Master's thesis in Mathematics: Differential Galois Theory, Hamiltonian Systems and the Spherical Pendulum

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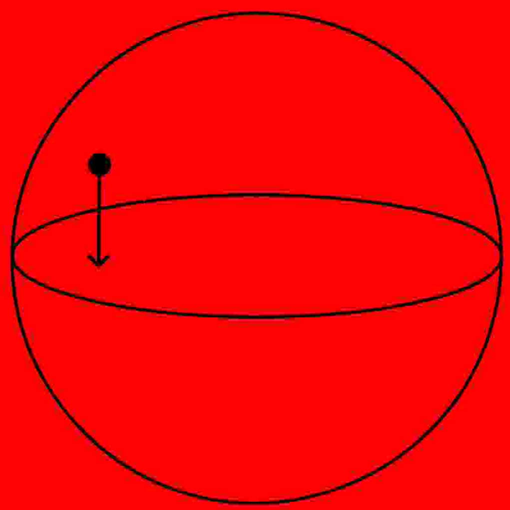

1 Introduction

The author's first encounter with the notion of first integrals was through the lecture notes [9] for the course "Symplectic Geometry" given by John Rawnsley at the university of Warwick. Its main objective was to give an overview of symplectic manifolds and the geometry of momentum maps to explain symplectic reduction. This reduction of dimensions helps with analyzing a specific problem and is a first step in checking if a Hamiltonian system is completely integrable, see [1].

In the recent work of Morales and Ramis [7], the existence of sufficient many first integrals was linked to a condition of the differential Galois group of the variational equation. The first real application was the three-body problem. The initial setup was done by Morales and Ramis but, the completion was given by Delphine Boucher in [3] for the planar three-body problem. The spherical pendulum, also a classic mechanics system, was already studied by Huygens [6] and that it is a completely integrable system has also been known for a long time. Different sources like [2] and [4] give it as an example but the whole analysis and the computation of the differential Galois group of the normal variational differential equation has not been published.

In this article, we present a brief introduction to the Morales-Ramis theorem as well as 2 computations of the differential Galois group. The differential galois group depends on the solution curve chosen for the linearization. In our first attempt the linearization is done along the Huygens solution curves and in the second we use the restriction of the spherical pendulum to the planar pendulum.

This article is a collaboration with my supervisor Marius van der Put, I am forever thankful for his guidance and continuous support.

2 From a Hamiltonian system to a differential Galois group

Let $\mathbf{M}$ be a complex manifold. Let X be a holomorphic vector field on $\mathbf{M}$. Locally, a solution of the differential equation is a holomorphic map $t \mapsto f(t)$, from an open set in $\mathbf{C}$ to $\mathbf{M}$, satisfying $\dot{f}(t) = X(f(t))$ (where the dot means the derivative w.r.t. t). An integral curve S is obtained as a maximal analytic continuation of a local solution. We assume that S is a Riemann surface and that the corresponding map $S \to \mathbf{M}$ is an embedding with normal crossings in $\mathbf{M}$.

The *variational equation* corresponding to X and S is the linearization of X along S. This can locally be described as follows:

Let S be locally described by $t \mapsto f(t) \in \mathbf{M}$ and let $\mathbf{M}$ be locally described by as a subspace of some $\mathbf{C}^m$. Then one considers holomorphic maps $t \mapsto g(t) \in \mathbf{C}^m$ such that $f(t) + \epsilon g(t) \in \mathbf{M}$, $\epsilon^2 = 0$ and $\dot{f}(t) + \epsilon \dot{g}(t)) = X(f(t) + \epsilon g(t))$. The first condition means that $g(t) \in T_{\mathbf{M},f(t)}$, the tangent space of $\mathbf{M}$ at $f(t)$. The second condition is equivalent to a linear differential equation for $g(t)$. Hence the variational equation is a linear differential equation on the restriction (or better pull back) $T_{\mathbf{M}}|_S$ of the tangent bundle of $\mathbf{M}$ to S. If one wants to forget the choices of local variables, then this amounts to a connection ∇ on the vector bundle $T_{\mathbf{M}}|_S$ on S.

Now we assume that S is already a compact Riemann surface, or that S and its connection have a natural extension to a compact Riemann surface $\hat{S}$ with a meromorphic connection $\hat{\nabla}$. Let $k := \mathbf{C}(\hat{S})$ denote the field of meromorphic functions on $\hat{S}$ and let M denote the vector space over k of finite dimension, consisting of the meromorphic sections of the vector bundle on $\hat{S}$. The connection induces an connection (again called ∇) of the following form

$$\nabla : M \rightarrow \Omega_{k/\mathbf{C}} \otimes M,$$

where $\Omega_{k/\mathbf{C}}$ is the universal differential module of $k/\mathbf{C}$. Now $\Omega_{k/\mathbf{C}}$ is a vector space of dimension 1 over k. Choose a non-trivial $\mathbf{C}$-linear differentiation $a \mapsto a'$ on k. Then k is a differential field with field of constants $\mathbf{C}$ and the connection ∇ translates into an additive map $\partial : M \rightarrow M$ that has the property $\partial(am) = a'm + a\partial m$ for $a \in k,\ m \in M$. In other words, (M, ∂) is a differential module over k.

The standard differential Galois theory can now be applied to (M, ∂). In particular, there is a Picard-Vessiot field $K \supset k$. The solution space V of M is equal to $\ker(\partial, K \otimes_k M)$. The differential Galois group G is the group of the differential automorphisms of K/k. This group acts faithfully on V and its image in $\mathrm{GL}(V)$ is an algebraic subgroup.

The Picard-Vessiot field K can be made somewhat more explicit. Suppose that $x_0 \in \hat{S}$ is a point where the differential equation is regular. Let t be an analytic local parameter at x_0. Then the field of functions, meromorphic at x_0, is the field of convergent Laurent series in t, denoted by $\mathbf{C}(\{t\})$. The kernel of ∂ on $\mathbf{C}(\{t\}) \otimes M$ can be identified with the solution space V. The field K is generated over k by all coordinates of all solutions in V with respect to some basis of M over k. In this way, K is embedded into $\mathbf{C}(\{t\})$.

A *first integral* F is a meromorphic function on $\mathbf{M}$ such that F is constant on every integral curve for the vector field X. Here, we only need that F is defined in some neighbourhood of $\hat{S}$. We want to show that the existence of F has consequences for the differential Galois group G. Locally, one has

$$F(f(t)+\epsilon g(t)) = F(f(t)) + \epsilon \sum_{i=1}^{n} g_i(t) \frac{\partial F}{\partial z_i}(f(t)).$$

Here $z_1, \ldots, z_n$ are local holomorphic coordinates on $\mathbf{M}$ and the $g_i(t)$ are the entries of the vector $g(t)$. Globally, we have an element $L := dF|_{\hat{S}}$ which is a meromorphic section of $T^*_{\mathbf{M}}|_{\hat{S}}$. In other words, $L : M \to k$ is a k-linear map and thus $L \in M^*$. Since F is a first integral, one has that $L(v) \in \mathbf{C}$ for every v in the solution space V of ∂.

Lemma 1 *$L = dF|_{\hat{S}} \in M^*$ satisfies $\partial(L) = 0$. In particular, L is an element in the dual V^* of the solution space and is moreover invariant under the action of G.*

Proof. We note that $\partial(L) \in M^*$ is defined by $\partial(L)(m) = L(\partial m) - L(m)'$ for all $m \in M$. As before, we take a point $x_0 \in \hat{S}$ that is a regular point for (M, ∂) and t is a local analytic parameter at x_0. Then L extends to a $\mathbf{C}(\{t\})$-linear map $\mathbf{C}(\{t\}) \otimes M \to \mathbf{C}(\{t\})$. This map will also be called L. Furthermore, $\mathbf{C}(\{t\}) \otimes_{\mathbf{C}(\hat{S})} M = \mathbf{C}(\{t\}) \otimes_{\mathbf{C}} V$. For $v \in V$ we have $\partial(L)(v) = L(\partial v) - L(v)'$. This is 0 since $\partial v = 0$ and $L(v) \in \mathbf{C}$. Then also $\partial(L) = 0$. □

It is possible that $dF|_{\hat{S}}$ is identical zero. In local coordinates (as above) this means that all $\frac{\partial F}{\partial z_i}(f(t))$ are zero. We repeat the ϵ-trick now with $\epsilon^3 = 0$ and one can write

$$F(f(t)+\epsilon g(t)) = F(f(t)) + \epsilon^2 \sum_{i,j=1}^{n} g_i(t) g_j(t) \frac{\partial^2 F}{\partial z_i \partial z_j}(f(t)).$$

Put $d^2F(f(t)) := \sum_{i,j} \frac{\partial^2 F}{\partial z_i \partial z_j}(f(t)) dz_i dz_j$. This is a local expression for an element $L_2 := d^2F|_{\hat{S}}$, which is a global meromorphic section of $sym^2(T^*_{\mathbf{M}}|_{\hat{S}})$. In other words, L_2 lies in $sym^2(M^*)$. Moreover $L_2(v \otimes v) \in \mathbf{C}$ for all $v \in V$.

As before, it follows that $\partial(L_2) = 0$ and therefore $L_2 \in sym^2(V^*)$ is an element invariant under the action of the differential Galois group G. More generally one has:

Corollary 2 *Let $k \geq 1$ denote the integer such that $d^i F|_{\hat{S}} = 0$ for $i = 1, \ldots, k-1$ and $L_k := d^k F|_{\hat{S}} \neq 0$. Then L_k lies in $sym^k M^*$ and $\partial(L_k) = 0$. In particular, $L_k \in sym^k V^*$ is invariant under the action of the differential Galois group G.*

Now we consider the *Hamiltonian case*. Let $\mathbf{A}$ denote a complex manifold of dimension n. Then $\mathbf{M} := T^*_{\mathbf{A}}$ is the cotangent bundle of $\mathbf{A}$ and is a complex symplectic manifold of dimension $2n$. Let $H : \mathbf{M} \to \mathbf{C}$ be a holomorphic function, called a *Hamilton function*. There is a standard way to attach to H a holomorphic vector field X_H. In local symplectic coordinates $q_1, \ldots, q_n, p_1, \ldots, p_n$ one writes $H = H(q,p)$ with $q = (q_1, \ldots, q_n)$ and $p = (p_1, \ldots, p_n)$. Now X_H is locally given by $X_H = \sum_{i=1}^n \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i} - \sum_{i=1}^n \frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i}$. The variational equation (attached to X_H and $\hat{S}$) is a differential module M over $\mathbf{C}(\hat{S})$ of dimension $2n$. As before, M consists of the global meromorphic sections of the vector bundle $T_{\mathbf{M}}|_{\hat{S}}$. The extra features are: M inherits a symplectic form $\langle\ ,\ \rangle : M \times M \to k$ and ∂ respects this form. The latter means that

$$\langle \partial m_1, m_2 \rangle + \langle m_1, \partial m_2 \rangle = \langle m_1, m_2 \rangle' \text{ for all } m_1, m_2 \in M.$$

The solution space V has dimension $2n$ over $\mathbf{C}$. It is equipped with a symplectic form $\langle\ ,\ \rangle$. Moreover the differential Galois group G respects this symplectic form. In other words, $G \subset \mathrm{Sp}(2n, \mathbf{C})$.

The Hamiltonian system is called *completely integrable* if there exists first integrals $F_1, \ldots, F_n$ such that the $dF_1, \ldots, dF_n$ are linearly independent and the corresponding vectors fields $X_{F_1}, \ldots, X_{F_n}$ commute. One form of the Morales-Ramis theorem is the following.

Theorem 3 *Suppose that the Hamiltonian system is completely integrable and that the integral curve $\hat{S}$ is not constant. Then the differential Galois group G has the property that its component of the identity G^o is abelian.*

Proof. We only sketch the proof. For convenience we identify V with V^*, via the symplectic form on V. Suppose that the $e_i := dF_i|_{\hat{S}}$, $i = 1, \ldots, n \in V$ are linearly independent over $\mathbf{C}$. Then they generate a maximal isotropic subspace A of V. Define elements $f_1, \ldots, f_n \in V$ with $\langle f_i, f_j \rangle = 0$ for all i, j and $\langle e_i, f_j \rangle = \delta_{i,j}$ for $i, j = 1, \ldots, n$. Put $B = \mathbf{C}f_1 + \cdots + \mathbf{C}f_n$. Then $V = A \oplus B$. Let $P : V \to V$ denote the projection onto B with kernel A. Any $\sigma \in G$ satisfies $\sigma e_i = e_i$ $i = 1, \ldots, n$. Since σ preserves $\langle\ ,\ \rangle$, one has that $\sigma f_i - f_i \in \mathbf{C}e_1 + \cdots \mathbf{C}e_n$. Thus G lies in the subgroup $id + \mathrm{Hom}(B, A) \circ P$ of $\mathrm{GL}(V)$. This subgroup is isomorphic to the direct sum of n^2 copies of the additive group $\mathbf{C}$. In particular, G is abelian.

In the general situation, $\mathbf{C}$-linear combinations of the dF_i can become zero when restricted to $\hat{S}$. One has to use corollary 2 to complete the proof. □

The variational equation on $T_{\mathrm{M}}|_{\hat{S}}$ has a 1-dimensional subbundle, namely $T_{\hat{S}}$, which is invariant under the connection. Thus M has a 1-dimensional subspace kv_0, invariant under ∂. Then $\tilde{M} := \{m \in M \mid \langle m, v_0 \rangle = 0\}$ is also invariant under ∂. The *normal variation equation* is by definition $N := \tilde{M}/kv_0$. This differential module of dimension $2n - 2$ inherits from M a symplectic form which is again respected by the ∂ of N. For a completely integrable Hamiltonian system, the normal variational equation N has again the property that the component of the identity of its differential Galois group is abelian. This is of interest for the case $n = 2$, since for a differential module of dimension 2 there are effective methods (Kovacic' algorithm and its refinements) for the computation of the differential Galois group.

3 Calculations on the spherical pendulum

Hamiltonian $H := \frac{1}{2}||p||^2 - \Gamma \cdot q$

$\Gamma = (0, 0, -1)$, $||q||^2 = 1$ and $p \cdot q = 0$.

Equations:

$$(1)\ \dot{q} = p \qquad (2)\ \dot{p} = \Gamma - (q \cdot \Gamma + ||p||^2)q$$

First we consider the integral curve S of Huygens. This curve is given by the formulas (with chosen a with $0 < a < 1$; and yet unknown b).

$$q(t) = (\sqrt{1 - a^2} \cos bt, \sqrt{1 - a^2} \sin bt, -a)$$

$$p(t) = (-b\sqrt{1 - a^2} \sin bt, b\sqrt{1 - a^2} \cos bt, 0)$$

Then

$$\dot{p}(t) = (-b^2\sqrt{1-a^2}\cos bt, -b^2\sqrt{1-a^2}\sin bt, 0)$$

$$\dot{p} = \Gamma - (q\cdot\Gamma + ||p||^2)q =$$
$$(0,0,-1) - (a + b^2(1-a^2))(\sqrt{1-a^2}\cos bt, \sqrt{1-a^2}\sin bt, -a)$$

yields $b = a^{-1/2}$. The integral curve S is then

$$q(t) = (\sqrt{1-b^{-4}}\cos bt, \sqrt{1-b^{-4}}\sin bt, -b^{-2})$$

$$p(t) = (-b\sqrt{1-b^{-4}}\sin bt, b\sqrt{1-b^{-4}}\cos bt, 0)$$

Variational equation:
$q+\epsilon Q,\ p+\epsilon P$ with $\epsilon^2 = 0$ should satisfy

$$(1),\ (2) \text{ and } (3)\ ||q+\epsilon Q||^2 = 1 \text{ and } \ (4)\ (p+\epsilon P)\cdot(q+\epsilon Q) = 0$$

One translates the equations:
(1) $\dot{Q} = P$, (3) $q\cdot Q = 0$, (4) $\dot{q}Q + q\dot{Q} = 0$, but this follows by differentiating (3), (2) reads

$$\dot{P} = (Q_3 - 2p_1P_1 - 2p_2P_2)q - b^2Q$$

Here $p = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}$, $Q = \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix}$ and so on.

Let k denote a suitable differential field. To start with we can take for $k = \mathbf{C}(\{t\})$ with differentiation $\dot{a} := \frac{da}{dt}$. Later on we will replace k by the function field of the compact Riemann surface, determined by S. This Riemann surface turns out to be the complex sphere.

The variational equation is now written in module form as follows: The elements of k^6 are denoted by $\begin{pmatrix} Q \\ P \end{pmatrix}$. Both Q and P are in k^3. Further we write $A\cdot B$ for $\sum_{i=1}^3 A_iB_i$, for $A, B \in k^3$. The symplectic form on k^6 is given by

$$\langle \begin{pmatrix} Q \\ P \end{pmatrix}, \begin{pmatrix} \tilde{Q} \\ \tilde{P} \end{pmatrix} \rangle = Q\cdot\tilde{P} - P\cdot\tilde{Q}.$$

Define $\partial : k^6 \to k^6$ by the formula

$$\partial \begin{pmatrix} Q \\ P \end{pmatrix} = \begin{pmatrix} \dot{Q} - P \\ \dot{P} - (Q_3 - 2p \cdot P) \cdot q + b^2 Q \end{pmatrix}.$$

And define

$$M := \{ \begin{pmatrix} Q \\ P \end{pmatrix} \in k^6 \mid q \cdot Q = 0 \text{ and } q \cdot P + p \cdot Q = 0 \}$$

Lemma 4 (1) *∂ maps M into itself.*
(2) *∂ respects the symplectic form, i.e.,*

$$\langle \partial \begin{pmatrix} Q \\ P \end{pmatrix}, \begin{pmatrix} \tilde{Q} \\ \tilde{P} \end{pmatrix} \rangle + \langle \begin{pmatrix} Q \\ P \end{pmatrix}, \partial \begin{pmatrix} \tilde{Q} \\ \tilde{P} \end{pmatrix} \rangle = \langle \begin{pmatrix} Q \\ P \end{pmatrix}, \begin{pmatrix} \tilde{Q} \\ \tilde{P} \end{pmatrix} \rangle'$$

(3) *The vector* $\begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ \dot{p}_1 \\ \dot{p}_2 \\ \dot{p}_3 \end{pmatrix}$ *lies in the kernel of ∂.*

Proof. Straightforward calculations. □

We introduce three vectors v_1, v_2, v_0 by the formulas

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ b^2 q_1 \\ 0 \\ -b \\ -2b^3 q_2 \end{pmatrix} \quad v_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ q_2 q_1^{-1} \\ (b^2 - b^{-2}) q_1^{-1} \end{pmatrix} \quad v_0 = \begin{pmatrix} q_2 \\ -q_1 \\ 0 \\ b q_1 \\ b q_2 \\ 0 \end{pmatrix}.$$

The vector v_0 is in fact equal to the vector of part (3) of the lemma (up to a constant).

Lemma 5 *v_1, v_2, v_0 is a basis of $\tilde{M} := \{m \in M \mid \langle m, v_0 \rangle = 0\}$. Moreover, $\langle v_1, v_2 \rangle = b^4$.*

Proof. Straightforward calculations. □

The normal variational equation
is (according to the literature) equal to the differential module $N := \tilde{M}/kv_0$. We note that N inherits from $\tilde{M}$ a symplectic structure, that ∂ on N commutes with this structure.

Lemma 6 *The operator ∂ on N has w.r.t. the basis of N, consisting of the images of v_1, v_2, the form*

$$\frac{d}{dt} - \begin{pmatrix} -bq_2q_1^{-1} & (1-b^{-4})q_1^{-2} \\ 3b^2q_1^2 - 2b^2 & bq_2q_1^{-1} \end{pmatrix}$$

Proof. Straightforward computation. □

The differential field that we want to work with is the function field of S, in this case the field $\mathbf{C}(s)$, derived from the parametrization of the circle. The formulas are:

$$q_1 = \sqrt{1-b^{-4}}\cos bt = \sqrt{1-b^{-4}}\frac{2s}{s^2+1},$$

$$q_2 = \sqrt{1-b^{-4}}\sin bt = \sqrt{1-b^{-4}}\frac{s^2-1}{s^2+1},$$

$$\frac{d}{dt} = \frac{b(s^2+1)}{2}\frac{d}{ds}.$$

The matrix differential equation associated to N over $\mathbf{C}(s)$ reads:

$$\frac{d}{ds} - \begin{pmatrix} \frac{1-s^2}{s(s^2+1)} & \frac{(s^2+1)}{2bs^2} \\ \frac{24(b-b^{-3})s^2}{(s^2+1)^3} - \frac{4b}{s^2+1} & \frac{s^2-1}{s(s^2+1)} \end{pmatrix}$$

Conjugation with a constant matrix $\begin{pmatrix} b^{-1/2} & 0 \\ 0 & b^{1/2} \end{pmatrix}$ improves the above somewhat:

$$\frac{d}{ds} - \begin{pmatrix} \frac{1-s^2}{s(s^2+1)} & \frac{(s^2+1)}{2s^2} \\ \frac{24(1-b^{-4})s^2}{(s^2+1)^3} - \frac{4}{s^2+1} & \frac{s^2-1}{s(s^2+1)} \end{pmatrix}$$

Again conjugation, now with the matrix $\begin{pmatrix} s^2+1 & 0 \\ 0 & (s^2+1)^{-1} \end{pmatrix}$ leads to a simpler matrix

$$\frac{d}{ds} - \begin{pmatrix} \frac{1-3s^2}{s(s^2+1)} & \frac{1}{2s^2(s^2+1)} \\ \frac{24(1-b^{-4})s^2}{(s^2+1)} - \frac{4(s^2+1)}{1} & \frac{3s^2-1}{s(s^2+1)} \end{pmatrix}$$

For a matrix differential equation $\frac{d}{ds} - \begin{pmatrix} a_1 & a_2 \\ a_3 & -a_1 \end{pmatrix}$ one computes the following differential operator

$$\partial^2 - \frac{a_3'}{a_3}\partial + (a_1' - a_1^2 - a_2 a_3 - a_1 \frac{a_3'}{a_3}).$$

A calculation of this shows that the only singular points are $\pm i, \infty$. These points are regular singular and for general values of b, the local exponents are not rational. Hence each local Galois group is $\mathbf{G}_m$ and the global Galois group is connected. If the global Galois group is commutative, then the above differential operator should factor in two ways. MAPLE does not give this answer, the programm only gives one factoriation. Looking at the formal solutions at the sigularities we find the above Galois group.

4 More calculations on the spherical pendulum

Again we have the given Hamiltonian $H := \frac{1}{2}||p||^2 - \Gamma \cdot q$ with $\Gamma = (0, 0, -1)$ and the restrictions $||q||^2 = 1$ and $p \cdot q = 0$. The Hamiltonian system is:

$$\begin{cases} \dot{q} = p \\ \dot{p} = \Gamma - (q \cdot \Gamma + ||p||^2) q \end{cases}$$

For the linearization we choose the solutions laying in a vertical plain through the unstable equilibrium point i.e. the point $(0,0,1)$. This integral curve is given by

$$\begin{cases} q(t) = (\cos f(t), 0, \sin f(t)) \\ p(t) = (-f'(t)\sin f(t), 0, f'(t)\cos f(t)) \end{cases}$$

The choice of $q_2 = 0$ is arbitrary, we could also have taken $q_1 = 0$ such that $p_2 = 0$ would follow. $f(t)$ is still an unknown function. Differentiating $p(t)$ again we find an expression for $\dot{p}(t)$

$$\dot{p}(t) = (-f''(t)\sin f(t) - f'(t)^2\cos f(t), 0, f''(t)\cos f(t) - f'(t)^2\sin f(t)).$$

Substituting these equation in the Hamiltonian system we obtain a second expression for $\dot{p}(t)$, symplifying the 2 expressions we obtain

$$\begin{cases} \dot{p} = (f''(t), 0, f''(t)) \quad \text{and} \\ \dot{p} = (-\cos f(t), 0, -\cos f(t)) \end{cases}$$

We can now solve for $f(t)$ with MAPLE and we find the solutions

$$\int^{f(t)} \frac{-1}{\sqrt{2\sin(a) + C1}} da - t - C2 = 0, \quad \int^{f(t)} \frac{1}{\sqrt{2\sin(a) + C1}} da - t - C2 = 0.$$

These solutions do not help us finding formal solution of our problem but, we do not need them to find the normal variational equation. The variational equation along these curves has to satisfy all the condition form the previous paragraph $||q + \epsilon Q||^2 = 1$ and $(p + \epsilon P) \cdot (q + \epsilon Q) = 0$. We will linearize the equation along the solution curves $q + \epsilon Q, p + \epsilon P$ with $\epsilon^2 = 0$.

Lemma 7 *The linearized system is*

$$\begin{cases} \dot{Q} = P \\ \dot{P} = -(||p||^2 - q_3)Q + 3qQ_3 \end{cases}$$

and define ∂ *by*

$$\partial \begin{pmatrix} Q \\ P \end{pmatrix} = \begin{pmatrix} \dot{Q} - P \\ \dot{P} + (||p||^2 - q_3)Q - 3qQ_3 \end{pmatrix}$$

$Q, P \in k^3$.

Proof. Straightforward calculation and using the rule $p \cdot P = -Q_3$ which finds its origin in that the Hamiltonian must be constant on all solution curves. So substituting $q+\epsilon Q, p+\epsilon P$ in $H(q,p) = \sum \frac{1}{2}||p_i||^2 + q_3 = \sum \frac{1}{2}(p_1 + \epsilon P_1)^2 + q_3 + \epsilon Q_3$ we find $\sum \frac{1}{2}p_i^2 + q_3 + \epsilon \sum p_i P_i + Q_3$. The whole last summation must be 0 producing $p \cdot P = -Q_3$. □

We introduce three new vectors v_0, v_1, v_2 by the formulas

$$v_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ q_1 \\ 0 \\ q_3 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -q_1 \\ 0 \\ -q_3 \\ -f'(t)q_3 \\ 0 \\ f'(t)q_1 \end{pmatrix}, \quad v_0 = \begin{pmatrix} -f'(t)q_3 \\ 0 \\ f'(t)q_1 \\ -f''(t)q_3 - f'(t)^2 q_1 \\ 0 \\ f''(t)q_1 - f'(t)^2 q_3 \end{pmatrix}$$

Lemma 8 *The vector v_0 is equal to the vector of part (3) of lemma 4 in the previous section, $\langle v_0, v_1 \rangle = \langle v_0, v_2 \rangle = 0$ and $\langle v_1, v_2 \rangle = 1$ this turns v_1, v_2, v_0 into a basic of $\tilde{M}$.*

Proof. Straightforward computations with the symplectic form on k^6 and the ∂ operator. □

Lemma 9 *The operator ∂ on N has the form*

$$\frac{d}{dt} - \begin{pmatrix} 0 & 2q_3 + f'(t)^2 \\ -1 & 0 \end{pmatrix}$$

Proof. Straightforward calculations. The normal variational equation is obtainted by expressing $\partial(v_1)$ and $\partial(v_2)$ as linear combinations of v_1 and v_2 modulo kv_0. □

This expression agrees (almost) with the normal variational equation given by Churchill in [4]. The computation of the differential Galois group is still impossible because of the form of $f(t)$. If we fix the total energy of the pendulum to 1 the described solution becomes homoclinic to the unstable equilibrium point. Substituting $p(t)$ and Γ into the Hamiltonian we obtain $f'(t)^2 + 2q_3 = 2$. Now ∂ can be written as

$$\frac{d}{dt} - \begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix}.$$

Now that we have a constant matrix we can compute the differential Galois group. Computing the eigenvalues $\pm\sqrt{-2}$ of the matrix and diagonalizing it, the differential solution of the operator becomes

$$\frac{d}{dt} - \begin{pmatrix} \sqrt{-2} & 0 \\ 0 & -\sqrt{-2} \end{pmatrix}.$$

So to include all the solutions of the operator in the differential field we extend $\mathbf{C}(t)$ to $\mathbf{C}(t, \sqrt{-2}t)$. The differential Galois group of this Picard-Vessiot extension is $\mathbf{C}^* = \mathbf{G}_m$. This is in accordance with the Morales-Ramis theorem that states that the identity component of the differential Galois group is abelian. We note what the time parameter is not a parametrization of the circle. In the next section we will use the parametrization of the circle.

5 Different approach

If we first parametrize the solution curves by

$$q(t) = (\sin f(t), 0, \cos f(t)) = (\frac{2s}{s^2+1}, 0, \frac{s^2-1}{s^2+1})$$

we need to find the jacobian $O(s)$ of the coordinate transformation

$$\frac{d}{dt}\sin f(t) = O(s)\frac{d}{ds}\frac{2s}{s^2+1}.$$

Applying the right substitutions and calculations we find $O(s) = -(s^2+1)^{\frac{1}{2}}$ and $\frac{d}{dt} = -(s^2+1)^{\frac{1}{2}}\frac{d}{ds}$. The matrix differential equation associated to N over the field $\mathbf{C}(s)$ is

$$\frac{d}{ds} - (s^2+1)^{-\frac{1}{2}} \begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix} \quad \text{or equivalently}$$

$$\frac{d}{ds} - (s^2+1)^{-\frac{1}{2}} \begin{pmatrix} \sqrt{-2} & 0 \\ 0 & -\sqrt{-2} \end{pmatrix}.$$

This is in fact a differential equation over $\mathbf{C}(s)$. In order to study its differential Galois group we will work over the field $\mathbf{C}(s, \sqrt{s^2+1}) = \mathbf{C}(u)$ with $\sqrt{s^2+1} = y$ such what we have $y^2 - s^2 = 1$. We will write $y = \frac{2u}{u^2+1}$, $s = \frac{i(u^2-1)}{u^2+1}$. So we have to find the coordinate transform $k(s)$ in $\frac{d}{ds} = k(s)\frac{d}{du}$.

Rigorous calculation reveals that $k(s) = \frac{-i(u^2+2)^2}{4u}$, transforming the operator in

$$\frac{i}{2}(u^2+1)\frac{d}{du} - \begin{pmatrix} i\sqrt{2} & 0 \\ 0 & -i\sqrt{2} \end{pmatrix} \quad \text{or equivalently}$$

$$\frac{d}{du}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{(u^2+1)}\sqrt{2} & 0 \\ 0 & \frac{-1}{(u^2+1)}\sqrt{2} \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

The differential Galois group is $\mathbf{G}_m$ which is easy to check from that the residues of $\frac{1}{(u^2+1)\sqrt{2}}du$ at $u = i$ equals $\frac{1}{2i\sqrt{2}} \notin \mathbf{Q}$
We have now obtained the cascade of field extensions $\mathbf{C}(s) \subset \mathbf{C}(u) \subset \mathbf{C}(u, y_1)$. The Picard-Vissiot extension is $\mathbf{C}(u, y_1)/\mathbf{C}(s)$ and its differential Galois group is $\mathbf{D}_\infty^{\mathbf{SL}_2} \subset \mathbf{SL}_2$. The identity components of $\mathbf{D}_\infty^{\mathbf{SL}_2}$ is again abelian.

6 Last remarks

We have verified the theorem of Morales-Ramis for the spherical pendulum by showing that the differential Galois group of the normal variational equation is always abelian. This is of course what we expected because we already know that the spherical pendulum is completely integral. Note that the differential Galois group depends on the solution curves we perform the initial linearization along and also its parametrization as can be see in the last section.

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