ASCENT, DESCENT, NULLITY AND DEFECT: A BACHELOR THESIS ON LINEAR RELATIONS

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ABSTRACT. For a linear operator $T : \mathfrak{H} \to \mathfrak{H}$ it is always the case that $\ker T^n \subset \ker T^{n+1}$ and $\operatorname{ran} T^{n+1} \subset \operatorname{ran} T^n$. If the growth (shrinkage) of the kernel (range) ceases for some nonnegative power of T, we call the lowest such power the ascent (descent) of T. In case no such number exists, the ascent (descent) of T is said to be infinite. Nullity and defect of a linear operator are the dimensions of its kernel and the complement of its range, respectively. These concepts can be generalized to linear *relations*. In this paper, theorems from a paper by A.E. Taylor about the aforementioned concepts in the case of linear operators [5] have been re-stated and "re-proved" in the parlance of relations. It turns out that the property of (partial) functionhood is fairly redundant here and that if we restrict ourselves to linear relations A such that $\mathfrak{R}_c(A) = \{0\}$, there are not many bumps on this particular road leading from operator to relation.

1. INTRODUCTION

In order to outline the statements made in this report, we first need several definitions. Throughout this report A will denote a linear subspace of $\mathfrak{H}^2$, $\mathfrak{H}$ being some complex linear space (i.e. $\mathfrak{H}^2$ is a cartesian product space). We call A a *linear relation* (as it is a subset of a cartesian product). We write $\{f, g\} \in A$ to indicate that the ordered pair f, g is an element of A.[1] We define the domain, range, kernel and multivalued part of A (all of which are subspaces of $\mathfrak{H}$) as follows:

$$\operatorname{dom} A := \{f \in \mathfrak{H} : \text{there is a } g \in \mathfrak{H} \text{ s.t. } \{f, g\} \in A\},$$

$$\operatorname{ran} A := \{g \in \mathfrak{H} : \text{there is a } f \in \mathfrak{H} \text{ s.t. } \{f, g\} \in A\},$$

$$\ker A := \{f \in \mathfrak{H} : \{f, 0\} \in A\},$$

$$\operatorname{mul} A := \{g \in \mathfrak{H} : \{0, g\} \in A\}.$$

Note that if we define $A^{-1} := \{\{f, g\} : \{g, f\} \in A\}$, we obtain $\operatorname{dom} A = \operatorname{ran} A^{-1}$ and $\operatorname{mul} A = \ker A^{-1}$. For two linear relations A and B, we define their product AB and sum $A + B$ as

$$AB := \{\{f, g\} \in \mathfrak{H}^2 : \text{there is an } h \in \mathfrak{H} : \{f, h\} \in B, \{h, g\} \in A\},$$

$$A + B := \{\{f, g + h\} \in \mathfrak{H}^2 : \{f, g\} \in A, \{f, h\} \in B\}.$$

Note that the relational product follows the notation of function composition, i.e. the relations are "applied" from right to left, in contrast with certain other areas of mathematics, where relational composition goes from left to right. In light of our (yet to be revealed) agenda, sticking to the functional notation is more convenient. Let $A^0 := I_{\mathfrak{H}}$, where $I_{\mathfrak{H}}$ is the 'identity relation' on $\mathfrak{H}$ (i.e. $I_{\mathfrak{H}} := \{\{x, x\} : x \in \mathfrak{H}\}$,

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[1]Because A contains only ordered pairs of elements of $\mathfrak{H}$, confusion with extensional representations of sets should hopefully remain within tolerable bounds.

most of the time the subscript will be omitted). For any nonnegative integer n let $A^{n+1} := AA^n = A^n A$ be inductively defined, then for any $r \in \mathbb{Z}$ we define

$$A^r := \begin{cases} A^r & \text{if } r \geq 0, \\ (A^{-1})^{-r} & \text{if } r < 0. \end{cases}$$

It can be easily seen that

$$(A^{-1})^r = (A^r)^{-1} \tag{1.1}$$

for any $r \in \mathbb{Z}$. If $\gamma \in \mathbb{C}$, then by $A - \gamma$ we denote $A - \gamma I$, where γI denotes $\{\{x, \gamma x\} : x \in \mathfrak{H}\}$. As a consequence, $\{x, y\} \in A - \gamma$ means that $\{x, y + \gamma x\} \in A$.

Lemma 1.1. *Let A be a linear relation and let n be a nonnegative integer. The following statements hold:*
(a) $\operatorname{dom} A^{n+1} \subset \operatorname{dom} A^n$ *and* $\operatorname{ran} A^{n+1} \subset \operatorname{ran} A^n$*;*
(b) $\ker A^{n+1} \supset \ker A^n$ *and* $\operatorname{mul} A^{n+1} \supset \operatorname{mul} A^n$*.*

Proof. (a). Suppose that $f \in \operatorname{dom} A^{n+1}$, so $\{f, g\} \in A^{n+1}$ for some g. Because $A^{n+1} = AA^n$, we have $\{f, h\} \in A^n$ for some h, whence $f \in \operatorname{dom} A^n$.

If $g \in \operatorname{ran} A^{n+1}$, so $\{f, g\} \in A^{n+1}$ for some f, then $\{g, f\} \in (A^{n+1})^{-1} = (A^{-1})^{n+1}$ (by (1.1)), so $g \in \operatorname{dom} (A^{-1})^{n+1}$. By the above, this implies that $g \in \operatorname{dom} (A^{-1})^n = \operatorname{dom} (A^n)^{-1} = \operatorname{ran} A^n$.

(b). Suppose that $f \in \ker A^n$, so $\{f, 0\} \in A^n$. Because $\{0, 0\} \in A$ as A is a linear subspace, $\{f, 0\} \in A^{n+1}$, whence $f \in \ker A^{n+1}$.

If $g \in \operatorname{mul} A^n$, then $g \in \ker (A^{-1})^n \subset \ker (A^{-1})^{n+1} = \operatorname{mul} A^{n+1}$. □

Now we are ready to define the notions allured to in the title of this report.

Lemma 1.2. *Let A be a linear relation. If for some nonnegative integer k we have* $\ker A^k = \ker A^{k+1}$ *then* $\ker A^n = \ker A^k$ *for all $n \geq k$.*

Proof. Assume $\ker A^{n+1} = \ker A^n$. If we can show that $\ker A^{n+2} = \ker A^{n+1}$ the statement will follow by induction. We have already established that $\ker A^{n+1} \subset \ker A^{n+2}$, so only the reverse inclusion remains to be proved. Let $f \in \ker A^{n+2}$, then $\{f, 0\} \in A^{n+2} = A^{n+1}A$, so $\{f, h\} \in A$ and $\{h, 0\} \in A^{n+1}$ for some h. Now $h \in \ker A^{n+1} = \ker A^n$ (by our induction hypothesis) so that $\{h, 0\} \in A^n$. But then $f \in \ker A^{n+1}$. □

Definition 1.3. If there is some $n \geq 0$ for which $\ker A^{n+1} = \ker A^n$ then we denote by $\alpha(A)$ the smallest such integer. If no such integer exists we write $\alpha(A) = \infty$. We call $\alpha(A)$ the *ascent* of A. By $\alpha_c(A) := \alpha(A^{-1})$ we denote the *co-ascent* of A.

Lemma 1.4. *Let A be a linear relation. If for some $k \geq 0$ we have* $\operatorname{ran} A^k = \operatorname{ran} A^{k+1}$ *then* $\operatorname{ran} A^n = \operatorname{ran} A^k$ *for all $n \geq k$.*

Proof. We assume that $\operatorname{ran} A^n = \operatorname{ran} A^{n+1}$. What needs to be shown to proof the lemma inductively is that $\operatorname{ran} A^{n+2} = \operatorname{ran} A^{n+1}$. Suppose $g \in \operatorname{ran} A^{n+1}$. Then $\{h, g\} \in A$ for some $h \in \operatorname{ran} A^n$. However, since $\operatorname{ran} A^n = \operatorname{ran} A^{n+1}$ we know that $\{f, h\} \in A^{n+1}$ for some f, which leads us to conclude that $g \in \operatorname{ran} A^{n+2}$ so that $\operatorname{ran} A^{n+1} \subset \operatorname{ran}{}^{n+2}$. The reverse inclusion holds because of Lemma 1.1 (a). □

Definition 1.5. If there is some $n \geq 0$ for which $\operatorname{ran} A^{n+1} = \operatorname{ran} A^n$ then we denote by $\delta(A)$ the smallest such integer. If no such integer exists we write $\delta(A) = \infty$. We call $\delta(A)$ the *descent* of A. By $\delta_c(A) := \delta(A^{-1})$ we denote the *co-descent* of A.

As it happens, $\alpha(A) = 0$ if and only if $\ker A = \{0\}$ and that $\delta(A) = 0$ if and only if $\operatorname{ran} A = \mathfrak{H}$. There are but two key words remaining for this subsection to tackle.

Definition 1.6. By $\mathrm{n}(A) := \dim \ker A$ we denote the *nullity* of A. Likewise, $\mathrm{n}_c(A) := \mathrm{n}(A^{-1})$ denotes the *co-nullity* of A.

Definition 1.7. $\mathrm{d}(A) := \dim(\mathfrak{H}/\operatorname{ran} A)$ is called the *defect* of A. Analogously, $\mathrm{d}_c(A) := \mathrm{d}(A^{-1})$ denotes the *co-defect* of A.

Note that the nullity and defect of a linear relation are not necessarily finite either, and that $\mathrm{n}(A) = 0$ and $\mathrm{d}(A) = 0$ are also logically equivalent with $\ker A = \{0\}$ and $\operatorname{ran} A = \mathfrak{H}$, respectively. What we call "defect" is sometimes also called "deficiency".

The reader may have noticed that words like kernel, range and domain also appear in functional analysis. In fact for any linear operator $T : S \to \mathfrak{H}$ (with $S \subset \mathfrak{H}$) the graph of T (i.e. the set of all $\{x, Tx\} \in \mathfrak{H}^2$ with $x \in S$) is a linear relation. Now the relation-domain of the graph of T is the same set as the operator-domain of T and the same correspondence exists for the range and the kernel. Readers may therefore find the above definitions to be perfectly compatible with their operator-siblings, in fact the latter can be regarded as special cases of the former. This is the principle at the heart of the research presented in this report: it is an attempt to prove the dual statements to established results in the theory of linear operators, thus generalizing them. The source material is a paper by Angus E. Taylor [5] on theorems relating the ascent, descent, nullity and defect of an operator.

We define the *root manifolds* of A at 0 and ∞ as $\mathfrak{R}_0(A) := \bigcup_{i=0}^{\infty} \ker A^i$ and $\mathfrak{R}_\infty(A) := \bigcup_{i=0}^{\infty} \operatorname{mul} A^i$. By Lemma 1.1 (b), $\mathfrak{R}_0(A)$ and $\mathfrak{R}_\infty(A)$ are linear subspaces of $\mathfrak{H}$, whence their intersection

$$\mathfrak{R}_c(A) := \mathfrak{R}_0(A) \cap \mathfrak{R}_\infty(A)$$

is also a subspace. We call $\mathfrak{R}_c(A)$ the *singular chain manifold* of A. Suppose $x \in \mathfrak{R}_c(A)$, then for certain $n, m \geq 0$, we have $x \in \ker A^n \cap \operatorname{mul} A^m$. This means that there exist $y_0, \ldots, y_k$ such that

$$\{0, y_0\}, \{y_0, y_1\}, \ldots, \{y_i, x\}, \{x, y_{i+1}\}, \ldots, \{y_k, 0\} \in A,$$

for some $0 \leq i \leq k \leq n + m - 2$, provided that $n, m \geq 1$ (otherwise we would immediately have $x = 0$). We call such a sequence of pairs a chain, and when it begins and terminates in 0, we call it a singular chain. The singular chain manifold of a relation is related to its point spectrum: in [3], Sandovici, De Snoo and Winkler show that $\mathfrak{R}_c(A) \neq \{0\}$ implies $\sigma_p(A) = \mathbb{C} \cup \{\infty\}$: see [3], Proposition 3.2. The converse is shown to hold in finite-dimensional spaces, see Theorem 4.4 in said paper.

One may regard the condition $\mathfrak{R}_c(A) = \{0\}$ as a reduct of function-hood. When A is the graph of an operator, we have $\operatorname{mul} A^n = \{0\}$ for all $n \geq 0$, whence $\mathfrak{R}_c(A) = \{0\}$, and as it turns out, almost every result from the first 4 sections of [5] generalizes from operators to relations, if we retain the condition that $\mathfrak{R}_c(A) = \{0\}$. Without this restriction, simple counterexamples can be found for some of the core results of this report.

Interestingly enough, it appears that there is something about the style of Taylor's proofs which makes it relatively easy to generalize them from operators to relations. A paper on the same subject by M.A. Kaashoek [2] offers much more resistance. The author claims that his methods "differ considerably" with those of [5], and the author concurs: the foundations of Kaashoek's results make heavy use of isomorphisms, which are constructed using a typical isomorphism-theorem from (linear) algebra: first show that a homomorphism (linear operator) is surjective and then turn it into an isomorphism by factoring out its kernel. In my experience, this technique does not apply to relations that well.

Linear relations are sometimes also called *multivalued linear operators*: see the book by that very name by Ronald Cross [1], for instance. There they are defined as mappings from $\mathfrak{H}$ to $2^{\mathfrak{H}} \setminus \emptyset$ (i.e. assigning sets of vectors to vectors) satisfying certain linearity conditions: a natural example is the inverse of a linear operator. When generalizing results about operators, retaining the operator notation has its aesthetical merits. It also invites a host of "classical" intuitions, some of which are useful, some of which are not. Anyway, the choice for the present notation/definition was not so much made by the author as by his supervisors, who also use it in their paper [3].

The results presented in this report are the fruits of research I conducted under the supervision and guidance of Henk de Snoo, Henrik Winkler and Adrian Sandovici, the latter of whom contributed several proofs presented in this report. Due to the nature of the research presented here and the frequent ease of generalization, certain results have almost been copied verbatim from [5]. However, in some cases simplification was possible. In section 2, we will lay down a few preliminary results. Section 3 contains a rudimentary lemma about relations between the nullity (defect) of a relation and that of its powers. Section 4 provides some basic results about the ascent and descent of a relation, most notably Theorem 4.4, which states that $\alpha(A) \leq \delta(A)$, provided that both quantities are finite – and that $\mathfrak{R}_c(A) = \{0\}$. Section 5 contains results relating ascent and descent to nullity and defect, and finally section 6 concerns itself with shifted relations, i.e. relations of the form $A - \gamma$ with $\gamma \in \mathbb{C}$. As an appendix, a table translating the present theorem numbers back to those of [5] is provided.

2. Preliminaries

In [5] we find two lemmas and some remarks about quotient spaces and complementary subspaces which we will also need in this paper. These results will be summarized only: for their proofs the reader is referred to [5].

Two linear subspaces $\mathfrak{M}_1, \mathfrak{M}_2$ of some linear space $\mathfrak{H}$ are called complementary (in $\mathfrak{H}$) if $\mathfrak{M}_1 \oplus \mathfrak{M}_2 = \mathfrak{H}$, i.e. if $\mathfrak{M}_1 \cap \mathfrak{M}_2 = \{0\}$ and $\mathfrak{M}_1 + \mathfrak{M}_2 = \mathfrak{H}$. If $\mathfrak{M}$ is a subspace of $\mathfrak{H}$ we denote by $\mathfrak{H}/\mathfrak{M}$ or $\frac{\mathfrak{H}}{\mathfrak{M}}$ the quotient space of all cosets $[x] = x + \mathfrak{M}$ (with $x \in \mathfrak{H}$). Note that if for some $\mathfrak{M}, \mathfrak{N}$, both subspaces of $\mathfrak{H}$, we have $\mathfrak{M} \oplus \mathfrak{N} = \mathfrak{H}$ then $\mathfrak{H}/\mathfrak{M}$ and $\mathfrak{N}$ are isomorphic as linear spaces.

Lemma 2.1. *Suppose that $\mathfrak{M}_1, \mathfrak{M}_2$ are subspaces of a linear space $\mathfrak{H}$ and $\mathfrak{M}_1 \subset \mathfrak{M}_2$. Then*

$$\dim \mathfrak{H}/\mathfrak{M}_1 = \dim \mathfrak{H}/\mathfrak{M}_2 + \dim \mathfrak{M}_2/\mathfrak{M}_1$$

(In case ∞ pops up, we embrace the following convention: $\infty + p = p + \infty = \infty = \infty + \infty$ for any nonnegative integer p.)

Lemma 2.2. *Suppose that $\mathfrak{N}$ and $\mathfrak{M}$ are subspaces of $\mathfrak{H}$ such that $\mathfrak{N} \cap \mathfrak{M} = \{0\}$ and $\dim \mathfrak{H}/\mathfrak{M} \leq \dim \mathfrak{N} < \infty$. Then $\mathfrak{H} = \mathfrak{N} \oplus \mathfrak{M}$.*

See Lemmas 2.1 and 2.2 in [5] for proofs.

Lemma 2.3. *Let $\mathfrak{M}, \mathfrak{N}$ be linear spaces, at least one of which is finite-dimensional. Then $\dim \mathfrak{M} \leq \dim \mathfrak{N}$ if the following implication holds for any nonnegative integer p:*

If there exist p linearly independent vectors in $\mathfrak{M}$, then $p \leq \dim \mathfrak{N}$.

Proof. If $\dim \mathfrak{N} = \infty$ then $\dim \mathfrak{M} \leq \dim \mathfrak{N}$ becomes a formality. On the other hand, suppose that $\dim \mathfrak{N} < \infty$ and that the above implication holds but that $\dim \mathfrak{M} > \dim \mathfrak{N}$. Then for any $p > \dim \mathfrak{N}$ take p linearly independent vectors in $\mathfrak{M}$ and we obtain a contradiction. $\square$

Lemma 2.4. *Let A, B be two linear relations on $\mathfrak{H}$ such that $A \subset B$. Then for any $n \geq 0$ we have*
(a) $A^n \subset B^n$;
(b) $\ker A^n \subset \ker B^n$, *whence also* $\operatorname{mul} A^n \subset \operatorname{mul} B^n$;
(c) $\operatorname{ran} A^n \subset \operatorname{ran} B^n$ *and* $\operatorname{dom} A^n \subset \operatorname{dom} B^n$.

Proof. (a). If $\{f, g\} \in A^n$, then there exist $h_1, \dots h_{n-1} \in \mathfrak{H}$ such that

$$\{f, h_1\}, \{h_1, h_2\}, \dots, \{h_{n-1}, g\} \in A \subset B,$$

whence $\{f, g\} \in B^n$.

(b). If $f \in \ker A^n$ then $\{f, 0\} \in A^n \subset B^n$, so that $f \in \ker B^n$. As $A \subset B$ implies $A^{-1} \subset B^{-1}$, we have $\operatorname{mul} A^n = \ker (A^n)^{-1} = \ker (A^{-1})^n \subset \ker (B^{-1})^n = \operatorname{mul} B^n$.

(c). If $g \in \operatorname{ran} A^n$, then $\{f, g\} \in A^n \subset B^n$ for some f, whence $g \in \operatorname{ran} B^n$. As with (b) above, $\operatorname{dom} A^n = \operatorname{ran} (A^n)^{-1} = \operatorname{ran} (A^{-1})^n \subset \operatorname{ran} (B^{-1})^n = \operatorname{dom} B^n$. □

Lemma 2.5. *Let A and B are two linear relations on $\mathfrak{H}$ such that $A \subset B$ and $\mathfrak{R}_c(B) = \{0\}$. Then $\alpha(A) \leq \alpha(B)$.*

Proof. The case $\alpha(B) = \infty$ is trivial, so assume that $\alpha(B) = p$ for some nonnegative integer p. Let $x \in \ker A^{p+1}$ be arbitrary, so for some y, we have $\{x, y\} \in A^p$ and $\{y, 0\} \in A$. Because $x \in \ker A^{p+1} \subset \ker B^{p+1} = \ker B^p$, we have $\{x, 0\} \in B^p$, whence $\{0, y\} \in B^p$. Because $\mathfrak{R}_c(B) = \{0\}$, we conclude that $y = 0$, whence $x \in \ker A^p$. □

The above lemma does not hold if we merely demand that $\ker A^n \subset \ker B^n$ for all $n \geq 0$, as the example below shows. In order to preserve legibility, we use the following notation: $\{\!\{e_1\}\!\} := \operatorname{span} \{e_1\}$.

Example 2.6. Let $\mathfrak{H} = \{\!\{e_1, e_2\}\!\}$ with e_1, e_2 linearly independent. Define $B = \{\!\{\{e_1, 0\}, \{e_2, 0\}\}\!\}$ and $A = \{\!\{\{e_2, e_1\}, \{e_1, 0\}\}\!\}$. Now $\ker B = \{\!\{e_1, e_2\}\!\}$, so $\alpha(B) = 1$. However, $\ker A = \{\!\{e_1\}\!\}$ and $\ker A^2 = \{\!\{e_1, e_2\}\!\}$, so $\ker B^n \supset \ker A^n$ for all $n \geq 0$, but $\alpha(A) = 2$.

3. Nullity and defect

The following lemma relates the nullity (defect) of a linear relation to that of its powers.

Lemma 3.1. *(a) If* $\mathrm{n}(A) < \infty$ *then* $\mathrm{n}(A^k) \leq k\,\mathrm{n}(A)$ *for* $k = 0, 1, 2, \dots$;
(b) If $\mathrm{d}(A) < \infty$ *then* $\mathrm{d}(A^k) \leq k\,\mathrm{d}(A)$ *for* $k = 0, 1, 2, \dots$.

Proof of (a). We know that $\ker A^k \subset \ker A^{k+1}$ for any nonnegative integer k. Let $\mathfrak{N}$ be a subspace of $\ker A^{n+1}$ such that $\ker A^{n+1} = \ker A^n \oplus \mathfrak{N}$ for some $n \geq 0$. If we can show that $\dim \mathfrak{N} \leq \mathrm{n}(A)$ then $\mathrm{n}(A^{n+1}) \leq \mathrm{n}(A^n) + \mathrm{n}(A)$ so that the statement follows by induction (remember that $\mathrm{n}(A^0) = 0$). The case $\dim \mathfrak{N} = 0$ is trivial, so assume that $\dim \mathfrak{N} > 0$. Let $x_1, x_2, \dots, x_p \in \mathfrak{N}$ be linearly independent $(1 \leq p \leq \dim \mathfrak{N})$. Then (because $\mathfrak{N} \subset \ker A^{n+1}$) there exist $y_1, y_2, \dots, y_p \in \ker A$ such that $\{x_1, y_1\}, \{x_2, y_2\}, \dots, \{x_p, y_p\} \in A^n$. Suppose that $\sum_{i=1}^p c_i y_i = 0$ for certain $c_1, c_2, \dots, c_p \in \mathbb{C}$. Then

$$\sum_{i=1}^p c_i \{x_i, y_i\} = \left\{ \sum_{i=1}^p c_i x_i, \sum_{i=1}^p c_i y_i \right\} = \left\{ \sum_{i=1}^p c_i x_i, 0 \right\} \in A^n$$

or in other words, $\sum_{i=1}^p c_i x_i \in \ker A^n \cap \mathfrak{N}$. But because $\mathfrak{N}$ and $\ker A^n$ are complementary spaces, their intersection is $\{0\}$ so that $\sum_{i=1}^p c_i x_i = 0$ which in turn implies that $c_1 = c_2 = \cdots = 0$. This means that for any p linearly independent vectors in $\mathfrak{N}$ there exist p linearly independent vectors in $\ker A$. Using Lemma 2.3 we conclude that $\dim \mathfrak{N} \leq \dim \ker A = \mathrm{n}(A)$. □

Proof of (b). Because $\mathrm{d}(A^0) = 0$ we will assume $k > 0$. Define

$$\mathfrak{M}_k := \operatorname{ran} A^{k-1}/\operatorname{ran} A^k.$$

Then it follows from Lemmas 1.1 (a) and 2.1 that

$$\begin{aligned}\mathrm{d}(A^k) &= \mathrm{d}(A^{k-1}) + \dim \mathfrak{M}_k = \mathrm{d}(A^{k-2}) + \dim \mathfrak{M}_{k-1} + \dim \mathfrak{M}_k = \\ &= \cdots = \dim \mathfrak{M}_1 + \dim \mathfrak{M}_2 + \cdots + \dim \mathfrak{M}_k.\end{aligned}$$

(Note that $\dim \mathfrak{M}_1 = \dim (\operatorname{ran} A^0/\operatorname{ran} A^1) = \dim (\mathfrak{H}/\operatorname{ran} A) = \mathrm{d}(A)$.) If we can now show that $\dim \mathfrak{M}_{n+1} \leq \dim \mathfrak{M}_n$ for any $n \geq 1$, the statement will follow by induction. Let $[y_1], [y_2], \dots, [y_p] \in \mathfrak{M}_{n+1}$ be linearly independent cosets. Then $y_i \in \operatorname{ran} A^n$ for $1 \leq i \leq p$ so there must exist $x_1, x_2, \dots, x_p \in \operatorname{ran} A^{n-1}$ so that $\{x_1, y_1\}, \{x_2, y_2\}, \dots, \{x_p, y_p\} \in A$ (even if $n = 1$, because $\operatorname{dom} A \subset \mathfrak{H} = \operatorname{ran} A^0$). Now if $\sum_{i=1}^p c_i[x_i] = [\sum_{i=1}^p c_i x_i] = [0]$ in $\mathfrak{M}_n$ for certain complex numbers c_i, then $\sum_{i=1}^p c_i x_i \in \operatorname{ran} A^n$. Consequently, $\sum_{i=1}^p c_i y_i \in \operatorname{ran} A^{n+1}$ so that

$$\sum_{i=1}^p c_i[y_i] = \Big[\sum_{i=1}^p c_i y_i\Big] = [0].$$

But then $c_1 = c_2 = \cdots = c_p = 0$. Because $\dim \mathfrak{M}_1 = \mathrm{d}(A) < \infty$ we can repeatedly apply Lemma 2.3 to see that $\dim \mathfrak{M}_{n+1} \leq \dim \mathfrak{M}_n$. □

4. Ascent and descent

Lemma 4.1. *(a) If there exists a nonnegative integer p such that* $\ker A \cap \operatorname{ran} A^p = \{0\}$ *then $\alpha(A) \leq p$.*

(b) Let A be a linear relation with $\mathfrak{R}_c(A) = \{0\}$. If $\alpha(A) \leq p$ for some nonnegative integer p then $\ker A^k \cap \operatorname{ran} A^p = \{0\}$ *for $k = 0, 1, 2, \dots$.*

Proof. Let $\ker A \cap \operatorname{ran} A^p = \{0\}$. Take any $x \in \ker A^{p+1}$. Now take any y such that $\{x, y\} \in A^p$ and $\{y, 0\} \in A$. Then $y \in \operatorname{ran} A^p \cap \ker A = \{0\}$, so $y = 0$ and therefore $x \in \ker A^p$, proving that $\alpha(A) \leq p$.

Now let $\mathfrak{R}_c(A) = \{0\}$ and $\alpha(A) \leq p$. Take $y \in \ker A^k \cap \operatorname{ran} A^p$, i.e. let $\{x, y\} \in A^p$ and $\{y, 0\} \in A^k$ for some x. (Note that if $k = 0$ then necessarily $y = 0$.) Then $x \in \ker A^{p+k} = \ker A^p$ (because $\alpha(A) \leq p$) so $\{x, 0\} \in A^p$. But then

$$\{x, y\} - \{x, 0\} = \{0, y\} \in A^p,$$

so that $y \in \mathfrak{R}_c(A) = \{0\}$ proving that $\ker A^k \cap \operatorname{ran} A^p = \{0\}$. □

If we allow non-trivial singular chains, it is easy to falsify Lemma 4.1 (b).

Example 4.2. Let $\mathfrak{H} := \{\!\{e_1\}\!\}$ with $e_1 \neq 0$, and let $A := \{\!\{\{e_1, 0\}, \{0, e_1\}\}\!\}$. We have $\ker A = \{\!\{e_1\}\!\} = \mathfrak{H}$, whence $\alpha(A) = 1$. However, $\ker A^1 \cap \operatorname{ran} A^1 = \{\!\{e_1\}\!\} \neq \{0\}$.

Lemma 4.3. *(a) Suppose that for some $q \geq 0$, $k \geq 1$ there exists a subspace $\mathfrak{M}_k$ such that $\mathfrak{M}_k \subset$* $\ker A^q$, $\mathfrak{M}_k \cap \operatorname{ran} A^q = \{0\}$ *and* $\operatorname{dom} A^q = (\operatorname{dom} A^q \cap \operatorname{ran} A^k) \oplus \mathfrak{M}_k$. *Then $\delta(A) \leq q$.*

(b) Suppose that $\delta(A) \leq q$. Then for every $k \geq 1$ there exists a subspace $\mathfrak{M}_k$ such that $\mathfrak{M}_k \subset$ $\ker A^q$, $\mathfrak{M}_k \cap \operatorname{ran} A^k = \{0\}$ *and* $\operatorname{dom} A^q = (\operatorname{dom} A^q \cap \operatorname{ran} A^k) \oplus \mathfrak{M}_k$.

Proof of (a). In order to prove that $\delta(A) \leq q$ we will show that $\operatorname{ran} A^q \subset \operatorname{ran} A^{q+k}$. Take $y \in \operatorname{ran} A^q$. Then $\{x, y\} \in A^q$ for some $x \in \operatorname{dom} A^q$. By hypothesis, there exist $u \in \operatorname{ran} A^k$ and $v \in \mathfrak{M}_k \subset \ker A^q$ such that $x = u + v$. Now

$$\{x, y\} - \{v, 0\} = \{u + v, y\} - \{v, 0\} = \{u, y\} \in A^q,$$

but then $y \in \operatorname{ran} A^{q+k}$. □

Proof of (b). Let $\delta(A) \leq q$. For some fixed $k \geq 1$ choose a subspace $\mathfrak{N}_k$ such that

$$\operatorname{dom} A^q = (\operatorname{dom} A^q \cap \operatorname{ran} A^k) \oplus \mathfrak{N}_k.$$

Let H be a Hamel basis[2] for $\mathfrak{N}_k$. Then for any $v \in H$ there exists a v' such that $\{v, v'\} \in A^q$. But because $\operatorname{ran} A^q = \operatorname{ran} A^{q+k}$ there also exists some $w \in \operatorname{ran} A^k$ such that $\{w, v'\} \in A^q$. Now $\{v, v'\} - \{w, v'\} = \{v - w, 0\} \in A^q$. Since for each $v \in H$ such a $w \in \operatorname{ran} A^k$ exists, we can let $\mathfrak{M}_k$ be the linear manifold generated by these differences $v - w$.

First of all, $\mathfrak{M}_k \subset \ker A^q$. Secondly, let us see if also $\mathfrak{M}_k \cap \operatorname{ran} A^k = \{0\}$. Take $y \in \mathfrak{M}_k \cap \operatorname{ran} A^k$. Then $y = \sum_{i=1}^n c_i(v_i - w_i)$ for certain $c_i \in \mathbb{C}$, $v_i \in H$, $w_i \in \operatorname{ran} A^k$, $1 \leq i \leq n$ for some n. But then

$$\sum_{i=1}^n c_i v_i = y + \sum_{i=1}^n c_i w_i \in \operatorname{ran} A^k \cap \mathfrak{N}_k = \{0\}.$$

Because the v_i are linearly independent it follows that $c_1 = c_2 = \cdots = c_n = 0$ so that $y = 0$. Thirdly and finally, we should determine whether $\operatorname{dom} A^q = (\operatorname{dom} A^q \cap \operatorname{ran} A^k) \oplus \mathfrak{M}_k$. Because $\mathfrak{M}_k \subset \ker A^q \subset \operatorname{dom} A^q$ it is certainly true that $\operatorname{dom} A^q \supset (\operatorname{dom} A^q \cap \operatorname{ran} A^k) \oplus \mathfrak{M}_k$, so the only remaining question is whether the reverse inclusion holds. Take any $x \in \operatorname{dom} A^q$, then $x = u + v$ for some $u \in \operatorname{ran} A^k$ and $v \in \mathfrak{N}_k$. We can write $v = \sum_{i=1}^n a_i v_i$ for certain $a_i \in \mathbb{C}$, $v_i \in H$, $1 \leq i \leq n$ for some n. For every v_i choose a $w_i \in \operatorname{ran} A^k$ such that $v_i - w_i \in \mathfrak{M}_k$. Then

$$x = u + v = u + \sum_{i=1}^n a_i(v_i - w_i) + \sum_{i=1}^n a_i w_i = u + \sum_{i=1}^n a_i w_i + \sum_{i=1}^n a_i(v_i - w_i)$$

with $u + \sum_{i=1}^n a_i w_i \in \operatorname{ran} A^k$ and $\sum_{i=1}^n a_i(v_i - w_i) \in \mathfrak{M}_k$, completing our proof. □

Theorem 4.4. *Let $\mathfrak{R}_c(A) = \{0\}$. If $\alpha(A), \delta(A) < \infty$, then $\alpha(A) \leq \delta(A)$. If also $\operatorname{dom} A = \mathfrak{H}$, then $\alpha(A) = \delta(A)$.*

Proof. Let $p = \alpha(A)$, $q = \delta(A)$. To show that $p \leq q$ we will assume $p > q$ and deduce a contradiction. Assume $p > q$. Then $\ker A^p \setminus \ker A^q \neq \varnothing$. Let $x \in \ker A^p \setminus \ker A^q$. Because $x \in \ker A^p \subset \operatorname{dom} A^p \subset \operatorname{dom} A^q$ we can apply Lemma 4.3 (b), so that $x = x_1 + x_2$ with $x_1 \in \operatorname{dom} A^q \cap \operatorname{ran} A^p$ and $x_2 \in \mathfrak{M}_p$. Then

$$x_1 = x - x_2 \in \ker A^p \cap \operatorname{ran} A^p$$

(because $x_2 \in \mathfrak{M}_p \subset \ker A^q \subset \ker A^p$). But then $x_1 = 0$ by Lemma 4.1 (b) and $x = x_2 \in \mathfrak{M}_p \subset \ker A^q$ contradicting our choice of x.

To show that $p = q$ if $\operatorname{dom} A = \mathfrak{H}$, we will show that $\operatorname{dom} A \neq \mathfrak{H}$ if $p < q$ and $q > 0$. Assume $\operatorname{dom} A = \mathfrak{H}$ and $0 < q$, $p < q$. Then $\operatorname{ran} A^p \setminus \operatorname{ran} A^q \neq \varnothing$. Let $x \in \operatorname{ran} A^p \setminus \operatorname{ran} A^q$. By Lemma 4.3 (b) $\mathfrak{H} = \operatorname{dom} A^q = \operatorname{ran} A^q \oplus \mathfrak{M}_q$ with $\mathfrak{M}_q \in \ker A^q$. So $x = x_1 + x_2$ with $x_1 \in \operatorname{ran} A^q$ and $x_2 \in \mathfrak{M}_q$. But

$$x_2 = x - x_1 \in \operatorname{ran} A^p \cap \ker A^q = \{0\}$$

(the latter equivalence courtesy of Lemma 4.1 (b)) so $x = x_1 \in \operatorname{ran} A^q$ contradicting our choice of x. This leaves us to conclude that $\operatorname{dom} A \neq \mathfrak{H}$ if $p < q$ and $q < 0$ which proves the second half of the theorem's statement. □

Again, the condition $\mathfrak{R}_c(A) = \{0\}$ prohibits the construction of very simple counterexamples.

Example 4.5. Let $\mathfrak{H} := \{\!\{e_1, e_2\}\!\}$ with e_1, e_2 linearly independent, and let $A := \{\!\{\{0, e_1\}, \{e_2, 0\}, \{e_1, e_2\}\}\!\}$. We have $\ker A = \{\!\{e_2\}\!\}$ and $\ker A^2 = \{\!\{e_1, e_2\}\!\}$, so $\alpha(A) = 2$. However, $\operatorname{ran} A = \{\!\{e_1, e_2\}\!\} = \operatorname{ran} A^0$, whence $\delta(A) = 0$.

[2]In general, the existence of such a basis depends on Zorn's Lemma. For further reading on these topics, see §1.72 and §1.7 of [4], or §I.9 and §I.11 of [6], for instance.

Theorem 4.6. *(a) Suppose that for some $r > 0$ the following equalities hold:* $\ker A^r \cap \operatorname{ran} A^r = \{0\}$ *and* $\operatorname{dom} A^r = (\operatorname{dom} A^r \cap \operatorname{ran} A^r) \oplus \ker A^r$. *Then $\alpha(A) \leq r$ and $\delta(A) \leq r$.*

(b) Suppose that $\alpha(A), \delta(A) < \infty$ and that $\mathfrak{R}_c(A) = \{0\}$. Let $q = \delta(A)$. Then $\ker A^q \cap \operatorname{ran} A^q = \{0\}$ *and* $\operatorname{dom} A^q = (\operatorname{dom} A^q \cap \operatorname{ran} A^q) \oplus \ker A^q$. [3]

Proof of (a). Because $\ker A \subset \ker A^r$ we have $\ker A \cap \operatorname{ran} A^r = \{0\}$, so we can apply Lemma 4.1 (a) which tells us that $\alpha(A) \leq r$.

Now we will show that $\operatorname{ran} A^r \subset \operatorname{ran} A^{2r}$ so $\delta(A) \leq r$. Take any $y \in \operatorname{ran} A^r$, then there must exist some $x \in \operatorname{dom} A^r$ such that $\{x, y\} \in A^r$ and by our hypothesis $x = x_1 + x_2$ with $x_1 \in \operatorname{ran} A^r$ and $x_2 \in \ker A^r$. Now

$$\{x, y\} - \{x_2, 0\} = \{x_1 + x_2, y\} - \{x_2, 0\} = \{x_1, y\} \in A^r,$$

so $y \in \operatorname{ran} A^{2r}$. □

Proof of (b). Let $p = \alpha(A)$. Note that because of Theorem 4.4, $p \leq q$. Now we can already conclude that $\ker A^q \cap \operatorname{ran} A^q = \{0\}$ using Lemma 4.1 (b).

Now let us consider the second half of the statement. First observe that $\operatorname{dom} A^q \supset (\operatorname{dom} A^q \cap \operatorname{ran} A^q) \oplus \ker A^q$ clearly holds. As for the reverse inclusion, we shall consider the cases $q = 0$ and $q \geq 1$. Suppose $q = 0$, then

$$\operatorname{dom} A^0 = \operatorname{dom} I = \mathfrak{H} \subset (\mathfrak{H} \cap \operatorname{ran} I) \oplus \ker I.$$

If on the other hand $q \geq 1$ we can apply Lemma 4.3 (b) so that

$$\operatorname{dom} A^q = (\operatorname{dom} A^q \cap \operatorname{ran} A^q) \oplus \mathfrak{M}_q$$

which proves the desired inclusion because (by definition) $\mathfrak{M}_q \subset \ker A^q$. □

5. Relating nullity and defect to ascent and descent

First we present some elementary relations between nullity and ascent and defect and descent, respectively.

Lemma 5.1. *Suppose there exists some nonnegative integer N such that $\mathrm{n}(A^k) \leq N$ for $k = 0, 1, 2, \ldots$. Then $\alpha(A) \leq N$.*

Proof. If $\alpha(A) = \infty$ then for any nonnegative integer k we have $\ker A^{k+1} \supsetneq \ker A^k$. But then the premise of our statement must be false. We can therefore assume that $\alpha(A) = p$ for some integer p. In case $p = 0$ the statement is trivial so what remains to be shown is that $p \leq N$ if $p > 0$. Note that $\{0\} = \ker A^0 \subsetneq \ker A \subsetneq \cdots \subsetneq \ker A^{p-1} \subsetneq \ker A^p$ (by Definition 1.3). Then

$$0 = \mathrm{n}(A^0) < \mathrm{n}(A) < \cdots < \mathrm{n}(A^{p-1}) < \mathrm{n}(A^p),$$

so that $(p-1) < \mathrm{n}(A^p) \leq N$, leaving us to conclude that $p \leq N$. □

Lemma 5.2. *Suppose that there is some nonnegative integer N such that $\mathrm{d}(A^k) \leq N$ for $k = 0, 1, 2, \ldots$. Then $\delta(A) \leq N$.*

Proof. If $\delta(A) = \infty$ then $\operatorname{ran} A^{k+1} \subsetneq \operatorname{ran} A^k$ for all $k \geq 0$ and consequently $\mathrm{d}(A^{k+1}) > \mathrm{d}(A^k)$, again contradicting our premise. We therefore assume $\delta(A) = q$ for some nonnegative integer q. Just as in the proof of Lemma 5.1 the case $q = 0$ is obvious so let $q > 0$. It follows from Lemmas 1.1 (a) and 2.1 that

$$0 = \mathrm{d}(A^0) < \mathrm{d}(A) < \cdots < \mathrm{d}(A^{q-1}) < \mathrm{d}(A^q)$$

(because $\dim(\operatorname{ran} A^k / \operatorname{ran} A^{k+1}) > 0$ for $k < q$). This implies that $q - 1 < \mathrm{d}(A^q) \leq N$ so that $q \leq N$. □

[3] In [5], part (b) of the theorem contains an additional claim which we will not pursue here.

Let $\mathfrak{M}$ be a linear subspace of $\mathfrak{H}$ and A a linear relation on $\mathfrak{H}$. Then the *invariant restriction of A to $\mathfrak{M}$* is a linear relation on $\mathfrak{M}$ which is defined as follows:

$$A|_{\mathfrak{M}} := A \cap (\mathfrak{M} \times \mathfrak{M})$$

(Because it is the intersection of two subspaces of one larger linear space (viz. $\mathfrak{H}^2$) $A|_{\mathfrak{M}}$ is well-defined.) $\mathfrak{M}$ is called *exactly invariant under A* if $\operatorname{ran} A|_{\mathfrak{M}} = \mathfrak{M}$. Note that $\operatorname{ran} A|_{\mathfrak{M}} \subset \mathfrak{M}$ holds by definition.

Theorem 5.3. *Assume* $\mathrm{n}(A) < \infty$ *and* $\mathfrak{R}_c(A) = \{0\}$. *Then* $\alpha(A) < \infty$ *if and only if for every subspace $\mathfrak{M}$ of $\mathfrak{H}$ which is exactly invariant under A we have* $\alpha(A|_{\mathfrak{M}}) = \delta(A|_{\mathfrak{M}}) = 0$.

Proof. Assume $\alpha(A) < \infty$. Let $\mathfrak{M}$ be exactly invariant under A. By Lemmas 2.4 (b) and 2.5, $\alpha(A|_{\mathfrak{M}}) \leq \alpha(A) < \infty$, as $A|_{\mathfrak{M}} \subset A$ and because we may embed $A|_{\mathfrak{M}}$ in $\mathfrak{H}^2$ (which makes no difference for the ascent). Because $\operatorname{ran} A|_{\mathfrak{M}} = \mathfrak{M}$ by our hypothesis, also $\delta(A|_{\mathfrak{M}}) = 0$ (here it does matter that $A|_{\mathfrak{M}}$ is a relation on $\mathfrak{M}$, not on $\mathfrak{H}$). But then by Theorem 4.4, $\alpha(A|_{\mathfrak{M}}) = 0$.

Conversely, assume that for any exactly A-invariant subspace $\mathfrak{M}$, it is the case that $\alpha(A|_{\mathfrak{M}}) = \delta(A|_{\mathfrak{M}}) = 0$. Consider the sequence of subspaces $\{\ker A \cap \operatorname{ran} A^n\}_{n\in\mathbb{N}}$. Notice that $\ker A \cap \operatorname{ran} A^{n+1} \subset \ker A \cap \operatorname{ran} A^n$ because $\operatorname{ran} A^{n+1} \subset \operatorname{ran} A^n$ so

$$0 \leq \cdots \leq \dim(\ker A \cap \operatorname{ran} A^{n+1}) \leq \dim(\ker A \cap \operatorname{ran} A^n) \leq \cdots \leq \mathrm{n}(A) < \infty.$$

This means that there must be some nonnegative integer r such that $\ker A \cap \operatorname{ran} A^r = \ker A \cap \operatorname{ran} A^n$ if $n \geq r$. Let

$$\mathfrak{M} := \bigcap_{i=0}^{\infty} \operatorname{ran} A^{r+i}.$$

Observe that $\mathfrak{M} = \bigcap_{i=0}^{\infty} \operatorname{ran} A^{r+i} = \bigcap_{i=j}^{\infty} \operatorname{ran} A^{r+i}$ for any $j \geq 0$ and that $\ker A \cap \mathfrak{M} = \ker A \cap \operatorname{ran} A^r$. Is $\mathfrak{M}$ exactly invariant under A, i.e. is it the case that $\mathfrak{M} \subset \operatorname{ran} A|_{\mathfrak{M}}$? Take $y \in \mathfrak{M}$. Then there exists a sequence $\{x_i\}_{i\geq 1}$ of elements in $\mathfrak{H}$ such that $\{x_i, y\} \in A^{r+i}$. For every x_i choose an x_i' such that

$$\{x_i, x_i'\} \in A^{r+i-1} \text{ and } \{x_i', y\} \in A.$$

Let $u_i = x_1' - x_i'$ (so that $\{u_i, 0\} = \{x_1', y\} - \{x_i', y\} \in A$), then $u_i \in \ker A \cap \operatorname{ran} A^r = \ker A \cap \operatorname{ran} A^{r+i-1}$. But then, since $x_i' \in \operatorname{ran} A^{r+i-1}$,

$$x_1' = u_i + x_i' \in \operatorname{ran} A^{r+i-1} \text{ for all } i \geq 1,$$

so $x_1' \in \mathfrak{M}$, i.e. $\{x_1', y\} \in A \cap \mathfrak{M}^2$ so that $y \in \operatorname{ran} A|_{\mathfrak{M}}$ proving that $\mathfrak{M}$ is indeed exactly A-invariant. Now by our hypothesis $\alpha(A|_{\mathfrak{M}}) = 0$ and therefore $\ker A \cap \operatorname{ran} A^r = \ker A \cap \mathfrak{M} = \ker A|_{\mathfrak{M}} = \{0\}$ which implies $\alpha(A) \leq r$ by Lemma 4.1 (a). □

Theorem 5.4. *Suppose that* $\mathfrak{R}_c(A) = \{0\}$, $\alpha(A) < \infty$ *and that either* $\mathrm{n}(A)$ *or* $\mathrm{d}(A)$ *is finite. Then* $\mathrm{n}(A) \leq \mathrm{d}(A)$.

Proof. [4] The logical disjunctive in our hypotheses makes that they comprise of three cases. If $\mathrm{d}(A) = \infty$, by our assumptions $\mathrm{n}(A)$ must be finite and there is nothing for us to prove. So henceforth let $\mathrm{d}(A) < \infty$. There are now two possible cases left, viz. $\mathrm{n}(A) = \infty$ and $\mathrm{n}(A) < \infty$. As it turns out, the former is an impossibility. Let $\alpha(A) = p$, then by Lemma 4.1 (b) $\ker A \cap \operatorname{ran} A^p = \{0\}$. From this we will conclude that $\mathrm{n}(A) \leq \mathrm{d}(A^p)$. Take $x_1, x_2, \ldots, x_k$ linearly independent in $\ker A$, then their corresponding cosets $[x_1], [x_2], \ldots, [x_k]$ in $\mathfrak{H}/\operatorname{ran} A^p$ are also linearly independent. For if we suppose that $c_1[x_1] + \cdots + c_k[x_k] = 0$ for certain $c_i \in \mathbb{C}$, $1 \leq i \leq k$, then

$$c_1 x_1 + \cdots + c_k x_k \in \operatorname{ran} A^p \cap \ker A = \{0\}$$

[4]This is a minor adaptation of a proof by Adrian Sandovici.

and therefore $c_1 = \cdots = c_k = 0$. By Lemma 2.3 we find that $\mathrm{n}(A) = \dim \ker A \leq \dim(\mathfrak{H}/\mathrm{ran}\, A^p) = \mathrm{d}(A^p)$, so by Lemma 3.1 (b), $\mathrm{n}(A) \leq p\,\mathrm{d}(A) < \infty$. If $p = 0$ or $p = 1$, we are now done. Henceforth let $p \geq 2$.

Let $\mathfrak{M}_k := \ker A \cap \mathrm{ran}\, A^k$, $k = 0, 1, \ldots, p$. Then $\mathfrak{M}_k \subset \mathfrak{M}_{k-1}$ (for $k > 0$), $\mathfrak{M}_0 = \ker A$ and $\mathfrak{M}_p = \{0\}$. Observe that $\dim \mathfrak{M}_k/\mathfrak{M}_p = \dim \mathfrak{M}_k$ (for $0 \leq k \leq p$) so that when we apply Lemma 2.1:

$$\dim \mathfrak{M}_{k-1}/\mathfrak{M}_p = \dim \mathfrak{M}_{k-1}/\mathfrak{M}_k + \dim \mathfrak{M}_k/\mathfrak{M}_p,$$

we get:

$$\dim \mathfrak{M}_{k-1} = \dim \mathfrak{M}_{k-1}/\mathfrak{M}_k + \dim \mathfrak{M}_k, \quad 1 \leq k \leq p. \tag{5.1}$$

If we write $m_k = \dim \mathfrak{M}_k/\mathfrak{M}_{k+1}$ (with $0 \leq k \leq p-1$) and combine the equations which result from (5.1) then we find that

$$\mathrm{n}(A) = m_0 + m_1 + \cdots + m_{p-1}. \tag{5.2}$$

Now, for $0 \leq k \leq p-1$, let $y_1^{(k)}, \ldots, y_{m_k}^{(k)} \in \mathfrak{M}_k$ such that their corresponding cosets $[y_1^{(k)}], \ldots, [y_{m_k}^{(k)}] \in \mathfrak{M}_k/\mathfrak{M}_{k+1}$ are linearly independent as elements of the quotient space. Because $\mathfrak{M}_k \subset \mathrm{ran}\, A^k$, there exist $x_j^{(k)} \in \mathrm{dom}\, A^k$ such that

$$\{x_j^{(k)}, y_j^{(k)}\} \in A^k, \quad 1 \leq j \leq m_k.$$

Consider the elements

$$x_1^{(0)}, \ldots, x_{m_0}^{(0)}, x_1^{(1)}, \ldots, x_{m_1}^{(1)}, \ldots, x_1^{(p-1)}, \ldots, x_{m_{p-1}}^{(p-1)}$$

(of which there are $\mathrm{n}(A)$) and their corresponding cosets $\left[x_j^{(k)}\right]$ in $\mathfrak{H}/\mathrm{ran}\, A$. If we can prove that the latter are linearly independent, we can conclude that $\mathrm{n}(A) \leq \mathrm{d}(A)$ which, of course, is what we want. To do this, we will show that if

$$\sum_{i=0}^{p-1} \sum_{j=1}^{m_i} c_{ij} \left[x_j^{(i)}\right] = [0] \in \mathfrak{H}/\mathrm{ran}\, A,$$

then $c_{ij} = 0$ for $0 \leq i \leq p-1$ and $1 \leq j \leq m_i$. Suppose that $\sum_{i=0}^{p-1} \sum_{j=1}^{m_i} c_{ij} \left[x_j^{(i)}\right] = [0]$, i.e. for some u

$$\left\{u, \sum_{i=0}^{p-1} \sum_{j=1}^{m_i} c_{ij} x_j^{(i)}\right\} \in A.$$

Let $w_i := \sum_{j=1}^{m_i} c_{ij} x_j^{(i)}$ and $\widetilde{w}_i := \sum_{j=1}^{m_i} c_{ij} y_j^{(i)}$, $0 \leq i \leq p-1$, then

$$\{w_i, \widetilde{w}_i\} \in A^i, \quad 0 \leq i \leq p-1 \tag{5.3}$$

and $\left\{u, \sum_{i=0}^{p-1} w_i\right\} \in A$. Because $\mathfrak{M}_i \subset \ker A$ we have $\widetilde{w}_i \in \ker A$, that is

$$\{\widetilde{w}_i, 0\} \in A, \quad 0 \leq i \leq p-1, \tag{5.4}$$

and therefore

$$w_i \in \ker A^{i+1} \subset \ker A^p, \quad 0 \leq i \leq p-1, \tag{5.5}$$

so that $w := \sum_{i=0}^{p-1} w_i \in \ker A^p$ and $u \in \ker A^{p+1} = \ker A^p$ (the latter equivalence courtesy of the fact that $p = \alpha(A)$). Let $\mathsf{u}^{(1)}, \ldots, \mathsf{u}^{(p-1)}, \mathsf{w}^{(1)}, \ldots, \mathsf{w}^{(p-1)} \in \mathfrak{H}$ such that

$$\begin{aligned}
&\{u, \mathsf{u}^{(1)}\}, \{\mathsf{u}^{(1)}, \mathsf{u}^{(2)}\}, \{\mathsf{u}^{(2)}, \mathsf{u}^{(3)}\}, \ldots, \{\mathsf{u}^{(p-2)}, \mathsf{u}^{(p-1)}\}, \{\mathsf{u}^{(p-1)}, 0\} \in A,\\
&\{u, w\} \in A,\\
&\{w, \mathsf{w}^{(1)}\}, \{\mathsf{w}^{(1)}, \mathsf{w}^{(2)}\}, \ldots, \{\mathsf{w}^{(p-3)}, \mathsf{w}^{(p-2)}\}, \{\mathsf{w}^{(p-2)}, \mathsf{w}^{(p-1)}\}, \{\mathsf{w}^{(p-1)}, 0\} \in A.
\end{aligned}$$

We now combine these elements in a cunning manner:

$$\begin{aligned}
&\{0, w - \mathsf{u}^{(1)}\},\\
&\{w - \mathsf{u}^{(1)}, \mathsf{w}^{(1)} - \mathsf{u}^{(2)}\},\\
&\{\mathsf{w}^{(1)} - \mathsf{u}^{(2)}, \mathsf{w}^{(2)} - \mathsf{u}^{(3)}\},\\
&\quad\vdots\\
&\{\mathsf{w}^{(p-3)} - \mathsf{u}^{(p-2)}, \mathsf{w}^{(p-2)} - \mathsf{u}^{(p-1)}\},\\
&\{\mathsf{w}^{(p-2)} - \mathsf{u}^{(p-1)}, \mathsf{w}^{(p-1)}\},\\
&\{\mathsf{w}^{(p-1)}, 0\} \ \in A
\end{aligned}$$

But then because $\mathfrak{R}_c(A) = \{0\}$ it must be the case that $\mathsf{w}^{(p-1)} = 0$ whence $w \in \ker A^{p-1}$, so that (remember (5.5)!)

$$w_{p-1} = w - (w_0 + \cdots + w_{p-2}) \in \ker A^{p-1} + \ker A^{p-1}$$

and therefore (because of (5.3))

$$\{w_{p-1}, \widetilde{w}_{p-1}\} - \{w_{p-1}, 0\} = \{0, \widetilde{w}_{p-1}\} \in A^{p-1}.$$

Using the fact that $\mathfrak{R}_c(A) = \{0\}$ once more in conjunction with (5.4) we can conclude that

$$\widetilde{w}_{p-1} = \sum_{j=1}^{m_{p-1}} c_{p-1,j} y_j^{(p-1)} = 0,$$

so $c_{p-1,1} = c_{p-1,2} = \cdots = c_{p-1,m_{p-1}} = 0$. Continuing this way, we are able to nullify the remaining c_{ij} as well. □

Despite its relative complexity, the theorem below generalized from operators to relations without the restriction on the singular chains.

Theorem 5.5. *Suppose that either* $\mathrm{n}(A)$ *or* $\mathrm{d}(A)$ *is finite, and that* $\delta(A) = q$ *is finite. Then*

$$\mathrm{d}(A) \leq \mathrm{n}(A) + \mathrm{d}_c(A^q).$$

In particular, $\mathrm{d}(A) \leq \mathrm{n}(A)$ *if* $\operatorname{dom} A = \mathfrak{H}$.

Proof. If $q = 0$, then $\operatorname{ran} A = \mathfrak{H}$ and hence $\mathrm{d}(A) = 0$ so there is nothing left to prove. Therefore, we assume $q \geq 1$. Let

$$\mathfrak{Q}_i = \ker A^i + \operatorname{ran} A, \quad 0 \leq i \leq q-1.$$

Observe that $\mathfrak{Q}_i \subset \mathfrak{Q}_{i+1}$. We shall show that

$$p_i = \dim \frac{\ker A^{i+1}}{\mathfrak{Q}_i \cap \ker A^{i+1}} < \infty, \quad 0 \leq i \leq q-1. \tag{5.6}$$

Choose numbers $n_0, \ldots, n_{q-1}$ so that we can let $x_1^{(i)}, x_2^{(i)}, \ldots, x_{n_i}^{(i)} \in \ker A^{i+1}$ such that their corresponding cosets $[x_j^{(i)}]$ in $\ker A^{i+1}/(\mathfrak{Q}_i \cap \ker A^{i+1})$ are linearly independent for every $0 \leq i \leq q-1$. We will show that the cosets of the $x_j^{(i)}$ in $\mathfrak{H}/\operatorname{ran} A$ are also linearly independent, so that

$$n_0 + n_1 + \cdots + n_{q-1} \leq \mathrm{d}(A). \tag{5.7}$$

Suppose $\sum_{i=0}^{q-1} \sum_{j=1}^{n_i} c_{ij} x_j^{(i)} \in \operatorname{ran} A$ for certain $c_{ij} \in \mathbb{C}$. Note that

$$\sum_{i=0}^{q-2} \sum_{j=1}^{n_i} c_{ij} x_j^{(i)} \in \ker A^1 + \cdots + \ker A^{q-1} \subset \ker A^{q-1}.$$

So, if we take the last term of the outer summation,

$$\sum_{j=1}^{n_{q-1}} c_{q-1,j} x_j^{(q-1)} = \sum_{i=0}^{q-1} \sum_{j=1}^{n_i} c_{ij} x_j^{(i)} - \sum_{i=0}^{q-2} \sum_{j=1}^{n_i} c_{ij} x_j^{(i)} \in \operatorname{ran} A + \ker A^{q-1} = \mathfrak{Q}_{q-1},$$

it turns out that it is in the 0-coset of $\ker\, A^q/(\mathfrak{Q}_{q-1}\cap \ker\, A^q)$ so by the choice of our $x_j^{(i)}$, it must be so that $c_{q-1,1}=\cdots=c_{q-1,n_{q-1}}=0$. We now have

$$\sum_{i=0}^{q-2}\sum_{j=1}^{n_i} c_{ij}x_j^{(i)} \in \operatorname{ran} A,$$

$$\sum_{j=1}^{n_{q-2}} c_{q-2,j}x_j^{(q-2)} \in \operatorname{ran} A + \ker\, A^{q-2} = \mathfrak{Q}_{q-2}$$

and the argument can be continued to show that all c_{ij} are 0, which proves (5.7).

Let $y_j^{(i)} \in \ker\, A$ such that

$$\{x_j^{(i)}, y_j^{(i)}\} \in A^i,\ \{y_j^{(i)}, 0\} \in A \tag{5.8}$$

(these elements exist because $x_j^{(i)} \in \ker\, A^{i+1}$). We shall now prove

$$n_0+n_1+\cdots+n_{q-1} \leq \dim \frac{\ker\, A}{\ker\, A\cap \operatorname{ran} A^q}\ \Big(\leq \mathrm{n}(A)\Big). \tag{5.9}$$

First of all, observe that $y_j^{(0)}=x_j^{(0)}$ for all $1\leq j\leq n_0$ and remember that for any $a_j\in\mathbb{C}$ $(1\leq j\leq n_0)$

$$\big[a_1x_1^{(0)}+a_2x_2^{(0)}+\cdots+a_{n_0}x_{n_0}^{(0)}\in \mathfrak{Q}_0\big] \Rightarrow \big[a_1=\cdots=a_{n_0}=0\big].$$

For any $1\leq m\leq q$ and $a_{ij}\in\mathbb{C}$ (with $0\leq i\leq m-1,\ 1\leq j\leq n_i$) we define proposition P_m as follows:

$$P_m :\Leftrightarrow \Big[\sum_{i=0}^{m-1}\sum_{j=1}^{n_i} a_{ij}y_j^{(i)} \in \operatorname{ran} A^m \Rightarrow a_{ij}=0,\quad 0\leq i\leq m-1,\ 1\leq j\leq n_i\Big]$$

Because $\mathfrak{Q}_0=\operatorname{ran} A$, we already know that P_1 holds. If we can also establish that P_m implies P_{m+1} (for any $1\leq m\leq q-1$) then we can conclude that P_q is true, which would prove (5.9). Assume P_m and $\sum_{i=0}^{m}\sum_{j=1}^{n_i} a_{ij}y_j^{(i)} = u \in \operatorname{ran} A^{m+1}$. Then

$$\sum_{i=0}^{m-1}\sum_{j=1}^{n_i} a_{ij}y_j^{(i)} = u-\sum_{j=1}^{n_m} a_{mj}y_j^{(m)} \in \operatorname{ran} A^{m+1}+\operatorname{ran} A^m \subset \operatorname{ran} A^m,$$

so $a_{ij}=0$ for $0\leq i\leq m-1,\ 1\leq j\leq n_i$ by P_m and therefore

$$u=\sum_{j=1}^{n_m} a_{mj}y_j^{(m)}.$$

Because $u\in \operatorname{ran} A^{m+1}$ there exists some $\widetilde{u}\in\operatorname{ran} A$ such that

$$\{\widetilde{u},u\}\in A^m. \tag{5.10}$$

Also (because of (5.8))

$$\Big\{\sum_{j=1}^{m} a_{mj}x_j^{(m)}, \sum_{j=1}^{m} a_{mj}y_j^{(m)}\Big\} \in A^m. \tag{5.11}$$

Subtracting (5.10) from (5.11) we find

$$\Big\{\sum_{j=1}^{m} a_{mj}x_j^{(m)}-\widetilde{u}, 0\Big\} \in A^m$$

and hence

$$\sum_{j=1}^{m} a_{mj}x_j^{(m)} = \Big(\sum_{j=1}^{m} a_{mj}x_j^{(m)}-\widetilde{u}\Big)+\widetilde{u} \in \ker\, A^m+\operatorname{ran} A=\mathfrak{Q}_m,$$

so $a_{m1} = \cdots = a_{m,n_m} = 0$, whence indeed $[P_m \Rightarrow P_{m+1}]$ and therefore P_q and (consequently) (5.9) are true.

Our assumption is that either $\mathrm{n}(A)$ or $\mathrm{d}(A)$ is finite. Hence, (5.6) is true for every i (using either (5.9) or (5.7) respectively). So let us choose the numbers n_i maximally (i.e. $n_i = p_i$). This gives us

$$p_0 + p_1 + \cdots + p_{q-1} \le \min\{\mathrm{n}(A), \mathrm{d}(A)\}. \tag{5.12}$$

Next, we shall prove that

$$p_0 + \cdots + p_{q-1} = \dim \frac{\ker A^q}{\operatorname{ran} A \cap \ker A^q}. \tag{5.13}$$

To prove that

$$p_0 + \cdots + p_{q-1} \le \dim \frac{\ker A^q}{\operatorname{ran} A \cap \ker A^q},$$

we can re-use the argument that proved (5.7) if we replace its initial supposition with

$$\text{“Suppose } \sum_{i=0}^{q-1} \sum_{j=1}^{p_i} c_{ij} x_j^{(i)} \in \operatorname{ran} A \cap \ker A^q \subset \operatorname{ran} A \text{ for } \ldots \text{ ”}$$

(because all the $x_j^{(i)}$ are elements of $\ker A^q$) and replace all instances of n_i with p_i. To prove the opposite inequality, observe that any element of $\ker A^{i+1}$ is expressible as a linear combination of $x_1^{(i)}, \ldots, x_{p_i}^{(i)}$ and an element of $\mathfrak{Q}_i$. Because $\mathfrak{Q}_i = \ker A^i + \operatorname{ran} A$ the latter can be expressed as a linear combination of $x_1^{(i-1)}, \ldots, x_{p_{i-1}}^{(i-1)}$, an element of $\mathfrak{Q}_{i-1}$ and an element of $\operatorname{ran} A \subset \mathfrak{Q}_{i-1}$. Applying this procedure repeatedly with $i = q-1, q-2, \ldots, 0$ we see that we can write any $x \in \ker A^q$ as

$$x = \sum_{i=0}^{q-1} \sum_{j=1}^{p_i} a_{ij} x_j^{(i)} + w,$$

where $w \in \operatorname{ran} A$. In fact, $w = x - \sum_{i=0}^{q-1} \sum_{j=1}^{p_i} a_{ij} x_j^{(i)} \in \operatorname{ran} A \cap \ker A^q$, so that $[x] = \left[\sum_{i=0}^{q-1} \sum_{j=1}^{p_i} a_{ij} x_j^{(i)} \right]$ and

$$\dim \frac{\ker A^q}{\operatorname{ran} A \cap \ker A^q} \le p_0 + \cdots + p_{q-1}.$$

And now at last the conclusion of this proof is drawing nigh. Since $\operatorname{ran} A \cap \operatorname{dom} A^q \subset \operatorname{dom} A^q \subset \mathfrak{H}$, we see by Lemma 2.1 that

$$\dim \frac{\mathfrak{H}}{\operatorname{ran} A \cap \operatorname{dom} A^q} = \dim \frac{\mathfrak{H}}{\operatorname{dom} A^q} + \dim \frac{\operatorname{dom} A^q}{\operatorname{ran} A \cap \operatorname{dom} A^q}.$$

Similarly, since also $\operatorname{ran} A \cap \operatorname{dom} A^q \subset \operatorname{ran} A \subset \mathfrak{H}$:

$$\dim \frac{\mathfrak{H}}{\operatorname{ran} A \cap \operatorname{dom} A^q} = \dim \frac{\mathfrak{H}}{\operatorname{ran} A} + \dim \frac{\operatorname{ran} A}{\operatorname{ran} A \cap \operatorname{dom} A^q}$$

and therefore

$$\dim \frac{\mathfrak{H}}{\operatorname{ran} A} + \dim \frac{\operatorname{ran} A}{\operatorname{ran} A \cap \operatorname{dom} A^q} = \mathrm{d}_c(A^q) + \dim \frac{\operatorname{dom} A^q}{\operatorname{ran} A \cap \operatorname{dom} A^q}. \tag{5.14}$$

(Recall that $\dim \mathfrak{H}/\operatorname{dom} A^q = \mathrm{d}_c(A^q)$.) By Lemma 4.3 we know that

$$\operatorname{dom} A^q = (\operatorname{ran} A \cap \operatorname{dom} A^q) \oplus \mathfrak{M}_1, \tag{5.15}$$

where

$$\mathfrak{M}_1 \subset \ker A^q \text{ and } \mathfrak{M}_1 \cap \operatorname{ran} A = \{0\}. \tag{5.16}$$

Consequently,

$$\dim \frac{\operatorname{dom} A^q}{\operatorname{ran} A \cap \operatorname{dom} A^q} = \dim \mathfrak{M}_1. \tag{5.17}$$

From (5.16) we see that

$$\dim \mathfrak{M}_1 \leq \dim \frac{\ker A^q}{\operatorname{ran} A \cap \ker A^q}. \tag{5.18}$$

By (5.17), (5.18), (5.13) and (5.12) we find that

$$\dim \frac{\operatorname{dom} A^q}{\operatorname{ran} A \cap \operatorname{dom} A^q} \leq \mathrm{n}(A).$$

So if we then drop the (nonnegative) term $\dim \operatorname{ran} A/(\operatorname{ran} A \cap \operatorname{dom} A^q)$ from (5.14) we find that

$$\begin{aligned} \dim \frac{\mathfrak{H}}{\operatorname{ran} A} = \mathrm{d}(A) &\leq \mathrm{d}_c(A^q) + \dim \frac{\operatorname{dom} A^q}{\operatorname{ran} A \cap \operatorname{dom} A^q} \\ &\leq \mathrm{d}_c(A^q) + \mathrm{n}(A). \end{aligned}$$

□

Corollary 5.6. *If $\mathfrak{R}_c(A) = \{0\}$, both $\alpha(A)$ and $\delta(A)$ are finite and if either $\mathrm{n}(A)$ or $\mathrm{d}(A)$ is finite, then*

$$\mathrm{n}(A) \leq \mathrm{d}(A) \leq \mathrm{n}(A) + \mathrm{d}_c(A^q),$$

where $q = \delta(A)$. Hence, if $\operatorname{dom} A = \mathfrak{H}$, we have $\mathrm{d}_c(A^q) = \mathrm{d}_c(A) = 0$ and therefore $\mathrm{n}(A) = \mathrm{d}(A)$.

This is simply a combination of Theorems 5.4 and 5.5.

Theorem 5.7. *(a) Suppose $\mathfrak{R}_c(A) = \{0\}$ and $\mathrm{n}(A)$ and $p = \alpha(A)$ are finite. Suppose also that*

$$\dim \frac{\operatorname{dom} A^p}{\operatorname{ran} A \cap \operatorname{dom} A^p} \leq \mathrm{n}(A). \tag{5.19}$$

Then $\delta(A) = \alpha(A)$. Furthermore, we actually have equality, rather than inequality, in (5.19). As a consequence, if follows that $\mathrm{n}(A) = \mathrm{d}(A)$ if $\operatorname{dom} A = \mathfrak{H}$.

(b) Suppose $\mathrm{n}(A)$ and $q = \delta(A)$ are finite. Suppose also that

$$\mathrm{n}(A) \leq \dim \frac{\ker A^q}{\operatorname{ran} A \cap \ker A^q}. \tag{5.20}$$

Then $\alpha(A) \leq q$. Furthermore, we actually have equality, rather than inequality, in (5.20). Also,

$$\dim \frac{\operatorname{dom} A^q}{\operatorname{ran} A \cap \operatorname{dom} A^q} \leq \mathrm{n}(A). \tag{5.21}$$

In case $\operatorname{dom} A = \mathfrak{H}$ we can conclude that $\mathrm{n}(A) = \mathrm{d}(A)$.

(c) Suppose that $\mathfrak{R}_c(A) = \{0\}$, that $\alpha(A)$ is finite and that $\mathrm{n}(A) = \delta(A) < \infty$. Then $\delta(A) = \alpha(A)$.

(d) Suppose that $\mathfrak{R}_c(A) = \{0\}$, that $\operatorname{dom} A = \mathfrak{H}$, that $\delta(A)$ is finite and that $\mathrm{n}(A) = \mathrm{d}(A) < \infty$. Then $\alpha(A) = \delta(A)$.

Proof of (a). We see that the hypotheses of Theorem 5.4 are satisfied. Consider the elements $x_j^{(i)} \in \ker A^p$ $(0 \leq i \leq p-1,\ 1 \leq j \leq m_i)$ introduced in the proof of said theorem, of which there are $\mathrm{n}(A)$ (recall (5.2)). Their corresponding cosets $[x_j^{(i)}]$ are linearly independent in $\mathfrak{H}/\operatorname{ran} A$. If we let

$$\mathfrak{M} := \operatorname{span} \{x_j^{(i)} : 0 \leq i \leq p-1,\ 1 \leq j \leq m_i\},$$

then $\mathfrak{M} \cap \operatorname{ran} A = \{0\}$ and $\dim \mathfrak{M} = \mathrm{n}(A)$. This enables us to conclude from (5.19) and Lemma 2.2 that

$$\operatorname{dom} A^p = (\operatorname{ran} A \cap \operatorname{dom} A^p) \oplus \mathfrak{M}. \tag{5.22}$$

This in turn allows us to appeal to Lemma 4.3 (a) (since $\mathfrak{M} \subset \ker A^p$) and conclude that $\delta(A) \leq p$. But then both $\alpha(A)$ and $\delta(A)$ are finite so $\alpha(A) \leq \delta(A)$, by Theorem 4.4. Hence $\alpha(A) = \delta(A)$.

Since (5.22),

$$\dim \mathfrak{M} = \dim \frac{\operatorname{dom} A^p}{\operatorname{ran} A \cap \operatorname{dom} A^p},$$

which leads to equality in (5.19). Also, in case $\operatorname{dom} A = \mathfrak{H}$,

$$\dim \frac{\operatorname{dom} A^p}{\operatorname{ran} A \cap \operatorname{dom} A^p} = \dim \frac{\mathfrak{H}}{\operatorname{ran} A} = \mathrm{d}(A).$$

Hence $\mathrm{n}(A) = \mathrm{d}(A)$ in this particular case. □

Proof of (b). Here we see that the hypotheses of Theorem 5.5 are satisfied and we consider the elements $y_j^{(i)} \in \ker A$ $(0 \leq i \leq q-1,\ 1 \leq j \leq p_i)$ introduced in its proof. There are $p_0 + \cdots + p_{q-1}$ of them and their corresponding cosets in $\ker A/\operatorname{ran} A^q$ are linearly independent (remember proposition P_q from the proof of said theorem). In view of (5.13) and our present hypothesis (5.20), it then follows that these elements $y_j^{(i)}$ generate $\ker A$ and therefore $\ker A \cap \operatorname{ran} A^q = \{0\}$. Hence, by Lemma 4.1 (a), $\alpha(A) \leq q$. Moreover $p_0 + \cdots + p_{q-1} = \mathrm{n}(A)$ so that (5.20) becomes an equality. Now (5.21) is a consequence of (5.17) and (5.18). Finally, in case $\operatorname{dom} A = \mathfrak{H}$, (5.21) implies that $\mathrm{d}(A) \leq \mathrm{n}(A)$ but then by Corollary 5.6, $\mathrm{d}(A) = \mathrm{n}(A)$. □

Proof of (c). Let $p = \alpha(A)$, then

$$\dim \frac{\operatorname{dom} A^p}{\operatorname{ran} A \cap \operatorname{dom} A^p} \leq \dim \frac{\mathfrak{H}}{\operatorname{ran} A} = \mathrm{d}(A),$$

merely because $\operatorname{dom} A^p \subset \mathfrak{H}$. Now as a consequence of the hypothesis that $\mathrm{n}(A) = \mathrm{d}(A)$, (5.19) holds and by Theorem 5.7 (a) we conclude that $\alpha(A) = \delta(A)$. □

Proof of (d). Let $q = \delta(A)$. Since $\operatorname{dom} A = \mathfrak{H}$, $\operatorname{dom} A^q = \mathfrak{H}$ also which (by Lemma 4.3 (b)) allows us to partition the whole space as $\mathfrak{H} = \operatorname{ran} A \oplus \mathfrak{M}$, where $\mathfrak{M} \cap \operatorname{ran} A = \{0\}$ and $\mathfrak{M} \subset \ker A^q$. Therefore

$$\mathrm{d}(A) = \dim \mathfrak{M} \leq \dim \frac{\ker A^q}{\operatorname{ran} A \cap \ker A^q}. \tag{5.23}$$

Now our current hypotheses allow us to pluck results from the proof of Theorem 5.5 and by (5.23), (5.12) and (5.13) we can conclude that

$$\mathrm{d}(A) = \dim \frac{\ker A^q}{\operatorname{ran} A \cap \ker A^q}.$$

Since $\mathrm{n}(A) = \mathrm{d}(A)$, it follows from Theorem 5.7 (b) that $\alpha(A) \leq \delta(A)$. But then $\alpha(A) = \delta(A)$ by Theorem 4.4. □

6. Shifted linear relations

In this section we will deal with shifted relations, i.e. relations of the form $A - \gamma$ with $\gamma \in \mathbb{C}$.

Lemma 6.1. *Let $\lambda, \mu \in \mathbb{C}$. Then $A - \lambda = (A - \mu) + (\mu - \lambda)$.*

Proof. [5] Let $\{x, y\} \in A - \lambda$, i.e. $\{x, y + \lambda x\} \in A$. Then we shift by μ to get $\{x, (y + \lambda x) - \mu x\} \in A - \mu$ which we regroup to $\{x, y - (\mu - \lambda)x\} \in A - \mu$. To cancel out the last term we shift by $-(\mu - \lambda)$: $\{x, y - (\mu - \lambda)x + (\mu - \lambda)x\} = \{x, y\} \in (A - \mu) + (\mu - \lambda)$. Because each step taken here can be reversed the reverse inclusion is also true, so that $A - \lambda = (A - \mu) + (\mu - \lambda)$. □

[5] Due to Adrian Sandovici.

Lemma 6.2. *Let $\gamma \in \mathbb{C}$, $\gamma \neq 0$. Then $\mathfrak{R}_c(A) = \{0\}$ if and only if $\mathfrak{R}_c(A-\gamma) = \{0\}$.*

Proof. [6] It will suffice to show that $\mathfrak{R}_c(A-\gamma) = \{0\}$ if $\mathfrak{R}_c(A) = \{0\}$ for the reverse implication would in that case follow if we considered $\mathfrak{R}_c((A-\gamma)-(-\gamma))$. Assume that $\mathfrak{R}_c(A) = \{0\}$. Towards a contradiction, assume that $\mathfrak{R}_c(A-\gamma) \neq \{0\}$. Then there exist $x_1, x_2, x_3, \dots, x_p \in \mathfrak{H}$ (which we may assume to be different from zero) such that

$$\{0, x_1\}, \{x_1, x_2\}, \dots, \{x_{p-1}, x_p\}, \{x_p, 0\} \in A - \gamma.$$

This means that

$$\{0, x_1\}, \{x_1, x_2 + \gamma x_1\}, \{x_2, x_3 + \gamma x_2\}, \dots, \{x_{p-2}, x_{p-1} + \gamma x_{p-2}\},$$
$$\{x_{p-1}, x_p + \gamma x_{p-1}\}, \{x_p, x_{p+1} + \gamma x_p\} \in A,$$

where $x_{p+1} = 0$. We define the following coefficients $z_{m,n} \in \mathbb{C}$ for $0 \leq n \leq m \leq p+1$:

$$z_{m,n} := (-1)^{m+n} \gamma^{m-n} \binom{p-n}{m-n}$$

which (as can be easily verified) have the following properties:

$$\begin{cases} z_{p+1,n} = 0 & 0 \leq n \leq p+1 \\ z_{m,m} = 1 & 0 \leq m \leq p \\ z_{m,n} + \gamma z_{m,n+1} = z_{m+1,n+1} & 0 \leq n \leq m \leq p \end{cases} \tag{6.1}$$

Now for any k such that $1 \leq k \leq p+1$ we add up

$$\begin{aligned} &\{0, x_1\} \cdot z_{k,0}, \\ &\{x_1, x_2 + \gamma x_1\} \cdot z_{k,1}, \\ &\quad \vdots \\ &\{x_{k-2}, x_{k-1} + \gamma x_{k-2}\} \cdot z_{k,k-2}, \\ &\{x_{k-1}, x_k + \gamma x_{k-1}\} \cdot z_{k,k-1}, \\ &\{x_k, x_{k+1} + \gamma x_k\} \cdot z_{k,k} \ \in A \end{aligned}$$

to find

$$\left\{ \sum_{i=1}^{k} z_{k,i} x_i, \sum_{i=1}^{k} (z_{k,i-1} + \gamma z_{k,i}) x_i + z_{k,k} x_{k+1} \right\} \in A \tag{6.2}$$

where $x_{p+1} = x_{p+2} = 0$. Define $y_k := \sum_{i=1}^{k} z_{k,i} x_i$ (where $1 \leq k \leq p+1$). Then it follows from (6.2) and (6.1) that

$$\{y_k, y_{k+1}\} \in A, \quad 1 \leq k \leq p$$

and $y_1 = x_1$ and $y_{p+1} = 0$. Therefore

$$\{0, y_1\}, \{y_1, y_2\}, \cdots, \{y_p, 0\} \in A,$$

contradicting our assumption that $\mathfrak{R}_c(A) = \{0\}$ (because $y_1 = x_1 \neq 0$). $\square$

Lemma 6.3. *Let M be a subspace of $\mathfrak{H}$. Then $\mathfrak{R}_c(A|_M) = \{0\}$ if $\mathfrak{R}_c(A) = \{0\}$.*

Proof. Suppose $\mathfrak{R}_c(A|_M) \neq \{0\}$, i.e. there exist $x_1, \dots, x_n \in \mathfrak{H}$ (all non-zero) such that $\{0, x_1\}, \{x_1, x_2\}, \dots, \{x_n, 0\} \in A|_M \subset A$, whence $\mathfrak{R}_c(A) \neq \{0\}$. $\square$

Lemma 6.4. *Let $\lambda, \mu \in \mathbb{C}$, $\lambda \neq \mu$, let j, k be positive integers and let $\mathfrak{R}_c(A) = \{0\}$. Then*

$$\ker (A - \lambda)^j \cap \ker (A - \mu)^k = \{0\}.$$

[6] Based on a proof by Adrian Sandovici.

Proof. [7] If we take $n = \max\{j,k\}$ and proof that $\ker\,(A-\lambda)^n \cap \ker\,(A-\mu)^n = \{0\}$ the statement will follow, because $\ker\,(A-\lambda)^j \subset \ker\,(A-\lambda)^n$ and $\ker\,(A-\mu)^k \subset \ker\,(A-\mu)^n$. If $n=0$ the proof is trivial, so assume $n \geq 1$. First we choose a few more convenient names: we define $B := A-\lambda$ and $\gamma := \lambda - \mu$ so that $A - \mu = B - \gamma$ (by Lemma 6.1). Note that Lemma 6.2 tells us that $\mathfrak{R}_c(B) = \{0\}$. What we now want to show is that $\ker B^n \cap \ker\,(B-\gamma)^n = \{0\}$. Let $z \in \ker B^n \cap \ker\,(B-\gamma)^n$, so there exist $x_1, x_2, \ldots, x_{n-1}, y_1, y_2, \ldots, y_{n-1} \in \mathfrak{H}$ such that

$$\begin{aligned}&\{z,x_1\},\{x_1,x_2\},\ldots,\{x_{n-1},0\} \in B,\\ &\{z,y_1\},\{y_1,y_2\},\ldots,\{y_{n-1},0\} \in B-\gamma,\end{aligned}$$

the latter of which unravels to

$$\{z, y_1+\gamma z\},\{y_1, y_2+\gamma y_1\},\ldots,\{y_{n-1}, 0+\gamma y_{n-1}\} \in B.$$

We will show that $z=0$. Let $x_0 := y_0 := z$ and $x_n := y_n := 0$. We define the following coefficients:

$$a_{p,i} := (-1)^{p+i+1}\gamma^{p-i}\binom{n}{p-i} \text{ and } b_{p,i} := (-1)^{p+i}\gamma^{p-i}\binom{n-1-i}{p-i},$$

with $0 \leq i,p \leq n$. Note that for $i > p$ and $p = n$, we have $b_{p,i} = 0$. We claim without proof that these numbers have the following properties:

$$a_{p,i} = a_{p-1,i-1}, \tag{6.3}$$

$$b_{p,i} = \gamma b_{p-1,i} + b_{p-1,i-1}, \tag{6.4}$$

$$\begin{aligned}&a_{0,0}+b_{0,0} = 0,\\ &a_{p,0}+b_{p,0} = \gamma b_{p-1,0},\end{aligned} \tag{6.5}$$

for $1 \leq i \leq p \leq n$, and also

$$b_{p,i} = 0 \quad \text{for } i > p \text{ and } p = n. \tag{6.6}$$

Let $0 \leq p \leq n-1$, then

$$\sum_{i=0}^{p} a_{p,i}\{x_i,x_{i+1}\} + \sum_{i=0}^{p} b_{p,i}\{y_i, y_{i+1}+\gamma y_i\} =$$
$$\left\{\sum_{i=0}^{p} a_{p,i}x_i + \sum_{i=0}^{p} b_{p,i}y_i, \sum_{i=0}^{p} a_{p,i}x_{i+1} + \sum_{i=0}^{p} b_{p,i}(y_{i+1}+\gamma y_i)\right\} \in B.$$

Let

$$u_p := \sum_{i=0}^{p} a_{p,i}x_i + \sum_{i=0}^{p} b_{p,i}y_i, \quad 0 \leq p \leq n.$$

We claim that $u_{p+1} = \sum_{i=0}^{p} a_{p,i}x_{i+1} + \sum_{i=0}^{p} b_{p,i}(y_{i+1}+\gamma y_i)$, so that $\{u_p, u_{p+1}\} \in B$ for all $0 \leq p \leq n-1$. After all,

$$\begin{aligned}u_{p+1} &= \sum_{i=0}^{p+1} a_{p+1,i}x_i + \sum_{i=0}^{p+1} b_{p+1,i}y_i\\ &= a_{p+1,0}x_0 + \sum_{i=1}^{p+1} a_{p+1,i}x_i + b_{p+1,0}y_0 + \sum_{i=1}^{p+1} b_{p+1,i}y_i.\end{aligned}$$

We perform a substitution $j = i-1$ to find

$$u_{p+1} = a_{p+1,0}x_0 + \sum_{j=0}^{p} a_{p+1,j+1}x_{j+1} + b_{p+1,0}y_0 + \sum_{j=0}^{p} b_{p+1,j+1}y_{j+1}.$$

[7] Based on a proof by Adrian Sandovici.

We apply (6.4) and the fact that $x_0 = y_0$ to find

$$\begin{aligned} u_{p+1} &= a_{p+1,0}x_0 + \sum_{j=0}^{p} a_{p+1,j+1}x_{j+1} + b_{p+1,0}y_0 + \sum_{j=0}^{p}(\gamma b_{p,j+1} + b_{p,j})y_{j+1} \\ &= \sum_{j=0}^{p} a_{p+1,j+1}x_{j+1} + (a_{p+1,0} + b_{p+1,0})y_0 + \sum_{j=0}^{p} \gamma b_{p,j+1}y_{j+1} + \sum_{j=0}^{p} b_{p,j}y_{j+1}. \end{aligned}$$

We locally revert our index back to $i = j + 1$ and apply (6.5) and find

$$\begin{aligned} u_{p+1} &= \sum_{j=0}^{p} a_{p+1,j+1}x_{j+1} + \gamma b_{p,0}y_0 + \sum_{i=1}^{p} \gamma b_{p,i}y_i + \gamma b_{p,p+1}y_{p+1} + \sum_{j=0}^{p} b_{p,j}y_{j+1} \\ &= \sum_{j=0}^{p} a_{p,j}x_{j+1} + \sum_{i=0}^{p} \gamma b_{p,i}y_i + \sum_{j=0}^{p} b_{p,j}y_{j+1}, \end{aligned}$$

using (6.3) and (6.6), thus proving that $u_{p+1} = \sum_{i=0}^{p} a_{p,i}x_{i+1} + \sum_{i=0}^{p} b_{p,i}(y_{i+1} + \gamma y_i)$. By (6.5), $u_0 = a_{0,0}x_0 + b_{0,0}y_0 = 0$, so we now have

$$\{0, u_1\}, \{u_1, u_2\}, \ldots, \{u_{n-1}, u_n\} \in B. \tag{6.7}$$

If we define

$$v_r := \sum_{i=0}^{n-r-1} a_{n,i}x_{i+r}, \quad 0 \le r \le n-1,$$

then $u_n = v_0$ (as $x_n = 0$). Now

$$\sum_{i=0}^{n-r-1} a_{n,i}\{x_{i+r}, x_{i+r+1}\} = \left\{ \sum_{i=0}^{n-r-1} a_{n,i}x_{i+r}, \sum_{i=0}^{n-(r+1)-1} a_{n,i}x_{i+r+1} \right\} \in B$$

for $0 \le r \le n-2$ as $x_n = 0$. Furthermore, because $v_{n-1} = a_{n,0}x_{n-1} \in \ker B$ we also have $\{v_{n-1}, 0\} \in B$, whence

$$\{v_0, v_1\}, \ldots, \{v_{n-1}, 0\} \in B, \tag{6.8}$$

so because of (6.7), the fact that $u_n = v_0$ and $\mathfrak{R}_c(B) = \{0\}$, we have $v_{n-1} = v_{n-2} = \cdots = v_0 = 0$, whence $x_{n-1} = 0$. However, because also $v_{n-2} = a_{n,0}x_{n-2} + a_{n,1}x_{n-1}$, we also have $x_{n-2} = 0$. Continuing this way, we conclude that $z = x_0 = 0$, whence $\ker B^n \cap \ker (B-\gamma)^n = \{0\}$. □

Lemma 6.5. *Let $\gamma \in \mathbb{C}$, $\gamma \neq 0$. Then*

$$\ker (A-\gamma)^k \subset \operatorname{ran} A^n \tag{6.9}$$

for any integers $n, k \ge 0$.[8]

Proof. [9] If either $k = 0$ of $n = 0$, the desired inclusion is trivial, so let $n, k \ge 1$. Our proof will be by induction on n. For the basis, we consider two cases: $k = 1$ and $k \ge 2$.

In case $k = 1$, $x_0 \in \ker (A-\gamma)$ implies that $\{x_0, 0\} \in A-\gamma$, so that $\{x_0, \gamma x_0\} \in A$ and therefore $x_0 \in \operatorname{ran} A$, as $\gamma \neq 0$. If, on the other hand, $k \ge 2$, we can also show that (6.9) holds for $n = 1$. Let $x_0 \in \ker (A-\gamma)^k$. Then there exist elements $x_1, x_2, \ldots, x_{k-1}$ such that $\{x_0, x_1\}, \{x_1, x_2\}, \ldots, \{x_{k-2}, x_{k-1}\}, \{x_{k-1}, 0\} \in A - \gamma$, which means that

$$\{x_0, x_1+\gamma x_0\}, \{x_1, x_2+\gamma x_1\}, \ldots \{x_{k-2}, x_{k-1}+\gamma x_{k-2}\}, \{x_{k-1}, \gamma x_{k-1}\} \in A. \tag{6.10}$$

[8]This is a stronger claim that the one found in the dual result in [5], for there the case is restricted to $n = \delta(A) < \infty$.

[9]An adaptation of a proof due to Adrian Sandovici

Now we take a suitable linear combination of these elements,

$$\begin{aligned}
&\{x_0, x_1 + \gamma x_0\} \cdot (-\gamma)^{k-1},\\
&\{x_1, x_2 + \gamma x_1\} \cdot (-\gamma)^{k-2},\\
&\quad\vdots\\
&\{x_{k-2}, x_{k-1} + \gamma x_{k-2}\} \cdot (-\gamma)^{1},\\
&\{x_{k-1}, \gamma x_{k-1}\} \cdot (-\gamma)^{0} \in A,
\end{aligned}$$

so that

$$\left\{ \sum_{i=0}^{k-1} (-\gamma)^{k-1-i} x_i, -(-\gamma)^k x_0 \right\} \in A$$

which shows us that $x_0 \in \operatorname{ran} A$ (as $\gamma \neq 0$).

For the induction, we assume $k \geq 1$ and we let (6.9) be our induction hypothesis. Take any $x_0 \in \ker\,(A - \gamma)$, then it is now for us to show that $x_0 \in \operatorname{ran} A^{n+1}$. Consider elements $x_1, \dots, x_{k-1}$ defined in the same manner as above (note that this is also possible if $k = 1$). Define $x_k := 0$. We take a linear combination of the first i lines of the resulting table (where $1 \leq i \leq k$):

$$\begin{aligned}
&\{x_0, x_1 + \gamma x_0\} \cdot \tbinom{i-1}{i-1}\gamma^{i-1},\\
&\{x_1, x_2 + \gamma x_1\} \cdot \tbinom{i-1}{i-2}\gamma^{i-2},\\
&\{x_2, x_3 + \gamma x_2\} \cdot \tbinom{i-1}{i-3}\gamma^{i-3},\\
&\quad\vdots\\
&\{x_{i-1}, x_i + \gamma x_{i-1}\} \cdot \tbinom{i-1}{0}\gamma^{0} \in A.
\end{aligned}$$

This gives us

$$\left\{ \sum_{j=0}^{i-1} \binom{i-1}{i-1-j} \gamma^{i-1-j} x_j, \sum_{j=0}^{i-1} \binom{i-1}{i-1-j} \gamma^{i-1-j} (x_{j+1} + \gamma x_j) \right\} \in A$$

So, if we define $z_i := \sum_{j=0}^{i-1} \binom{i-1}{j} \gamma^j x_{i-1-j}$, it can be seen that

$$\{z_i, z_{i+1}\} \in A \quad (1 \leq i \leq k). \tag{6.11}$$

Observe that z_{k+1} is well-defined and that all $z_i \in \operatorname{ran} A^n$. After all, $x_0 \in \operatorname{ran} A^n$ (by our induction hypothesis) and $\{x_0, x_1 + \gamma x_0\} \in A$ together imply that $x_1 = x_1 + \gamma x_0 - \gamma x_0 \in \operatorname{ran} A^n$. In the same vein it can be demonstrated that all $x_i \in \operatorname{ran} A^n$. Even our tag-along x_k complies, because trivially $0 \in \operatorname{ran} A^n$. Define

$$c_i := (-1)^{k-i} \gamma^{k-i} \binom{k}{k-i} \; 1 \leq i \leq k.$$

Note that these coefficients have been purposefully chosen so that

$$\sum_{i=1}^{k} c_i z_{i+1} = (-1)^{k+1} \gamma^k x_0 + x_k = (-1)^{k+1} \gamma^k x_0.$$

But as by (6.11), $\sum_{i=0}^{k} c_i \{z_i, z_{i+1}\} \in A$, this means that $(-1)^{k+1}\gamma^k x_0 \in \operatorname{ran} A^{n+1}$, which implies $x_0 \in \operatorname{ran} A^{n+1}$ as $\gamma \neq 0$. □

Lemma 6.6. *Suppose* $\operatorname{dom} A = \mathfrak{H}$ *and* $\gamma \neq 0$. *Also let* $\mathfrak{H}_1 = \operatorname{ran} A$ *and* $A_1 := A|_{\mathfrak{H}_1}$. *Then:*

(a) $\alpha(A - \lambda) = \alpha(A_1 - \lambda)$ *and* $\mathrm{n}(A - \lambda) = \mathrm{n}(A_1 - \lambda)$.

(b) If $\mathfrak{M}$ *is a subspace of* $\mathfrak{H}_1$ *such that* $\mathfrak{H}_1 = \operatorname{ran}(A_1 - \lambda) \oplus \mathfrak{M}$, *then also* $\mathfrak{H} = \operatorname{ran}(A - \lambda) \oplus \mathfrak{M}$. *Therefore* $\mathrm{d}(A - \lambda) = \mathrm{d}(A_1 - \lambda)$.

(c) Suppose that $\mathfrak{R}_c(A) = \{0\}$. *If* $\mathrm{n}(A_1 - \lambda) = \mathrm{d}(A_1 - \lambda) < \infty$ *and if either* $\alpha(A_1 - \lambda)$ *or* $\delta(A_1 - \lambda)$ *is finite, then* $\alpha(A - \lambda) = \mathrm{d}(A - \lambda) < \infty$.

Proof of (a). We shall show that $\ker(A_1-\lambda)^m = \ker(A-\lambda)^m$ for $m = 0,1,2,\ldots$. Clearly this is true for $m=0$, so assume $m \geq 1$. To show that $\ker(A_1-\lambda)^m \subset \ker(A-\lambda)^m$, take any $x_1 \in \ker(A_1-\lambda)^m$. Then:

$$\left\{\begin{array}{l}\{x_1,x_2\}\\ \{x_2,x_3\}\\ \quad\vdots\\ \{x_{m-1},x_m\}\\ \{x_m,0\}\end{array}\right. \in A_1-\lambda \quad\Rightarrow\quad \left\{\begin{array}{l}\{x_1,x_2+\lambda x_1\}\\ \{x_2,x_3+\lambda x_2\}\\ \quad\vdots\\ \{x_{m-1},x_m+\lambda x_{m-1}\}\\ \{x_m,\lambda x_m\}\end{array}\right. \in A_1$$

And because A_1 is merely a restriction of A,

$$\left\{\begin{array}{l}\{x_1,x_2+\lambda x_1\}\\ \{x_2,x_3+\lambda x_2\}\\ \quad\vdots\\ \{x_{m-1},x_m+\lambda x_{m-1}\}\\ \{x_m,\lambda x_m\}\end{array}\right. \in A \quad\Rightarrow\quad \left\{\begin{array}{l}\{x_1,x_2\}\\ \{x_2,x_3\}\\ \quad\vdots\\ \{x_{m-1},x_m\}\\ \{x_m,0\}\end{array}\right. \in A-\lambda,$$

so that $x_1 \in \ker(A-\lambda)$. Now for the reverse inclusion, let $x_1 \in \ker(A-\lambda)^m$ so that, by the previous argument, for certain $x_2, x_3, \ldots, x_m \in \mathfrak{H}$ we have:

$$\{x_1,x_2+\lambda x_1\},\{x_2,x_3+\lambda x_2\},\ldots,\{x_{m-1},x_m+\lambda x_{m-1}\},\{x_m,\lambda x_m\} \in A$$

The last pair tells us that $\lambda x_m \in \operatorname{ran} A$ so that also $x_m \in \operatorname{ran} A$ (recall that $\lambda \neq 0$). Because also $x_m + \lambda x_{m-1} \in \operatorname{ran} A$ by the penultimate pair, $\lambda x_{m-1} \in \operatorname{ran} A$. Repeating this argument we can conclude that all the elements $x_1, \ldots, x_m$ are in $\operatorname{ran} A = \mathfrak{H}_1$, whence all the pairs listed above are also in A_1 so that $x_1 \in \ker(A_1-\lambda)^m$. The truth of (a) follows at once. □

Proof of (b). First of all, $\operatorname{ran}(A_1-\lambda) = \operatorname{ran}(A-\lambda)\cap\mathfrak{H}_1$. For, let $y \in \operatorname{ran}(A-\lambda)\cap\mathfrak{H}_1$. Then for some $x \in \mathfrak{H}$, $\{x, y+\lambda x\} \in A$, i.e. $y + \lambda x \in \mathfrak{H}_1 = \operatorname{ran} A$. But then $\lambda x \in \mathfrak{H}_1$ so that $\{x, y+\lambda x\} \in A_1$ and therefore $y \in \operatorname{ran}(A_1-\lambda)$. The reverse inclusion is now also evident. Secondly, we see that $\mathfrak{M}\cap\operatorname{ran}(A-\lambda) = \mathfrak{M}\cap\mathfrak{H}_1\cap\operatorname{ran}(A-\lambda) = \mathfrak{M}\cap\operatorname{ran}(A_1-\lambda) = \{0\}$, since $\mathfrak{M}$ is a subspace of $\mathfrak{H}_1$ which is complementary to $\operatorname{ran}(A_1-\lambda)$. Thirdly and finally, we shall show that $\mathfrak{H} \subset \operatorname{ran}(A-\lambda)\oplus\mathfrak{M}$. Let $x \in \mathfrak{H}$, then because $\operatorname{dom}(\lambda - A) = \operatorname{dom} A = \mathfrak{H}$ there is some $y \in \operatorname{ran}(A-\lambda)$ such that $\{x,y\} \in A-\lambda$, i.e. $\{x, y+\lambda x\} \in A$. Then $y + \lambda x \in \mathfrak{H}_1$ and by our hypothesis there exist $u \in \operatorname{ran}(A_1-\lambda) \subset \operatorname{ran}(A-\lambda)$ and $v \in \mathfrak{M}$ such that $y + \lambda x = u + v$. Now

$$x = \frac{u-y}{\lambda} + \frac{v}{\lambda} \in \operatorname{ran}(A-\lambda) + \mathfrak{M},$$

which proves (b). □

Proof of (c). Observe that by Lemmas 6.3 and 6.2 $\mathfrak{R}_c(A_1-\lambda) = \{0\}$ so if $\delta(A_1-\lambda)$ is finite, we can apply Theorem 5.7 (d) to $A_1-\lambda$ to show that $\alpha(A_1-\lambda) = \delta(A_1-\lambda)$. Then by part (a) of the present lemma, $\alpha(A-\lambda)$ is finite. Moreover, as a consequence of both parts (a) and (b), $\mathrm{n}(A-\lambda) = \mathrm{d}(A-\lambda) < \infty$. Because by Lemma 6.2 also $\mathfrak{R}_c(A-\lambda) = \{0\}$, we can now use Theorem 5.7 (c) to conclude that $\delta(A-\lambda) = \alpha(A-\lambda)$.

If on the other hand $\alpha(A_1-\lambda)$ is finite, we apply Theorem 5.7 (c) to $A_1-\lambda$ to see that $\delta(A_1-\lambda) = \alpha(A_1-\lambda)$. The previous argument now shows that $\alpha(A-\lambda) = \delta(A-\lambda) < \infty$. □

Theorem 6.7. *Suppose* $\mathfrak{R}_c(A) = \{0\}$, $\operatorname{dom} A = \mathfrak{H}$, $\lambda \neq 0$ *and* $\dim\operatorname{ran} A < \infty$. *Then* $\mathrm{n}(A-\lambda) = \mathrm{d}(A-\lambda) < \infty$ *and* $\alpha(A-\lambda) = \delta(A-\lambda) < \infty$.

Proof. Let A_1 and $\mathfrak{H}_1$ be as in Lemma 6.6. Because in our present case $\dim \mathfrak{H}_1 < \infty$, it must necessarily be so that $\mathrm{n}(A_1 - \lambda)$ and $\mathrm{d}(A_1 - \lambda)$ are finite (for they cannot 'grow' or 'shrink' indefinitely in a finite-dimensional space, to put it in rather unprecise words). Also $\alpha(A_1 - \lambda)$ and $\delta(A_1 - \lambda)$ are finite. Then, by Corollary 5.6, $\mathrm{n}(A_1-\lambda) = \mathrm{d}(A_1-\lambda)$. It now follows from Lemma 6.6 that $\mathrm{n}(A-\lambda) = \mathrm{d}(A-\lambda) < \infty$ and $\alpha(A - \lambda) = \delta(A - \lambda) < \infty$. □

Appendix: Theorem translation

In this section the reader can find which of the theorems and lemmas in this report are dual to which results in [5].

Taylor	This report
Lemma 2.1	Lemma 2.1
Lemma 2.2	Lemma 2.2
Lemma 3.1	Lemma 5.1
Lemma 3.2	Lemma 5.2
Lemma 3.3	Lemma 3.1
Lemma 3.4	Lemma 4.1
Lemma 3.5	Lemma 4.3
Theorem 3.6	Theorem 4.4
Theorem 3.7	Theorem 4.6

Taylor	This report
Lemma 3.8	Lemma 6.4
Lemma 3.9	Lemma 6.5
Theorem 4.1	Theorem 5.3
Theorem 4.2	Theorem 5.4
Theorem 4.3	Theorem 5.5
Corollary 4.4	Corollary 5.6
Theorem 4.5	Theorem 5.7
Lemma 4.6	Lemma 6.6
Theorem 4.7	Theorem 6.7

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