Representation Theory and Quarks

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Abstract

This thesis introduces representation theory. It gives the necessary mathematics for its applications on the quark model. These applications are discussed, especially the pentaquark is analyzed on its symmetries.

Contents

1 Introduction

Each chapter of mathematics has an application in physics. This thesis will go into one of those chapters, representation theory, and one of its applications in physics, quarks. Quarks have been hypothesized to be the building blocks of many particles, called hadrons. These hadrons have a symmetry, i.e., they can be transformed into each other with a set of matrices SU_n. Hadrons can not be transformed into arbitrary hadrons, but they form groups, called multiplets. A multiplet is a group of particles which is invariant under the transformations SU_n. For the study of hadrons it is thus of interest to study the invariant groups of SU_n.

To study the groups describing the symmetries of hadrons knowledge of the symmetric groups S_n is needed. These groups will be studied with the help of the mathematical representation theory. Chapters 2 through 4 will introduce this theory at the hand of many examples. In chapter 5 it is explained how representation theory can be used as a model for quarks. One of the important goals of this thesis is to introduce a general mathematical formalism that is needed to study the quark model, which is done in chapter 6. Most of the mathematics in this chapter is given without proof but still a few examples are given to clarify the theory. In chapter 7 the formalism is applied to the quark model. The formalism nicely explains the need of a new symmetry called colour. It will also be shown it what manner one can construct exotic hadrons.

One of the exotic hadrons that can be constructed is the pentaquark. As an application of the mathematical formalism introduced in this thesis, this particle is studied in chapter 8. The pentaquark is an exotic particle that is neither a baryon nor a meson; it consists of 4 quarks and an antiquark. It has a complete different structure compared to the hadrons discovered so far. Because of this it will give more insight on the forces described by Quantum Chromo Dynamics. These particles are thus of great interest for development of QCD. A lot of experiments have been done to find the pentaquark, in particular the Θ particle. For an overview of these experiments see [1]. So far there is no extraordinary proof for the extraordinary claim of the pentaquark.

The analysis of the pentaquark will be given in this thesis only as an application of the mathematical formalism. To minimize the computations needed, the spin of quarks won't be considered during this analysis. Nevertheless, it will be shown that the flavour and colour symmetries give limits on how can make a pentaquark. More direct and complete analyses of the pentaquarks have been done, [2]. Although the model here is incomplete, it is still comparable with those described in the literature.

2 Representation Theory

Representation theory is a way to describe groups. It is assumed that the reader is acquainted with group theory. In representation theory one starts with a map from the group to a space of linear maps. Each element of the group is thus associated with an invertible linear transformation. This is done in a way such that properties of the group are conserved, in other words, the map is a homomorphism. One can wonder whether the maps that can be constructed in this way have a certain ordering. This leads to the concept of irreducible representations. These representations are in a way the basis of all representations. How these look like depends of course on the group one starts with. But first a more formal definition for representations is given.

2.1 Constructing Representations

The formal definition of a representation of a group is:

Definition 2.1. *A representation of a group G on a finite dimensional complex vector space V is a map ρ from G to the group of automorphisms of V:*

$$\rho : G \to GL(V),$$

which is a homomorphism.

The requirement of homomorphism in definition 2.1 explicitly means:

$$\rho(a \times b) = \rho(a)\rho(b), \tag{1}$$

where $a, b \in G$.

Example 2.1. Take an arbitrary group G and let the vector space V be the 1-dimensional complex space $\mathbb{C}$. Now let ρ map all elements of G to the map $\rho(g)z = z$, in other words, the identity map. It can easily be checked that this is a representation. It is called the trivial representation.

In example 2.1 it is shown that representations can be very simple, in the next example another simple 1-dimensional representation will be constructed.

Example 2.2. Take a symmetric group S_n and let again the vector space V be the 1-dimensional complex space $\mathbb{C}$. Now map all the even permutations to the identity map (1) and map all the odd permutations to the negative identity map (-1). In other words $\rho(g)v = \operatorname{sgn}(g)v$ for all $v \in V$. Because for all $a, b \in G$:

$$\operatorname{sgn}(a \times b) = \operatorname{sgn}(a) \cdot \operatorname{sgn}(b), \tag{2}$$

the map is homomorphic. This representation is called the alternating representation.

The next theorem will be used further on in the next chapter. It is stated here to give more insight on the alternating representation.

Theorem 2.2. *Given a finite group G with a 1-dimensional representation ρ. The subset $S \subset G$ on which ρ is trivial: $S = \{g|\rho(g) = 1\}$ is a normal subgroup whose quotient is cyclic.*

Proof. Part if this theorem is actually a special case of a lemma in group theory stating: If the map $\phi: \ G_1 \to G_2$ is a group homomorphism, then $\text{Ker}(\phi)$ is a normal subgroup. This can now be applied to ρ from which follows that S is a normal subgroup of G.

Now take two elements $g, \ h \in G \backslash S$ and state that $\rho(g) = \rho(h)$. This is equivalent to saying that $\exists s \in S$ s.t. $g = sh$ from which follows that g and h are equivalent in G/S. Because G/S has finite number of elements ($\#G/S = n$), the elements of $\rho(\bar{g}) \in G/S$ are n-th roots of unity. Because $\rho(\bar{g}) \neq \rho(\bar{h})$ if $\bar{g} \neq \bar{h}$ the set of transformation must be all the n-th unity roots, which is a cyclic group. From this it follows that G/S must also be a cyclic group. $\square$

Example 2.3. The group S_3 has two normal subgroups whose quotient is cyclic: the group itself and the alternating subgroup A_3. Therefore, the group S_3 can have only two different 1-dimensional representations. One which is trivial on the whole group, the trivial representation. And one which is trivial only on the alternating subgroup group A_3, that would be the alternating representation of example 2.2. For S_3 the alternating representation is:

g	(e)	(12)	(13)	(23)	(123)	(132)
$\rho(g)$	1	-1	-1	-1	1	1

According to the definition of a representation, one refers to the map when one speaks of a representation. If it is clear what the underlying map is one usually refers to the space V as being the representation of the group. When this is the case one can speak of the dimension of the representation, which is equal to the dimension of the space V. The trivial and the alternating representations are both 1-dimensional representations. In the next example a higher dimensional representation will be constructed.

Example 2.4. Take again a symmetric group S_n and let the transformation $\rho(g), \ g \in G$ work on an n-dimensional representation in the following way:

$$\rho(g)\vec{x} = \rho(g) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_{g^{-1}(1)} \\ x_{g^{-1}(2)} \\ \vdots \\ x_{g^{-1}(n)} \end{pmatrix}. \tag{3}$$

In S_3 the group elements will be mapped to the following matrices:

$$\rho(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho(123) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \rho(132) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\rho(12) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho(23) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \rho(13) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

I will call this representation the permutation representation. To check whether this actually is a representation, whether it satisfies the holomorphic condition one can do the following calculation:

$$\rho(g_1)\rho(g_2)\vec{x} = \begin{pmatrix} (\rho(g_2)x)_{g_1^{-1}(1)} \\ (\rho(g_2)x)_{g_1^{-1}(2)} \\ \vdots \\ (\rho(g_2)x)_{g_1^{-1}(n)} \end{pmatrix} \tag{4}$$

$$= \begin{pmatrix} x_{g_2^{-1} \times g_1^{-1}(1)} \\ x_{g_2^{-1} \times g_1^{-1}(2)} \\ \vdots \\ x_{g_2^{-1} \times g_1^{-1}(n)} \end{pmatrix} = \begin{pmatrix} x_{(g_1 \times g_2)^{-1}(1)} \\ x_{(g_1 \times g_2)^{-1}(2)} \\ \vdots \\ x_{(g_1 \times g_2)^{-1}(n)} \end{pmatrix} = \rho(g_1 \times g_2)\vec{x}. \tag{5}$$

Example 2.5. In this example the regular representation will be constructed. Take an arbitrary finite group G with n elements. The representation R is n dimensional, and has the following basis vectors: $\{e_g | g \in G\}$. Thus each basis vector is associated with a group element A transformation works on a arbitrary basis vector in the following way:

$$\rho(g)e_h = e_{gh}. \tag{6}$$

If one orders the elements of S_3 as: $\{e,\ (123),\ (132),\ (12),\ (23),\ (13)\}$ the element (12) will be mapped to the matrix:

$$\rho(12) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

To check whether the map is homomorphic, it is sufficient to check whether equation (1) holds for an arbitrary basis vector:

$$\rho(g)\rho(h)e_i = \rho(g)e_{hi} \tag{7}$$

$$= e_{ghi} = \rho(g \times h)e_i. \tag{8}$$

2.2 Combining Representations

As was shown in the examples in the previous section, representations can be constructed in different ways. It is also possible to construct more representations from given representations. Before this is done the representations constructed so far will be given the following symbols.

Representation	Symbol
Trivial	U
Alternating	U'
Permutation	P
Regular	R

Table 1: names and symbols of four different representations

The symbols refer to the vector spaces. One way to construct new representations is to sum representations. If S is a sum of two vector spaces ($S = V \oplus W$) the representation coinciding with S can be defined in the following manner:

$$\rho_S(g)\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \rho_V(g)(a) \\ \rho_W(g)(b) \end{pmatrix}. \tag{9}$$

Because things are getting a little abstract, it is useful to look closely at the notation. ρ_S is the representation corresponding to the vector space S. Thus ρ_S is by definition a map from the group G to a space of transformations. Because $g \in G$, $\rho_S(g)$ is a transformation, working on the vector space S: $\rho_S(g) \in \text{End}(S)$. Furthermore, $(a, b)^T \in S$.

So summing two representations is actually summing two vector spaces. The first representation describes how to transform the first part of the vector space and the second representation describes how to transform the second part of the vector space. Looking at the construction of this representation should be enough to convince the reader that it is indeed a representation, that the defining map ρ_S is indeed homomorphic.

Example 2.6. Take for example the trivial representation U and the alternating representation U'. The sum representation $U \oplus U'$ of the group S_2 will take form as the following matrices:

$$\rho(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho(12) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The same trick can be performed with tensor product spaces: $Pr = V \otimes W$. In this case the defining equation for the representation coinciding with the product space is:

$$\rho_{Pr}(g)[a \otimes b] = [\rho_V(g)(a) \otimes \rho_W(g)(b)]. \tag{10}$$

Here $a \in V$ and $b \in W$. To describe this equation in further detail one can construct a basis for the product space $V \otimes W$. The standard basis is constructed as follows: $\{v_i \otimes w_j\}$ where v_i is the basis for V and w_j the basis for W. So written out the basis for Pr becomes:

$$\begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_m \\ p_{m+1} \\ \vdots \\ p_{m \times n} \end{pmatrix} = \begin{pmatrix} v_1 \otimes w_1 \\ v_1 \otimes w_2 \\ \vdots \\ v_1 \otimes w_m \\ v_2 \otimes w_1 \\ \vdots \\ v_n \otimes w_m \end{pmatrix}. \tag{11}$$

It can easily be shown that the map defined by equation (10) is homomorphic:

$$\begin{aligned} \rho_{Pr}(g_2)\rho_{Pr}(g_1)[a \otimes b] &= \rho_{Pr}(g_2)[\rho_V(g_1)(a) \otimes \rho_W(g_1)(b)] \\ &= \rho_V(g_2)\rho_V(g_1)(a) \otimes \rho_W(g_2)\rho_W(g_1)(b) \\ &= \rho_V(g_2 \times g_1)(a) \otimes \rho_W(g_2 \times g_1)(b) \\ &= \rho_{Pr}(g_2 \times g_1)[a \otimes b]. \end{aligned}$$

Example 2.7. Look at the product space of the representations of S_2: $Pr = (U \oplus U') \otimes P$. In table 1 it is described to what representations these symbols refer. A little calculation shows that the dimension of this new representation is 4. For clarity, let e be the basis vector of U, let f be the basis vector for $U'$and let $\{e_{(e)}, e_{(12)}\}$ be the basis vectors for P. As an example of how to calculate the matrix elements of this new representation, look at $\rho_{Pr}(12)(p_3)$:

$$\begin{aligned} \rho_{Pr}(12)(p_3) &= \rho_{Pr}(12)[f \otimes e_{(e)}] \\ &= [\rho_{U \oplus U'}(12)(f) \otimes \rho_P(12)(e_{(e)})] \\ &= [-(f) \otimes (e_{(12)})] \\ &= -p_4. \end{aligned}$$

The other matrix elements are also given:

$$\rho_{Pr}(e) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \rho_{Pr}(12) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

The next way to create more representations that is going to be introduced is the second symmetric power $\mathrm{Sym}^2 V$. The space $\mathrm{Sym}^2 V$ consists of pairs of vectors $(v \otimes w)$, for $v, w \in V$ in quite the same way as a product space. The difference is that the vector pairs are symmetric, meaning that the component $(e_1 \otimes e_2)$ is equal to $(e_2 \otimes e_1)$. The basis of $\mathrm{Sym}^2 V$ thus consists only of those pairs $(e_i \otimes e_j)$ where $i < j$. The space $\mathrm{Sym}^2 V$ is $n(n+1)/2$ dimensional where n is the dimension of V. Given a representation on V, the defining equation for the representation coinciding with the symmetric power $\mathrm{Sym}^2 V$ takes the same form as with the tensor product, see equation (10):

$$\rho_{\mathrm{Sym}^2 V}(g)(v \otimes w) = (X(w) \otimes X(v)), \tag{12}$$

$v, w \in V$ and $X = \rho_V(g)$. In the same way as with the product space it can be shown that this map is also homomorphic.

Example 2.8. In this example the second symmetric power of the projection representation of A_3 will be taken. To explicitly calculate matrix elements one first has to choose a basis. Here the following basis has been chosen: $\{(e_1 \otimes e_1), (e_1 \otimes e_2), (e_1 \otimes e_3), (e_2 \otimes e_2), (e_2 \otimes e_3), (e_3 \otimes e_3)\}$. The matrices describing the transformations are going to be:

$$\rho_{\mathrm{Sym}^2 V}(e) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \rho_{\mathrm{Sym}^2 V}(123) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

$$\rho_{\mathrm{Sym}^2 V}(132) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

In the same way as the symmetric power was just defined, it is also possible to define a kind of antisymmetric power, called exterior power. The second exterior power $\wedge^2 V$ is an $n(n-1)/2$ dimensional space. Again it consists of pairs of vectors $(v \otimes w)$ for $v, w \in V$ but this time the component $(e_i \otimes e_j)$ is equal to $-(e_j \otimes e_i)$. In particular, this implies $(v \otimes v) = 0$ for all $v \in V$ The map defining the representation is given by:

$$X(v \otimes w) = 1/2[(X(w) \otimes X(v)) - (X(v) \otimes X(w))], \tag{13}$$

$v, w \in V$ and $X = \rho_V(g)$. As an example the matrix elements of the exterior power of the projection representation of A_3 are given:

Example 2.9. With the following basis for $\wedge^2 Pr$: $\{(e_1 \otimes e_2), (e_1 \otimes e_3), (e_2 \otimes e_3)\}$ the matrix elements of the second exterior power of the projection representations will be:

$$\rho_{\wedge^2 Pr}(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho_{\wedge^2 Pr}(123) = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\rho_{\wedge^2 Pr}(132) = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}.$$

A totally different way to create more representations is by means of the dual V^* of a representation V. This is the space of linear maps from $\phi: V \to \mathbb{C}$. The defining map for this representation is:

$$\rho_{V^*}(g)(\phi) = \phi \rho_V(g^{-1}). \tag{14}$$

So in this case, not just vectors are being transformed but maps. The space of maps is still linear so there is no problem in doing this. Because $\rho_V(g^{-1})$ is a map from V to V and ϕ is a map from V to $\mathbb{C}$, the map $\rho_{V^*}(g)(\phi)$ in equation (14) being a map from V to $\mathbb{C}$ is well defined. One can also easily check if the map is homomorphic:

$$\begin{aligned} \rho_{V^*}(g_1)\rho_{V^*}(g_2)\phi &= \rho_{V^*}(g_1)\phi\rho_V(g_2^{-1}) \\ &= \phi\rho_V(g_2^{-1})\rho_V(g_1^{-1}) \\ &= \phi\rho_V((g_1 \times g_2)^{-1}) = \rho_{V^*}(g_1 \times g_2)\phi. \end{aligned}$$

Example 2.10. This is an example of how the map $\phi: V \to \mathbb{C}$ works. Let V be the permutation representation P of the group S_3 which is worked out in example 2.4. Let ϕ be the map $-6e_1 + ie_2 + 2e_3$. Then the element (123) transforms this map to:

$$\rho_{V^*}(123)\phi = (-6,\ i,\ 2)\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = (2,\ -6,\ i). \tag{15}$$

The dual representation V is actually a special case of $\mathrm{Hom}(V, W)$, where V and W are representations of the same group G. $\mathrm{Hom}(V, W)$ is the notation for all linear maps $\phi: V \to W$. The associated map of the representation $\mathrm{Hom}(V, W)$ is:

$$\rho_{\mathrm{Hom}(V,W)}g(\phi) = \rho_W(g)\phi\rho_V(g^{-1}). \tag{16}$$

Checking that this map is well defined and is homomorphic is very similar to the case of the dual map:

$$\begin{aligned} g_1 g_2 \phi &= g_1[\rho_W(g_2)\phi\rho_V(g_2^{-1})] \\ &= \rho_W(g_1)\rho_W(g_2)\phi\rho_V(g_2^{-1})\rho_V(g_1^{-1}) \\ &= \rho_W(g_1 \times g_2)\phi\rho_V(g_1 \times g_2)^{-1} = (g_1 \times g_2)\phi. \end{aligned}$$

Comparing definitions one can see that when W is the trivial representation $\mathbb{C}$ the following equation holds: $V^* = \mathrm{Hom}(V, \mathbb{C})$. This can be generalized to the following equation:

$$\mathrm{Hom}(V, W) \sim V^* \otimes W. \tag{17}$$

The equivalence between these two representations is not trivial. Let v be a basis for V and w a basis for W. The equivalence of the two representations is shown in figure 1. It can be seen by associating to each element $(v^* \otimes w) \in V^* \otimes W$ a map $\phi_{v,w} \in \mathrm{Hom}(V, W)$ which maps v to w and all other vectors to zero. Now transforming the tensor gives the new tensor $(v^*\rho(g^{-1}) \otimes \rho(g)w)$. In the same way this tensor should be associated with the map $\phi_{(\rho(g)v,\rho(g)w)} = \rho(g)\phi_{(v,w)}\rho(g^{-1})$ which is exactly the map $\rho(g)\phi$.

$$\begin{array}{ccc} v^* \otimes w & \overset{\rho(g)}{\rightarrow} & v^*\rho(g^{-1}) \otimes \rho(g)w \\ \updownarrow & & \updownarrow \\ \phi_{v,w} & \overset{\rho(g)}{\rightarrow} & \rho(g)\phi_{(v,w)}\rho(g^{-1}) \end{array}$$

Figure 1: a sketch of relation (17)

2.3 Irreducible Representations

As is described in the previous section, a group can be represented in many ways. Irreducible representations are representations that are used to seek order in all the ways a group can be represented. Later it shall be shown that irreducible representations are kind of the basis vectors of all the representations. But first a more formal description of irreducible representations is given:

Definition 2.3. *A vector space W is a subrepresentation of a representation V if W is a subspace of V, $W \neq V$ and if W is an invariant subspace under transformations defined by the representation V, i.e.,*

$$\rho_V(g)(w) \in W, \tag{18}$$

for all $w \in W$ and all $g \in G$.

One can show that for each subrepresentation W of V there exists a complementary subrepresentation W' such that $V = W \oplus W'$. See [3]-pg 6.

Example 2.11. For an example of a subrepresentation look at example 2.6. Here it is easy to see that the sum representation constructed has two subrepresentations: the two representations it was made out of. For a less trivial example, take the permutation representation of S_3, which is the vector space $\mathbb{C}^3$. Now construct the subspace W:

$$W = \mathbb{C}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \tag{19}$$

This space is invariant under transformations of the projection representation. Thus W is a new 1-dimensional representation. It also interesting to mention that W is the trivial representation because:

$$\rho(g)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \forall\ g \in S_3. \tag{20}$$

In this way each representation can be split up into subrepresentations. Of course, for any finite dimensional representation, this can be done only a finite number of times, because the dimension of a subrepresentation W is always less then that of the representation V it was part of. The trivial and the alternating are examples of representations that can't be decomposed any further in subrepresentations. This leads to the definition of irreducible representations:

Definition 2.4. *An irreducible representation is a representation which has no subrepresentations.*

So the trivial and the alternating representations are irreducible representations. Each representation can be split up in irreducible representations. This is called complete reducibility and is a consequence of every subrepresentation having a complementary subrepresentation.

2.4 Schur's Lemma

It is interesting to investigate if each representation decomposes into irreducible representations in a unique way, just like natural numbers splitting up into prime factors in a unique way. A first starting point is to introduce the concept of G-module homomorphism:

Definition 2.5. *Let V and W be two representations of the group G. The map $\phi : V \to W$ is a G-module homomorphism if the following relation holds:*

$$\phi(gv) = g\phi(v) \quad \forall\ g \in G,\ \forall\ v \in V. \tag{21}$$

In equation (21) the notation $\rho_V(g)$ and $\rho_W(g)$ have both been shortened to just g. In the future, when it is clear that g is not a group element but a transformation representing that element this shorthand notation will be used. Two representations are G-homomorphic to each other when there exist a map between the vector spaces which is a G-module homomorphism. This definition is introduced to describe more formally when two representations are similar. In figure 2 relation (21) has been worked out.

$$
\begin{array}{ccc}
V & \stackrel{\rho_V(g)}{\rightarrow} & V \\
\downarrow & & \downarrow \\
W & \stackrel{\rho_W(g)}{\rightarrow} & W
\end{array}
$$

Figure 2: a sketch of relation (21)

So in the case that V and W are G-homomorphic to each other, in figure 2 it wouldn't matter if one takes the top/right route or the left/bottom route from V to W. If V and W are also both irreducible representations then Schur's lemma has something to say about the similarity between V and W.

Schur's Lemma 2.6. *If V and W are irreducible representations of G and $\phi : V \to W$ is a G-module homomorphism, then:*

1. *Either ϕ is an isomorphism, or $\phi = 0$.*
2. *If $V = W$, then $\phi = \lambda I$ for some $\lambda \in \mathbb{C}$.*

Only the first part of Schur's Lemma will be proved here, for a proof of the second part see [3]. For the first statement of Schur's Lemma to be true, W doesn't need to be irreducible.

Proof. Note that kernel of ϕ is a invariant subspace of V:

$$\phi(v) = 0 \quad \Rightarrow \quad g\phi(v) = \phi(gv) = 0 \tag{22}$$

$$\Rightarrow \quad g\text{Ker } \phi \subset \text{Ker } \phi \tag{23}$$

The fact that ϕ is a G-module homomorphism is used. From the fact that V is an irreducible representation it follows that Ker ϕ is either $\{0\}$ or V. In the latter case $\phi = 0$. If Ker $\phi = 0$ meaning that ϕ has an inverse. This inverse is also a G-module homomorphism: $\phi^{-1}(gw) = g\phi^{-1}w$. This implies that ϕ is an isomorphism. □

Corollary 2.7. *Decompositions of representations of groups are unique.*

Proof. Begin with two decompositions of a representation V:

$$V = V_1 \oplus \ldots \oplus V_k = W_1 \oplus \ldots \oplus W_l, \tag{24}$$

and consider the identity map $Id : V_i \to W_1 \oplus \ldots \oplus W_l$ which is of course a G-module homomorphism. First look at where a subrepresentation V_i can be mapped to. If it was mapped to more than one subrepresentation W_j then V_i can't be a irreducible representation. So V_i can only be mapped to one irreducible representation W_j. From Schur's Lemma it follows that $V_i = W_j$. (because ϕ is in this case an isomorphic identity map.) □

A direct consequence of Schur's Lemma considers Albelian groups. Take an representation V of an Abelian group G. For the map ϕ used in Schur's Lemma use the map $\rho(g) : V \to V$. Because g commutes with all elements of the group, $\rho(g)$ defines a G-module homomorphism, thus part two of Schur's Lemma can be used. This part states that $\rho(g)$ is equal to a multiple of the identity map. This holds for all $g \in G$ from which follows $\rho(g) = \lambda I \ \forall \ g \in G$ implying that V can be decomposed into 1-dimensional subrepresentations. Abelian groups thus only have 1-dimensional irreducible representations.

Example 2.12. As an example consider the projection representation of the Abelian group A_3:

$$\rho(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad , \rho(123) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad , \rho(132) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

The three irreducible representations which this representation consists of are 1-dimensional:

$$V_1 = Span\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\} \quad , V_2 = Span\{\begin{pmatrix} 1 \\ w \\ w^2 \end{pmatrix}\} \quad , V_3 = Span\{\begin{pmatrix} 1 \\ w^2 \\ w \end{pmatrix}\}.$$

where $w = e^{2\pi i/3}$. The proof of this pudding is in the eating, or so the English say.

3 Character Theory

So far we have looked at representations of groups. Each representation of each group can be uniquely decomposed into the smallest subrepresentations called irreducible representations. Do there exist infinitely many different kinds of irreducible representations . And if not, is it possible to give a classification of irreducible representations? Is it also possible to determine the decomposition of a arbitrary representation? Character theory is a very useful theory and gives some answers to these questions.

Definition 3.1. *Given a representation V of a group G, the character of the representation V is a complex valued function on the group defined by:*

$$\chi_V(g) = Tr(g|_V). \tag{25}$$

One of the first observation that can be made is that the character is a class function on a group G. It is an element of the set of class functions. This set is denoted by $\mathbb{C}_{class}(G)$:

$$\begin{aligned} \chi_V(hgh^{-1}) &= Tr(hgh^{-1}) &(26)\\ &= Tr(h^{-1}hg) = Tr(g). &(27)\end{aligned}$$

Example 3.1. From example 2.4 the character of the projection representation of the group S_3 can easily be calculated.

Class	(e)	(12)	(123)
χ_P	3	1	0

A few properties of characters that are needed for further discussion are stated here without proof:

$$\begin{aligned} \chi_{V\oplus W}(g) &= \chi_V(g) + \chi_W(g), &(28)\\ \chi_{V\otimes W}(g) &= \chi_V(g)\cdot\chi_W(g), &(29)\\ \chi_V(g^{-1}) = \chi_{V^*}(g) &= \overline{\chi_V(g)}. &(30)\end{aligned}$$

To further exploit the concept of character two other concepts have to be introduced: $\mathrm{Hom}^G(V,W)$ and the first projection formula. The meaning of the first notation can be generalized to:

$$V^G = \{v \in V \mid gv = v \ \forall g \in G\}. \tag{31}$$

Here V is a representation of G. By definition V^G is a direct sum of trivial subrepresentations of V. For the permutation representation of the group S_3,

P^{S_3} is already calculated in example 2.11: $P^{S_3} = \mathbb{C}(1,1,1)^T$.

Now $\mathrm{Hom}^G(V,W)$ is a direct sum of trivial subrepresentation of $\mathrm{Hom}(V,W)$. One can show that $\mathrm{Hom}^G(V,W)$ consists of all G-module homomorphisms between V and W. According to Schur's Lemma, if V and W are irreducible representations, the dimension of this representation is:

$$\dim\ \mathrm{Hom}^G(V,W) = \begin{cases} 1 \text{ if } V \simeq W \\ 0 \text{ if } V \neq W. \end{cases} \tag{32}$$

Also, with the help of equations (29) and (30), it is possible to calculate the character of $\mathrm{Hom}^G(V,W)$. The easiest way to do this is using the dual of V: $\mathrm{Hom}(V,W) = V^* \otimes W$. The character can now directly be seen to be: $\chi_{\mathrm{Hom}^G(V,W)} = \overline{\chi_V}\chi_W$. The character of the map in question can also be calculated using basic linear algebra techniques. Start by defining a basis for V ($\{e_1 \ldots e_n\}$) and W ($\{f_1 \ldots f_m\}$). A basis of $\mathrm{Hom}(V,W)$ is thus: $\{e_i f_j\}$ which maps the basis vector e_i to the basis vector f_j (and all other basis vectors $e \neq e_i$ to zero). These "basis maps" can also be denoted by the matrices δ_i^j. To calculate the trace of a transformation with these "basis maps" as basis vectors, one needs to calculate in what quantity these maps are mapped to their selves:

$$\begin{aligned} \chi_{\mathrm{Hom}(V,W)}(g) &= \text{Trace } \rho_{\mathrm{Hom}(V,W)}(g) \\ &= \sum_{ij} [\rho_W(g)\delta_i^j \rho_V(g^{-1})]_{ij} \\ &= \sum_{ij}\sum_{kl} [\rho_W(g)]_{ik}[\delta_i^j]_{kl}[\rho_V(g^{-1})]_{lj} \\ &= \sum_{ij} [\rho_W(g)]_{ii}[\rho_V(g^{-1})]_{jj} \\ &= \sum_{i} [\rho_W(g)]_{ii} \sum_{j} [\rho_V(g^{-1})]_{jj} \\ &= \chi_W(g)\chi_V(g^{-1}), \end{aligned}$$

$$\text{so} \quad \chi_{\mathrm{Hom}(V,W)}(g) = \chi_W(g)\overline{\chi_V(g)}. \tag{33}$$

As was said before, a way to exploit the concept of characters is by means of the first projection formula given by equation (34):

$$\phi = \frac{1}{|G|}\sum_{g\in G} g. \tag{34}$$

Here $|G|$ is a notation for the number of elements of the group G. The transformation ϕ is kind of average of all the transformations g. The reason why this is called a projection formula is because ϕ maps V into V^G for any representation V of G. Thus with an appropriate basis, ϕ can take on the following form:

$$\phi v = \left(\begin{array}{c|c} I_m & \mathrm{A} \\ \hline 0 & 0 \end{array} \right) \left(\begin{array}{c} v_1 \\ v_2 \end{array} \right). \tag{35}$$

Here I_m is the m-dimensional identity matrix, $m = \mathrm{dim}V^G$, $v_1 \in V^G$ and $v_2 \in V \backslash V^G$. The trace of the matrix of the transformation ϕ is thus m. But this is also equal to:

$$m = \mathrm{dim}V^G = \mathrm{Trace}(\phi) = \frac{1}{|G|} \sum_{g \in G} \mathrm{Trace}(g) = \frac{1}{|G|} \sum_{g \in G} \chi_V(g). \tag{36}$$

Now replace in equation (36) the representation V by the representation $\mathrm{Hom}(V, W)$ for arbitrary irreducible representations V, W. The dimension m of $\mathrm{Hom}^G(V, W)$ is already calculated, see equation (32). Also the character $\chi_{\mathrm{Hom}(V,W)}(g)$ has been calculated, see equation (33). Filling this in gives the striking result:

$$\frac{1}{|G|} \sum_{g \in G} \chi_W(g) \overline{\chi_V(g)} = \left\{ \begin{array}{l} 1 \text{ if } V \simeq W \\ 0 \text{ if } V \neq W \end{array} \right. . \tag{37}$$

To see what the consequences of these equations are consider the linear space: $\mathbb{C}_{class}(G)$. This is a space of all maps from conjugacy classes of the group G to the complex numbers. In the beginning of this section it was shown that the character $\chi_V(g)$ of a representation is an element of this space for all representations V. Now define an inner product on this linear space:

$$(a, b) = \frac{1}{|G|} \sum_{g \in G} a(g) \overline{b(g)}. \tag{38}$$

The dimension of the space $\mathbb{C}_{class}(G)$ is equal to the number of conjugacy classes. $\chi_V(g)$ is an element of this space and due to equation (37) all characters of irreducible representations are orthogonal elements of the space $\mathbb{C}_{class}(G)$. Thus

Corollary 3.2. *The number of different irreducible representations of a group G is equal to or smaller than the number of conjugacy classes of that group.*

Another consequence of equation (37) is:

Corollary 3.3. *If V is any representation with a decomposition $V = V_1^{\oplus a_1} \oplus \ldots \oplus V_k^{\oplus a_k}$, then the multiplicities a_i can be calculated using the inner product:*

$$a_i = (\chi_V, \chi_{V_i}). \tag{39}$$

The last consequences that will be discussed concerns the calculation of inner product (χ_V, χ_V) for any representation. First look at the regular representation R of an arbitrary group G, which was introduced in example 2.5. Any element $g \in G$ will map basis vectors (coinciding with elements of the group) to other basis vectors. Only the unit element e is an exception in this case. The character of R can thus be expressed by:

$$\chi_R(g) = \begin{cases} 0 & \text{if} \quad g \neq e \\ |G| & \text{if} \quad g = e \end{cases}. \tag{40}$$

In general R is not an irreducible representation but has a decomposition $R = V_1^{\oplus a_1} \oplus \ldots \oplus V_k^{\oplus a_k}$. Here all irreducible representations of G have been taken into this decomposition. The multiplicities a_i can be calculated with the use of corollary 3.3:

$$a_i = \frac{1}{|G|} \sum_{g \in G} \chi_{V_i}(g)\overline{\chi_R(g)} = \frac{1}{|G|}\chi_{V_i}(e)|G| = \text{dim}V_i. \tag{41}$$

One can already conclude that any irreducible representations V is part of the regular representation R. Now comparing the dimensions of R and its decomposition:

Corollary 3.4. *For any group G, the number of elements is equal to:*

$$|G| = \sum_i dim(V_i)^2. \tag{42}$$

The sum is over all possible irreducible representations of G.

Corollary 3.5. *A representation V is irreducible if and only if $(\chi_V, \chi_V) = 1$.*

This last corollary follows from corollary 3.3 and equation (37). With the results achieved so far it is possible to make character tables. These are tables which describe the characters of all irreducible representations of a group.

3.1 The Characters of S_3

Consider for example the group S_3. The characters of the trivial representation U and the alternating representation U' are easily calculated, they are shown in table 2. These characters can be checked by calculating their norm which should be equal to: $(\chi_U, \chi_U) = (\chi_{U'}, \chi_{U'}) = 1$.

The character of the permutation representation was already calculated in example 3.1. Its norm $(\chi_P, \chi_P) = 1/6(9 \cdot 1 + 1 \cdot 3 + 0 \cdot 2) = 2$. Thus P is not an irreducible representation. Indeed, in example 2.11 it was calculated that P has a trivial representation as a subrepresentation. One can show that every permutation representation of a symmetric group S_n has a trivial subrepresentation. The complementary representation is called the standard representation S. So $P = U \oplus S$ from which the character of S can be calculated.

$$\chi_S = \chi_P - \chi_U. \tag{43}$$

Once the character of S is known one can easily calculate its norm: $(\chi_S, \chi_S) = 1/6(4 \cdot 1 + 0 \cdot 3 + 1 \cdot 2) = 1$. Table 2 gives a list of all characters calculated so far. Note that on the top line the number of elements in the conjugacy classes are shown. These are needed to quickly check the norms of the characters.

	1	3	2
	(e)	(12)	(123)
U	1	1	1
U'	1	-1	1
S	2	0	-1

Table 2: character table of the group S_3

From corollary 3.2 it can be concluded that all irreducible representations have been discovered. This can also be checked with equation (42). It is not a coincidence that the number of irreducible representations is equal to the number of conjugacy classes. It can be shown that this is the case for all groups.

With the help of characters one can easily check whether a representation is irreducible. Also it gives a way to check whether all the irreducible representations are known. Moreover it gives a way to find the complete decomposition of a representation into its irreducible representations. It is thus a very useful theory to analyze representations. Let us enhance this theory by working out another example.

3.2 The Characters of S_6

In this example the much larger group S_6 will be analyzed. It will be a cumbersome calculation, but with all the techniques discussed so far it is going to work. The analysis will start with the conjugacy classes of S_6 and a calculation of their number of elements. Once these are known the first irreducible representations can be filled in the character table.

	1	15	40	90	144	120	45	15	120	90	40
	(e)	(12)	(123)	(1234)	(12345)	(123456)	(12)(34)	(12)(34)(56)	(12)(345)	(12)(3456)	(123)(456)
U	1	1	1	1	1	1	1	1	1	1	1
U'	1	-1	1	-1	1	-1	1	-1	-1	1	1
S	5	3	2	1	0	-1	1	-1	0	-1	-1

The character of the standard representation is calculated using equation (43). A calculation of its norm shows that it is indeed an irreducible representation. To find other representations one can take either symmetric powers or exterior powers of known representations. Their characters are given by:

$$\chi_{\wedge^2 V}(g) = 1/2[\chi_V(g)^2 - \chi_V(g^2)], \tag{44}$$

$$\chi_{\text{Sym}^2 V}(g) \quad = \quad 1/2[\chi_V(g)^2 + \chi_V(g^2)]. \tag{45}$$

These two relations can be proven by analyzing the eigenvalues of transformations coinciding with V, $\wedge^2 V$ and $\text{Sym}^2 V$. First of all look at $\wedge^2 S$. By calculating its character and taking its norm one can see that this is an irreducible representation of dimension 10. Also one can calculate the character of $\text{Sym}^2 S$. But calculating its norm shows that it isn't an irreducible representation. To see what its subrepresentations are one can calculate inner products of $\text{Sym}^2 S$ with known irreducible representations, hoping to find an new irreducible representation in this way. Doing the calculations shows that:

$$(\chi_{\text{Sym}^2 V}, \chi_U) \quad = \quad 1 \tag{46}$$

$$(\chi_{\text{Sym}^2 V}, \chi_S) \quad = \quad 1. \tag{47}$$

So $\text{Sym}^2 s$ has rhe decomposition: $\text{Sym}^2 s = U \oplus S \oplus V$. The character of the new representation V is: $\chi_V = \chi_{\text{Sym}^2 S} - \chi_S - \chi_U$. This new representation turns out to be irreducible. Another trick to find other irreducible representations is to construct product spaces of known irreducible representations. With symmetric groups it turns out that the product space of the alternating representation and another representation $V' = V \otimes U'$ is irreducible when V is irreducible. In this way another 3 irreducible representations can be found.

	1 (e)	15 (12)	40 (123)	90 (1234)	144 (12345)	120 (123456)	45 (12)(34)	15 (12)(34)(56)	120 (12)(345)	90 (12)(3456)	40 (123)(456)
U	1	1	1	1	1	1	1	1	1	1	1
U'	1	-1	1	-1	1	-1	1	-1	-1	1	1
S	5	3	2	1	0	-1	1	-1	0	-1	-1
$S \otimes U'$	5	-3	2	-1	0	1	1	1	0	-1	-1
$\wedge^2 S$	10	2	1	0	0	1	-2	-2	-1	0	1
$\wedge^2 S \otimes U'$	10	-2	1	0	0	-1	-2	2	1	0	1
V	9	3	0	-1	-1	0	1	3	0	1	0
$V \otimes U'$	9	-3	0	1	-1	0	1	-3	0	1	0

One can wonder if not all irreducible representations have been found yet. This is not the case: as mentioned before the number of irreducible representations is equal to the number of conjugacy classes. Also equation (42) must hold. Up to now, the right side of this equations sums op to 414 which is by far not the number of elements of S_6. Finding the remaining irreducible representations one can again take products of known representations. Consider for example the following product space: $S \otimes \wedge^2 S$. This is a $5 \times 10 = 10$-dimensional representation, which character can be calculated using equation (29). Again one can take inner products:

$$(\chi_{\text{Sym}^2 V \otimes S}, \chi_S) \quad = \quad 1 \tag{48}$$

$$
\begin{aligned}
(\chi_{\mathrm{Sym}^2 V\otimes S}, \chi_V) &= 1 &(49)\\
(\chi_{\mathrm{Sym}^2 V\otimes S}, \chi_{\wedge^2 S}) &= 1 &(50)\\
(\chi_{\mathrm{Sym}^2 V\otimes S}, \chi_{\wedge^2 S\otimes U'}) &= 1. &(51)
\end{aligned}
$$

This representation turns out to be reducible in at least the following subrepresentations: $S \otimes \wedge^2 S = S \oplus V \oplus \wedge^2 S \oplus \wedge^2 S \otimes U' \oplus W$ where W is a new representation of dimension 16. This suggestion wouldn't be mentioned if W didn't turn out to be an irreducible representation. Also one can again try to construct another irreducible representation: $W \otimes U'$. But this representation turns out to be the same as W.

Now equation (42) is going to be fully exploited. Written out the equation is in this case:

$$720 = 1^2 + 1^2 + 5^2 + 5^2 + 10^2 + 10^2 + 9^2 + 9^2 + 16^2 + n_1^2 + n_2^2. \qquad (52)$$

A look at this equation tells that $n_1^2 + n_2^2 = 50$. There are exactly 2 remaining irreducible representations. There are 2 possibilities for their dimensions: $n_1 = 7,\ n_2 = 1$ or $n_1 = n_2 = 5$. If there was another 1-dimensional irreducible representation problems would occur. According to theorem 2.2 it is going to be trivial on normal subgroup of S_6 whose quotient group is cyclic. There are only two such normal groups: the group itself and the group A_6. The representations coinciding with these normal groups are the trivial group and the alternating group. The conclusion is that the dimension of the last two remaining representations must be equal to 5.

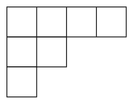
Known is that the character of one of the two remaining irreducible representations Z looks like $(5, a_2, a_3, \ldots, a_{11})$. An educated guess for the remaining representation could be $Z' = U' \otimes Z$ with its character $(5, -a_2, a_3, \ldots, a_{11})$. First check that $Z \neq Z'$. If $Z = Z'$ then $a_2 = a_4 = a_6 = a_8 = a_9 = 0$, which leaves 5 unknown variables. The character of Z should satisfy the $(\chi_Z, \chi_U) = (\chi_Z, \chi_S) = (\chi_Z, \chi_{\wedge^2 S}) = (\chi_Z, \chi_V) = (\chi_Z, \chi_W) = 0$. These 5 equations give a unique solution for a: $(a_3, a_5, a_7, a_{10}, a_{11}) = (-5.7473, 1.9827, 0.4730, -1.1663, 1.3866)$. It is known that characters of symmetric groups are always natural numbers thus it can be concluded that $Z \neq Z'$. These remaining two characters must be orthogonal to all other characters. This is enough information to complete the character table:

	1 (e)	15 (12)	40 (123)	90 (1234)	144 (12345)	120 (123456)	45 (12)(34)	15 (12)(34)(56)	120 (12)(345)	90 (12)(3456)	40 (123)(456)
U	1	1	1	1	1	1	1	1	1	1	1
U'	1	-1	1	-1	1	-1	1	-1	-1	1	1
S	5	3	2	1	0	-1	1	-1	0	-1	-1
$S \otimes U'$	5	-3	2	-1	0	1	1	1	0	-1	-1
$\wedge^2 S$	10	2	1	0	0	1	-2	-2	-1	0	1
$\wedge^2 S \otimes U'$	10	-2	1	0	0	-1	-2	2	1	0	1
V	9	3	0	-1	-1	0	1	3	0	1	0
$V \otimes U'$	9	-3	0	1	-1	0	1	-3	0	1	0
W	16	0	-2	0	1	0	0	0	0	0	-2
Z	5	1	-1	-1	0	0	1	-3	1	-1	2
$Z \otimes U'$	5	-1	-1	1	0	0	1	3	-1	-1	2

4 Young Diagrams

Up to now representations have been discussed. It turned out that these can be decomposed into irreducible representations. Character theory gives ways to check whether a representation is irreducible. But it doesn't give a method to find all the irreducible representations of a group, only by trial and error one is capable of finding all the irreducible representations (see section 3.2) and there is no reason to assume that this techniqu will work in all cases. Young diagrams give a way to systematically give all the irreducible representations of a group, but it only works for symmetric groups S_n.

The number of irreducible representations of a group is equal to the number of conjugacy classes. For a symmetric group S_n this is equal to the number of partitions of n. A partition of n is a way to split up n in a sum of natural numbers. A partition of 7 could be $4+2+1$ which corresponds to the conjugacy class of the group element (1234)(56). For each partition $n = \lambda_1 + \ldots \lambda_n$ a Young diagram can be constructed. A Young diagram is a number of boxes. The i^{th} row consists of λ_i boxes. The Young diagram of $7 = 4 + 2 + 1$ looks like:



A shorthand notation for a Young diagram is $(\lambda_1, \ldots, \lambda_n)$. The diagram above would be (421). One can also fill in the boxes with the numbers $\{1, \ldots, n\}$. The picture that is constructed in this way is called a Young tableau. The canonical tableau of the diagram (421) is:

1	2	3	4
5	6		
7			

Before going on with this story, the group algebra $\mathbb{C}G$ of a group G needs to be introduced. This is a linear vector space with basis e_g corresponding to the elements in G. The multiplication in this algebra is defined in a logical way:

$$e_g \cdot e_h = e_{g \times h}. \tag{53}$$

The group algebra thus contains the structure of the underlying group. The advantage is that it is possible to create expressions like: $(234) + (23) \in \mathbb{C}S_4$. The basis elements of the space $\mathbb{C}G$ are not denoted with e_g but with the same notation as a group element. This notation will be used when it is clear what the underlying set or space is. The group can be seen as a space of maps from the group algebra to the group algebra. This is done by left-multiplying all the basis vectors of the group algebra with an element of the group. See the next example:

Example 4.1. Take the following elements: $g = (12) \in S_3$ and $c = (e) - 4 \cdot (13) \in \mathbb{C}S_3$. Then the map:

$$g \in G : \mathbb{C}S_3 \to \mathbb{C}S_3, \tag{54}$$

maps g to $g(c) = (12) \times [(e) - 4 \cdot (13)] = (12) - 4 \cdot (132)$.

In this way it is possible to construct representations of the group G. Actually, the representation $\mathbb{C}G$ is the regular representation discussed in example 2.5. The group algebra itself can also be seen as a space of maps, in the sense that $\mathbb{C}G : \mathbb{C}G \to \mathbb{C}G$. This works in exactly the same way as with the map in example 4.1.

A representation of the group S_n that is needed for this discussion but has not yet been introduced is the n^{th} tensor product of an arbitrary vector space V: $V^{\otimes n}$. The coinciding map is defined as follows:

$$\sigma(e_{i_1} \otimes \ldots \otimes e_{i_n}) = e_{i_{\sigma^{-1}(1)}} \otimes \ldots \otimes e_{i_{\sigma^{-1}(n)}}. \tag{55}$$

This equation can also be used to define a map $\mathbb{C}S_n \to \mathrm{End}(V^{\otimes n})$. An example of such a map is given in example 4.2.

Example 4.2. Take an element $c \in \mathbb{C}S_3$: for example $c = (e) - 4 \cdot (13)$: Then c defines the map:

$$c(v_{ijk}) = v_{ijk} + 4v_{kji}. \tag{56}$$

Here the shorthand notation for the basis vectors is used: $v_{ijk} = e_i \otimes e_j \otimes e_k \in V^{\otimes 3}$.

Because it is not trivial why the inverse of σ is used in equation 55 it will be shown that it does define a homomorfism. Consider $gh(e_{i_1} \otimes \ldots \otimes e_{i_n})$:

$$\begin{aligned} gh(e_{i_1} \otimes \ldots \otimes e_{i_n}) &= g(e_{i_{h^{-1}(1)}} \otimes \ldots \otimes e_{i_{h^{-1}(n)}}) \\ &= e_{j_{g^{-1}(1)}} \otimes \ldots \otimes e_{j_{g^{-1}(n)}}, \end{aligned}$$

where $j_k = i_{h^{-1}(k)}$ for all indices k. Thus $j_{g^{-1}(n)} = i_{h^{-1}g^{-1}(n)}$. Thus:

$$\begin{aligned} e_{j_{g^{-1}(1)}} \otimes \ldots \otimes e_{j_g^{-1}(n)} &= e_{i_{h^{-1}g^{-1}(1)}} \otimes \ldots \otimes e_{i_{h^{-1}g^{-1}(n)}} \\ &= (gh)(e_{i_1} \otimes \ldots \otimes e_{i_n}) \end{aligned}$$

For certain group elements $c \in \mathbb{C}S_n$ this map will turn out to be very useful. Using Young tableaux one can construct these elements called Young symmetrizers. To construct a Young symmetrizer coinciding with a Young tableau one first starts with the following subsets of the group S_n

$$P_\lambda = \{g \in S_n \mid g \text{ preserves each row}\}, \tag{57}$$
$$Q_\lambda = \{g \in S_n \mid g \text{ preserves each column}\}, \tag{58}$$

where the subscript λ refers to the Young tableau.

Example 4.3. For the canonical Young tableau of the Young diagram (421) of S_7, the subgroups P and Q of $\mathbb{C}G$ are:

$$P_\lambda = \{(e),\ (12),\ (13),\ (14),\ (1234),\ (56),\ \ldots\}, \tag{59}$$
$$Q_\lambda = \{(e),\ (15),\ (17),\ (57),\ (157),\ (26),\ \ldots\}. \tag{60}$$

Now construct the following elements of $\mathbb{C}S_n$:

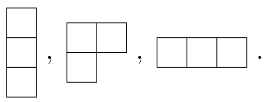
$$a_\lambda = \sum_{g \in P_\lambda} e_g \quad \text{and} \quad b_\lambda = \sum_{g \in Q_\lambda} \operatorname{sgn}(g) \cdot e_g \tag{61}$$

and set $c_\lambda = a_\lambda \cdot b_\lambda \in \mathbb{C}S_n$. This is called the Young symmetrizer. It is has the following astonishing property.

Theorem 4.1. *For each Young diagram λ and arbitrary corresponding Young tableau λ, the image of Young symmetrizer $c_\lambda(\mathbb{C}S_n) \in \mathbb{C}S_n$ is an irreducible representation V_λ of S_n with corresponding map $\rho_{V_\lambda}(g) = g$. Moreover, each irreducible representation of S_n can be found in this way.*

This theorem will not be proven in this thesis, for a proof see [3]-pg52. Instead the following example will make clear how this theorem can be used:

Example 4.4. To keep the calculations simple, consider the group S_3. The number 3 has exactly 3 partitions (corresponding to 3 irreducible representations) which lead to 3 Young diagrams:



First start with the first Young diagram ($\lambda = (111)$). For this diagram it does not matter which corresponding tableau is chosen. $P_{(111)}$ consists of the unit element only while $Q_{(111)} = S_3$, thus:

$$c_{(111)} = (e) - (12) + (123) - (13) + (132) - (23). \tag{62}$$

Now to calculate the image of $c_{(111)}(\mathbb{C}S_3)$ one can calculate where $c_{(111)}$ maps the basis vectors e_g of $\mathbb{C}S_3$. Because $gc_{(111)} = \pm c_{(111)}$ this image is equal to the one dimensional space spanned by $c_{(111)}$. So this space is according to theorem 4.1 an irreducible representation. The map corresponding to this representation is the map $\rho_g(c_{(111)}) = \operatorname{sgn}(g) \cdot c_{(111)}$ which shows that it is the alternating representation. In general the Young diagram (1^n) corresponds to the alternating representation of S_n.

Now look at the third Young diagram ($\lambda = (3)$). Again it does not matter what corresponding tableau is chosen. $Q_{(3)}$ is just the unit element while $P_{(3)} = S_3$, thus now $c_{(3)}$ becomes:

$$c_{(3)} = (e) + (12) + (123) + (13) + (132) + (23). \tag{63}$$

This Young symmetrizer looks even simpler then $c_{(111)}$. A quick argumentation would be to say that it must correspond to the most simple representation, the trivial representation. $\rho_g(c_{(3)}) = c_{(3)}$ shows that this is indeed the case.

The last Young diagram should coincide with the standard two dimensional representation. It will take some more work to show this. Start with an arbitrary Young tableau, for example:

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$$

The corresponding a_λ and b_λ become: $a_{(12)} = ((e) + (12))$ and $b_{(12)} = ((e) - (13))$. This yields the Young symmetrizer:

$$c_{(12)} = (e) + (12) - (13) - (132). \tag{64}$$

Now one has to calculate the image of $c_{(12)}$. A first glance shows that $(e)c_{(12)} = (12)c_{(12)} = c_{(12)}$. But expressions of images of other basis vectors are not as easy: $(13)c_{(12)} = (123)c_{(12)} = (13) + (123) - (e) - (23)$. These two elements are thus mapped to a vector independent of $c_{(12)}$, call this vector a. This shows that the image of $c_{(12)}$ is indeed at least 2-dimensional. The remaining two basis vectors are mapped into the space spanned by a and $c_{(12)}$: $(23)c_{(12)} = (132)c_{(12)} = -(a + c_{(12)})$. Thus according to the theory $V_{(12)}$ is a 2-dimensional irreducible representation. If one takes a and $c_{(12)}$ as basis vectors of $V_{(12)}$ one can explicitly calculate the corresponding map ρ:

$$\rho(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho(123) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \quad \rho(132) = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix},$$

$$\rho(12) = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}, \quad \rho(23) = \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}, \quad \rho(13) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

It is now easy to calculate the character of the representation $V_{(12)}$. Comparing it to the character of the standard representation shows that these two are indeed equivalent.

5 $SU(n)$ and Quarks

Quarks are the building blocks of a lot of particles, the most popular ones being the neutron and the proton. They were introduced to explain the existence of the many subatomic particles, called hadrons, that were discovered. Also, quarks were hypothesized to have certain symmetries. Not only does this hypothesis lead to an ordering of quarks, it also gives predictions on what particles may still be discovered. To explain the relation between quarks and SU_n symmetry in more detail the concept of isospin will be discussed. This discussion can be found in many standard textbooks such as [7] and [6].

5.1 Isospin

It was discovered that many hadrons have approximately the same mass, see table 3 for a few of these particles.

Hadron	Mass (MeV)	Charge
p	939.6	0
n	938.3	1
π^-	139.6	-1
π^0	135	0
π^+	139.6	1
Σ^-	1197.2	-1
Σ^0	1192.3	0
Σ^+	1189.4	1

Table 3: mass and charge of a few hadrons

For example, the neutron and the proton have approximately the same mass. The theory of isospin states that this mass difference is due to the difference in charge. If one was able to create a world without electromagnetic forces, the proton and neutron would actually be the same particle. It is equivalent to stating that a spin-up electron and a spin-down electron are different states of the same electron. In the same way, the neutron and the proton can be seen as different states of the same hadron.

Particles are seen as vectors in $\mathbb{C}^n$. Thus with an appropriate basis, the neutron can be seen as $n = e_1$ and the proton can be seen as $p = e_2$. Now if electromagnetic forces were to be shut down, these two particles would be actually different states of the same hadron. Any transformation which mixes these two basis vectors would yield a new state (a linear combination of the p and n) of the same particle. The allowed transformations are mathematically SU_2, the group of 2 by 2 special unitary matrices.

The group of special unitary matrices SU_2 are all the matrices U with det $U = 1$ and $UU^* = 1$. This set forms a group under matrix multiplication. The space spanned by p and n is invariant under SU_2 transformations. In the language of representation theory, this space is an representation of the SU_2 group. In a sense this representation is irreducible. This is only the case when one sees SU_2 not as the group but as the underlying transformations. Because in this case one can not construct for example a trivial subrepresentation, for there is an SU_2 matrix that maps e_1 to e_2. Physically it is saying that one always considering all transformation and not only a subset of the transformations.

So isospin groups particles with approximately the same mass. It states that particles have an underlying symmetry, in other words, that certain particles are nearly the same. Nearly, because this symmetry is broken by the electromagnetic forces. Isospin shows that all hadrons are tensors belonging to an irreducible representation of a group, a group determining its symmetry. In the case of isospin symmetry it is SU_2. The other way around one can say that a tensor can only represent a particle if it is an element of an irreducible representation. Another conclusion is that the tensors spanning the space of an irreducible representation of a symmetry, form a set of particles which are equivalent with respect to this symmetry.

This symmetry of SU_2 seemed to work but it was not satisfying. There were still to many small groups of particles. That is why in 1962 Gell-Mann proposed a higher symmetry, that of SU_3. He called this theory of higher symmetry the Eightfold-Way. But to analyze this symmetry, a little more mathematics needs to be discussed.

6 Link between S_n and $SU(n)$

In chapters 2, 3 and 4 representations of symmetric groups have extensively been analyzed. To analyze quarks it is of great interest to know the representations of SU_n. Not only will these be discussed in this chapter, also decomposing tensor products and other mathematical tricks will be discussed. All theorems will be stated here without proof but only some remarks.

6.1 Irreducible Representations of SU_n

Say a quark has n freedoms (due to spin, isospin, flavour or any other internal property). It can then be seen as a vector in $\mathbb{C}^n$. Particles are mostly made out of either 2 or 3 quarks. In this discussion this is generalized to the statement that a hadron can be made out of d quarks. It is thus an element of the tensor product $(\mathbb{C}^n)^{\otimes d}$. An example of an element of this product space is the neutron which consists of two down quarks and an up quark. If one only looks at three flavour freedoms of a quark (up, down and strange) the neutron can be seen as an element of $(\mathbb{C}^3)^{\otimes 3}$. With an appropriate basis this becomes explicitly:

$$n = ddu = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \tag{65}$$

The space $(\mathbb{C}^n)^{\otimes d}$ can be seen as a representation of SU_n. A matrix $U \in SU_n$ transforms a tensor $v_1 \otimes \ldots \otimes v_d \in (\mathbb{C}^n)^{\otimes d}$ by transforming all the components of the tensor:

$$Uv = Uv_1 \otimes Uv_2 \otimes \ldots \otimes Uv_d. \tag{66}$$

This can also be denoted using the Einstein summation convention. Using this convention a tensor in $(\mathbb{C}^n)^{\otimes d}$ is denoted using the basis tensors $v_{i_1 i_2 \ldots i_d}$ where the indices $i_1,\ i_2, \ldots, i_d$ run from 1 to n. An arbitrary tensor $T \in (\mathbb{C}^n)^{\otimes d}$ can be is denoted as:

$$T = T^{i_1 i_2 \ldots i_d} v_{i_1 i_2 \ldots i_d}, \tag{67}$$

where $T^{i_1 i_2 \ldots i_d} \in \mathbb{C}$. Here the same indices denote a summation, which implies that the different basis vectors are summed together. The tensor T is said to have d lower indices. The equation for the neutron (see equation 74) is denoted as $n = e_2 \otimes e_2 \otimes e_1 = v_{221}$. A transformation of an arbitrary tensor T with d lower indices is denoted as:

$$(UT)^{j_1 \ldots j_d} = U^{j_1}_{i_1} U^{j_2}_{i_2} \ldots U^{j_d}_{i_d} T^{i_1 i_2 \ldots i_d}. \tag{68}$$

The numbers $U^{i_k}_{j_k}$ are the matrix elements U_{ij}. Furthermore, the convention states that one again sums over similar indices, in this equation one sums over all i indices. This equation thus defines the transformation for the constants

$T^{i_1 i_2 \dots i_d} \in \mathbb{C}$.

The question is thus how the representation $(\mathbb{C}^n)^{\otimes d}$ splits up in irreducible representations. The Young symmetrizers will help answer this question. To see this recall the map $c : \mathbb{C}S_d \to \mathrm{End}(V^{\otimes d})$ which was defined for all group algebra elements c. A Young symmetrizer is such element and thus defines such a map. The image of the map defined by the Young symmetrizer is denoted by:

$$\mathbb{S}_\lambda V = \mathrm{Im}(c_\lambda|_{V^{\otimes d}}). \tag{69}$$

Where λ denotes the corresponding Young diagram. In a sense the space $\mathbb{S}_\lambda V$ is not only dependent of the Young diagram λ but also of the Young tableau. This is not embedded in the notation for if one uses different Young tableaux to calculate the spaces $\mathbb{S}_\lambda V$, these spaces turn out to be equivalent. When discussing quarks V would be the space $\mathbb{C}^n$.

Example 6.1. In this example the spaces $\mathbb{S}_\lambda V$ will be constructed with V the space $\mathbb{C}^3$ and $d = 3$. These will be the spaces of three quarks (the baryons) with each three freedoms. Throughout this example the shorthand notation for basis tensors will be used. The Young symmetrizers coinciding with the three Young diagrams where already calculated in example 4.4. Now start with the easiest Young symmetrizer $c_{(3)}$ belonging to the symmetric Young tableau:

1	2	3

The Young symmetrizer is $c_{(3)} = (e) + (123) + (132) + (12) + (13) + (23)$. Now to calculate the space $\mathbb{S}_{(3)}\mathbb{C}^3$ one can calculate the image of all the basis tensors of $\mathbb{C}^3$. There are three kinds of basis tensors of $\mathbb{C}^3$: v_{iii}, v_{iij} and v_{ijk}. One has that $c_{(3)}v_{iii} = 6v_{iii}$ and $c_{(3)}v_{ijk} = v_{ijk} + v_{kij} + v_{jki} + v_{jik} + v_{kji} + v_{ikj}$ for all $i \neq j \neq k \neq i$. Finally, $c_{(3)}v_{iij} = c_{(3)}v_{iji} = c_{(3)}v_{jii} = 2v_{iij} + 2v_{iji} + 2v_{jii}$ for all $i \neq j$. The space $\mathbb{S}_{(3)}\mathbb{C}^3$ is thus an 10-dimensional subspace of $(\mathbb{C}^n)^{\otimes d}$.

The next symmetrizer that is going to by analyzed is the one belonging to the totally antisymmetric Young diagram. Note that, like the totally symmetric diagram, it does not matter which tableau is taken. In this case the canonical one is taken:

1
2
3

The Young symmetrizer belonging to this tableau is $c_{(111)} = (e) + (123) + (132) - (12) - (13) - (23)$. One has that $c_{(111)}v_{iii} = c_{(111)}v_{iij} = c_{(111)}v_{iji} = c_{(111)}v_{jii} = 0$ for all $i \neq j$. And $c_{(111)}v_{ijk} = v_{ijk} + v_{kij} + v_{jki} - v_{jik} - v_{kji} - v_{ikj} = \pm(v_{123} + v_{312} + v_{231} - v_{213} - v_{321} - v_{132})$. The space $\mathbb{S}_{(111)}\mathbb{C}^3$ is thus 1-dimensional.

The last symmetrizer comes with a few complications. Now it does matter which Young tableau is taken if one wants to explicitly calculate basis vectors of $\mathbb{S}_{(12)}\mathbb{C}^3$. The calculations here are done with the following Young tableau

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

The Young symmetrizer of this tableau is $c_{(12)} = (e) + (12) - (13) - (132)$. Again one has that $c_{(12)}v_{iii} = c_{(12)}v_{iji} = 0$ for all i. Furthermore, $c_{(12)}v_{iij} = -c_{(12)}v_{jii} = 2v_{iij} - v_{jii} - v_{iji}$. Running over all i, j $(i \neq j)$, these tensors span a 6-dimenionsal space. Running over all i,j and k, $c_{(12)}v_{ijk}$ denotes six different tensors which span a 2-dimensional space. The space $\mathbb{S}_{(12)}\mathbb{C}^3$ is thus 8-dimensional. A thorough calculation of this tensor space can be found in Appendix 1.

It is also possible to calculate the space $\mathbb{S}_{(12)}\mathbb{C}^3$ when one starts with a different Young tableau. As said before the space one would get would be equivalent to the space already calculated.

Now the spaces $\mathbb{S}_\lambda\mathbb{C}^n$ which were calculated in the above example turn out to be the irreducible representations of SU_n. Here this statement is explained with a minor discussion. For more thorough mathematical discussion about this statement see [3] and [4]. With a few calculations one can see that the space $\mathbb{S}_\lambda\mathbb{C}^n$ is an invariant subspace under SU_n transformations. This can be generalized to the following statement. Let W be a representation of an arbitrary group G. $W^{\otimes d}$ is a representation of G as well as S_d. Now for all $c \in \mathbb{C}S_d$, the space $c \cdot W^{\otimes d}$ is a subrepresentation of G. This follows from the fact that elements from S_d and G commute. Let $g \in G$, $\sigma \in S_n$ and $w \in W^{\otimes d}$:

$$\begin{aligned} g\sigma(w_{i_1} \otimes \ldots \otimes w_{i_d}) &= g(w_{i_{\sigma^{-1}(1)}} \otimes \ldots \otimes w_{i_{\sigma^{-1}(d)}}) \\ &= gw_{i_{\sigma^{-1}(1)}} \otimes \ldots \otimes gw_{i_{\sigma^{-1}(d)}} \\ &= \sigma(gw_{i_1} \otimes \ldots \otimes gw_{i_d}) = \sigma g(w_{i_1} \otimes \ldots \otimes w_{i_d}) \end{aligned}$$

Physically one can say that a SU_n transformations conserves symmetries

The dimensions of the spaces $\mathbb{S}_\lambda\mathbb{C}^n$ can be calculated as in example 6.1, by calculating explicitly a basis for this space. There is a far easier way though, with the help of the Hook product. For each box in a Young diagram one can associate a Hook length: h_{ij} = (number of boxes under box (i, j)+number of boxes to the right of box $(i, j) + 1$). Here box (i, j) is the box in the i^{th} row and j^{th} column, counting from the upper right corner. As an example, the Hook length of each box of the Young diagram (421) has been calculated:

$$\begin{array}{|c|c|c|c|}\hline 6 & 4 & 2 & 1 \\ \hline 3 & 1 & \multicolumn{2}{c}{} \\ \cline{1-2} 1 & \multicolumn{3}{c}{} \\ \cline{1-1}\end{array}$$

The Hook product of a Young diagram is the product of all its Hook lengths: $h_\lambda = \prod_{ij} h_{ij}$. The Hook product of the Young diagram (421) is thus 144. The dimension of $\mathbb{S}_\lambda \mathbb{C}^n$ is now given by:

$$\dim\, \mathbb{S}_\lambda \mathbb{C}^n = \prod_{ij} \frac{n-i+j}{h_{ij}}. \tag{70}$$

The dimensions of the irreducible representations which were calculated in example 6.1 can be checked using this formula.

$$\begin{aligned}
\dim\, \mathbb{S}_{(3)}\mathbb{C}^n &= (3*4*5)/(3*2*1) = 10 \\
\dim\, \mathbb{S}_{(21)}\mathbb{C}^n &= (3*4*2)/(3*1*1) = 8 \\
\dim\, \mathbb{S}_{(111)}\mathbb{C}^n &= (3*2*1)/(3*2*1) = 1
\end{aligned}$$

The space of 3 quarks with 3 symmetries used in example 6.1 splits up in these irreducible representations: $(\mathbb{C}^3)^{\otimes 3} = \mathbb{S}_{(3)}\mathbb{C}^3 \oplus (\mathbb{S}_{(12)}\mathbb{C}^3)^{\oplus 2} \oplus \mathbb{S}_{(111)}\mathbb{C}^3$. A check on this kind of expressions is to verify that both sides of the equality have the same dimensions. It is easy to see that both spaces are 27-dimensional. The equation can also be expressed in terms of Young diagrams:

$$\square \otimes \square \otimes \square = \ydiagram{1,1,1} \oplus 2 \cdot \ydiagram{2,1} \oplus \ydiagram{3} \tag{71}$$

6.2 Decomposition of Representations (Part 1)

Equation (71) is actually an example of decomposing product spaces into irreducible representations. In chapter 3 it was shown that character theory gave useful formulas for decomposing representations, see corollary 3.3. The problem here is that the characters of representations of SU_n are quite hard to describe because there are infinitely many conjugacy classes. There is however a link between the decomposition of a representation of SU_n and that of a similar decomposition of S_d:

Theorem 6.1. *If $c \in \mathbb{C}S_d$ and $c \cdot (\mathbb{C}S_d) = \bigoplus_\lambda V_\lambda^{\oplus r_\lambda}$ as representations of S_d, then there is a corresponding decomposition of $GL(V)$-spaces:*

$$c \cdot V^{\otimes d} = \bigoplus_\lambda \mathbb{S}_\lambda V^{\oplus r_\lambda}. \tag{72}$$

A proof of this theorem can be found in [3]-pg 84. In [4] it is explained that this theorem not only holds for GL(V)-spaces but also for $SU(V)$ spaces, where $V = \mathbb{C}^n$ in our case. Note that the decomposition of $V^{\otimes d}$ is independent of the dimension of V. The next example will demonstrate the use of this theorem by giving a derivation of equation (71).

Example 6.2. In this example the decomposition of $\square \otimes \square \otimes \square$ will be calculated. Again assume the number of freedoms for a quark n to be 3. One must thus look for a decomposition of $(\mathbb{C}^3)^{\otimes 3} = (\mathbb{C}^3)^{\otimes 3} \cdot (e)$. Now use theorem 6.1 with $c = (e)$. The representation $(\mathbb{C}S_3) \cdot (e)$ is just the regular representation R of S_3. A glance at example 2.5 gives the character of this representation: $\chi_R(e) = 6$ and $\chi_R(123) = \chi_R(12) = 0$. Taking inner products with characters of irreducible representations gives: $(\mathbb{C}S_3) \cdot (e) = V_{(111)} \oplus (V_{(12)})^{\oplus 2} \oplus V_{(3)}$

This example can be generalized to the following theorem:

Theorem 6.2. *The decomposition of $(\mathbb{C}^n)^{\otimes d}$ as a representation of SU_n is:*

$$(\mathbb{C}^n)^{\otimes d} = \bigoplus_{\lambda} \mathbb{S}_\lambda V^{\oplus n_\lambda}, \tag{73}$$

where n_λ is the dimension of the representation coinciding with the Young diagram λ of the group S_d.

Example 6.3. This example will make clear that the method just explained works but is cumbersome to use and that logical thinking can sometimes be a lot quicker. Look at the representation $\ydiagram{2} \otimes \ydiagram{1}$, which is the representation of a symmetric hadron consisting of 2 quarks combined with another quark. A first calculation of this representation shows that it is $6 * 3 = 18$-dimensional. Because this product space consists of a representation of 3 quarks the possible irreducible representations it could consists of are $\ydiagram{3}$, $\ydiagram{2,1}$ and $\ydiagram{1,1,1}$. At least 2 of the three quarks are symmetric thus the last Young diagram does not appear. A further dimensional analysis shows that the decomposition must be:

$$\ydiagram{2} \otimes \ydiagram{1} = \ydiagram{2,1} \oplus \ydiagram{3}, \tag{74}$$

since $\dim(\ydiagram{3}) = 10$ and $\dim(\ydiagram{2,1}) = 8$. This is by far a faster method then using theorem 6.1. However, the theorems does show the close relation between the symmetric groups S_n and the groups SU_n and for that reason it will be utilized here. The slower method using theorem 6.1 starts by writing down the group algebra element $c = (e) + (12)$. The second step is calculating the character of the representation $\mathbb{C}S_3 \cdot ((e) + (12))$ of the group S_3: $\chi(e) = 3$, $\chi(123) = 0$ and $\chi(12) = 1$. Calculating this character is most of the work of the whole calculation and is omitted here. For the calculation see appendix 2. It is equivalent to the calculation is example 4.4. Now only the inner products of this character with characters of different irreducible representations must be taken to get to equation (74).

6.3 Decomposition of Representations (Part 2)

When considering hadrons the first question one should always ask is what freedoms the quark has. In the last section the quarks in question always had one kind of freedom, in both examples the quarks only had a flavour freedom (up, down or strange). But quarks can have more degrees of freedoms. And each kind of freedom can be analyzed separately to give different symmetries. In the next example it is explained how these symmetries can be put together.

Example 6.4. Take a hadron consisting of 3 quarks. First consider only its spin freedom, which is an SU_2 symmetry. The tensor representing the spin of the particle should be an element of an irreducible representation of SU_2, say $\yng(2,1)$. The quark could also have a flavour freedom. This would be an SU_3 symmetry because only the flavours up, down and strange are taken into consideration. Again the tensor describing the flavour of the quark should be an element of an irreducible representation of SU_3, say $\yng(2,1)$. The two tensors describing spin and flavour are elements of the spaces $\mathbb{S}_{(21)}\mathbb{C}^2$ and $\mathbb{S}_{(21)}\mathbb{C}^3$ respectively. To see how these two tensors combine, look at the combination of these to spaces:

$$\left(\mathbb{S}_{(21)}\mathbb{C}^2\ ,\ \mathbb{S}_{(21)}\mathbb{C}^3\right) = \bigoplus_{\lambda} (\mathbb{S}_\lambda \mathbb{C}^6)^{\oplus a_\lambda}. \tag{75}$$

Here the coefficients a_λ are called *Clebsch-Gordan*-coefficients. In this new space quarks have both spin and flavour freedoms. It is thus a representation of the larger group $SU_2 \cdot SU_3$ which is a subset of SU_6. To see that this is true one can first generalize the statement to $SU_n \cdot SU_m \subset SU_{nm}$. Take $A \in SU_n$ and $B \in SU_m$ and define the map $A \otimes B \in \mathrm{End}(\mathbb{C}^n \otimes C^m)$ as:

$$A \otimes B(v \otimes w) = Av \otimes Bw. \tag{76}$$

Now with this equation one can show that because A and B are simple unitary, the new transformation $A \otimes B$ is also simple unitary. Return to the discussion concerning $SU_2 \cdot SU_3 \subset SU_6$. An appropriate basis for the new underlying space $\mathbb{C}^6$ could in this example be:

$$\{e_{\uparrow u},\ e_{\downarrow u},\ e_{\uparrow d},\ e_{\downarrow d},\ e_{\uparrow s},\ e_{\downarrow s}\}, \tag{77}$$

where the arrows denote the two different spin states and the letters u, d and s the three different flavour states (up, down and strange). Combining the freedoms in this way gives a restriction on the symmetry the new tensor can have, or in other words an equation for CG-coefficients. The equation for the CG-coefficients is [4]-pg 79:

$$V_{(21)} \otimes V_{(21)} = \bigoplus_{\lambda} (V_\lambda)^{\otimes a_\lambda}. \tag{78}$$

Here V_λ is the representation of S_n corresponding to the Young diagram λ. This can be understood as follows. Only those irreducible representations are part of the new representation, which represent a combined symmetry, combined from the two different kind of freedoms. Now the CG-coefficients a_λ can easily be calculated with the help of character theory. The decomposition in terms of Young diagrams now becomes:

$$\left(\yng(2,1)\,,\,\yng(2,1)\right) = \yng(2,1) \oplus \yng(3) \oplus \yng(1,1,1) \tag{79}$$

The decomposition explained in example 6.4 can be generalized to the following theorem:

Theorem 6.3. *Let $\mathbb{S}_\mu\mathbb{C}^{n_1}$ be an irreducible representation of the group SU_{n_1} and $\mathbb{S}_\nu\mathbb{C}^{n_2}$ an irreducible representation of the group SU_{n_2}. Then the combination of these two representation as a representation of $SU_{n_1} \cdot SU_{n_2} \subset SU_n$ $(n = n_1 \cdot n_2)$ is:*

$$(\mathbb{S}_\mu\mathbb{C}^{n_1}\ ,\ \mathbb{S}_\nu\mathbb{C}^{n_2}) = \bigoplus_\lambda (\mathbb{S}_\lambda\mathbb{C}^n)^{\oplus a_{\mu\nu\lambda}} \tag{80}$$

where the CG-coefficients are given by the inner product $(\chi_\mu\chi_\nu,\ \chi_\lambda)$ defined in equation (38). The symbols μ, ν and λ denote irreducible representations of the groups S_{n_1}, S_{n_2} and S_n respectively.

The decomposition just described could also work the other way around. Take a particle consisting of three quarks with spin and flavour symmetry. Say this particle is totally symmetric, so it is an element of the irreducible representation $\square\square\square$ of the group SU_6. What kind of spin and flavour symmetries can it have? The next theorem answers this question [4]-pg 79:

Theorem 6.4. *Let $\mathbb{S}_\lambda\mathbb{C}^n$ be an irreducible representation of the group SU_n and $n = n_1 \cdot n_2$. Then as a representation of the group $SU_{n_1} \cdot SU_{n_2}$ the decomposition of this representation is:*

$$(\mathbb{S}_\lambda\mathbb{C}^n) = \bigoplus_{\mu\nu} (\mathbb{S}_\mu\mathbb{C}^{n_1}\ ,\ \mathbb{S}_\nu\mathbb{C}^{n_2})^{\oplus a_{\nu\mu\lambda}} \tag{81}$$

where the CG-coefficients are given by the inner product $(\chi_\mu\chi_\nu,\ \chi_\lambda)$. The symbols μ, ν and λ denote irreducible representations of the groups S_{n_1}, S_{n_2} and S_n respectively.

According to this theorem the decomposition of the irreducible representation $\square\square\square$ as representation of the group $SU_2 \cdot SU_3$ in terms of Young diagrams becomes:

$$\square\square\square = \left(\begin{smallmatrix}\square&\square\\\square&\end{smallmatrix}\,,\,\begin{smallmatrix}\square&\square\\\square&\end{smallmatrix}\right) \oplus \left(\square\square\square\,,\,\square\square\square\right). \tag{82}$$

Actually, when one just calculates the CG-coefficients one would find that $(\chi_{(111)}\chi_{(111)},\ \chi_3) = 1$ meaning that one could expect a term $\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix} \otimes \begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}$. But since $\mu = (111)$ is not a valid Young diagram with the group SU_2 (because $\mathbb{S}_{(111)}\mathbb{C}^2 = (0)$), this term is omitted.

6.4 Invariant Tensors

Up to now a few examples of hadrons and their symmetries have been discussed. The hadrons discussed always consisted of quarks which transform with matrices U. Hadrons can also consist of anti-quarks, which transform with U^*. If a quark is represented by a vector in the space V, than an anti-quark should be represented by a vector in its dual V^*. For example a meson consisting of a quark q and an antiquark $\bar{q}$ transforms like:

$$U(q_1 \otimes \bar{q_2}) = U(q_1) \otimes (\bar{q_2})U^*, \tag{83}$$

where $q_1 \in V$ and $\bar{q_2} \in V^*$. Using the Einstein summation convention this becomes:

$$(UT)^{j_1}_{j_2} = U^{j_1}_{i_1} U^{*i_2}_{j_2} T^{i_1}_{i_2}. \tag{84}$$

Here $T = q_1 \otimes \bar{q_2}$ and $T^{i_1}_{i_2} = (q_1)_{i_1} \otimes (\bar{q_2})_{i_2}$. Because the quark and the anti-quark transform with different matrices, this transformation does not conserve symmetry properties. Moreover, because of its upper index, the tensor isn't an element of the space $\mathbb{S}_\lambda \mathbb{C}^n$. To overcome this problem one can use an invariant tensor to get rid of the upper index in the tensor T. An invariant tensor which can do the job is the Levi-Civita tensor ϵ. The Levi-Civita tensor is a tensor with either n upper indices or n lower indices. All indices can run from 1 to n. Furthermore:

$$\epsilon_d = \sum_\sigma \mathrm{sgn}(\sigma) e_{\sigma(1)} \otimes \ldots \otimes e_{\sigma(n)} = \sum_\sigma \mathrm{sgn}(\sigma) v_{\sigma(1),\ldots,\sigma(n)} \tag{85}$$

$$\epsilon_u = \sum_\sigma \mathrm{sgn}(\sigma) e^*_{\sigma(1)} \otimes \ldots \otimes e^*_{\sigma(n)} = \sum_\sigma sgn(\sigma) v^{\sigma(1),\ldots,\sigma(n)} \tag{86}$$

For $n = 3$ the Levi-Civita tensors become:

$$\epsilon_d = v_{123} + v_{231} + v_{312} - v_{213} - v_{321} - v_{132} \tag{87}$$

$$\epsilon_u = v^{123} + v^{231} + v^{312} - v^{213} - v^{321} - v^{132} \tag{88}$$

A quick glance at equations (87) shows that the Levi-Civita tensor with lower indices spans the totally antisymmetric irreducible representation $\mathbb{S}_{(111)}\mathbb{C}^3$. In general the Levi-Civita tensor with lower indices of rank n spans the 1-dimensional representation $\mathbb{S}_{(1^n)}\mathbb{C}^n$. From this follows that this tensor is always transformed to a multiple of itself with an SU_n transformation. A standard calculation (see [4]-pg 51) shows that $U\epsilon = \epsilon$ which implies that the Levi-Civita tensor is indeed an invariant tensor.

The Levi-Civita tensor will be used to lower or raise indices. To see how this works, first take a tensor T with an upper and a lower index:

$$T = T^i_j v^j_i \in V \otimes V^*, \tag{89}$$

where v^j_i is a basis tensor of the space $V \otimes V^*$. The tensor T needs te be transformed to a tensor with three lower indices:

$$T' = T'^{\ ijk} v_{ijk} \in V^{\otimes 3}. \tag{90}$$

Now the constants $T'^{\ ijk}$ need to be calculated. First consider the tensor product $T' = \epsilon_d T$, this can be written out in components:

$$(T')^{jkm} = (\epsilon_d)^{ijk} T_i^m \tag{91}$$

Here $(\epsilon_d)^{ijk}$ denotes the constants corresponding to the ϵ_d tensor. Equation (87) implies that $(\epsilon_d)^{ijk} = \mathrm{sgn}(\epsilon)$. Equation 91 can also be understood as follows. Loosely speaking the ϵ tensor consists of three vectors in V. The T tensor consist of a vector part and part which maps vectors to $\mathbb{C}$ (according to the equation $e^{*j}(e_i) = \delta_{ij}$). The notation in equation 91 implies that the first vector of the ϵ tensor is mapped to $\mathbb{C}$. As always when calculating a tensor element $(T')^{jkm}$ using such an equation, one has to sum over the index i.

Note that using an Levi Civita tensor with upper indices ϵ_u one can in the same way transform the tensor T back to T'. This gives the inverse of the Levi Civita tensor. Now because the Levi Civita tensor is an invariant tensor the following equation holds:

$$UT' = U(\epsilon T) = (U\epsilon)(UT) = \epsilon(UT) \tag{92}$$

for an arbitrary SU_n transformation U. This equation implies that the Levi Civita tensor is an homomorphism between the tensors $T \in V \otimes V^*$ and the tensors $T' \in V^{\otimes 3}$. In fact it is an isomorphism, because the Levi Civita tensor has an inverse which is also homomorphic. One can wonder if thus the spaces $V^{\otimes 3}$ and V^* are isomorphic, but this is not the case because their dimensions are not equal. But note that the transformed tensor T' is always antisymmetric in its first two indices:

$$(T')^{jkm} = (\epsilon_d)^{ijk} T_i^m = -(\epsilon_d)^{ikj} T_i^m = -(T')^{kjm} \tag{93}$$

This implies that for all $T,\ T' \in V^{\otimes 3}((e) - (12))$. Now both spaces are n^2-dimensional where n is the dimension of V. Because these two spaces are isomorphic, properties such as invariant subspaces are the same in both spaces. To conclude one can say that the Levi Civita tensor can be used to transforms tensors with upper indices to equivalent tensors with only lower indices. This transformation is needed to be able to use the theories in the previous two sections. Thus if one ones to analyze for example a meson, one first lowers its upper index, analyzes the equivalent tensor on its irreducible representations and finally raises the lower index again.

One can generalize this strategy to a tensor T with d_1 lower indices and d_2 upper indices. The space V with dimension n is still the underlying space. For each upper index one uses an ϵ tensor of rank n to get an equivalent tensor with an extra $n-1$ antisymmetric lower indices. The transformed tensor T' is thus a tensor with $d_1 + d_2(n-1)$ lower indices:

$$T'^{j_1 \cdots j_{d_1} i_{(1,1)} \cdots i_{(d_2,(n-1))}} = \epsilon^{i_1 i_{(1,1)} \cdots i_{(1,(n-1))}} \dots \epsilon^{i_{d_2} i_{(d_2,1)} \cdots i_{(d_2,(n-1))}} T^{j_1 j_2 \dots j_{d_1}}_{i_1 i_2 \dots i_{d_2}} \tag{94}$$

7 Colour

Before Colour was introduced the quarks were hypothesized to have an SU_3 flavour symmetry. This was proposed in 1962 by Gell-Mann. Only three flavours were known in that time: up, down and strange. Nowadays it is known that there are quarks of three other flavours: charm, top and bottom. Putting these flavours into the model will give a higher symmetry SU_6. In the following discussion, like in the examples of the last chapter, only the lighter three flavours up, down and strange will be taken into consideration.

The quarks are the building blocks of the hadrons. The different kinds of hadrons split up into two groups: baryons, consisting of three quarks, and mesons, consisting of a quark and an antiquark. Now we have all tools for analyzing these groups, starting with the mesons.

7.1 Mesons

Only the spin and the up, down and strange flavour of the quarks are taken into consideration. A quark has six freedoms and transforms with SU_6 matrices. One quark is a tensor in the irreducible representations $\square$, which is 6-dimenional. An antiquark can be represented by a tensor with an upper index. In section 6.4 it is explained why and how to lower this index. This tensor is equivalent to a tensor with 5 antisymmetric lower indices, which is an element of the irreducible representation (1^5). Now mesons can be represented by tensors in the product space of these two representations, which decomposes into:

$$\square \otimes (1^5) = (2,1^4) \oplus (1^6). \tag{95}$$

A check on the dimension of this decomposition: $(6 * 6 = 35 + 1)$. To further analyze these mesons these representations must be split up into its representations of $SU_2 \cdot SU_3$. For the 1-dimensional representation (1^6) this can be easily done:

$$(1^6) = \left((1^2), (1^3) \right). \tag{96}$$

This irreducible representation is also 1-dimensional and can directly be associated with a particle. It must be totally antisymmetric in both spin and flavour. There is however a complication. This is an analysis of the tensor v_{ijk} with 2 antisymmetric indices, and not of the equivalent tensor representing the mesons. The totally antisymmetric tensor A, which is the basis tensor of the irreducible representation corresponding to the Young diagram (111), can be transformed back into the tensor representing a meson with an ϵ tensor.

$$\begin{aligned}(\epsilon_u A)_i^m &= (\epsilon_u)_{ijk}(A)^{mjk}\\ &= \text{sgn}(\sigma_1)\text{sgn}(\sigma_2) = 1\end{aligned}$$

The last equality follows from the fact that $\sigma_1 = \sigma_2$ which follows from the fact that the last two indices of the ϵ tensor must be equal two the last two indices of the A tensor. Now one can explicitly write down the particle that corresponds to this one dimensional irreducible representations:

$$(\epsilon_u A) = (e_1 \otimes e_1^*) + (e_2 \otimes e_2^*) + (e_3 \otimes e_3^*) \Rightarrow u\bar{u} + d\bar{d} + s\bar{s} \tag{97}$$

where u, s, d denote quarks of the three different flavours, and $\bar{u}$, $\bar{u}$, $\bar{u}$ denote antiquarks of the three different flavours. The 35-dimensional representation $(2, 1^4)$ can also be decomposed into irreducible representations of $SU_3 \cdot SU_2$. How this is done is explained in theorem 9:

$$\yng(2,1,1,1,1) = \left(\,\yng(2,1)\,,\,\yng(1,1)\,\right) + \left(\,\yng(2,1)\,,\,\yng(2)\,\right) + \left(\,\yng(1,1,1)\,,\,\yng(2)\,\right). \tag{98}$$

Again a check on the dimension of this decomposition: $35 = 8*1+8*3+1*3$. These representations coincide with three different multiplets: the spin-0 octet, the spin-1 octet and a spin-1 singlet. Octets are a group of eight particles that have been measured to have the same properties. In the same way that isospin grouped particles, octets are groups of particles that are the same under SU_3 transformations. Like with isospin, their symmetry (the particles being the same) is broken by electromagnetic forces and weak forces. So the SU_3 model nicely orders particles to an ordering coinciding with experiment. This was one of the first successes of this model. For a list of the particles found in these octets and for more on properties of multiplets, see [6].

7.2 Baryons

The same analyses can be done with baryons. In this case there is no complication concerning antiparticles because a baryon consists of three quarks and no antiquarks. Just as with the mesons the baryons can be denoted with tensors which are elements of the following product space:

$$\yng(1) \otimes \yng(1) \otimes \yng(1) = \yng(3) \oplus 2 \cdot \yng(2,1) \oplus \yng(1,1,1). \tag{99}$$

Just to be sure, a check a dimensions gives: $6*6*6 = 56+2*70+20$. First decompose the representation $\lambda = (3)$ further into irreducible representations of $SU_3 \cdot SU_2$. This decomposition has already been calculated just after theorem 9:

$$\yng(3) = \left(\,\yng(2,1)\,,\,\yng(2,1)\,\right) \oplus \left(\,\yng(3)\,,\,\yng(3)\,\right), \tag{100}$$

which can again be checked on dimensions: $56 = 10*4 = 8*2$. The resulting decomposition thus gives a 10-dimenional space in flavour, a decaplet, combined with a 4-dimensional space in spin, meaning spin-3/2 particles. And it gives an octet of spin-1/2 particles. These 18 particles are exactly all the baryons that have been discovered. Not all of them were discovered before Gell-Mann proposed the Eightfold-Way, but the predicted 18-th particle was discovered in 1964 making this theory a big success.

But how about the other two irreducible representations of SU_6? Don't they represent particles? It seems that all particles are already represented by the representation $\yng(3)$. The decomposition of the other two representations are given here for completeness:

$$\yng(2,1) = \left(\yng(3), \yng(2,1)\right) \oplus \left(\yng(2,1), \yng(3)\right) \oplus \left(\yng(1,1,1), \yng(2,1)\right) \oplus \left(\yng(2,1), \yng(2,1)\right), \quad (101)$$

$$\yng(1,1,1) = \left(\yng(1,1,1), \yng(3)\right) \oplus \left(\yng(2,1), \yng(2,1)\right). \quad (102)$$

Moreover, when one considers the Pauli exclusion principle things become even more complicated. The total wave function of the particles discovered (flavour and spin together) is symmetric while the Pauli exclusion principle insists this wave function to be antisymmetric. During the discussion of mesons this was not considered since it consisted of a quark and an antiquark. The exclusion principle only insists similar particles to be antisymmetric.

This is why the extra freedom of colour was introduced. It is postulated that each quark has an extra colour symmetry. A quark can be red, green or blue. The colour symmetry is, just like the flavour symmetry, an SU_3 symmetry. It was furthermore postulated that hadrons can only exist if they are invariant under an SU_3-colour transformation, insisting that they should be represented by a tensor which is in a 1-dimensional irreducible representation. This property is called colour confinement. It is also explained as the condition that a hadron should be colourless.

The analysis should be redone, but now with colour. The total number of freedoms of a quark is now $3 * 2 * 3 = 18$. Equation (100) still holds, but now the dimensions of the representations are different because the Young diagrams represent irreducible representations in SU_{18} instead of SU_6. The dimensions become: $5832 = 1140 + 2 * 1938 + 816$. The Pauli exclusion principle states that only the 816-dimensional antisymmetric representation can represent particles. Decomposing this representation into irreducible representations of $SU_3^{colour} \cdot SU_6^{flavour/spin}$:

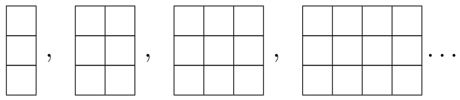
$$\yng(1,1,1) = \left(\yng(1,1,1), \yng(3)\right) \oplus \left(\yng(3), \yng(1,1,1)\right) \oplus \left(\yng(2,1), \yng(2,1)\right). \quad (103)$$

The dimensions of this decomposition are: $816 = 1*56+10*20+8*70$. Now colour confinement states that only the irreducible representations $\left(\yng(1,1,1), \yng(3)\right)$ can represent particles. From here our former analysis can be picked up. Colour confinement thus explains neatly why flavour/spin-wave function must be symmetric while the total wave function is antisymmetric. In the next chapter exotic particles will be analyzed on colour as well as spin and flavour.

8 Consequences

8.1 Different Hadrons

Colour confinement means that a hadron must be invariant under colour transformations. The tensor representing the colour symmetry must thus be an element of a 1-dimensional irreducible representation of SU_3. All the 1-dimensional irreducible representations of SU_3 can be represented by the following Young diagrams:



(104)

Their dimension can be calculated with equation (70). It can also be shown that these are the only 1-dimensional irreducible representations of SU_3. From this fact the conclusion can be taken that if hadrons consist of only quarks, they must consist of $3m$ quarks, where m is a natural number. If a hadron consists of an antiquark, its representing tensor has as upper index. The equivalent tensor obtained by lowering this index has $(n-1)$ extra lower indices, where n is the number of freedoms of a quark. Because a quark always has three colour freedoms, n is always divisible by three. The total number of lower indices should be a multiple of 3 for the tensor to be an element of a 1-dimenional representation. If a hadron thus contains an antiquark it should have $3n+1$ quarks.

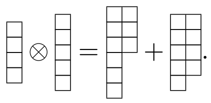
This argument can be generalized to a hadron with d_2 antiquarks. Lowering all these indices the equivalent tensor gets an extra $(d_2n - d_2)$ lower indices where n is divisible by 3. The hadron thus should have $3m + d_2$ quarks for it to have $(3m + d_2n)$ lower indices, which is divisible by 3. The different kinds of hadrons that can be constructed are thus those that have d_1 quark and d_2 antiquarks where $|d_1 - d_2|$ is divisible by 3.

$$\begin{array}{cccc} qqq & q\bar{q} & qq\bar{q}\bar{q} & \bar{q}\bar{q}\bar{q} \\ qqqqqq & qqqq\bar{q} & qqqqq\bar{q}\bar{q} & qqq\bar{q}\bar{q}\bar{q} \\ qqqqqqqqq & qqqqqqq\bar{q} & qqqqqqqq\bar{q}\bar{q} & qqqqqq\bar{q}\bar{q}\bar{q} \end{array} \tag{105}$$

In this table one can clearly see the mesons and the baryons. Note that this discussion does not depend on the number of other freedoms of the quarks in question. In theory, also hadrons consisting of either a number of mesons or a number of baryons can be allowed, which is not very surprising. Also a particle consisting of a meson and a baryon is allowed to exist. This particle, having 5 quarks, is called a pentaquark. In the next paragraph it will be investigated what flavours these pentaquarks can consist of.

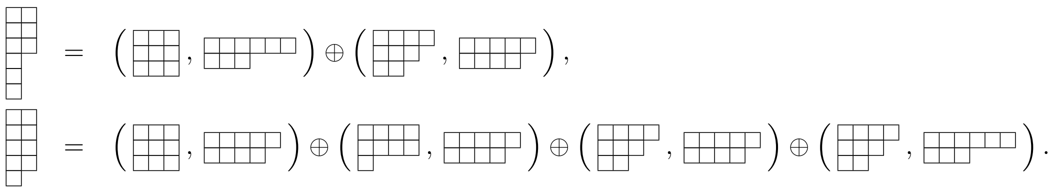
8.2 Pentaquark (two flavours)

The discussion about the pentaquark will start simple. Only two flavours and three different colours will be take into consideration. The pentaquark can thus be represented by a tensor v^m_{ijkl}, with a SU_6 symmetry. The first step is lowering the upper index m. Doing this with the appropriate Levi-Civita tensor one obtains a tensor with nine lower indices. Four of the nine indices are antisymmetric because all the quarks are antisymmetric to one and other. And the other five are antisymmetric to one and other for they stem from the Levi-Civita symbol. The equivalent tensor of the pentaquark is thus an element of the following product space:



(106)

And of course a dimension check: $15 * 6 = 20 + 70$. Both of these representations should first be decomposed into irreducible representations of $SU_3 \cdot SU_2$. In appendix 3 it is explained how these bigger and more complex Young diagrams are decomposed into irreducible representations.



And again a check on dimensions: $20 = 1 * 4 + 8 * 2$ and $70 = 1 * 2 + 10 * 2 + 8 * 2 + 8 * 4$. Both decompositions contain an irreducible representation with a 1-dimensional component as a colour representation. They both need to by analyzed, starting with the decomposition of the (222111) representation. Its flavour component coinciding with the 1-dimensional colour component is the 2-dimensional (63) representation. One can try to calculate two basis tensors for this space via the Young Symmetrizer as explained in chapter 4. But the Young symmetrizer is an sum of $6! \cdot 3! \cdot 8$ elements of S_9. This is a time consuming calculation and is out of the scope of this project. But with a bit of deduction a few conclusions can never the less be taken. First construct the following basis of the space $\mathbb{C}^6$

$$\{u_r,\ u_b,\ u_g,\ d_r,\ d_b,\ d_g\}, \tag{107}$$

where the tensor $v_1 = u_r$ denotes a red up quark, etcetera. Assume that the pentaquark consist of an anti-up quark of the colour red. The colour of this antiquark is irrelevant for the discussion. The tensor T representing this pentaquark becomes:

$$T = T_i^{jkmn} v^i_{jkmn}. \tag{108}$$

Thus $T_i^{jkmn} \neq 0$ only when $i = 1$. Thus for the tranformed tensor T',

$$(T')^{abcdejkmn} = (\epsilon_d)^{abcde} T_i^{jkmn}, \tag{109}$$

the following holds: $(T')^{abcdejkmn} \neq 0$ only when the indices a, b, c, d, e are a permutation of the numbers $2, \ldots, 5$. Physically this means that the transformed tensor consist of three down quarks (one of each colour) and two up quarks. Because eventually the flavour part of the equivalent tensor should be an element of the irreducible representation (63), the equivalent tensor should either consist of three up quarks and six down quarks or three down quarks and six up quarks. The four possibilities for a pentaquark in this irreducible representation are listed in table 4.

$\bar{u}uuuu,$	$\bar{d}dddd,$
$\bar{u}dddu,$	$\bar{d}uuud.$

Table 4: possible quarks in (63) considering 2 flavours

This same discussion can be done when considering the second irreducible representation of SU_6: (22221). Because the colour part must be a singlet, the flavour part of the tensor must be an element of the irreducible representation (54). Thus the equivalent tensor can consist of either five up quarks and four down quarks, or five down quarks and four up quarks. Again considering that the index representing the antiquark must be lowered one has the following four possibilities:

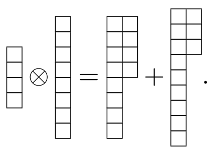
$\bar{u}uuud,$	$,\bar{d}dddu$
$\bar{u}uudd,$	$\bar{d}dduu.$

Table 5: possible quarks in (54) considering 2 flavours

The pentaquarks that have not been listed in both tables are the quarks $\bar{d}uuuu$ or $\bar{u}dddd$. According to this model it can be concluded that they are forbidden.

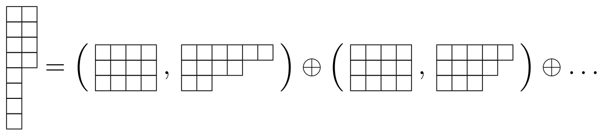
8.3 Pentaquark (three flavours)

The discussion of the previous section will now be extended to three flavours. The quarks now have nine freedoms. Lowering the upper index of the antiquark will give a tensor of 12 lower indices, which is an element of the following product space:

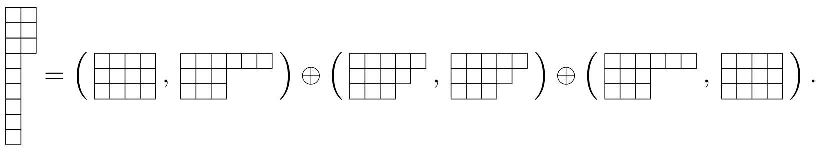


(110)

The dimensions keep getting larger: $126 * 9 = 1050 + 84$. The first Young diagram decomposes as representation of $SU_3^{colour} \cdot SU_3^{flavour}$ into 12 different irreducible representations. Of course, not all 12 are relevant, only those who have a 1-dimensional colour component:


(111)

The equation giving the dimensions of the 12 different irreducible representations is $(1050 = 1*27+1*8+8*1+2(8*8)+8*10+8*10+8*27+10*8+10*10+10*8+27*127*8)$. The second Young diagram decomposes into only three irreducible representations:


(112)

And for the last time, this can also be checked on dimensions: $841*10+64+10*1$. Like with only two flavours, it is out of the scope of this project to calculate basis tensors for the flavour component of the irreducible representations in question. A similar reasoning as the previous section can be used to derive some conclusions. First look at the irreducible representations (642). Six indices should be symmetric to one and other, all of them being antisymmetric to four other symmetric indices, being again antisymmetric to two other symmetric indices. This can be accomplished by stating that in the equivalent tensor their should be six quarks of one flavour, four of another and two of the third tensor. Pentaquarks which satisfy these conditions are:

$$\bar{u}ddds, \quad \bar{u}sssd, \quad \bar{d}uuus, \quad \bar{d}sssu \quad \bar{s}dddu, \quad \bar{s}uuud.$$

Table 6: possible quarks in (642) considering 3 flavours

Quarks belonging to the other two representations become;

$\bar{u}dduu$,	$\bar{u}ssuu$,	$\bar{d}uudd$,	$\bar{d}ssdd$	$\bar{s}ddss$,	$\bar{s}uuss$,
$\bar{u}duuu$,	$\bar{u}suuu$,	$\bar{d}uddd$,	$\bar{d}sddd$	$\bar{s}usss$,	$\bar{s}dsss$,
$\bar{u}ddsu$,	$\bar{u}ssdu$,	$\bar{d}uusd$,	$\bar{d}ssud$	$\bar{s}uuds$,	$\bar{s}ddus$.

Table 7: possible quarks in (543) considering 3 flavours

Pentaquarks of the type $\bar{u}dddd$, $\bar{u}ssdd$ and $\bar{u}sduu$ can not be found in one of the three tables meaning that according to this model, they are forbidden.

$\bar{u}dddu$,	$\bar{u}sssu$,	$\bar{d}uuud$,	$\bar{d}sssd$	$\bar{s}uuus$,	$\bar{s}ddds$,
$\bar{u}uuuu$,		$\bar{d}dddd$,		$\bar{s}ssss$,	.

Table 8: possible quarks in (633) considering 3 flavours

It is not surprising that the pentaquark $\bar{u}dddd$ was already forbidden when describing only two flavours. Do note that this is only a very simple first model. To actually prove that pentaquarks of this type are forbidden a deeper mathematical analysis of the flavour tensors is needed. But as was said before, this is beyond the scope of this project.

One can also try to insert spin into this simple model. One would then be working in a $SU_3^{colour} \cdot SU_3^{flavour} \cdot SU_2^{spin} \subset SU_{18}$. Lowering the upper index representing the antiquark one would get a tensor with 21 indices. If one want to split this into irreducible representations of $SU_3^{colour} \cdot SU_3^{flavour} \cdot SU_2^{spin}$ a knowledge of the characters of the symmetric group S_{21} is needed. This is a group of 21! elements and 792 different irreducible representations. Calculations with this group are also beyond the scope of this project.

The results from table 6, 7 and 8 can be compared with results from other analyses such as [2]. The big difference between both studies is that in this thesis spin isn't included. The 18 pentaquarks that follow from the analysis in [2] are given in table 9

$\bar{u}dduu$,	$\bar{u}ddss$,	$\bar{u}ssuu$,	$\bar{u}ssdu$	$\bar{u}uuds$,	$\bar{u}ddsu$,
$\bar{s}dduu$,	$\bar{s}ddss$,	$\bar{s}ssuu$,	$\bar{s}ssdu$	$\bar{s}uuds$,	$\bar{s}ddsu$,
$\bar{d}dduu$,	$\bar{d}ddss$,	$\bar{d}ssuu$,	$\bar{d}ssdu$	$\bar{d}uuds$,	$\bar{d}ddsu$,

Table 9: possible quarks according to [2]

Comparing these results with those from the analysis in this thesis one immediately sees that quarks from the representations (642) and (633) are not given in table 9. The 18-plet given in this table is comparable with the 18-plet in table 7. There are differences, one of the big one being that the Θ particle ($\bar{s}uudd$), the particle which is searched for in experiments isn't mentioned in table 7 (nor in table 6 or 8).

It must thus be emphasized that the analysis done in here are only an application of the mathematical formalism described in chapter 6. The results do not correctly predict the pentaquarks that can be formed because spin hasn't been included.

9 Discussion and Conclusions

As mentioned in the introduction, this thesis describes how representation theory can be applied to particle physics. One of the main conclusions that be can taken is that representation theory can be very useful when analyzing quarks. It not only gives more understanding on what quarks are, but also gives predictions as to what kind of hadrons quarks can form. One of the most important conclusions that can be taken from the mathematical formalism is the following: colour confinement insists a hadron to consist of d_1 quarks and d_2 antiquarks such that $|d_1 - d_2|$ is divisible by three. This statement is independent on the number of other freedoms and thus also holds if one considers SU_6 symmetries

One such hadron is the pentaquark. It is investigated how this exotic particle can be made out of quarks. This investigation illustrates how symmetries can be analyzed and how this analysis can lead to a description of what particles can be constructed. Comparison with other studies on pentaquarks shows that this first investigation isn't sufficient. To actually use the mathematical formalism described in chapter 6 to predict how pentaquarks can be constructed, a more thorough analysis is needed. An analysis which at least also includes the spin of the particles has to be done. Also the basis tensors spanning the irreducible representations describing different groups of pentaquarks need te be calculated using the theory in chapter 4 to verify conclusions taken in chapter 8.

References

[1] Kenneth H. Hicks, Prog. in Part. Nucl. Phys. **55**, 647-676 (2005)

[2] Wybourne B.G., arXiv, 07170 (2003)

[3] W. Fulton and J. Harris *Representation Theory.* Springer-Verlag 1991

[4] L.J.A.M. Somers *Unitary Symmetries, The Symmetric Group and Multiquark Systems.* Krips Repro 1984

[5] T.P Cheng and L.F. Li *Gauge Theory of Elementary Particle Physics.* Oxford Science Publications 1984

[6] L.H. Ryder *Elementary Particles and Symmetries.* Gorden and Breach 1975

[7] K. Huang *Quarks, Leptions & Gauge Fields.* World Scientific 1982

[8] Wieb Bosma, John Cannon, and Catherine Playoust *The Magma algebra system. I. The user language.* J. Symbolic Comput. 1997

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Appendices

A.1 - Calculations on the space $\mathbb{S}_{(21)}\mathbb{C}^n$

Calculation of the Young symmetrizer belonging to $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$:

$$\begin{array}{rcl} P_{(21)} & = & \{(e),(12)\} \\ Q_{(21)} & = & \{(e),(13)\} \end{array} \Rightarrow \begin{array}{rcl} a_{(21)} & = & (e)+(12) \\ b_{(21)} & = & (e)-(13) \end{array}$$
$$\Rightarrow \quad c_{(21)} = (e)+(12)-(13)-(132)$$

Calculation of the space $\mathbb{S}_{(21)}\mathbb{C}^n$:

$$\begin{array}{rcl}
c_{(21)}(v_{iii}) & = & v_{iii}+v_{iii}-v_{iii}-v_{iii}=0 \\
c_{(21)}(v_{iji}) & = & v_{iji}+v_{jii}-v_{iji}-v_{jii}=0 \\
c_{(21)}(v_{iij}) & = & 2v_{iij}-v_{jii}-v_{iji} \quad (i \neq j \text{ and } i,j \in \{1,2,3\} \Rightarrow 6 \text{ tensors}) \\
c_{(21)}(v_{jii}) & = & v_{jii}+v_{iji}-2v_{iij} \quad (i \neq j \text{ and } i,j \in \{1,2,3\} \Rightarrow 6 \text{ tensors}) \\
c_{(21)}(v_{ijk}) & = & v_{ijk}+v_{jik}-v_{kji}-v_{jki} \quad (i \neq j \neq k \text{ and } i,j,k \in \{1,2,3\} \Rightarrow 6 \text{ tensors})
\end{array}$$

Calculation on linear dependency: The tensors $c_{(21)}(v_{iij}) = -1 \cdot c_{(21)}(v_{jii})$ are all are linear independent. This is not the case for the tensors (v_{ijk}):

$$\begin{array}{lll}
c_{(21)}(v_{123}) & = v_{123}+v_{213}-v_{321}-v_{231} & \equiv v^1 \\
c_{(21)}(v_{132}) & = v_{132}+v_{312}-v_{231}-v_{321} & \equiv v^2 \\
c_{(21)}(v_{213}) & = v_{213}+v_{123}-v_{312}-v_{132} & = v^1-v^2 \\
c_{(21)}(v_{231}) & = v_{231}+v_{321}-v_{132}-v_{312} & = -v^2 \\
c_{(21)}(v_{312}) & = v_{312}+v_{132}-v_{213}-v_{123} & = v^2-v^1 \\
c_{(21)}(v_{321}) & = v_{321}+v_{231}-v_{123}-v_{213} & = -v^1
\end{array}$$

A.2 - Calculations on the product space $(2) \otimes (1)$

The Young symmetrizer coinciding with the representation $\square\square \otimes \square$

$$c = ((e)+(12)) \cdot (e) = (e)+(12)$$

Calculation of the space $\mathbb{C}S_3((e)+(12))$

$$\begin{array}{rll}
(e)c & = (e)+(12) & \equiv c_1 \\
(12)c & = (12)+(e) & = c_1 \\
(13)c & = (13)+(123) & \equiv c_2
\end{array}$$

$$\begin{aligned}
(23)c &= (23) + (132) &\equiv c_3 \\
(123)c &= (123) + (13) &= c_2 \\
(132)c &= (132) + (23) &= c_3
\end{aligned}$$

The elements c_i are linear independent and can thus be seen as a basis for $\mathbb{C}S_3(c)$: With respect to this basis the matrices $\rho(g)$ become:

$$\rho(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho(123) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \rho(132) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\rho(12) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \rho(13) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho(23) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The characters are:

Class	(e)	(12)	(13)
$\chi_{\mathbb{C}S_3(c)}$	3	1	0

$$\left.\begin{aligned}
(\chi_{\mathbb{C}S_3(c)}\ ,\ \chi_U) &= 1 \\
(\chi_{\mathbb{C}S_3(c)}\ ,\ \chi_{U'}) &= 0 \\
(\chi_{\mathbb{C}S_3(c)}\ ,\ \chi_S) &= 1
\end{aligned}\right\} \Rightarrow \mathbb{C}S_3(c) = U \oplus S$$

Now from theorem the decomposition for □□⊗□ follows.

A.3 - Calculations on characters of S_9 and S_{12}

The character table of these two groups have been calculated with the program Magma [8] using the code: SymmetricCharacterTable(d), where d is either 9 or 12. The CG-coefficients have been calculated with the Matlab.