On the Parametrization over $\mathbb{Q}$ of Cubic Surfaces

René Pannekoek
Supervisor: prof. dr. J. Top

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1 Introduction

In this thesis a central role is played by cubic surfaces in $\mathbb{P}^3$. Informally speaking, a cubic surface consists of triples $(x, y, z) \in \mathbb{C}^3$ satisfying $f(x, y, z) = 0$, where f is a third degree polynomial with coefficients in some field. In this thesis, we will often look at surfaces arising from polynomials over $\mathbb{Q}$, the field of rational numbers. We say that these surfaces are *defined over* $\mathbb{Q}$.

The geometry of cubic surfaces has been an object of study since the 19^{th} century. In 1849 it was discovered by Cayley and Salmon that every non-singular (or smooth) cubic surface contains exactly 27 straight lines. These lines, along with their intersection properties, contain a lot of information about the surface itself.

The interest of cubic surfaces also lies in their connection with number theory. For instance, the following questions are directly related to the geometry of cubic surfaces.

Question 1.1. Can every rational number be expressed as the sum of three cubes of rational numbers?

Question 1.2. Find an integer that is expressible as the sum of two cubes of integers in two distinct ways.

Question 1.3. Find an infinite number of rational solutions to $f(x, y, z) = 0$, where f is a third-degree polynomial (not necessarily homogeneous).

An important property of cubic surfaces is that they can be *parametrized* with rational functions, or stated in a slightly more technical way, for any cubic surface S defined over $\mathbb{Q}$ there is a birational map $\phi : \mathbb{P}^2(\overline{\mathbb{Q}}) \to S$. But for a problem like Question 1.3, this is not enough. Not only do we want a parametrization in terms of rational functions, but these rational functions have to be defined over $\mathbb{Q}$, that is, we don't want them to contain any non-rational numbers. We are thus led to pose the following question: for which cubic surfaces S does there exist a birational map $\phi : \mathbb{P}^2(\overline{\mathbb{Q}}) \to S$ that is defined over $\mathbb{Q}$?

1.1 Outline of the thesis

In this section I will give a brief outline of this thesis.

In Chapter 2, the reader finds some facts from the algebraic geometry of surfaces (blow-ups, Picard group, intersection form) and their applications to cubic surfaces. Also, the symmetry of the 27 lines is examined, resulting in a brief discussion of the Weyl group $W(\mathrm{E}_6)$.

In Chapter 3, the main result of Swinnerton-Dyer is stated: this result gives a necessary and sufficient criterion for a smooth cubic surface S over a number field K to be birationally trivial over K; that is, to allow a K-birational map to $\mathbb{P}^2$. This is followed by a classification of birationally trivial smooth cubic surfaces. Finally, we turn to the problem of explicitly finding K-rational and K-birational maps, distinguishing three cases according to the classification mentioned.

In Chapter 4, some birationally trivial cubic surfaces are presented. Here it is shown that all types of the classification are indeed represented by a smooth cubic surface. Also, we exhibit smooth cubic surfaces that are birationally trivial over $\mathbb{Q}$, but not blow-ups over $\mathbb{Q}$.

In Chapter 5, we turn to the computational aspects of Swinnerton-Dyer's criterion. A Maple algorithm to find all 27 lines on a smooth cubic surface S is presented. It turns out that, modulo the difficulties of working in high-degree number fields, the criterion can be easily checked. The case of the twisted Fermat cubic surface $x^3 + y^3 + z^3 + 2w^3 = 0$ is done as an example.

Chapter 6 is devoted to birationally non-trivial surfaces and the Galois orbits of straight lines lying on them. We explore some different types of orbits and establish the fact that an orbit must have cardinality 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 16, 18, 24 or 27, and that moreover all these cardinalities do indeed occur. Next, we discuss a possible way of finding a cubic surface without any rational points, but with an orbit of 6 pairwise skew lines.

1.2 About notation and conventions

Throughout the thesis, K denotes a number field.

By $\mathbb{P}^n(K)$ is meant the set $(K^n - (0, 0, \ldots, 0)) / \sim$ where $(a_1, a_2, \ldots, a_n) \sim (b_1, b_2, \ldots, b_n)$ if and only if there is $\lambda \in K^*$ such that $b_i = \lambda a_i$ for all i. When the field K is apparent from context, we will just write $\mathbb{P}^n$.

Whenever we speak of a curve or surface, and a ground field is not mentioned, the ground field is understood to be just $\mathbb{Q}$.

To avoid clutter, we occasionally use the same notation for divisors and their classes in the Picard group. When there is any danger of confusion, we denote the divisor class of C as $\widehat{C}$.

When studying a cubic surface S, we sometimes use the existence of a morphism $\pi : S \to \mathbb{P}^2$ that blows down 6 lines. It is sometimes useful to swap back and forth between S and its image under π. In these cases, the image of any subvariety ℓ of S under π will be denoted $\overline{\ell}$.

For $n \in \mathbb{N}$, we will sometimes call a set of n elements an n-set. Likewise, when discussing group actions, we will refer to an orbit consisting of n elements as an n-orbit.

Throughout the thesis, S denotes a **smooth** cubic surface and L its set of 27 lines. L' usually denotes a distinguished subset of L. We will reserve the calligraphic letter ℓ, with or without primes and subscripts, for lines on S.

2 Prerequisites about cubic surfaces

We now come to the definition of a cubic surface, as it will be used in this thesis.

Definition 2.1. A cubic surface over a number field K is a set

$$\left\{(x:y:z:w) \in \mathbb{P}^3(\overline{\mathbb{Q}}) : F(x,y,z,w) = 0\right\} \subset \mathbb{P}^3(\overline{\mathbb{Q}}),$$

where $F \in K[x,y,z,w]$ is irreducible and homogeneous of degree 3.

If we do not specify the ground field K, the cubic surface is understood to be defined over $\mathbb{Q}$.

2.1 The 27 lines on a cubic surface

We will mainly study the geometry of a cubic surface by the lines on it: there are always 27 of them.

Theorem 2.2. There are 27 lines on any smooth cubic surface.

Proof. We will not give the details, for which plenty of references exist, e.g. [6]. The first step is to demonstrate that any smooth cubic surface contains at least one line. This can simply be done by "counting constants" (see [9, Ch. 1, pp. 79-80]). The next step is to prove that every line is contained in 5 distinct tritangent planes, i.e. planes that intersect the cubic surface in a union of three lines. The last step is to fix a tritangent plane H, then count the number of lines intersecting one of the three lines in H, and lastly showing that there are no more lines than the ones already found. □

Remark 2.3. On a surface with only isolated singularities, there are still lines, but their number is strictly less than 27.

In order to establish the fact that smooth cubic surfaces over a number field K can be parametrised over its algebraic closure $\overline{\mathbb{Q}}$, the concept of a blow-up is convenient.

Theorem 2.4. Let S be a smooth surface and let $x \in S$. There exists a surface $\mathrm{Bl}_x S$ and a morphism $\phi : \mathrm{Bl}_x S \to S$, unique up to isomorphism, such that

1. $\phi^{-1}(x) \cong \mathbb{P}^1$
2. $\phi : \mathrm{Bl}_x S \backslash \phi^{-1}(x) \to S \backslash \{x\}$ is an isomorphism

(By a surface we mean any two-dimensional projective variety, in particular it does not have to be embeddable in $\mathbb{P}^3$.) We call $\mathrm{Bl}_x S$ the *blow-up* of S in the point x. The map ϕ is called the *blow-down morphism*, while the (restriction of its) inverse ϕ^{-1} is called the *blow-up map.* The inverse image $\phi^{-1}(x)$ is the *exceptional divisor* of the blow-up $\mathrm{Bl}_x S$. Finally, for every curve $C \subset S$, we define the *strict transform* of C to be $\overline{\phi^{-1}(C - \{x\})} \subset \mathrm{Bl}_x S$, where the bar denotes the Zariski closure.

Throughout the rest of this thesis, we will need the notion of a set of n points being "in general position". For simplicity, and since we will not need anything more, we will only treat the case $n = 6$.

Definition 2.5. Six points in a projective plane $\mathbb{P}^2(F)$, where F is any algebraically closed field, are said to be in general position if no three lie on a line, and not all six on a conic.

In 1871, Alfred Clebsch established the fact that every smooth cubic surface can be realised as a blow-up of the plane in the union of six points in general position. Of course, being a classical geometer of the 19$^{\text{th}}$ century, Clebsch only established this for surfaces defined over $\mathbb{C}$. By virtue of the Lefschetz principle, we can translate both of his results into geometry over $\overline{\mathbb{Q}}$. His results then read as follows:

Theorem 2.6. Let $p_1, \ldots, p_6 \in \mathbb{P}^2(\overline{\mathbb{Q}})$ be six points in general position and let $\{f_1, f_2, f_3, f_4\}$ be a basis for the $\overline{\mathbb{Q}}$-vector space of cubic curves vanishing in all of the p_i. Then the rational map $\phi : \mathbb{P}^2 \to \mathbb{P}^3$ given by $\phi(P) = (f_1(P) : f_2(P) : f_3(P) : f_4(P))$ is a blow-up of $\mathbb{P}^2$ in the six points p_i, and the Zariski closure $\overline{\phi(\mathbb{P}^2 - \{p_1, \ldots, p_6\})} \subset \mathbb{P}^3$ is a smooth cubic surface.

As a converse to this, we have:

Theorem 2.7. Every smooth cubic surface over $\overline{\mathbb{Q}}$ is isomorphic over $\overline{\mathbb{Q}}$ to $\mathbb{P}^2(\overline{\mathbb{Q}})$ blown up in six points in general position.

The intersection properties of the 27 lines on a cubic surface S can be investigated by considering S as a blow-up of the projective plane. We have the following:

Theorem 2.8. Let S be a smooth cubic surface and $\pi : S \to \mathbb{P}^2$ a blow-down morphism, and let p_i $(1 \leq i \leq 6)$ be the images of the exceptional divisors. Then the image of the 27 lines under π are:

1. the six points $p_i =: \overline{\ell_i}$
2. the fifteen lines $\overline{\ell_{ij}}$ connecting p_i and p_j $(1 \leq i < j \leq 6)$
3. the six conics $\overline{\ell_i'}$ passing through all p_j except p_i

Proof. The proof is an explicit verification, using the fact that S is a blow-up of the plane. We will do a sample case; the rest of the cases go similarly. Let S be a smooth cubic surface arising as a blow-up of $\mathbb{P}^2$ in $p_1, \ldots, p_6$. We will check the case of the conic not passing through p_1, whose defining polynomial we denote C_1. Let F, G be a basis of linear forms vanishing in p_1, then FC_1,GC_1 are linearly independent cubic forms vanishing in all p_i. Let H, J be cubic forms vanishing in the p_i such that FC_1, GC_1, H, J are linearly independent. We will now choose the blow-up map ϕ given by $\phi(P) = (FC_1(P) : GC_1(P) : H(P) : J(P))$; any other choice would amount to a projective linear transformation in $\mathbb{P}^3$. The image is clearly contained in the line $X = Y = 0$, and since ϕ is an isomorphism outside the p_i, it must be the whole line. $\square$

A very classical and useful geometric description of the 27 lines on a cubic surface is furnished by Schläfli's *double six*. It is immediately linked to the description of the lines as listed in Theorem 2.8.

Definition 2.9. A double six is a set of 12 lines (on a cubic surface S), which we shall denote by $\{\ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6, \ell'_1, \ell'_2, \ell'_3, \ell'_4, \ell'_5, \ell'_6\}$, such that no two of the ℓ_i intersect, no two of the ℓ'_i intersect and ℓ_i intersects ℓ'_j if and only if $i \neq j$.

Furthermore, the remaining 15 lines can be labeled ℓ_{ij}, for $1 \leq i < j \leq 6$, in a unique way such that ℓ_i meets ℓ_{jk} if and only if $i = j$ or $i = k$, ℓ'_i meets ℓ_{jk} if and only if $i = j$ or $i = k$ and ℓ_{ij} meets ℓ_{km} if and only if $\{i, j\} \cap \{k, m\} = \varnothing$.

Remark 2.10. The notation used in Definition 2.9 already suggests one possibility for a double-six: again, regard S as the blow-up of $\mathbb{P}^2$ in $p_1, \ldots, p_6$. Let the ℓ_i be the exceptional divisors corresponding to the points p_i, let ℓ'_i be the strict transforms of the conics not passing through p_i and let ℓ_{ij} be the strict transforms of the lines through p_i and p_j.

We can now easily check that these choices do indeed give rise to a double-six. I will work out two cases, the other four are even easier.

First I prove $\ell'_i \cap \ell'_j = \varnothing$ if $i \neq j$. Fix two conics $\overline{\ell'_i}$, $\overline{\ell'_j}$. These meet in the four distinct points $p_1, \ldots, \hat{p}_i, \ldots, \hat{p}_j, \ldots, p_6$, so by Bézout they cannot have a double contact in any of the remaining p_k: equivalently, they intersect each p_k at different slopes. By an elementary property of blow-ups, this means that their strict transforms intersect ℓ_k at different points. Since the blow-up map is an isomorphism outside the p_k, the strict transforms ℓ'_i, ℓ'_j do not intersect.

Now the proof that ℓ'_i meets ℓ_{jk} if and only if $i = j$ or $i = k$. First, fix a line $\overline{\ell_{ij}}$ and a conic $\overline{\ell_k}$, where $k \notin \{i, j\}$. Their points of intersection are p_i and p_j. By Bézout, these intersections are single contacts, and so the strict transforms are disjoint on S. Secondly, fix a line $\overline{\ell_{ij}}$ and a conic $\overline{\ell_j}$. These meet in p_i and none other of p_k: either it has a double contact at p_i, or they intersect outside the union of the p_k; either way, the strict transforms intersect.

To see what is "behind" the double-six configuration, and to describe the symmetry of the 27 lines (which for instance lead to more double-sixes), we need to delve a little further into the geometry of surfaces and examine the concept of the *Picard group* of a surface.

2.2 The symmetry of the 27 lines

> Their elegant symmetry both enthralls and at the same time irritates; what use is it to know, for instance, the number of coplanar triples of such lines (forty-five) or the number of double Schläffli sixfolds (thirty-six)?
>
> Manin ([5, p. 112])

The 27 lines possess a remarkably high degree of symmetry. Not only does every of the 27 lines intersect exactly 10 of the other lines, but also does every pair of skew lines intersect 15 other lines, and every triple of skew lines intersect 18 other lines, and so on. We are thus led to ask ourselves what the symmetry group of the lines is, that is, the group of all permutations that preserve the incidence relations among the lines. This is formalized in the notion of the *collineation group* of a set of lines:

Definition 2.11. Let L be a collection of lines (for example in $\mathbb{P}^3$) with intersection map $i : L \times L \to \{0, 1\}$ (0 denoting crossing lines and 1 denoting intersecting or equal lines). Let $\mathrm{Sym}(L)$ be the permutation group on the elements of L. The collineation group of L, denoted G_L, is the subgroup of all elements $\sigma \in \mathrm{Sym}(L)$ satisfying $i(\sigma(\ell_1), \sigma(\ell_2)) = i(\ell_1, \ell_2)$ for all pairs $\ell_1, \ell_2 \in L$.

One way to get a good grip on the collineation group of the 27 lines is the double-six. The first thing to establish is the answer to the following question: *how many double-sixes are there on a cubic surface?*

Proposition 2.12. There are 36 double-sixes on a smooth cubic surface S.

Proof. We start out with any double-six, denoted in the same way as in Definition 2.9. Claim: there are 72 sets of 6 pairwise disjoint lines on S. This can be verified by just listing them all:

- 2 sets A_1 and A_2 given by $A_1 := \{\ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6\}$ and $A_2 := \{\ell'_1, \ell'_2, \ell'_3, \ell'_4, \ell'_5, \ell'_6\}$,
- 30 sets B_{ij} determined by a pair i, j satisfying $1 \leq i \neq j \leq 6$, consisting of the 2 lines ℓ_i and ℓ'_i, and the 4 lines ℓ_{jk}, where $k \in \{1, \dots, 6\} \backslash \{i, j\}$,
- the 20 sets C_{ijk} determined by a triple $1 \leq i < j < k \leq 6$, consisting of the 3 lines ℓ_i, ℓ_j, ℓ_k and the 3 lines ℓ_{mn}, where $m, n \notin \{i, j, k\}$
- the 20 sets C'_{ijk} determined by a triple $1 \leq i < j < k \leq 6$, consisting of the 3 lines ℓ'_i, ℓ'_j, ℓ'_k and the 3 lines ℓ_{mn}, where $m, n \notin \{i, j, k\}$

It is easy to see that there is no other way to get 6 pairwise skew lines on S. These 72 sets correspond to 36 double sixes in an obvious way: A_1 goes with A_2, B_{ij} goes with B_{ji} and C_{ijk} goes with C'_{ijk}. □

With the above explicit description of the double-sixes on a smooth cubic surface S, it is easy to derive the order of the collineation group G_L. First off, any element $\sigma \in G_L$ sends a double-six to a double-six, giving us 36 choices. This has to be multiplied by the number of elements that sends the double-six to itself. Now, G_L can act on a double-six by interchanging the two sets $\{\ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6\}$ and $\{\ell'_1, \ell'_2, \ell'_3, \ell'_4, \ell'_5, \ell'_6\}$ or by any permutation of the elements of $\{\ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6\}$, inducing the corresponding permutation on $\{\ell'_1, \ell'_2, \ell'_3, \ell'_4, \ell'_5, \ell'_6\}$. This gives us a total of $36 \cdot 2 \cdot 6! = 51{,}840$ elements in G_L.

Important Remark 2.13. Using the above reasoning, we can now exploit the symmetries of the lines on S in the following way. As we have seen, given 6 pairwise skew lines on S, there is a double-six on S that contains those lines. *Therefore, whenever we encounter 6 lines that are pairwise skew, we can "embed" them into a double-six.* From there, we can make use of everything we know about double-sixes.

Even more is true. For $1 \leq n \leq 4$, if we have n pairwise skew lines, we may embed those into a double-six as the lines $\ell_1, \ldots, \ell_n$! (The only thing to check is that we can supply a set of 4 pairwise skew lines on a cubic surface with two more lines to make a set of 6 pairwise skew lines.) This gives us a very convenient tool to settle all kinds of enumerative questions concerning the 27 lines. As a little example, we take the following question: *how many lines intersect 2 given skew lines on a smooth cubic surface?* Taking the 2 skew lines to be ℓ_1, ℓ_2, we see that there are 5 lines intersecting both ℓ_1 and ℓ_2, namely the lines $\ell_{12}, \ell_3, \ell_4, \ell_5, \ell_6$.

2.3 The 27 lines and their images in the Picard group

Having defined and to some extent investigated the collineation group of the 27 lines, we are still in search of its precise identity. We will define the *Picard group* associated to a smooth cubic surface S, and show how to embed the 27 lines as elements of that group. The collineation group acts on the images of the 27 lines in the Picard group, and this action will turn out to be very easy to describe. Using these ideas, the collineation group G_L will be identified as a well-known finite group, a so-called *Weyl group* going by the name $W(\mathrm{E}_6)$.

2.3.1 Some definitions

We will set out to define the Picard group, but first there are a number of preliminary definitions to be made.

Definition 2.14. Let C be the set of irreducible curves lying on S. The divisor group of S (denoted $\mathrm{Div}(S)$) is then the free abelian group on C, or equivalently, the group of formal sums $n_1C_1 + n_2C_2 + \ldots + n_kC_k$ where $n_i \in \mathbb{Z}$ and $C_i \in C$.

Definition 2.15. An effective divisor on S is a divisor D that can be written as $D = \sum n_i \cdot C_i$, where $n_i > 0$.

Remark 2.16. In other words, effective divisors correspond with codimension 1 subvarieties in a one-to-one way, counting irreducible components with multiplicity.

The divisor group is too large to be any kind of useful invariant, so we want to divide out a large subgroup. The subgroup we are aiming for consists of the so-called *principal divisors*. Principal divisors arise from functions on S. Let f be a function on S. Then its associated principal divisor, denoted (f), is constructed in the following way: the positive terms correspond to the irreducible curves on S where f is identically zero and the negative terms correspond to the irreducible curves where f is everywhere undefined; the coefficient attached to an irreducible curve appearing in (f) equals the multiplicity of the zero or pole.

Example 2.17. Let S be the smooth cubic surface given by $x^3 + y^3 + z^3 + w^3 = 0$ and consider the function $f = xy/z^2$ on S. Passing to the open affine set S' by intersecting with $w \neq 0$ and passing to the new variables $t = x/w$, $u = y/w$, $v = z/w$, we can write S' as $t^3 + u^3 + v^3 + 1 = 0$ and f becomes tu/v^2. Roughly speaking, the "zeros" of the numerator correspond to the zeros of f, while the zeros of the denominator are the poles of f, as long as we count multiplicities. Setting the numerator equal to zero gives us the union of the irreducible curves $C_1 := Z(t, u^3 + v^3 + 1)$ and $C_2 := Z(u, t^3 + v^3 + 1)$. The denominator vanishes doubly on $C_3 := Z(v, t^3+u^3+1)$. So we see that $(f) = C_1+C_2-2C_3$.

The group of all principal divisors on S is denoted $\mathrm{PDiv}(S)$.

Definition 2.18. The Picard group of S, denoted $\mathrm{Pic}(S)$, is defined as $\mathrm{Div}(S)/\mathrm{PDiv}(S)$.

The elements of $\mathrm{Pic}(S)$ are equivalence classes of divisors on S and are hence called *divisor classes*. We denote the canonical map by $\widehat{\ } : \mathrm{Div}(S) \to \mathrm{Pic}(S)$. Two divisors D_1, D_2 are called *linearly equivalent* if they map to the same image in the Picard group under the canonical map: we will write this as $D_1 \sim D_2$, which is equivalent to $\widehat{D_1} = \widehat{D_2}$.

Example 2.19. In Example 2.17, and using the same notation, we saw that $C_1+C_2-2C_3$ is a principal divisor. Its divisor class is therefore the identity element in the Picard group. Written as a formula, this is $C_1+C_2-2C_3 \sim 0$, or, what comes to the same thing, $C_1 + C_2 \sim 2C_3$, denoting the identity element in the Picard group simply by 0.

Example 2.20. We have $\mathrm{Pic}(\mathbb{P}^2) \cong \mathbb{Z}$. (See for instance [9, Ch. 3, p. 154].) The isomorphism simply sends the divisor class of the curve $\{F = 0\}$, where F is an irreducible homogeneous polynomial, to $\deg F \in \mathbb{Z}$. This means that all curves of the same degree have the same class in the Picard group. We can convince ourselves of this in the following way: let $C_1 := \{F = 0\}$ and $C_2 := \{G = 0\}$ be two curves such that F, G are irreducible and of the same degree. Then $f := F/G$ defines a function on $\mathbb{P}^2$ and its divisor (f) is equal to $(f) = C_1 - C_2$. Equivalently, $C_1 = C_2 + (f)$, so C_1 and C_2 are linearly equivalent. (More generally, $\mathrm{Pic}(\mathbb{P}^n) \cong \mathbb{Z}$.)

The Picard group on any surface comes equipped with an intersection form, that is, a bilinear map $i : \mathrm{Pic}(S) \times \mathrm{Pic}(S) \to \mathbb{Z}$. This intersection form does exactly what its name suggests: if C_1, C_2 are curves on S meeting transversally and $\widehat{C}_1, \widehat{C}_2$ are their classes in $\mathrm{Pic}(S)$, then $i(\widehat{C}_1, \widehat{C}_2)$ denotes the number of points of the intersection $C_1 \cap C_2$, counting multiplicities.

Example 2.21. On $\mathbb{P}^2$, the intersection form satisfies $i(\ell, \ell) = 1$, where ℓ is the divisor class of a line. Note that this makes sense, since two distinct lines meet in exactly one point. Also, if C_m is a curve of degree m and C_n is a curve of degree n, then their classes in $\mathrm{Pic}(\mathbb{P}^2)$ are $m\ell$ and $n\ell$ respectively by Example 2.20. So their intersection product becomes $i(\widehat{C}_m, \widehat{C}_n) = i(m\ell, n\ell) = mn$ by the bilinearity of i, meaning that C_1 and C_2 meet mn times counting multiplicities. Of course, we already knew this from Bézout's theorem.

2.3.2 The Picard group of a cubic surface

To describe $\mathrm{Pic}(S)$, it is again useful to keep the blow-down morphism $\pi : S \to \mathbb{P}^2$ in mind. We use the same notation as in Theorem 2.8.

Theorem 2.22. Let S be a smooth cubic surface. Then $\mathrm{Pic}(S) \cong \mathbb{Z}^7$. Furthermore, $\mathrm{Pic}(S)$ is freely generated by $\ell, e_1, e_2, e_3, e_4, e_5, e_6$, where ℓ is the divisor class of the strict transform of a general line (not equal to any of the $\overline{\ell_{ij}}$) and the e_i are the classes of the $\overline{\ell_i}$.

Proof. I will only give the general idea. The result follows from the fact that we can realise S as the result of six successive blow-ups of $\mathbb{P}^2$, which has Picard group $\mathbb{Z}$, each blow-up adding a direct summand $\mathbb{Z}$, corresponding to the class of its exceptional curve. For the details, see Hartshorne, ([4, p. 401]). □

The intersection form on S is completely defined by the following relations: $i(\ell, \ell) = 1$, $i(\ell, e_i) = 1$ where $1 \leq i \leq 6$ and $i(e_i, e_j) = -\delta_{ij}$, where δ_{ij} is the Kronecker delta. (This too is established in Hartshorne's book ([4, pp.401-2]), but it is again an easy consequence of the fact that S is a blow-up.) Using the intersection form on S, we may deduce the divisor classes of the remaining 21 lines. Denote the divisor class of the ℓ_{ij} by e_{ij} and the divisor class of the ℓ'_i by e'_i.

Lemma 2.23. The divisor classes of the remaining 21 lines are as follows: (1) the class of ℓ_{ij} is $\ell - e_i - e_j$ $(1 \leq i < j \leq 6)$ and (2) the class of ℓ'_i is $2\ell + e_j - \sum_{j=1}^{6} e_j$ $(1 \leq i \leq 6)$.

Proof. By our discussion of the intersection form, the classes of the e_{ij} and e'_i are determined by the number of times they intersect the curves representing $\ell, e_1, \ldots, e_6$. The line ℓ_{ij} intersects both ℓ_i and ℓ_j, which means that $i(e_{ij}, e_i) = 1$, $i(e_{ij}, e_j) = 1$ and $i(e_{ij}, e_k) = 0$ for $k \neq i, j$. Furthermore, ℓ_{ij} intersects the strict transform of a general line in $\mathbb{P}^2$ exactly once, so $i(e_{ij}, \ell) = 1$. These 7 equations combined yield that $e_{ij} = \ell - e_i - e_j$. In the same way, we find that $e'_i = 2\ell + e_i - \sum_j e_j$. □

Remark 2.24. By applying the previous Lemma, we get that $i(e_{ij}, e_{ij}) = -1$ for all i, j and $i(e'_i, e'_i) = -1$ for all i. This means that all 27 lines on S are exceptional curves by Castelnuovo's Contractibility Criterion ([1, Ch. 2, p. 21]).

2.3.3 The Weyl group

In what follows, we will identify the Picard group of S with $\mathbb{Z}^7$, which inherits the bilinear form i from $\mathrm{Pic}(S)$. Let $\omega := (-3, 1, 1, 1, 1, 1, 1) \in \mathbb{Z}^7$, or equivalently, $\omega = -3\ell + \sum_i e_i$. (This happens to be the canonical class of a smooth cubic surface, hence the notation.) For the results in this subsection, which are stated entirely without proof, we refer to Manin's book ([5, Ch. 4, §23-26]).

Definition 2.25. We define the *root system* $\mathrm{E}_6 \subset \mathbb{Z}^7$ as the subset of all elements $v \in \mathbb{Z}^7$ satisfying $i(v, \omega) = 0$ and $i(v, v) = -2$.

The set E_6 is finite and has 72 elements. Of these, we define the elements $v_1 := (1, 1, 1, 1, 0, 0, 0)$, $v_2 := (0, 1, -1, 0, 0, 0, 0)$, $v_3 := (0, 0, 1, -1, 0, 0, 0)$, $v_4 := (0, 0, 0, 1, -1, 0, 0)$, $v_5 := (0, 0, 0, 0, 1, -1, 0)$ and $v_6 := (0, 0, 0, 0, 0, 1, -1)$. Furthermore, for any $w \in \mathbb{Z}^7$ we define $\phi_w : \mathbb{Z}^7 \to \mathbb{Z}^7$, the *reflection through* w, by

$$\phi_w(a) := a - 2\frac{i(w, a)}{i(w, w)} w \tag{1}$$

This is a linear transformation leaving the hyperplane $H_w := \{v \in \mathbb{Z}^7 : i(v, w) = 0\}$ fixed. Let the set Φ consist of the reflections through the v_i, so $\Phi := \{\phi_{v_1}, \phi_{v_2}, \phi_{v_3}, \phi_{v_4}, \phi_{v_5}, \phi_{v_6}\}$.

Proposition 2.26. All elements of E_6 can be obtained by taking one of the v_i and applying finitely many reflections of Φ. Furthermore, the reflections of Φ send elements of E_6 to elements of E_6.

Viewing $\mathbb{Z}$ as a subset of $\mathbb{Q}$, we can view Φ as a subset of $\mathrm{GL}_7(\mathbb{Q})$. Therefore, the elements of Φ generate a subgroup of $\mathrm{GL}_7(\mathbb{Q})$ leaving E_6 invariant. Also, as we can check using Equation 1, the elements of Φ leave the divisor classes of the 27 lines invariant.

We are ready for the main theorem. Here, I_6 is the set of divisor classes of the 27 lines.

Theorem 2.27. The following three groups are isomorphic:

- the subgroup of $\mathrm{GL}_7(\mathbb{Q})$ sending $\mathbb{Z}^7$ to $\mathbb{Z}^7$ and preserving ω and $i(\cdot, \cdot)$
- the group of permutations of the elements of I_6 preserving their pairwise intersection products given by i (this is exactly the collineation group of the 27 lines)
- the subgroup of $\mathrm{GL}_7(\mathbb{Q})$ generated by the elements of Φ, also known as the Weyl group $W(\mathrm{E}_6)$

We have now realized the collineation group of the 27 lines as a linear group acting faithfully on a number of structures in 7-dimensional space, including the set of lines itself as represented by the set I_6. This very remarkable series of facts extends to blow-ups of $\mathbb{P}^2$ in not just 6, but a given number of points (but from 9 points on strange things begin to happen). More about root systems, Weyl groups and their application to blow-ups of $\mathbb{P}^2$ are to be found in Manin's book ([5]).

We close this chapter with a little application of the results found so far.

2.4 Orbits of the 27 lines under Galois

Let S be a smooth cubic surface defined over a number field k. The absolute Galois group $G = \mathrm{Gal}(\overline{\mathbb{Q}}/k)$ acts on S, but also on L, the set of lines on S, as lines have to go to lines. Moreover, it preserves all incidence properties between the elements of L. This means that the action of G on L factors through $W(\mathrm{E}_6)$. This presents us with an easy corollary:

Theorem 2.28. Let S be a cubic surface. S does not contain a G-orbit of 7, 11, 13, 14, 17, 19, 21, 22, 23, 25 or 26 lines.

Proof. Since the cardinality of any orbit would have to divide 51,840. □

The above result is pretty strong: orbits of all other cardinalities occur, except for one consisting of 20 lines. This is stated and proven in Theorem 6.8.

3 The Galois action on the 27 lines

As we will see, given a number field K, the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$ on the 27 lines determines whether or not the surface is birational to $\mathbb{P}^2$ over K. Assuming some knowledge about the birational geometry of surfaces, we can already see why this is true in a special case: Suppose a smooth cubic surface S defined over K contains a K-rational point and a set of 6 skew lines defined over K. Using the fact that the exceptional curves on S are precisely the 27 lines lying on it, we see that we can blow down these 6 lines over K. The resulting blown-down surface is isomorphic to $\mathbb{P}^2$ over $\overline{K} = \overline{\mathbb{Q}}$; moreover, it contains a K-rational point, so it is isomorphic to $\mathbb{P}^2$ over K.

However, as we will see, a cubic surface does not have to be a blow-up of $\mathbb{P}^2$ over K to be birational to $\mathbb{P}^2$ over K (we will establish this in Chapter 4). The precise conditions for birationality to $\mathbb{P}^2$ are given by a theorem of Swinnerton-Dyer ([10]), as we will see in the next section.

3.1 Swinnerton-Dyer's theorem

Definition 3.1. We call a cubic surface birationally trivial over K if it is birationally equivalent to $\mathbb{P}^2(K)$. Occasionally, when the ground field K is evident from the context, we will just say that a cubic surface is birationally trivial if it is birationally equivalent to $\mathbb{P}^2(K)$.

Theorem 3.2. *Let S be a smooth cubic surface defined over a number field K. S is birationally trivial if and only if (a) S contains a point defined over K and (b) S contains a $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$-stable set of 2, 3 or 6 pairwise skew lines.*

Proof. The proof consists of exhibiting a two-dimensional linear system of curves on S satisfying certain properties; it mainly relies on the Riemann-Roch theorem and the adjunction formula. See Swinnerton-Dyer's article ([10, pp. 12-15]). □

3.1.1 Some consequences of the theorem

Using Swinnerton-Dyer's theorem, we can partition the set of birationally trivial cubic surfaces into five types. This is almost more of a semantical than a mathematical business, but it simplifies the task of finding a parametrization if we look at one Type at a time.

We need a little lemma, which illustrates an important way of reasoning we shall use over and over again:

Lemma 3.3. *Suppose a smooth cubic surface S, defined over K, contains a stable set of 5 pairwise skew lines, such that there exists at least one line on S not intersected by any of them. Then it contains two skew rational lines.*

Proof. We can assume that the five skew lines are $L := \cup_{i=1}^{5} \ell_i$. Then we see that there is a unique line intersecting none lines in L, namely ℓ_6. For any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/K)$, the image of ℓ_6 under σ must be a line not intersecting any of the lines in σL, which is again L. So we must have that $\sigma\ell_6 = \ell_6$, in other words, ℓ_6 is rational.

Furthermore, there is a unique line intersecting *all* of the lines in L, namely ℓ'_6. By the same argument, ℓ'_6 is also rational. So we have the two rational lines ℓ_6 and ℓ'_6 on S, and these are skew. □

Proposition 3.4. (i) If a smooth cubic surface S is birationally trivial to K, it falls into one of the following types of smooth cubic surfaces containing a K-rational point:

no.	smooth cubic surfaces containing a K-rational point and
I	2 skew rational lines
II	an orbit of 2 skew lines, and no set of 2 skew rational lines
III	an orbit of 3 skew lines and no stable set of two or six pairwise skew lines
IV	6 skew lines forming two orbits of order 3
V	an orbit of 6 skew lines

(ii) A cubic surface can only be of one type.

Remark 3.5. Before we give the proof, a short remark is in order. Of course the type of a cubic surface depends on the field K over which one works. By "a cubic surface of Type N" we will mean a cubic surface of Type N over $\mathbb{Q}$.

Proof. (i) First, we prove that the above types exhaust all smooth birationally trivial cubic surfaces. Let S be a smooth birationally trivial cubic surface, so it has a K-rational point and it contains a $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$-stable set of 2, 3 or 6 lines.

Suppose that S contains a stable set of 6 lines. Then these lines form a set of full Galois orbits according to one of the following partitions of 6: $6 = 5+1 = 4+2 = 4+1+1 = 3+3 = 3+2+1 = 3+1+1+1 = 2+2+2 = 2+2+1+1 = 2+1+1+1+1 = 1+1+1+1+1+1$. We see that all partitions correspond to Type I, II, IV or V cubic surfaces (for $5+1$ we use Lemma 3.3).

We may now suppose that S does not contain a stable set of 6 lines. So S must contain a stable set of 2 or 3 skew lines. First assume that S does contain a stable set of 2 skew lines. Then S obviously falls into Type I or II. If S does *not* contain a stable set of 2 or 6 skew lines, it has to contain a stable set of 3 skew lines, which must be a 3-orbit. This is precisely Type III.

(ii) So we have proven that Types I-V exhaust the class of birationally trivial cubic surface. Now for their pairwise disjointness.

That Type III is disjoint from any of the others is obvious by the last condition in its definition.

Also, we can check that Type V is disjoint from all the others by going through all the possible Galois actions on a 6-orbit L_6 (there are 16 of them, corresponding to the 16 transitive subgroups of the symmetric group S_6). A given Galois action on L_6 completely determines the action on all 27 lines: the 6 skew lines uniquely determine a double six, which shows that a line on S is completely determined by how it intersects with the lines in L_6. We can now check if there is a Galois action giving rise to a configuration falling under one of the Types I-IV, and it turns out that there isn't. (The checking is done in Remark 3.7.)

That Types I and II are disjoint is also obvious from the way they're defined, so the only non-trivial part of this Proposition is the fact that Type IV is disjoint from Types I and II. For this, take a smooth cubic surface S with the orbits $\{\ell_1, \ell_2, \ell_3\}$ and $\{\ell_4, \ell_5, \ell_6\}$. Now every orbit on S has cardinality ≥ 3: the stable set $\{\ell_{12}, \ell_{23}, \ell_{13}\}$ must be an orbit precisely because $\{\ell_1, \ell_2, \ell_3\}$ is: for instance if $\sigma\ell_1 = \ell_2$, then $\sigma\ell_{23} = \ell_{13}$, etc. The set $\{\ell_{45}, \ell_{56}, \ell_{46}\}$ is an orbit for the same reason. The set $\{\ell_{14}, \ell_{24}, \ell_{34}, \ell_{15}, \ell_{25}, \ell_{35}, \ell_{16}, \ell_{26}, \ell_{36}\}$ seems to allow for a lot of different possible Galois actions, but at least we have that any line intersecting ℓ_1 should be conjugate to a line intersecting ℓ_2 and to another line intersecting ℓ_3, etc. From this we see that here too, every orbit has to have cardinality ≥ 3. This shows that we can't have a stable set of cardinality 2 in this case, so we're done. $\square$

Remark 3.6. Moreover, over $\mathbb{Q}$, every type is indeed represented by a smooth cubic surface. We will show this by exhibiting examples of each type in Chapter 4.

Remark 3.7. In the table below I have computed the subdivision of the 27 lines on S in distinct Galois orbits in the case where S contains an orbit of 6 skew lines $\{\ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6\}$. In that case, the set $\{\ell'_1, \ell'_2, \ell'_3, \ell'_4, \ell'_5, \ell'_6\}$ is also an orbit, so the only lines left to consider are the ℓ_{ij} for $1 \leq i < j \leq 6$. The table is based on the well-known numbering first used in the article [2]. For all groups 6T*NN*, I have started from a set of generators and computed the conjugates of all lines ℓ_{ij}. This leads to a subdivision of the lines ℓ_{ij} into orbits.

One more note on notation: the symbols $a^m b^n c^p \dots$ mean: an orbit of a within which every line intersects m others, etc. From this information we may conclude that no 3-orbits of skew lines arise.

G	orbits on S	G	orbits on S	G	orbits on S	G	orbits on S
6T1	$6^0\, 6^0\, 6^3\, 6^3\, 3^2$	6T5	$6^0\, 6^0\, 9^4\, 6^3$	6T9	$6^0\, 6^0\, 9^4\, 6^3$	6T13	$6^0\, 6^0\, 9^4\, 6^3$
6T2	$6^0\, 6^0\, 6^3\, 3^2\, 3^2\, 3^2$	6T6	$6^0\, 6^0\, 12^5\, 3^2$	6T10	$6^0\, 6^0\, 9^4\, 6^3$	6T14	$6^0\, 6^0\, 15^6$
6T3	$6^0\, 6^0\, 6^3\, 6^3\, 3^2$	6T7	$6^0\, 6^0\, 12^5\, 3^2$	6T11	$6^0\, 6^0\, 12^5\, 3^2$	6T15	$6^0\, 6^0\, 15^6$
6T4	$6^0\, 6^0\, 12^5\, 3^2$	6T8	$6^0\, 6^0\, 12^5\, 3^2$	6T12	$6^0\, 6^0\, 15^6$	6T16	$6^0\, 6^0\, 15^6$

3.2 Finding parametrizations

Having established the existence of a K-birational map $f : S \to \mathbb{P}^2$, the next question is of course: can we find such an f explicitly? In an abstract way, these maps can be defined using linear systems associated to certain divisors on S, but this cannot be done without a computer algebra system, and it does not admit of a nice geometric description.

The problem is then: *for each birationally trivial smooth cubic surface S, find a K-birational map $f : S \to \mathbb{P}^2$ arising from a purely geometric construction.*

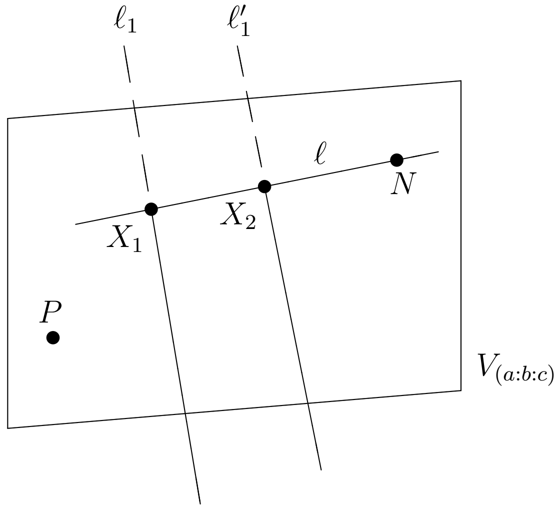
In this section, we will give partial results on this problem. We divide the problem into three cases: we do Types I-II in Subsection 3.2.1, Types IV and V in Subsection 3.2.2 and Type III in Subsection 3.2.3.

3.2.1 Cubic surfaces of Type I or II.

This is the easiest case. Let S be a smooth cubic surface containing a stable 2-set of lines. Manin ([5, Ch. 4, §31, pp. 191]) observes that it automatically has a point defined over K (and hence infinitely many). To see this, assume ℓ_1, ℓ_2 are skew lines on S and form an orbit under Galois (if they are rational, we are done). Intersect the S with a rational plane to find a Galois orbit of two points on S. Consider the line through these points: if the line is contained in S, it is rational; if not, it intersects S in a single, and hence rational point. We state this as a lemma:

Lemma 3.8. If a smooth cubic surface S contains a set of two skew lines which is defined over $\mathbb{Q}$, then it has a rational point. Hence, it is birational to $\mathbb{P}^2$.

This observation also helps us in constructing a birational map $\phi : \mathbb{P}^2 \to S$. Let $\{\ell_1, \ell_2\}$ be a stable set of skew lines. Fix a point P on S, not on a line, and let F_1, F_2, F_3 be linearly independent linear forms vanishing in P. Then to every $M := (a : b : c) \in \mathbb{P}^2$, we associate the plane $V_{(a:b:c)}$ given by $aF_1 + bF_2 + cF_3 = 0$. This gives us a bijection between $\mathbb{P}^2$ and the planes passing through P. The plane $V_{(a:b:c)}$ intersects ℓ_1 and ℓ_2, say in the points X_1 and X_2. Then the line ℓ intersects S in X_1, X_2 and a third point N, which we take to be $\phi(M)$. (The only way in which this can go wrong is if $V_{(a:b:c)}$ contains ℓ_1 or ℓ_2, which happens for two choices of $(a : b : c)$, or if ℓ happens to be a line on S, which happens for five more choices of $(a : b : c)$. So ϕ is well-defined outside these seven points.)



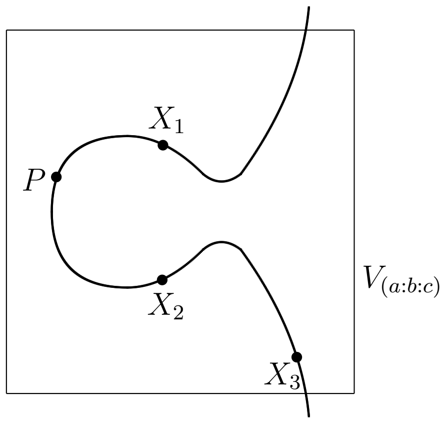
It is obvious from the geometric way in which ϕ is defined that ϕ is rational. But ϕ also has an inverse: given Q, we can find the line ℓ by the elementary fact that there is a unique line passing through Q and intersecting ℓ_1 and ℓ_2. Then take the plane through P and ℓ, and we get our point $(a:b:c)$ back. So ϕ is a birational map.

3.2.2 Cubic surfaces of Type IV-V.

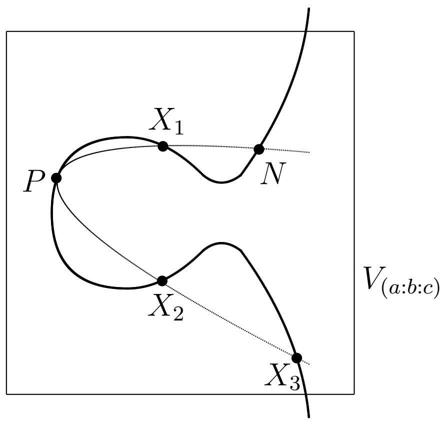
This case can be simply dealt with "in principle", but not yet in practice. Let S be of Type IV or V. Then we know that S is a blow-up over $\mathbb{Q}$: this is because S contains a set of 6 pairwise skew lines which is defined over $\mathbb{Q}$, so these lines can be blown down over $\mathbb{Q}$. This means that we know that there is a birational map $\mathbb{P}^2 \to S$ given by four cubic polynomials in x, y, z: this leaves us only 40 coefficients to determine! In his PhD thesis ([7]), Josef Schicho points out that this method of constructing a (bi)rational map is completely intractable from a computational point of view. He does, however, suggest an alternative, which might or might not help us out, but due to a lack of time I have been unable to sort this out.

3.2.3 Cubic surfaces of Type III.

Type III are not blow-ups by their definition. We try something similar to what we did for Type I/II surfaces. Let S be a cubic surface of Type III and let $\{\ell_1, \ell_2, \ell_3\}$ be a 3-orbit of pairwise skew lines. We will construct a rational map $\phi : \mathbb{P}^2 \to S$ as follows.



Again, we fix a point P on S, again not lying not on a line, and we consider the set of planes through P, which can be naturally identified with $\mathbb{P}^2$ in the same way as before. For a point $M := (a:b:c) \in \mathbb{P}^2$, let $V_{(a:b:c)}$ again be the corresponding plane through P. Then $V_{(a:b:c)}$ intersects S in a cubic curve C which is smooth for "most" points M by Bertini's theorem. Now, $V_{(a:b:c)}$ intersects the lines ℓ_1, ℓ_2, ℓ_3 in X_1, X_2, X_3 respectively. The points X_i also lie on C. Let C' be the conic lying in $V_{(a:b:c)}$, tangent to C at P and passing through X_1, X_2, X_3. Then by Bézout, C' intersects C in six points counting multiplicities, so apart from P, X_1, X_2, X_3 there is one remaining point of intersection. Call it N. Then we define $N := \phi(M)$.



The main question now is: *is ϕ a birational map*? This does not seem to be the case. For a specific Type III cubic surface, namely the one to be constructed in Subsection 4.3.1, we have computed a number of values of ϕ, since it was already impossible for MAPLE to find a general expression for ϕ. For all points $M \in \mathbb{P}^2$, we found that there was exactly one other point $M' \in \mathbb{P}^2$ satisfying $\phi(M) = \phi(M')$. It seems, then, that the maps ϕ constructed according to this method are 2:1 (so in particular, its image is Zariski dense in S), but a rigorous proof of this I have not yet been able to find.

To clarify this situation, I see the following approaches. Let us assume for the moment that the rational map has degree 2, as conjectured.

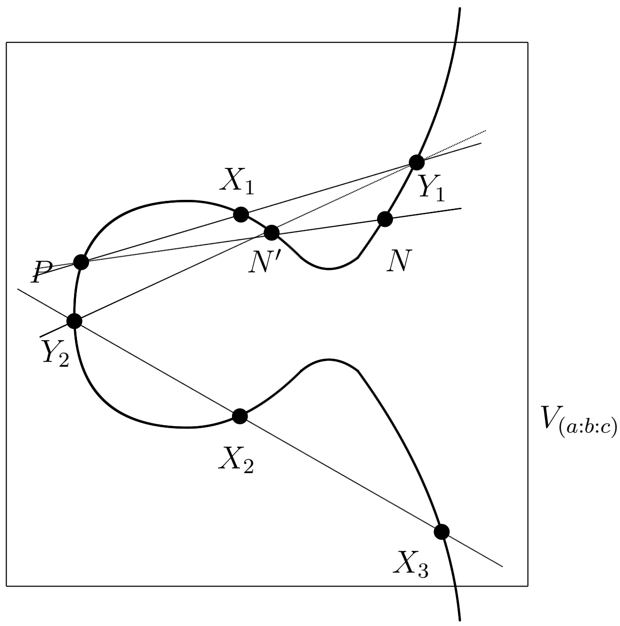
1. There exists an involution $i : \mathbb{P}^2 \to \mathbb{P}^2$ such that $\phi = \phi \circ i$. Computing this involution for some values of M suggests that it is not of too complicated a nature, but it seems a hard problem to determine it explicitly. Suppose it can be done, however, then we have an explicit group $\Gamma := \{1, i\} \subset \mathrm{Aut}(\mathbb{P}^2)$ and a birational map $\widetilde{\phi} : \mathbb{P}^2/\Gamma \to S$ such that $\phi = \widetilde{\phi} \circ \kappa$ where $\kappa : \mathbb{P}^2 \to \mathbb{P}^2/\Gamma$ is the natural map. My question is: can we use κ to construct an $\mathbb{Q}$-birational map $\kappa' : \mathbb{P}^2 \to \mathbb{P}^2/\Gamma$ so that $\widetilde{\phi} \circ \kappa' : \mathbb{P}^2 \to S$ is birational?

2. Projection from P to a plane in $\mathbb{P}^3$ gives a 2:1 rational map $\pi : S \to \mathbb{P}^2$. Together with ϕ, this induces us a tower of inclusions: $\mathbb{Q}(\mathbb{P}^2) \hookrightarrow \mathbb{Q}(S) \hookrightarrow \mathbb{Q}(\mathbb{P}^2)$. This means that $\mathbb{Q}(S)$ is squeezed in between two purely transcendental field extensions of $\mathbb{Q}$: $\mathbb{Q}(s,t) \hookrightarrow \mathbb{Q}(S) \hookrightarrow \mathbb{Q}(u,v)$. Questions: is $\mathbb{Q}(s,t) \subset \mathbb{Q}(u,v)$ a Galois extension? Can we use this picture to find two explicit generators for $\mathbb{Q}(S)$ over $\mathbb{Q}$? Does Galois theory help?

3. I have also tried a more "classical" approach. Given a point N on S, we would like a purely geometric description of its preimages under ϕ. This doesn't seem a dead end by any means, and I have the distinct impression that it should work out. For now it doesn't however. To get some grip on what is going on, I considered two surfaces Q_1 and Q_2 defined in the following way. Fix $N \in S$. We then define Q_1 the union of all conics which (i) lie in a plane V containing the line NP; (ii) intersect

N and P and (iii) intersect the lines ℓ_1, ℓ_2, ℓ_3. Next, define Q_2 as as the the union of all conics which (i) lie in a plane V containing the line NP; (ii) are tangent to $V \cap S$ in P and (iii) intersect the lines ℓ_1, ℓ_2, ℓ_3. Then the intersection of Q_1 and Q_2 is a finite union of curves in $\mathbb{P}^3$, and contains the conics corresponding to the preimages of N.

4. Finally, I want to mention a different way of defining ϕ. In effect, we replace the conic curve in our previous construction, which was generally smooth, by a degenerate one. For simplicity, it is useful to employ the following concept from Manin's book ([5, Ch. 1, §1]): *for any x, y on a cubic curve (or cubic surface) C, let ℓ be the line through x and y: we define the composition law $x \oplus y$ to be the third point of intersection of $S \cap \ell$, if this exists and is unique.* Now we can proceed: again we consider a plane V through P and we consider the cubic curve $V \cup S$, which is generally smooth. On $V \cap S$, we define X_1, X_2, X_3 as before. What we do next is, basically, we draw some lines, obtaining points of intersection, and draw more lines through these: we define $Y_1 := P \oplus X_1$, $Y_2 := X_2 \oplus X_3$, $N' := Y_1 \oplus Y_2$ and $N := P \oplus N'$. If $V \cup S$ is smooth, this N is the same one as we got before. This can be proven by considering the divisor classes of the above points in $\mathrm{Pic}(V \cap S)$, but we will not do that here.



4 Constructions of birationally trivial cubic surfaces

In this chapter we will mainly construct examples of birationally trivial cubic surfaces. Among other things, we show that every type mentioned in Proposition 3.4 is indeed represented by a smooth cubic surface.

Our examples will serve to answer another question that has come up: is a birationally trivial surface over K also a blow-up defined over K? It will turn out that there are cubic surfaces of Type I, II and III over $\mathbb{Q}$ that are not blow-ups over $\mathbb{Q}$. (Of course, Type III are never blow-ups by their definition, but it remains to show that they exist.)

4.1 Possible types of orbits on S

If a cubic surface is, say, of Type II, we know that it has an orbit of 2 skew lines. But more can be said: trivially, it has to have a stable set of 25 lines. If we look even closer, these stable set of 25 lines falls apart into stable sets of 5, 10 and 10 lines. But this is a more complete characterization of Type II cubic surfaces: we have now divided the full set of 27 lines into stable sets. In this section, we will do the same for all types (we did Type V already did in Remark 3.7).

In this section, I will show what can be done using only the elementary combinatorial properties of the 27 lines. The fundamental idea that we will constantly use is: if a line ℓ has some intersection properties with respect to a Galois stable set of lines, all conjugates of ℓ will have those same properties. Let S be defined over K, then there are the following lemmas:

Lemma 4.1. Let L be a stable set of lines on S. Let ℓ be a line intersecting m lines of L. Then for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/K)$, $\sigma\ell$ also intersects m lines of L.

Proof. If ℓ intersects $\ell_1, \dots, \ell_m$ and does not intersect $\ell_{m+1}, \dots, \ell_n$, then $\sigma\ell$ intersects $\sigma\ell_1, \dots, \sigma\ell_m$ and does not intersect $\sigma\ell_{m+1}, \dots, \sigma\ell_n$. □

Lemma 4.2. Let $\mathcal{O}, \mathcal{O}'$ be two orbits of lines on S. Then there is m such that for any $\ell \in \mathcal{O}$, ℓ intersects exactly m lines of $\mathcal{O}'$.

Proof. Pick any $\ell \in \mathcal{O}$ and let m be the number of lines of $\mathcal{O}'$ it intersects. Then $\sigma\ell$ also intersects m lines of $\mathcal{O}'$ by the preceding Lemma. □

Lemma 4.3. Let L be a Galois stable set of lines on S. Then the lines on S intersecting m lines in L form a stable set.

Proof. Denote the set of lines intersecting m lines in L by $\mathcal{O}$. Take $\ell \in \mathcal{O}$ arbitrary. Then for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/K)$, $\sigma\ell$ intersects m lines in L, so $\sigma\ell \in \mathcal{O}$. □

Using these lemmas, we can now prove the following.

Proposition 4.4. Let S be a cubic surface containing 2 skew rational lines. Then the lines on S can be partitioned in 6 stable sets of lines, with cardinalities 1, 1, 5, 5, 5 and 10.

Proof. We will use Lemma 4.3 in combination with the double-six formalism. Assume that ℓ_1 and ℓ_2 are rational and skew lines on S. Then, following Important Remark 2.13, they form part of a double six, employing the usual notation for the other 25 lines on S. Let L_0 be the set of lines not intersecting ℓ_1 or ℓ_2. Then L_0 is a Galois stable set by Lemma 4.3. Similarly, let L_1 be the set of lines intersecting ℓ_1 but not ℓ_2, L_1' the set of lines intersecting ℓ_2 but not ℓ_1 and L_2 the set of lines intersecting both ℓ_1 and ℓ_2. All these sets are Galois stable, and moreover we can identify the members of all sets using the standard double-six notation. The table below lists all six stable sets and their members, showing that the cardinalities of the stable sets are as claimed.

set	#	property	line(s)
	1		ℓ_1
	1		ℓ_2
L_0	10	$\#(\ell \cap \{\ell_1, \ell_2\}) = 0$	$\ell_3, \ell_4, \ell_5, \ell_6, \ell_{34}, \ell_{35}, \ell_{36}, \ell_{46}, \ell_{45}, \ell_{56}$
L_1	5	$\#(\ell \cap \ell_1) = 1, \#(\ell \cap \ell_2) = 0$	$\ell_{13}, \ell_{14}, \ell_{15}, \ell_{16}, \ell_2'$
L_1'	5	$\#(\ell \cap \ell_1) = 0, \#(\ell \cap \ell_2) = 1$	$\ell_{23}, \ell_{24}, \ell_{25}, \ell_{26}, \ell_1'$
L_2	5	$\#(\ell \cap \{\ell_1, \ell_2\}) = 2$	$\ell_{12}, \ell_3', \ell_4', \ell_5', \ell_6'$

□

What follows is a series of results analogous to Proposition 4.4. Their proofs follow the same pattern entirely.

Proposition 4.5. Let S be a cubic surface containing an orbit of 2 skew lines. Then the lines on S can be partitioned in 4 stable sets of lines, with cardinalities 2, 5, 10 and 10.

Proof. Let ℓ_1, ℓ_2 be skew lines on S forming a Galois orbit and forming part of a double-six denoted in the usual way. For $0 \leq i \leq 2$, let L_i be the set of lines intersecting i lines of $\{\ell_1, \ell_2\}$. Then we have the following subdivision of the 27 lines into stable sets:

set	#	property	lines
	2		ℓ_1, ℓ_2
L_0	10	$\#(\ell \cap \{\ell_1, \ell_2\}) = 0$	$\ell_3, \ell_4, \ell_5, \ell_6, \ell_{34}, \ell_{35}, \ell_{36}, \ell_{46}, \ell_{45}, \ell_{56}$
L_1	10	$\#(\ell \cap \{\ell_1, \ell_2\}) = 1$	$\ell_{13}, \ell_{14}, \ell_{15}, \ell_{16}, \ell_{23}, \ell_{24}, \ell_{25}, \ell_{26}, \ell_1', \ell_2'$
L_2	5	$\#(\ell \cap \{\ell_1, \ell_2\}) = 2$	$\ell_{12}, \ell_3', \ell_4', \ell_5', \ell_6'$

□

Proposition 4.6. Let S be a cubic surface containing an orbit of 3 skew lines. Then the lines on S can be partitioned in 5 stable sets of lines, with cardinalities 3, 3, 6, 6 and 9.

Proof. Let ℓ_1, ℓ_2, ℓ_3 be skew lines on S forming a Galois orbit and forming part of a double-six denoted in the usual way. For $0 \leq i \leq 3$, let L_i be the set of lines intersecting i lines of $\{\ell_1, \ell_2, \ell_3\}$. Then we have the following subdivision of the 27 lines into stable sets:

set	#	property	lines
	3		ℓ_1, ℓ_2, ℓ_3
L_0	6	$\#(\ell \cap \{\ell_1, \ell_2, \ell_3\}) = 0$	$\ell_4, \ell_5, \ell_6, \ell_{45}, \ell_{46}, \ell_{56}$
L_1	9	$\#(\ell \cap \{\ell_1, \ell_2, \ell_3\}) = 1$	$\ell_{14}, \ell_{15}, \ell_{16}, \ell_{24}, \ell_{25}, \ell_{26}, \ell_{34}, \ell_{35}, \ell_{36}$
L_2	6	$\#(\ell \cap \{\ell_1, \ell_2, \ell_3\}) = 2$	$\ell_{12}, \ell_{13}, \ell_{23}, \ell'_1, \ell'_2, \ell'_3$
L_3	3	$\#(\ell \cap \{\ell_1, \ell_2, \ell_3\}) = 3$	$\ell'_4, \ell'_5, \ell'_6$

□

Proposition 4.7. Let S be a cubic surface containing an orbit of 3 skew lines and another orbit of 3 skew lines, such that none of these six lines intersects another. Then the lines on S can be partitioned in 7 stable sets of lines, with cardinalities 3, 3, 3, 3, 3, 3 and 9.

Proof. Let ℓ_1, ℓ_2, ℓ_3 be skew lines on S forming a Galois orbit, and let ℓ_4, ℓ_5, ℓ_6 be skew lines on S forming a Galois orbit. They are part of a double-six, which we write in the usual way. Then we have the following subdivision of the 27 lines into stable sets:

set	#	property	lines
L'	3		ℓ_1, ℓ_2, ℓ_3
L''	3		ℓ_4, ℓ_5, ℓ_6
L_1	3	$\#(\ell \cap L') = 1$, $\#(\ell \cap L'') = 0$	$\ell_{12}, \ell_{23}, \ell_{13}$
L'_1	3	$\#(\ell \cap L') = 0$, $\#(\ell \cap L'') = 1$	$\ell_{45}, \ell_{56}, \ell_{46}$
L_2	9	$\#(\ell \cap L') = 1$, $\#(\ell \cap L'') = 1$	$\ell_{14}, \ell_{15}, \ell_{16}, \ell_{24}, \ell_{25}, \ell_{26}, \ell_{34}, \ell_{35}, \ell_{36}$
L_5	3	$\#(\ell \cap L') = 2$, $\#(\ell \cap L'') = 3$	$\ell'_1, \ell'_2, \ell'_3$
L'_5	3	$\#(\ell \cap L') = 3$, $\#(\ell \cap L'') = 2$	$\ell'_4, \ell'_5, \ell'_6$

□

More obviously, we have the following version for a stable set of 6 skew lines:

Proposition 4.8. Let S be a cubic surface containing an orbit of 6 skew lines. Then the lines on S can be partitioned in 3 stable sets of lines, with cardinalities 6, 6 and 15.

Proof. In fact, we have already established this a while back in Remark 3.7. □

Using some of the results obtained in this section, we can prove the following:

Theorem 4.9. A smooth cubic surface does not contain a Galois orbit of 20 lines.

Proof. Suppose the contrary. Denote the set of all lines on S by L, denote the set of 20 lines forming a single orbit by L_1, and let L_2 denote the stable set formed by the other 7 lines. We focus our attention on the stable set L_2 which has cardinality 7. Since a 7-orbit does not occur, we only have to exclude the further possibilities of a 1-orbit, a 2-orbit or a 3-orbit, which we will do now.

Suppose first L_2 contains a rational line ℓ. Since ℓ intersects a total of 10 lines on S, it has to intersect one of the lines of L_1. But then, by the fact that all the lines of L_1 are conjugate, ℓ has to intersect them all. Contradiction, so there is no rational line.

If L_2 contains an 2-orbit $\mathcal{O}_2$, then the two lines belonging to it must be skew, since otherwise they would determine a rational line (i.e. the one that intersects them both). But in this case, we know from Proposition 4.5 that the maximum cardinality of an orbit is 10. So there is no 2-orbit.

Suppose next that L_2 contains an orbit $\mathcal{O}_3$ consisting of 3 lines. These are either skew, or they are coplanar. If they are coplanar, then each line in $L \backslash \mathcal{O}_3$ intersects exactly one of the lines in $\mathcal{O}_3$. Furthermore, any line in $\mathcal{O}_3$ should intersect the same number of lines in L_1. This would imply that $\# L_1$ is a multiple of 3, so again, contradiction. If the lines are skew, we know from Proposition 4.6 that the maximum cardinality of an orbit is 9 in this case. □

4.1.1 4 or 5 skew lines on a cubic surface

In this short subsection, we consider the case of an orbit of 4 or 5 pairwise skew lines. This will establish that cubic surfaces containing 4 or 5 skew lines are birationally trivial, since we can find a stable set of 2 skew lines on them. (See tables.)

Example 4.10. Let L' be a set containing 4 skew lines $\ell_1, \ell_2, \ell_3, \ell_4$ on a cubic surface S, forming part of a double-six in the usual way. (There always exists a cubic surface S containing these lines, since requiring S to contain a line imposes 4 conditions, and the general cubic has 20 coefficients.) The table below, following the pattern of those in Section 4.1, is easily verified:

set	#	property	lines
L'	4		$\ell_1, \ell_2, \ell_3, \ell_4$
L_0	3	$\#(\ell \cap L') = 0$	$\ell_5, \ell_6, \ell_{56}$
L_1	8	$\#(\ell \cap L') = 1$	$\ell_{15}, \ell_{25}, \ell_{35}, \ell_{45}, \ell_{16}, \ell_{26}, \ell_{36}, \ell_{46}$
L_2	6	$\#(\ell \cap L') = 2$	$\ell_{12}, \ell_{13}, \ell_{14}, \ell_{23}, \ell_{24}, \ell_{34}$
L_3	4	$\#(\ell \cap L') = 3$	$\ell'_1, \ell'_2, \ell'_3, \ell'_4$
L_4	2	$\#(\ell \cap L') = 4$	ℓ'_5, ℓ'_6

Finally, consider 5 pairwise skew lines on a smooth cubic surface. These can be part of a double-six in two ways: either we can take them to be $\ell_1, \ell_2, \ell_3, \ell_4, \ell_5$, or else $\ell_1, \ell_2, \ell_3, \ell_4, \ell_{56}$. We deal with both cases in a table:

set	#	property	lines
L'	5		$\ell_1, \ell_2, \ell_3, \ell_4, \ell_5$
L_0	1	$\#(\ell \cap L') = 0$	ℓ_6
L_1	5	$\#(\ell \cap L') = 1$	$\ell_{16}, \ell_{26}, \ell_{36}, \ell_{46}, \ell_{56}$
L_2	10	$\#(\ell \cap L') = 2$	$\ell_{12}, \ell_{13}, \ell_{14}, \ell_{15}, \ell_{23}, \ell_{24}, \ell_{25}, \ell_{34}, \ell_{35}, \ell_{45}$
L_3	0	$\#(\ell \cap L') = 3$	
L_4	5	$\#(\ell \cap L') = 4$	$\ell'_1, \ell'_2, \ell'_3, \ell'_4, \ell'_5$
L_5	1	$\#(\ell \cap L') = 5$	ℓ'_6

set	#	property	lines
L'	5		$\ell_1, \ell_2, \ell_3, \ell_4, \ell_{56}$
L_0	0	$\#(\ell \cap L') = 0$	
L_1	10	$\#(\ell \cap L') = 1$	$\ell_5, \ell_6, \ell_{15}, \ell_{25}, \ell_{35}, \ell_{45}, \ell_{16}, \ell_{26}, \ell_{36}, \ell_{46}$
L_2	0	$\#(\ell \cap L') = 2$	
L_3	10	$\#(\ell \cap L') = 3$	$\ell_{12}, \ell_{13}, \ell_{14}, \ell_{23}, \ell_{24}, \ell_{34}, \ell'_1, \ell'_2, \ell'_3, \ell'_4$
L_4	0	$\#(\ell \cap L') = 4$	
L_5	2	$\#(\ell \cap L') = 5$	ℓ'_5, ℓ'_6

4.2 A blow-up of $\mathbb{P}^2(\mathbb{Q})$

In order to exhibit cubic surfaces falling into Type V, we will construct a blow-up of the projective plane $\mathbb{P}^2(\overline{\mathbb{Q}})$. Actually, we will work over the ground field $\mathbb{F}_2$ and then lift our results back to $\mathbb{Q}$. Note that surfaces arising from this method necessarily fall into Type V, so we cannot use it to complete the proof of Lemma 3.4.

Given six points in general position (see Definition 2.5) in the projective plane $\mathbb{P}^2(\overline{\mathbb{F}}_2)$, Theorem 2.6 by Clebsch hands us an explicit way of constructing the blow-up of the plane in these points. We will choose these six points to be a full $\mathrm{Gal}(\overline{\mathbb{F}}_2/\mathbb{F}_2)$-orbit, so that the blow-up is defined over $\mathbb{F}_2$, and the six exceptional divisors resulting from the blow-up are a single Galois orbit consisting of 6 lines.

4.2.1 The calculation

Since we are looking for a point in $\mathbb{P}^2(\overline{\mathbb{F}}_2)$ of degree 6 over $\mathbb{F}_2$, we will work over $\mathbb{F}_{64} = \mathbb{F}_{2^6}$. This is a Galois extension of $\mathbb{F}_2$, with its Galois group $\cong \mathbb{Z}/6\mathbb{Z}$ generated by the Frobenius automorphism ($x \mapsto x^2$). Denote $G := \mathrm{Gal}(\mathbb{F}_{64}/\mathbb{F}_2)$. To be able to do explicit calculations, we fix a $\zeta \in \overline{\mathbb{F}}_2$ satisfying $\zeta^6 + \zeta + 1 = 0$ and use the isomorphism $\mathbb{F}_{64} \cong \mathbb{F}_2(\zeta)$.

Next, we pick an arbitrary point $P := (1 : \zeta^3 : \zeta^5)$. Consider its set of G-conjugates, i.e.

$$G \cdot P = \{(1 : \zeta^3 : \zeta^5), (1 : \zeta + 1 : \zeta^5 + \zeta^4), (1 : \zeta^2 + 1 : \zeta^5 + \zeta^4 + \zeta^3 + \zeta^2),$$
$$(1 : \zeta^4 + 1 : \zeta^5 + \zeta^3 + \zeta^2 + \zeta + 1), (1 : \zeta^3 + \zeta^2 + 1 : \zeta^5 + \zeta^2 + 1), (1 : \zeta^4 + \zeta : \zeta^5 + \zeta^2)\}$$

The first thing to do is to check that no three of these points are on a line and not all six are on a conic. This is equivalent to checking a finite set of matrices for invertibility. Given a set of three points $P_i = (X_i : Y_i : Z_i)$, $1 \leq i \leq 3$, they are collinear if and only if there is a triple $(a, b, c) \in \overline{\mathbb{F}}_2^3$ such that $aX_i + bY_i + cZ_i = 0$ for $1 \leq i \leq 3$, i.e. if and only if the matrix $\begin{pmatrix} X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \\ X_3 & Y_3 & Z_3 \end{pmatrix}$ has non-zero kernel. Let us check that the first three points in the enumeration above are non-collinear. We check that the following matrix:

$$\begin{pmatrix} 1 & \zeta^3 & \zeta^5 \\ 1 & \zeta + 1 & \zeta^5 + \zeta^4 \\ 1 & \zeta^2 + 1 & \zeta^5 + \zeta^4 + \zeta^3 + \zeta^2 \end{pmatrix}$$

is invertible, so the points $(1 : \zeta^3 : \zeta^5), (1 : \zeta + 1 : \zeta^5 + \zeta^4), (1 : \zeta^2 + 1 : \zeta^5 + \zeta^4 + \zeta^3 + \zeta^2)$ do not lie on a straight line. If we continue this for the other $\binom{6}{3} - 1 = 14$ lines, we find that no three of the six conjugates of P lie on a straight line.

The check that six given points $P_i = (X_i : Y_i : Z_i)$, $1 \leq i \leq 6$ lie off any conic, follows a similar pattern. The general equation of a conic is $aX^2 + bY^2 + cZ^2 + dXY + eXZ + fYZ = 0$, so we should check that the matrix whose ith row is

$$X_i^2 \quad Y_i^2 \quad Z_i^2 \quad X_iY_i \quad X_iZ_i \quad Y_iZ_i$$

has a kernel consisting only of the zero vector. This conditioned too is satisfied in the case of the set $G \cdot P$, so our six points are indeed in general position.

We proceed according to the construction suggested by Clebsch, so we determine the $\mathbb{F}_{64}$-vector space V_P of cubic forms vanishing on $G \cdot P$. We will use a little short-cut, namely, instead of determining the forms defined over $\mathbb{F}_{64}$ that vanish on all the conjugates of P, where the vanishing on each separate point imposes another linear condition, we determine the forms defined over $\mathbb{F}_2$ that vanish on P alone: since, however, such a form is defined over $\mathbb{F}_2$, it also vanishes on all conjugates of P. Using Maple, we find that a basis for V_P is given by the four forms

$$f_1 := x^2z + xy^2 + y^2z, f_2 := x^2y + x^2z + z^3, f_3 := x^3 + x^2y + xy^2 + yz^2,$$
$$f_4 := x^2y + x^2z + xz^2 + y^3$$

The blow-up map is then the rational map $\phi : \mathbb{P}^2_{\mathbb{F}_p} \to \mathbb{P}^3_{\mathbb{F}_p}$ given by $(\xi : \eta : \zeta) \mapsto (f_1(\xi, \eta, \zeta) : f_2(\xi, \eta, \zeta) : f_3(\xi, \eta, \zeta) : f_4(\xi, \eta, \zeta))$. A different choice of basis leads to a projective linear transformation of the $\mathbb{P}^3$.

We want to find the equation belonging to the image of ϕ, or more precisely of its closure. This we can do, at least in principle, by considering the general equation of a cubic surface, which is $a_0x^3 + a_1x^2y + a_2x^2z + \ldots + a_{19}z^3$ and then substituting $x = f_1(\xi, \eta, \zeta)$, etc. The delicate point here is solving the resulting 55 equations in 20 unknowns over the finite field $\mathbb{F}_2$. The point is that if we lift all 55 equations to $\mathbb{Q}$, a solution may no longer exist, since the rank of the system may increase! The correct way to proceed is to find a subset of 19 equations which has full rank: if we lift this to $\mathbb{Q}$ it still has full rank, and so there is a unique non-zero solution up to scalar multiplication, which we can lift back to the unique solution over $\mathbb{F}_2$. In the cases I considered it turned out just fine by selecting the top-most 19 equations.

This finally gives us the equation (lifted to $\mathbb{Q}$) of the cubic surface S_1 as $-x^2y+xz^2-z^3-x^2w+y^2w+zw^2=0$. S_1 contains a rational point, $(1:1:1:1)$, which was not guaranteed by its construction. Its 27 lines are contained in three orbits of 6, 6 and 15 lines, as can be checked by the MAPLE algorithm given in Section 5.2. Hence, S_1 is of Type 5. Also, we have a birational map $\phi : \mathbb{P}^2 \to S_1$ defined over $\mathbb{Q}$. The problem of explicitly determining ϕ^{-1}, i.e. a blow-down morphism, is still open.

The resulting surface is smooth by construction, hence it automatically lifts to a smooth surface over $\mathbb{Q}$ (since singular points go to singular points when reducing to $\mathbb{F}_2$) and sets of $\mathrm{Gal}(\overline{\mathbb{F}}_2/\mathbb{F}_2)$-conjugate lines lift to sets of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-conjugate lines.

4.2.2 Another blow-up of $\mathbb{P}^2(\mathbb{Q})$

To obtain a surface of Type IV, we should blow up $\mathbb{P}^2$ in 6 points forming two Galois orbits of order 3. Fixing the isomorphism $\mathbb{F}_8 \cong \mathbb{F}_2[\zeta]$, where $\zeta \in \overline{\mathbb{F}}_2$ satisfies $\zeta^3+\zeta+1=0$, we pick the points $(1:\zeta:\zeta^2)$ and $(1:\zeta^2:\zeta)$ and their conjugates $(1:\zeta^2:\zeta^2+\zeta)$, $(1:\zeta^2+\zeta:\zeta)$, $(1:\zeta^2+\zeta:\zeta^2)$, $(1:\zeta:\zeta^2+\zeta)$. We do the same computation as above, and obtain the smooth cubic surface $-x^3-w^3+yzw-x^2w+y^2z+z^2y=0$ over $\mathbb{Q}$. It has orbits of cardinality 3, 3, 3, 3, 3, 3 and 9 and contains the rational point $(1:1:1:1)$. Hence, it is of Type IV.

4.3 Some cubic surfaces that are not blow-ups

We now turn to cubic surfaces over $\mathbb{Q}$ belonging to Types I, II and III. Also, we will construct a cubic surface that does not contain a stable set of six skew lines, hence is not a blow-up over $\mathbb{Q}$.

4.3.1 An orbit of 3 skew lines

We want to construct cubic surfaces containing a stable set of three skew lines. It turns out that one can actually require S to contain three arbitrary skew lines, the only drawback is that smoothness can only be checked afterwards. However, in all our constructions, there is enough freedom in the choice of equation for us to try around and make sure that S is smooth, has a $\mathbb{Q}$-rational point and has other desirable properties.

First, in the presence of a 3-orbit, it is possible to say something about the decomposition of the 27 lines into stable sets. As was shown in Proposition 4.6, there are at least 5 stable sets, having cardinalities 3, 3, 6, 6 and 9 We reproduce the table from Proposition 4.6 for convenience:

set	#	property	lines
L'	3		ℓ_1, ℓ_2, ℓ_3
L_0	6	$\#(\ell \cap L') = 0$	$\ell_4, \ell_5, \ell_6, \ell_{45}, \ell_{46}, \ell_{56}$
L_1	9	$\#(\ell \cap L') = 1$	$\ell_{14}, \ell_{15}, \ell_{16}, \ell_{24}, \ell_{25}, \ell_{26}, \ell_{34}, \ell_{35}, \ell_{36}$
L_2	6	$\#(\ell \cap L') = 2$	$\ell_{12}, \ell_{13}, \ell_{23}, \ell'_1, \ell'_2, \ell'_3$
L_3	3	$\#(\ell \cap L') = 3$	$\ell'_4, \ell'_5, \ell'_6$

Let $G := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and denote by H the subgroup $H \subset G$ acting trivially on the $\{\ell_1, \ell_2, \ell_3\}$. Then $G/H \cong \mathbb{Z}/3\mathbb{Z}$ or $G/H \cong S_3$. In the latter case, the stable set S_0 breaks up in two stable sets of three lines each, i.e. $\{\ell_4, \ell_5, \ell_6\}$ and $\{\ell_{45}, \ell_{46}, \ell_{56}\}$. This would imply that we have a stable set of six skew lines, so we would end up with a surface in Type IV. So we need $G/H \cong \mathbb{Z}/3\mathbb{Z}$. So the action of the Galois group has to be cyclic, therefore, we need to work over a cyclic extension of $\mathbb{Q}$ of degree 3. The field $E := \mathbb{Q}(\zeta_7)$, where ζ_7 is primitive 7th root of unity has a degree 3 subfield, so we will work over E.

Let ζ_7 be a primitive 7th root of unity, i.e. ζ_7 satisfies $\zeta_7^6+\zeta_7^5+\zeta_7^4+\zeta_7^3+\zeta_7^2+\zeta_7+1 = 0$. The field E has Galois group G cyclic of order 6. As a generator, we choose the element σ satisfying $\sigma : \zeta_7 \mapsto \zeta_7^3$.

The procedure is as follows. We will first pick a more-or-less arbitrary plane $\{F_0 = 0\}$ in $\mathbb{P}^3$, such that F_0 has 5 conjugates $\{\sigma^i(F_0)\}_{i=1}^5$ other than itself in $\mathbb{Q}(\zeta_7)[x, y, z, w]$. For $1 \leq i \leq 5$, let $F_i := \sigma^i F_0$. Then consider the three lines $\{\ell_i\}_{i=1}^3$ determined by $F_i = \sigma^3 F_i = 0$. By construction these make up a full Galois orbit. Also, it is easy to find some (possibly singular) cubic surfaces, defined over $\mathbb{Q}$, containing the ℓ_i, for instance the polynomials

$$f_1 := F_0\sigma^2(F_0)\sigma^4(F_0) + \sigma(F_0)\sigma^3(F_0)\sigma^5(F_0) \qquad (2)$$

and

$$f_2 := F_0\sigma(F_0)\sigma^2(F_0) + \sigma(F_0)\sigma^2(F_0)\sigma^3(F_0) + \ldots + F_0\sigma(F_0)\sigma^5(F_0) \qquad (3)$$

We can take linear combinations of these to avoid singular surfaces and make sure of rational points.

We pick $F_0 := X + \zeta_7 Y + \zeta_7^2 Z + \zeta_7^3 W$. Then the conjugates of F_0 are

$$\begin{aligned} F_1 &:= X + \zeta_7^3 Y + \zeta_7^6 Z + \zeta_7^2 W \\ F_2 &:= X + \zeta_7^2 Y + \zeta_7^4 Z + \zeta_7^6 W \\ F_3 &:= X + \zeta_7^6 Y + \zeta_7^5 Z + \zeta_7^4 W \\ F_4 &:= X + \zeta_7^4 Y + \zeta_7 Z + \zeta_7^5 W \\ F_5 &:= X + \zeta_7^5 Y + \zeta_7^3 Z + \zeta_7 W \end{aligned}$$

Consider the lines defined by $\ell_1 := \{F_0 = F_3 = 0\}$, $\ell_2 := \{F_1 = F_4 = 0\}$, $\ell_3 := \{F_2 = F_5 = 0\}$. Our choice was lucky, since ℓ_1, ℓ_2, ℓ_3 turn out to be pairwise skew. Hence, ℓ_1, ℓ_2, ℓ_3 form a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable set of three skew lines.

With these F_i, we can compute the cubic forms f_1 and f_2 (see equations 2 and 3) which contain the designated lines. With the help of Maple, we find:

$f_1 := 2X^3 - X^2Z - YZ^2 - X^2W - Z^2W - XZ^2 - XW^2 - X^2Y - Y^2W - YW^2 - ZW^2 + 5XZW - 2YZW - 2XYZ + 5XYW - XY^2 + 2W^3 + 2Z^3 + 2Y^3 - Y^2Z$
$f_2 := YZW + 6X^3 + 8XYZ + XYW + XZW - 3X^2Y - 3X^2Z - 3X^2W - 3XY^2 - 3XZ^2 - 3XW^2 + 4Y^2Z - 3Y^2W - 3YZ^2 + 4YW^2 + 4Z^2W - 3ZW^2 - Y^3 - Z^3 - W^3$

We compute values of f_1 and f_2 for integral values of X, Y, Z, W and use these to find a linear combination of f_1 and f_2 containing an integral point. We embark on the cubic surface S_{III} given by $337f_1 + 44f_2 = 0$. A check with MAPLE learns that S_{III} is smooth, and that it contains Galois orbits of order 3, 3, 6, 6 and 9 lines, corresponding to the table above.

In particular, S_{III} does not contain a stable set of 6 skew lines, so it can not be a blow-up defined over $\mathbb{Q}$.

4.3.2 An orbit of 2 skew lines

Just as we can require S to contain 3 skew lines, we can require it to contain 2 skew lines. Take the conjugate lines $\ell_1 := (1 : t : \sqrt{2} : \sqrt{2}t)$ and $\ell_2 := (1 : t : -\sqrt{2} : -\sqrt{2}t)$ which are obviously skew. The equation of ℓ_1 is $F_0 = Z - \sqrt{2}X = 0, F_1 = W - \sqrt{2}Y = 0$, the equation of ℓ_2 is $F_2 = Z + \sqrt{2}X, F_3 = W + \sqrt{2}Y = 0$. From this, we get four quadrics containing ℓ_1 and ℓ_2: $Q_1 : Z^2 - 2X^2$, $Q_2 : W^2 - 2Y^2$, $Q_3 : ZW - 2XY$, $Q_4 : XW - YZ$.

We can construct a cubic surface containing ℓ_1 and ℓ_2 by taking a linear combination of the 16 terms XQ_1, YQ_1, $\ldots$, WQ_4. I ended up with $F = 3XQ_1 + 9XQ_2 + 3YQ_2 - 46ZQ_3 + 46XQ_4 + 138YQ_4 + 230ZQ_4 + 46WQ_4$. As the cubic surface S_2 given by $F = 0$ has a stable set of 2 lines, it has a rational point; furthermore, a quick check learns that it is smooth. Applying the algorithm from Section 5.1, we find that S_2 contains Galois orbits of 2, 5, 10 and 10 lines each. From Proposition 4.5, we reproduce the table that gives the stable sets (which are exactly the orbits in this case) explicitly:

set	#	property	lines
L'	2		ℓ_1, ℓ_2
L_0	10	$\#(\ell \cap L') = 0$	$\ell_3, \ell_4, \ell_5, \ell_6, \ell_{34}, \ell_{35}, \ell_{36}, \ell_{46}, \ell_{45}, \ell_{56}$
L_1	10	$\#(\ell \cap L') = 1$	$\ell_{13}, \ell_{14}, \ell_{15}, \ell_{16}, \ell_{23}, \ell_{24}, \ell_{25}, \ell_{26}, \ell'_1, \ell'_2$
L_2	5	$\#(\ell \cap L') = 2$	$\ell_{12}, \ell'_3, \ell'_4, \ell'_5, \ell'_6$

The conclusion is that S_2 falls into Type II.

4.3.3 Two rational lines

We construct a cubic surface containing the two rational lines ℓ_1 given by $X = Y = 0$ and ℓ_2 given by $Z = W = 0$. This is achieved by taking any linear combination of the terms $X^2Z, XYZ, XZ^2, XZW, X^2W, XYW, XW^2, Y^2Z, YZ^2, YZW, Y^2W, YW^2$.

We take S_1 to be the cubic surface satisfying the (admittedly rather outlandish) equation: $175959X^2Z + 518643XYZ - 131841XZ^2 + 19XZW + 27X^2W + 400653XYW +$

$121068XW^2 + 52326Y^2Z + 11799YZ^2 + 383211YZW + 235467Y^2W + 108243YW^2$. It is smooth and contains a rational point, as in the previous computation. It contains six orbits of 1, 1, 5, 5, 5 and 10 lines each. From Proposition 4.4, we reproduce the table that gives the stable sets (which are exactly the orbits in this case) explicitly:

set	#	property	line(s)
	1		ℓ_1
	1		ℓ_2
L_0	10	$\#(\ell \cap \{\ell_1, \ell_2\}) = 0$	$\ell_3, \ell_4, \ell_5, \ell_6, \ell_{34}, \ell_{35}, \ell_{36}, \ell_{46}, \ell_{45}, \ell_{56}$
L_1	5	$\#(\ell \cap \ell_1) = 1, \#(\ell \cap \ell_2) = 0$	$\ell_{13}, \ell_{14}, \ell_{15}, \ell_{16}, \ell'_2$
L'_1	5	$\#(\ell \cap \ell_1) = 0, \#(\ell \cap \ell_2) = 1$	$\ell_{23}, \ell_{24}, \ell_{25}, \ell_{26}, \ell'_1$
L_2	5	$\#(\ell \cap \{\ell_1, \ell_2\}) = 2$	$\ell_{12}, \ell'_3, \ell'_4, \ell'_5, \ell'_6$

In particular, S_1 is of Type I.

5 Finding lines on a cubic surface

5.1 An algorithm

Given a cubic surface S defined by the polynomial F, we know that it contains a line. We will now turn to the problem of actually finding it. In general, we might use the observation made before that the condition for a line ℓ being on S can be expressed algebraically in terms of the Plücker coordinates of ℓ ([9, Ch. 1, pp. 78-79]). But this will give us not just one line, but all lines at once, while if we are given one line, it is easy to find them all. Let us therefore try to make use of the fact that we only have to find one line.

Let $(p_{00} : \cdots : p_{23})$ be the Plücker coordinates of the line ℓ we are looking for. First observe that $p_{00} = 0$ if and only if ℓ intersects the line $X = Y = 0$, so we can assume $p_{00} \neq 0$, which will only exclude up to 11 lines. But this means that if $P_0 = (x_0 : y_0 : z_0 : w_0)$ and $P_1 = (x_1 : y_1 : z_1 : w_1)$ are two points on ℓ, then $\det \begin{pmatrix} x_0 & y_0 \\ x_1 & y_1 \end{pmatrix} = p_{00} \neq 0$, so by taking a suitable linear combination $\mu P_0 + \nu P_1$ of the two points, we find that two points of the form $(1 : 0 : a : b)$ and $(0 : 1 : c : d)$ lie on S. Parametrizing, we find that we can represent ℓ as $(1 : \lambda : a + \lambda c : b + \lambda d)$. In order for ℓ to be contained in the surface S a necessary and sufficient condition is that $F(1 : \lambda : a + \lambda c : b + \lambda d) \equiv 0$. Expanding this and equating to zero coefficients of λ^i yields four equations in the four unknowns a, b, c, d.

Example 5.1. We want to find a line on the Fermat cubic $X^3 + Y^3 + Z^3 + W^3 = 0$. We substitute $X = 1$, $Y = t$, $Z = a + ct$, $W = b + dt$, obtaining

$$(1 + c^3 + d^3)t^3 + (3ac^2 + 3bd^2)t^2 + (3a^2c + 3b^2d)t + 1 + a^3 + b^3 = 0$$

This yields the system

$$\begin{aligned} 1 + c^3 + d^3 &= 0 \\ 3ac^2 + 3bd^2 &= 0 \\ 3a^2c + 3b^2d &= 0 \\ 1 + a^3 + b^3 &= 0 \end{aligned}$$

Without further manipulation, we notice solutions like $a = d = 0$, $b = c = -1$, which gives us the line $(1 : 0 : t : -1)$ or $X + W = Y = 0$.

5.2 Implementation in Maple

The following algorithm for MAPLE finds the lines on a cubic surface given by $F = 0$, following the discussion in the previous section. We need to equate the coefficients of t^i, which are polynomials in a, b, c, d, to zero. This means in other words that we have to find the zero locus of an ideal $I_F \subset \mathbb{Q}[a, b, c, d]$ in $\mathbb{A}^4_{\mathbb{Q}}$. To simplify the calculation, we first determine a Groebner basis for I_F, from which we can already read off the cardinality of Galois orbits of the lines, for instance. The computation of the Groebner basis, and the subsequent solution of the resulting system, are done in the last two lines.

The packages PolynomialTools and Groebner are used.

```
with(PolynomialTools): with(Groebner):
F:=x^3+y^3+z^3+w^3: # select the Fermat cubic
rule:={x=1,y=t,z=a+c*t,w=b+d*t}:
    # search for lines of the form (1 : t : a + ct : b + dt)
F1a:=expand(subs(rule,F)):
F2a:=collect(F1a,t):
vect:=CoefficientVector(F2a,t):
F3a:=[vect[1],vect[2],vect[3],vect[4]]:
G1a:=gbasis(F3a,plex(a,b,c,d)):
sol:=solve({seq(G1a[i],i=1..4)},{a,b,c,d});
```

5.3 Worked example: the twisted Fermat

We now turn to the twisted Fermat cubic F_2 defined by $x^3 + y^3 + z^3 + 2w^3 = 0$. This surface has been investigated by Manin ([5, Ch. 4, §23, p. 113]), who shows that it is a minimal surface, so it does not contain any Galois orbits of pairwise skew lines. Following the algorithm described above, we find that the lines on F_2 fall into six orbits: three of order 3 and three of order 6. Choose elements $\omega_1, \eta_1 \in \overline{\mathbb{Q}}$ satisfying $\omega_1^2 - \omega_1 + 1 = 0$ (which makes ω_1 a primitive 6th root of unity) and $2\eta_1^3 + 1 = 0$. The conjugates of both elements are $\omega_2 = 1 - \omega_1 = -\omega_1^2$ and $\eta_2 = -\eta_1\omega_1$ and $\eta_3 = -\eta_1\omega_2$. Furthermore, since ω_1, ω_2 are roots of $X^2 - X + 1$, their product satisfies $\omega_1\omega_2 = 1$.

Let $\mathrm{inv}(L) \subset \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ be the subgroup of elements that leave all the lines on F_2 fixed. Then $\overline{\mathbb{Q}}^{\mathrm{inv}(L)} \supset \mathbb{Q}$ is a finite Galois extension. The Galois group $G := \mathrm{Gal}(\overline{\mathbb{Q}}^{\mathrm{inv}(L)}/\mathbb{Q})$ is isomorphic to S_3, and we have $G \cong \langle \sigma, \tau \rangle$ where σ permutes the η_i cyclically ($\sigma\eta_1 = \eta_2$, etc.) and τ interchanges η_1 and η_2, leaving η_3 fixed (which implies that τ also interchanges the ω_i). For convenience, we describe the isomorphism $\phi : S_3 \xrightarrow{\sim} G$ explicitly:

S_3	G
$\pi_{(123)}$	σ
$\pi_{(132)}$	σ^2
$\pi_{(12)}$	τ
$\pi_{(23)}$	$\tau\sigma = \sigma^2\tau$
$\pi_{(13)}$	$\tau\sigma^2 = \sigma\tau$

G acts faithfully on the set $\{\eta_1, \eta_2, \eta_3, \omega_1, \omega_2\}$ and an element g of G is determined by specifying $g(\eta_i)$ and $g(\omega_j)$ for some choice of i and j.

We now specify all Galois orbits of lines on F_2, along with representatives of each orbit. We have listed them in a table:

orbit	#orbit	representative
a	3	$l_1 = (1 : t : -t : \eta_1)$
b	3	$l_2 = (1 : t : -1 : \eta_1 t)$
c	3	$l_3 = (1 : -1 : t/\eta_1 : t)$
A	6	$l_4 = (1 : t : \omega_1 t : \eta_1)$
B	6	$l_5 = (1 : t : \omega_1 : \eta_1 t)$
C	6	$l_6 = (1 : \omega_1 : t/\eta_1 : t)$

In the following, we will investigate the intersection properties of the 27 lines listed above. We will make some good use of the fact that G takes intersecting lines to intersecting lines. For instance, we only have to find out which lines are intersected by the six l_i.

1. The lines intersected by $l_1 = (1 : t : -t : \eta_1)$. In orbit (a), the two lines σl_1 and σl_2. In orbit (b), the one line l_2. In orbit (c), the one line l_3. In orbit (A), tbe two lines $(1 : t' : \omega_1 t' : \eta_1)$ and $(1 : t' : \omega_2 t' : \eta_1)$, so l_4 and $(\tau\sigma)l_4$. In orbit (B), tbe two lines $(1 : t' : \omega_1 : \eta_3 t')$ and $(1 : t' : \omega_2 : \eta_2 t')$, i.e. $\sigma^2 l_5$ and τl_5. In orbit (C), tbe two lines $(1 : \omega_1 : t'/\eta_3 : t')$ and $(1 : \omega_2 : t'/\eta_2 : t')$, i.e. $\sigma^2 l_6$ and τl_6.

2. The lines intersected by $l_2 = (1 : t : -1 : \eta_1 t)$. In orbit (a), the one line l_1. In orbit (b), the two lines σl_2, $\sigma^2 l_2$. In orbit (c), the one line l_3. In orbit (A), tbe two lines $(1 : t' : \omega_1 t' : \eta_3)$ and $(1 : t' : \omega_2 t' : \eta_2)$, so $\sigma^2 l_4$ and τl_4. In orbit (B), tbe two lines $(1 : t' : \omega_1 : \eta_1 t')$ and $(1 : t' : \omega_2 : \eta_1 t')$, i.e. l_5 and $(\tau\sigma)l_5$. In orbit (C), tbe two lines $(1 : \omega_1 : t'/\eta_2 : t')$ and $(1 : \omega_2 : t'/\eta_3 : t')$, i.e. σl_6 and $(\tau\sigma^2)l_6$.

3. The lines intersected by $l_3 = (1 : -1 : t/\eta_1 : t)$. In orbit (a), the one line l_1. In orbit (b), the one line l_2. In orbit (c), the two lines σl_3, $\sigma^2 l_3$. In orbit (A), the two lines $(1 : t' : \omega_1 t' : \eta_2)$ and $(1 : t' : \omega_2 t' : \eta_3)$, so σl_4 and $(\tau\sigma^2)l_4$. In orbit (B), tbe two lines $(1 : t' : \omega_1 : \eta_2 t')$ and $(1 : t' : \omega_2 : \eta_3 t')$, i.e. σl_5 and $(\tau\sigma^2)l_5$. In orbit (C), tbe two lines $(1 : \omega_1 : t'/\eta_1 : t')$ and $(1 : \omega_2 : t'/\eta_1 : t')$, i.e. l_6 and $(\tau\sigma)l_6$.

4. The lines intersected by $l_4 = (1 : t : \omega_1 t : \eta_1)$. In orbit (A), tbe three lines $\sigma l_4 = (1 : t' : \omega_1 t' : \eta_2)$, $\sigma^2 l_4 = (1 : t' : \omega_1 t' : \eta_3)$ and $(\tau\sigma)l_4 = (1 : t' : \omega_2 t' : \eta_1)$. In orbit (B) we have the lines $l_5 = (1 : t' : \omega_1 : \eta_1 t')$ and $(\tau\sigma^2)l_5 = (1 : t' : \omega_2 : \eta_3 t')$. In orbit (C), the lines $\sigma l_6 = (1 : \omega_1 : t'/\eta_2 : t')$ and $(\tau\sigma)l_6 = (1 : \omega_2 : t'/\eta_1 : t')$.

5. The lines intersected by $l_5 = (1 : t : \omega_1 : \eta_1 t)$. Under point 4 we saw that l_4 intersected l_5 and $(\tau\sigma^2)l_5$, we reverse this to get that l_5 intersects l_4 and $(\tau\sigma^2)l_4$. Furthermore, l_5 intersects σl_5, $\sigma^2 l_5$ and $(\tau\sigma)l_5$. As to orbit (C), l_5 intersects $l_6 = (1 : \omega_1 : t/\eta_1 : t)$ and $\tau l_6 = (1 : \omega_2 : t/\eta_2 : t)$.

6. The lines intersected by $l_6 = (1 : \omega_1 : t/\eta_1 : t)$. We can just reverse the relevant parts of the last two steps: as l_4 intersects σl_6 and $(\tau\sigma)l_6$, we see that l_6 intersects $\sigma^2 l_4$ and $(\tau\sigma)l_4$. Similarly, as l_5 intersects l_6 and τl_6, l_6 intersects l_5 and τl_5. Lastly, in orbit (C), l_6 intersects σl_6, $\sigma^2 l_6$ and $(\tau\sigma)l_6$.

Theorem 5.2. F_2 is not birational to $\mathbb{P}^2(\mathbb{Q})$.

Proof. In the above calculations we saw that all orbits contain intersecting lines (in fact, each line l' on F_2 intersects $\sigma l'$ and $\sigma^2 l'$), so theorem 3.2 with $k = \mathbb{Q}$ yields that $F_2/\mathbb{Q}$ is not birational to $\mathbb{P}^2(\mathbb{Q})$. □

Question 5.3. What can we say over $\mathbb{Q}(\eta_1)$ and $\mathbb{Q}(\omega_1)$?

As we are able to check using the table below, the two lines $(1 : t : \omega_1 t : \eta_2)$ and $(1 : t : \omega_2 t : \eta_3)$ are skew and form a full $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\eta_1))$-orbit on F_2. So F_2 is birational to $\mathbb{P}^2(\mathbb{Q}(\eta_1))$.

Over $\mathbb{Q}(\omega_1)$ however, we obtain 9 orbits of order 3 each, all of which contain intersecting lines. So F_2 is not birational to $\mathbb{P}^2(\mathbb{Q}(\omega_1))$.

line	intersects
l_1	$\sigma l_1, \sigma^2 l_1, l_2, l_3, l_4, (\tau\sigma)l_4, \sigma^2 l_5, \tau l_5, \sigma^2 l_6, \tau l_6$
σl_1	$l_1, \sigma^2 l_1, \sigma l_2, \sigma l_3, \sigma l_4, \tau l_4, l_5, (\tau\sigma^2)l_5, l_6, (\tau\sigma^2)l_6$
$\sigma^2 l_1$	$l_1, \sigma l_1, \sigma^2 l_2, \sigma^2 l_3, \sigma^2 l_4, (\tau\sigma^2)l_4, \sigma l_5, (\tau\sigma)l_5, \sigma l_6, (\tau\sigma)l_6$
l_2	$l_1, \sigma l_2, \sigma^2 l_2, l_3, \sigma^2 l_4, \tau l_4, l_5, (\tau\sigma)l_5, \sigma l_6, (\tau\sigma^2)l_6$
σl_2	$\sigma l_1, l_2, \sigma^2 l_2, \sigma l_3, l_4, (\tau\sigma^2)l_4, \sigma l_5, \tau l_5, \sigma^2 l_6, (\tau\sigma)l_6$
$\sigma^2 l_2$	$\sigma^2 l_1, l_2, \sigma l_2, \sigma^2 l_3, \sigma l_4, (\tau\sigma)l_4, \sigma^2 l_5, (\tau\sigma^2)l_5, l_6, \tau l_6$
l_3	$l_1, l_2, \sigma l_3, \sigma^2 l_3, \sigma l_4, (\tau\sigma^2)l_4, \sigma l_5, (\tau\sigma^2)l_5, l_6, (\tau\sigma)l_6$
σl_3	$\sigma l_1, \sigma l_2, l_3, \sigma^2 l_3, \sigma^2 l_4, (\tau\sigma)l_4, \sigma^2 l_5, (\tau\sigma)l_5, \sigma l_6, \tau l_6$
$\sigma^2 l_3$	$\sigma^2 l_1, \sigma^2 l_2, l_3, \sigma l_3, l_4, \tau l_4, l_5, \tau l_5, \sigma^2 l_6, (\tau\sigma^2)l_6$
l_4	$l_1, \sigma l_2, \sigma^2 l_3, \sigma l_4, \sigma^2 l_4, (\tau\sigma)l_4, l_5, (\tau\sigma^2)l_5, \sigma l_6, (\tau\sigma)l_6$
σl_4	$\sigma l_1, \sigma^2 l_2, l_3, l_4, \sigma^2 l_4, \tau l_4, \sigma l_5, (\tau\sigma)l_5, \sigma^2 l_6, \tau l_6$
$\sigma^2 l_4$	$\sigma^2 l_1, l_2, \sigma l_3, l_4, \sigma l_4, (\tau\sigma^2)l_4, \sigma^2 l_5, \tau l_5, l_6, (\tau\sigma^2)l_6$
τl_4	$\sigma l_1, l_2, \sigma^2 l_3, (\tau\sigma)l_4, (\tau\sigma^2)l_4, \sigma l_4, \tau l_5, \sigma^2 l_5, (\tau\sigma)l_6, \sigma l_6$
$(\tau\sigma)l_4$	$l_1, \sigma^2 l_2, \sigma l_3, \tau l_4, (\tau\sigma^2)l_4, l_4, (\tau\sigma)l_5, \sigma l_5, (\tau\sigma^2)l_6, l_6$
$(\tau\sigma^2)l_4$	$\sigma^2 l_1, \sigma l_2, l_3, \tau l_4, (\tau\sigma)l_4, \sigma^2 l_4, (\tau\sigma^2)l_5, l_5, \tau l_6, \sigma^2 l_6$
l_5	$\sigma l_1, l_2, \sigma^2 l_3, l_4, (\tau\sigma^2)l_4, \sigma l_5, \sigma^2 l_5, (\tau\sigma)l_5, l_6, \tau l_6$
σl_5	$\sigma^2 l_1, \sigma l_2, l_3, \sigma l_4, (\tau\sigma)l_4, l_5, \sigma^2 l_5, \tau l_5, \sigma l_6, (\tau\sigma^2)l_6$
$\sigma^2 l_5$	$l_1, \sigma^2 l_2, \sigma l_3, \sigma^2 l_4, \tau l_4, l_5, \sigma l_5, (\tau\sigma^2)l_5, \sigma^2 l_6, (\tau\sigma)l_6$
τl_5	$l_1, \sigma l_2, \sigma^2 l_3, \tau l_4, \sigma^2 l_4, (\tau\sigma)l_5, (\tau\sigma^2)l_5, \sigma l_5, \tau l_6, l_6$
$(\tau\sigma)l_5$	$\sigma^2 l_1, l_2, \sigma l_3, (\tau\sigma)l_4, \sigma l_4, \tau l_5, (\tau\sigma^2)l_5, l_5, (\tau\sigma)l_6, \sigma^2 l_6$
$(\tau\sigma^2)l_5$	$\sigma l_1, \sigma^2 l_2, l_3, (\tau\sigma^2)l_4, l_4, \tau l_5, (\tau\sigma)l_5, \sigma^2 l_5, (\tau\sigma^2)l_6, \sigma l_6$
l_6	$\sigma l_1, \sigma^2 l_2, l_3, \sigma^2 l_4, (\tau\sigma)l_4, l_5, \tau l_5, \sigma l_6, \sigma^2 l_6, (\tau\sigma)l_6$
σl_6	$\sigma^2 l_1, l_2, \sigma l_3, l_4, \tau l_4, \sigma l_5, (\tau\sigma^2)l_5, l_6, \sigma^2 l_6, \tau l_6$
$\sigma^2 l_6$	$l_1, \sigma l_2, \sigma^2 l_3, \sigma l_4, (\tau\sigma^2)l_4, \sigma^2 l_5, (\tau\sigma)l_5, l_6, \sigma l_6, (\tau\sigma^2)l_6$
τl_6	$l_1, \sigma^2 l_2, \sigma l_3, (\tau\sigma^2)l_4, \sigma l_4, \tau l_5, l_5, (\tau\sigma)l_6, (\tau\sigma^2)l_6, \sigma l_6$
$(\tau\sigma)l_6$	$\sigma^2 l_1, \sigma l_2, l_3, l_4, \tau l_4, (\tau\sigma)l_5, \sigma^2 l_5, l_6, (\tau\sigma^2)l_6, \tau l_6$
$(\tau\sigma^2)l_6$	$\sigma l_1, l_2, \sigma^2 l_3, (\tau\sigma)l_4, \sigma^2 l_4, (\tau\sigma^2)l_5, \sigma l_5, \tau l_6, (\tau\sigma)l_6, \sigma^2 l_6$

Using this table, we can find a set $\{\ell_i\}_{i=1}^6$ of six skew lines as follows (equivalently, six lines generating the subgroup of the Picard group generated by all 27 lines). Finding four skew lines is not difficult: first off, ℓ_1 can be any line, then ℓ_2 can be any line skew to ℓ_1 (there are 16 of these), ℓ_3 can be any line skew to ℓ_1 and ℓ_1; for ℓ_4 still six choices are left. Finally, there are now three lines left that are skew to $\ell_1, \ldots, \ell_4$, of which one intersects both of the others: this is ℓ_{56}, which makes the other two ℓ_5 and ℓ_6.

Following this procedure with $\ell_1 := l_4$, we select $\ell_2 := l_2$, $\ell_3 := \sigma l_3$, $\ell_4 := (\tau\sigma^2)l_4$. The three lines skew to these four are τl_5, σl_5 and l_6. The first one intersects the second and third, so we end up with $\ell_5 := \sigma l_5$ and $\ell_6 := l_6$. We can now write all the other lines in terms of the $\{\ell_i\}_{i=1}^6$:

ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5	ℓ_6	ℓ_{12}	ℓ_{13}	ℓ_{14}
l_4	l_2	σl_3	$(\tau\sigma^2)l_4$	σl_5	l_6	l_1	$\sigma^2 l_3$	$(\tau\sigma^2)l_5$
ℓ_{15}	ℓ_{16}	ℓ_{23}	ℓ_{24}	ℓ_{25}	ℓ_{26}	ℓ_{34}	ℓ_{35}	ℓ_{36}
σl_4	$(\tau\sigma)l_6$	$(\tau\sigma)l_5$	τl_4	$(\tau\sigma^2)l_6$	$\sigma^2 l_2$	τl_6	$\sigma^2 l_5$	σl_1
ℓ_{45}	ℓ_{46}	ℓ_{56}	ℓ'_1	ℓ'_2	ℓ'_3	ℓ'_4	ℓ'_5	ℓ'_6
$\sigma^2\ell_1$	$\sigma^2 l_6$	τl_5	l_3	$(\tau\sigma)l_4$	l_5	σl_6	$\sigma^2\ell_4$	σl_2

The above table was compiled in a routine manner: we go through the complete list of lines, and for each line we check which of the $\{\ell_i\}_{i=1}^6$ are intersected by it.

6 Some birationally non-trivial cubic surfaces

To get some perspective on how large the set of birationally non-trivial surfaces is, we will take a closer look at them in this chapter. We will begin by constructing some examples, and then prove a general result on the possible cardinalities of Galois orbits of lines on S.

6.1 The general cubic surface containing a set of lines

We have previously established that for $n \in \{7, 11, 13, 14, 17, 19, 20, 21, 22, 23, 25, 26\}$, an n-orbit of lines on a smooth cubic surface does not exist. For all other n, a cubic surface S with an n-orbit can be constructed. I will now outline my approach to doing this.

Fix a set L' of lines in $\mathbb{P}^3$. (Usually, the lines in L' are chosen so as to have some specific Galois action.) Requiring S to contain the lines in L' means imposing a set of linear conditions on the coefficients of its defining polynomial F. Therefore, the polynomials F that we are looking for form a vector space. We shall denote this vector space by $V_{L'}$. Carrying out the ideas of the previous section, the lines in L' give rise to stable sets of lines on S whose cardinalities can be determined. The precise way in which this is done depends a little on L', but it is pretty straightforward. Next, we choose $F \in V_{L'}$ and let S be the corresponding cubic surface. If we are lucky, S is smooth, and the mentioned stable sets of lines on S are actually orbits. It appears that if we choose F in a "sufficiently general" way, this is indeed the case, but this is not something that I shall attempt to prove.

Remark 6.1. As is clear from the above discussion, the cubic surface resulting from our little procedure is not guaranteed to contain an orbit of the desired order. (It is not guaranteed to be smooth, either.) We really have to check that our S has the wanted properties.

This section I will mainly spend giving examples of my procedure, producing smooth cubic surfaces with orbits of 8, 12, 16, 18 and 24 lines. But first, its most obvious application is the one where L' is the empty set:

Example 6.2. If L' is the empty set, we do not require anything of S. Then $V_{L'}$ is just the vector space of homogeneous third-degree polynomials in x, y, z, w. We fix on the following polynomial, which certainly looks sufficiently general: $F_{27} = yzw - 2x^3 + 3xyz - 4xyw + 5xzw - 6x^2y + 7x^2z - 8x^2w + 9xy^2 - 10xz^2 + 11xw^2 - 12y^2z + 13y^2w - 14yz^2 + 15yw^2 - 16z^2w + 17zw^2 - 18y^3 + 19z^3 - 20w^3$. The corresponding cubic surface S_{27} is smooth. I claim furthermore that its 27 lines are contained in a single Galois orbit. This can be checked by the algorithm described in Section 5.1.

Example 6.3. Let L' be a set of 3 coplanar lines forming a Galois orbit. If S is a smooth cubic surface containing L', then the remaining lines cannot be distinguished by their intersection properties relative to L': all 24 lines intersect exactly one line in L'. This suggests that a "general cubic surface" containing L' has an orbit of 24 lines.

To show this, we first have to find a suitable set L'. For this, let $\eta, \omega \in \overline{\mathbb{Q}}$ be algebraic integers satisfying $\eta^3 - 2 = 0$ and $\omega^2 + \omega + 1 = 0$. We then define lines ℓ_1, ℓ_2, ℓ_3 as $\ell_1 := (1 : t : t : \eta)$, $\ell_2 := (1 : t : t : \eta\omega)$ and $\ell_3 := (1 : t : t : \eta\omega^2)$. These lines make up a full set of conjugates under $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and are all contained in the plane $y - z = 0$, so they indeed form a coplanar 3-orbit.

Now, consider the general cubic polynomial $F := a_0x^3 + a_1x^2y + a_2x^2z + \ldots + a_{19}w^3$. To require that the surface given by $F = 0$ contains ℓ_1, for example, we substitute the parametric equation of ℓ_1 into F. We then get 4 equations in the coefficients a_i. We do the same for ℓ_2 and ℓ_3, so that the ensuing system of linear conditions is $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant, and we can expect a solution over $\mathbb{Q}$. Letting MAPLE do all the hard work, we get the following solution: $a_0x^3 - a_4yx^2 + (-a_5 - a_7)y^2x + (-a_9 - a_8 - a_6)y^3 + a_4zx^2 + a_5zyx + a_6zy^2 + a_7z^2x + a_8z^2y + a_9z^3 - a_{13}wyx + (-a_{14} - a_{15})wy^2 + a_{13}wzx + a_{14}wzy + a_{15}wz^2 - a_{18}w^2y + a_{18}w^2z - \frac{1}{2}a_0w^3$. Choosing some wildly arbitrary values for the a_i, we land on the polynomial $F_{24} = 222x^3 - 1110yx^2 - 3108y^2x - 5774y^3 + 1110zx^2 + 1332zyx + 1554zy^2 + 1776z^2x + 1998z^2y + 2222z^3 - 11110wyx - 28886wy^2 + 11110wzx + 13332wzy + 15554wz^2 - 22222w^2y + 22222w^2z - 111w^3$. This gives a smooth cubic surface containing a 24-orbit.

We thus find that a "general" cubic surface S containing a coplanar 3-orbit contains a large orbit. After Proposition 4.6, where a 3-orbit of pairwise skew lines breaks the remaining lines up into 5 stable sets, this could come as somewhat of a surprise. Nevertheless, it could put us in mind of the following idea. We could suppose that, perhaps, *two* coplanar 3-orbits could produce another large orbit. We investigate this next.

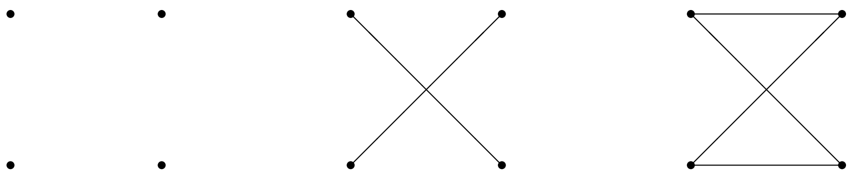
Example 6.4. Let L' be a set of 3 coplanar lines forming a Galois orbit and let L'' be another such set. (Observe that not all combinations of L' and L'' are allowed. In the previous example, let S contain L', then all remaining lines intersect one of the lines in L'. So if the lines in L'' do not intersect lines in L', there is no smooth cubic surface containing the lines in L' and L''.) If every line in L' intersects exactly one line in L'', however, the construction does work (see Proposition 6.9). In this case, consider the remaining 21 lines on a cubic surface containing the lines in L' and L''. These 21 lines all intersect exactly two lines in $L' \cup L''$: one line in L' and one in L''. However, looking closer, 3 of the 21 lines intersect two lines that intersect each other (forming tritangent planes), while the other 18 intersect two lines that are skew.

This time fix elements η, ω satisfying $2\eta^3 + 1 = 0$ and $\omega^2 - \omega + 1 = 0$. For the lines L', we pick $(1 : t : -t : \eta)$ and its conjugates $(1 : t : -t : -\eta\omega)$ and $(1 : t : -t : -\eta\omega^2)$. For L'', take $(1 : t : -1 : \eta t)$ and its conjugates $(1 : t : -1 : -\eta\omega t)$ and $(1 : t : -1 : -\eta\omega^2 t)$. The sets L' and L'' are admissible in the sense that there is a cubic surface containing all their lines. (I did not pull these sets out of thin air: they lie on the twisted Fermat cubic surface described in Section 5.3. This surface does not contain an 18-orbit, however.)

Then the general cubic surface containing the lines of L' and L'' is given by the third-degree polynomial $a_3x^3 + a_3y^3 + (2a_{11} - 2a_{16} + 2a_{10} + a_3)z^3 + 2a_3w^3 + 2(-a_{11} + a_{16})x^2y + 2(-a_{11} + a_{16})x^2z + 2(a_{16} - a_{10})y^2x + 2(a_{16} - a_{10})y^2z + 2a_{10}z^2x + 2a_{11}z^2y + 2a_{19}z^2w + 2a_{16}xyz + 2a_{19}xyw + 2a_{19}xzw + 2a_{19}wyz$.

Let $F_{18} := 51x^3 + 51y^3 + 2z^3 + 102w^3 - 170x^2y - 170x^2z + 238y^2x + 238y^2z - 219z^2x + 189z^2y - 374z^2w + 19xyz - 374xyw - 374xzw - 374wyz$. Then F_{18} determines a smooth cubic surface containing an 18-orbit.

Example 6.5. For an orbit of 12 lines, we have to find a new approach. In Manin's book ([5, §31, pp. 176-177]), we find the description of an element $\sigma \in W(\mathrm{E}_6)$ for which the action of the group $\langle\sigma\rangle$ on the 27 lines has 5 orbits of orders 1, 4, 4, 6 and 12. This suggests that L' should be a stable set of 1 or 4 lines. A rational line determines stable sets of 1, 10 and 16 lines each, as can be easily checked by now, so that doesn't work. This suggests that we take L' to be a Galois stable set of four lines. There are still some possibilities here, as can be seen in the figure below (the vertices of the graphs represent the lines; a pair of intersecting lines is joined by an edge).



The rightmost one turns out to be the one we are looking for; indeed, it divides the 27 lines into 5 stable sets of 1, 4, 4, 6 and 12 lines each. So now, we have to construct a Galois orbit of four lines with these intersection properties. First, let ζ be a 5^{th} root of unity. We will work over $\mathbb{Q}(\zeta)$ and let $\sigma \in \mathrm{Aut}_{\mathbb{Q}}\mathbb{Q}(\zeta)$ be the automorphism sending $\zeta \mapsto \zeta^2$. Now, take any plane plane H whose minimal field of definition is $\mathbb{Q}(\zeta)$, or equivalently, which has an orbit of order 4 under the action of $\langle\sigma\rangle$. Then let $\ell := H \cap \sigma H$. Then ℓ and $\sigma\ell$ both lie in σH, so they intersect. But ℓ and $\sigma^2\ell$ do not necessarily intersect, so for an arbitrary H satisfying our previous conditions, we expect ℓ and $\sigma^2\ell$ to be skew. (If H is given by the linear form G, an equivalent condition for this is that the 4×4-matrix, whose ith row consist of the coefficients of σ^iG, is invertible. This is the case if and only if the coefficients of G form a basis for the $\mathbb{Q}$-vector space $\mathbb{Q}(\zeta)$.) We choose H to be given by the linear form $G := x + \zeta y + \zeta^2 z + \zeta^3 w$. Next, we set $\ell_1 := H \cap \sigma H$, $\ell_2 := \sigma H \cap \sigma^2 H$, $\ell_3 := \sigma^2 H \cap \sigma^3 H$ and $\ell_4 := H \cap \sigma^3 H$. These have the required intersection properties.

Again, we take F to be the general third-degree polynomial and substitute parametric equations of the ℓ_i into F to derive the linear conditions on its coefficients. The result is the following expression, whose main attraction is its sheer length: $(-2a_{18} - 2a_{19} + 2a_{16} + 6a_{10} - 2a_9 + 2a_8)x^3 + (2a_{19} + 2a_{18} + 2a_9 + 4a_8 + 12a_1 + 4a_{16})yx^2 + (a_{18} - 4a_{16} - 3a_{10} + a_9 + 3a_{17} + 2a_8 - 2a_{19})zx^2 + (-a_{18} - 3a_{17} - 8a_{16} - 9a_{10} + 5a_9 - 8a_8 - 6a_1 + 2a_{19})wx^2 + (-3a_{18} - 3a_{17} + 3a_{10} - 6a_1 - 3a_9)xy^2 + 6a_{16}xyz + 6a_{17}xyw + 6a_{10}xz^2 + 6a_{18}xzw + (6a_{10} - 6a_9 - 6a_1 - 6a_{19})xw^2 + 6a_1y^3 + 6a_9y^2w + 6a_8y^2z + (-3a_{18} - 6a_8 - 3a_{17} - 3a_{10} - 3a_9)yz^2 + 6a_{19}yzw + (2a_{18} - 2a_{16} - 4a_9 - 4a_{19} - 2a_8)yw^2 + (-2a_{19} + 6a_1 + a_9 + 3a_{10} + 2a_{16} + 3a_{17} + a_{18} + 8a_8)z^3 + (-8a_8 - 2a_{16} + 2a_9 - 4a_{18} - 6a_1 - 6a_{10} + 2a_{19} - 6a_{17})wz^2 + (6a_{10} + 12a_8 + 6a_{17} + 6a_{16} + 12a_1)zw^2 + (-a_{18} + 2a_{19} - 2a_{16} - 2a_8 + 5a_9 - 3a_{10} - 3a_{17})w^3$.

We pick $F_{12} := 15x^3 - 11y^3 + 10z^3 - 11w^3 - 20x^2y + 4x^2z - 17x^2w + 12y^2x + 7y^2z + 3y^2w + 12z^2x - 18z^2y - 18z^2w + 21w^2x - 5w^2y + 15w^2z - 2xyz + 13xyw - 6xzw - wyz$, which looks a lot more civilized. This gives a smooth cubic surface with orbits of 1, 4, 4, 6 and 12 lines, as desired.

Example 6.6. In passing, we remarked that a rational line on a cubic surface induces a stable set of order 16. Let L' consist of just one rational line ℓ, say $x = y = 0$. The cubic polynomials giving rise to surfaces containing ℓ are just the cubic polynomials composed of monomials containing at least one factor x or y. If we choose $F_{16} := x^3 - 2x^2y + 3x^2z - 4x^2w + 5xyz - 6xyw + 7xzw - 8xy^2 + 9xz^2 - 10xw^2 + 11y^3 - 12y^2z + 13y^2w - 14yz^2 + 15yzw - 16yw^2$, the result is a smooth cubic surface containing a 16-orbit.

The number we haven't yet encountered as the cardinality of a stable set is 8. This can be obtained in a number of ways, the following probably being the easiest.

Example 6.7. Let L' be a set consisting of 3 coplanar rational lines. For instance, take ℓ_1 given by $x = y = 0$, ℓ_2 given by $x = z = 0$ and ℓ_3 given by $x = w = 0$. These all lie in the plane $x = 0$. Furthermore, it is easy to describe the vector space of general cubic polynomials belonging to this problem: it is generated by cubic monomials either containing x as a factor, or being equal to yzw. So the general cubic polynomial we want is $a_0x^3 + a_1x^2y + a_2x^2z + a_3x^2w + a_4xyz + a_5xyw + a_6xzw + a_7xy^2 + a_8xz^2 + a_9xw^2 + a_{10}yzw$. If we just choose $F_8 := x^3 - 2x^2y + 3x^2z - 4x^2w + 5xyz - 6xyw + 7xzw - 8xy^2 + 9xz^2 - 10xw^2 + 11yzw$, the result is a smooth cubic surface containing three 8-orbits.

From all the examples that we have seen in this thesis, together with Theorems 2.28 and 4.9, we can now infer the following:

Theorem 6.8. There is smooth cubic surface containing an orbit of n lines if and only if $n \in \{1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 16, 18, 24, 27\}$.

Proof. The negative part has previously been established. For $n \leq 6$, we can take a blow-up in 6 points in $\mathbb{P}^2$, which can have any Galois action of our choosing. For $n = 8, 12, 16, 18, 24, 27$ we just encountered examples. For $n = 9, 10, 15$, we refer to Section 4.3.1, Section 4.3.2 and Section 4.2 respectively. (Although for $n = 9$ or 15 we could also take a blow-up of $\mathbb{P}^2$ over $\mathbb{Q}$ in a 6-orbit of points: see the table under Remark 3.7.) □

6.1.1 A further way of constructing cubic surfaces

As mentioned in Example 6.4, we can also prove the existence of a surface that contains "two prescribed tritangent" planes, that is: six prescribed lines partitioned into two triples, such that (1) the lines making up a triple all lie in one plane, and (2) each of the six lines intersects exactly one line of the other triple (this condition is always satisfied by six lines lying in two tritangent planes of a cubic surface).

Proposition 6.9. Let $l_1, l_2, l_3, l_1', l_2', l_3'$ be six lines with the following intersection properties: l_1, l_2, l_3 lie in the same plane; l_1', l_2', l_3' lie in the same plane; l_1 intersects l_1'; l_2 intersects l_2' and l_3 intersects l_3'. Then there is a cubic surface containing $l_1, l_2, l_3, l_1', l_2', l_3'$.

Proof. Suppose the cubic surface is given by $F = 0$. After a possible change of coordinates, we may assume that the tritangent planes are $X = 0$, $Y = 0$, that is, they are given by $F(0, Y, Z, W) = 0$ and $F(X, 0, Z, W) = 0$. Then $F(0, Y, Z, W) = l_1 l_2 l_3$ where the l_i are linear polynomials of the form $a_i Y + b_i Z + c_i W$. Similarly we write $F(X, 0, Z, W) = l'_1 l'_2 l'_3$ and $l'_i = \alpha_i X + \beta_i Z + \gamma_i W$. We know that l_1 intersects l'_1, and l_1 is given by $X = 0$, $a_1 Y + b_1 Z + c_1 W = 0$; l'_1 is given by $Y = 0$, $\alpha_1 X + \beta_1 Z + \gamma_1 W$. So:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_1 & b_1 & c_1 \\ 0 & 1 & 0 & 0 \\ \alpha_1 & 0 & \beta_1 & \gamma_1 \end{pmatrix}$$

is singular. By some elementary row operations we find that, also,

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & b_1 & c_1 \\ 0 & 0 & \beta_1 & \gamma_1 \end{pmatrix}$$

is singular, or equivalently, $\det \begin{pmatrix} b_1 & c_1 \\ \beta_1 & \gamma_1 \end{pmatrix} = 0$. Similarly we derive $\det \begin{pmatrix} b_2 & c_2 \\ \beta_2 & \gamma_2 \end{pmatrix} = \det \begin{pmatrix} b_3 & c_3 \\ \beta_3 & \gamma_3 \end{pmatrix} = 0$.

Now take the product $l_1 l_2 l_3$, which is a polynomial in Y, Z, W, and consider the sum of the terms not containing Y, these are given by $(b_1 Z + c_1 W)(b_2 Z + c_2 W)(b_3 Z + c_3 W)$. Similarly, the sum of the terms in the product $l'_1 l'_2 l'_3$ not containing X is $(\beta_1 Z + \gamma_1 W)(\beta_2 Z + \gamma_2 W)(\beta_3 Z + \gamma_3 W)$. The three conditions on the $\det \begin{pmatrix} b_i & c_i \\ \beta_i & \gamma_i \end{pmatrix} = 0$ imply that each $(b_i Z + c_i W)$ is a multiple of $(\beta_i Z + \gamma_i W)$, so the product $(b_1 Z + c_1 W)(b_2 Z + c_2 W)(b_3 Z + c_3 W)$ is a multiple of the product $(\beta_1 Z + \gamma_1 W)(\beta_2 Z + \gamma_2 W)(\beta_3 Z + \gamma_3 W)$. After possibly dividing some l_i by a constant (which does not alter the line that is determined by it), we get that $l_1 l_2 l_3 - l'_1 l'_2 l'_3$ has no terms in Z and W alone.

In order for a cubic surface given by $F = 0$ to contain tritangent planes given by $F(0, Y, Z, W) = l_1 l_2 l_3$, $F(X, 0, Z, W) = l'_1 l'_2 l'_3$, we must have $F = X q_1 + l_1 l_2 l_3 = Y q_2 + l'_1 l'_2 l'_3$, where q_1 and q_2 are quadratic polynomials. This is equivalent with the existence of quadratic polynomials q_1, q_2 satisfying $-X q_1 + Y q_2 = l_1 l_2 l_3 - l'_1 l'_2 l'_3$, which is equivalent with $l_1 l_2 l_3 - l'_1 l'_2 l'_3$ not containing any terms containing only constants, Z and W. $\square$

Remark 6.10. We are even left with some freedom: we can add to q_1 an arbitrary polynomial of the form $Y(aX + bY + cZ + dW)$.

6.2 A cubic surface without rational points

In this section, I will describe my unsuccesful attempt to construct a cubic surface with a stable set of 6 skew lines, *but without a $\mathbb{Q}$-rational point.* It is expected that a similar approach should work, but with a different choice of group instead of A_4.

To carry out the construction, I follow a suggestion from Manin's book ([5, Ch. 4, §31, pp. 191-2]). The following definition provides us with the key concept of a Severi-Brauer surface:

Definition 6.11. A *Severi-Brauer surface* over $\mathbb{Q}$ is a surface which is isomorphic to $\mathbb{P}^2$ over $\overline{\mathbb{Q}}$.

The following proposition is a well-known fact about Severi-Brauer surfaces:

Proposition 6.12. Let X be a Severi-Brauer surface. If X has a point over $\mathbb{Q}$, then X is isomorphic to $\mathbb{P}^2$ over $\mathbb{Q}$.

We will call such a Severi-Brauer surface *trivial*.

To obtain a smooth cubic surface without rational points, but with a stable set of 6 pairwise skew lines, we have to take a non-trivial Severi-Brauer surface X and blow this up in a Galois stable set of 6 points. (Over $\overline{\mathbb{Q}}$, this is just a blow-up of $\mathbb{P}^2$.) In this section, we will use some results about Severi-Brauer surfaces, for instance found in [3].

General facts about twists of $\mathbb{P}^2$

Fix the coordinates x, y, z on $\mathbb{P}^2$. Consider the Veronese embedding $\phi : \mathbb{P}^2 \to \mathbb{P}^9$ given by $(x : y : z) \mapsto (x^3 : x^2y : x^2z : xy^2 : xyz : xz^2 : y^3 : y^2z : yz^2 : z^3)$. Denote the image of $\mathbb{P}^2$ in $\mathbb{P}^9$ by P (plane) and the coordinates on P by $u_0, \ldots, u_9$. It is known ([3]) that any Severi-Brauer surface P_{twist} can be embedded in $\mathbb{P}^9$ in such a way that P and P_{twist} are projectively isomorphic, that is, there exists $M \in \mathrm{PGL}_{10}(\overline{\mathbb{Q}})$ such that $(u_0 : \ldots : u_9) \mapsto M \cdot (u_0 : \ldots : u_9)$ is an isomorphism which we will also denote by $M : P \xrightarrow{\sim} P_{\text{twist}}$ for convenience. (Obviously, if P_{twist} is a non-trivial twist, M can't be an element of $\mathrm{PGL}_{10}(\mathbb{Q})$.) So we may without loss of generality restrict the definition of a Severi-Brauer surface to subsets of $\mathbb{P}^9$ that are projectively isomorphic (over $\overline{\mathbb{Q}}$) to P.

Blowing up a Severi-Brauer surface

We now give an explicit description of a blow-up of a Severi-Brauer surface P_{twist}. We write the coordinates on P_{twist} as $v_0, \ldots, v_9$. Via the above discussion, we have a map $\psi : \mathbb{P}^2 \to P_{\text{twist}}$ defined by $\psi := M \circ \phi$ (this is just the Veronese embedding composed with a projective transformation), which is not defined over $\mathbb{Q}$ if P_{twist} is a non-trivial twist.

According to Clebsch's result, to realise the blow-up of $\mathbb{P}^2$ as a cubic surface in $\mathbb{P}^3$, we have to use functions in the vector space of cubic forms in x, y, z. Now ψ identifies this vector space with the space of *linear* forms on P_{twist}. So to construct a blow-up of P_{twist} in the points p_i', we need to look for forms $\sum_i a_i v_i$ vanishing in the p_i'. Using the map ψ, we can pull these functions back to cubic polynomials in x, y, z that vanish in the six points $\psi^{-1}(p_i')$. Of course, in general, these cubic polynomials are not defined over $\mathbb{Q}$ anymore: in this way, the rational map $\pi : P_{\text{twist}} \to \mathbb{P}^3$ is defined over $\mathbb{Q}$, but not the pull-back $\pi^* : \mathbb{P}^2 \to \mathbb{P}^3$.

An embedding $A_4 \hookrightarrow \mathbf{PGL}_3(\mathbb{Q})$

To define the needed twist, we need a suitable finite subgroup of $\mathrm{PGL}_3(\overline{\mathbb{Q}})$. We start from the standard representation of A_4 on $V := \{(x_1, x_2, x_3, x_4) \in \mathbb{Q}^4 : x_1 + x_2 + x_3 + x_4 = 0\}$. An element $\pi \in A_4$ acts on $\mathbb{Q}^4$ by $\pi\,(x_1, x_2, x_3, x_4) = \big(x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)}, x_{\pi(4)}\big)$. Furthermore, the action of A_4 leaves V invariant, so we get an embedding $A_4 \hookrightarrow \mathrm{GL}_3(\mathbb{Q})$ which under the quotient map $\mathrm{GL}_3(\mathbb{Q}) \to \mathrm{PGL}_3(\mathbb{Q})$ gives an embedding $i : A_4 \hookrightarrow \mathrm{PGL}_3(\mathbb{Q})$.

Here follows an explicit description of the embedding i.

element of A_4	image in $\mathrm{PGL}_3(\mathbb{Q})$	element of A_4	image in $\mathrm{PGL}_3(\mathbb{Q})$
id	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	(234)	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{pmatrix}$
(12)(34)	$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & -1 & -1 \end{pmatrix}$	(243)	$\begin{pmatrix} 1 & 0 & 0 \\ -1 & -1 & -1 \\ 0 & 1 & 0 \end{pmatrix}$
(13)(24)	$\begin{pmatrix} 0 & 0 & 1 \\ -1 & -1 & -1 \\ 1 & 0 & 0 \end{pmatrix}$	(124)	$\begin{pmatrix} 0 & 1 & 0 \\ -1 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$
(14)(23)	$\begin{pmatrix} -1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	(142)	$\begin{pmatrix} -1 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
(123)	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$	(134)	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & -1 & -1 \end{pmatrix}$
(132)	$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$	(143)	$\begin{pmatrix} -1 & -1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

Let G denote the image of A_4 in $\mathrm{PGL}_3(\mathbb{Q})$.

Constructing $P_{\mathbf{twist}}$

We will use some general facts about twists of algebraic varieties to make a twist of $\mathbb{P}^2$. For this, we identify $\mathbb{P}^2$ with P. The action of G on $\mathbb{P}^2$ induces an action of G on P. We will therefore understand G to be a subgroup of $\mathrm{Aut}(P)$ as well.

$G \subset \mathrm{Aut}(P)$ is a subgroup isomorphic to A_4. Let $t_1, t_2 \in \overline{\mathbb{Q}}$ be two arbitrary zeros of the polynomial $f = t^4 + 4t^3 + 12t^2 + 24t + 24$, which has Galois group A_4. Then the remaining two roots are: $t_3 := -\frac{1}{3}t_2^2t_1 + \frac{2}{3}t_2t_1 + \frac{1}{6}t_2t_1^2 - \frac{1}{12}t_1^3t_2^2 - \frac{1}{2}t_2^2 - t_1 - \frac{1}{6}t_1^2t_2^2 - 4 - \frac{1}{6}t_1^2 - \frac{1}{6}t_1^3$, $t_4 := \frac{1}{2}t_2^2 + \frac{1}{12}t_1^3t_2^2 + \frac{1}{6}t_1^2 + \frac{1}{6}t_1^3 + \frac{1}{6}t_1^2t_2^2 - t_2 + \frac{1}{3}t_2^2t_1 - \frac{2}{3}t_2t_1 - \frac{1}{6}t_2t_1^2$. So $\mathrm{Gal}(\mathbb{Q}(t_1, t_2)/\mathbb{Q})$ is the splitting field of the polynomial f.

Consider the group isomorphism $c : \mathrm{Gal}(\mathbb{Q}(t_1, t_2)/\mathbb{Q}) \to G$ arising from the fact that each element $\sigma \in \mathrm{Gal}(\mathbb{Q}(t_1, t_2)/\mathbb{Q})$ is an even permutation of the 4-tuple (t_1, t_2, t_3, t_4).

Since the Galois action on the elements of G is trivial, the map c is actually a 1-cocycle determining an element $[c] \in H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathrm{Aut}(P))$. We now define an action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\overline{\mathbb{Q}}[\{u_i\}] = \mathbb{Q}[\{u_i\}] \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$, the homogeneous coordinate ring of $\mathbb{P}^9$, as follows: let $\widetilde{\sigma} \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and let σ be the element of $\mathrm{Gal}(\mathbb{Q}(t_1, t_2)/\mathbb{Q})$ obtained by restriction. Then define the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ as $\widetilde{\sigma}(f \otimes \alpha) := c(\sigma)(f) \otimes \widetilde{\sigma}(\alpha)$.

Now according to the theory, we may obtain a full set of invariants in $\overline{\mathbb{Q}}[\{u_i\}]$ under the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. We will produce invariants by just adding to an element of $\overline{\mathbb{Q}}[\{u_i\}]$ its conjugates under the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$:

$$v_0 := \sum_{\sigma \in \mathrm{Gal}(\mathbb{Q}(t_1,t_2)/\mathbb{Q})} c(\sigma)(u_0)$$

$$v_1 := \sum_{\sigma \in \mathrm{Gal}(\mathbb{Q}(t_1,t_2)/\mathbb{Q})} \sigma(t_1) \otimes c(\sigma)u_0$$

$$v_2 := \sum_{\sigma \in \mathrm{Gal}(\mathbb{Q}(t_1,t_2)/\mathbb{Q})} \sigma(t_1^2) \otimes c(\sigma)u_0$$

$$v_3 := \sum_{\sigma \in \mathrm{Gal}(\mathbb{Q}(t_1,t_2)/\mathbb{Q})} \sigma(t_1^3) \otimes c(\sigma)u_0$$

$$v_4 := \sum_{\sigma \in \mathrm{Gal}(\mathbb{Q}(t_1,t_2)/\mathbb{Q})} \sigma(t_2) \otimes c(\sigma)u_1$$

$$v_5 := \sum_{\sigma \in \mathrm{Gal}(\mathbb{Q}(t_1,t_2)/\mathbb{Q})} \sigma(t_1 t_2) \otimes c(\sigma)u_1$$

$$v_6 := \sum_{\sigma \in \mathrm{Gal}(\mathbb{Q}(t_1,t_2)/\mathbb{Q})} \sigma(t_1^2 t_2) \otimes c(\sigma)u_1$$

$$v_7 := \sum_{\sigma \in \mathrm{Gal}(\mathbb{Q}(t_1,t_2)/\mathbb{Q})} \sigma(t_1^3 t_2) \otimes c(\sigma)u_1$$

$$v_8 := \sum_{\sigma \in \mathrm{Gal}(\mathbb{Q}(t_1,t_2)/\mathbb{Q})} \sigma(t_1 t_2^2) \otimes c(\sigma)u_1$$

$$v_9 := \sum_{\sigma \in \mathrm{Gal}(\mathbb{Q}(t_1,t_2)/\mathbb{Q})} \sigma(t_1^3 t_2^2) \otimes c(\sigma)u_1$$

One can check that the above linear transformation is invertible with determinant $2^{26}3^{13}$, so that the above equations determine an element of $\mathrm{PGL}_{10}(\mathbb{Q}(t_1, t_2))$. This means that we have obtained a birational map $M_0 : P \to P_{\mathrm{twist}}$ (actually its inverse), defined over $\mathbb{Q}(t_1, t_2)$, given by the above equations.

We have now arrived at our Severi-Brauer surface P_{twist}: just take the image of $\mathbb{P}^2$ under the composition $M_0 \circ \phi$. We will not bother about the defining equations of P_{twist}, the birational map will actually be all that we need.

The result

To check that the image W of M_0 is actually a trivial Severi-Brauer surface, we can just blow W up in six points, without bothering about the points being in general position: as long as we have a $\mathbb{Q}$-birational map to a cubic surface S' in $\mathbb{P}^3$, we can check for the presence or absence of $\mathbb{Q}$-rational points on W by considering S' instead.

Starting from the points $p_1 := (1:0:0)$ and its conjugates under G, so $p_2 := (0:1:0)$, $p_3 := (0:0:1)$, $p_4 := (1:-1:0)$, $p_5 := (1:0:-1)$, $p_6 := (0:1:-1)$, we determine the linear forms vanishing on the points $(M_0 \circ \phi)(p_i)$ on W. This gives four linear forms $L_1 := v_6 + v_8, L_2 := v_1 - 3v_4, L_3 := v_0, L_4 := v_5$, which is definitely what we want.

Via ψ we can pull back the forms L_i to cubic forms F_i on $\mathbb{P}^2$. As we already noted, these are no longer defined over $\mathbb{Q}$, so we will not give them explicitly. From here, we just have to substitute the $X = F_1, Y = F_2, Z = F_3, W = F_4$ in the general equation of a cubic surface, that is in $a_0X^3 + a_1X^2Y + \ldots + a_{19}W^3$, and solve for the a_i. We get 55 equations in 20 unknowns with coefficients in $\mathbb{Q}(t_1, t_2)$, but MAPLE turns up with a solution anyway. It is

$$-3X^3 + 12XY^2 + 8Y^3 - 30X^2Z + 48Y^2Z - 96XZ^2 - 128Z^3 - 18X^2W - 36XYW - 24Y^2W - 96XZW - 192YZW - 192Z^2W + 96ZW^2 + 24W^3$$

The cubic surface S' corresponding to the above cubic form is non-singular, and a verification using the line finding algorithm reveals that it indeed contains a Galois orbit of 6 lines, none of which intersects another. However, considering the singular primes $p = 2$ and $p = 3$ shows that all points over $\mathbb{F}_p$ can be lifted to $\mathbb{Q}_p$, so the Hasse principle, which holds for these surfaces, tells us that S' has a rational point.

Comments and suggestions

I will now say something about the problems that occur when trying the above kind of approach. This will make it clearer why I chose to embed A_4 in $\mathrm{GL}_3(\mathbb{Q})$, and not some other group. The first thing to note is: a subgroup H of $\mathrm{GL}_3(\mathbb{Q})$ with rational eigenvectors is no good, because the image of H in $\mathrm{PGL}_3(\mathbb{Q})$ leaves at least one point of $\mathbb{P}^2(\mathbb{Q})$ fixed. This means that the image of $\mathbb{P}^2$ under M_0 would contain a rational point, which we don't want.

This already rules out some of the simplest choices of groups. For instance, any cyclic group H has an eigenvector, and for small order cyclic groups this is always a rational eigenvector. The irreducible representations of S_3 have dimensions 1, 1 and 2, so the image of any embedding $S_3 \to \mathrm{GL}_3(\mathbb{Q})$ has a 1-dimensional invariant subspace containing a rational eigenvector.

This means that we are forced to look for finite groups of higher order. As we saw, our choice of embedding for A_4 didn't work either. Perhaps a different embedding would work. The following question, too, is worth considering:

Question 6.13. Fix a cocycle $c : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{PGL}_3(\overline{\mathbb{Q}})$, factoring over a number field K, and let ϕ be the associated rational map to P_{twist}. Is there a systematic way to look for rational points on P_{twist}?

This can be answered just by looking at c and points of $\mathbb{P}^2(K)$, for the usual action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on P_{twist} induces a "twisted" kind of Galois action on $\mathbb{P}^2$. Take a basis $1, \xi_1, \ldots, \xi_m$ for K, then we can write a point of $\mathbb{P}^2(K)$ as $(\sum a_i\xi_i : \sum b_i\xi_i : \sum c_i\xi_i)$. If we require this point to be stable under an element $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/K)$, what kind of equations does that give us? This doesn't seem hard to find out, but I haven't done it yet.

7 Summary

My thesis concerns smooth cubic surfaces S over a number field K for which there is a birational map $f : S \to \mathbb{P}^2$ where f is defined over K. There is no material difficulty involved in finding out whether or not an arbitrary S admits such a K-birational map to $\mathbb{P}^2$. To construct such an f in a purely geometric way, however, is not so easy. We have shown how it can be done for smooth cubic surfaces containing a Galois stable set of 2 skew lines, and for cubic surfaces containing a stable set of 6 skew lines, we know that there is a blow-down f defined over K. This leaves out the set of cubic surfaces containing a stable set of 3 skew lines, but no stable sets of 2 or 6 lines. This set we have shown to be non-empty.

Another focus point of this thesis are the different types of orbits and combinations of orbits that arise on cubic surfaces. It was shown here that 20-orbits of lines do not exist (over any ground field). Any other divisor ≤ 27 of 51,840 does occur as the cardinality of an orbit of lines on a cubic surface.

An interesting phenomenon is formed by cubic surfaces which satisfy Swinnerton-Dyer's criterion, except for the existence of a K-rational point. The last part of the thesis contains a possible approach to construct such surfaces.

7.1 Possibilities for further research

- It is possible (and to me seems probable) that the rational map for Type III surfaces constructed in subsection 3.2.3 could lead to a birational map in one way or the other. In subsection 3.2.3, I sketched many possible approaches, but it remains to be seen how far one can get with these.

- In a similar vein, this construction could work a surface S of Type IV or V: take a rational plane through a rational point $P \in S$ and consider the plane cubic $V \cap S$. This cubic is intersected by 6 pairwise skew lines (forming a stable set) in $X_1, \ldots, X_6$ and contains P. Take the plane cubic curve through $P, X_1, \ldots, X_6$, which intersects $V \cap S$ in X_7 and X_8 in addition to the points already mentioned. Then the line through X_7 and X_8 intersects $V \cap S$ in a rational point. **Question:** What kind of a map is this? Can we do anything that parallels our discussion for Type III surfaces?

- My construction of a Severi-Brauer surface, which could be used to obtain a non-trivial cubic surface with a Galois stable set of 6 pairwise skew lines, remains unfinished. Is it possible to "fix" my approach?

- For Type IV and V surfaces over $\mathbb{Q}$, we would like to construct an explicit blow-down morphism over $\mathbb{Q}$. The easiest case would be, given an explicit blow-up map, to find its inverse. This can be done by an general inversion procedure for birational maps, for instance found in [8]. However, we would like to have a more transparent approach that uses the specific geometric properties of the cubic surface.

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