}} \\ &= (AA^*+BB^*)^{\frac{-1}{2}} \begin{pmatrix} AA^*+BB^* & -AB^*+BA^* \\ -BA^*+AB^* & BB^*+AA^* \end{pmatrix} (AA^*+BB^*)^{\frac{-1}{2}} = I_{2n}. \end{aligned}$$

and W is an operator on a finite-dimensional space, W is unitairy. Therefore, with $X=(AA^*+BB^*)^{-1}$, we have that

$$\begin{aligned} I_{2n} = W^*W &= \begin{pmatrix} A^* & -B^* \\ B^* & A^* \end{pmatrix} (AA^*+BB^*)^{-1} \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \\ &= \begin{pmatrix} A^*XA+B^*XB & A^*X^{-1}B-B^*X^{-1}A \\ B^*XA-A^*XB & A^*X^{-1}A+B^*X^{-1}B \end{pmatrix}. \end{aligned}$$

Using the above equalities we have that

$$\begin{aligned} &W^* \begin{pmatrix} 0 & -iI_n \\ iI_n & 0 \end{pmatrix} W = \begin{pmatrix} A^* & -B^* \\ B^* & A^* \end{pmatrix} (AA^*+BB^*)^{\frac{-1}{2}} \begin{pmatrix} 0 & -iI_n \\ iI_n & 0 \end{pmatrix} \\ &\times (AA^*+BB^*)^{\frac{-1}{2}} \begin{pmatrix} A & B \\ -B & A \end{pmatrix} = \begin{pmatrix} A^* & -B^* \\ B^* & A^* \end{pmatrix} (AA^*+BB^*)^{-1} \begin{pmatrix} iB & -iA \\ iA & iB \end{pmatrix} \\ &= \begin{pmatrix} i\left[A^*XB-B^*XA\right] & i\left[-A^*XA-B^*XB\right] \\ i\left[B^*XB+A^*XA\right] & i\left[-B^*XA+A^*XB\right] \end{pmatrix} = \begin{pmatrix} 0 & -iI_n \\ iI_n & 0 \end{pmatrix}, \end{aligned}$$

which proves that W satisfies the conditions of Proposition 2.2.7. □

For the canonical self-adjoint extension A_Θ of $T_{\min}$ we have the following special representations of the resolvent of A_Θ.

Theorem 4.4.3. *Let Θ be a self-adjoint relation in $\mathbb{C}^n$ and let $\{A,B\}$ be a symmetric Nevanlinna pair representative, i.e. $\Theta = \{\{-Bg, Ag\} : g \in \mathbb{C}^n\}$ where $A^*=A$ and $B^*=B$. Then the resolvent of A_Θ can be written as*

(i) $(A_\Theta-\lambda)^{-1} = H_\lambda + Y_{\lambda,\Theta}\gamma^*_{\bar\lambda,\Theta}$.

(ii) $(A_\Theta-\lambda)^{-1} = H_{\lambda,\Theta} + Y_{\lambda,\Theta}M_\Theta(\lambda)Y^*_{\bar\lambda,\Theta}$.

Here

$$H_\lambda = G_\lambda - \frac{1}{2}Y_\lambda J Y^*_{\bar\lambda}$$

and

$$H_{\lambda,\Theta} = H_\lambda - \frac{1}{2}Y_\lambda\left[(JA-B)B(AA+BB)^{-1}J-(JA-B)A(AA+BB)^{-1}\right]Y^*_{\bar\lambda}$$

are entire in λ. Furthermore, γ_λ and $M_\Theta(\lambda)$ are as in Lemma 4.4.2 and $Y_{\lambda,\Theta}$ is defined as

$$Y_{\lambda,\Theta} = Y_\lambda \frac{JA-B}{\sqrt{2}}(AA+BB)^{\frac{-1}{2}}.$$

Proof. Combining Proposition 2.2.6 with Lemma 4.1.4 for the resolvent of A_0 and Proposition 4.1.2 for the structure of γ_λ we obtain the following expression for the resolvent of A_Θ

$$(A_\Theta - \lambda)^{-1} = G_\lambda + Y_\lambda \Omega(\lambda) Y_{\bar{\lambda}}^*.$$

where

$$\Omega(\lambda) = -J\frac{M(\lambda)}{2}J - 2(I_n + Y_\lambda(b))^{-1}B(M(\lambda)B + A)^{-1}(I_n + Y_{\bar{\lambda}}(b))^{-*}.$$

By the formula for $M(\lambda)$ see Proposition 4.1.2 we have that

$$(I_n + Y_\lambda)^{-1} = \frac{JM(\lambda) + I_n}{2},$$

therefore $\Omega(\lambda)$ can also be written, using the Nevanlinna properties of $M(\lambda)$, as

$$\Omega(\lambda) = \frac{1}{2}\Big[-JM(\lambda)J - (JM(\lambda) + I_n)B(M(\lambda)B + A)^{-1}(I_n - M(\lambda)J)\Big].$$

Since

$$\begin{aligned}
&JM(\lambda)B(M(\lambda)B + A)^{-1}(I_n - M(\lambda)J) \\
=&J(M(\lambda)B + A)(M(\lambda)B + A)^{-1}(I_n - M(\lambda)J) - JA(M(\lambda)B + A)^{-1}(I_n - M(\lambda)J) \\
=&-JA(M(\lambda)B + A)^{-1}(I_n - M(\lambda)J) + J(I_n - M(\lambda)J),
\end{aligned}$$

$\Omega(\lambda)$ can be rewritten as

$$\Omega(\lambda) = \frac{1}{2}\Big[-J + (JA - B)(M(\lambda)B + A)^{-1}(I_n - M(\lambda)J)\Big].$$

which proves the first representation of the resolvent of A_Θ. For the second representation define X as

$$X = \frac{1}{2}\Big[J + (JA - B)B(AA + BB)^{-1}J - (JA - B)A(AA + BB)^{-1}\Big],$$

then we have that

$$\begin{aligned}
&2\,[\Omega(\lambda) + X] \\
=&(JA - B)(M(\lambda)B + A)^{-1}\Big[-M(\lambda)\,[AA + BB] + [M(\lambda)B + A]\,B\Big](AA + BB)^{-1}J \\
&+(JA - B)(M(\lambda)B + A)^{-1}\Big[[AA + BB] - [M(\lambda)B + A]\,A\Big](AA + BB)^{-1} \\
=&(JA - B)(M(\lambda)B + A)^{-1}\Big[-M(\lambda)AA + AB\Big](AA + BB)^{-1}J \\
&+(JA - B)(M(\lambda)B + A)^{-1}\Big[-M(\lambda)BA + BB\Big](AA + BB)^{-1} \\
=&(JA - B)(M(\lambda)B + A)^{-1}(M(\lambda)A - B)(AA + BB)^{-1}(-AJ - B) \\
=&(JA - B)(M(\lambda)B + A)^{-1}(M(\lambda)A - B)(AA + BB)^{-1}(JA - B)^*.
\end{aligned}$$

Here we used the fact that A and B commute, which is a consequence of the fact that they are self-adjoint combined with Proposition 2.5.1. Because $X^* = X$ and $\Omega(\lambda) = \Omega(\bar{\lambda})^*$, $X + \Omega(\lambda) = X^* + \Omega(\bar{\lambda})^*$, therefore we have that

$$\begin{aligned}
2\left[\Omega(\lambda) + X\right] &= 2\left[\Omega(\bar{\lambda}) + X\right]^* \\
&= (JA - B)(AA + BB)^{-1}(M(\bar{\lambda})^*A - B)(M(\bar{\lambda})^*B + A)^{-1}(JA - B)^* \\
&= (JA - B)(AA + BB)^{-1}(M(\lambda)A - B)(M(\lambda)B + A)^{-1}(JA - B)^*,
\end{aligned}$$

which proves the second representation of the resolvent of A_Θ. □

Let Z be defined as

$$Z = \operatorname{span}\{\{\varphi, \psi\}\} = \operatorname{span}\{\{\varphi_1, \psi_1\}, \ldots, \{\varphi_p, \psi_p\}\}$$

be an p-dimensional subspace of $\mathfrak{H} \times \mathfrak{H}$ such that $A_\Theta \cap Z = \{0, 0\}$. Without loss of generality it may be assumed that Z is symmetric, cf. Lemma 2.6.2. Define the p-dimensional restriction of A_Θ by $S_\Theta = A_\Theta \cap Z^*$, then S_Θ is a closed symmetric relation in $\mathfrak{H}$ with defect indices (p, p) by Theorem 2.6.3. We recall its adjoint, boundary mappings, γ-field and Weyl function.

$S_\Theta^* = A_\Theta \hat{\dot{+}} Z$ and each element $\{f, g\} \in S_\Theta^*$ can be decomposed uniquely as

$$\{f, g\} = \{f_0, g_0\} + \{\varphi, \psi\}c,$$

where $\{f_0, g_0\} \in A_\Theta$ and $c \in \mathbb{C}^n$. The mappings

$$\check{\Gamma}_0\{f, g\} = c, \quad \check{\Gamma}_1\{f, g\} =< \{f_0, g_0\}, \{\varphi, \psi\} >_{\mathfrak{H}},$$

are boundary mappings for S_Θ^* with associated γ-field χ_λ

$$\chi_\lambda = \varphi + (A_\Theta - \lambda)^{-1}(\lambda\varphi - \psi), \tag{4.14}$$

and Weyl function $Q(\lambda)$

$$Q(\lambda) = (\chi_\lambda(\cdot), \bar{\lambda}\varphi(\cdot) - \psi(\cdot))_{\mathfrak{H}^2}, \tag{4.15}$$

for $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Using the above boundary triplet for S_Θ^* the Kreĭn-Naimark formula, see (2.7), has the following form

$$(S_{\Theta, -\check{\tau}(\lambda)} - \lambda)^{-1} = (A_\Theta - \lambda)^{-1} - \chi_\lambda(Q(\lambda) + \check{\tau}(\lambda))^{-1}\chi_{\bar{\lambda}}^* \tag{4.16}$$

for Nevanlinna families $\check{\tau}(\lambda)$ in $\mathbb{C}^p$.

Note that $\{f_0 + \varphi c, g\} \in S_\Theta^* - \lambda$, $f_0 \in \operatorname{dom} S_\Theta^*$, if and only if $\{f_0 + \varphi c, g + \lambda f_0 - \psi c + \lambda\varphi c + \psi c\} = \{f_0, g + \lambda f_0 - \psi c + \lambda\varphi c\} + \{\varphi c, \psi c\} \in S_\Theta^*$. Therefore by definition $\{f_0 + \varphi c, g\} \in S_\Theta^* - \lambda$ if and only if $\Gamma\{f_0, g + \lambda f_0 + [\lambda\varphi - \psi]\, c\} \in \Theta$ and

$$Jf_0'(t) - [\lambda\Delta(t) + H(t)]\, f_0(t) = \Delta(t)\,[\lambda\varphi - \psi]\, c + \Delta(t)g(t), \quad t \in (a, b).$$

For $j = 1, \ldots, p$, let Π_λ be such that $\Pi_\lambda e_j$, $z \in \mathbb{C}^n$, is the unique solution of the inhomogeneous differential equation

$$(4.17)\quad Jf'(t) - [\lambda\Delta(t) + H(t)]\, f(t) = \Delta(t)\, [\lambda\varphi_j(t) - \psi_j(t)] + \Delta(t)g(t), \quad t \in (a, b),$$

satisfying the initial conditions $\Pi_\lambda(a)e_j = 0$. Note that such a solution exist by Proposition 3.4.2. With this notation the following lemma shows how the γ-field χ_λ in (4.14) can be expressed in terms of $Y_{\lambda,\Theta}$.

Lemma 4.4.4. *χ_λ and Π_λ are connected by*

$$\chi_\lambda = \varphi + \Pi_\lambda + Y_{\lambda,\Theta}C(\lambda),$$

where the $n \times p$ matrix $C(\lambda)$ is defined as

$$C(\lambda) = \gamma^*_{\bar\lambda,\Theta}\, [\lambda\varphi - \psi]\,. \tag{4.18}$$

Here $Y_{\lambda,\Theta}$ is as defined in Theorem 4.4.3 and $\gamma_{\lambda,\Theta}$ is as in Lemma 4.4.2.

Proof. Both $[\chi_\lambda - \varphi]\, e_j$ and $[\Pi_\lambda + Y_{\lambda,\Theta}C(\lambda)]\, e_j$, $1 \le j \le p$, are solutions of (4.17), if they also satisfy the same initial condition they must be equal by Theorem 3.1.1 proving the statement.

Since $\Pi_\lambda(a)e_j$ is by assumption zero, we need only show that $[\chi_\lambda(a) - \varphi(a)]\, e_j = [Y_{\lambda,\Theta}(a)C(\lambda)]\, e_j$, $1 \le j \le p$. Use the structure of $(A_\Theta - \lambda)^{-1}$ given by Theorem 4.4.3 in (4.14) to obtain, with $R_\lambda = \lambda\varphi - \psi$, the following result

$$\begin{aligned}[\chi_\lambda(a) - \varphi(a)]\, e_j = &\left[(G_\lambda R_\lambda)(a) - \frac{1}{2}Y_\lambda(a)JY^*_{\bar\lambda}R_\lambda + Y_{\lambda,\Theta}(a)\gamma^*_{\bar\lambda}R_\lambda\right] e_j \\ =&Y_{\lambda,\Theta}(a)\gamma^*_{\bar\lambda,\Theta}R_\lambda e_j = Y_{\lambda,\Theta}(a)C(\lambda)e_j,\end{aligned}$$

which proves the lemma. □

The functions $\Pi_{\lambda,j} = \Pi_\lambda e_j$ are entire in λ by Theorem 3.1.1, hence the $n \times 1$ matrix functions $\Upsilon_{\lambda,j}$ defined as

$$\Upsilon_{\lambda,j} = \varphi_j + \Pi_{\lambda,j}, \tag{4.19}$$

are also entire in λ. In terms of these functions the decomposition in (4.18) can be rewritten as

$$\chi_\lambda = \Upsilon_\lambda + Y_{\lambda,\Theta}C(\lambda), \tag{4.20}$$

where $\Upsilon_\lambda = \begin{pmatrix}\Upsilon_{\lambda,1} & \ldots & \Upsilon_{\lambda,p}\end{pmatrix}$ is the $n \times p$ matrix containing the factors $\Upsilon_{\lambda,j}$ as defined in (4.19). By means of $Y_{\lambda,\Theta}$ and $\Upsilon_{\lambda,j}$ we define F_λ by

$$F^*_{\bar\lambda} f = \begin{pmatrix} Y^*_{\bar\lambda,\Theta} \\ \Upsilon^*_{\bar\lambda,1} \\ \vdots \\ \Upsilon^*_{\bar\lambda,n} \end{pmatrix} f = \begin{pmatrix} \int_a^b Y_{\bar\lambda,\Theta}(t)^*\Delta(t)f(t)\mathrm{d}t \\ \int_a^b \Upsilon_{\bar\lambda,1}(t)^*\Delta(t)f(t)\mathrm{d}t \\ \vdots \\ \int_a^b \Upsilon_{\bar\lambda,p}(t)^*\Delta(t)f(t)\mathrm{d}t \end{pmatrix}. \tag{4.21}$$

Obviously, F_λ is entire in λ as each component is an entire function in λ. Using the above defined transformation, we have the following alternative expression for the resolvent of an extension $S_{\Theta,-\check{\tau}(\lambda)}-$, $\check{\tau}(\lambda)$ a Nevanlinna family, of S_Θ, see [22].

Proposition 4.4.5. *For a Nevanlinna family $\check{\tau}(\lambda)$ in $\mathbb{C}^p$ the Kreĭn-Naimark formula, see* (4.16)*, can be written as*

$$(S_{\Theta,-\check{\tau}(\lambda)} - \lambda)^{-1} = H_{\lambda,\Theta} + F_\lambda \Omega(\lambda) F_{\bar{\lambda}}^*,$$

where the $(n+p)\times(n+p)$ matrix function $\Omega(\lambda)$ is given by

$$\Omega(\lambda) = \begin{pmatrix} M_\Theta(\lambda) - C(\lambda)(Q(\lambda)+\check{\tau}(\lambda))^{-1}C(\bar{\lambda})^* & -C(\lambda)(Q(\lambda)+\check{\tau}(\lambda))^{-1} \\ -(Q(\lambda)+\check{\tau}(\lambda))^{-1}C(\bar{\lambda})^* & -(Q(\lambda)+\check{\tau}(\lambda))^{-1} \end{pmatrix},$$

the $n\times p$ matrix function $C(\lambda)$ is given by (4.18) *and $H_{\lambda,\Theta}$ is as in Theorem 4.4.3.*

Proof. In the Kreĭn-Naimark formula (4.16) use the decomposition (4.20) of χ and the structure of $(A_\Theta - \lambda)^{-1}$ given by Theorem 4.4.3. □

Note that the matrix $\Omega(\lambda)$ involves the Weyl functions $M_\Theta(\lambda)$, $Q(\lambda)$, the connecting factors $C(\lambda)$, and the Nevanlinna family $\check{\tau}(\lambda)$ describing the exit space extension. Next we will show that $\Omega(\lambda)$ is a Nevanlinna function,see [22]. Therefore define the $(n+p)\times(n+p)$ matrix function $W(\lambda)$ as

$$W(\lambda) = \begin{pmatrix} M_\Theta(\lambda) & C(\lambda) \\ C(\bar{\lambda})^* & Q(\lambda) \end{pmatrix}, \tag{4.22}$$

where $C(\lambda) = \begin{pmatrix} C_1(\lambda) & \dots & C_p(\lambda)\end{pmatrix}$ is the $n\times p$ matrix containing the connecting factors in (4.18), $M_\Theta(\lambda)$ is the Weyl function corresponding to the boundary triplet for A_{Θ^*}, see Lemma 4.4.2, and $Q(\lambda)$ is the Weyl function corresponding to the boundary triplet for S_{Θ^*}, see (4.15).

Lemma 4.4.6. *The $(n+p)\times(n+p)$ matrix function $W(\lambda)$ in* (4.22) *is a Nevanlinna function.*

Proof. First observe that $W(\lambda)$ is holomorphic for $\lambda \in \mathbb{C}\setminus\mathbb{R}$, and that the symmetry property $W(\lambda)^* = W(\bar{\lambda})$ holds. Next consider the Nevanlinna kernel for the function $W(\lambda)$

$$\begin{aligned} \frac{W(\lambda)-W(\mu)^*}{\lambda-\bar{\mu}} &= \begin{pmatrix} \frac{M_\Theta(\lambda)-M_\Theta(\mu)^*}{\lambda-\bar{\mu}} & \frac{C(\lambda)-C(\bar{\mu})}{\lambda-\bar{\mu}} \\ \frac{C(\bar{\lambda})^*-C(\mu)^*}{\lambda-\bar{\mu}} & \frac{Q(\lambda)-Q(\mu)^*}{\lambda-\bar{\mu}} \end{pmatrix} \\ &= \begin{pmatrix} \gamma_{\mu,\Theta}^*\gamma_{\lambda,\Theta} & \frac{C(\lambda)-C(\bar{\mu})}{\lambda-\bar{\mu}} \\ \frac{C(\bar{\lambda})^*-C(\mu)^*}{\lambda-\bar{\mu}} & \chi_\mu^*\chi_\lambda \end{pmatrix}, \end{aligned}$$

see Proposition 2.2.5. From the identity

$$\gamma_{\bar{\lambda},\Theta} = \left(I_{\mathfrak{H}} + (\bar{\lambda}-\mu)(A_\Theta - \bar{\lambda})^{-1}\right)\gamma_{\mu,\Theta},$$

see Proposition 2.2.5, and the definition of $C(\lambda)$, see (4.18), it follows that

$$\begin{aligned} C(\lambda) - C(\bar{\mu}) &= \gamma^*_{\bar{\lambda},\Theta}\,[\lambda\varphi - \psi] - \gamma^*_{\mu,\Theta}\,[\bar{\mu}\varphi - \psi] \\ &= \left[\left(I_{\mathfrak{H}} + (\bar{\lambda} - \mu)(A_\Theta - \bar{\lambda})^{-1}\right)\gamma_{\mu,\Theta}\right]^*[\lambda\varphi - \psi] - \gamma^*_{\mu,\Theta}\,[\bar{\mu}\varphi - \psi] \\ &= \gamma^*_{\mu,\Theta}\,[\lambda - \bar{\mu}]\,\varphi - \gamma^*_{\mu,\Theta}\,[\lambda - \bar{\mu}]\,(A_\Theta - \lambda)^{-1}\,[\lambda\varphi - \psi] \\ &= [\lambda - \bar{\mu}]\,\gamma^*_{\mu,\Theta}\big[\varphi + (A_\Theta - \lambda)^{-1}\,[\lambda\varphi - \psi]\big] = [\lambda - \bar{\mu}]\,\gamma^*_{\mu,\Theta}\chi_\lambda \end{aligned}$$

This implies that

$$\frac{C(\lambda) - C(\bar{\mu})}{\lambda - \bar{\mu}} = \gamma^*_{\mu,\Theta}\chi_\lambda,$$

and, by symmetry, also that

$$\frac{C(\bar{\lambda})^* - C(\mu)^*}{\lambda - \bar{\mu}} = \chi^*_\mu\gamma_{\lambda,\Theta}.$$

Hence, the Nevanlinna kernel of $W(\lambda)$ is given by

$$\frac{W(\lambda) - W(\mu)^*}{\lambda - \bar{\mu}} = \begin{pmatrix} \gamma^*_{\mu,\Theta}\gamma_{\lambda,\Theta} & \gamma^*_{\mu,\Theta}\chi_\lambda \\ \chi^*_\mu\gamma_{\lambda,\Theta} & \chi^*_\mu\chi_\lambda \end{pmatrix},$$

which shows that $\operatorname{Im} W(\lambda)/\operatorname{Im}\lambda \geq 0$ for $\lambda \in \mathbb{C}\setminus\mathbb{R}$. This completes the proof. □

Proposition 4.4.7. *The $(n+p)\times(n+p)$ matrix function $\Omega(\lambda)$ in Proposition 4.4.5 is a Nevanlinna function.*

Proof. Observe that the function $\Omega(\lambda)$ is holomorphic for $\lambda \in \mathbb{C}\setminus\mathbb{R}$, and that the symmetry condition $\Omega(\lambda)^* = \Omega(\bar{\lambda})$ holds. Thus we only need to show that $\frac{\Omega(\lambda)-\Omega(\lambda)^*}{\lambda-\bar{\lambda}} \geq 0$. Define the $(n+p)\times(n+p)$ matrix function $\Xi(\lambda)$ by

$$\Xi(\lambda) = \begin{pmatrix} I_n & -C(\bar{\lambda})(Q(\bar{\lambda}) + \tau(\bar{\lambda}))^{-1} \\ 0 & -(Q(\bar{\lambda}) + \check{\tau}(\bar{\lambda}))^{-1} \end{pmatrix}.$$

A straightforward calculation shows that

$$\frac{\Omega(\lambda) - \Omega(\lambda)^*}{\lambda - \bar{\lambda}} = \Xi(\lambda)\,(W(\lambda) - W(\lambda)^*)\,\Xi(\lambda)^* + \begin{pmatrix} C(\bar{\lambda}) \\ I_p \end{pmatrix}(R(\lambda,\lambda) - R(\lambda,\lambda)^*)\begin{pmatrix} C(\bar{\lambda}) \\ I_p \end{pmatrix}^*, \tag{4.23}$$

where the $p\times p$ matrix function $R(\lambda,\mu)$ is defined as

$$R(\lambda,\mu) = (Q(\bar{\mu}) + \check{\tau}(\bar{\mu}))^{-1}\left[I_p - Q(\lambda)(Q(\lambda) + \check{\tau}(\lambda))^{-1}\right].$$

Now note The first term in the righthand side of (4.23) is nonnegative for by Lemma 4.4.6. For the second term we have that

$$R(\lambda,\mu) - R(\lambda,\mu)^* = (Q(\bar{\mu}) + \check{\tau}(\bar{\mu}))^{-1}\,(\check{\tau}(\lambda) - \check{\tau}(\mu)^*)\,(Q(\lambda) + \check{\tau}(\lambda))^{-1}.$$

Hence the second term in the righthand side of (4.23) is also nonnegative, because $\check{\tau}(\lambda)$ is a Nevanlinna family. Therefore $\Omega(\lambda)$ is a Nevanlinna function. □

4.5 Regular Hamiltonian canonical systems with an interface condition

In this section, unless mentioned otherwise, Y_λ is such that the associated fundamental matrix $Y_\lambda(\cdot)$ satisfies $Y_\lambda(c) = I_{2m}$, $c \in (a,b)$. Additionally, we assume that the canonical system restricted to (a,c) or (c,b) is again a canonical system. Furthermore, we will use the notation $c+ = \lim_{t\downarrow c} t$ and $c- = \lim_{t\uparrow c} t$.

Consider functions on the interval (a,b), whose first m components vanish at both endpoints and allow a discontinuity at the point $c \in (a,b)$, i.e. we consider the relation

$$T = \{\{f,g\} \in \mathfrak{H}^2 : Jf' - Hf = \Delta g, \quad f \in \mathrm{AC}^{2m}_{\mathrm{loc}}((a,b)\setminus\{c\}) \quad f_1(a) = f_1(b) = 0\}. \tag{4.24}$$

Theorem 4.5.1. *Let $c \in (a,b)$ and let the relation $\breve{A}$ be defined as*

$$\breve{A} = \{\{f,g\} \in T_{max} : f(c) = f_1(a) = f_1(b) = 0\}.$$

Then $\breve{A}$ is a closed symmetric relation in $\mathfrak{H}$ with defect numbers $n_+(\breve{A}) = n_-(\breve{A}) = 2m$ and its adjoint $\breve{A}^*$ is T. A boundary triplet $\{\mathbb{C}^{2m}, \breve{\Gamma}_0, \breve{\Gamma}_1\}$ for $\breve{A}^*$ is given by

$$\breve{\Gamma}_0\{f,g\} := f(c+) - f(c-) \quad \text{and} \quad \breve{\Gamma}_1\{f,g\} = \begin{pmatrix} f_2(c-) \\ -f_1(c+) \end{pmatrix}, \quad \{f,g\} \in \breve{A}^*.$$

Here $\breve{A}_0 := \ker \breve{\Gamma}_0$ corresponds with the self-adjoint relation A_0 as in Theorem 4.2.1.

Proof. Note that the boundary mappings $\breve{\Gamma}_0$ and $\breve{\Gamma}_1$ are well-defined by Theorem 3.1.1, because (3.1) is regular at the point c. The fact that the defect numbers are $(2m, 2m)$ is a consequence of the fact that (3.1) considered on the interval (a,c) or (c,b) with m boundary conditions has defect numbers m, see Proposition 4.3.1.

To prove the remainder of the theorem we will use Theorem 2.2.4, therefore we need to show that T and $\breve{\Gamma}$ satisfty its condtions. The self-adjointness of $\ker \breve{\Gamma}_0 = A_0$ is evident by Theorem 4.2.1 and the surjectivity of the mapping $\breve{\Gamma}$ follows from Proposition 3.4.2 (iii) applied to the intervals (a,c) and (c,b), because the system (3.1) is assumed to be regular on (a,c) and (c,b). Finally, we prove that Γ satisfies Green's identity. Let $\{f,g\},\{h,k\} \in T$, then

$$\begin{aligned}(g,h)_{\mathfrak{H}} - (f,k)_{\mathfrak{H}} &= \int_a^{c-} [h(t)^* J f(t)]'\,\mathrm{d}t + \int_{c+}^{b} [h(t)^* J f(t)]'\,\mathrm{d}t \\ &= h(c-)^* J f(c-) - h(c+)^* J f(c+) \\ &= h_2(c-)^* f_1(c-) - h_1(c-)^* f_2(c-) - h_2(c+)^* f_1(c+) + h_1(c+)^* f_2(c+)\end{aligned}$$

while on the other hand

$$
\begin{aligned}
&(\breve{\Gamma}_1\{f,g\},\breve{\Gamma}_0\{h,k\}) - (\breve{\Gamma}_0\{f,g\},\breve{\Gamma}_1\{h,k\}) \\
&= [h(c+) - h(c-)]^* \begin{pmatrix} f_2(c-) \\ -f_1(c+) \end{pmatrix} - \begin{pmatrix} h_2(c-) \\ -h_1(c+) \end{pmatrix}^* (f(c+) - f(c-)) \\
&= [h_1(c+) - h_1(c-)]^* f_2(c-) - [h_2(c+) - h_2(c-)]^* f_1(c+) \\
&-h_2(c-)^* [f_1(c+) - f_1(c-)] + h_1(c+)^* [f_2(c+) - f_2(c-)] \\
&= h_2(c-)^* f_1(c-) - h_1(c-)^* f_2(c-) - h_2(c+)^* f_1(c+) + h_1(c+)^* f_2(c+),
\end{aligned}
$$

which proves that $\breve{\Gamma}$ satisfies the Green's identity. □

Proposition 4.5.2. *The γ-field associated with the boundary triplet defined in Theorem 4.5.1 is*

$$
\breve{\gamma}_\lambda = \begin{cases} \breve{\gamma}_{\lambda,1} = Y_\lambda \alpha(\lambda) = Y_\lambda \begin{pmatrix} X(\lambda)Z(b,\lambda) & X(\lambda) \\ -Z(a,\lambda)X(\lambda)Z(b,\lambda) & -Z(a,\lambda)X(\lambda) \end{pmatrix}, & a < t < c, \\ \breve{\gamma}_{\lambda,2} = Y_\lambda \beta(\lambda) = Y_\lambda \begin{pmatrix} X(\lambda)Z(a,\lambda) & X(\lambda) \\ -Z(b,\lambda)X(\lambda)Z(a,\lambda) & -Z(b,\lambda)X(\lambda) \end{pmatrix}, & c < t < b, \end{cases}
$$

for $\lambda \in \mathbb{C} \setminus \mathbb{R}$, where

$$
Z(t,\lambda) = Y_\lambda^{12}(t)^{-1} Y_\lambda^{11}(t) \quad \text{and} \quad X(\lambda) = (Z(a,\lambda) - Z(b,\lambda))^{-1}.
$$

Its Weyl function $\breve{M}(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, is given by

$$
\breve{M}(\lambda) = \begin{pmatrix} -Z(a,\lambda)X(\lambda)Z(b,\lambda) & -Z(a,\lambda)X(\lambda) \\ -X(\lambda)Z(a,\lambda) & -X(\lambda) \end{pmatrix}.
$$

Proof. Clearly, by Lemma 3.3.1 applied to the intervals (a,c) and (c,b) $Z(a,\lambda)$ and $Z(b,\lambda)$ are well-defined operators for $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Next we will show that $X(\lambda)$ is also well-defined. Therefore look at

$$
(Z(a,\lambda) - Z(b,\lambda)) = Y_\lambda^{12}(a)^{-1} \left[Y_\lambda^{11}(a) - Y_\lambda^{12}(a) Y_\lambda^{12}(b)^{-1} Y_\lambda^{11}(b) \right].
$$

Because $Y_\lambda^{12}(a)^{-1}$ and $Y_\lambda^{12}(b)^{-1}$ are invertible, the above expression is invertible if and only if

$$
\left[Y_\lambda^{11}(a) - Y_\lambda^{12}(a) Y_\lambda^{12}(b)^{-1} Y_\lambda^{11}(b) \right] g = 0,
$$

for $g \in \mathbb{C}^{2m}$ implies that $g = 0$. If g has the stated property, then

$$
f_\lambda = Y_\lambda(\cdot) \begin{pmatrix} I_m \\ -Y_\lambda^{12}(b)^{-1} Y_\lambda^{11}(b) \end{pmatrix} g,
$$

is an element of $\operatorname{dom}(A_0)$ and $\mathfrak{N}_\lambda(T_{\max})$ and thus $\{f_\lambda, \lambda f_\lambda\} \in A_0 \cap \widehat{\mathfrak{N}}_\lambda(T_{\max})$. Since A_0 is self-adjoint its intersection with $\widehat{\mathfrak{N}}_\lambda(T_{\max})$ is trivial for $\lambda \in \rho(A_0)$, see Lemma 2.2.3, therefore g must be zero and $(Z(a,\lambda) - Z(b,\lambda))$ is invertible, i.e. $X(\lambda)$ is well-defined.

To prove that $\breve{\gamma}_\lambda$ is the γ-field we need to show that $\breve{\Gamma}_0\hat{\breve{\gamma}}_\lambda g = g$, for $g \in \mathbb{C}^{2m}$, and that $\breve{\gamma}_\lambda g \in \mathfrak{N}_\lambda(\breve{A}^*)$. We begin by proving the latter condition, let $g \in \mathbb{C}^{2m}$, then

$$\begin{aligned}
[\breve{\gamma}_\lambda(a)g]_1 &= \left[Y_\lambda(a)\begin{pmatrix} X(\lambda)Z(b,\lambda) & X(\lambda) \\ -Z(a,\lambda)X(\lambda)Z(b,\lambda) & -Z(a,\lambda)X(\lambda)\end{pmatrix} g\right]_1 \\
&= \left(\left[Y_\lambda^{11}(a) - Y_\lambda^{12}(a)Z(a,\lambda)\right] X(\lambda)Z(b,\lambda) \quad \left[Y_\lambda^{11}(a) - Y_\lambda^{12}(a)Z(a,\lambda)\right] X(\lambda)\right) g \\
&= \left(\left[Y_\lambda^{11}(a) - Y_\lambda^{11}(a)\right] X(\lambda)Z(b,\lambda) \quad \left[Y_\lambda^{11}(a) - Y_\lambda^{11}(a)\right] X(\lambda)\right) g = 0.
\end{aligned}$$

The calculation to prove that $[\breve{\gamma}_\lambda(b)g]_1 = 0$ is similar to the above calculation, therefore we have proven that $\breve{\gamma}_\lambda g \in \mathfrak{N}_\lambda(\breve{A}^*)$. Now only the former condition is left;

$$\begin{aligned}
\breve{\Gamma}_0\hat{\breve{\gamma}}_\lambda g &= Y_\lambda(c)\left[\beta(\lambda) - \alpha(\lambda)\right] g = Y_\lambda(c)\left[\begin{pmatrix} X(\lambda)Z(a,\lambda) & X(\lambda) \\ -Z(b,\lambda)X(\lambda)Z(a,\lambda) & -Z(b,\lambda)X(\lambda)\end{pmatrix}\right. \\
&\quad \left. - \begin{pmatrix} X(\lambda)Z(b,\lambda) & X(\lambda) \\ -Z(a,\lambda)X(\lambda)Z(b,\lambda) & -Z(a,\lambda)X(\lambda)\end{pmatrix}\right] g \\
&= \begin{pmatrix} X(\lambda)\left[Z(a,\lambda) - Z(b,\lambda)\right] & 0 \\ Z(a,\lambda)X(\lambda)Z(b,\lambda) - Z(b,\lambda)X(\lambda)Z(a,\lambda) & \left[Z(a,\lambda) - Z(b,\lambda)\right] X(\lambda)\end{pmatrix} g \\
&= I_{2m} g = g.
\end{aligned}$$

Here it was used that

$$\begin{aligned}
Z(a,\lambda)X(\lambda)Z(b,\lambda) &= (Z(a,\lambda) - Z(b,\lambda))X(\lambda)Z(b,\lambda) + Z(b,\lambda)X(\lambda)Z(b,\lambda) \\
&= Z(b,\lambda) - Z(b,\lambda)X(\lambda)(Z(a,\lambda) - Z(b,\lambda)) + Z(b,\lambda)X(\lambda)Z(a,\lambda) \\
&= Z(b,\lambda) - Z(b,\lambda) + Z(b,\lambda)X(\lambda)Z(a,\lambda) = Z(b,\lambda)X(\lambda)Z(a,\lambda).
\end{aligned}$$

$\breve{M}(\lambda)$ is by definition given by

$$\begin{aligned}
\breve{M}(\lambda)g = \Gamma_1\hat{\breve{\gamma}}_\lambda g &= \begin{pmatrix} \left[Y_\lambda(c)\begin{pmatrix} X(\lambda)Z(b,\lambda) & X(\lambda) \\ -Z(a,\lambda)X(\lambda)Z(b,\lambda) & -Z(a,\lambda)X(\lambda)\end{pmatrix} g\right]_2 \\ -\left[Y_\lambda(c)\begin{pmatrix} X(\lambda)Z(a,\lambda) & X(\lambda) \\ -Z(b,\lambda)X(\lambda)Z(a,\lambda) & -Z(b,\lambda)X(\lambda)\end{pmatrix} g\right]_1 \end{pmatrix} \\
&= \begin{pmatrix} -Z(a,\lambda)X(\lambda)Z(b,\lambda) & -Z(a,\lambda)X(\lambda) \\ -X(\lambda)Z(a,\lambda) & -X(\lambda)\end{pmatrix} g.
\end{aligned}$$

□

The resolvents of extensions of the closed symmetric relation $\breve{A}$ are characterized by Nevanlinna families $\breve{\tau}(\lambda)$ in $\mathbb{C}^{2m}$, via the Kreĭn-Naimark formula, see (2.7), with γ-field $\breve{\gamma}_\lambda$ and Weyl function $\breve{M}(\lambda)$. Explicitly, with $\mathfrak{H}_1 = L^2_\Delta((a,c))$ and $\mathfrak{H}_2 = L^2_\Delta((c,b))$, the Kreĭn-Naimark formula can be written as

$$(\breve{A}_{-\breve{\tau}(\lambda)} - \lambda)^{-1} = (\breve{A}_0 - \lambda)^{-1} - \Theta_\lambda, \tag{4.25}$$

where

$$\Theta_\lambda = \begin{cases} \breve{\gamma}_{\lambda,1}\breve{B}(\lambda)\breve{\Lambda}(\lambda)\breve{\gamma}^*_{\bar{\lambda},1}|_{\mathfrak{H}_1} + \breve{\gamma}_{\lambda,1}\breve{B}(\lambda)\breve{\Lambda}(\lambda)\breve{\gamma}^*_{\bar{\lambda},2}|_{\mathfrak{H}_2}, & a < t < c, \\ \breve{\gamma}_{\lambda,2}\breve{B}(\lambda)\breve{\Lambda}(\lambda)\breve{\gamma}^*_{\bar{\lambda},1}|_{\mathfrak{H}_1} + \breve{\gamma}_{\lambda,2}\breve{B}(\lambda)\breve{\Lambda}(\lambda)\breve{\gamma}^*_{\bar{\lambda},2}|_{\mathfrak{H}_2}, & c < t < b. \end{cases} \tag{4.26}$$

Here $\breve{\gamma}_{\lambda,1}$ and $\breve{\gamma}_{\lambda,2}$ are as defined in Proposition 4.5.2 and $\breve{\Lambda}(\lambda) = (\breve{M}(\lambda)\breve{B}(\lambda) - \breve{A}(\lambda))^{-1}$.

To obtain an expression for the resolvent of $\breve{A}_{-\breve{\tau}(\lambda)}$, the resolvent of $\breve{A}_0$ has to be determined and decomposed in accordance with Θ_λ.

Lemma 4.5.3. *For $\lambda \in \rho(\breve{A}_0)$ the resolvent of $\breve{A}_0$ is given by*

$$(\breve{A}_0 - \lambda)^{-1} = G_\lambda + Y_\lambda \frac{Q(\lambda)}{2} Y_{\bar{\lambda}}^*,$$

wherer G_λ is entire in λ and $Q(\lambda)$ is given by

$$\begin{aligned} Q(\lambda) = &-\beta(\lambda)\begin{pmatrix} -Z(a,\lambda)^{-1} & 0 \\ 0 & Z(a,\lambda)\end{pmatrix}\alpha(\bar{\lambda})^* - \alpha(\lambda)\begin{pmatrix} -Z(b,\lambda)^{-1} & 0 \\ 0 & Z(b,\lambda)\end{pmatrix}\alpha(\bar{\lambda})^* \\ &+ \alpha(\lambda)\begin{pmatrix} -Z(b,\lambda)^{-1} & 0 \\ 0 & Z(b,\lambda)\end{pmatrix}\beta(\bar{\lambda})^* + \beta(\lambda)\begin{pmatrix} -Z(a,\lambda)^{-1} & 0 \\ 0 & Z(a,\lambda)\end{pmatrix}\beta(\bar{\lambda})^*. \end{aligned}$$

Proof. First, note that elements f in the resolvent of $\breve{A}_0$ are absolutely continuous at c and satisfy $f_1(a) = f_1(b) = 0$. By Proposition 3.1.4 the general form of the solution of the canonical system (3.1) is given by

$$(H_\lambda g)(t) = (G_\lambda g)(t) + Y_\lambda(t) d_\lambda(g), \tag{4.27}$$

where G_λ is entire in λ. Here d_λ, a $B(\mathfrak{H}, \mathbb{C}^{2m})$-valued function on $\rho(A_0)$, needs to be determined such that $(H_\lambda g)(t) \in \operatorname{dom}(\breve{A}_0)$ for all $g \in \mathfrak{H}$. Since $(H_\lambda g)(t) \in \operatorname{dom}(\breve{A}_0)$ if and only if $[(H_\lambda g)(a)]_1 = 0$ and $[(H_\lambda g)(b)]_1 = 0$, we have the following conditions on $d_\lambda = \begin{pmatrix} d_{\lambda,1} & d_{\lambda,2}\end{pmatrix}^T$

$$0 = [(H_\lambda g)(a)]_1 = Y_\lambda^{11}(a) d_{\lambda,1}(g) + Y_\lambda^{12}(a) d_{\lambda,2}(g) + \frac{1}{2}\begin{pmatrix} Y_\lambda^{12}(a) & -Y_\lambda^{11}(a)\end{pmatrix}(g(\cdot), Y_{\bar{\lambda}}(\cdot))_{\mathfrak{H}},$$

$$0 = [(H_\lambda g)(b)]_1 = Y_\lambda^{11}(b) d_{\lambda,1}(g) + Y_\lambda^{12}(b) d_{\lambda,2}(g) - \frac{1}{2}\begin{pmatrix} Y_\lambda^{12}(b) & -Y_\lambda^{11}(b)\end{pmatrix}(g(\cdot), Y_{\bar{\lambda}}(\cdot))_{\mathfrak{H}}.$$

After premultiplying the first equality by $Y_\lambda^{12}(a)^{-1}$ and the second equality by $Y_\lambda^{12}(b)^{-1}$ the following system of equations is obtained

$$\begin{aligned} Z(a,\lambda) d_{\lambda,1}(g) + d_{\lambda,2}(g) &= \frac{1}{2}\begin{pmatrix} -I_m & Z(a,\lambda)\end{pmatrix}(g(\cdot), Y_{\bar{\lambda}}(\cdot))_{\mathfrak{H}}, \\ Z(b,\lambda) d_{\lambda,1}(g) + d_{\lambda,2}(g) &= \frac{1}{2}\begin{pmatrix} I_m & -Z(b,\lambda)\end{pmatrix}(g(\cdot), Y_{\bar{\lambda}}(\cdot))_{\mathfrak{H}}. \end{aligned}$$

Here $Z(t,\lambda)$, $t \in (a,b)$, is as defined in Proposition 4.5.2. The above equations yield the following expressions for $d_{\lambda,1}$ and $d_{\lambda,2}$;

$$\begin{aligned} d_{\lambda,1} &= \frac{1}{2}\begin{pmatrix} -2X(\lambda) & X(\lambda)(Z(a,\lambda) + Z(b,\lambda))\end{pmatrix} Y_{\bar{\lambda}}^*, \\ d_{\lambda,2} &= \frac{1}{2}\begin{pmatrix} (Z(a,\lambda) + Z(b,\lambda)X(\lambda) & -2Z(a,\lambda)X(\lambda)Z(b,\lambda)\end{pmatrix} Y_{\bar{\lambda}}^*. \end{aligned}$$

If we use this result in (4.27) we obtain the indicated form of the resolvent of A_0 with $Q(\lambda)$ given by

$$\begin{aligned} Q(\lambda) &= \frac{1}{2}\begin{pmatrix} -2X(\lambda) & X(\lambda)(Z(a,\lambda)+Z(b,\lambda)) \\ (Z(a,\lambda)+Z(b,\lambda)X(\lambda) & -2Z(a,\lambda)X(\lambda)Z(b,\lambda) \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} I_m & 0 \\ 0 & Z(a,\lambda) \end{pmatrix}\begin{pmatrix} -X(\lambda) & X(\lambda) \\ X(\lambda) & -X(\lambda) \end{pmatrix}\begin{pmatrix} I_m & 0 \\ 0 & Z(b,\lambda) \end{pmatrix} \\ &+ \frac{1}{2}\begin{pmatrix} I_m & 0 \\ 0 & Z(b,\lambda) \end{pmatrix}\begin{pmatrix} -X(\lambda) & X(\lambda) \\ X(\lambda) & -X(\lambda) \end{pmatrix}\begin{pmatrix} I_m & 0 \\ 0 & Z(a,\lambda) \end{pmatrix}. \end{aligned} \tag{4.28}$$

To combine the resolvent of A_0 with formula (4.25), Y_λ in $Q(\lambda)$ must be rewritten in terms of $\breve{\gamma}_{\lambda,1}$ and $\breve{\gamma}_{\lambda,2}$. Therefore we need to introduce factors $\alpha(\lambda)$ and $\beta(\lambda)$ in $Q(\lambda)$. Note that $\alpha(\lambda)$, see Proposition 4.5.2, can be written as

$$\alpha(\lambda) = \begin{pmatrix} I_m & 0 \\ 0 & Z(a,\lambda) \end{pmatrix}\begin{pmatrix} -X(\lambda) & X(\lambda) \\ X(\lambda) & -X(\lambda) \end{pmatrix}\begin{pmatrix} -Z(b,\lambda) & 0 \\ 0 & I_m \end{pmatrix}$$

A similar result holds for $\beta(\lambda)$, namely with a replaced by b and vica versa. Rewrite $Q(\lambda)$ as in (4.28) using the above formula for $\alpha(\lambda)$ and $\beta(\lambda)$ as

$$\begin{aligned} Q(\lambda) &= \frac{1}{2}\alpha(\lambda)\begin{pmatrix} Z(b,\lambda)^{-1} & 0 \\ 0 & Z(b,\lambda) \end{pmatrix} \\ &+ \frac{1}{2}\beta(\lambda)\begin{pmatrix} Z(a,\lambda)^{-1} & 0 \\ 0 & Z(a,\lambda) \end{pmatrix}. \end{aligned}$$

Since $\beta(\lambda) - \alpha(\lambda) = I_{2m}$, see the calculations made in Proposition 4.5.2, $\beta(\bar{\lambda})^* - \alpha(\bar{\lambda})^* = I_{2m}$. If we now multiply the above expression for $Q(\lambda)$ from the right with $I_{2m} = \beta(\bar{\lambda})^* - \alpha(\bar{\lambda})^*$, we obtain the form of $Q(\lambda)$ as in the statement of the lemma. □

Note that compared to the four terms in (4.26) $Q(\lambda)$ in Lemma 4.5.3 contains eight terms, because to be able to compare the two formulas $Y_{\bar{\lambda}}^*$ needs to be written as $Y_{\bar{\lambda}}^*|_{\mathfrak{H}_1} + Y_{\bar{\lambda}}^*|_{\mathfrak{H}_2}$, where $\mathfrak{H}_1 = L^2_\Delta((a,c))$ and $\mathfrak{H}_2 = L^2_\Delta((c,b))$.

To simplify notation define $\kappa_\lambda(t)$ as

$$\kappa_\lambda(t) = \begin{pmatrix} -Z(t,\lambda)^{-1} & 0 \\ 0 & Z(t,\lambda) \end{pmatrix}.$$

Rewriting (4.25) using Lemma 4.5.3 and (4.27) gives the following result.

Theorem 4.5.4. *The resolvent of an extension $\breve{A}_{-\breve{\tau}(\lambda)}$ of $\breve{A}$ is given by*

$$(\breve{A}_{-\breve{\tau}(\lambda)} - \lambda)^{-1} = G_\lambda - Y_\lambda\frac{\Omega_1(\lambda)}{2}Y_{\bar{\lambda}}^*|_{\mathfrak{H}_1} - Y_\lambda\frac{\Omega_2(\lambda)}{2}Y_{\bar{\lambda}}^*|_{\mathfrak{H}_2}, \tag{4.29}$$

for $\lambda \in \rho(\breve{A}_{-\breve{\tau}(\lambda)}) \cap \rho(\breve{A}_0)$. *Here* G_λ *is entire in* λ *and, with* $\{\breve{A}(\lambda), \breve{B}(\lambda)\}$ *a symmetric Nevalinna pair representation of the Nevanlinna family* $\breve{\tau}(\lambda)$, $\breve{\Lambda}(\lambda)$ *is defined as*

$$\breve{\Lambda}(\lambda) = (\breve{M}(\lambda)\breve{B}(\lambda) - \breve{A}(\lambda))^{-1}.$$

Finally, with $\mathfrak{H}_1 = L^2_\Delta((a,c))$, $\mathfrak{H}_2 = L^2_\Delta((c,b))$ $\Omega_1(\lambda)$ *and* $\Omega_2(\lambda)$ *are given by*

$$\begin{aligned}\Omega_1(\lambda) = \alpha(\lambda) &\left[\left[\kappa_\lambda(b)\breve{M}(\lambda) + 2I_{2m}\right]\breve{B}(\lambda) - \kappa_\lambda(b)\breve{A}(\lambda)\right]\breve{\Lambda}(\lambda)\alpha(\lambda)^* \\ - \alpha(\lambda) &\left[\kappa_\lambda(b)\breve{M}(\lambda)\breve{B}(\lambda) - \kappa_\lambda(b)\breve{A}(\lambda)\right]\breve{\Lambda}(\lambda)\beta(\lambda)^* \\ - \beta(\lambda) &\left[\kappa_\lambda(a)\breve{M}(\lambda)\breve{B}(\lambda) - \kappa_\lambda(a)\breve{A}(\lambda)\right]\breve{\Lambda}(\lambda)\left[\beta(\bar{\lambda})^* - \alpha(\bar{\lambda})^*\right]\end{aligned}$$

and

$$\begin{aligned}\Omega_2(\lambda) = \alpha(\lambda) &\left[\kappa_\lambda(b)\breve{M}(\lambda)\breve{B}(\lambda) - \kappa_\lambda(b)\breve{A}(\lambda)\right]\breve{\Lambda}(\lambda)\alpha(\lambda)^* \\ - \alpha(\lambda) &\left[\left[\kappa_\lambda(b)\breve{M}(\lambda) - 2I_{2m}\right]\breve{B}(\lambda) - \kappa_\lambda(b)\breve{A}(\lambda)\right]\breve{\Lambda}(\lambda)\beta(\lambda)^* \\ - \beta(\lambda) &\left[\kappa_\lambda(a)\breve{M}(\lambda)\breve{B}(\lambda) - \kappa_\lambda(a)\breve{A}(\lambda)\right]\breve{\Lambda}(\lambda)\left[\beta(\bar{\lambda})^* - \alpha(\bar{\lambda})^*\right].\end{aligned}$$

for $a < t < c$. *While for* $c < t < b$ $\Omega_1(\lambda)$ *and* $\Omega_2(\lambda)$ *are given by*

$$\begin{aligned}\Omega_1(\lambda) = \beta(\lambda) &\left[\left[\kappa_\lambda(a)\breve{M}(\lambda) + 2I_{2m}\right]\breve{B}(\lambda) - \kappa_\lambda(a)\breve{A}(\lambda)\right]\breve{\Lambda}(\lambda)\alpha(\lambda)^* \\ - \beta(\lambda) &\left[\kappa_\lambda(a)\breve{M}(\lambda)\breve{B}(\lambda) - \kappa_\lambda(a)\breve{A}(\lambda)\right]\breve{\Lambda}(\lambda)\beta(\lambda)^* \\ - \alpha(\lambda) &\left[\kappa_\lambda(b)\breve{M}(\lambda)\breve{B}(\lambda) - \kappa_\lambda(b)\breve{A}(\lambda)\right]\breve{\Lambda}(\lambda)\left[\beta(\bar{\lambda})^* - \alpha(\bar{\lambda})^*\right]\end{aligned}$$

and

$$\begin{aligned}\Omega_2(\lambda) = \beta(\lambda) &\left[\kappa_\lambda(a)\breve{M}(\lambda)\breve{B}(\lambda) - \kappa_\lambda(a)\breve{A}(\lambda)\right]\breve{\Lambda}(\lambda)\alpha(\lambda)^* \\ - \beta(\lambda) &\left[\left[\kappa_\lambda(a)\breve{M}(\lambda) - 2I_{2m}\right]\breve{B}(\lambda) - \kappa_\lambda(a)\breve{A}(\lambda)\right]\breve{\Lambda}(\lambda)\beta(\lambda)^* \\ - \alpha(\lambda) &\left[\kappa_\lambda(b)\breve{M}(\lambda)\breve{B}(\lambda) - \kappa_\lambda(b)\breve{A}(\lambda)\right]\breve{\Lambda}(\lambda)\left[\beta(\bar{\lambda})^* - \alpha(\bar{\lambda})^*\right].\end{aligned}$$

Chapter 5: Examples

This chapter contains some examples of (systems of) differential equations which can be interpreted as canonical systems. Here the first example shows that the maximal and minimal relations, $T_{\max}$ and $T_{\min}$, associated with a canonical system can be relations with finite dimensional domains, see [31]. The second example is the well-known Sturm-Liouville differential equation. The last example shows that in fact a large class of differential equations can be interpreted as canonical systems, see [33]. For more in depth analysis of specific canonical system see for instance [35] and [36].

Example 5.1. Consider the system (3.1) on (a,b), $-\infty < a < b < \infty$, where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad H(t) = 0 \quad \text{and} \quad \Delta(t) = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, & a < t < c, \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, & c < t < b. \end{cases}$$

To prove that the above regular system of differential equations is a canonical system we need only prove that the system is definite.

Let $f = \begin{pmatrix} f_1 & f_2 \end{pmatrix}^T \in \mathrm{AC}^2_{\mathrm{loc}}((a,b))$ satisfy $Jf' = 0$ and $\Delta f = 0$ a.e. on (a,b). Since f_1 and f_2 are assumed to be locally absolutely continuous the former condition implies that f_1 and f_2 are constant. While the latter condition implies that f_1 is zero on (a,c) and f_2 is zero on (c,b). We conclude that f is the zero function which proves that system of differential equations is definite.

To understand the nature of the elements of the space $\mathfrak{H} = L^2_\Delta((a,b))$ we will determine its zero equivalance class, denoted by $\begin{pmatrix} \bar{0} & \tilde{0} \end{pmatrix}^T$. A staight-forward calculation shows that $\bar{0}$ contains all functions on (a,b) which are zero on (a,c), while $\tilde{0}$ contains all functions on (a,b) which are zero on (c,b).

Finally, we determine the domain of $T_{\max}$, see (3.11). Let $\{f,g\} \in T_{\max}$ and let $\tilde{f} = \begin{pmatrix} \tilde{f}_1 & \tilde{f}_1 \end{pmatrix}^T$ be the unique locally absolutely continuous representative of f and $\tilde{g} = \begin{pmatrix} \tilde{g}_1 & \tilde{g}_2 \end{pmatrix}^T$ a representative of g. Then $\{f,g\} \in T_{\max}$ if and only if

$$\tilde{f}_1'(t) = \begin{cases} 0, & a < t < c, \\ \tilde{g}_2(t), & c < t < b, \end{cases}$$
$$\tilde{f}_2'(t) = \begin{cases} -\tilde{g}_1(t), & a < t < c, \\ 0, & c < t < b \end{cases}$$

for some $\tilde{g} \in g$. From the above differential equation it follows that $\{f, g\} \in T_{\max}$ if and only if there exists numbers γ and δ and a representative $\tilde{g}$ of g such that

$$\tilde{f}_1(t) = \begin{cases} \gamma, & a < t < c, \\ \gamma + \int_c^t \tilde{g}_2(s)\mathrm{d}s, & c < t < b, \end{cases}$$
$$\tilde{f}_2(t) = \begin{cases} \delta - \int_t^c \tilde{g}_1(s)\mathrm{d}s, & a < t < c, \\ \delta, & c < t < b. \end{cases}$$

If we now define $\hat{f}$ as $\hat{f} = \begin{pmatrix} \gamma & \delta \end{pmatrix}^T$, then $\hat{f} - \tilde{f}$ is an element which is in the zero equivalence class of $\mathfrak{H}$. Thus we conclude that the unique locally absolutely continuous representative of f is in fact given by $\hat{f}$ and this allows us to conclude that the domain of $T_{\max}$ has dimension 2. Since the canonical system is regular it now follows that $T_{\min}$, see Proposition 3.4.2 (v), is such that the dimension of its domain is zero, i.e. $T_{\min}$ is a pure relation.

Example 5.2. Consider the Sturm-Liouville differential equation

$$-\left[p(t)f'(t)\right]' + q(t)f(t) = r(t)g(t), \quad t \in (a, b)$$

with p, q and r real-valued locally L^1-functions, $r(t) \geq 0$ a.e. on (a, b), $r(t) > 0$ on $[\alpha, \beta]$ and $p(t) \neq 0$ on (a, b). The Sturm-Liouville differential equation can be written as a first order differential system in the following manner;

$$\begin{pmatrix} f \\ pf' \end{pmatrix}' = \begin{pmatrix} 0 & \frac{1}{p} \\ q(t) & 0 \end{pmatrix} \begin{pmatrix} f \\ pf' \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ r & 0 \end{pmatrix} \begin{pmatrix} g \\ \tilde{g} \end{pmatrix}, \tag{5.1}$$

where $\tilde{g}$ is any function. The above system can be rewritten as

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} f \\ pf' \end{pmatrix}' = \begin{pmatrix} -q & 0 \\ 0 & \frac{1}{p} \end{pmatrix} \begin{pmatrix} f \\ pf' \end{pmatrix} + \begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} g \\ \tilde{g} \end{pmatrix}.$$

From which we see that the Sturm-Liouville differential equation can be interpreted as a canonical system with

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad H(t) = \begin{pmatrix} -q(t) & 0 \\ 0 & \frac{1}{p(t)} \end{pmatrix} \quad \text{and} \quad \Delta(t) = \begin{pmatrix} r(t) & 0 \\ 0 & 0 \end{pmatrix},$$

if we can show that the system is definite. Let therefore $f = \begin{pmatrix} f_1 & f_2 \end{pmatrix}^T \in \mathrm{AC}^2_{\mathrm{loc}}((a, b))$ satisfy $Jf' = Hf$ and $\Delta f = 0$ a.e. on (a, b). By the latter condition $rf_1 = 0$ a.e. on $[\alpha, \beta]$ which implies that $f_1 = 0$ on $[\alpha, \beta]$, because f_1 is locally absolutely continuous. The former condition implies that $f_2' = qf_1$ and $pf_1' = f_2$ on $[\alpha, \beta]$, from the second equality it follows that f_2 must be zero on $[\alpha, \beta]$ as well. Now on the intervals $(a, \alpha]$ and $[\beta, b)$ f is a solution of $Jf' = Hf$ with $f(\alpha) = 0$ and $f(\beta) = 0$. Therefore by the uniqueness of locally absolutely continuos solutions of this differential system, see Theorem 3.1.1, f is zero on $(a, \alpha]$ and $[\beta, b)$ as well, which allows us to conclude that the system is definite.

Here the space $\mathfrak{H}$ consists of elements $\begin{pmatrix} f_1 & f_2 \end{pmatrix}^T$, where the equivalence class f_2 contains all elements, and can therefore be identified with 0, while the equivalence class f_1 contains only one locally absolutely continuous function.

Example 5.3. An extension of the previous differential equation is the following differential equation of order $2m$,

$$\begin{aligned}&\sum_{j=0}^{m}(-1)^j(m_j(t)y^{(j)}(t))^{(j)}+i\sum_{j=0}^{m-1}\left[(g_j(t)y^{(j+1)})^{(j)}+(g_j(t)y^{(j)})^{(j+1)}\right]\\ &=\lambda\sum_{j=0}^{m-1}(-1)^j(n_j(t)y^{(j)}(t))^{(j)},\quad t\in(a,b).\end{aligned} \tag{5.2}$$

Here the coefficient $m_j(t)$, $g_j(t)$ and $n_j(t)$, $1\le j\le 2m$, are all real-valued functions with $m_{2n}(t)\neq 0$ a.e. on (a,b). $m_j(t), n_j(t)\in C^j((a,b))$ for $1\le j\le 2m$ and $g_j(t)\in C^{j+1}((a,b))$ for $1\le j\le 2m-1$. Finally, $n_j(t)\ge 0$, $1\le j\le 2m-1$, and $n_0(t)>0$ a.e. on (a,b).

With $[f_1(t)]_m = y(t)$ and $m=2n$, $n\in\mathbb{N}$, the differential equation (5.2) is equivalent to following first order differential system

$$\begin{pmatrix}y_1(t)\\ y_2(t)\end{pmatrix}'-\begin{pmatrix}D_{12}^*(t) & D_{22}(t)\\ -D_{11}(t) & -D_{12}(t)\end{pmatrix}\begin{pmatrix}y_1(t)\\ y_2(t)\end{pmatrix}=\lambda\begin{pmatrix}0 & 0\\ E_{11}(t) & 0\end{pmatrix}\begin{pmatrix}y_1(t)\\ y_2(t)\end{pmatrix},$$

where

$$D_{11}(t)=\begin{pmatrix} m_{2n-1}(t)-g_{2n-1}^2(t)m_{2n}^{-1}(t) & -ig_{2n-2}(t) & & & 0\\ ig_{2n-2}(t) & m_{2n-2}(t) & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & m_1(t) & -ig_0(t)\\ 0 & & & -ig_0(t) & m_0(t)\end{pmatrix},$$

$$D_{12}(t)=\begin{pmatrix} -ig_{2n-1}(t)m_{2n}^{-1}(t) & 1 & & 0\\ & 0 & \ddots & \\ & & \ddots & 1\\ 0 & & & 0\end{pmatrix}, D_{22}(t)=\begin{pmatrix} -m_{2n}^{-1}(t) & & & 0\\ & 0 & & \\ & & \ddots & \\ 0 & & & 0\end{pmatrix},$$

$$E_{11}(t)=\begin{pmatrix} n_{2n-1}(t) & & 0\\ & \ddots & \\ 0 & & n_0(t)\end{pmatrix}.$$

The above system of differential equations can be rewritten as

$$J\begin{pmatrix}y_1(t)\\ y_2(t)\end{pmatrix}-H(t)\begin{pmatrix}y_1(t)\\ y_2(t)\end{pmatrix}=\lambda\Delta(t)\begin{pmatrix}y_1(t)\\ y_2(t)\end{pmatrix},$$

where

$$J=\begin{pmatrix}0 & -I_m\\ I_m & 0\end{pmatrix},\quad H(t)=\begin{pmatrix}D_{11}(t) & D_{12}(t)\\ D_{12}^*(t) & D_{22}(t)\end{pmatrix}\quad\text{and}\quad \Delta(t)=\begin{pmatrix}E_{11}(t) & 0\\ 0 & 0\end{pmatrix}.$$

This shows that the differential equation in (5.2) can be interpreted as is a canonical differential equation, if we can show that the above system is definite.

To prove that the system is definite let $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in \mathrm{AC}^{2n}((a,b))$, $f_1, f_2 \in \mathrm{AC}^{n}((a,b))$, satisfy $Jf' = Hf$ and $\Delta f = 0$ a.e. on (a,b). Since $n_0(t) > 0$ a.e. on (a,b) the latter condition implies that $[f_1(t)]_n = 0$. Then the last n equations of $Jf' = Hf$ show that $f_1(t) = 0$ and $[f_2(t)]_1 = 0$. Finally, the first $n-1$ equations of $Jf' = Hf$ show that $[f_2(t)]_j = 0$, $2 \le j \le n$, proving the definiteness.

In case $m = 2n+1$, $n \in \mathbb{N}$, in addition to the earlier assumptions on the coefficients of (5.2) assume that $g_m > 0$ a.e. on (a,b). Then the differential equation (5.2) can be written as

$$Jf'(t) - H(t)f(t) = \Delta(t)f(t),$$

where

$$J = \begin{pmatrix} 0 & 0 & I_n \\ 0 & i & 0 \\ -I_n & 0 & 0 \end{pmatrix},$$

$$H = -\begin{pmatrix}
m_{m-1} & -ig_{m-2} & & 0 & i\frac{g_{2m-1}}{\sqrt{2g_m}} & 0 & 1 & & 0 \\
ig_{m-2} & \ddots & \ddots & & 0 & & \ddots & \ddots & \\
& \ddots & \ddots & -ig_0 & \vdots & & & \ddots & 1 \\
0 & & ig_0 & m_0 & 0 & 0 & & & 0 \\
-i\frac{g_{2m-1}}{\sqrt{2g_m}} & 0 & \cdots & 0 & \frac{m_m}{2g_m} & \frac{1}{\sqrt{2g_m}} & 0 & \cdots & 0 \\
0 & & & 0 & \frac{1}{\sqrt{2g_m}} & & & & \\
1 & \ddots & & & 0 & & 0 & & \\
& \ddots & \ddots & & \vdots & & & & \\
0 & & 1 & 0 & 0 & & & &
\end{pmatrix},$$

$$\Delta = -\begin{pmatrix}
n_{m-1} & & & 0 & 0 & \\
& \ddots & & & \vdots & 0 \\
& & \ddots & & \vdots & \\
0 & & & n_0 & 0 & \\
0 & \cdots & \cdots & 0 & \frac{n_m}{2g_m} & 0 \\
& 0 & & & 0 & 0
\end{pmatrix}.$$

Here the dependence on t of the quantities is not made explicit.

To prove that the system is definite, note that since $n_0(t) > 0$ a.e. on (a,b) by $\Delta f = 0$ we know that $[f(t)]_n = 0$, if $f \in \mathrm{AC}_{\mathrm{loc}}((a,b))$. Then the last n equations of $Jf' - Hf = 0$ show that $[f(t)]_i = 0$, $1 \le i \le n$ and $[f(t)]_{n+1} = 0$. By the middle equation of $Jf' - Hf = 0$ if follows that $[f(t)]_{n+2} = 0$ and finally, the first $n-1$ equations of $Jf' = Hf$ show that $[f(t)]_j = 0$, $n+3 \le j \le 2n+1$, proving the definiteness.

Appendix A: Existence and uniqueness proof

In this appendix Theorem 3.1.1 will be proven. Recall that we investigate the following system of differential equations

(A.1)
$$f'(t,\lambda) = F(t,f,\lambda) = \tilde{F}(t,\lambda)f(t,\lambda) + \hat{F}(t) = [H(t) + \lambda\Delta(t)]\, f(t,\lambda) + \Delta(t)g(t)$$

in $\mathbb{C}^n$ with initial condition $f(\tau,\lambda) = \xi(\lambda)$, $\tau \in (a,b)$. Note that here the dependence of the solution of (3.1) on λ is made explicit. Since we allow F to be discontinuous, we look for a solution $f(t,\lambda)$ of

$$f(t,\lambda) = \xi(\lambda) + \int_\tau^t F(s, f(s,\lambda), \lambda)\mathrm{d}s. \tag{A.2}$$

Here we recall that the following conditions are assumed to hold; $\hat{F} = \Delta g \in L^1_{\text{loc}}((a,b))$, $\tilde{F}(\cdot,\lambda) = [H(\cdot) + \lambda\Delta(\cdot)] \in L^1_{\text{loc}}((a,b))$ and $\xi(\lambda)$ are entire functions in λ with values in $\mathbb{C}^n$. Finally, the $\mathbb{C}^n$-valued function F is defined on the domain $D = ((a,b), \mathbb{C}^n, \mathbb{C})$ and satisfies

$$|F(t,x,\lambda) - F(t,\tilde{x},\lambda)| \le k(t,\lambda)|x - \tilde{x}| \tag{A.3}$$

on that domain for the function $k(\cdot,\lambda) = ||H(\cdot) + \lambda\Delta(\cdot)||_\infty$. Furthermore, $k(\cdot,\lambda) \in L^1_{\text{loc}}((a,b))$, because

$$\begin{aligned} \int_\alpha^\beta |k(t,\lambda)|\mathrm{d}t &\le \sum_{i,j=1}^n \int_\alpha^\beta |H(t) + \lambda\Delta(t)|_{ij}\,\mathrm{d}t \\ &\le \sum_{i,j=1}^n \left[\int_\alpha^\beta |H(t)|_{ij}\mathrm{d}t + |\lambda| \int_a^b |\Delta(t)|_{ij}\mathrm{d}t \right] \end{aligned}$$

and by the condition (C2) H and Δ are locally integrable.

We will prove Theorem 3.1.1 in multiply steps. First a lemma will be proven, which will be used to prove the uniqueness of the local solution of a system of differential equations as in (A.1). After this initial step, the local existence and uniqueness of a locally absolutely continuous solution of (A.1) will be proven using a Picard iteration method. Finally, we will show that the local absolutely continuous solution can be extended to a solution on the whole of (a,b), which is absolutely continuous at a boundary point if the boundary point is regular.

Lemma A.1. Let $\phi_1(t,\lambda)$, $\phi_2(t,\lambda)$ be two continuous functions in t on $[\alpha,\beta] \subset (a,b)$ such that $(t,\phi_i(t,\lambda),\lambda) \in D$, $1 \le i \le 2$, for $t \in [\alpha,\beta]$ and $\lambda \in \mathbb{C}$. For $\tau \in [\alpha,\beta]$

write $\phi_i(t,\lambda)$ as

(A.4) $$\phi_i(t,\lambda) = \phi_i(\tau,\lambda) + \int_\tau^t F(s,\phi_1(s),\lambda)\mathrm{d}s + E_i(t,\lambda), \quad 1 \le i \le 2.$$

Then with $|\phi_1(\tau,\lambda) - \phi_2(\tau,\lambda)| \le \delta(\lambda)$ and $E(t,\lambda) = |E_1(t,\lambda)| + |E_2(t,\lambda)|$ the following estimate holds for $\tau \le t \le \beta$

$$|\phi_1(t,\lambda) - \phi_2(t,\lambda)| \le \delta(\lambda)e^{\int_\tau^t k(s)\mathrm{d}s} + E(t,\lambda) + \int_\tau^t E(s,\lambda)k(s,\lambda)e^{\int_s^t k(u,\lambda)\mathrm{d}u}\mathrm{d}s$$

and a similar estimate holds for $\alpha \le t \le \tau$.

Proof. Consider the case that $\tau \le t < \beta$, similar arguments hold if $\alpha < t \le \tau$. Subtracting (A.4) with $i = 2$ from (A.4) with $i = 1$, we obtain

$$\begin{aligned}\phi_1(t,\lambda) - \phi_2(t,\lambda) = \phi_1(\tau,\lambda) - \phi_2(\tau,\lambda) + \int_\tau^t \left[F(s,\phi_1(s),\lambda) - F(s,\phi_2(s),\lambda)\right]\mathrm{d}s \\ + E_1(t,\lambda) - E_2(t,\lambda),\end{aligned}$$

Taking norms we obtain

$$|\phi_1(t,\lambda)-\phi_2(t,\lambda)| \le |\phi_1(\tau,\lambda)-\phi_2(\tau,\lambda)| + \int_\tau^t k(s,\lambda)|\phi_1(s,\lambda)-\phi_2(s,\lambda)|\mathrm{d}s + E(t,\lambda).$$

With the notation $\psi(\cdot,\lambda) = |\phi_1(\cdot,\lambda)-\phi_2(\cdot,\lambda)|$, the above equation can be rewritten as

(A.5)
$$\psi(t,\lambda) \le \psi(\tau,\lambda) + \int_\tau^t k(s,\lambda)\psi(s,\lambda)\mathrm{d}s + E(t,\lambda) \le \delta(\lambda) + \int_\tau^t k(s,\lambda)\psi(s,\lambda)\mathrm{d}s + E(t,\lambda).$$

Define

$$R(t,\lambda) = \int_\tau^t k(s,\lambda)\psi(s,\lambda)\mathrm{d}s,$$

then, since $k(\cdot,\lambda)$ is locally absolutely integrable and $\psi(\cdot,\lambda)$ continuous $R(\cdot,\lambda)$ is absolutely continuous, and

$$R'(t,\lambda) = k(t,\lambda)\psi(t,\lambda).$$

Because $k(\cdot,\lambda)$ is nonnegative the function $R(\cdot,\lambda)$ allows us to write (A.5) as

$$R'(t,\lambda) - k(t,\lambda)R(t,\lambda) \le k(t,\lambda)\left[\delta(\lambda) + E(t,\lambda)\right]$$

or, equivalently,

$$\left[e^{-\int_\tau^t k(u,\lambda)\mathrm{d}u}R(t,\lambda)\right]' \le e^{-\int_\tau^t k(u,\lambda)\mathrm{d}u}k(t,\lambda)\left[\delta(\lambda) + E(t,\lambda)\right].$$

Integrating both sides we obtain

$$e^{-\int_\tau^t k(u,\lambda)\mathrm{d}u}R(t,\lambda) \le \int_\tau^t e^{-\int_\tau^s k(u,\lambda)\mathrm{d}u}k(s,\lambda)\left[\delta(\lambda) + E(s,\lambda)\right]\mathrm{d}s,$$

where we used the fact that $R(\tau, \lambda) = 0$. The above inequality implies that

$$\begin{aligned}
R(t,\lambda) &\leq e^{\int_\tau^t k(u,\lambda)\mathrm{d}u} \int_\tau^t e^{-\int_\tau^s k(u,\lambda)\mathrm{d}s} k(s,\lambda)\left[\delta(\lambda) + E(s,\lambda)\right] \mathrm{d}s \\
&= \int_\tau^t e^{\int_s^t k(u,\lambda)\mathrm{d}u} k(s,\lambda)\left[\delta(\lambda) + E(s,\lambda)\right] \mathrm{d}s \\
&= \int_\tau^t -\frac{\mathrm{d}}{\mathrm{d}s}\left[e^{\int_s^t k(u,\lambda)\mathrm{d}u}\right] \mathrm{d}s\, \delta(\lambda) + \int_\tau^t e^{\int_s^t k(u,\lambda)\mathrm{d}u} k(s,\lambda) E(s,\lambda)\mathrm{d}s \\
&= \left[e^{\int_\tau^t k(u,\lambda)\mathrm{d}u} - 1\right] \delta(\lambda) + \int_\tau^t e^{\int_s^t k(u,\lambda)\mathrm{d}u} k(s,\lambda) E(s,\lambda)\mathrm{d}s.
\end{aligned}$$

Using this result in (A.5) proves the lemma. □

Proposition A.2. For $\tau \in [\alpha, \beta] \subset (a, b)$, the successive approximations ϕ_k defined as

$$\begin{aligned}
\phi_0(t,\lambda) &= \xi(\lambda), \\
\phi_n(t,\lambda) &= \xi(\lambda) + \int_\tau^t F(t, \phi_{n-1}(s), \lambda)\mathrm{d}s,
\end{aligned} \tag{A.6}$$

exist on $[\alpha, \beta]$ as continuous functions in t and entire in λ. Moreover, they converge uniformly on $[\alpha, \beta]$ to the unique solution $\phi(t, \lambda)$ of (A.1), which is absolutely continuous in t and analytic in λ.

Proof. Assume that $t \leq \tau \leq \beta$, then $\phi_0(t, \lambda)$ is bounded in t, continuous in t and analytic in λ, furthermore $(t, \phi_0(t, \lambda), \lambda) \in D$. Assume that $\phi_{n-1}(t, \lambda)$ is bounded in t, continuous in t, entire in λ and $(t, \phi_{n-1}(t, \lambda), \lambda) \in D$. By the last property it follows from (A.6) that $\phi_n(t, \lambda)$ is well-defined, entire and continuous in t. Moreover,

$$\begin{aligned}
|\phi_n(t,\lambda) - \xi(\lambda)| &= \left|\int_\tau^t \tilde{F}(s,\lambda)\phi_{n-1}(s,\lambda) + \hat{F}(s)\mathrm{d}s\right| \\
&\leq \int_\tau^t |\tilde{F}(s,\lambda)\phi_{n-1}(s,\lambda)| + \int_\tau^t |\hat{F}(s)|\mathrm{d}s \leq K(t)||\phi_{n-1}(t,\lambda)||_\infty + L(t).
\end{aligned}$$

Here

$$K(t) = \int_\tau^t |\tilde{F}(s,\lambda)|\mathrm{d}s \quad \text{and} \quad L(t) = \int_\tau^t |\hat{F}(s)|\mathrm{d}s$$

are bounded by assumption, because $\tilde{F}(\cdot, \lambda)$ and $\hat{F}(\cdot)$ are elements of $L^1_{\mathrm{loc}}((a, b))$. We conclude that $\phi_n(\cdot, \lambda)$ is bounded and therefore $(t, \phi_n(t, \lambda), \lambda) \in D$. By induction we have now proven that all $\phi_n(t, \lambda)$ are well-defined, continuous in t and analytic in λ.

Let $\delta_n(t, \lambda)$ be defined as

$$\delta_n(t,\lambda) = |\phi_{n+1}(t,\lambda) - \phi_n(t,\lambda)|,$$

then

$$\delta_0(t,\lambda) = \left|\phi_1(t,\lambda) - \phi_0(t,\lambda)\right| = \left|\int_\tau^t \tilde{F}(s,\lambda)\phi_0(s,\lambda) + \hat{F}(s)\mathrm{d}s\right|$$
$$\leq \int_\tau^t |\tilde{F}(s,\lambda)\xi(\lambda)| + \int_\tau^t |\hat{F}(s)|\mathrm{d}s \leq ||\xi(\lambda)||_\infty K(t) + L(t).$$

Assume that $\delta_k(t,\lambda) \leq (1+||\xi(\lambda)||_\infty)\frac{[K(t)+L(t)]^{k+1}}{(k+1)!}$, then we have already shown that this formula is correct for $k=0$. Assume that the formula is correct for $k \leq n-1$, then

$$\delta_n(t,\lambda) = |\phi_{n+1}(t,\lambda) - \phi_n(t,\lambda)| = \left|\int_\tau^t \tilde{F}(s,\lambda)\left[\phi_n(s,\lambda) - \phi_{n-1}(s,\lambda)\right]\mathrm{d}s\right|$$
$$\leq \int_\tau^t \left[|\tilde{F}(s,\lambda)| + |\hat{F}(s)|\right]\left[1 + ||\xi(\lambda)||_\infty\right]\frac{[K(s)+L(s)]^n}{n!}\mathrm{d}s$$
$$= \left[1 + ||\xi(\lambda)||_\infty\right]\int_\tau^t \frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{[K(s)+L(s)]^{n+1}}{(n+1)!}\right]\mathrm{d}s = \left[1 + ||\xi(\lambda)||_\infty\right]\frac{[K(t)+L(t)]^{n+1}}{(n+1)!},$$

which proves the induction hypothesis. It follows that

$$\sum_{i=1}^\infty \delta_i(t,\lambda) \leq \sum_{i=0}^\infty \left[1 + ||\xi(\lambda)||_\infty\right]\frac{[K(t)+L(t)]^{n+1}}{(n+1)!} \leq \left[1 + ||\xi(\lambda)||_\infty\right]e^{[K(t)+L(t)]}.$$

Since $K(t)+L(t)$ is increasing on $[\tau,\beta]$ and bounded for $t=\beta$, the above inequality shows that the series $\sum_i \delta_i(\cdot,\lambda)$ is uniformly convergent on $[\tau,\beta]\times K$ for every compact subset K of $\mathbb{C}$. This implies that the series

$$\phi_0(t,\lambda) + \sum_{i=0}^\infty \left[\phi_{i+1}(t,\lambda) - \phi_0(t,\lambda)\right]$$

is absolutely and uniformly convergent on $[\tau,\beta]\times K$, consequently the partial sums, $\phi_n(t,\lambda)$, tend uniformly on $[\tau,\beta]\times\mathbb{C}$ to a function $\phi(t,\lambda)$ which is continuous in t and entire in λ.

Next we show that $\phi(t,\lambda)$ satisfies

$$\phi(t,\lambda) = \xi(\lambda) + \int_\tau^t F(s,\phi(s,\lambda),\lambda)\mathrm{d}s, \tag{A.7}$$

from which we can conclude that $\phi(t,\lambda)$ is absolutely continuous on $[\tau,\beta]$, by taking the limit in (A.6) with respect to n. Similar arguments show that $\phi(\cdot,\lambda)$ is an absolutely continous solution of (A.1) on $[\alpha,\tau]$. This proof will proceed in multiple steps; first we need show that the righthand side in (A.7) is well-defined, i.e. we need to show that for all $t\in[\tau,\beta]$ $(t,\phi(t,\lambda),\lambda)\in D$. Therafter we will show that the righthand side of (A.6) converges to the righthand side of (A.7), which proves that $\phi(t,\lambda)$ satisfies (A.7) because $\phi_n(t,\lambda)$ converges uniformly to $\phi(t,\lambda)$.

It is clear that $(t,\phi(t,\lambda),\lambda) \in D$ for $t \in [\tau,\beta]$, because for every $t \in [\tau,\beta]$ $(t,\phi_n(t,\lambda),\lambda) \in D$ for the open domain D combined with the uniform convergence, on compact subsets of $(a,b) \times \mathbb{C}$, of $\phi_n(t,\lambda)$ to $\phi(t,\lambda)$. For the second statement note that

$$\begin{aligned}
&\lim_{n\to\infty} \left| \int_\tau^t F(s,\phi(s,\lambda),\lambda) - F(s,\phi_n(s,\lambda),\lambda)\mathrm{d}s \right| \\
&\leq \lim_{n\to\infty} \int_\tau^t ||\tilde{F}(s,\lambda)|_\infty |\phi(s,\lambda) - \phi_n(s,\lambda)|\mathrm{d}s \leq \lim_{n\to\infty} \int_\tau^t k(s,\lambda)|\phi(s,\lambda) - \phi_n(s,\lambda)|\mathrm{d}s \\
&= \int_\tau^t k(s,\lambda) \lim_{n\to\infty} |\phi(s,\lambda) - \phi_n(s,\lambda)|\mathrm{d}s = 0,
\end{aligned}$$

where we used the dominated convergence theorem. We were allowed to used that theorem, because $k(\cdot,\lambda)|\phi(\cdot,\lambda) - \phi_n(\cdot,\lambda)|$ is dominated by absolutely integrable function $k(\cdot,\lambda)C$ for some constant C. Such a constant C exists as a consequence of the uniform convergence of $\phi_n(\cdot,\lambda)$.

To prove uniqueness assume that both $\phi(\cdot,\lambda)$ and $\psi(\cdot,\lambda)$ satisfy (A.2) on $[\alpha,\beta]$. Then $\chi(\cdot,\lambda) = \phi(\cdot,\lambda) - \psi(\cdot,\lambda)$ satisfies

$$\chi'(t,\lambda) = \tilde{F}(t,\lambda)\chi(t,\lambda), \quad t \in [\alpha,\beta],$$

with initial condition $\chi(\tau,\lambda) = 0$. But since the zero function on $[\alpha,\beta]$ also satisfies the same differential equation with the same initial condition as $\chi(\cdot,\lambda)$, Lemma A.1 shows that $\chi(\cdot,\lambda) = 0$ on $[\alpha,\beta]$ proving the uniqueness. □

Theorem A.3. On (a,b) there exists an unique locally absolutely function $f(\cdot,\lambda)$ satisfying (A.1). Moreover, if the endpoint a (or b) is regular, then $f(\cdot,\lambda)$ is locally absolutely continuous on $[a,c)$ (or $(a,b]$).

Proof. By Proposition A.2 there exists an unique absolutely continuous solution $\hat{f}(\cdot,\lambda)$ satisfying $\hat{f}(\tau,\lambda) = \xi(\lambda)$ of (A.1) on every compact subinterval $[\alpha,\beta]$ of (a,b) which contains τ. It remains to show that absolutely continuous solution $\hat{f}(\cdot,\lambda)$ can be extended.

Define $\tilde{\xi}(\lambda)$ as $\tilde{\xi}(\lambda) = \hat{f}(\beta,\lambda)$, then there exist an unique absolutely function $\tilde{f}(\cdot,\lambda)$ which is a solution of (A.1) on the compact interval $[\gamma,\delta]$, $\gamma < \beta < \delta$, by proposition A.2. Now define $f(\cdot,\lambda)$ as

$$f(t,\lambda) = \begin{cases} \hat{f}(t,\lambda), & \alpha \leq t \leq \beta, \\ \tilde{f}(t,\lambda), & \beta \leq t \leq \delta, \end{cases}$$

then $f(\cdot,\lambda)$ is the unique absolutely continuous solution of (A.1) on $[\alpha,\gamma]$ which satisfies $f(\tau,\lambda) = \xi(\lambda)$. Using the above procedure the absolutely continuous solution $f(\cdot,\lambda)$ can be extended to an absolutely continuous solution on any compact interval ι of (a, b).

If a is regular, we can consider (A.1) on $[a, \beta]$, $\beta < b$ and by Proposition A.2 there exists an absolutely continuous solution of (A.1) on $[a, \beta]$. By the above arguments we can extend this absolutely continuous solution to an absolutely continuous solution on any compact subset ι of $[\alpha, b)$. The remainder of the theorem is now clear. $\square$

Bibliography

[1] J. Behrndt and H.S.V. de Snoo, *Extension theory of symmetric operators in Hilbert spaces*, Lecture notes, manuscript, 2007.

[2] J. Behrndt, S. Hassi, and H.S.V. de Snoo, ”Boundary relations, unitary colligations, and functional models”, manuscript, 2007.

[3] E.A. Coddington, ”Extension theory of formally normal and symmetric subspaces”, Mem. Amer. Math. Soc., 134 (1973).

[4] E.A. Coddington, ”Self-adjoint subspace extensions of nondensely defined symmetric operators”, Adv. in Math., 14 (1974), 309–332.

[5] E.A. Coddington, ”Self-adjoint problems for non-densely defined ordinary differential operators and their eigenfunction expansions”, Adv. in Math., 15 (1974), 1–40.

[6] E.A. Coddington and A. Dijksma, ”Self-adjoint subspaces and eigenfunction expansions for ordinary differential subspaces”, J. Differential Equations, 20 (1976), 473–526.

[7] E.A. Coddington and N. Levinson, *Theory of ordinary differential equations*, McGraw-Hill, New York, 1955.

[8] V.A. Derkach and M.M. Malamud, ”Generalized resolvents and the boundary value problems for hermitian operators with gaps”, J. Funct. Anal., 95 (1991), 1-95.

[9] V.A. Derkach and M.M. Malamud, ”The extension theory of hermitian operators and the moment problem”, J. Math. Sciences, 73 (1995), 141-242.

[10] V.A. Derkach, S. Hassi, M.M. Malamud, and H.S.V. de Snoo, ”Generalized resolvents of symmetric operators and admissibility”, Report of the Department of Mathematics, 252, University of Helsinki, 2000.

[11] V.A. Derkach, S. Hassi, M.M. Malamud, and H.S.V. de Snoo, ”Generalized resolvents of symmetric operators and admissibility”, Methods of Functional Analysis and Topology, 6 (2000), no.3, 24–55.

[12] V.A. Derkach, S. Hassi, M.M. Malamud, and H.S.V. de Snoo, ”Boundary relations and their Weyl families”, Trans. Amer. Math. Soc., 358 (2006), 5351–5400.

[13] V.A. Derkach, S. Hassi, M.M. Malamud, and H.S.V. de Snoo, ”Boundary relations and generalized resolvents of symmetric operators”, 2007.

[14] A. Dijksma, A.A. El Sabbagh, and H.S.V. de Snoo, "Selfadjoint extensions of regular canonical systems with Stieltjes boundary conditions", J. Math. Anal. Appl., 152 (1990), 546–583.

[15] A. Dijksma, H. Langer, and H.S.V. de Snoo, "Hamiltonian systems with eigenvalue depending boundary conditions", Operator Theory Adv. Appl., 35 (1988), 37–83.

[16] A. Dijksma and H.S.V. de Snoo, "Self-adjoint extensions of symmetric subspaces", Pacific Journal of Mathematics, 54 (1974), 71-100.

[17] A. Dijksma and H.S.V. de Snoo, "Eigenfunction expansions for nondensely defined differential operators", J. Differential Equations, 17 (1975), 198–219.

[18] J. Elstrodt, *Mass-und Integrationstheorie*, 4. korrigierte Auflage, Springer Verlag, 2005.

[19] V.I. Gorbachuk and M.L. Gorbachuk, *Boundary value problems for operator differential equations*, Mathematics and its Applications (Soviet Series), 48, Kluwer Academic Publishers, Dordrecht, 1991.

[20] D.B. Hinton and J.K. Shaw, "On Titchmarsh-Weyl $m(\lambda)$-functions for linear Hamiltonian systems", J. Differential Equations, 40 (1981), 316-342.

[21] S. Hassi, H.S.V. de Snoo, and H. Winkler, "Boundary-value problems for two-dimensional canonical systems", Integral Equations Operator Theory, 36 (2000), 445–479.

[22] S. Hassi, H.S.V. de Snoo, A.E. Sterk, and H. Winkler, "Nonstandard boundary conditions for a class of Sturm-Liouville operators", Revue Roumaine de Mathematiques Pures et Appliquees, 51 (2006), 641–653.

[23] S. Hassi, H.S.V. de Snoo, A.E. Sterk, and H. Winkler, "Finite-dimensional graph perturbations of selfadjoint Sturm-Liouville operators", Tiberiu Constantinescu Memorial Volume, to appear.

[24] V.I. Kogan and F.S. Rofe-Beketov, "On square-integrable solutions of symmetric systems of differential equations of arbitrary order", Proc. Roy. Soc. Edinburgh Sect. A, 74 (1976), 5-40.

[25] H. Langer and B. Textorius, "On generalized resolvents and Q-functions of symmetric linear relations", Pacific J. Math., 72 (1977), 135–165.

[26] H. Langer, and B. Textorius, "L-resolvent matrices of symmetric linear relations with equal defect numbers; appliccations to canonical differential relations", Integral Equations and Operator Theory, 5 (1982), 208-243.

[27] V.E. Lyantse and O.G. Storozh, *Methods of the theory of unbounded operators*, Naukova Dumka, Kiev, 1983 (Russian).

[28] O.Ya. Mylyo, *Differential-boundary operators with integral boundary conditions and corresponding operator theory models*, Thesis for the Candidate Degree, Lviv, 1998.

[29] O.Ya. Mylyo and O.G. Storozh, "Selfadjointness and maximal dissipativeness conditions for a class of finite-dimensional perturbations of a positive definite operator", Mat. Stud., 7 (1997), 97–102.

[30] H.D. Niessen, "Singuläre S-hermitische Rand-Eigenwertprobleme", manuscripta math., 3 (1970), 35–68.

[31] B.C. Orcutt, *Canonical Differential Equations*, Dissertation, University of Virginia, 1969.

[32] M. Schechter, *Principles of functional analysis*, AMS, 2nd edition, 2001.

[33] A. Schneider, "Zur Einordnung selbstadjungierter Rand-Eigenwertprobleme bei gewöhnlichen Differentialgleichungen in die Theorie S-hermitescher Rand-Eigenwertprobleme", Math. Annalen, 178 (1968), 277-294.

[34] A. Schneider, "Weitere Untersuchungen ber singulre reelle S-hermitesche Differentialgleichungssysteme im Normalfall", Math. Z., 109 (1969), 153–168.

[35] H.S.V. de Snoo and H. Winkler, "Canonical systems of differential equations with selfadjoint interface conditions on graphs", Proc. Roy. Soc. Edinburgh, 135A (2005), 297–315.

[36] H.S.V. de Snoo and H. Winkler, "Two-dimensional trace-normed canonical systems of differential equations and selfadjoint interface conditions", Integral Equations Operator Theory, 51 (2005), 73–108.

[37] O.G. Storozh, *Extension theory methods and differential-boundary operators*, Doctor of Science thesis, Lviv, 1995.