## Abelian Sandpiles

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#### Abstract

Summary Sandpiles models have nice physical properties. However, they can also be studied from an algebraic point of view. In this thesis we consider the abelian sandpile model. We begin by examining the group structure of this model. The elements of this group, configurations of the sandpile, can be added by placing them on top of each other and subsequently letting them topple to a stable configuration. We show this operation is well defined. It turns out that the sandpile groups are in fact finitely generated. The structure of the group can be determined using the minor method. We define four equivalence relations on the elements of the sandpile group. Each of these is generated by a different interpretation of the idea that configurations can be toppled into each other and are thus 'equivalent'. We compare these equivalence classes and show that some of them are in fact equivalent.


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## Chapter 1

## Introduction

Imagine building a sand castle of loose sand. You start with a steady base and slowly you make it higher and higher. Eventually the slope of what was meant to be a castle, becomes too large and the pile of sand will collapse. This illustrates the concept of self-organized criticality, which was first introduced in 1987 by Bak, Tang and Wiesenfeld [1]. A self-organized critical system reaches a critical state, not because some parameter is fine-tuned, but by its intrinsic dynamics. In case of the sandpile the same thing happens; when grains of sand are rained upon a sandpile, after a while it will not get steeper but, because of small avalanches, the pile will reach its critical slope.
Bak, Tang and Wiesenfeld studied self-organized criticality by a mathematical sandpile model. Since then sandpile models have raised the interest of mathematical physicists, experimentalists, probabilists, combinatorists and algebraists [8]. In this thesis I will focus on a subclass of sandpile models: abelian sandpile models. I will do this from an algebraic point of view.

### 1.1 The model

The concept of the abelian sandpile model (ASM) is as follows. We consider a certain number of sites, some of which are connected. Every site has a corresponding capacity. When this capacity is exceeded, the site loses grains to neighbouring sites. This is called a toppling. When a site topples, it can of course cause neighbouring sites to topple as well. This might cause the first site to topple once more. To prevent the system from toppling indefinitely, we introduce sinks. Grains can enter these sinks, but can never leave them. We can thus think of them as sites without restrictions on the capacity; loose ends where grains just perish.
If we think of sites as vertices and of the connections between those sites as edges, we can consider the ASM in terms of graphs [2]. Let $V=\{1, \ldots, n\}$ be a set of vertices and denote by $e_{i j}$ the number of edges from vertex $i$ to vertex $j$. The graph is undirected if $e_{i j}=e_{j i}$ for all $i, j \in V$. Otherwise we speak of a directed graph. From now on we will restrict ourselves to undirected graphs. Moreover, we assume that the graphs contain no loops, that is $e_{i i}=0$ for any $i$.


Figure 1.1: Unstable configuration $u=(2,0,4,2)$.

A vertex may have one or more sinks attached to it. Let $s_{i}$ denote the number of sinks attached to vertex $i$. In graphs we represent these sinks as 'dangling' edges, that are connected to only one site. We call the combination of vertices, edges and sinks a toppling graph. On this graph we can define configurations. A configuration $u=\left(u_{1}, \ldots, u_{n}\right)$ is a vector of nonnegative integers. Therefore $\mathbb{Z}_{\geq 0}^{n}$ is the set of all possible configurations. Integer $u_{i}$ can be interpreted as the number of grains at site $i$. As mentioned before, every site or vertex $i$ has a certain capacity. This capacity depends on the degree $d e g_{i}$ of vertex $i$, which is equal to the number of outgoing edges plus the number of sinks attached to that edge, i.e. $\operatorname{deg}_{i}=s_{i}+\sum_{j=1}^{n} e_{i j}$. The capacity of a vertex is its degree minus one.

If the number of grains at a certain vertex exceeds its capacity, we call a configuratation unstable at this vertex. We prefer this over the term 'unstable vertex', which has become common in the literature, since being stable is not an intrinsic property of a vertex. If the number of grains at a vertex does not exceed its capacity, a configuration is stable at that vertex. A stable configuration is a configuration that is stable at all of its vertices. We denote by $\Omega_{n}$ the set of all stable configurations, that is

$$
\Omega_{n}=\left\{u \in \mathbb{Z}_{\geq 0}^{n} \mid u_{i}<d e g_{i} \text { for all } i\right\} .
$$

If a configuration is unstable at a vertex $i$, that vertex topples. This means that its number of grains $u_{i}$ is decreased by its degree $\operatorname{deg}_{i}$, while the number of grains at neighbouring vertices $j$ are increased by $e_{i j}$. In this way we can define $n$ toppling operations $T_{i}$, one for each vertex. The sequence of topplings, that causes an initial configuration $u$ to stabilize, is called an avalanche. We denote the stabilized configuration by $\mathcal{T}(u)$, where $\mathcal{T}$ is a mapping $\mathcal{T}: \mathbb{Z}_{\geq 0}^{n} \rightarrow \Omega_{n}$.

Example 1.1. Consider figure 1.1. We see a toppling graph, consisting of four vertices, five edges and four sinks. The initial configuration is $u=(2,0,4,2)$. Vertex 3 contains four grains, while its capacity is $\operatorname{deg}_{3}-1=4-1=3$, so $u$ is unstable at this vertex and will topple. Therefore it has been marked gray in the figure. The toppling causes vertex 1, 2 and 4 to gain one grain each. The fourth grain disappears into a sink. As we can see in figure 1.2, both vertex 1 and 4 have become unstable. We choose to topple vertex 1 first, the result of which is shown in figure 1.3 , and after that we topple vertex 4 . Figure 1.4 shows that we have now reached the stable configuration $v=(0,3,2,0)$. We say that $\mathcal{T}(u)=v$.


Figure 1.2: Ustable configuration ( $3,1,0,2$ ) obtained by toppling vertex 3 in configuration (2, 0, 4, 2).


Figure 1.3: Ustable configuration $(0,2,1,3)$ obtained by toppling vertex 1 in configuration (3, 1, 0, 2).

### 1.2 Toppling operator $\mathcal{T}$

In example 1.1 we could have decided to first topple vertex 4 and then vertex 1. In that way we change the order of the topplings. One might think that this could lead to a different final state. This however is not the case. The resulting state is independent of the order in which the topplings are executed, since toppling operations commute on unstable configurations [9]. This result is called the abelian property, which is why our sandpile model is called abelian. The proof of this property is almost trivial, if we keep in mind that it only concerns two topplings, instead of the full stabilization of a configuration at two sites.

Theorem 1.2. Topplings of individual vertices commute.

Proof. Let a configuration $u$ be unstable at both vertex $i$ and vertex $j(i \neq j)$, that is $u_{i} \geq d e g_{i}$ and $u_{j} \geq d e g_{j}$. We first consider $T_{i} \circ T_{j}$. Let $v=T_{j}(u)$, then

$$
\begin{aligned}
v_{j} & =u_{j}-\operatorname{deg}_{j} \\
v_{k} & =u_{k}+e_{j k} \quad k \neq j
\end{aligned}
$$



Figure 1.4: Stable configuration $(0,3,2,0)$ obtained by toppling vertex 4 in configuration ( $0,2,1,3$ ).

We now apply $T_{i}$ to $v$ and let $w=T_{i}(v)$,

$$
\begin{aligned}
w_{i} & =u_{i}+e_{j i}-d e g_{i} \\
w_{j} & =u_{j}-d e g_{j}+e_{i j} \\
w_{k} & =u_{k}+e_{j k}+e_{i k} \quad k \notin\{i, j\}
\end{aligned}
$$

In the same way we can derive configuration $w^{\prime}$ as a result of $T_{j} \circ T_{i}$ :

$$
\begin{aligned}
w_{i}^{\prime} & =u_{i}-d_{i}+e_{j i} \\
w_{j}^{\prime} & =u_{j}+e_{i j}-d e g_{j} \\
w_{k}^{\prime} & =u_{k}+e i k+e_{j k} \quad k \notin\{i, j\}
\end{aligned}
$$

We conclude that $w=w^{\prime}$. In other words, $T_{i} \circ T_{j}(u)=T_{j} \circ T_{i}(u)$.

In physics literature it is often stated that theorem 1.2 directly implies that operator $\mathcal{T}$ is well defined. This means that it always assigns to a configuration in $\mathbb{Z}_{>0}^{n}$ a unique configuration in $\Omega_{n}$. Meester, Redig, and Znamenski rightly say that there is a bit more to this. They address the following problem. Assume configuration $u$ is unstable at vertices $i$ and $j$. A toppling of vertex $i$ may cause $u$ to become unstable at another vertex $k$. This possibly would never have happened if vertex $j$ had been toppled first. So different toppling orders could lead to different stable configurations.

This is not the only problem we encounter. It is also not trivial, that the toppling operator converges. Maybe we can find a configuration $u$ such that repeatedly toppling does not cause $u$ to stabilize. Let $u^{k}(k \geq 0)$ denote configuration $u$ after toppling $k$ times. We say that configuration $u$ does not stabilize, if $u^{k}$ is unstable for all $k \geq 0$. Since the total amount of grains $\sum_{i=1}^{n} u_{i}^{k}$ is non-increasing and we assume $u$ does not stabilize, this number has to remain constant after a certain number of topplings $K$. The only way this can happen is if there exists a nontrivial cycle of unstable states, for there only exists a finite number of unstable configurations $v$ that contain $\sum_{i=1}^{n} u_{i}^{K}$ grains. We will prove that the existence of such a nontrivial cycle is impossible [5].

Theorem 1.3. The toppling operator $\mathcal{T}$ converges.
Proof. We will show that there does not exist a nontrivial cycle of unstable states. Let $\operatorname{dis}(i)$ be the minimal distance from vertex $i$ to a vertex that is connected to a sink. For vertices $i$, that have sinks attached to them $\left(s_{i}>0\right)$, we thus have that $\operatorname{dis}(i)=0$. We call these vertices the boundary vertices of the toppling graph.

First note that the toppling of a vertex $i$ in the 'interior' of the toppling graph, that is $\operatorname{dis}(i)>0$, does not change total amount of grains $\sum_{i=1}^{n} u_{i}^{k}$. However, a toppling of a boundary vertex decreases this sum, because one or more grains disappear into the sink. So the sum is nonincreasing and only decreases, if a vertex on the boundary topples. A nontrivial cycle of unstable states therefore can not contain a toppling of a boundary vertex.

Now let us assume that there does exist a nontrivial cycle, so we can consequtively topple vertices $i_{1}, i_{2}, \ldots, i_{k}$ over and over again. Let $n_{j}=\operatorname{dis}\left(i_{j}\right)$ for
$j=1, \ldots, k$. Now take $n=\min \left\{n_{1}, \ldots, n_{k}\right\}$. Every time a vertex with distance $n$ topples in the cycle, the number of grains in a vertex $m$ with $\operatorname{dis}(m)=n-1$ increases. After at most $d e g_{m}$ cycles the configuration is also unstable at vertex $m$, so this vertex has to be included in the cycle as well.
Repeating this argument we find that some vertices with distances $n-2$ to 0 have to be included in the cycle. However, we have seen that a nontrivial cycle can not contain a toppling of a boundary vertex. This yields that there can not exist a nontrivial cycle and therefore toppling operator $\mathcal{T}$ converges.

Now we have solved our second problem, but we still do not know whether or not $\mathcal{T}$ is well-defined. It could still be that the toppling process does not uniquely determine a final, stable configuration for every unstable initial configuration. We will show that every vertex always topples the same number of times, irrespective of the order in which we execute the topplings, since this implies that the toppling order does not affect our final configuration.

Theorem 1.4. The operator $\mathcal{T}$ is well-defined.
Proof. This proof is nearly identical to the one given by Meester et al. [9]. If $u$ is stable then there is nothing to prove. Let $u$ be an unstable configuration and suppose that

$$
T_{i_{N}} \circ \ldots \circ T_{i_{2}} \circ T_{i_{1}}(u)
$$

and

$$
T_{j_{M}} \circ \ldots \circ T_{j_{2}} \circ T_{j_{1}}(u)
$$

are both stable. We have to show two things: firstly that $M=N$ and secondly that sequences $i$ and $j$ are permutations of each other. If these propositions holds, every vertex always topples the same number of times and $\mathcal{T}$ is welldefined.
We will prove by induction with respect to $N$ that the two propositions hold. If $N=1$ there is nothing to prove. Suppose that $N>1$ and assume that both propositions hold for all configurations having some toppling sequence of length less than $N$. Let $j=j_{1}, j_{2}, \ldots, j_{M}$ also be a sequence for $u$. Since vertex $i_{1}$ is the first to topple according to sequence $i$, the initial configuration $u$ has to be unstable at that vertex. Therefore $i_{1}$ has to occur at least once in sequence $j$. Let $k$ be minimal such that $j_{k}=i_{1}$. We claim that

$$
\begin{equation*}
T_{j_{M}} \circ \ldots \circ T_{j_{k+1}} \circ T_{i_{1}} \circ T_{j_{k-1}} \circ \ldots \circ T_{j_{2}} \circ T_{j_{1}}(u) \tag{1.1}
\end{equation*}
$$

and

$$
T_{j_{M}} \circ \ldots \circ T_{j_{k+1}} \circ T_{j_{k-1}} \circ T_{i_{1}} \circ \ldots \circ T_{j_{2}} \circ T_{j_{1}}(u)
$$

are the same. We can show this as follows. Let $u^{\prime}=T_{j_{k-2}} \circ \ldots \circ T_{j_{2}} \circ T_{j_{1}}(u)$. Vertex $i_{1}$ has not yet toppled at this point, so $u_{i_{1}}^{\prime} \geq d_{i_{1}}$. Since we also know that $u_{j_{k-1}}^{\prime} \geq d_{j_{k-1}}$, we may interchange $T_{i_{1}}$ and $T_{j_{k-1}}$. By repeating this argument we can slide $T_{i_{1}}$ to the right completely. Thus configuration (1.1) and

$$
T_{j_{M}} \circ \ldots \circ T_{j_{k+1}} \circ T_{j_{k-1}} \circ \ldots \circ T_{j_{1}} \circ T_{i_{1}}(u)
$$

denote the same stable configuration. Now let $v=T_{i_{1}}(u)$, then both

$$
T_{i_{N}} \circ \ldots \circ T_{i_{2}}(v)
$$

and

$$
T_{j_{M}} \circ \ldots \circ T_{j_{k+1}} \circ T_{j_{k-1}} \circ \ldots \circ T_{j_{1}}(v)
$$

are stable and the corresponding sequences are minimal. Using the assumption we made at the beginning of this proof, we know that $M-1=N-1$ and that sequences $i_{2}, \ldots, i_{N}$ and $j_{1}, \ldots, j_{k-1}, j_{k+1}, \ldots, j_{M}$ are permutations of each other. Consequently, $M=N$ and sequence $i$ and $j$ are permutations of each other as well, which is what we had to prove.

### 1.3 Toppling matrices

We have introduced the ASM in terms of toppling graphs. Every graph can induce an ASM, under the condition that it contains at least one sink. Another way to consider the ASM is by means of toppling matrices [10].

Definition 1.5. Let $V=\{1, \ldots, n\}$. An $n \times n$ integer matrix $\Delta$ is a toppling matrix if it satisfies the following conditions.

1. For all $i \in V: \Delta_{i i} \geq 1$ and for all $i, j \in V, i \neq j: \Delta_{i j} \leq 0$
2. For all $i, j \in V, i \neq j: \Delta_{i j}=\Delta_{j i}$ (symmetry)
3. For all $i \in V: \sum_{j \in V} \Delta_{i j} \geq 0$ (dissipativity)
4. $\sum_{i, j \in V} \Delta_{i j}>0$ (strict dissippativity)

This definition only describes toppling matrices of undirected graphs. If we replace the symmetry property by property $2^{*}$, another dissipativity property, we define toppling matrices of both directed and undirected graphs.

$$
2^{*} \text {. For all } j \in V: \sum_{i \in V} \Delta_{i j} \geq 0 \text { (dissipativity) }
$$

However, we will restrict ourselves to undirected graphs and symmetric toppling matrices. There exists a bijection between these two sets, which is defined by

$$
\Delta_{i j}= \begin{cases}d e g_{i} & \text { if } i=j  \tag{1.2}\\ -e_{i j} & \text { else }\end{cases}
$$

Notice that this expression can also be written as $\Delta=D-E$, where $D$ is defined by $D_{i j}=\operatorname{deg}_{i} \delta_{i j}$ and $E$ by $E_{i j}=e_{i j}$, with $i, j=1, \ldots, n$.

Example 1.6. The toppling matrix that corresponds to the graph in figure 1.1, is given by:

$$
\Delta=\left(\begin{array}{cccc}
3 & -1 & -1 & 0 \\
-1 & 4 & -1 & -1 \\
-1 & -1 & 4 & -1 \\
0 & -1 & -1 & 3
\end{array}\right)
$$

Toppling operations can be expressed in terms of the toppling matrix. Toppling a vertex $i$ in a configuration $u=\left(u_{1}, \ldots, u_{n}\right)$ is equivalent to substracting the $i$ th row of $\Delta$ from $u$.


Figure 1.5: Toppling graph: $d_{1}=d_{2}=2$.

### 1.4 Recurrent configurations

The space of all configurations on $n$ vertices is $\mathbb{Z}_{\geq 0}^{n}$. Only $d e g_{1} \cdot \operatorname{deg}_{2} \cdots \operatorname{deg}_{n}$ of them are stable. All the others can through toppling return to a stable configuration. Some stable configurations are more probable to be returned to than others. Consider for example the configuration $z=(0, \ldots, 0)$. There does not exist any configuration $u$ with $z$ as its stabilized result, that is $\mathcal{T}(u)=z$. Therefore we say that it occurs with zero probability. Only a small subset of all stable configurations occurs with nonzero probability. These are called recurrent configurations [2]. Since we however would like to avoid a probabilistic definition, we use the following one.

Definition 1.7. A configuration $u$ is recurrent if it is stable and there exists a (nontrivial) configuration $v$ such that $\mathcal{T}(u+v)=u$.

In this definition, + denotes the vertexwise addition of two configurations. We denote the set of recurrent configurations by $\mathcal{R}$. Dhar introduced in [7] the socalled 'burning algorithm', that determines whether a configuration is in $\mathcal{R}$ or not. Speer generalized this into his 'script algorithm' [12]. However, for really small graphs we can determine the recurrent configurations by hand.

Example 1.8. Consider the toppling graph in figure 1.5. The set of all stable configurations is $\left\{u_{1}, u_{2}, u_{3}, u_{4}\right\}=\{(0,0),(0,1),(1,0),(1,1)\}$, since $d_{1}=2$ and $d_{2}=2$. Which of these configurations are recurrent? Certainly, $(0,0)$ is not, since in the sequence of topplings leading to the stable configuration $(0,0)$ the last configuration before $(0,0)$ would be either $(-1,2)$ or $(2,-1)$. These configurations are not in $\mathbb{Z}_{\geq 0}^{2}$. Now let us consider the other three:

$$
\begin{aligned}
& (0,1)+(1,1)=(1,2) \rightarrow(2,0) \rightarrow(0,1) \\
& (1,0)+(1,1)=(2,1) \rightarrow(0,2) \rightarrow(1,0) \\
& (1,1)+(1,1)=(2,2) \rightarrow(0,3) \rightarrow(1,1)
\end{aligned}
$$

So $\mathcal{T}\left(u_{2}+u_{4}\right)=u_{2}, \mathcal{T}\left(u_{3}+u_{4}\right)=u_{3}$ and $\mathcal{T}\left(u_{4}+u_{4}\right)=u_{4}$. Therefore the configurations $u_{2}, u_{3}$ and $u_{4}$ are recurrent.

On $\mathcal{R}$ we can define an operator $\oplus$ by $x \oplus y=\mathcal{T}(x+y)$. In chapter 2 we will prove that the set $\mathcal{R}$ combined with this operator forms an abelian group, that is isomorphic to $\mathbb{Z}^{n} / \Delta \mathbb{Z}^{n}$.

Group $\mathbb{Z}^{n} / \Delta \mathbb{Z}^{n}$ is a finitely generated abelian group. It can be decomposed into a direct product of cyclic groups, see chapter 3, but this structure of $\mathbb{Z}^{n} / \Delta \mathbb{Z}^{n}$ is not trivial. In chapter 4 we present a way to determine this structure.

Several authors have tried to approach the abelian sandpile model in a different manner than we presented. Nevertheless, the focus remained on the recurrent
configurations. In order to say something about these configurations, they defined equivalence classes, which we will discuss in chapter 5.

We end this introduction with an example, that was introduced by Redig [10], called 'The crazy office'.

Example 1.9. In the crazy office $n$ persons are sitting at a long table, person 1 at the left end and person $n$ and the right end. We think of them as commissioners who have to treat files. All of them can treat at most one file. However, in reality the commisioners do not treat files at all. In normal circumstances the files just lay in front of them, but if a commissioner has two or more files he does the following. He passes on one file to his left neighbour and one to his right neighbour at the same time and will repeat this, until he has only one or zero files in front of him. The persons at the left and right end of the table do things somewhat differently. They also pass on one file to a neighbour, but throw the other file out of the window.

We present a slightly changed version of this example. To impress their boss the commissioners have told her that they are able to treat at most three files, instead of one. In reality they drilled two holes in the floor, one in front of the table and one behind it. When a commisioner has four or more files in front of him now, he passes one file on to each neighbour and throws one file in each of the holes. In that way they lose four files at once. Again he repeats this if necessary, until he has less than four files in front of him and he can relax.

One day commisioner 1 reads something about abelian sandpile models and decides to examine the group structure of his coworkers (read: recurrent configurations of files on the table). He realizes that this must have something to do with $F_{n}$, the $n \times n$ file dispenser matrix.

$$
F_{n}=\left(\begin{array}{ccccc}
4 & -1 & 0 & \cdots & 0 \\
-1 & 4 & -1 & & \vdots \\
0 & -1 & \ddots & \ddots & 0 \\
\vdots & & \ddots & 4 & -1 \\
0 & \cdots & 0 & -1 & 4
\end{array}\right)
$$

Before they drilled holes in the floor, this matrix would have looked as follows:

$$
\left(\begin{array}{ccccc}
2 & -1 & 0 & \cdots & 0 \\
-1 & 2 & -1 & & \vdots \\
0 & -1 & \ddots & \ddots & 0 \\
\vdots & & \ddots & 2 & -1 \\
0 & \cdots & 0 & -1 & 2
\end{array}\right)
$$

However, commisioner 1 is not satisfied with his results and would like to know more. To be continued...

## Chapter 2

## Toppling groups

The set of recurrent configurations $\mathcal{R}$ combined with operator $\oplus$ appears to have a very nice properties. We can already see this in example 1.8. Consider the addition table, corresponding to the recurrent configurations in this example, and compare it to the table of $\mathbb{Z} / 3 \mathbb{Z}$.

| $\oplus$ | $(0,1)$ | $(1,0)$ | $(1,1)$ |
| :---: | :---: | :---: | :---: |
| $(0,1)$ | $(1,0)$ | $(1,1)$ | $(0,1)$ |
| $(1,0)$ | $(1,1)$ | $(0,1)$ | $(1,0)$ |
| $(1,1)$ | $(0,1)$ | $(1,0)$ | $(1,1)$ |


| + | $\overline{2}$ | $\overline{1}$ | $\overline{0}$ |
| :---: | :---: | :---: | :---: |
| $\overline{2}$ | $\overline{1}$ | $\overline{0}$ | $\overline{2}$ |
| $\overline{1}$ | $\overline{0}$ | $\overline{2}$ | $\overline{1}$ |
| $\overline{0}$ | $\overline{2}$ | $\overline{1}$ | $\overline{0}$ |

We conclude that in this case $\mathcal{R}$ is isomorphic to the abelian group $\mathbb{Z} / 3 \mathbb{Z}$. Remind that we added configuration $(1,1)$ to each of the three configurations to show that they were recurrent. This was no arbitrary choice. Under addition of this configuration all elements in $\mathcal{R}$ remain the same. Therefore this is the unit of $\mathcal{R}$.
We will show that any set of recurrent configurations is isomorphic to an abelian group. This proof and the remainder of this chapter are based mainly on the work of Redig [10] and Meester, Redig and Znamenski [9].
Our plan is as follows. We will first introduce another set $G$ endowed with a suitable operation and show that this forms a group. Then we will find a bijection between this group $G$ and $\mathcal{R}$.
In order to define $G$, we first have to define the addition operations $a_{i}: \Omega_{n} \rightarrow \Omega_{n}$ by

$$
a_{i}(u)=\mathcal{T}\left(u+\delta_{i}\right), \quad \text { for } i=1, \ldots, n
$$

Here $\delta_{i}$ represents the configuration containing one grain at vertex $i$ and none at the other vertices. The fact that $\mathcal{T}$ is well-defined (theorem 1.4) implies that the addition operators are well-defined as well and that abelianness holds. This means that for all $i, j \in\{1, \ldots, n\}$ and $u \in \Omega_{n}$,

$$
a_{i} a_{j}(u)=a_{j} a_{i}(u)=\mathcal{T}\left(u+\delta_{i}+\delta_{j}\right)
$$

Now we are ready to introduce set $G$ :

$$
G=\left\{\prod_{i=1}^{n} a_{i}^{k_{i}}: k_{i} \in \mathbb{N}\right\}
$$

The following theorem states that this set, with multiplication as operation, is an abelian group.

Theorem 2.1. Set $G$, viewed as operator on $\mathcal{R}$, defines an abelian group.
Proof. If a configuration $u$ is recurrent we can by definition find a (non-trivial) configuration $v$ such that $\mathcal{T}(u+v)=u$. Since $v$ is nontrivial, some of the $v_{i}$ are nonzero. Now suppose we add $v$ to $u$ very often. As a result we end up with at least one site with a large surplus of grains. Performing topplings from this vertex outwards will distribute these grains to all vertices. Since $u+v+\ldots+v$ topples to $u$, any configuration obtained by topplings of $u+v+\ldots+v$ also has $u$ as stabilized result. Now we are sure that we can find a configuration $m$ such that $m_{i}>0$ for all $i$ and $\mathcal{T}(u+m)=u$.
In terms of set $G$ this means that there exist integers $m_{i}>0$ such that

$$
\prod_{i=1}^{n} a_{i}^{m_{i}}(u)=u
$$

Let $e_{u}=\prod_{i=1}^{n} a_{i}^{m_{i}}$ and consider $A_{u}=\left\{v \in \mathcal{R}: e_{u}(v)=v\right\}$. This set is not empty, since $u \in A$. Moreover, if $g=\prod_{i=1}^{n} a_{i}^{k_{i}}$, where $k_{i} \in \mathbb{N}$ and $v \in A$, then by abelianness

$$
e_{u}(g(v))=g\left(e_{u}(v)\right)=g(v)
$$

So $v \in A$ implies $g(v) \in A$.
Now suppose $u=\left(\operatorname{deg}_{1}-1, \ldots\right.$, deg $\left._{n}-1\right)$, which is the maximal stable configuration, and let $e_{u}$ correspond to this $u$. For all $w \in \mathcal{R}$ we can find a function $g$ such that $g(u)=w$, by a similar argument as used in the first paragraph of this proof, so $w \in A$. Therefore $\mathcal{R} \subset A$. By definition $A \subset \mathcal{R}$, so we have that $A=\mathcal{R}$.

Note that this shows that $e_{u}$ is a neutral element; it sends every element to itself. Since $m_{i}>0$ we can find an inverse for addition operator $a_{i}$ :

$$
a_{i}^{-1}=a_{i}^{m_{i}-1} \prod_{j=1, j \neq i}^{n} a_{j}^{m_{j}} .
$$

Then indeed $a_{i} \cdot a_{i}^{-1}=a_{i}^{-1} \cdot a_{i}=e_{u}$. From this we can derive that every element in the set

$$
G=\left\{\prod_{i=1}^{n} a_{i}^{k_{i}}: k_{i} \in \mathbb{N}\right\}
$$

acting on $\mathcal{R}$, has an inverse. We have already seen that $G$ contains a neutral element and when two elements from $G$ are multiplied, the result again is in $G$. The order in which we execute this multiplication does not affect this result, since the $a_{i}$ 's commute. This proves that $G$ defines an abelian group.

Theorem 2.2. There exists a bijection between $G$ and $\mathcal{R}$.
Proof. We will prove that for all $u \in \mathcal{R}$, the map $\Psi_{u}: G \rightarrow \mathcal{R}$ defined by $\Psi_{u}(g)=g(u)$ is a bijection between $G$ and $\mathcal{R}$.

Following the same argument used in the proof of theorem 2.1 we can find for every element $v \in \mathcal{R}$ a function $g$ such that $\Psi_{u}(g)=g(u)=v$. Thus map $\Psi_{u}$ is surjective. Now it only remains to prove that $\Psi_{u}$ is injective.
Suppose that $\Psi_{u}(g)=\Psi_{u}\left(g^{\prime}\right)$, that is $g(u)=g^{\prime}(u)$, for some $g, g^{\prime} \in G$. Then by abelianness $h(g(u))=h\left(g^{\prime}(u)\right)$ gives that

$$
g(h(u))=g^{\prime}(h(u))
$$

for any $h \in G$. Consider the set $\{h(u): h \in G\}$. This set contains $u$ and we can reach every element in $\mathcal{R}$ by choosing a suitable $h$ and applying it to $u$, so $\{h(u): h \in G\}=\mathcal{R}$. This implies that $g(v)=g^{\prime}(v)$ for any $v \in \mathcal{R}$ and thus $g=g^{\prime}$. This proves that $\Psi_{u}$ is injective.

Combining the results of the previous theorems, we find that $\mathcal{R}$ is an abelian group. The following theorem shows that its structure is completely determined by the toppling matrix $\Delta$.

Theorem 2.3. Group $\mathcal{R}$ is isomorphic to $\mathbb{Z}^{n} / \Delta \mathbb{Z}^{n}$.
Proof. We will show that $G \cong \mathbb{Z}^{n} / \Delta \mathbb{Z}^{n}$. Then, by theorem 2.2 , we also know that $\mathcal{R} \cong \mathbb{Z}^{n} / \Delta \mathbb{Z}^{n}$. First consider the map $\Psi: \mathbb{Z}^{n} \rightarrow G$, defined by

$$
\Psi(m)=\prod_{i=1}^{n} a_{i}^{m_{i}}
$$

This map is a homomorphism, since clearly for $m, s \in \mathbb{Z}^{n}, \Psi(m+s)=\Psi(m) \Psi(s)$. Furthermore, $\Psi$ is surjective, so by the first isomorphism theorem $G \cong \mathbb{Z}^{n} / K$, where $K$ is the kernel of $\Psi$. We will prove that this kernel is equal to $\Delta \mathbb{Z}^{n}$, where

$$
\Delta \mathbb{Z}^{n}=\left\{\Delta m: m \in \mathbb{Z}^{n}\right\}
$$

by showing that both $\Delta \mathbb{Z}^{n} \subset K$ and $K \subset \Delta \mathbb{Z}^{n}$.
The toppling rules imply that for all $i=1, \ldots, n$,

$$
a_{i}^{\operatorname{deg}_{i}}=\prod_{j=1}^{n} a_{j}^{e_{i j}}
$$

and therefore

$$
a_{i}^{d e g_{i}} \prod_{j=1}^{n} a_{j}^{-e_{i j}}=e
$$

where $e$ is the neutral element of $G$, given by $\prod_{i=1}^{n} a_{i}^{0}$. We can write this in terms of the toppling matrix as

$$
\prod_{j=1}^{n} a_{j}^{\Delta_{i j}}=e
$$

This implies that we have that for all $m \in \mathbb{Z}^{n}$,

$$
\prod_{i=1}^{n} \prod_{j=1}^{n} a_{j}^{m_{i} \Delta_{i j}}=\prod_{j=1}^{n} a_{j}^{\sum_{i=1}^{n} m_{i} \Delta_{i j}}=e
$$

Since $\sum_{i=1}^{n} m_{i} \Delta_{i j}=(\Delta m)_{j}$ and $\Psi(\Delta m)=\prod_{i=1}^{n} a_{i}^{(\Delta m)_{i}}=e$ for all $m \in \mathbb{Z}^{n}$, we know that $\Delta \mathbb{Z}^{n} \subset K$.
Now let us prove that $\mathbb{Z}^{n} \subset K$. Suppose that $\Psi(m)=e$ for some $m \in \mathbb{Z}^{n}$. If we write $m=m^{+}-m^{-}$with $m_{i}^{+} \geq 0$ and $m_{i}^{-} \geq 0$, we have

$$
\prod_{i=1}^{n} a_{i}^{m_{i}^{+}}=\prod_{i=1}^{n} a_{i}^{m_{i}^{-}}
$$

Adding $m^{+}$to a recurrent configuration $u$ yields the same result after stabilization as adding $m^{-}$to $u$. Let us call this result $v$, then

$$
\begin{aligned}
& u+m^{+}=v+\Delta k^{+} \\
& u+m^{-}=v+\Delta k^{-}
\end{aligned}
$$

where $k_{i}^{+}$represents the number of topplings at vertex $i$ during the stabilization of $u+m^{+}$. In a similar way we define $k_{i}^{-}$. If we substract the two equations, we get

$$
m=m^{+}-m^{-}=\Delta\left(k^{+}-k^{-}\right)
$$

so $K \subset \Delta \mathbb{Z}^{n}$. We thus can conclude that $G$ is isomorphic to $\mathbb{Z}^{n} / \Delta \mathbb{Z}^{n}$ and $\mathcal{R}$ is too.

## Chapter 3

## Finitely generated abelian groups

We have seen that $\mathcal{R}$ is isomorphic to $\mathbb{Z}^{n} / \Delta \mathbb{Z}^{n}$. In this chapter we will learn what it means that this group is finitely generated and we will determine the structure of $\mathbb{Z}^{n} / \Delta \mathbb{Z}^{n}$ more precisely.

Some groups are simple in the sense that they can be generated by a few elements. Take for example $\mathbb{Z}^{3}$. Every element in this space is a linear combination of the vectors $(1,0,0),(0,1,0)$ and $(0,0,1)$, so the group $\mathbb{Z}^{3}$ is finitely generated. Let us define this property more precisely [13].

Definition 3.1. A group $G$ is finitely generated if there exists a finite number of elements $g_{1}, \ldots, g_{n}$ that generate $G$.

By means of these elements, called generators, we can write every finitely generated abelian group $G$ in the following way:

$$
G=\mathbb{Z} g_{1}+\ldots+\mathbb{Z} g_{n}
$$

Note that we have made a condition here: $G$ has to be an abelian group. As a result the order of the $g_{i}$ 's does not matter. We now define a very intuitive $\operatorname{map} \phi: \mathbb{Z}^{n} \rightarrow G$ by $\phi\left(m_{1}, \ldots, m_{n}\right)=m_{1} g_{1}+\ldots m_{n} g_{n}$. Map $\phi$ defines an onto homomorphism. Let $K$ be the kernel of $\phi$. This is a subgroup of $\mathbb{Z}^{n}$ and by the first isomorphism theorem $G \cong \mathbb{Z}^{n} / K$.

We want to determine the structure of $G$. However, it has now become possible to determine the structure of $K$ instead. To do this we first need the following theorem.

Theorem 3.2. Every subgroup of $\mathbb{Z}^{n}$ is finitely generated (by at most $n$ elements).

Proof. We will prove this by induction. Let $K$ be a subgroup of $\mathbb{Z}^{n}$. The case $n=0$ is trivial. If $n=1$, then we can find the least positive element $m$ in $K$. All the other elements are multiples of $m$. If not, there would have existed
an $h$ in $K$ not divisible by $m$. Then an element $m^{\prime}$ could be constructed as a linear combination of $m$ and $h$ such that $0 \leq m^{\prime}<m$. This contradicts our assumption about the minimality of $K$. So if $n=1$, then $K=m \mathbb{Z}, m \geq 0$.

Now consider the case $n \geq 2$. Assume that the theorem holds for every subgroup of $\mathbb{Z}^{n-1}$. Let $F$ consist of the first components of the elements of $K$. This is a subgroup of $\mathbb{Z}$. Choose the least positive element $f$ in $F$ (or $f=0$ if $F$ is trivial) and select $\left(f, c_{2}, \ldots, c_{n}\right) \in K$. Every $k_{1} \in F$ can now be written as $k_{1}=s f$, $s \in \mathbb{Z}$. So for all elements in $K$

$$
\left(k_{1}, \ldots, k_{n}\right)=s\left(f, c_{2}, \ldots, c_{n}\right)+\left(0, k_{2}-s c_{2}, \ldots, k_{n}-s c_{n}\right)
$$

The set of all elements $\left(k_{2}-s c_{2}, \ldots, k_{n}-s c_{n}\right)$ is a subgroup of $\mathbb{Z}^{n-1}$ and therefore by induction, finitely generated by at most $n-1$ elements. When we add $\left(f, c_{2}, \ldots, c_{n}\right)$ to the generators of this set, we find a set of at most $n$ generators, that generate $K$. This proves the theorem.

Kernel $K$ of $\phi$ is a subgroup of $\mathbb{Z}^{n}$ and therefore by theorem 3.2 finitely generated. Let $k_{1}, \ldots, k_{m}(m \leq n)$ be generators of $K$ and let $e_{1}, \ldots, e_{n}$ be generators of $\mathbb{Z}^{n}$. Write $k_{i}=a_{i 1} e_{1}+\ldots+a_{i n} e_{n}, 1 \leq i \leq m$ and set

$$
A=\left(\begin{array}{ccc}
a_{11} & \cdots & a_{m 1} \\
\vdots & & \vdots \\
a_{1 n} & \cdots & a_{n m}
\end{array}\right)
$$

Kernel $K$ can now be written as $A \mathbb{Z}^{n}$. By manipulating and rewriting generators, we can reduce this matrix to its Smith normal form $B$, as shown in equation (3.1). See section 3.1 for the construction of this form.

$$
B=\left(\begin{array}{ccc}
d_{1} & &  \tag{3.1}\\
& \ddots & \\
& & d_{m} \\
0 & \ldots & 0 \\
\vdots & & \vdots \\
0 & \ldots & 0
\end{array}\right)
$$

We will see that the non-negative integers $d_{1}, \ldots, d_{n}$ uniquely define the structure of $K$ and therefore also the structure of $G$. The proof of the fundamental theorem of finitely generated abelian groups, which we will state hereafter, is based mainly on this reduction and the fact that $A \mathbb{Z}^{n} \cong B \mathbb{Z}^{n}$. The proof of this theorem can be found in [11] and [14].

Theorem 3.3 (Fundamental theorem of Finitely Generated Abelian Groups). Let $G$ be a finitely generated abelian group. Then there is an isomorphism

$$
G \cong \mathbb{Z}^{r} \times \mathbb{Z} / d_{1} \mathbb{Z} \times \ldots \times \mathbb{Z} / d_{k} \mathbb{Z}
$$

where $d_{i}>1$ and $d_{1}|\ldots| d_{k-1} \mid d_{k}$. Furthermore, the integers $d_{i}$ and $r \geq 0$ are uniquely defined by $G$.

Definition 3.4. Let $G$ be a finitely generated abelian group. Then the integer $r$ in theorem 3.3 is called the rank and $d_{1}, \ldots, d_{k}$ are called the elementary divisors of $G$.

Example 3.5. Now consider $G=\mathbb{Z} / 2 \mathbb{Z} \times \mathbb{Z} / 9 \mathbb{Z}$. The rank of $G$ is 0 . The elementary divisors are somewhat harder to discover: 2 and 9 seem to be obvious options, but remember that $d_{1}$ should divide $d_{2}$. So first let us take a closer look at $G$ :

$$
\mathbb{Z} / 2 \mathbb{Z} \times \mathbb{Z} / 9 \mathbb{Z} \cong \mathbb{Z} / 18 \mathbb{Z}
$$

The elementary divisors of $G$ are $d_{1}=1$ and $d_{2}=18$, since $d_{1} \mid d_{2}$.

In the previous example we considered a quite simple and small group. We were able to determine the elementary divisors in a straightforward way. It will not be easy to apply this method to a group, that is much larger.

### 3.1 Smith normal form

We have already mentioned the Smith normal form briefly. We can reduce any integer matrix to this form by repeatedly applying the following steps:

1. Interchange two rows (or columns).
2. Multiply a row (or column) by -1 .
3. Add a multiple of one row (or column) to another.

Note that, unlike Gaussian elimination, a matrix reduction to Smith normal form requires both elementary row and elementary column operations. These operations should however not be executed in some random order. We can turn a matrix $A$ in finitely many steps into its Smith normal form, using the following algorithm [14]. We define $\left|a_{i j}\right|$ to be the size of entry $a_{i j}$.

Step 1. If $A$ is the zero matrix, then we are done. If not, pick a non-zero entry of $A, a_{i j}$, of least size. Interchange the first and the $i$ th row and the first and the $j$ th column.

Step 2. Repeatedly add the first column to or substract it from other columns, reducing the size $r$ of the entries $a_{1 j}$ to $0 \leq r<\left|a_{11}\right|$. If $r$ is non-zero, interchange the first and $j$ th column. In this way all entries in the first row, except for $a_{11}$, are turned zero.

Step 3. Turn all entries in the first column, except for $a_{11}$, zero in the same way.

Step 4. If $a_{11}$ does not divide every entry of matrix A, say it does not divide $a_{i j}$, then add the first row to the $i$ th row and then add or substract the first column a sufficient number of times to the $j$ th column. The size of $a_{i j}$ now is smaller than the size of $a_{11}$. Return to step 1 .

Step 5. Let $B$ be the matrix obtained by deleting the first row and first column of $A$. At this point $a_{11}$ divides all entries of $B$. Return to step 1 to follow the same steps for matrix $B$.

Step 6. By multiplying the rows with $\pm 1$, we finally find the Smith normal form of $A$.

Example 3.6. Let us reexamine example 3.5 from the previous section. We were looking for the elementary divisors of $\mathbb{Z} / 2 \mathbb{Z} \times \mathbb{Z} / 9 \mathbb{Z}$. Another way to write this group is $\mathbb{Z}^{2} / A \mathbb{Z}^{2}$, where

$$
A=\left(\begin{array}{ll}
2 & 0 \\
0 & 9
\end{array}\right)
$$

Running the algorithm on this matrix yields the following. The numbers above the arrows denote which step is executed.

$$
\left(\begin{array}{ll}
2 & 0 \\
0 & 9
\end{array}\right) \xrightarrow{4}\left(\begin{array}{cc}
2 & -8 \\
2 & 1
\end{array}\right) \xrightarrow{1}\left(\begin{array}{cc}
1 & 2 \\
-8 & 2
\end{array}\right) \xrightarrow{2}\left(\begin{array}{cc}
1 & 0 \\
-8 & 18
\end{array}\right) \xrightarrow{2}\left(\begin{array}{cc}
1 & 0 \\
0 & 18
\end{array}\right) .
$$

So the elementary divisors of $\mathbb{Z} / 2 \mathbb{Z} \times \mathbb{Z} / 9 \mathbb{Z}$ are 1 and 18 , as we saw before.

### 3.2 Number of elements

The Smith normal form of matrix $A$ does not only give us the full structure of $\mathbb{Z}^{n} / K$. It also tells us whether this group is finite or not. If the group is finite, then its number of elements is completely determined too.

Theorem 3.7. Let $\left\{e_{1}, \ldots, e_{n}\right\}$ be a basis of $\mathbb{Z}^{n}$. Let $K$ be a subset of $\mathbb{Z}^{n}$, generated by $k_{1}, \ldots, k_{n}$, where $k_{i}=a_{i 1} e_{1}+\ldots+a_{i n} e_{n}$. Denote by $A=\left(a_{i j}\right)$ the corresponding $n \times n$ matrix. Group $\mathbb{Z}^{n} / K$ is finite only if $\operatorname{det}(A) \neq 0$. Moreover, if $\operatorname{det}(A) \neq 0$, then $\# \mathbb{Z}^{n} / K=|\operatorname{det}(A)|$.

Proof. We have already seen that we can reduce matrix $A$ to Smith normal form, with integers $d_{1}, \ldots, d_{n}$ on its diagonal. Note that during the process of reducing, the determinant does not change except possibly for the sign, so $|\operatorname{det}(A)|=\left|d_{1} \cdot \ldots \cdot d_{n}\right|$. From theorem 3.3 we know that $\mathbb{Z}^{n} / K \cong \mathbb{Z} / d_{1} \mathbb{Z} \times \ldots \times$ $\mathbb{Z} / d_{n} \mathbb{Z}$. Some of the $d_{i}$ may be be zero. In fact, in theorem 3.3 we know that $r$ of the $d_{i}$ 's are zero, which results in the factor $\mathbb{Z}^{r}$. Therefore $\mathbb{Z}^{n} / K$ is a finite group only if none of the $d_{i}$ 's is zero, that is, if $\operatorname{det}(A) \neq 0$. If this is true, then $\# \mathrm{Z}^{n} / K=\left|d_{1} \cdot \ldots \cdot d_{n}\right|=|\operatorname{det}(A)|$.

## Chapter 4

## Minors-method

In chapter 1 we have introduced the toppling matrix $\Delta$. We have shown that this matrix uniquely determines the structure of $\mathcal{R}$ by means of its elementary divisors. One way to compute these elementary divisors is to determine the Smith normal form of $\Delta$. However, this is a quite elaborate job. Therefore we present in this chapter another way to find the elementary divisors.

The minors of $\Delta$ are used in this method. These are the determinants of the square submatrices of $\Delta$. For example, the $2 \times 2$ minors of

$$
A=\left(\begin{array}{lll}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9
\end{array}\right)
$$

are the determinants of the matrices that can be obtained by eliminating one row and one column of $A$,

$$
\left(\begin{array}{ccc}
1 & 2 & \square \\
\square & \square & \square \\
7 & 8 & \square
\end{array}\right) \Rightarrow B=\left(\begin{array}{ll}
1 & 2 \\
7 & 8
\end{array}\right)
$$

So $B$ is a submatrix of $A$ and its determinant -6 is a minor of $A$. Theorem 4.1 states the connection between the elementary divisors and the minors of a matrix. Since we are only interested in toppling matrices, which are square, we only state and prove this theorem for square matrices. However, theorem 4.1 holds in general, for all $m \times n$ integer matrices.

Theorem 4.1. Let $A$ be an $n \times n$ integer matrix and let $d_{1}, \ldots, d_{n}$ denote its elementary divisors. The product $\delta_{i}=d_{1} \cdots d_{i}$ equals the (non-negative) greatest common divisor of the $i \times i$ minors of $A$.

Proof. Any integer matrix can be reduced into Smith normal form. Since it is more easy to prove the theorem for matrices in this form, we will do this first. Then we will show that the reduction process does not affect the greatest common divisor (gcd) of the minors.

Let $B$ be the Smith normal form of $A$ :

$$
B=\left(\begin{array}{cccc}
d_{1} & 0 & \cdots & 0 \\
0 & d_{2} & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & d_{n}
\end{array}\right)
$$

Any $i \times i$ minor of $B$ is zero or the product of $i$ distinct $d_{j}$. Since $d_{1}\left|d_{2}\right| \ldots \mid d_{n}$ these minors will be divisible by the product of the first $i d_{j}$. This product is also the greatest common divisor of the minors, since $d_{1} \cdots d_{i}$ is an $i \times i$ minor itself. So we have proved the theorem for a matrix in Smith normal form.
Now we need to show the following: if $M$ is an integer matrix where $d_{1} \cdots d_{i}$ is the gcd of all of the $i \times i$ minors, then applying an elementary row or column operation to this matrix, will not affect this property.

1. Interchange a row (column) $I$ by a row (column) $J$.

For any $i \times i$ minor that does not involve $I$ or $J$, the minor is unchanged. If a submatrix contains both $I$ and $J$, its determinant is multiplied by -1 , which does not affect the gcd. The other minors, that involve only one row (column), are just permuted and are possibly multiplied by -1 . All this does not affect the gcd.
2. Multiply a row (column) by -1 .

By this operation only the signs of the minors can be changed, so this will not affect the gcd.
3. Add a multiple of row (column) $I$ to row (column) $J$.

For any $i \times i$ minor that does not involve $I$ or $J$, the minor is unchanged. If the minor contains both $I$ and $J$, the minor will remain the same as well. If only one row (column) is present in a submatrix, then the ggd of the minors remains the same, which follows from the fact that $\operatorname{ggd}(a+b, a)=$ $\operatorname{ggd}(a, b)$.

We saw that matrix $B$ satisfies the theorem. By reversing the elementary operations we reobtain $A$ without changing the greatest common divisor of the minors. This proves the theorem.

Example 4.2. Again take a look at example 1.1. We already saw that the matrix, corresponding to the graph in figure 1.1, is

$$
\Delta=\left(\begin{array}{cccc}
3 & -1 & -1 & 0 \\
-1 & 4 & -1 & -1 \\
-1 & -1 & 4 & -1 \\
0 & -1 & -1 & 3
\end{array}\right)
$$

Theorem 4.1 has provided us with an easy way to determine the structure of $\mathcal{R} \cong \mathbb{Z}^{4} / \Delta \mathbb{Z}^{4}$. We only have to consider the minors of $\Delta$. Since -1 is an entry of $\Delta, d_{1}$ is equal to 1 . Also the second elementary divisor $d_{2}$ is 1 , since

$$
\left(\begin{array}{cc}
-1 & 0 \\
-1 & -1
\end{array}\right)
$$

is a $2 \times 2$ submatrix of $\Delta$ and its determinant is 1 . Computer calculations show that $d_{3}=5$. The determinant of $\Delta$ is 75 , so $\delta_{4}=1 \cdot 1 \cdot 5 \cdot d_{4}=75$ and $d_{4}=15$. Now we have determined the structure of $\mathcal{R}$.

$$
\begin{align*}
\mathcal{R} & \cong \mathbb{Z} / 1 \mathbb{Z} \times \mathbb{Z} / 1 \mathbb{Z} \times \mathbb{Z} / 5 \mathbb{Z} \times \mathbb{Z} / 15 \mathbb{Z}  \tag{4.1}\\
& \cong \mathbb{Z} / 5 \mathbb{Z} \times \mathbb{Z} / 15 \mathbb{Z}
\end{align*}
$$

Example 4.3. We return to example 1.9, where we again meet commissioner 1 , who is very busy to determine the structure of his coworkers. Using theorem 4.1 he finds it rather easy to do this, for he notices that for $1 \leq i \leq n-1$ there alway exists an $i \times i$ lower triangular submatrix of $F_{n}$ with diagonal entries all -1 . The minor corresponding to this submatrix is $(-1)^{i}$, so for $1 \leq i \leq n-1$ the greatest common divisor of the $i \times i$ submatrices is 1 . So commisioner 1 knows that the group structure of his coworkers is cyclic, $\mathbb{Z} / \operatorname{det}\left(F_{n}\right) \mathbb{Z}$. The determinant of matrix $F_{n}$ is given by

$$
\operatorname{det}\left(F_{n}\right)=\frac{1}{2 \sqrt{3}}\left[(2+\sqrt{3})^{n+1}-(2-\sqrt{3})^{n+1}\right]
$$

A derivation of this formula is given in the appendix.
Sadly it did not take long before all the commisioners in the crazy office were fired. If only they had known that their boss's office was one floor below...

## Chapter 5

## Equivalence classes

In the previous chapter we presented a way to determine elementary divisors. The results were based directly on the toppling matrix. However, other methods to consider the structure of a toppling group have been discovered. These methods are generally more laborious.

A first method, described by Dhar, Ruelle, Sen and Verma [6], divides the space of all possible configurations into equivalence classes. By means of special functions, called toppling invariants, they label these classes. Each equivalence class contains exactly one recurrent configuration. So the invariants label the recurrent configurations as well. The toppling invariants provide us with full information about the structure of the toppling group.
A second method, described by Cori, Rossin and Salvy [3], deals with configurations and toppling operations in a polynomial setting. To every configuration they assign a monomial and the set of toppling operations gives rise to an ideal. Cori et al. define two equivalence relations, one in the regular and one in the polynomial setting. They claim that these relations are the same. In doing so they can say a lot about the toppling group, for example about its unit element.
We see that both methods heavily depend on equivalence relations. In this chapter we will take a closer look at these relations. We will state and restate the definitions used by the several authors. Moreover, we will try to find the relations between the several equivalences.
The terms we use for the different equivalences are not generally used in the literature.

### 5.1 Delta equivalence

Dhar, Ruelle, Sen and Verma [6] define equivalence classes, using the following definition.
Definition 5.1. Two configurations $\left\{z_{i}\right\}$ and $\left\{z_{i}^{\prime}\right\}$ are equivalent (under toppling) if there exist $N$ integers $n_{j}$ such that

$$
z_{i}^{\prime}=z_{i}-\sum_{j} \Delta_{i j} n_{j}, \quad \text { for all } i
$$

We can rewrite this definition into the following one, which is somewhat easier to work with.

Definition 5.2. Two configurations $u$ and $v$ are delta equivalent if

$$
u \in v+\mathbb{Z}^{n} \Delta
$$

Dhar et al. prove that each equivalence class contains exactly one recurrent configuration. This in fact proves that $\mathcal{R} \cong \mathbb{Z}^{n} / \Delta \mathbb{Z}^{n}$.

### 5.2 Toppling equivalence

Cori, Rossin and Salvy [3] use the notation $u \rightarrow v$ if $v$ is obtained from $u$ by toppling a vertex. The transitive closure of this operation is denoted by $\xrightarrow{*}$, so $u \xrightarrow{*} v$ means that $v$ is obtained from $u$ by a sequence of topplings. Note that this sequence does not have to be stabilizing, so $\xrightarrow{*}$ truly differs from operator $\mathcal{T}$. Cori et al. define equivalence in the following way.

Definition 5.3. Two configurations $u$ and $v$ are equivalent if $u \equiv v$, where $\equiv$ denotes the symmetric closure of $\xrightarrow{*}$.

This symmetric closure corresponds to allowing topplings and reverse topplings. A reverse toppling can be consdered as follows: when all neighbours $j$ of a vertex $i$ are non-empty, they can each give that vertex $e_{i j}$ grains. In case one or more sinks are attached to that vertex, the vertex receives an extra $s_{i}$ grains. Let us denote a reverse toppling by $\rightarrow$. We can restate definition 5.3 in the following way.

Definition 5.4. Two configurations $u$ and $v$ are toppling equivalent if $\mathcal{T}(u)=$ $\mathcal{T}(v)$.

The fact that definitions 5.3 and 5.4 are equivalent, that is $u \equiv v$ if and only if $\mathcal{T}(u)=\mathcal{T}(v)$, can be shown as follows. If $u \rightarrow v$, then it is clear that $\mathcal{T}(u)=$ $\mathcal{T}(v)$. If $u \rightarrow v$, then $v \rightarrow u$ and again $\mathcal{T}(u)=\mathcal{T}(v)$. So all configurations $w$, that we can derive from $u$ by applying a sequence of topplings $\rightarrow$ and reverse topplings $-\rightarrow$, have the property that $\mathcal{T}(u)=\mathcal{T}(w)$. Therefore, if $u \equiv v$, then $\mathcal{T}(u)=\mathcal{T}(v)$. The reverse statement is true as well. We can find sequences of topplings and reverse topplings $S$ and $T$, that respectively derive $\mathcal{T}(u)$ from $u$ and $\mathcal{T}(v)$ from $v$. If we combine $S$ with the reverse of $T$, we have found a sequence that derives $v$ from $u$. So $\mathcal{T}(u)=\mathcal{T}(v)$ implies $u \equiv v$.

### 5.3 Ideal equivalence

Cori, Rossin and Salvy [3] define another kind of equivalence as well. In order to understand this equivalence, we first have to take a closer look at the polynomial setting mentioned in the introduction of this chapter.

### 5.3.1 Polynomial setting

Let $X$ denote the finite set $\left\{x_{1}, \ldots, x_{n}\right\}$ and $\mathbb{Q}[X]$ the ring of polynomials with variables in $X$ and coefficients in $\mathbb{Q}$. We can link a configuration $u=$ $\left(u_{1}, \ldots, u_{n}\right) \in \mathbb{Z}_{\geq 0}^{n}$ to a monomial in this space. To this end we consider $u$ as a degree vector and put $x^{u}=x_{1}^{u_{1}} \cdots x_{n}^{u_{n}}$. The addition of two configurations, $u$ and $v$, results in a multiplication in the polynomial setting:

$$
x^{u+v}=x_{1}^{u_{1}+v_{1}} \cdots x_{n}^{u_{n}+v_{n}}=x_{1}^{u_{1}} \cdots x_{n}^{u_{n}} \cdot x_{1}^{v_{1}} \cdots x_{n}^{v_{n}}=x^{u} \cdot x^{v} .
$$

This helps us to translate topplings into $\mathbb{Q}[X]$. Recall that if we topple a vertex 1 configuration $u$ changes into

$$
v=\left(u_{1}-d_{1}, u_{2}+e_{12}, \ldots, u_{n}+e_{1 n}\right)
$$

Let $v$ be a degree vector. Then

$$
x^{v}=x_{1}^{u_{1}-d_{1}} \cdot x_{2}^{u_{2}+e_{12}} \cdots x_{n}^{u_{n}+e_{1 n}}=x_{1}^{-d_{1}} \prod_{j=1}^{n} x_{j}^{e_{1 j}} \cdot x^{u}
$$

Note that we have added an extra term $x_{1}^{e_{11}}$ to the product. This does not matter, since $e_{11}=0$. Now let us put this more in general. A toppling of a configuration $u$ at a site $i$ corresponds to the multiplication of the monomial $x^{u}$ by $x_{i}^{-d_{i}} \prod_{j=1}^{n} x_{j}^{e_{i j}}$ in $\mathbb{Q}[X]$.

### 5.3.2 Toppling ideal

First let us recall the definition of an ideal in $\mathbb{Q}[X]$.
Definition 5.5. A subset $I \in \mathbb{Q}[X]$ is an ideal if it satisfies the following conditions:
(I1) If $f, g \in I$, then $f+g \in I$
(I2) If $f \in I$ and $h \in \mathbb{Q}[X]$, then $h f \in I$.
We can construct an ideal in $\mathbb{Q}[X]$ by taking $s$ polynomials from this space and setting

$$
\left\langle f_{1}, \ldots, f_{s}\right\rangle=\left\{\sum_{i=1}^{s} h_{i} f_{i}: h_{i} \in \mathbb{Q}[X]\right\}
$$

Then $\left\langle f_{1}, \ldots, f_{s}\right\rangle$ is called the ideal generated by $f_{1}, \ldots, f_{s}[4]$. We can define a really meaningful ideal in $\mathbb{Q}[X]$ when it comes to configuration polynomials.

Definition 5.6. The toppling ideal $\mathcal{I}$ is the ideal generated by

$$
\left\{x_{i}^{d_{i}}-\prod_{j=1}^{n} x_{j}^{e_{i j}}: i=1, \ldots, n\right\} .
$$


(a)

(b)

Figure 5.1: Toppling graph without and with sink

However, this is not the ideal that was defined by Cori, Rossin and Salvy. They add an extra vertex to a toppling graph. Vertex 0 is the 'sink vertex' and satisfies the properties $e_{0 i}=e_{i 0}=s_{i}$ for all $i=1, \ldots, n$ and $e_{00}=0$. The degree of this vertex is $d e g_{0}=\sum_{i=1}^{n} s_{i}$. This degree however does not say anything about the capacity of the sink vertex, which is in fact infinite. Cori et al. present a toppling ideal, that is slightly different from the one in definition 5.6.

Definition 5.7. The toppling ideal $\mathcal{I}_{s}$ is the ideal generated by $x_{0}-1$ and

$$
\left\{x_{i}^{d_{i}}-\prod_{j=1}^{n} x_{j}^{e_{i j}}: i=0, \ldots, n\right\}
$$

We can now define two new equivalences.
Definition 5.8. Two configurations $u$ and $v$ are ideal equivalent if $x^{u}-x^{v} \in \mathcal{I}$.
Definition 5.9. Two configurations $u$ and $v$ are $S$-ideal equivalent if $x^{u}-x^{v} \in$ $\mathcal{I}_{s}$.

### 5.4 Relations between equivalences

We now have four equivalence relations. We wonder in what way these relations are connected. Cori, Rossin and Salvy state the following theorem.

Theorem 5.10. Two configurations $u$ and $v$ are toppling equivalent if and only if $x^{u}-x^{v} \in \mathcal{I}_{s}$.

We can easily disprove this by showing that we can find configurations $u$ and $v$ such that $x^{u}-x^{v} \in \mathcal{I}_{s}$, though they are not toppling equivalent. Recall that $x^{u}-x^{v} \in \mathcal{I}_{s}$ if and only if $x^{u}-x^{v}$ reduces to zero in $\mathbb{Q}[X] / \mathcal{I}_{s}$. No let us consider the toppling graph in figure 5.1 (a), which with a sink vertex can be represented as in figure 5.1(b).
Here $\mathcal{I}_{s}=\left\langle x^{2}-y z, y^{2}-x z, z^{2}-x y, z-1\right\rangle$. We have that

$$
\mathbb{Q}[x, y, z] / \mathcal{I}_{s} \cong \mathbb{Q}[x, y] /\left\langle x^{2}-y, y^{2}-x, 1-x y\right\rangle
$$

Now if $u=(0,0)$ and $v=(1,1)$, then $x^{u}-x^{v}=1-x y$ reduces to zero in $\mathbb{Q}[x, y, z] / \mathcal{I}_{s}$. This would mean that $u$ and $v$ are toppling equivalent. However, since $\mathcal{T}(u)=u \neq v=\mathcal{T}(v)$ this is not true.


Figure 5.2: Toppling graph without and with sink

The question arises whether there exist connections between the several equivalence relations or not. Intuitively, one would say this is the case. To get a firmer hold on this problem we consider the following example.

Example 5.11. Look at the toppling graph in figure 5.2(a). On this graph, $u=(0,1)$ and $v=(0,0)$ are the only stable configurations. We are going to answer the question 'Are $u$ and $v$ equivalent?' for each of the four equivalence relations.

- Toppling equivalence. Answer: no.

Configurations $u$ and $v$ are stable and not the same, so $\mathcal{T}(u) \neq \mathcal{T}(v)$.

- Delta equivalence. Answer: yes.

The toppling matrix, corresponding to the toppling grapg in figure 5.2(a), is given by

$$
\Delta=\left(\begin{array}{cc}
0 & -1 \\
-1 & 1
\end{array}\right)
$$

We have that $u=v-1 \cdot(0,-1)+0 \cdot(-1,1)$, so $u \in v+\mathbb{Z}^{2} \Delta$.

- Ideal equivalence. Answer: no.

The ideal corresponding to the graph in figure $5.2(\mathrm{a})$ is $\mathcal{I}=\left\langle x-y, y^{2}-x\right\rangle$ and $(x, y)^{u}-(x, y)^{v}=y-1$. We have that

$$
\begin{aligned}
y-1 \bmod \mathbb{Q}[x, y] / \mathcal{I} & \equiv x-1 \bmod \mathbb{Q}[x] /\left\langle x^{2}-x\right\rangle \\
& \not \equiv 0 \bmod \mathbb{Q}[x] /\left\langle x^{2}-x\right\rangle
\end{aligned}
$$

Thus $y-1 \notin \mathcal{I}$.

- S-ideal equivalence. Answer: yes.

Figure $5.2(\mathrm{~b})$ shows the toppling graph, extended with a sink vertex. Now the ideal becomes $\mathcal{I}_{s}=\left\langle x-y, y^{2}-x z, z-y, z-1\right\rangle$. Since we have that

$$
\mathbb{Q}[x, y, z] / \mathcal{I}_{s} \cong \mathbb{Q}[x, y] /\left\langle x-y, y^{2}-x, 1-y\right\rangle,
$$

we know that $y-1 \in \mathcal{I}_{s}$.
From this example we can derive that neither delta nor S-ideal equivalence can imply toppling or ideal equivalence. Other implications still remain possible. Consider for example theorems 5.12 and 5.13 , that show two simple connections between the several equivalence relations.
Theorem 5.12. If two confgurations are toppling equivalent, the they are also delta equivalent.

Proof. If we topple a configuration $u$ at a vertex $i$, we can write the result as $u-\Delta_{i}$, wheere $\Delta_{i}$ denotes the $i$ th row of $\Delta$. We therefore have for $\mathcal{T}(u)$ that

$$
\begin{equation*}
\mathcal{T}(u)=u-\sum_{i=1}^{n} m_{i} \Delta_{i} \tag{5.1}
\end{equation*}
$$

where $m_{i}$ is the number of topplings over vertex $i$ during the stabilization of $u$. In the same way we have for $\mathcal{T}(v)$ that

$$
\begin{equation*}
\mathcal{T}(v)=v-\sum_{i=1}^{n} k_{i} \Delta_{i} \tag{5.2}
\end{equation*}
$$

Configurations $u$ and $v$ are toppling equivalent, so $\mathcal{T}(u)=\mathcal{T}(v)$. Now if we combine the two equations (5.1) and (5.2), we find that

$$
u=v+\sum_{i=1}^{n} m_{i} \Delta_{i}-\sum_{i=1}^{n} k_{i} \Delta_{i}=v+\sum_{i=1}^{n}\left(m_{i}-k_{i}\right) \Delta_{i} \in v+\mathbb{Z}^{n} \Delta .
$$

So $u$ and $v$ are delta equivalent.
Theorem 5.13. If two confgurations are ideal equivalent, the they are also $S$-ideal equivalent.

Proof. This follows directly from the fact that $\mathcal{I} \subset \mathcal{I}_{s}$.
At this point we know both relations between toppling equivalence and delta equivalence. The first implies the second, but the second does not imply the first. The same holds for ideal equivalence and S-ideal equivalence. We now would like to compare an equivalence relation in polynomial setting to such a relation in our regular setting. Example 5.11 already gives us an idea about the possibilities.

## Chapter 6

## Discussion

We have seen in chapter 2 that the set of recurrent configurations forms a group, that is isomorphic to $\mathbb{Z}^{n} / \Delta \mathbb{Z}^{n}$. Group $\mathbb{Z}^{n} / \Delta \mathbb{Z}^{n}$ is a finitely generated abelian group, of which the structure is uniquely determined by its elementary divisors. We can determine the elementary divisors by computing the Smith normal form of the toppling matrix $\Delta$. We can also apply the minors-method, described in chapter 4 , that uses the minors of $\Delta$ to determine the elementary divisors.
In chapter 5 I have focussed specifically on equivalence classes. Although I have a strong feeling that the following two statements hold, I have not been able to prove them:

- Two configurations are delta equivalent if and only if they are S-ideal equivalent.
- Two configurations are toppling equivalent if and only if they are ideal equivalent.

These conjectures are supported by the following two facts:

- According to [3] all S-ideal equivalence classes contain exactly one recurrent configuration. According to [6] the same holds for the delta equivalence classes.
- For some small graphs I have found that $|\mathbb{Q}[X] / \mathcal{I}|$ is equal to the number of stable configurations. This is also the number of equivalence classes under $\mathcal{T}$.


## Appendix

Our aim is to find the determinant of the $n \times n$ matrix

$$
F_{n}=\left(\begin{array}{ccccc}
4 & -1 & 0 & \cdots & 0 \\
-1 & 4 & -1 & & \vdots \\
0 & -1 & \ddots & \ddots & 0 \\
\vdots & & \ddots & 4 & -1 \\
0 & \cdots & 0 & -1 & 4
\end{array}\right)
$$

Let this determinant be denoted by $u_{n}$. If we expand the determinant along the first row, we find a recurrence relation for $u_{n}$.

$$
\begin{aligned}
u_{n} & =4 u_{n-1}+\operatorname{det}\left(\begin{array}{ccccc}
-1 & -1 & 0 & \cdots & 0 \\
0 & 4 & -1 & & \vdots \\
0 & -1 & \ddots & \ddots & 0 \\
\vdots & & \ddots & 4 & -1 \\
0 & \cdots & 0 & -1 & 4
\end{array}\right) \\
& =4 u_{n-1}-u_{n-2} .
\end{aligned}
$$

We could choose to take the initial conditions

$$
u_{1}=\operatorname{det} F_{1}=4 \quad \text { and } \quad u_{2}=\operatorname{det} F_{2}=15,
$$

but since $u_{2}=4 u_{1}-1 \cdot 1$, we can simplify these conditions by taking

$$
u_{0}=1 \quad \text { and } \quad u_{1}=4
$$

We can rewrite our recurrence relation into matrix form:

$$
\left(\begin{array}{cc}
4 & -1 \\
1 & 0
\end{array}\right)\binom{u_{n-1}}{u_{n-2}}=\binom{u_{n}}{u_{n-1}}
$$

This form gives rise to the following matrix equality:

$$
\left(\begin{array}{cc}
4 & -1 \\
1 & 0
\end{array}\right)^{n-1}\binom{u_{1}}{u_{0}}=\binom{u_{n}}{u_{n-1}}
$$

By diagonalizing matrix

$$
A=\left(\begin{array}{cc}
4 & -1 \\
1 & 0
\end{array}\right)
$$

it becomes easy to compute the $n-1$ th power of this matrix. The eigenvalues of matrix $A$ are $\lambda_{1}=2+\sqrt{3}$ and $\lambda_{2}=2-\sqrt{3}$. Let $S$ be the matrix whose columns are the eigenvectors of $A$ then

$$
S=\left(\begin{array}{cc}
2+\sqrt{3} & 2-\sqrt{3} \\
1 & 1
\end{array}\right) \quad \text { and } \quad S^{-1}=\frac{1}{2 \sqrt{3}}\left(\begin{array}{cc}
1 & -2+\sqrt{3} \\
-1 & 2+\sqrt{3}
\end{array}\right) .
$$

Let $D$ be the diagonal matrix with the eigenvalues of $A$ on its diagonal. Then

$$
A^{n-1}=S D^{n-1} S^{-1}=S\left(\begin{array}{cc}
(2+\sqrt{3})^{n-1} & 0 \\
0 & (2-\sqrt{3})^{n-1}
\end{array}\right) S^{-1}
$$

Determining the first entry of the column vector

$$
A^{n-1}\binom{u_{1}}{u_{0}}
$$

results in the following formula for the determinant of $F_{n}$ :

$$
\operatorname{det}\left(F_{n}\right)=u_{n}=\frac{1}{2 \sqrt{3}}\left[(2+\sqrt{3})^{n+1}-(2-\sqrt{3})^{n+1}\right]
$$

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