Solving the undecidability of the continuum hypothesis:

a short summary of the results since 1963

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1 Introduction

In 1891 Georg Cantor proved that there exist multiple size of infinity. In particular, the size of the natural numbers, $\aleph_0$, is not the same as that of the reals. This of course begs the question: is there a set larger than the natural numbers, but smaller than the reals? The Continuum Hypothesis (CH) is the statement that there is no such set. Since the reals can be seen as the powerset of the natural numbers, CH can be written as '$2^{\aleph_0} = \aleph_1$'. Cantor tried to prove CH, but he failed (obviously, as would later be shown). CH then made it on the list of Hilbert's 23 problems (problem 1). A first step to a solution for CH came in 1940, by Kurt Gödel, who proved that CH is consistent with the axioms of Zermelo-Fraenkel set theory and the axiom of choice (ZFC) [4]. At that point however, it was already clear that the best solution was probably ¬CH. In 1947 Gödel published an article where he concluded that ¬CH is probably consistent with ZFC too, and an important task in set theory would be to find an extra axiom that would decide the problem in favor of ¬CH [5]. The first part of Gödels prediction came true in 1963 when Paul Cohen proved the consistency of ¬CH with ZFC [2]. The second part of Gödels prediction is being worked on.

I'll discuss part of Cohen's method to prove the consistency of ¬CH with ZFC, and some of the results in finding a new axiom that would solve the continuum hypothesis. This article is based mostly on the Bourbaki Lecture *'Progrès récents sur l'hypothèse du continu [d'apres Woodin]'* by Patrick Dehornoy in 2003 [3], in which results by Hugh Woodin on solving the continuum hypothesis are discussed.

2 Extensions

Cohen's method is called forcing. Forcing is used to guarantee that in a certain extension of a model the axioms of ZFC still hold.

We start with a countable transitive model (ctm) M of ZFC. M being a model means that M is a collection of sets such that all axioms of ZFC hold for the sets in M. M is therefore a 'subuniverse', we can see it as a part of a greater set-theoretical universe V, or as a universe on its own, where no sets other than those in M exist. M being countable means that M and all sets in M are either finite or countable. M being transitive means that if a∈M, then for any b∈a, b∈M. In M, like in any model of ZFC, there exist cardinalities $\aleph_0$, $\aleph_1$, $\aleph_2$, and so on. Like in the set-theoretical universe V we will call $\aleph_0$ countable (in M), and larger cardinalities uncountable (in M). The important thing to realise here is that uncountable in M really is different from uncountable in V. After all, M is a ctm, so any set in M is countable. Therefore, there exists a bijective function f_{ij} from $\aleph_i$ to $\aleph_j$ for any i and j. However, a function is also a set (of ordered pairs of originals and their images), and the set f_{ij} is not an element of M. So, as far as M is concerned, there is no bijection between $\aleph_i$ and $\aleph_j$ for $i \neq j$, so the $\aleph$'s really are different cardinals (in M). (Basically, f_{ij} isn't in M because for M to be a model of ZFC it really needs different cardinalities.)

Something similar happens with $2^{\aleph_0}$. We know $2^{\aleph_0}$ is uncountable in M, but countable in V. Now, suppose we want to make ¬CH true. Then all we have to do is take the bijection f between

$2^{\aleph_0}$ and $\aleph_2$ in V, and add f to M. (In fact, we'll create a model M_P[G] where $\aleph_2 \subset 2^{\aleph_0}$, but a bijection need not exist. This inclusion however, is enough to prove ¬CH.) Of course this is all a bit harder than it sounds. Adding just the single set f to M will not result in a model of ZFC. In order to guarantee that our extension M_P[G] of M is again a ctm of ZFC we will have to do a bit of work.

In order to make this extension of M we will need an element P of M, and a subset G of P. We will also need some restrictions on P and G, but I'll leave those until we need them. From here on, however, I will for the ease of notation assume P and G to have the desired properties, even for P and G used in definitions. We start by recursively defining P-names.

Definition 1 *A set x* is called a* P-name *if all elements of x* are ordered sets* $<y^*,p>$ *with y* a P-name, and* $p \in P$.

$\emptyset$ is a P-name, so $\{<\emptyset,p>\}$ is a P-name for p∈P. But also $\{<\emptyset,p>|p\in P\}$. Then $\{<\{<\emptyset,p_2>\},p_1>\}$ is also a P-name for $p_1,p_2\in P$. Every P-name will yield an element of M_P[G], though multiple P-names might yield the same element. G however, will decide which element of M_P[G] a given P-name defines. In order to do this we will define the value of a P-name.

Definition 2 *val(x*,G)=*$\{val(y^*,G)|\exists g \in G$ *with* $<y^*,g>\in x^*\}$ *is the* value *of a P-name x* under G.*

Once again a recursive definition. The value of $\emptyset$ is $\emptyset$, the value of $\{<\emptyset,p>\}$ is $\{\emptyset\}$ if p ∈G, and $\emptyset$ otherwise, the value of $\{<\{<\emptyset,p_2>\},p_1>\}$ is $\emptyset$ if $p_1 \notin$G, $\{\emptyset\}$ if $p_1 \in$G and $p_2 \notin$G, and $\{\{\emptyset\}\}$ if $p_1,p_2 \in$G. Note that different P-names will often have the same value. From here on I'll use x*,y* et cetera for P-names, and x,y et cetera for their corresponding values. With P-names and values of P-names we have what we need to define a generic extension of M.

Definition 3 *M_P[G] is a* generic extension *of M if M_P[G]=*$\{val(x^*,G)|x^*\in M$*, and x* a P-name*$\}$.

Specifically note that any P-name we use has to be an element of M. This restriction will make sure that some elements are not in M_P[G]. After all, if we could just pick any P-name we can think of, the entire structure of values of P-names would become trivial. After all, we could change any element p∈P that isn't in G to an element that is in G, and the result would still be a P-name. Unfortunately it is not possible to give a simple example of a P-name that isn't included in M, because we don't know enough about M, and because the axiom of pairing ensures that any P-name with a finite number of elements will be in M.

Now let's take a look at what's really happening here. We have an element P of M. P is in M, so it can be 'understood within M'. G however, is a subset of P, and need not be an element of M. So, all P-names in M can be 'understood within M', but the values of those P-names can't be 'understood in M', because it requires knowledge of G.

We want M_P[G] to be an extension of M, so M⊂M_P[G]. For this we need our first restriction on G (and thereby P): G has to be nonempty, and contain an element we'll call "1". This restriction gives us M⊂M_P[G], G∈M_P[G] and ensures that M_P[G] satisfies some of the axioms of ZFC.

For the other axioms we will need forcing.

Definition 4 *An element p∈P* forces *a statement* $\psi(x_1^*, x_2^*, ...)$ *if* $\forall G \subset P$, $p \in G \Rightarrow$ *(*$\psi(x_1, x_2, ...)$ *is true in M_P[G]). We write* $p \models \psi(x_1^*, x_2^*, ...)$.

Forcing gives a way to understand values inside M. Since we don't know G in M, we can't give the values of names. We can however sometimes understand statements like 'If G satisfies property A, then M_P[G] satisfies property B'. For example, for any m∈M we know that 'if 1∈G, then G∈M_P[G]'. Using forcing, we write this as 1⊨G*∈M_P[G].

In the end, we want any statement ψ which is true in M_P[G] to be forced by some element of G. Without other restrictions on P and G this is of course impossible. After all, a single element of G doesn't give that much information. In order to guarantee that a single elements of G gives us enough information we'll have to restrict P to having a partial order $\leq$ with 1 a maximal element, and G to being a generic filter.

Definition 5 *G is called a* filter *if:*

1. $1 \in G$
2. $x \in G \to \forall y \in P(x \leq y \to y \in G)$
3. $\forall x, y \in G\, \exists z \in G$ *with* $z \leq x$ *and* $z \leq y$

Definition 6 *A subset A of P is called* dense *if* $\forall p \in P\, \exists a \in A$ *with* $a \leq p$.

Definition 7 *A filter G is called* generic *if it intersects any dense subset of P.*

I'll give an example of a P and G satisfying the desired properties later on when constructing the model satisfying $\neg$CH. For now I'll get to the point of all these restrictions: we will interpret $a \leq b$ as "a is more informative than b". After all, $a \in G \to b \in G$ because G is a filter, but not the other way around. This way, if $a \leq b$, a will be 'more likely' to force a statement than b. If G is a *generic* filter G will always contain elements 'informative enough'.

Theorem 1 *Suppose M is a ctm of ZFC, P∈M has a partial order $\leq$ with maximal element 1, $G \subset P$ with G a generic filter. Then for any statement $\psi(x_1, x_2, ...)$ which is true in $M_P[G]$ there exists a $g \in G$ such that $g \models \psi(x_1^*, x_2^*, ...)$.*

This result is required for the proof of the next theorem.

Theorem 2 *If M is a ctm of ZFC, then for any $P \in M$ with a partial order $\leq$ with maximal element 1 and any $G \subset P$ with G a generic filter, $M_P[G]$ is a ctm of ZFC. $M_P[G]$ is called a* generic extension *of M.*

The proof of this theorem is both rather straightforward and quite complicated. One simply takes an axiom of ZFC, and uses the fact that the axiom is true in M to prove that it its true in M_P[G]. For some axioms this is quite easy. For some other axioms however, this is quite difficult. This is mainly caused by the fact that a P-name x* may have a lot of elements (y*,p) that disappear when the value is taken, because p∉G. By using forcing one can construct another P-name x̃* with the same value, but without disappearing elements that cause trouble.
Now we have that (Given M, P and G with the required properties):

1. M_P[G] is a ctm of ZFC
2. M⊂M_P[G]
3. G∈M_P[G]

This is enough to create a model satisfying $\neg$CH. Take M a ctm of ZFC. M contains $\aleph_0, \aleph_2$ and 2. Now take P to be the set of all finite functions $f_i : \aleph_2 \times \aleph_0 \longrightarrow 2$. That is, an element f_i of P is a finite set of ordered pairs $< x, y >$ with $x \in \aleph_2 \times \aleph_0$ and $y \in 2$, and if $< x, y > \in f_i$ and $< x, z > \in f_i$ then $y = z$. As partial order we use reverse inclusion, that is $f_i \leq f_j$ if $f_j \subset f_i$. The unique maximal element is simply the empty set. For G we take any generic filter.

Take D_y the set of all elements of P which have y in their domain. Then clearly D_y is dense. Therefore, G intersects D_y, so for any y, G contains at least one function which is defined on y. Now suppose that G contains two elements defined on y, f_i and f_j. Say $f_i(y) = a$, $f_j(y) = b$. G is a filter, so G contains an element f_k with $f_k \leq f_i$ and $f_k \leq f_j$. With our definition of $\leq$ this means that f_k contains both $< y, a >$ and $< y, b >$. However, $f_k \in P$, so $a = b$. This implies that $h = \bigcup G$ is a complete function from $\aleph_2 \times \aleph_0$ to 2.

Now we're going to use the $\aleph_2$ part as an index, $h_i(y) := h(< i, y >)$. Suppose $h_i = h_j$, so $(\forall y \in \aleph_0)\ h(i, y) = h(j, y)$. Suppose $i \neq j$.
Take $D_{ij} = \{f \in P | \exists n \in \aleph_0 (\forall x \leq n(f(< i, x >) = f(< j, x >)) \wedge f(< i, n+1 >) \neq f(< j, n+1 >))\}$. So basicly, D_{ij} is the set of all functions that agree (on indices i and j) on the values of all elements in $\aleph_0$ up to a certain point, after which they disagree at elast once. Since P contains only finite functions, D_{ij} is dense. After all, if $f \in P$ has $f(< i, x >) \neq f(< j, x >)$ for some x, then $f \in D_{ij}$. If $f(< i, x >) = f(< j, x >)$ for all x on which f is defined, then one can simply make $\tilde{f}$ defined over a larger domain where in one of the extra domain elements $\tilde{f}(< i, y >) \neq \tilde{f}(< j, y >)$. However, G doesn't intersect D_{ij}, since for any element of D_{ij} there exists an l such that $f(< i, l >) \neq f(< j, l >)$. Therefore, we have $i = j$, so all h_i are distinct. Thus, we have $\aleph_2$ distinct functions from $\aleph_0$ to 2. This means that $2^{\aleph_0}$ has at least $\aleph_2$ elements, so ¬CH is true in M_P[G].

Note that instead of $\aleph_2$ we could have picked any $\aleph_i$, so this even shows that there is no upper limit in the $\aleph_i$'s for the cardinality of $2^{\aleph_0}$.
Combined with Gödels result this shows CH is undecidable in ZFC.

3 Finding a solution for CH

So CH is undecidable in ZFC. At this point we could just move on, and start considering other problems. That, however, is not the only way to deal with this undecidability. Another option is to try and find an extra axiom (or perhaps extra axioms) that will decide CH. This is what many people have been trying to do for some time now.

The most obvious solution, simply adding ¬CH or CH to ZFC would work (they are consistent with ZFC after all), but is not very elegant. We don't want to specific axioms for all undecidable statements, after all. Instead, we want an axiom that decides pretty much everything. Since this is quite hard to do in V, we'll focus on smaller parts of V.

Definition 8 *H_i is the set of all sets with less than $\aleph_i$ elements.*

We'll consider H_0 (finite sets), H_1 (countable sets) and H_2. Now to define exactly what we want to find:

Problem 1 *Find an axiom that makes $(H_i, \in)$ inside M sufficiently complete, invariant under forcing and is consistent with the existence of large cardinals.*

Sufficiently complete because we want to answer CH and not have to do it all over again for any other undecided problem, but really complete isn't possible. Invariant under forcing means that we can't change the truth of a statement by choosing an extension of M to calculate $(H_i, \in)$ in. Consistency with the existence of large cardinals is a property some people consider to be nice, and will prove to be useful since we'll need some large cardinals, so they might as well exist.

While searching for axioms satisfying problem 1, we will take a slightly different course of action than used to decide on the axioms of ZFC. The axioms of ZFC were accepted because the axioms themselves seemed reasonable to assume. The axioms represent out intuitive idea of set theory as closely as possible. Then we accept the consequences of the axioms, even if they are somewhat counterintuitive. The axioms used to solve problem 1 work the other way around. We want a certain result, so we accept an axiom that guarantees that result, even if the axiom itself isn't immediately clear to be true intuitively.

For H_0 finding a solution to problem 1 isn't too hard. In fact, ZFC itself will do. H_0 is sufficiently complete in ZFC, generic extensions can't add finite sets so H_0 doesn't change when going from M to M[G], and ZFC is consistent with the existence of large cardinals.

3.1 Problem 1 for H_1

Unfortunately, $(H_1, \in)$ isn't invariant under forcing, nor is it sufficiently complete. There is an axiom that (almost) solves problem 1 for $(H_1, \in)$ though. This axiom works by interpreting $(H_1, \in)$

as the projective subsets of $\{0,1\}^{\mathbb{N}}$ and therefore the projective subsets of $\mathbb{R}$.

Axiom 1 (Projective Determinacy (PD)) *Any projective subset of [0,1] is determined.*

Now for what this axiom really means.

Definition 9 *Take X a completely metrisable topological space. A subset of X^p is called* projective *if it can be obtained by applying*

- *switching to the complement*
- *taking a projection from $X^l \longrightarrow X^m$ with $l > m$*

a finite amount of times, starting from a Borel set in X^{p+k}.

'Nice' sets (that is, most sets that can be easily defined, like open and closed sets) are always projective sets. Some less nice sets are also projective though, and those are the really interesting sets for PD.

Definition 10 *A subset A of [0,1] is called* determined *if for $e_i \in \{0,1\}$ either*
$\exists e_1 \forall e_2 \exists e_3 ... (\sum_i e_i 2^{-i} \in A)$ *or*
$\forall e_1 \exists e_2 \forall e_3 ... (\sum_i e_i 2^{-i} \notin A)$

Determinacy can best be seen as the existence of a winning strategy for one player in a two player game. Player one chooses e_i for i odd, player two chooses e_i for i even. Player one tries to get $(\sum_i e_i 2^{-i} \in A)$, player two tries to get $(\sum_i e_i 2^{-i} \notin A)$. If there is a winning strategy for player one, he can choose an e_1 such that whatever player two chooses as e_2, player one can choose an e_3 such that... and so on, where player one wins, so $\exists e_1 \forall e_2 \exists e_3 ... (\sum_i e_i 2^{-i} \in A)$. Likewise, if there is a winning strategy for player two, $\forall e_1 \exists e_2 \forall e_3 ... (\sum_i e_i 2^{-i} \notin A)$.

Like for projectivity, nice sets (say, the union of a finite number of intervals, or $\mathbb{Q} \cap$ [0,1]) are determined. There are sets that are not determined, however, since the assumption that any set in [0,1] is projective contradicts the Axiom of Choice. Assuming that all projective sets are determined (so PD) doesn't contradict ZFC however.

Also, ZFC+PD makes $(H_1, \in)$ sufficiently complete. For invariance under forcing we need slightly more. There is a certain type of large cardinals called Woodin Cardinals. What the exact definition of a Woodin cardinal is isn't very important at this point. What is important is that

Theorem 3 *If there exists a proper class of Woodin cardinals, $(H_1, \in)$ is invariant under forcing in ZFC+PD.*

A proper class is something 'too large to be a set', it has elements, but cannot be the element of anything. They're a way to deal with a lot of the paradoxes arising from 'naive set theory', where for example $S = \{x | x \notin x\}$ could be a set. By making $S\{x | x \notin x\}$ a class, the problem is solved. $S \notin S$, simply because S cannot be element of anyhting. We'll see the 'proper class of Woodin cardinals' assumption more often. For now, all that remains is to prove that PD is consistent with the existence of large cardinals. This is quite nicely solved by the next theorem.

Theorem 4 *If there exists an infinite amount of Woodin cardinals, PD is true.*

Since the existence of an infinite amount of Woodin cardinals is weaker than the existence of a proper class of Woodin cardinals this proves that PD is consistent with the existence of large cardinals. In fact, we could just add 'There exists a proper class of Woodin cardinals' to ZFC to solve problem 1. Note that neither of those assumptions would be accepted as axiom the way the axioms of ZFC are. They cannot even be considered the 'parallel postulate for ZFC'. Sure, like the parallel postulate it is independent of the other axioms, and a lot more complicated. But unlike the parallel postulate, PD and the existence of a proper class of Woodin cardinals wouldn't be accepted as axioms just for being intuitive.

3.2 Problem 1 for H_2

$(H_2, \in)$ is of specific interest, since it is the smallest H_i where the continuum hypothesis can be formulated.

The first idea is of course to see if PD, or the existence of a proper class of Woodin cardinals solves problem 1 for H_2. It doesn't. In fact, no axiom that is like those two will work. The axioms that might work are the 'axioms of forcing', connected to the concept of stationarity.

Definition 11 *A subset of the well ordered set $\aleph_1$ with the order topology is called stationary if it intersects any closed and unbounded subset of $\aleph_1$.*

So being stationary for a subset of $\aleph_1$ is quite comparable to being generic for a filter.

The forcing axioms extend the idea of the Baire Category Theorem, which assures that in a locally compact Hausdorff space a countable union of nowhere dense closed sets is itself nowhere dense. The forcing axioms try to give restrictions under which the union of $\aleph_1$ nowhere dense closed sets is nowhere dense. The first of those axioms is Martin's Axiom, MA.

Axiom 2 (Martin's Axiom (MA)) *If X is a locally compact Hausdorff space where for any collection Y of open sets which is pairwise disjoint, Y is countable, then any union of $\aleph_1$ nowhere dense closed sets is nowhere dense.*

There exist many variants of this axiom. For one of those, the Maximal Martin Axiom (MM), it is even proven that it is consistent with the existence of large cardinals. For H_2 however, we're more interested in a weaker form of MM, called the Bounded Maximal Martin Axiom (BMM). BMM can be formulated resembling MA, but also as a statement considering bounded statements. A statement is bounded if the only quantifiers it contains are $\exists x \in z$ and $\forall x \in z$. A statement of the type $\forall\exists$ is a statement $\forall x \exists y \psi(x, y)$ with ψ bounded. MMB guarantees that any property of H_2 that can be expressed by a statement of the type $\forall\exists$ isn't contradicted under stationarity-preserving forcing. That's not enough to solve problem 1, however, since this is a conditional invariance under forcing.

To solve this there is a variant of BMM, the Woodins Maximal Martin Axiom (WMM).

Theorem 5 *Suppose there exists a proper class of Woodin Cardinals. Then ZFC+WMM makes $(H_2, \in)$ sufficiently complete and invariant under forcing.*

However, unlike BMM, WMM isn't a weaker version of MM. In order to have WMM be the solution to problem 1 we'll therefore have to show WMM to be consistent with the existence of large cardinals.

In order to do this we'll use Ω-logic.

4 Ω-logic

4.1 Overview of Ω-logic.

The failure in proving certain statements from ZFC led to a new - weaker - type of proof, in the Ω-logic. For a theorem to be proven from ZFC, it needs to be true in any model of ZFC. In the Ω-logic a statement can be Ω-proven if it is true in *certain* models of ZFC. If this '*certain* models' can be made restrictive enough for the Ω-logic to be sufficiently weaker than normal logic - so some statements that cannot be proven from ZFC can be Ω-proven - but broad enough to include most useful models - so an Ω-proof is actually of some importance - the Ω-logic might give a partial answer to some interesting statements, like CH.

The Ω-logic works as follows. Take a set B, satisfying certain properties. What properties B should satisfy I'll explain later. Models with a certain property related to B are called B-closed. B will be called an Ω-proof for a statement ψ if ψ is satisfied in all B-closed countable transitive models of ZFC. So far, an Ω-proof is pretty useless, since B-closedness is dependent on B. For this the notion of Ω-validity is introduced. A statement ψ is Ω-valid if ψ is true in any model V_α with a certain property. There are a lot of models with this property, and it is not dependent on B, so these model will have to do. The trick is that is has been proven that - if a proper class of Woodin-cardinals exists - any Ω-provable statement is also Ω-valid.

Now to fill in the technical details. The sets B among which the Ω-proofs will be chosen should be universally Baire.

Definition 12 *A subset B of $\mathbb{R}^p$ is called* universally Baire *if for any compact K and any continuous function $f : K \to \mathbb{R}^p$, $f^{-1}(B)$ has the Baire property in K.*

(Reminder: a set W has the Baire property if there exists an open set U such that the symmetric difference between W en U is meager, that is, the symmetric difference is a countable union of nowhere dense sets).

All Borel-sets are universally Baire, and if there exists a proper class of Woodin cardinals, all projective sets are universally Baire. There are also much more complicated sets that are universally Baire.

B-closedness is a bit harder to explain. Any universally Baire set B can be described as the projection of a certain set $\tilde{\text{B}}$. When going from V to a generic extension V[G] of V, the set $\tilde{\text{B}}$ will not change. However, the projection of $\tilde{\text{B}}$ can be larger than B, since it may contain elements of V[G] which are not in V. The projection of $\tilde{\text{B}}$ in V[G] will be called B_G.

Now we can give the definition of B-closed.

Definition 13 *A transitive model M of ZFC is* B-closed *if for any generic extension V[G] of V the set $B_G \cup$M[G] is an element of M[G].*

Note that this definition is not trivial. $B_G \cup$M[G] is always a subset of M[G], but M[G] being transitive guarantees that any element of M[G] is a subset of M[G], not the other way around.

If B is a Borel-set, B-closedness is rather trivial however. In that case any transitive countable model of ZFC is B-closed. If B is a more complicated set, B-closedness is a stronger condition.

Definition 14 *If there exists a proper class of Woodin cardinals, a universally Baire set B is an* Ω-proof *for a statement ψ if for any countable B-closed model M, ψ is true in M.*
A statement ψ is Ω-provable *if there exists an Ω-proof for ψ.*

In order to Ω-prove a statement ψ that is not provable in ZFC, we should therefore pick rather complicated universally Baire sets B. In that case there will be relatively few countable B-closed models of ZFC, so ψ has to be true on relatively few models.

Note that Ω-proofs are automatically invariant under forcing, since generic extensions are 'covered in the definition of B-closedness'. Therefore, if ψ has been Ω-proven in V, $\neg\psi$ cannot be true in any generic extension of V.

Then we turn to Ω-validity. We take V_α to be the set obtained by starting with $\emptyset$ and taking the powerset at most α times. If α is an unreachable cardinal (that is, if $\beta < \alpha$, then $2^\beta < \alpha$), $(V_\alpha, \in)$ is a model of ZFC. We'd like to see the V_α as an estimation of V, so basically $\lim_{\alpha\to\infty} V_\alpha = V$. Unfortunately, we lack both limits and an appropriate infinity, so this doesn't work. The models of the type $(V_\alpha, \in)$ are a nice, large class of models though.

Definition 15 *A statement ψ is* Ω-valid *if ψ is true in any model $(V_\alpha, \in)$ of ZFC.*

The interesting thing is, that any Ω-provable statement turns out to be Ω-valid.

Theorem 6 *Any Ω-provable statement is Ω-valid.*

This way, we get a situation much like 'normal' logic. Statements can be true, and in order to guarantee they are true, they can be proven. However, being true does not automatically imply being provable.

4.2 Ω-logic and problem 1

Instead of searching for an axiom that makes H_2 sufficiently complete, we'll search for an Ω-axiom for H_2.

Definition 16 ***A*** *is an* Ω-axiom *for (H,$\in$) if for any statement ψ definable in (H,$\in$) either '**A** $\Rightarrow$ (H,$\in$) satisfies ψ' or '**A** $\Rightarrow$ (H,$\in$) satisfies $\neg$ ψ' is Ω-provable.*

So an Ω-axiom 'Ω-decides' all properties of (H,$\in$). It is immediately clear that Ω-logic is an extension of normal logic if one looks at H_1. In normal logic, an extra axiom is required to make $(H_1, \in)$ sufficiently complete. In the Ω-logic, $(H_1, \in)$ is sufficiently complete in ZFC without any extra axioms. In other words, '0=0' is an Ω-axiom for $(H_1, \in)$.

Unfortunately, for $(H_2, \in)$, '0=0' isn't an Ω-axiom. In the search for an Ω-axiom we return to MMW. If a proper class of Woodin-cardinals exists, MMW is equivalent to:

Axiom 3 *Take A to be a subset of $\mathbb{R}$ with $A \in L(\mathbb{R})$, and I_{NS} to be the set of all non-stationary subsets of $\mathbb{R}$. Then for any statement ζ of the type $\forall\exists$, ζ formulated in $(H_2, I_{NS}, A, \in)$ and $\neg\zeta$ not Ω-provable, $(H_2, I_{NS}, A, \in)$ satisfies ζ.*

So for any statement ζ of a certain form in a certain extension of H_2 which isn't satisfied, $\neg\zeta$ is Ω-provable. If there exists a proper class of Woodin cardinals, then MMW is an ω-axiom for $(H_2, \in)$.

This, however, still doesn't prove that MMW is consistent with the existence of large cardinals. MMW being an Ω-axiom gives us that for any statement ψ either ψ or $\neg\psi$ is Ω-provable from ZFC+MMW, and therefore Ω-valid. Suppose ψ is Ω-provable. Then ψ is true in any model V_α. That would guarantee consistence with the existence of large cardinals, if it weren't for the fact that ψ could still be true. After all, Ω-validity doesn't imply Ω-provability, so even if $\neg\psi$ isn't Ω-provable, it can still be Ω-valid.

This problem is as of yet unsolved. However, there is a conjecture called the Ω-conjecture that would solve it:

Conjecture 1 (Ω-conjecture) *Any Ω-valid statement is Ω-provable.*

Note that the Ω-conjecture implies the existence of a proper class of Woodin-cardinals, since that is a perquisite for the existence of an Ω-proof. The Ω-conjecture implies that any Ω-axiom is consistent with the existence of large cardinals. So, in particular, MMW is consistent with the existence of large cardinals. Since we already knew that MMW makes $(H_2, \in)$ invariant under forcing and sufficiently complete, the Ω-conjecture implies that MMW is a solution to problem 1.

4.3 Results for CH

By (almost) answering problem 1 for H_2 we haven't fully solved CH. We know that if problem 1 is solved for H_2 by an axiom **A**, either CH or $\neg$CH is provable from ZFC+**A**. Now we want to know which one. The answer comes by the study of Ω-recursive sets. I won't discuss the precise definition of Ω-recursive here, only the main result.

Theorem 7 *Suppose there exists a proper class of Woodin cardinals, and T is an Ω-recursive set. Then either T is definable in $(H_2, \in)$, or there exists a surjection from $\mathbb{R}$ onto $\aleph_2$.*

The use of this theorem will of course be to use an Ω-recursive set not definable in $(H_2, \in)$ to prove $\neg$CH. Take an Ω-axiom **A**. Then the set T of (the numbers of) all statements ψ such that "$(H_2, \in)$ satisfies ψ" is Ω-provable from ZFC+**A** is Ω-recursive. For models V_α, T is equal to the set of all statements satisfied in $(H_2, \in)$, since Ω-provable statements are true in models V_α. However, the set of the (numbers of) statements that are satisfied in $(H_2, \in)$ cannot be defined in $(H_2, \in)$, so we get that CH is false for any model V_α of ZFC+**A**. If **A** solves problem 1 for H_2, this will extend to all models of ZFC+**A**, since ZFC+**A** would make $(H_2, \in)$ sufficiently complete.

Theorem 8 *Suppose there exists a proper class of Woodin cardinals. Then any* Ω*-axiom for* H_2 *which is a solution for problem 1 implies* $\neg CH$*.*

In other words,

Theorem 9 *If the* Ω*-conjecture is true, then in any set theory obtained by adding an axiom that makes* $(H_2, \in)$ *sufficiently complete, is consistent with the existence of large cardinals and makes* $(H_2, \in)$ *invariant under forcing to ZFC,* $\neg CH$ *is satisfied.*

Since the Ω-conjecture also implies that MMW is a solution to problem 1 for $(H_2, \in)$, the Ω-conjecture would enable us to solve the continuum hypothesis in a reasonably nice way: there exists a solution to problem 1, and any solution to problem 1 will make $\neg$CH true. In particular, MMW implies $2^{\aleph_0} = \aleph_2$.

5 Conclusion

The continuum hypothesis is undecided in ZFC. In fact, there is no index $i \in \mathbb{N}$ such that ZFC guarantees $2^{\aleph_0} \leq \aleph_i$. However, by adding an axiom to ZFC the continuum hypothesis might be solved. The problem of doing this is that unlike the axioms of ZFC, such a new axiom wouldn't be accepted because the axiom is intuitively believable, but because the consequences of the axiom seem nice and reasonable. Since we don't want to add CH or $\neg$CH to ZFC directly, we look for axioms that make certain structures within V sufficiently complete, invariant under forcing and consistent with the existence of large cardinals.

For H_0 we don't really need an extra axiom, 0=0 will do. For H_1 however we do need an extra axiom. One that works for H_1 is PD, which says that any projective set is determined. Finding a suitable extra axiom for H_2 is harder. The axiom MMW works, but only if the Ω-conjecture is true. In that case, $\neg$CH is true in ZFC+MMW. Of course this replaces one unsolved problem by another. The Ω-conjecture does seem plausible though, making this a real step towards solving CH, even if the result is not conclusive yet.

Should the Ω-conjecture be proven to be true, there is a solution to CH. That however would still not mean that the problem of what to do with CH is fully solved. After all, the Ω-conjecture only implies that any axiom that makes $(H_2, \in)$ sufficiently complete, invariant under forcing and consistent with the existence of large cardinals makes $\neg$CH true. There might still be other suitable criteria for an axiom, which might yield another result.

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