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Mathematical Brass

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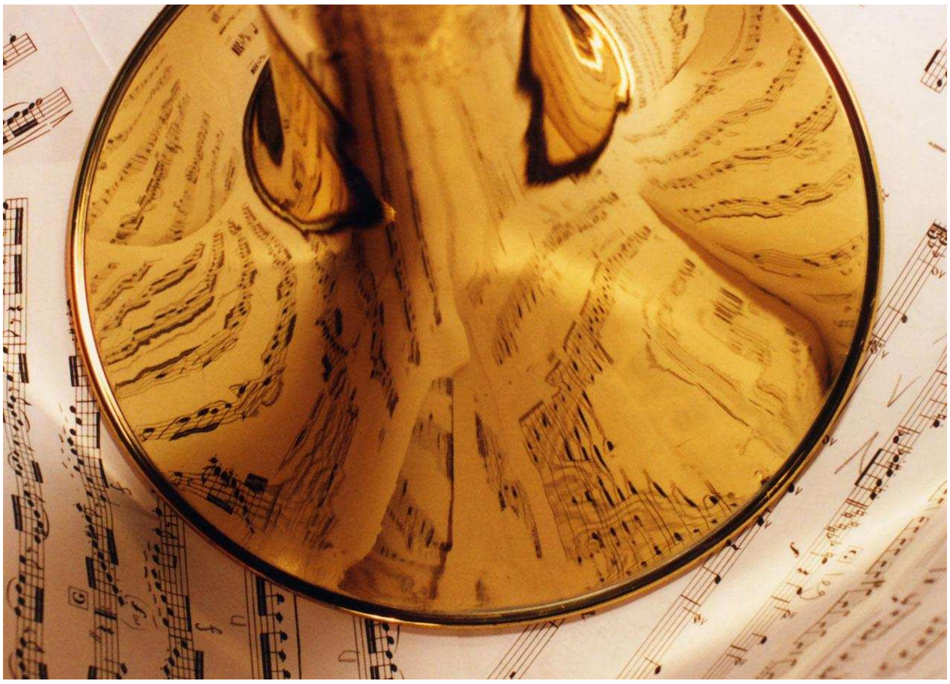

Bachelor Thesis in Applied Mathematics

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Chapter 1

INTRODUCTION

Nowadays, you just can't imagine a society without music. Everybody knows the background music in shopping areas and on television. Also, many people of many different ages play instruments for fun or to earn their living.
These musicians use different kinds of instruments, we can divide those into three important groups: strings, wind and percussion instruments. During this research we'll try to find out several things about brass instruments. In particular we'll try to answer the following questions:

- What physical process describes the production of sound waves by brass instruments?
- Which simple representation can we use to model brass instruments and what are the differences with real instruments?
- What are the solutions of the wave equation analytically and via a Finite Element Method? How do these solutions differ from reality?

When a musician plays his instrument, for example the trombone, people nearby the musician can hear the sound. This is because the trombonist creates sound waves with his instrument. Those waves travel through the air and everybody nearby can hear the sound. At this point a question arises: what physical process creates those sound waves and what's the main function of the instrument during this process? This is the first question we try to answer with this paper. This question will be answered in chapter 2, by examining the most important parts of brass instruments.
A simple representation of a wind instrument is just a tube. From high school physics we already know that some waves are able to travel through such a straight, cylindrical or conical tube. The selection of those waves strongly depends on the length of the tube. Musicians influence the pitch of a tone by changing the length of the tube, this is obvious when you see a trombonist playing. In this case the frequency directly depends on the length of the tube.

We also know from high school that a certain tube has a fundamental tone and several overtones or harmonics. Wind musicians can use this physical aspect by tensing their lips in different ways. By using some theory about those fundamental tones and corresponding harmonics we can calculate the frequencies of these tones. But, of course wind instruments differ from straight cylindrical or conical tubes. Most instruments obviously have bends and a bell. In chapter 3 we'll find out which tube gives the best representation of brass instruments. In chapters 4 and 5 we'll use this representation to do some experiments with a Finite Element Method.

In chapter 3 we'll also study the wave equation. The wave equation is a partial differential equation which has solutions describing a wave. By analytically solving this equation with several special boundary conditions the solution for a sound wave through a straight cylindrical tube can be found. This is the same equation we solved in the last chapters by using a Finite Element Method.

Chapter 2

PARTS OF BRASS INSTRUMENTS AND THEIR PHYSICS

When a musician plays an instrument, for example the trombone or the trumpet, people nearby can hear the created sound. Hereto, the musician creates sound waves with his instrument. Those waves travel through the air and reach the audience. At this point a question arises: what physical process causes these sound waves and what's the main function of the instrument during this process? We'll answer this question by examining four important topics: the lips, the mouthpiece, the tube and the bell.

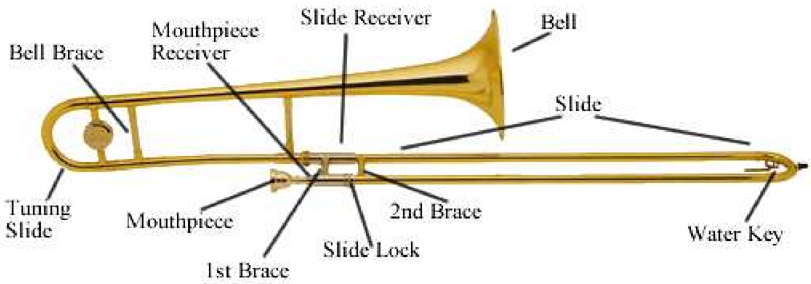

Figure 2.1: The most important parts of the trombone.

2.1 Lips

Let's first have a look at the lips. Brass players are able to create sound with just their lips. They can do this by closing their mouth and pulling their lips back, what causes a kind of tension in the lips. At that point the air in the mouth is at a pressure above atmospheric. When they start blowing, the air rush opens the lips, but because of the tension the lips also want to close again. This combination of tension and air flow causes an oscillation which provides a sound. These sounds by themselves don't have any musical value, but by blowing into an instrument these oscillations can be forced into a fixed shape and that will create a useful sound.

2.2 Mouthpiece

The first part of the instrument is called the mouthpiece. The mouthpiece has a rounded section that comfortably fits against the lips, an enclosed volume (cup volume) of air, a narrow constriction (throat) and a conical part that widens out towards the bore of the instrument, as can be seen in the figure below.

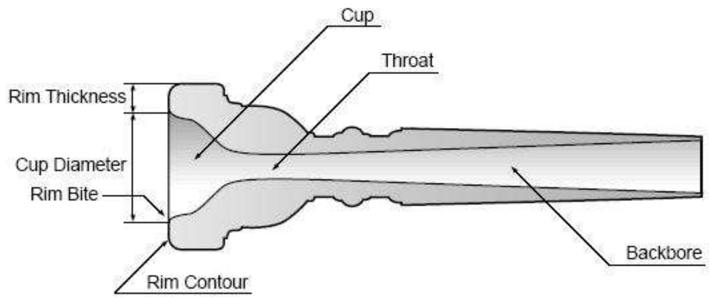

Figure 2.2: Design of a mouthpiece.

The mouthpiece, although very small, seems to be a very important part of the instrument. Inside the mouthpiece the greatest sound pressure is created and therefore its acoustic qualities - dependent on for example rim thickness and shape - are very important.

Figure 2.3: Mouthpieces of different brass instruments.

The mouthpiece has several effects. Of course it allows the musician to connect the instrument to a large section of the lips (this is important, because the lips create the vibration). Another important effect concerns the resonances of the instrument. The narrowest part of the mouthpiece, and therefore the narrowest part of the whole instrument, is the throat of the mouthpiece (see figure 2.2). At that point the sound pressure reaches a maximum, and that's why this part has such a great influence on the resonances, which determine the timbre of the instrument. Sound waves with high frequencies propagate much easier through a long, narrow throat than through a short, wider throat. So, in general a long narrow throat creates

a sound which has a relatively small spectrum with many high harmonics and is therefore brighter.
Finally we'll briefly discuss the effect of differences in cup volume (see figure 2.2). The inner shape of the mouthpiece is very important with respect to the cup volume. In general a larger cup volume (often accompanied by a wider throat) leads to more pressure differences throughout the instrument, so louder tones can be produced. On the other hand, with a smaller cup volume it's easier to play in the high register.

2.3 Tube

In general, the tube is the largest part of the instrument. Every tube can produce several different tones, but definitely not all tones. Or, when we state this different: every tube has several eigenfrequencies. With the lips or the mouthpiece we can generate almost every frequency, but when the tube is added, the number of possible sound waves decreases immediately. So, the tube has a sort of selective function. This effect of the tube will be discussed extensively in chapter 3.

2.4 Bell

Last but not least: the bell. The bell actually has two different effects. First of all, in front of the instrument's bell there's a conical section. At this point the flared section ends in a bell. Because of this conical section the set of harmonics (overtones) is different from the set of harmonics according to a straight cylindrical tube. This effect will also be discussed in section 3.
Another important effect of the bell is that the short waves (according to high frequencies) are better able to travel into the rapidly widening bell. This phenomenon is described by the effective length of the instrument (Pyle, 1975). The effective length of the instrument changes with frequency: the effective length increases as frequency increases. Figure 2.4, from an article of R.J. Pyle Jr. in the Journal of the Acoustical Society of America, shows this effect by measured data. The measurements are taken from two instruments (french horns) with a bell and different bore. Without the attachment of a bell the points should form a straight, horizontal line. Apparently after the attachment of a bell, the effective length starts to increase if frequency increases. The same article describes an analytic way to derive the relation between effective length and frequency. The further these high frequencies go into the bell, the easier it is for them to escape into the outside air.

So, the higher frequencies are more efficiently radiated as sound and that's why we can hear these overtones much better from instruments with a large bell. This phenomenon causes the bright sound of brass instruments: a large bell prevents a strong decrease of power of the higher harmonics.

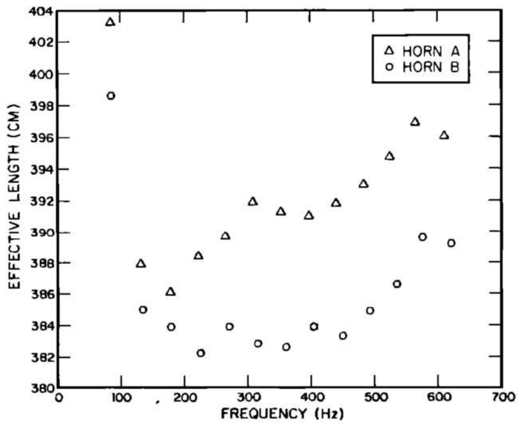

Figure 2.4: Effective length of the French horns in F, determined from measured resonance frequencies. Horn A has a large bore, Horn B a medium bore.

Because of the addition of a bell we can hear more harmonics. An increasing number of audible frequencies creates a louder tone, as can be seen in the Figure 2.5. Brass instruments are normally built with a large bell, which makes brass players able to play very loud.

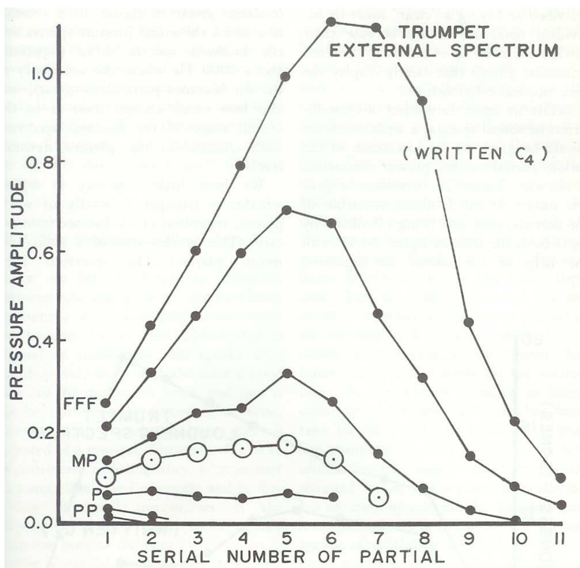

Figure 2.5: External spectra of a trumpet.

Note that the abbreviations in figure 2.5 are the same as used by musicians:

pp : pianissimo, very soft
p : piano, soft
mp : mezzo-piano, moderately quiet
mf : mezzo-forte, moderately loud
f : forte, loud
ff : fortissimo, very loud
fff : fortississimo, extremely loud

Chapter 3

WAVES FOR SIMPLE GEOMETRIES

To examine properties of brass instruments we should use a simple model of the instruments. In this chapter we'll review three different models to see which one suits best:

- A half-open tube model; at first sight this seems plausible, after all it's obvious that the bell end is open and we could model the mouthpiece end as a closed end. But, later on we'll see why this representation doesn't satisfy. From now we'll refer to this model as the *half-open model*.
- A conical tube model; most brass instrument consist of a cylindrical and conical part, so modelling with a conical tube is also a possibility we should discuss. But, unfortunately this leads to a little bit more complicated calculations. From now we'll refer to this model as the *conical model*.
- An open tube model; at first sight probably a non-trivial choice to model brass instruments. But, this representation appears to have some of the same properties as the conical model and can be used with less complicated calculations. From now we'll refer to this model as the *open model*.

Section 3.1 describes all three models by using some high school techniques. After that, section 3.2 will derive the wave equation and the last section of this chapter will deal with the latter two models from an analytic point of view.

3.1 Sound waves through straight tubes

Tubes resonate at harmonically related frequencies. Typical aspects of standing sound waves are the amplitude (A) and wavelength (λ).
The wavelength is directly dependent on the dimensions of the tube, especially its length. When we examine a tube which has a closed and open end, called a half-open tube, we find two boundary conditions: there has to be a nod at the closed end and an antinod at the open end.

Further explanation about these boundary conditions can be found in section 3.3 (although from figures 3.1 and 3.3 the latter condition can be deduced intuitively). These conditions cause a strong decrease of the number of possible waves travelling through the tube.

The largest possible wavelength will be four times the length of the tube (see figure 3.1). This wavelength $\lambda = 4L$ is the wavelength corresponding to the fundamental tone. Figure 3.1 also shows the first three harmonics, having wavelengths $\lambda = \frac{4}{3}L$, $\lambda = \frac{4}{5}L$ and $\lambda = \frac{4}{7}L$. In general, every possible wavelength can be written as

$$\lambda = \tfrac{4}{(2k+1)}L \quad \text{for some } k \in \mathbb{N}.$$

Using these wavelengths we can calculate the frequencies of the fundamental tone and the first four harmonics. Sound waves normally travel with velocity $c = 343$ meters/second (under conditions of normal pressure $P_0 = 1,01325 \cdot 10^5$ Pa and temperature 20° Celsius). The period T of the wave can be found by calculating λ/c. The reciprocal of T gives the frequency f, which satisfies

$$f = \frac{c}{\lambda}.$$

	λ (m)	P (s)	f (Hz)
Fundamental tone	12	$3,499 \cdot 10^{-2}$	28, 58
First harmonic	4	$1,166 \cdot 10^{-2}$	85, 75
Second harmonic	2, 4	$6,997 \cdot 10^{-3}$	142, 92
Third harmonic	1, 71	$4,998 \cdot 10^{-3}$	200, 08
Fourth harmonic	1, 33	$3,887 \cdot 10^{-3}$	257, 25

Table 3.1: Wavelengths (λ), periods (P) and frequencies (f) for the half-open tube.

Table 3.1 and figure 3.1 give wavelengths, periods and frequencies for the fundamental tone and a number of harmonics through a straight, half-open tube with a length of 3 meters.

When carefully examining this table, we note that the frequency of the first harmonic is about 3 times the frequency of the fundamental tone. This means that the distance between the fundamental tone and the first harmonic is an octave plus a fifth. However, we do know about brass instruments that the distance between the fundamental tone and the first harmonic should just be one octave, hence a factor two between the frequencies. This is why the half-open model doesn't satisfy our purpose.

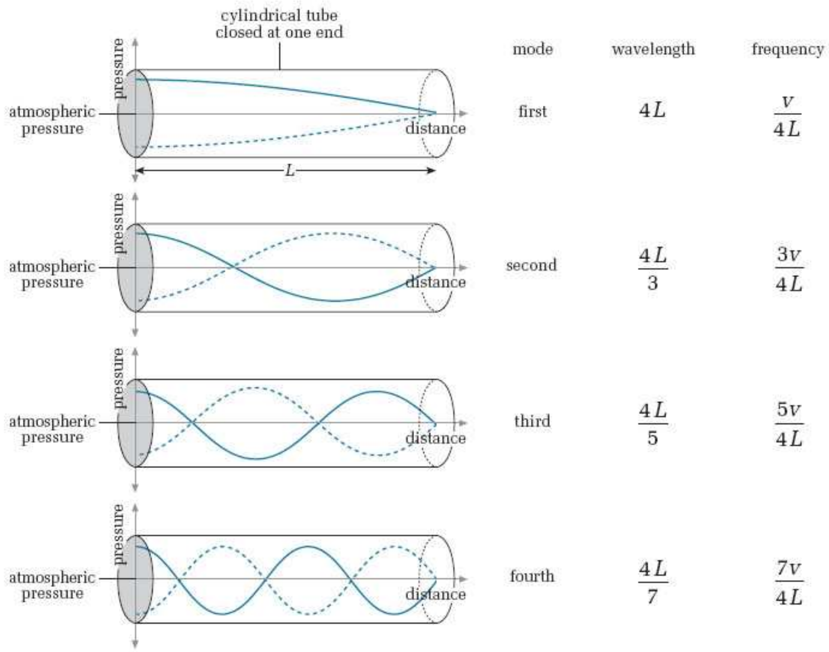

Figure 3.1: Standing waves through a half-open tube.

As said before, the second model is about a straight, conical tube. In high school, this isn't really one of the topics teachers explain. But, the idea behind these conical tubes appears to be the same as behind cylindrical open tubes. Figure 3.2 illustrates the kind of pressure waves corresponding to a conical tube. Again we just have to use the fact that there'll be an antinod at the open end. Furthermore, as we expect, there is a pressure maximum or minimum at the closed end (the top of the cone). The only difference with an open tube is a decrease in the amplitude of the pressure function. For our simple representation we won't investigate this property of the conical tube.

Thus, apparently conical tubes and open tubes have the same eigenfrequencies, so we can use our knowledge about open tubes. In the latter case, the largest wavelength will be twice the length of the tube (see figure 3.3). This wavelength is the one corresponding to the fundamental tone. The figure also shows the first three harmonics in this case, having wavelengths $\lambda = L$, $\lambda = \frac{2}{3}L$,$\lambda = \frac{2}{4}L$ and $\lambda = \frac{2}{5}L$. In general, every possible wavelength can be written as

$$\lambda = \tfrac{2}{k}L \text{ for some } k \in \mathbb{N}.$$

As in the first case, from these wavelengths we can calculate the frequencies of the fundamental tone and the harmonics by the formula

$$f = \frac{c}{\lambda} = \frac{cn}{2L} \tag{3.1}$$

with $n \in \mathbb{N}$ and $c = 343$ meters/second.

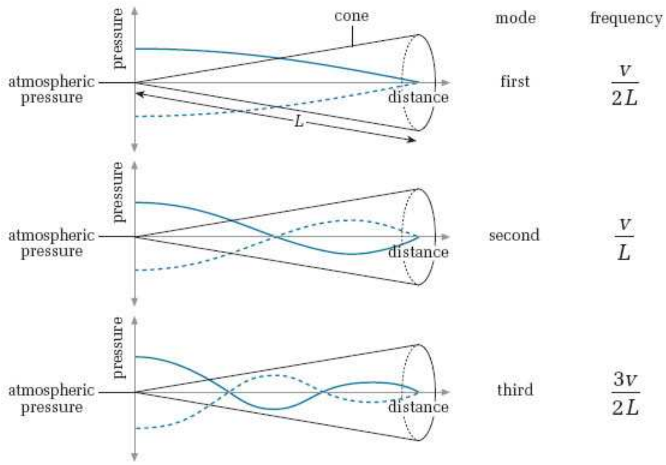

Figure 3.2: Standing waves through a conical tube.

	λ (m)	P (s)	f (Hz)
Fundamental tone	6	$1,749 \cdot 10^{-2}$	$57,17$
First harmonic	3	$8,746 \cdot 10^{-3}$	$114,33$
Second harmonic	2	$5,831 \cdot 10^{-3}$	$171,50$
Third harmonic	$1,5$	$4,373 \cdot 10^{-3}$	$228,66$
Fourth harmonic	$1,2$	$3,499 \cdot 10^{-3}$	$285,833$

Table 3.2: Wavelengths (λ), periods (P) and frequencies (f) for the open tube.

Table 3.2 and figure 3.3 give wavelengths, periods and frequencies for the fundamental tone and a number harmonics through a straight tube with a length of 3 meters. This time we note that the frequency corresponding to the first harmonic is twice the frequency of the fundamental tone. This is the same as for brass instruments, so that's why we prefer using the open model to represent the instruments.

Note that, because we're just interested in the frequencies of the different harmonics, we can also use a tube which has both ends closed. Such a tube has the same set of harmonics (later on we'll call these harmonics the eigenfrequencies) as an open tube. Although this idea of using a closed tube seems rather unusual, it really isn't that strange.
As said before, it seems plausible to model the instrument's end with the mouthpiece by a closed end. Also, from literature (as can be found in *Fundamentals of Musical Acoustics*, Arthur H. Benade) we know that sound waves travel a lot better through a narrow instrument than through a huge surrounding space. This is caused by a huge difference in acoustical impedance between the instrument and the space around the instrument. This acoustical impedance is comparable with electrical impedance. Where electrical impedance gives information about the effect of a force on a signal, the acoustical impedance gives information about the ability of sound waves to travel through a certain geometry. Now, because of the huge difference in impedance between the instrument and its surrounding space the sound waves reflect almost totally at the end of the instrument. Normally these reflected

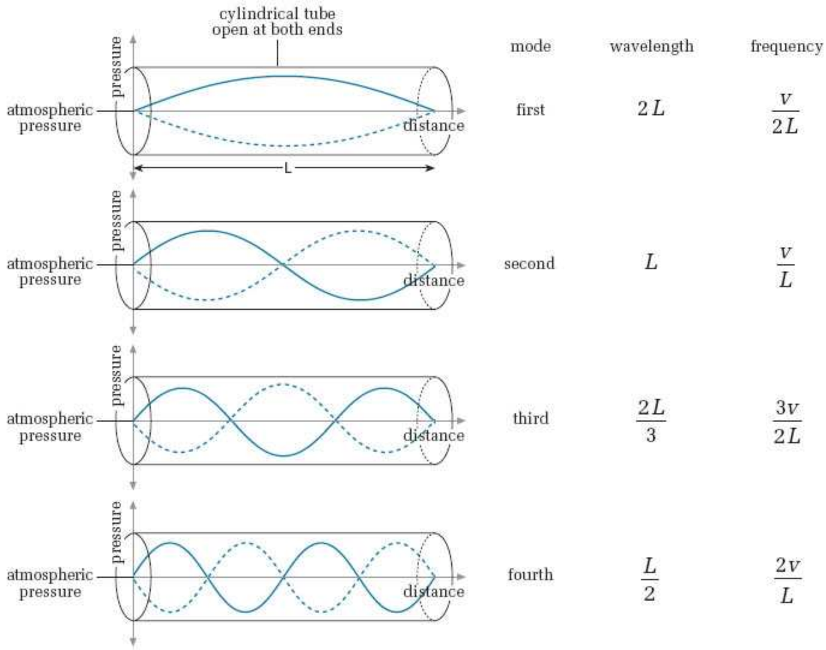

Figure 3.3: Standing waves through an open tube.

waves resonate with waves coming from the mouthpiece. Then, the amplitude of the wave increases and every time such an enforced wave reaches the end of the tube a little bit escapes to the outside air, this creates the sound we can hear. But, as said before the major part of the waves reaching the end is reflected and starts to build up an eigenfrequency. This phenomenon explains why modelling by a closed tube to get an approximation for the eigenfrequencies isn't that strange. Note that, we can only use this representation to approximate the eigenfrequencies, after all a closed tube won't create any sound. At the end of the next chapter we'll use the above idea again.

3.2 The wave equation: derivation

The wave equation is a partial differential equation which has solutions describing a wave. In various areas of science, such as electromagnetics, fluid dynamics and of course the study of acoustics, you'll find the wave equation. By using the right boundary conditions we are able to find solutions for our case. In general the wave equation is given by

$$\frac{\partial^2 P}{\partial t^2} = c^2 \Delta P \tag{3.2}$$

with P and c representing the pressure and speed of the wave. This differential equation can be found by carefully studying three important equations: the continuity equation

$$\rho_t + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{3.3}$$

Euler's equations of motion

$$\mathbf{v}_t + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla P + F \tag{3.4}$$

and the relation between the air pressure P and density ρ of the medium in an adiabatic process (which is denoted by s)

$$\frac{1}{c^2} = (\frac{\partial \rho}{\partial P})_S. \tag{3.5}$$

Here, $\mathbf{v}$ is the fluid velocity and F an external force. The ∇-operator is called the *nabla*-operator and is defined by

$$\nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, ..., \frac{\partial}{\partial x_n})$$

and thus, the operator ∇^2 represents the same as the *Laplace*-operator:

$$\nabla^2 = \Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}.$$

Assume $\mathbf{v}$ to be very small, as is the case with sound waves, and ignore all terms depending on $\mathbf{v}$ itself. Notice that $\mathbf{v}$ very is small doesn't imply $\mathbf{v}_t$ is very small and, since we don't have any external influences, F should be zero. Hence, equation (3.4) reduces to

$$\rho \mathbf{v}_t = -\nabla P. \tag{3.6}$$

Now, we use equation (3.5) to rewrite ρ_t:

$$\rho_t = (\frac{\partial \rho}{\partial P})_S \cdot P_t = \frac{1}{c^2} P_t,$$

and use the continuity equation

$$\frac{1}{c^2} P_t + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{3.7}$$

which, after differentiating with respect to t, leads to

$$\frac{1}{c^2} \rho_{tt} + \nabla(\rho_t \mathbf{v} + \mathbf{v}_t \rho) = 0. \tag{3.8}$$

Again using the assumption that $\mathbf{v}$ is very small and after substitution of equation (3.6) we find the wave equation

$$P_{tt} = c^2 \Delta P. \tag{3.9}$$

3.3 The wave equation: analytical solutions

In this subsection we'll derive the analytic solution for the wave equation through an open, straight tube and a conical tube. To simplify notation we use spherical coordinates in the latter case.

3.3.1 The wave equation: 1D

Let's first consider the open straight tube. To find the analytic solution for this case we just consider the 1D wave equation. In that case, the wave equation reads as

$$\frac{1}{c^2}\frac{\partial^2 P}{\partial t^2} = \frac{\partial^2 P}{\partial x^2} \tag{3.10}$$

General solutions of this differential equation are Bessel functions. For large r, however, the can be approximated as

$$P(x,t) = A\cos(\omega t + a)\sin(\kappa r + b), \tag{3.11}$$

where A represents the amplitude of the wave, κ and ω are related through $\omega = \kappa c$ (here κ is called the wave number) and a, b are arbitrary constants. The wavelength of this solution can be found by computing $\lambda = 2\pi\backslash\kappa$ and the frequency by computing $f = c\backslash\lambda = c\kappa\backslash 2\pi = \omega\backslash 2\pi$. At this point we have to use some boundary conditions to determine a particular solution. Useful boundary conditions can be found by examining the pressure. The pressure should be zero at the open ending, let this be at $x = 0$ and $x = L$, of the tube. Actually the pressure should be equal to the atmospheric pressure at the open ending, but to simplify the equations we use the following boundary conditions

$$P(0,t) = P(L,t) = 0 \ \ \forall t. \tag{3.12}$$

From the first condition we get

$$P(0,t) = A\cos(\omega t + a)\sin(b) = 0 \ \ \forall t,$$

hence $b = n\pi$, with $n \in \mathbb{N}$. Substituting the second boundary condition (with $n = 0$) leads to

$$P(L,t) = A\cos(\omega t + a)\sin(\kappa L) = 0 \ \ \forall t,$$

hence $\kappa L = m\pi$ with $m \in \mathbb{N}$. Thus, the solution now reads as

$$P(x,t) = A\cos(\frac{m\pi c}{L}t + a)\sin(\frac{m\pi}{L}x). \tag{3.13}$$

Finally, from general wave-theory we know that the a from equation (3.13) is the phase. The variable a represents the phase at $t = 0$, so we can easily choose $a = 0$:

$$P(x,t) = A\cos(\frac{m\pi c}{L}t)\sin(\frac{m\pi}{L}x). \tag{3.14}$$

The next figure shows the results for $A = 0.02$ meter, $c = 343$ meter/second, L is 3 meters, a is zero and $m = 1, 2, 3, 4, 5$ at $t = T_1 = \frac{\lambda}{c} = \frac{2L}{nc}$ for x from 0 to L. Note that we found exactly the same behaviour as shown in figure 3.3.

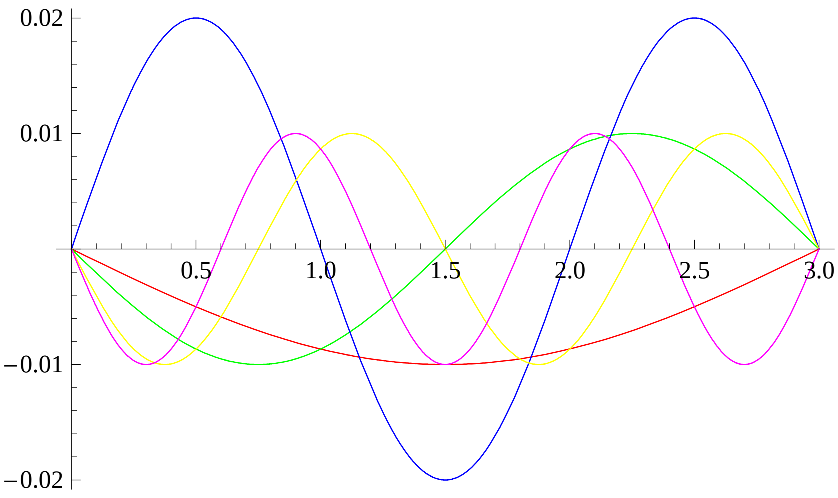

Figure 3.4: Results for $m = 1$ (red), $m = 2$ (green), $m = 3$ (blue), $m = 4$ (yellow) en $m = 5$ (purple).

3.3.2 Spherical wave

Now, we'll consider the conical tube and use spherical coordinates to derive a solution. Again we write the wave equation in a different, but now using r, θ and ϕ:

$$\frac{1}{c^2}\frac{\partial^2 P}{\partial t^2} = \frac{\partial^2 P}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 P}{\partial \theta^2} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 P}{\partial \phi^2} + \frac{2}{r}\frac{\partial P}{\partial r} + \frac{\cos\theta}{r^2\sin^2\theta}\frac{\partial P}{\partial \theta}$$

Again we assume the pressure wave only varies in the direction of r, where r is the distance from the closed end of the cone see figure 3.5.

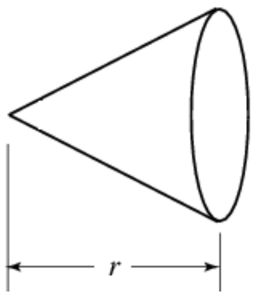

Figure 3.5: r represents the distance from the closed ending of the cone.

The previous equation simplifies to

$$\frac{1}{c^2}\frac{\partial^2 P}{\partial t^2} = \frac{\partial^2 P}{\partial r^2} + \frac{2}{r}\frac{\partial P}{\partial r} \tag{3.15}$$

We reformulate this equation a little bit by some standard techniques

$$0 = \frac{1}{c^2}\frac{\partial^2 P}{\partial t^2} - \frac{\partial^2 P}{\partial r^2} - \frac{2}{r}\frac{\partial P}{\partial r} \tag{3.16}$$

$$
\begin{aligned}
\Rightarrow 0 &= \frac{r}{c^2}\frac{\partial^2 P}{\partial t^2} - r\frac{\partial^2 P}{\partial r^2} - 2\frac{\partial P}{\partial r} \qquad (3.17)\\
&= \frac{r}{c^2}\frac{\partial^2 P}{\partial t^2} - (\frac{\delta P}{\delta r} + \frac{\delta P}{\delta r} + r\frac{\delta^2 P}{\delta r^2})\\
&= \frac{r}{c^2}\frac{\delta^2 P}{\delta t^2} - \frac{\delta}{\delta r}(P + r\frac{\delta P}{\delta r})\\
&= \frac{r}{c^2}\frac{\delta^2 P}{\delta t^2} - \frac{\delta}{\delta r}(\frac{\delta}{\delta r}rP)\\
\Rightarrow 0 &= \frac{\delta^2(rP)}{\delta t^2} - c^2\frac{\delta^2(rP)}{\delta r^2}
\end{aligned}
$$

Note that this formulation only holds for $r \neq 0$, because we multiplied by r in order to get (3.17). So, also in this case we can use the harmonic solution of the wave equation. First substitute $\Phi(r,t) = rP(r,t)$. The general harmonic solution for Φ is given by

$$\Phi(r,t) = \frac{A}{\kappa}\cos(\omega t + a)\sin(\kappa r + b). \qquad (3.18)$$

This non-trivial solution of the differential equation can be found by using a technique called separation of variables. The general solution implies

$$P(r,t) = \frac{A}{\kappa r}\cos(\omega t + a)\sin(\kappa r + b), \qquad (3.19)$$

and we're left with determining the constants ω and κ. As usual, to determine these constants we use some boundary conditions. As said before, the conical tube has an open end, where the pressure should be zero, and a closed end, where the velocity should be zero. At this closed end the pressure will reach a maximum or a minimum. Define the closed to be at $r = 0$ and the open end at $r = L$, then use the following boundary conditions

$$P(L,t) = 0 \ \ \forall t \qquad (3.20)$$

$$v(0,t) = 0 \ \ \forall t \qquad (3.21)$$

To implement to second boundary condition we use equation 3.7. Integration of this equation with respect to t yields

$$\mathbf{v} = \int_0^t -\frac{1}{\rho}\nabla P(\tau)\ \mathrm{d}\tau, \qquad (3.22)$$

which leads to

$$
\begin{aligned}
v(r,t) &= \int_0^t -\frac{1}{\rho}\frac{\partial P}{\partial r}(r,\tau)\ \mathrm{d}\tau\\
&= -\frac{1}{\rho}\frac{A\kappa r\cos(\kappa r + b) - A\sin(\kappa r + b)}{\kappa r^2}\int_0^t \cos(\omega\tau + a)\ \mathrm{d}\tau\\
&= -\frac{1}{\rho}\frac{A\kappa r\cos(\kappa r + b) - A\sin(\kappa r + b)}{\omega\kappa r^2}\cdot[\sin(\omega t + a) - \sin(a)]
\end{aligned}
$$

in spherical coordinates.

Using the first boundary condition we conclude that

$$v(0,t) = \lim_{r\to 0}\frac{1}{\rho}\frac{A\kappa r\cos(\kappa r + b) - A\sin(\kappa r + b)}{\omega\kappa r^2}\cdot[\sin(a) - \sin(\omega t + a)]$$

should be zero for all t, implying the condition

$$\lim_{r\to 0} \frac{A[\sin(a) - \sin(\omega t + a)][\kappa r \cos(\kappa r + b) - \sin(\kappa r + b)]}{\rho\omega\kappa r^2} = 0.$$

That means, we have to choose b such that

$$\lim_{r\to 0} \frac{A\kappa r[\sin(a) - \sin(\omega t + a)]\cos(\kappa r + b)}{\rho\omega\kappa r^2}$$
$$= \lim_{r\to 0} \frac{A[\sin(a) - \sin(\omega t + a)]\sin(\kappa r + b)}{\rho\omega\kappa r^2}.$$

Thus, with $\sin(a) = \hat{a}$

$$\lim_{r\to 0} \kappa r[\hat{a} - \sin(\omega t + a)]\cos(\kappa r + b) = \lim_{r\to 0} [\hat{a} - \sin(\omega t + a)]\sin(\kappa r + b),$$

which, for $r \to 0$, implies $\sin(b) = 0$, so $b = n\pi$ with $n \in \mathbb{N}$. Using this result, with $n = 0$ and equations (3.19) and (3.20), we get

$$p(L, t) = \frac{A}{\kappa L}\cos(\omega t + a)\sin(\kappa L) = 0 \;\; \forall t$$

and thus $\sin(\kappa L) = 0$. This implies $\kappa L = m\pi$ with $m \in \mathbb{N}$, so κ satisfies $\kappa = \frac{m\pi}{L}$ for any $m \in \mathbb{N}$ and thus for $P(r, t)$ we get

$$P(r, t) = \frac{AL}{mr\pi}\cos(\omega t + a)\sin(\frac{m\pi}{L}r) \;\; \forall m \in \mathbb{N}$$

and because $\omega = \kappa c = \frac{mc\pi}{L}$ our result becomes

$$P(r, t) = \frac{AL}{mr\pi}\cos(\frac{mc\pi}{L}t + a)\sin(\frac{m\pi}{L}r) \;\; \forall m \in \mathbb{N}. \tag{3.23}$$

Finally, there's just one unknown variable left. We just didn't have enough boundary conditions approximate a. Now, by the same reasoning as before we choose $a = 0$.

The next figure shows the results for $A = 0.02$ meter, $c = 343$ meters/second, L is 3 meters, a is zero and $m = 1, 2, 3, 4, 5$ at $t = T_1 = \frac{\lambda}{c} = \frac{2\pi}{1\pi c}$ for r from 0 to L. Note that we found exactly the same behaviour as shown in figure 3.2.

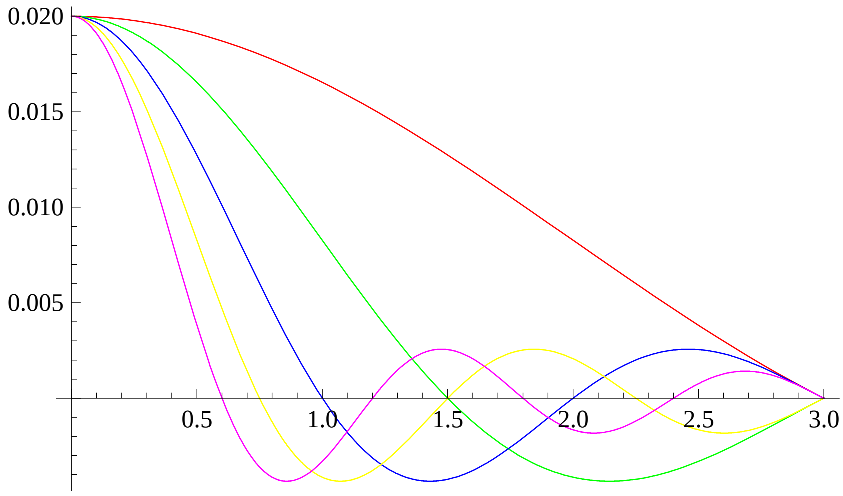

Figure 3.6: Results for $m = 1$ (red), $m = 2$ (green), $m = 3$ (blue), $m = 4$ (yellow) en $m = 5$ (purple).

Chapter 4

METHODS OF RESEARCH

This chapter consists of two short sections. The first describing a method to calculate the fundamental tone and harmonics for the trombone. This can be done for every slide position in a straightforward way. Later on, in the next chapter we'll see the measured eigenfrequencies of the real trombone. The second paragraph mainly includes the ideas for the experiments we'll do with a Finite Element Method. The results of these simulations can be found in the next chapter.

4.1 Fundamental tone and harmonics for the trombone

Which waves a straight cylindrical tube can produce is strongly related to the length of the tube. A very illustrative example is the trombone (see figure 2.1). The trombone is a brass instrument which mechanism is clearly based on the relation between frequency and length of the tube. The trombonist creates an oscillation with his lips, which causes a standing wave through the instrument. By changing the tension of his lips, the trombonist is able to use the different available eigenfrequencies of the tube.

This chapter explains the research we're going to do to find out the differences between the theory of waves through a straight, conical tube and the trombone. Trombonists divide their slide in seven positions (see figure 4.1), which is comparable to the trumpet. The trumpet has three valves (see figure 4.2): the first one is twice as long as the second and the third as long as the first and second together. The first valve lowers the pitch by one whole step, the second by a half step and the third by one and a half step. The next table shows there are seven possible combinations.

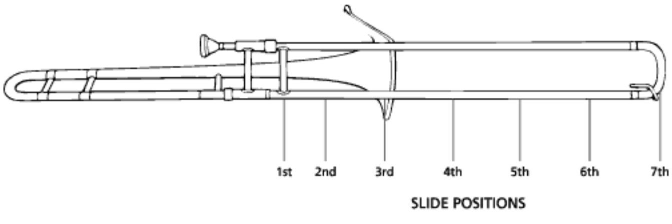

Figure 4.1: Seven positions on the slide of the trombone

Figure 4.2: The second and third valve are connected with extensions of respectively half and one and a half the length of the extension connected with the first valve.

Combination of valves	**Position**	**Result**	**Example**
None	First	None	B♭ or A♯
2	Second	Lowers a half step	A
1	Third	Lowers a whole step	A♭ or G♯
3 or 1-2	Fourth	Lowers one and a half step	G
2-3	Fifth	Lowers two whole steps	G♭ or F♯
1-3	Sixth	Lowers two and a half step	F
1-2-3	Seventh	Lowers three whole steps	E

Table 4.1: All possible positions and valve-combinations for the trombone and trumpet.

The research in this paper includes the following. The total length of the trombone is about $2,90$ meter if the slide is in the first position. Using the theory about fundamental tones and harmonics from the previous chapter we can calculate these tones. For that we use the relation between the frequency and the number of the partials given by equation 3.1:

$$f = \frac{cn}{2L}$$

Equivalently, we can do this for the second till the seventh position, we just have to use another length L. After this, we'll compare the calculated frequencies to the real ones by playing the trombone. Table 4.2 shows the theoretically calculated frequencies. For creating this table we assumed the trombone to be able to play a B♭ in tune by using the first position. This B♭ should have a frequency $f = 55 \cdot 2^{\frac{1}{12}} = 58,27$ Hz (assuming equal temperament). This implies the length of the trombone should be

$$L = \frac{cn}{2f} = \frac{343 \cdot 1}{2 \cdot 58,27} = 2,943 \text{ meter.}$$

From this length we calculated the fundamental tones and harmonics for the first position. We used a similar calculation to find out the length of the trombone using the second position and it's harmonics. The next table shows the results of these calculations.

	1st	**2nd**	**3rd**	**4th**	**5th**	**6th**	**7th**
Length (m)	2,943	3,118	3,304	3,500	3,708	3,929	4,162
Fundamental tone (Hz)	58,27	55	51,91	49,00	46,259	43,65	41,20
First harmonic (Hz)	116,54	110	103,83	98,00	92,50	87,31	82,41
Second harmonic (Hz)	174,81	165	155,74	147,00	138,75	130,90	123,61
Third harmonic (Hz)	233,08	220	207,65	196,00	185,00	174,61	164,81
Fourth harmonic (Hz)	291,35	275	259,57	195	231,25	218,27	206,02
Fifth harmonic (Hz)	349,62	330	311,48	294,00	277,50	261,92	247,22
Sixth harmonic (Hz)	407,89	385	363,39	343,00	323,75	305,58	288,42
Seventh harmonic (Hz)	466,16	440	415,31	392,00	369,99	349,23	329,63

Table 4.2: Theoretical lengths and frequencies for the seven positions of the trombone.

4.2 The wave equation: FEM

This section includes two short parts. First we'll see another derivation of equation 3.1. After that the ideas for the experiments with a Finite Element computer program will be shown. The frequencies corresponding to the fundamental tone and the harmonics of a certain tube are called the eigenfrequencies of the tube. So, when one creates a signal at the left end of the tube which has a frequency equal to the eigenfrequency this signal will be reinforced because of reflections. The idea behind this is that a sound wave travelling through the tube will be almost totally (later on we'll assume exact totally) reflected at the right end. This is by the same reasoning as in section 2.3, using the impedance. After this reflection the sound wave travels in opposite direction to the left end. At this end the sound wave reflects again, so again the wave travels forward. At that moment we have two waves travelling from the left tot the right: the wave which is twice reflected and a new wave which originates directly from the signal. You can imagine these two waves don't have to travel exactly in phase, but this should be the case when the two waves reinforce each other. This phenomenon restricts the number of possible sound wave through the tube which strengthen enormous. We can find these possible sound waves by simple reasoning.

Assume that the signal at the left end of the tube creates a wave with velocity c, which satisfies $c = \omega/\kappa$. Then, for a tube of length L, we can calculate the time Θ which is needed to travel from the left to the right. This will be given by $\Theta = L/c$. After this time Θ the wave will be (totally) reflected and start travelling back to the origin. Again it will take a time Θ before the wave reaches the origin. That means exactly after a time 2Θ a new pulse should be emitted. Then 2Θ should be equal to αT with $\alpha = 1, 2, 3, \cdots$:

$$\alpha T = 2\Theta = 2\frac{L}{c}$$

$$\Rightarrow T = \frac{2}{\alpha}\frac{L}{c}$$

$$\Rightarrow \lambda = \frac{c}{f} = c\,T = \frac{2}{\alpha}L, \text{ with } \alpha = 1, 2, 3, \cdots$$

So, we get exactly the same result as in section 2.3, but this time with a little bit more insight in why these frequencies are the eigenfrequencies.

To use a Finite Element Method we used a programme called Comsol. This is a standard program for using a Finite Element Method for physical problems. The idea for the simulations of the solution of the wave equation (through the tube) consists of three steps.
First we find out the eigenfrequencies (by FEM) of an empty area. After that we place the tube in the middle of this area and calculate again the eigenfrequencies. Now, the results will give some extra eigenfrequencies of this new geometry. From these new frequencies we'll try to find the eigenfrequencies of the tube.

Chapter 5

RESULTS

5.1 The trombone

As already announced in chapter 3, we did some research with the trombone. While a trombonist played we measured the frequencies. At first, by using the tuning pipe we made sure the trombone did have a length of about $2,94$ meters. After that we started measuring frequencies of a number of partials corresponding to the first position. Because of practical reasons we only measured the second till the seventh partial. Table 5.1 shows the results.

	1st	**2nd**	**3rd**	**4th**	**5th**	**6th**	**7th**
Length (m)	2,943	3,118	3,304	3,500	3,708	3,929	4,162
Second harmonic (Hz)	166	158	152	143	136	129	121
Third harmonic (Hz)	223	210	199	190	180	168	160
Fourth harmonic (Hz)	284	269	254	240	224	211	199
Fifth harmonic (Hz)	339	324	308	294	277	257	243
Sixth harmonic (Hz)	394	381	359	340	322	303	288
Seventh harmonic (Hz)	454	427	408	387	368	345	327

Table 5.1: Experimentally determined frequencies for the seven positions of the trombone.

Note that there are at least two important differences with the theory. First, only a few of these frequencies are almost equal to the theoretically calculated ones. Almost every measured frequency is lower than we expected. These differences are mostly larger than 2 Hz, which almost everyone can perceive. However, when two brass instruments are producing tones with between them a difference of 0,5 Hz, people recognise these tones as unequal. This is because of the increasing difference between the harmonics of both instruments: the first harmonics differ already 1 Hz, the third harmonics differ 2 Hz, which can be perceived.
Second, the distance between the third and the seventh partial should be an octave. This implies the frequency of the seventh partial should be twice the frequency corresponding to the third partial. Every position shows the distance between those two harmonics seems to be a little bit more than an octave. Further conclusions can be found in the next chapter.

5.2 The wave equation: solutions by FEM

In this section we will describe the experiments we did and results we found by using a Finite Element approach, via Comsol Multiphysics. For these experiments we only used 2D-geometries. We started with creating an open tube, with length $l = 2.94$ and an inner-diameter $d = 0.05$ (all dimensions in meters), and a surrounding area. We used an ellipse shaped area with semi-axis A and B given by $A = 4$ and $B = 2$. This geometry is shown by Figure 5.1.

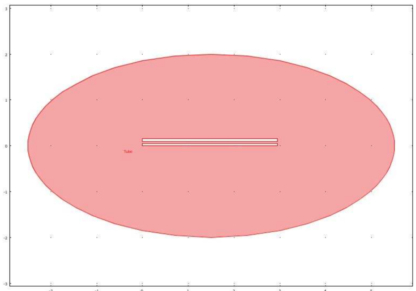

Figure 5.1: Geometry consisting of a tube and surrounding ellipse.

In this geometry, on the walls of the tube the pressure satisfies a hard boundary condition. This condition is given by a Neumann-condition: $\partial p \ / \ \partial n = 0$. At the boundary of the surrounding area I used a soft boundary condition. This condition is given by a Dirichlet-condition: $p = 0$ at the boundary. After creating a geometry, the next step in a Finite Element Method is to create a mesh. This can be done in several ways, for example using triangles or quadrilaterals. We used triangles, which resulted in a mesh shown by the next figure.

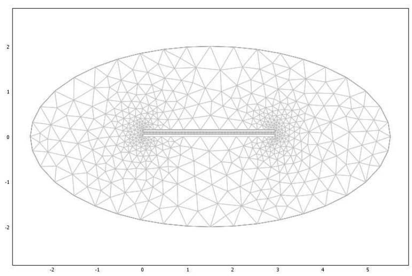

Figure 5.2: Mesh created by triangles.

From this point the program is capable of doing an eigenfrequency analysis for the given geometry. Of course the surrounding area itself has also a lot of eigenfrequencies, so after this analysis we had to find out which eigenfrequencies did correspond to the eigenfrequencies of the tube. Figures 5.3 clearly shows a frequency that is an eigenfrequency of the surrounding area, but doesn't resonate inside the tube. The tube's eigenfrequencies can be recognised almost by inspection of the surface plots. An important property of the eigenfrequencies of the tube is that the pressure waves should be strongly reinforced inside the tube. It would be very accidental that this specific frequency also causes resonances in the surrounding area. That means, we expect a clear resonance pattern inside the tube and almost zero pressure in the surrounding area. The following pictures (in 5.4) show the 2D pressure plots for the fundamental tone and first three harmonics.

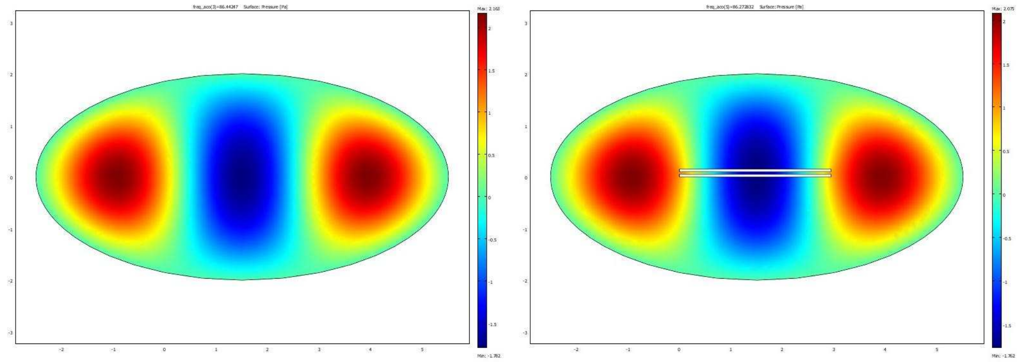

Figure 5.3: Frequency which resonates in the area, but doesn't inside the tube.

Another way to see that these frequencies, given by $f_0 = 57,23$, $f_1 = 112,93$, $f_2 = 173,45$ and $f_3 = 227,55$ are really disturbing the pressure pattern of the surrounding area is by examining the 3D surface plot. This is a normal graph with the pressure on the z-axis. By figure 5.5 it's very clear $f_3 = 227,55$ really is an eigenfrequency of the tube.

From these first experiments we can already conclude several things. At first sight we see, that the frequencies found by Comsol are lower than the eigenfrequencies for open tube calculated in the previous chapter (later on I will refer to these values as the theoretical values). Which is equivalent to mention that these waves apparently have a longer wavelength. Also, from the pressure plots we can see some pressure differences near the boundary of the surrounding area. For example, this phenomenon is very clear in case of the second harmonic. These kind of irregularities give rise to the idea that the boundary of the area influences the eigenfrequencies, or at least these conditions influence the pressure waves at the endings of the tube.
Moreover, when examining a cross-section plot of the pressure through the tube we observe this behaviour again. This cross-section plot is given by figure 5.6.

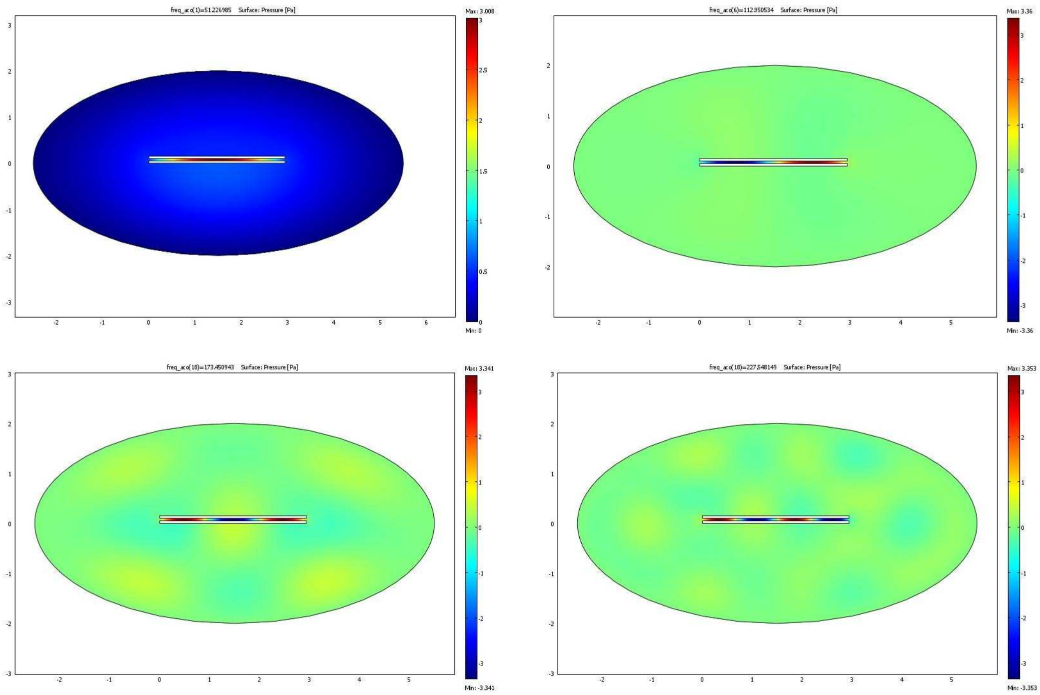

Figure 5.4: Pressure plots for the fundamental tone and first three harmonics.

The frequencies of the previous chapter are calculated for an open tube which has antinods (points where the pressure is zero) at both endings. As mentioned before, in this case the eigenfrequency turns out to be a little bit smaller, which indicates a longer wavelength. Figure 5.6 shows that antinods can be found at $x = -0,195$ and $x = 3,095$. This indicates a sort of effective length of about 3,29 meters. Nevertheless the tube has a length of only 2,94 meters, which is 34 centimeters shorter. But, we can also examine the coordinates of the nods (points where the pressure reaches a maximum or minimum) and antinods in the middle section of the tube. Here, the middle section of the tube means the part inside the tube between the first and fourth extrema. All of these points in the middle section of the tube have in-between distances which indicate an effective length of about 3,02 meters. For example, the distance between the first and second antinod in the middle part of the tube is 0,754 meters and should be equal to λ / 2. For the third harmonic (see figure 3.3) the length of the tube is equal to two whole wavelengths. That means, according to these two antinods, an effective length of the tube equal to $4 \cdot 0,754 = 3,016$ meters. Apparently the antinods at the endings of the tube are somewhat shifted outward. This shift is probably an effect of the surrounding area and the boundary conditions for this area.

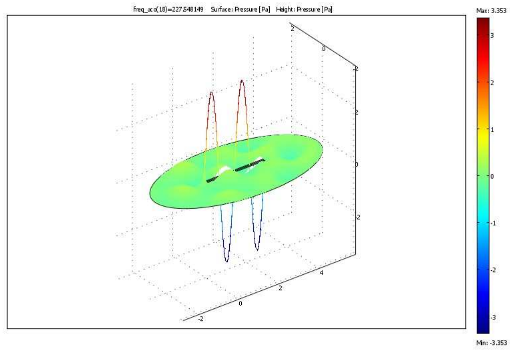

Figure 5.5: 3D surface plot for the third harmonic.

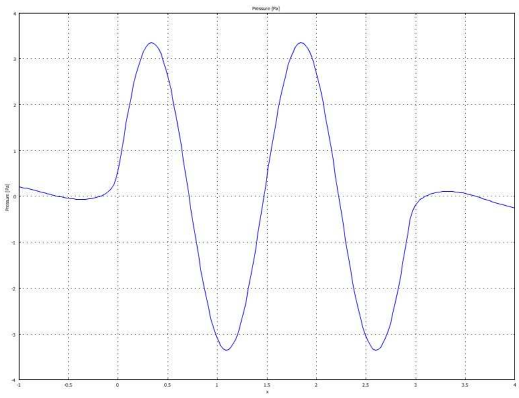

Figure 5.6: The cross-section plot of the pressure through the tube.

Now, we would expect these antinods to move somewhat towards the tube if we decrease the surrounding area. After all, when we would decrease the area until the tube fits exactly, the eigenfrequencies will be given by the eigenfrequencies for a closed tube. On the other hand, by increasing the surrounding area we would expect to eliminate the effect of the boundary conditions for the area. That means, in that case we would also expect eigenfrequencies which satisfy the values we found earlier.

The results from Comsol for a closed tube, an open tube in a very small area and an open tube in a very wide area are shown by the next table.
Note that I didn't try to find the frequencies of the fundamental tone, first and second partial for the latter case. This is because of practical reasons: such a very big geometry has many eigenfrequencies, so it's a lot of work to find out which one corresponds to an eigenfrequency of the tube. As you can see, I only did this for the third harmonic.

	Closed tube	**Decreased area**	**Increased area**
Fundamental tone (Hz)	57,33	57,35	-
First harmonic (Hz)	116,67	114,70	-
Second harmonic (Hz)	175,00	172,04	-
Third harmonic (Hz)	233,33	229,38	225,3545

Table 5.2: Eigenfrequencies determined by FEM for three different situations: a closed tube, a tube in a very small area (with $A = 1,52$ and $B = 0,5$) and a tube in a very wide area ($A = 50$ and $B = 25$).

Furthermore, note that the eigenfrequencies for the closed tube approximate the theoretical values very well. The eigenfrequencies for the tube inside a very small area seem to converge to these values when we keep decreasing the surrounding area.
Also, figure 5.7 shows the third harmonic for the first to cases. The same figure also shows the third harmonic for the latter case. Here the resonance pattern does indeed look like the resonance pattern we found in previous simulations, but more important are the pressure differences in the surrounding area. It looks like there are only some pressure differences just around the tube and close to the boundary there are no pressure waves active. At least this confirms this frequency isn't an eigenfrequency of the surrounding area, but also indicates that there isn't any influence of the boundary on the pressure waves through the tube.

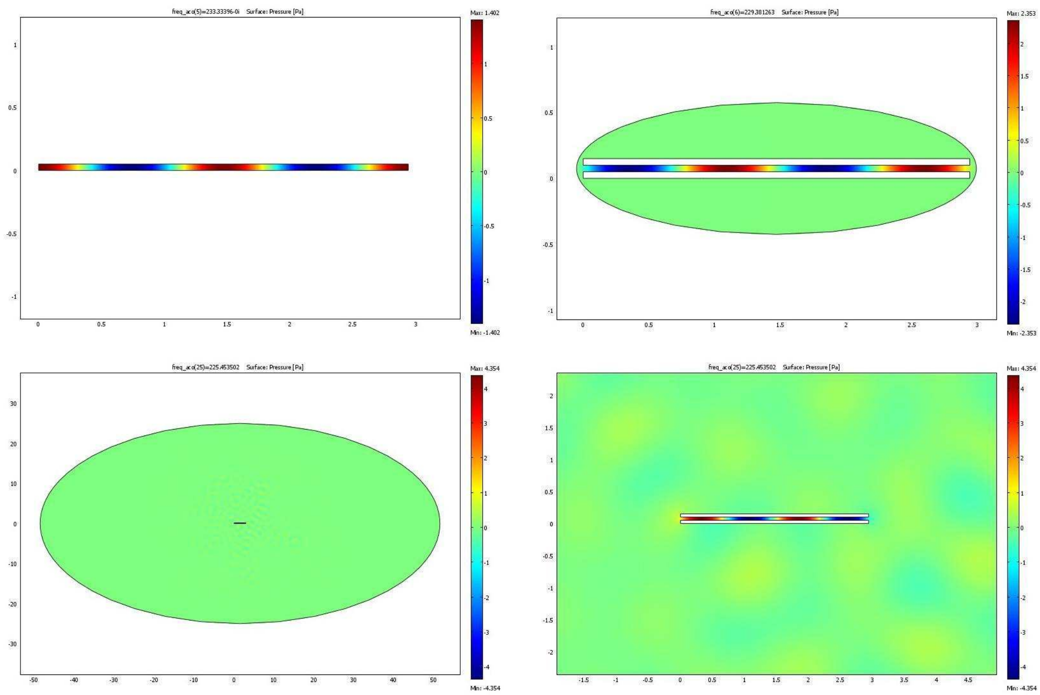

Figure 5.7: On the first row the pressure plots for a closed tube and an open tube in a very small area. The second row gives the plots for an open tube in a very wide area (normal and zoomed on the tube).

As can be concluded from table 5.2 the eigenfrequency corresponding to the third harmonic is even lower in the case with an enormous surrounding area. This actually enervates the idea that the difference between the theoretical values and the eigenfrequencies via Comsol is an effect of wrong boundary conditions. Although at this point I changed the boundary condition of the surrounding area into a Sommerfeld boundary condition. Basically this is an absorbing boundary condition which imitates an infinite area (which of course can't be simulated numerical). So, it absorbs the waves which reach the boundary as if the waves travel just forward outside the area. By using this condition there won't be any reflection from the boundary of the area, so this won't influence the pressure waves through the tube anymore.
First I implemented this boundary condition for the wide area. This already gave much better results: the third harmonic now had an eigenfrequency of 230,1495 Hz.

After this improvement I could only think of one other adjustment to the model, namely decreasing the thickness of the tube. The length of the trombone is indeed about 2,94 meters, but the thickness of the tube is about 1,5 centimetres more than 60%. I set d at 0,015 and did the experiments again for the area I started with. This time I got the following results:

	Frequency
Fundamental tone (Hz)	57,61
First harmonic (Hz)	114,46
Second harmonic (Hz)	174,02
Third harmonic (Hz)	231,27

Table 5.3: Eigenfrequencies determined by FEM, using a Sommerfeld condition for the surrounding area (with $A = 4$ and $B = 2$) and a tube with thickness $d = 0,015$.

Apparently the thickness of the tube does really influence the eigenfrequencies of the tube. The effect of using a thinner or thicker tube can be seen by examining a contour plot. The next figure gives a contour plots for both endings of a thin tube with $d = 0.015$ on the first row and a thick tube with $d = 0.05$ on the second row.
From these four pictures we can easily see that for the thicker tube the contour line which indicates $p = 0$ is shifted further outward than the contour line corresponding to the thinner tube. Also note that in case of a pressure maximum before an end of the tube (which holds for the right endings) this contour line is shifted even more than in case of a pressure minimum (which holds for the left endings). Apparently the distance between the last maximum in the middle section of the tube and the next antinod is larger for the thick tube. This is probably an effect of the impedance as described at the end of chapter 2. The wave impedance I of certain geometry approximately behaves as $I \sim 1/A$, where A is the surface of the cross section of the area. For our 2D case this surface is just the thickness of the tubes or the height of the surrounding area. So, increasing the thickness by a factor 50/15 means decreasing the wave impedance by a factor 15/50. On the other hand, the wave impedance for the surrounding area is much smaller, so $I_a > I_b > I_c$, where I_a denotes the impedance of the thin tube, I_b the impedance of the thick tube and I_c the impedance of the surrounding area. This implies the transition of impedances between the thick tube and the area is smaller than the transition of impedances between the thin tube and the area. So, waves which travel optimal through the

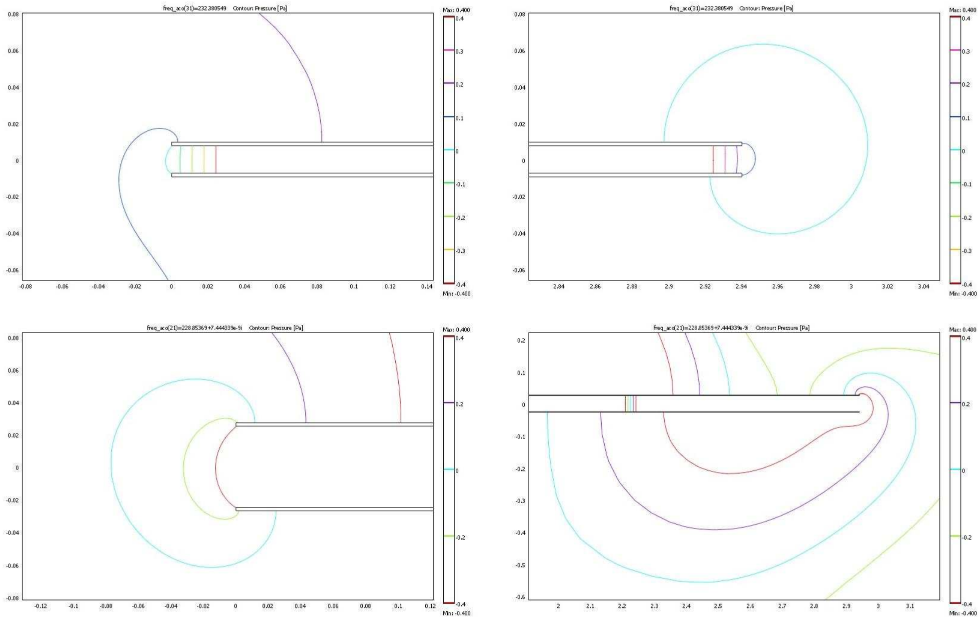

Figure 5.8: Contour plots for the endings of the two different tubes. Note that the x-scale of the fourth figure is different.

thick tube won't be as much reflected as by the surrounding space than waves which travel optimal through a thin tube. That's why the first antinod can be found a little bit further away from the tube, what causes a lower eigenfrequency.
Actually, by using the same argument we can explain why all eigenfrequencies we found by using Comsol are a bit lower than the calculated ones. That is, the calculated frequencies are based on the idea of total reflection and the open ends. Total reflection will only occur if the thickness of the tube is infinitesimal (then the impedance difference will be infinite). In that case the pressure will be exactly zero at the end of the tube. But, of course in reality the thickness of the tube isn't infinitesimal, so there isn't total reflection, otherwise we would never hear anything. This also explains why in reality the contour line $p = 0$ can never be found exactly at the end of the tube, but a little bit more outward.

Chapter 6

CONCLUSION

This last chapter will describe the way we did our research. In short we will describe the important parts of a brass instrument, the models we used, their disadvantages and results. After we'll have a look at the experiments we did, which include the results the trombone gave us and the results of using Comsol. Finally the last part of this chapter will explain the differences between the models and experiments and show that there is still enough matter for doing some extra research.

We started this paper with a short description of the sound producing mechanism of brass instruments. The most important factors in this process are the lips of the musician, the mouthpiece, the tube and the bell of the instrument. Each of those has its own function: the lips create a vibration, the mouthpiece has substantial effect on the timbre, the tube determines more or less the frequency and finally the bell takes care of the radiation and decreases the frequencies a little bit. After this short description we started focussing on the tube. The length of the tube more or less has a selection function: brass instruments use only the eigenfrequencies of their instruments, because these are the only frequencies which resonate. It seemed to be important to find the eigenfrequencies of the instruments. But, because instruments have very complicated geometries we had to use a simple representation as a model to find the eigenfrequencies. We found several options to do this: using a half-open cylindrical tube, an open cylindrical tube and a conical tube. The first didn't satisfy our purpose, because we knew from practise the difference between the fundamental tone and first harmonic, which wasn't the same as for the half-open model. Then we tried the conical model, which seemed quite hard to study by itself, but somehow its eigenfrequencies were the same as for the open tube model. Those observations implied we could use the open model as a simplified model of brass instruments.

The second part of the paper dealt with calculating the eigenfrequencies of an open tube. We did these calculations in many different ways. First we used a standard high school technique discussing eigenmodes of open air columns. The important relation between the length of the tube and the eigenfrequencies was given by

$$f = \frac{v}{\lambda} = \frac{vn}{2L}$$

with $n \in \mathbb{N}$ and $v = 343$ meters/second. The same equation could be found by studying the behaviour of the sound waves at the open endings. At an open end of the tube the wave impedance suddenly decreases enormously. This decrease causes almost total reflection of the sound wave, so we could as well examine the eigenfrequencies of a closed tube. That argument confirmed the relation we already had. A third theoretical way to find the eigenfrequencies of a certain tube is solving the wave equation. First we derived this equation by using the continuity equation, Euler's equations of motion and the relation between the pressure and the density. By using some standard techniques we finally found the wave equation

$$\frac{\partial^2 P}{\partial t^2} = c^2 \Delta P$$

with P the pressure. We solved this equation for the an open cylindrical tube and a conical tube. From the general solution of this equation and the boundary conditions we constructed a particular solution for both cases. These solutions also led to the same relation between the eigenfrequencies and the length of the tube and again confirmed conical and cylindrical tubes have the same eigenfrequencies.

After this theoretical point of view we started doing some experiments. For the experiments we used a trombone, which gave us some information about the reality, and a computer program called Comsol. We used the trombone and measured its real eigenfrequencies. Note that this isn't independent of the musician who plays the trombone. The results may differ a little for each player. However, the measured frequencies denoted tones which had a lower pitch than we would expect for an open cylindrical tube of the same length. This was also the case when using the computer program. Comsol can solve many different physical problems (including eigenvalue problems) by using a Finite Element Method. Also from these experiments we found the eigenfrequencies a bit smaller than calculated by our theory. Although, the results were closer to the theoretical values than the results we got by using the real trombone. The main reason for lower eigenfrequencies should be the interaction between the surrounding space and the tube. Apparently, it depends on the thickness of the tube what the exact eigenfrequency is, that means the thicker the tube, the lower the eigenfrequency. This effect is a result of a relatively small difference in impedance between a thick tube and the surrounding area. As said before, the results for the real trombone differ even more from the theoretical values. These differences can actually be perceived by most people.

In sum, we can conclude that modelling by an open cylindrical tube gives a nice indication of the eigenfrequencies of a brass instrument. But, the results aren't very accurate. For instance, the theory about standing wave through air columns doesn't take in account the interaction between the tube and the surrounding area. Apparently because of the sudden change of impedances at the end of a tube the eigenfrequencies differ a bit from the theoretical values. The model we used is actually based on a state of zero impedance of the surrounding area of the tube. This zero impedance causes total reflection of the sound wave (as with a closed tube). In case of total reflection there will be a pressure antinod at the open end. Now, in reality zero impedance is impossible (impedance I behaves as $1/A$, where A is the surface of the cross section of the area). That difference rules out the possibility of total reflection. For that reason the antinod is shifted a little bit outward, which causes a lower eigenfrequency. Also we did the observation that this effect is even more present for a thicker tube. As said before, using a thicker tube decreases the difference in impedances, which causes even less reflection, so an even lower eigenfrequency. At last we have to mention the difference between reality and our model together with the experiments. The low eigenfrequencies for the trombone can be caused by many different factors. For example we didn't take in account the friction between the tube and sound waves. Also, from a geometrical point of view, we didn't take in account the mouthpiece, the curves of the tube, the conical section, the rapidly widening bell and the material. All these factors can have their own influence on the eigenfrequencies of the trombone.

I would like to end this thesis with stating that, although it was an interesting research, I finished this thesis with even more questions than I started. For example during the research I bumped into the so-called principle of effective length, a phenomenon which really deserves more attention. Also, in the beginning of this paper we shortly investigated the different parts of brass instruments, but to understand the exact behaviour and physical meaning of the different parts we would need to do some more research. Thinking of these issues I could come up with an endless list of new, unanswered questions, such as:

- What is the exact influence of the bell on the frequencies of the radiated sound? What physics can describe the phenomenon?
- What is the exact influence of the mouthpiece on the timbre? And on the eigenfrequencies of the instrument?
- How can we explain the principle of effective length?
- What is the influence of the curves of the tube on the sound? Is there any influence on the radiated frequencies?

I hope these questions give rise to some new, interesting research in the future.

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